

Renormalization of stochastic continuity equations on Riemannian manifolds[☆]

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Abstract

We consider the initial-value problem for stochastic continuity equations of the form

$$\partial_t \rho + \operatorname{div}_h \left[\rho \left(u(t, x) + \sum_{i=1}^N a_i(x) \circ \frac{dW^i}{dt} \right) \right] = 0,$$

defined on a smooth closed Riemannian manifold M with metric h , where the Sobolev regular velocity field u is perturbed by Gaussian noise terms $\dot{W}_i(t)$ driven by smooth spatially dependent vector fields $a_i(x)$ on M . Our main result is that weak (L^2) solutions are renormalized solutions, that is, if ρ is a weak solution, then the nonlinear composition $S(\rho)$ is a weak solution as well, for any “reasonable” function $S : \mathbb{R} \rightarrow \mathbb{R}$. The proof consists of a systematic procedure for regularizing tensor fields on a manifold, a convenient choice of atlas to simplify technical computations linked to the Christoffel symbols, and several DiPerna–Lions type commutators $\mathcal{C}_\varepsilon(\rho, D)$ between (first/second order) geometric differential operators D and the regularization device (ε is the scaling parameter). This work, which is related to the “Euclidean” result in Punshon–Smith (0000), reveals some structural effects that noise and nonlinear domains have on the dynamics of weak solutions.

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1. Introduction

For a number of years many researchers appended new effects and features to partial differential equations (PDEs) in fluid mechanics in order to better account for various physical phenomena. An interesting example arises when a hyperbolic PDE is posed on a curved manifold instead of a flat Euclidean domain, in which case the curvature of the domain makes nontrivial alterations to the solution dynamics [2,7,35]. Relevant applications include geophysical flows and general relativity. Another example is the rapid rise in the use of stochastic processes to extend the scope of hyperbolic PDEs (on Euclidean domains) in an attempt to achieve better understanding of turbulence. Randomness can enter the PDEs in different ways, such as through stochastic forcing or in uncertain system parameters (fluxes). Generally speaking, the mathematical literature for stochastic partial differential equations (SPDEs) on manifolds is at the moment in short supply [13,18,21,22]. In this paper we consider stochastic continuity equations with a non-regular velocity field that is perturbed by Gaussian noise terms powered by spatially dependent vector fields. In contrast to the existing literature, the main novelty is indeed that we pose these equations on a curved manifold, being specifically interested in the combined effect of noise and nonlinear domains on the dynamics of weak solutions.

Fix a d -dimensional ($d \geq 1$) smooth Riemannian manifold M , endowed with a metric h . We assume M to be compact, connected, oriented, and without boundary. We are interested in the initial-value problem for the stochastic continuity equation

$$d\rho + \operatorname{div}_h(\rho u) dt + \sum_{i=1}^N \operatorname{div}_h(\rho a_i) \circ dW^i(t) = 0 \quad \text{on } [0, T] \times M, \tag{1.1}$$

where W^1, \dots, W^N are independent Wiener processes, a_1, \dots, a_N are smooth vector fields on M (i.e., first order differential operators on M), the symbol \circ refers to the Stratonovich interpretation of stochastic integrals, $u : [0, T] \times M \rightarrow TM$ is a time-dependent $W^{1,2}$ vector field on M (a rough velocity field), div_h is the divergence operator linked to the manifold (M, h) , and $\rho = \rho(\omega, t, x)$ is the unknown (density of a mass distribution) that is sought up to a fixed final time $T > 0$. Eq. (1.1) is supplemented with initial data $\rho(0) = \rho_0 \in L^2$ on M .

In the deterministic case ($a_i \equiv 0, M = \mathbb{R}^d$), the well-posedness of weak solutions follows from the theory of renormalized solutions due to DiPerna and Lions [10]. A key step in this theory relies on showing that weak solutions are renormalized solutions, i.e., if ρ is a weak solution, then $S(\rho)$ is a weak solution as well, for any “reasonable” nonlinear function $S : \mathbb{R} \rightarrow \mathbb{R}$. The validity of this chain rule property depends on the regularity of the velocity field u . DiPerna and Lions proved it in the case that u is $W^{1,p}$ -regular in the spatial variable, while Ambrosio [1] proved it for BV velocity fields. An extension of the DiPerna–Lions theory to a class of Riemannian manifolds can be found in [14] (we will return to this paper below).

The well-posedness of stochastic transport/continuity equations with “Lipschitz” coefficients (defined on Euclidean domains) is classical in the literature and has been deeply analyzed in Kunita’s works [9,23]. In [3] the renormalization property is established for stochastic transport equations with irregular (BV) velocity field u and “constant” noise coefficients ($a_i \equiv 1$). Moreover, they proved that the renormalization property implies uniqueness without the usual L^∞ assumption on the divergence of u , thereby providing an example of the so-called “regularization by noise” phenomenon. In recent years “regularization by noise” has been a recurring theme in many papers on the analysis of stochastic transport/continuity equations, a significant part of it motivated by [16], see e.g. [6,11,15,17,27,29,30,36].

Recently [33,35] the renormalization property was established for stochastic continuity equations with spatially dependent noise coefficients, written in Itô form and defined on an Euclidean domain. In the one-dimensional case and without a “deterministic” drift term, the equations analyzed in [33] take the form

$$\partial_t \rho + \partial_x (\sigma \rho) \frac{dW(t)}{dt} = \partial_{xx}^2 \left(\frac{\sigma^2}{2} \rho \right), \quad (t, x) \in [0, T] \times \mathbb{R}, \tag{1.2}$$

where $\sigma = \sigma(x)$ is an irregular coefficient that belongs to $W_{loc}^{1, \frac{2p}{p-2}}$, while ρ is an L^p weak solution ($p \geq 2$). The derivation of the (renormalized) equation satisfied by $F(\rho)$, for any sufficiently smooth $F : \mathbb{R} \rightarrow \mathbb{R}$, is based on regularizing (in x) the weak solution ρ by convolution with a standard mollifier sequence $\{J_\varepsilon(x)\}_{\varepsilon>0}$, $\rho_\varepsilon := J_\varepsilon \star \rho$, using the Itô (temporal) and classical (spatial) chain rules to compute $F(\rho_\varepsilon)$, and deriving commutator estimates to control the regularization error. A key insight in [33], also needed in one of the steps in our renormalization proof for (1.1), is the identification of a “second order” commutator, which is crucial to conclude that the regularization error converges to zero, without having to assume some kind of “parabolic” regularity like $\sigma \partial_x \rho \in L^2$ — the nature of the SPDE (1.2) is hyperbolic not parabolic, so this regularity is not available (at variance with [24]). To be a bit more precise, the “second order” commutator in [33] takes the form

$$\begin{aligned} C_2(\varepsilon; \varrho, \sigma) &:= \frac{\sigma^2}{2} \partial_{xx}^2 \varrho_\varepsilon - \sigma \partial_{xx}^2 (\sigma \varrho)_\varepsilon + \partial_{xx}^2 \left(\frac{\sigma^2}{2} \varrho \right)_\varepsilon \\ &= \frac{1}{2} \int_{\mathbb{R}} \partial_{xx}^2 J_\varepsilon(x-y) (\sigma(x) - \sigma(y))^2 \varrho(y) dy, \end{aligned}$$

where $\varrho \in L^p_{loc}(\mathbb{R})$ and $\sigma = \sigma(x) \in W^{1,q}_{loc}(\mathbb{R})$, $p, q \in [1, \infty]$. It is proved in [33] that, as $\varepsilon \rightarrow 0$, $C_2(\varepsilon; \varrho, \sigma) \rightarrow (\partial_x \sigma)^2 \varrho$ in $L^r_{loc}(\mathbb{R})$ with $\frac{1}{r} = \frac{1}{p} + \frac{2}{q}$.

Modulo a deterministic drift term (which we do not include), Eq. (1.2) can also be written in the form

$$\partial_t \rho + \partial_x (\sigma \rho) \frac{dW(t)}{dt} = \partial_x \left(\frac{\sigma^2}{2} \partial_x \rho \right). \tag{1.3}$$

This particular equation is similar to the equation studied in [18], which arises in the kinetic formulation of stochastically forced hyperbolic conservation laws (on manifolds). The uniqueness proof in [18] relies on writing the equation satisfied by $F(\rho) = \rho^2$. In the Euclidean setting, one is lead to control the following error term, linked to the second order differential operator in (1.3) and the “Itô correction”:

$$\mathcal{R}(\varepsilon) := \left| \int \partial_x \varrho_\varepsilon (\sigma^2 \partial_x \varrho)_\varepsilon - \left((\sigma \partial_x \varrho)_\varepsilon \right)^2 dx \right|,$$

again without imposing a condition like $\sigma \partial_x \varrho \in L^2$. Nevertheless, in the kinetic formulation of conservation laws one has access to additional structural information, namely that $\partial_x \rho$ is a bounded measure. In [18] we use this, and the observation

$$\mathcal{R}(\varepsilon) = \frac{1}{2} \int (\sigma(y) - \sigma(\bar{y}))^2 (\partial_x \varrho)(y) (\partial_x \varrho)(\bar{y}) J_\varepsilon(x-y) J_\varepsilon(x-\bar{y}) dy d\bar{y} dx,$$

to establish that $\mathcal{R}(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. The detailed handling of error terms like $\mathcal{R}(\varepsilon)$ becomes significantly more complicated on a curved manifold, cf. [18] for details.

Let us return to Eq. (1.1). Our main result is the renormalization property for weak L^2 solutions, roughly speaking under the assumption that $u(t, \cdot)$ is a $W^{1,2}$ vector field on M ,

whereas a_1, \dots, a_N are smooth vector fields on M . As corollaries, we deduce the uniqueness of weak solutions and an a priori estimate, under the additional (usual) condition that $\operatorname{div}_h u \in L^1_t L^\infty$.

The complete renormalization proof is long and technical, with the “Euclidean” discussion above shedding some light on one part of the argument in a simplified situation. A key technical part of the proof concerns the regularization of functions via convolution using a mollifier. In the Euclidean case mollification commutes with differential operators and the regularization error (linked to a commutator between the derivative and the convolution operator) converges as the mollification radius tends to zero. These properties are not easy to engineer if the function in question is defined on a manifold. On a Riemannian manifold there exist different approaches for smoothing functions, including (i) the use of partition of unity combined with Euclidean convolution in local charts (see e.g. [12]), (ii) the so-called “Riemannian convolution smoothing” [20] that is better at preserving geometric properties, and (iii) the heat semigroup method (see e.g. [14]). In [14], the authors employ the heat semigroup to regularize functions as well as vector fields on manifolds. As an application, they extend the DiPerna–Lions theory (deterministic equations) to a class of Riemannian manifolds. One of the results in [14] says that the DiPerna–Lions commutator converges in L^1 . It is not clear to us how to improve this to L^2 convergence, which is required by our argument to handle the regularization error coming from the second order differential operators (arising when passing from Stratonovich to Itô integrals), cf. the discussion above.

In the present work we need to regularize functions as well as *tensor fields*. We will make use of an approach based on “pullback, Euclidean smoothing, and then extension”, in the spirit of [18]. When applied to functions our approach reduces to (i). Our regularizing procedure consists of three main steps: (I) a localization step based on a partition of unity; (II) transportation of tensor fields from M to \mathbb{R}^d and vice versa via pushforwards and pullbacks to produce “intrinsic” geometric objects; (III) a convenient choice of atlas that allows us to work (locally) with the standard d -dimensional volume element dx instead of the Riemannian volume element dV_h , which in local coordinates equals $|h|^{\frac{1}{2}} dx^1 \cdots dx^d$ (presumably not essential, but it dramatically simplifies some computations). Although our approach shares some similarities with the mollifier smoothing method found in Nash’s celebrated work [28] on embeddings of manifolds into Euclidean spaces, there are essential differences. The most important one is that Nash regularizes tensor fields on Riemannian manifolds by embedding the manifold into an Euclidean space and then convolve the tensor field with a mollifier defined on the ambient space. Since the mollifier lives in the larger Euclidean space, we cannot easily use it as a test function in the weak formulation of (1.1) to derive a similar SPDE for ρ_ε , the regularized version of the weak solution ρ .

Roughly speaking, our proof starts off from the following Itô form of (1.1) (cf. Section 3 for details):

$$d\rho + \operatorname{div}_h(\rho u) dt + \sum_{i=1}^N \operatorname{div}_h(\rho a_i) dW^i(t) = \frac{1}{2} \sum_{i=1}^N \Lambda_i(\rho). \tag{1.4}$$

Recall that for a vector field X (locally given by $X^j \partial_j$), the divergence of X is given by $\operatorname{div}_h X = \partial_j X^j + \Gamma_{ij}^j X^i$, where Γ_{ij}^k are the Christoffel symbols associated with the Levi-Civita connection ∇ of the metric h (the Einstein summation convention over repeated indices is used throughout the paper). For a smooth function $f : M \rightarrow \mathbb{R}$, we have $X(f) = (X, \operatorname{grad}_h f)_h$ (which locally becomes $X^j \partial_j f$). Moreover, $X(X(f)) = (\nabla^2 f)(X, X) + (\nabla_X X)(f)$, where

$\nabla^2 f$ is the covariant Hessian of f and $\nabla_X X$ is the covariant derivative of X in the direction X . In the Itô SPDE (1.4) we denote by $\Lambda_i(\cdot) := \operatorname{div}_h(\operatorname{div}_h(\rho a_i) a_i)$ the formal adjoint of $a_i(a_i(\cdot))$. Later we prove that the second order differential operator $\Lambda_i(f)$ may be recast into the form $\operatorname{div}_h^2(f \hat{a}_i) - \operatorname{div}_h(f \nabla_{a_i} a_i)$, where $\operatorname{div}_h^2(S)$ is defined by $\operatorname{div}_h(\operatorname{div}_h(S))$ for any symmetric $(0, 2)$ -tensor field S . Further, \hat{a}_i is the symmetric $(0, 2)$ -tensor field whose components are locally given by $\hat{a}_i^{kl} = a_i^k a_i^l$. We refer to an upcoming section for relevant background material in differential geometry.

Fixing a smooth partition of the unity $\{\mathcal{U}_\kappa\}_{\kappa \in \mathcal{A}}$ subordinate to a conveniently chosen atlas \mathcal{A} , cf. (III) above, we utilize our regularization device to derive a rather involved equation for each piece $(\rho(t)\mathcal{U}_\kappa)_\varepsilon$. A global SPDE for $\rho_\varepsilon := \sum_\kappa (\rho(t)\mathcal{U}_\kappa)_\varepsilon$ is then obtained by summing up the local equations. We subsequently use the Itô and classical chain rules to arrive at an equation for $F(\rho_\varepsilon)$, $F \in C^2$ with F, F', F'' bounded, which contains numerous remainder terms coming from the regularization procedure, some of which can be analyzed in terms of first order commutators related to the differential operators $\operatorname{div}_h(\cdot u)$, $\operatorname{div}_h(\cdot \nabla_{a_i} a_i)$ and second order commutators related to $\operatorname{div}_h^2(\cdot \hat{a}_i)$. In addition, we must exploit specific cancellations coming from some quadratic terms linked to the covariation of the martingale part of Eq. (1.4) and the second order operators Λ_i . The localization part of the regularization procedure generates a number of error terms as well, some of which are easy to control whereas others rely on the identification of specific cancellations. At long last, after sending the regularization parameter ε to zero, we arrive at the renormalized equation

$$\begin{aligned} \partial_t F(\rho) + \operatorname{div}_h(F(\rho)u) - \sum_{i=1}^N \operatorname{div}_h(G_F(\rho)\bar{a}_i) + G_F(\rho) \operatorname{div}_h u \\ + \sum_{i=1}^N \operatorname{div}_h(F(\rho)a_i) \dot{W}^i + \sum_{i=1}^N G_F(\rho) \operatorname{div}_h a_i \dot{W}^i(t) \\ = \frac{1}{2} \sum_{i=1}^N \Lambda_i(F(\rho)) - \frac{1}{2} \sum_{i=1}^N \Lambda_i(1)G_F(\rho) + \frac{1}{2} \sum_{i=1}^N F''(\rho) (\rho \operatorname{div}_h a_i)^2, \end{aligned}$$

where $G_F(\rho) = \rho F'(\rho) - F(\rho)$, $\bar{a}_i = (\operatorname{div}_h a_i) a_i$, $\Lambda_i(1) = \operatorname{div}_h^2(\hat{a}_i) - \operatorname{div}_h(\nabla_{a_i} a_i)$.

The remaining part of this paper is organized as follows: In Section 2 we collect the assumptions that are imposed on the “data” of the problem, and present background material from differential geometry and stochastic analysis. The definitions of solution and the main results are stated in Section 3. Section 4 is dedicated to an informal outline of the proof of the renormalization property, while a rigorous proof is developed in Section 5. Corollaries of the main result (uniqueness and a priori estimate) are proved in Section 6. Finally, in the Appendix we bring together a few basic results used throughout the paper.

2. Background material and hypotheses

In an attempt to make this paper more self-contained and to fix relevant notation, we briefly review some basic aspects of differential geometry and stochastic analysis. Furthermore, we collect the precise assumptions imposed on the coefficients u, a_i appearing in the stochastic continuity equation (1.1).

2.1. Geometric framework

We refer to [4,25] for basic definitions and facts concerning manifolds. Consider a d -dimensional smooth Riemannian manifold M , which is closed, connected, and oriented (for instance, the d -dimensional sphere). Moreover, M is endowed with a smooth (Riemannian) metric h . By this we mean that h is a positive-definite 2-covariant tensor field, which thus determines for every $x \in M$ an inner product h_x on $T_x M$. Here, $T_x M$ denotes the tangent space at x , whereas $TM = \coprod_{x \in M} T_x M$ denotes the tangent bundle. For two arbitrary vectors $X_1, X_2 \in T_x M$, we will henceforth write $h_x(X_1, X_2) =: (X_1, X_2)_{h_x}$ or even $(X_1, X_2)_h$ if the context is clear. We set $|X|_h := (X, X)_h^{1/2}$. Recall that in local coordinates $x = (x^i)$, the partial derivatives $\partial_i := \frac{\partial}{\partial x^i}$ form a basis for $T_x M$, while the differential forms dx^i determine a basis for the cotangent space $T_x^* M$. Therefore, in local coordinates, h reads

$$h = h_{ij} dx^i dx^j, \quad h_{ij} = (\partial_i, \partial_j)_h.$$

We will denote by (h^{ij}) the inverse of the matrix (h_{ij}) .

We denote by dV_h the Riemannian density associated to h , which in local coordinates takes the form

$$dV_h = |h|^{1/2} dx^1 \dots dx^d,$$

where $|h|$ is the determinant of h . Integration with respect to dV_h is done in the following way: if $f \in C^0(M)$ has support contained in the domain of a single chart $\Phi : U \subset M \rightarrow \Phi(U) \subset \mathbb{R}^d$, then

$$\int_M f(x) dV_h(x) = \int_{\Phi(U)} (|h|^{1/2} f) \circ \Phi^{-1} dx^1 \dots dx^d,$$

where (x^i) are the coordinates associated to Φ . If $\text{supp } f$ is not contained in a single chart domain, then the integral is defined as

$$\int_M f(x) dV_h(x) = \sum_{i \in \mathcal{I}} \int_M (\alpha_i f)(x) dV_h(x),$$

where $(\alpha_i)_{i \in \mathcal{I}}$ is a partition of unity subordinate to some atlas \mathcal{A} . Throughout the paper, we will assume for convenience that

$$\text{Vol}(M, h) := \int_M dV_h = 1.$$

For $p \in [1, \infty]$, we denote by $L^p(M)$ the usual Lebesgue spaces on (M, h) . Always in local coordinates, the gradient of a function $f : M \rightarrow \mathbb{R}$ is the vector field given by the following expression

$$\text{grad}_h f := h^{ij} \partial_i f \partial_j.$$

A smooth k -dimensional real vector bundle is a pair of smooth manifolds E (the total space) and V (the base), together with a surjective map $\pi : E \rightarrow V$ (the projection), satisfying the following three conditions: (i) each set $E_x := \pi^{-1}(x)$ (called the fiber of E over x) is endowed with the structure of a real vector space; (ii) for each $x \in V$, there exists a neighborhood U of x and a diffeomorphism $\phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$, called a local trivialization of E , such that $\pi_1 \circ \phi = \pi$ on $\pi^{-1}(U)$, where π_1 is the projection onto the first factor; (iii) the restriction of ϕ to each fiber, $\phi : E_x \rightarrow \{x\} \times \mathbb{R}^k$, is a linear isomorphism.

Given a smooth vector bundle $\pi : E \rightarrow V$ over a smooth manifold V , a section of E is a section of the map π , i.e., a map $\sigma : V \rightarrow E$ satisfying $\pi \circ \sigma = \text{Id}_V$.

For an arbitrary finite-dimensional real vector space H , we use $\mathcal{T}^m(H)$, $\mathcal{T}_l(H)$, and $\mathcal{T}_l^m(H)$ to denote the spaces of covariant m -tensors, contravariant l -tensors, and mixed tensors of type (m, l) on H , respectively. For an arbitrary smooth manifold V , we define the bundles of covariant m -tensors, contravariant l -tensors, and mixed tensors of type (m, l) on V respectively by

$$\mathcal{T}^m(V) = \coprod_{x \in V} \mathcal{T}^m(T_x V), \quad \mathcal{T}_l(V) = \coprod_{x \in V} \mathcal{T}_l(T_x V), \quad \mathcal{T}_l^m(V) = \coprod_{x \in V} \mathcal{T}_l^m(T_x V).$$

Note the natural identifications $\mathcal{T}^1(V) = T^*V$ and $\mathcal{T}_1(V) = TV$.

Let $F : V \rightarrow \bar{V}$ be a diffeomorphism between two smooth manifolds V, \bar{V} . The symbols F_* , F^* denote the smooth bundle isomorphisms $F_* : \mathcal{T}_l^m(V) \rightarrow \mathcal{T}_l^m(\bar{V})$ and $F^* : \mathcal{T}_l^m(\bar{V}) \rightarrow \mathcal{T}_l^m(V)$ satisfying

$$F_* S(X_1, \dots, X_m, \omega^1, \dots, \omega^l) = S(F_*^{-1} X_1, \dots, F_*^{-1} X_m, F^* \omega^1, \dots, F^* \omega^l),$$

for $S \in \mathcal{T}_l^m(V)$, $X_i \in T\bar{V}$, $\omega^j \in T^*\bar{V}$, and

$$F^* S(X_1, \dots, X_m, \omega^1, \dots, \omega^l) = S(F_* X_1, \dots, F_* X_m, F^{-1*} \omega^1, \dots, F^{-1*} \omega^l),$$

for $S \in \mathcal{T}_l^m(\bar{V})$, $X_i \in TV$, $\omega^j \in T^*V$ (for further details see [25, Chapter 11]).

The symbol ∇ refers to the Levi-Civita connection of h , namely the unique linear connection on M that is compatible with h and is symmetric. The Christoffel symbols associated to ∇ are given by

$$\Gamma_{ij}^k = \frac{1}{2} h^{kl} (\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij}).$$

In particular, the covariant derivative of a vector field $X = X^\alpha \partial_\alpha$ is the $(1, 1)$ -tensor field which in local coordinates reads

$$(\nabla X)_j^\alpha := \partial_j X^\alpha + \Gamma_{kj}^\alpha X^k.$$

The divergence of a vector field $X = X^j \partial_j$ is the function defined by

$$\text{div}_h X := \partial_j X^j + \Gamma_{kj}^j X^k.$$

For any vector field X and $f \in C^1(M)$, we have $X(f) = (X, \text{grad}_h f)_h$, which locally takes the form $X^j \partial_j f$. We recall that for a (smooth) vector field X , the following integration by parts formula holds:

$$\int_M X(f) dV_h = \int_M (\text{grad}_h f, X)_h dV_h = - \int_M f \text{div}_h X dV_h,$$

recalling that M is closed (all functions are compactly supported).

Given a smooth vector field X on M , we consider the norms

$$\|X\|_{L^p(M)}^p := \int_M |X|_h^p dV_h, \quad p \in [1, \infty), \quad \|X\|_{L^\infty(M)} := \| |X|_h \|_{L^\infty(M)}.$$

The closure of the space of the smooth vector fields on M with respect to the norm $\|\cdot\|_{L^p(M)}$ is denoted by $\overrightarrow{L^p(M)}$. We define the Sobolev space $\overrightarrow{W^{1,p}(M)}$ in a similar fashion. Indeed, consider the norm

$$\|X\|_{\overrightarrow{W^{1,p}(M)}}^p := \int_M |X|_h^p + |\nabla X|_h^p dV_h, \quad p \in [1, \infty),$$

$$\|X\|_{\overrightarrow{W^{1,\infty}(M)}} := \| |X|_h + |\nabla X|_h \|_{L^\infty(M)},$$

where locally $|\nabla X|_h^2 = (\nabla X)_j^i h_{ik} h^{jm} (\nabla X)_m^k$. The closure of the space of the smooth vector fields with respect to this norm is $\overrightarrow{W^{1,p}(M)}$. For more operative definitions, $\overrightarrow{L^p(M)}$ and $\overrightarrow{W^{1,p}(M)}$ can be seen as the spaces of vector fields whose components in any arbitrary chart belong to the corresponding Euclidean space.

Given a smooth vector field X , consider the second order differential operator $X(X(\cdot))$. We have

Lemma 2.1 (Geometric Identity). *For any smooth vector field X and $\psi \in C^2(M)$,*

$$X(X(\psi)) = (\nabla^2 \psi)(X, X) + (\nabla_X X)(\psi),$$

where $\nabla^2 \psi$ denotes the covariant Hessian of ψ and $\nabla_X X$ denotes the covariant derivative of X in the direction X .

Proof. In any coordinate system, we have

$$\begin{aligned} (\nabla^2 \psi)(X, X) &= \partial_{lm} \psi X^l X^m - \Gamma_{lm}^j \partial_j \psi X^l X^m, \\ (\nabla_X X)(\psi) &= X^m \partial_m X^l \partial_l \psi + \Gamma_{lm}^j \partial_j \psi X^l X^m. \end{aligned}$$

On the other hand, $X(X(\psi)) = \partial_{lm} \psi X^l X^m + X^m \partial_m X^l \partial_l \psi$. \square

In the following, we will consistently write $(\nabla^2 \cdot)(a_i, a_i) + (\nabla_{a_i} a_i)(\cdot)$ instead of $a_i(a_i(\cdot))$, thereby highlighting the presence of the Hessian.

Let us introduce the following second order differential operators associated to the vector fields $\{a_i\}_{i=1}^N$:

$$A_i(\psi) := \operatorname{div}_h(\operatorname{div}_h(\psi a_i) a_i), \quad \psi \in C^2(M), \quad i = 1, \dots, N. \tag{2.1}$$

We will need to write these operators in a more appropriate form. To this end, we will first make a short digression into some concepts from differential geometry.

Given a smooth symmetric $(0, 2)$ -tensor field S on M , we can compute $\operatorname{div}_h S$, which is the smooth vector field whose local expression is given by

$$\operatorname{div}_h S := \nabla_j S^{ij} \partial_i = \left\{ \partial_j S^{ij} + \Gamma_{lj}^i S^{lj} + \Gamma_{lj}^j S^{il} \right\} \partial_i, \tag{2.2}$$

where, obviously, $S = S^{ij} \partial_i \otimes \partial_j$ (since S is symmetric, it is irrelevant which index we contract). Because $\operatorname{div}_h S$ is a vector field, it can operate on functions by differentiation. Moreover, we can compute its divergence. Henceforth, we set

$$\operatorname{div}_h^2(S) := \operatorname{div}_h(\operatorname{div}_h(S)). \tag{2.3}$$

Given any vector field X on M , we can canonically construct a symmetric $(0, 2)$ -tensor field on M in the following fashion: we consider the endomorphism induced by X on the tangent bundle TM ,

$$Y_p \mapsto (X_p, Y_p)_h X_p, \quad p \in M, \quad Y = \text{vector field}.$$

This endomorphism can be canonically identified with a $(1, 1)$ -tensor field. Besides, rising an index via the metric h produces a symmetric $(0, 2)$ -tensor field \hat{X} , whose components are locally given by

$$\hat{X}^{jk} = X^j X^k.$$

Remark 2.1. In what follows, we use the symbols $\hat{a}_1, \dots, \hat{a}_N$ to denote the smooth symmetric $(0, 2)$ -tensor fields obtained by applying the procedure defined above to the vector fields a_1, \dots, a_N .

We may now state

Lemma 2.2 (Alternative Expression for Λ_i). For $\psi \in C^2(M)$,

$$\Lambda_i(\psi) = \operatorname{div}_h^2(\psi \hat{a}_i) - \operatorname{div}_h(\psi \nabla_{a_i} a_i), \quad i = 1, \dots, N. \tag{2.4}$$

Proof. In any coordinates, from the definition of the divergence of a vector field,

$$\begin{aligned} \left(\operatorname{div}_h(\psi a_i) a_i^\beta \right) \partial_\beta &= \left[\partial_\ell (\psi a_i^\ell) a_i^\beta + \Gamma_{\ell k}^k a_i^\ell a_i^\beta \right] \partial_\beta \\ &= \left[\partial_\ell (\psi \hat{a}_i^{\ell\beta}) + \Gamma_{\ell k}^k \hat{a}_i^{\ell\beta} \psi - \psi a_i^\ell \partial_\ell a_i^\beta \right] \partial_\beta \\ &= \left[\partial_\ell (\psi \hat{a}_i^{\ell\beta}) + \Gamma_{\ell k}^k \hat{a}_i^{\ell\beta} \psi - \psi a_i^\ell \partial_\ell a_i^\beta - \psi \Gamma_{jk}^\beta \hat{a}_i^{jk} + \psi \Gamma_{jk}^\beta \hat{a}_i^{jk} \right] \partial_\beta. \end{aligned}$$

Therefore, recalling that locally (2.2) and $\nabla_{a_i} a_i = \left[a_i^\ell \partial_\ell a_i^\beta + \Gamma_{jk}^\beta \hat{a}_i^{jk} \right] \partial_\beta$ hold, we obtain the following identity between vector fields:

$$\operatorname{div}_h(\psi a_i) a_i = \operatorname{div}_h(\psi \hat{a}_i) - \psi \nabla_{a_i} a_i.$$

We apply div to this equation to obtain (2.4). \square

Remark 2.2 (Adjoint of Λ_i). The adjoint of $\Lambda_i(\cdot)$ is $a_i(a_i(\cdot))$, i.e. $\forall \psi, \phi \in C^2(M)$,

$$\int_M \Lambda_i(\psi) \phi \, dV_h = \int_M \psi a_i(a_i(\phi)) \, dV_h = \int_M \psi \left((\nabla^2 \phi)(a_i, a_i) + (\nabla_{a_i} a_i)(\psi) \right) \, dV_h.$$

The following lemma turns out to be an extremely useful instrument in the proof of Theorem 3.2. It allows us to introduce a special atlas on M , in whose charts the determinant of the metric h will be constant. It turns out that this atlas significantly simplifies several terms in some already long computations; in broad strokes, the underlying reason is we can work locally with the standard d -dimensional Lebesgue measure dz instead of the Riemannian volume element dV_h .

Lemma 2.3 (Convenient Choice of Atlas). On the manifold M there exists a finite atlas $\mathcal{A} = \{ \kappa : X_\kappa \subset M \rightarrow \tilde{X}_\kappa \subset \mathbb{R}^d \}$ such that, for any $\kappa \in \mathcal{A}$, the determinant of the metric written in that chart is equal to one: $|h_\kappa| \equiv 1$. In particular, we have

$$\Gamma_{mj}^m = 0 \text{ on } X_\kappa, \text{ for any } j = 1, \dots, d. \tag{2.5}$$

Proof. Fix $x \in M$ and consider a chart Φ around x , whose induced coordinates are named (u^i) and whose range is the open unit cube in \mathbb{R}^d , $(0, 1)^d$. Then, $(\Phi^{-1})^* dV_h = f \, du^1 \wedge \dots \wedge du^d$, where $f = |h_\Phi|^{1/2}$, where $(\Phi^{-1})^*$ is defined in Section 2.1, and \wedge denotes the wedge product between forms. Without loss of generality, we can assume from the beginning that

$f \in C^\infty([0, 1]^d)$. Consider the following map from $(0, 1)^d$ to \mathbb{R}^d :

$$\Psi : \begin{cases} z^1 = \int_0^{u^1} f(\zeta, u^2, \dots, u^d) d\zeta \\ z^2 = u^2 \\ \vdots \\ z^d = u^d. \end{cases}$$

One can check that Ψ is smooth and invertible onto its image (recall $f > 0$). Moreover, $|\Psi'| = f(u^1, \dots, u^d) > 0$. By the inverse function theorem and the fact that Ψ admits a global inverse, we infer that Ψ is a diffeomorphism of $(0, 1)^d$ onto its image, and

$$((\Psi \circ \Phi)^{-1})^* dV_h = (\Psi^{-1})^*(\Phi^{-1})^* dV_h = dz^1 \wedge \dots \wedge dz^d.$$

We set $\kappa_x := \Psi \circ \Phi$. We repeat this procedure for any $x \in M$, and by compactness of M we end up with a finite atlas $\mathcal{A} = \{\kappa : X_\kappa \subset M \rightarrow \tilde{X}_\kappa \subset \mathbb{R}^d\}$ with the desired property. In general, $\Gamma_{mj}^m = \partial_j \log |h_\kappa|^{\frac{1}{2}}$ [4, page 106]. Hence, (2.5) follows. \square

Remark 2.3. A different proof of Lemma 2.3, which requires much more baggage, can be found in [5].

Finally, we discuss the conditions imposed on the vector field u . Firstly,

$$u \in L^1\left([0, T]; \overrightarrow{W^{1,2}(M)}\right). \tag{2.6}$$

In particular, we have $u \in L^1\left([0, T]; \overrightarrow{L^2(M)}\right)$, which is sufficient to conclude that for $\rho \in L_t^\infty L_{\omega,x}^2$ and $\psi \in C^\infty(M)$, $t \mapsto \int_0^t \int_M \rho(s)u(s)(\psi) dV_h ds$ is absolutely continuous, \mathbb{P} -a.s., and hence is not contributing to cross-variations against W^i . These cross-variations appear when passing from Stratonovich to Itô integrals in the SPDE (1.1), consult the upcoming Lemma 3.1.

For the uniqueness result (cf. Corollary 3.3), we must also assume

$$\operatorname{div}_h u \in L^1([0, T]; L^\infty(M)). \tag{2.7}$$

Remark 2.4. In the following, for a function $f : M \rightarrow \mathbb{R}$ and a vector field X , we will freely jump between the different notations

$$f(x)X(x), \quad f(x)X, \quad (fX)(x), \quad x \in M,$$

for the vector field obtained by pointwise scalar multiplication of f and X .

2.2. Stochastic framework

We refer to [31,34] for relevant notation, concepts, and basic results in stochastic analysis. From beginning to end, we fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a complete right-continuous filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$. Without loss of generality, we assume that the σ -algebra \mathcal{F} is countably generated. Let $W = \{W_i\}_{i=1}^N$ be a finite sequence of independent one-dimensional Brownian motions adapted to the filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$. We refer to $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P}, W)$ as a (Brownian) *stochastic basis*.

Consider two real-valued stochastic processes Y, \tilde{Y} . We call \tilde{Y} a *modification* of Y if, for each $t \in [0, T]$, $\mathbb{P}(\{\omega \in \Omega : Y(\omega, t) = \tilde{Y}(\omega, t)\}) = 1$. It is important to pick good modifications of

stochastic processes. Right (or left) continuous modifications are often used (they are known to exist for rather general processes), since any two such modifications of the same process are indistinguishable (with probability one they have the same sample paths). Besides, they necessarily have left-limits everywhere. Right-continuous processes with left-limits are referred to as *càdlàg*.

An $\{\mathcal{F}_t\}_{t \in [0, T]}$ -adapted, *càdlàg* process Y is an $\{\mathcal{F}_t\}_{t \in [0, T]}$ -semimartingale if there exist processes F, M with $F_0 = M_0 = 0$ such that

$$Y_t = Y_0 + F_t + M_t,$$

where F is a finite variation process and M is a local martingale. In this paper we will only be concerned with *continuous* semimartingales. The quantifier “local” refers to the existence of a sequence $\{\tau_n\}_{n \geq 1}$ of stopping times increasing to infinity such that the stopped processes $\mathbf{1}_{\{\tau_n > 0\}} M_{t \wedge \tau_n}$ are martingales.

Given two continuous semimartingales Y and Z , we can define the Fisk–Stratonovich integral of Y with respect to Z by

$$\int_0^t Y(s) \circ dZ(s) = \int_0^t Y(s) dZ(s) + \frac{1}{2} \langle Y, Z \rangle_t,$$

where $\int_0^t Y(s) dZ(s)$ is the Itô integral of Y with respect to Z and $\langle Y, Z \rangle$ denotes the quadratic cross-variation process of Y and Z . Let us recall Itô’s formula for a continuous semimartingale Y . Let $F \in C^2(\mathbb{R})$. Then $F(Y)$ is again a continuous semimartingale and the following chain rule formula holds:

$$F(Y(t)) - F(Y(0)) = \int_0^t F'(Y(s)) dY(s) + \frac{1}{2} \int_0^t F''(Y(s)) d \langle Y, Y \rangle_s.$$

Martingale inequalities are generally important for several reasons. For us they will be used to bound Itô stochastic integrals in terms of their quadratic variation (which is easy to compute). One of the most important martingale inequalities is the Burkholder–Davis–Gundy inequality. Let $Y = \{Y_t\}_{t \in [0, T]}$ be a continuous local martingale with $Y_0 = 0$. Then, for any stopping time $\tau \leq T$,

$$\mathbb{E} \left(\sup_{t \in [0, \tau]} |Y_t| \right)^p \leq C_p \mathbb{E} \sqrt{\langle Y, Y \rangle_\tau}, \quad p \in (0, \infty), \tag{2.8}$$

where C_p is a universal constant. We use (2.8) with $p = 1$, in which case $C_p = 3$.

Finally, the vector fields driving the noise in (1.1) satisfy

$$a_1, \dots, a_N \in C^\infty(M). \tag{2.9}$$

3. Weak solutions and main results

Inspired by [16], we work with the following concept of solution for (1.1).

Definition 3.1 (*Weak L^2 Solution*). Given $\rho_0 \in L^2(M)$, a weak L^2 solution of (1.1) with initial datum $\rho|_{t=0} = \rho_0$ is a function $\rho \in L^\infty([0, T]; L^2(\Omega \times M))$ such that for all $\psi \in C^\infty(M)$ the stochastic process $(\omega, t) \mapsto \int_M \rho(t) \psi dV_h$ has a continuous modification which is an

$\{\mathcal{F}_t\}_{t \in [0, T]}$ -semimartingale and \mathbb{P} -a.s., for all $t \in [0, T]$,

$$\int_M \rho(t)\psi dV_h = \int_M \rho_0\psi dV_h + \int_0^t \int_M \rho(s)u(\psi) dV_h ds + \sum_{i=1}^N \int_0^t \int_M \rho(s)a_i(\psi) dV_h \circ dW^i(s). \tag{3.1}$$

Remark 3.1. Since each vector field a_i is smooth, cf. (2.9), the corresponding stochastic process $(\omega, t) \mapsto \int_M \rho(s)a_i(\psi) dV_h$ has a continuous modification that is an $\{\mathcal{F}_t\}_{t \in [0, T]}$ -semimartingale.

The first result brings (1.1) into its equivalent Itô form. The result is analogous to Lemma 13 in [16].

Lemma 3.1 (Stratonovich–Itô Conversion). *Let ρ be a weak L^2 solution of (1.1), according to Definition 3.1. Then Eq. (3.1) is equivalent to*

$$\int_M \rho(t)\psi dV_h = \int_M \rho_0\psi dV_h + \int_0^t \int_M \rho(s)u(\psi) dV_h ds + \sum_{i=1}^N \int_0^t \int_M \rho(s)a_i(\psi) dV_h dW^i(s) + \frac{1}{2} \sum_{i=1}^N \int_0^t \int_M \rho(s)a_i(a_i(\psi)) dV_h ds. \tag{3.2}$$

Proof. Let us commence from (3.1). The Stratonovich integrals can be written as

$$\sum_{i=1}^N \int_0^t \int_M \rho(s)a_i(\psi) dV_h \circ dW^i(s) = \sum_{i=1}^N \int_0^t \int_M \rho(s)a_i(\psi) dV_h dW^i(s) + \frac{1}{2} \sum_{i=1}^N \left\langle \int_M \rho(s)a_i(\psi) dV_h, W^i \right\rangle_t.$$

where $\langle \cdot, \cdot \rangle$ denotes the cross-variation between stochastic processes. Using (3.1) with $a_i(\psi) \in C^\infty(M)$ as test function, we infer

$$\begin{aligned} \left\langle \int_M \rho a_i(\psi) dV_h, W^i \right\rangle_t &= \sum_{j=1}^N \left\langle \int_0^\cdot \int_M \rho a_j(a_i(\psi)) dV_h \circ dW^j, W^i \right\rangle_t \\ &= \sum_{j=1}^N \left\langle \int_0^\cdot \int_M \rho a_j(a_i(\psi)) dV_h dW^j, W^i \right\rangle_t \\ &\quad + \frac{1}{2} \sum_{j=1}^N \left\langle \left\langle \int_M \rho a_j(a_i(\psi)) dV_h, W^j \right\rangle_t, W^i \right\rangle_t, \end{aligned}$$

where we have exploited that the time-integral is absolutely continuous and thus not contributing to the cross-variation against W^i , which follows from (2.6) and the fact that ρ belongs \mathbb{P} -a.s. to $L^2([0, T] \times M)$.

Since $a_j(a_i(\psi)) \in C^\infty(M)$, the stochastic process $(\omega, t) \mapsto \int_M \rho a_j(a_i(\psi)) dV_h$ is a continuous semimartingale by assumption. It follows from [23, Theorem 2.2.14] that the variation process $\left\langle \int_M \rho a_j(a_i(\psi)) dV_h, W^j \right\rangle$ is continuous and of bounded variation. Hence,

$\langle \langle \cdot, \cdot \rangle, \cdot \rangle = 0$. Therefore,

$$\left\langle \int_M \rho a_i(\psi) dV_h, W^i \right\rangle_t = \sum_{j=1}^N \left\langle \int_0^t \int_M \rho a_j(a_i(\psi)) dV_h dW^j, W^i \right\rangle_t.$$

Since $\rho \in L^\infty([0, T]; L^2(\Omega \times M))$, we clearly have $\int_M \rho a_j(a_i(\psi)) dV_h \in L^2([0, T])$, \mathbb{P} -a.s., and so by [23, Theorem 2.3.2] we obtain

$$\begin{aligned} \left\langle \int_M \rho a_i(\psi) dV_h, W^i \right\rangle_t &= \sum_{j=1}^N \int_0^t \int_M \rho a_j(a_i(\psi)) dV_h d\langle W^j, W^i \rangle_s \\ &= \sum_{j=1}^N \int_0^t \int_M \rho a_j(a_i(\psi)) dV_h \delta^{ji} ds = \int_0^t \int_M \rho a_i(a_i(\psi)) dV_h ds, \end{aligned}$$

and the sought equation (3.2) follows. Finally, we can repeat this argument, starting with (3.2) and working our way back to (3.1). This concludes the proof. \square

In view of Lemma 3.1, we have an equivalent concept of solution.

Definition 3.2 (Weak L^2 Solution, Itô Formulation). A weak L^2 -solution of (1.1) with initial datum $\rho|_{t=0} = \rho_0 \in L^2(M)$ is a function $\rho \in L^\infty([0, T]; L^2(\Omega \times M))$ such that for any $\psi \in C^\infty(M)$ the stochastic process $(\omega, t) \mapsto \int_M \rho(t)\psi dV_h$ has a continuous modification which is $\{\mathcal{F}_t\}_{t \in [0, T]}$ -adapted and satisfies the following equation \mathbb{P} -a.s., for all $t \in [0, T]$:

$$\begin{aligned} \int_M \rho(t)\psi dV_h &= \int_M \rho_0\psi dV_h + \int_0^t \int_M \rho(s)u(\psi) dV_h ds \\ &+ \sum_{i=1}^N \int_0^t \int_M \rho(s)a_i(\psi) dV_h dW^i(s) + \frac{1}{2} \sum_{i=1}^N \int_0^t \int_M \rho(s)a_i(a_i(\psi)) dV_h ds. \end{aligned}$$

Definition 3.3 (Renormalization Property). Let ρ be a weak L^2 solution of (1.1) with initial datum $\rho|_{t=0} = \rho_0 \in L^2(M)$. We say that ρ is renormalizable if, for any $F \in C^2(\mathbb{R})$ with F, F', F'' bounded, and for any $\psi \in C^\infty(M)$, the stochastic process $(\omega, t) \mapsto \int_M F(\rho(t))\psi dV_h$ has a continuous modification that is $\{\mathcal{F}_t\}_{t \in [0, T]}$ -adapted and satisfies the following SPDE weakly (in x) \mathbb{P} -a.s.:

$$\begin{aligned} dF(\rho) + \operatorname{div}_h(F(\rho)u) dt + G_F(\rho) \operatorname{div}_h u dt \\ + \sum_{i=1}^N \operatorname{div}_h(F(\rho)a_i) dW^i(t) + \sum_{i=1}^N G_F(\rho) \operatorname{div}_h a_i dW^i(t) \\ = \frac{1}{2} \sum_{i=1}^N \Lambda_i(F(\rho)) dt - \frac{1}{2} \sum_{i=1}^N \Lambda_i(1)G_F(\rho) dt \\ + \frac{1}{2} \sum_{i=1}^N F''(\rho) (\rho \operatorname{div}_h a_i)^2 dt + \sum_{i=1}^N \operatorname{div}_h(G_F(\rho)\bar{a}_i) dt, \end{aligned} \tag{3.3}$$

where the second order differential operator A_i is defined in (2.4),

$$\bar{a}_i := (\operatorname{div}_h a_i) a_i, \quad A_i(1) = \operatorname{div}_h^2 (\hat{a}_i) - \operatorname{div}_h (\nabla_{a_i} a_i), \tag{3.4}$$

and

$$G_F(\xi) = \xi F'(\xi) - F(\xi), \quad \xi \in \mathbb{R}. \tag{3.5}$$

Eq. (3.3) is understood in the space-weak sense, that is, for all test functions $\psi \in C^\infty(M)$ and for all $t \in [0, T]$, \mathbb{P} -a.s.,

$$\begin{aligned} \int_M F(\rho(t)) \psi \, dV_h &= \int_M F(\rho_0) \psi \, dV_h + \int_0^t \int_M F(\rho(s)) u(\psi) \, dV_h \, ds \\ &+ \sum_{i=1}^N \int_0^t \int_M F(\rho(s)) a_i(\psi) \, dV_h \, dW^i(s) + \frac{1}{2} \sum_{i=1}^N \int_0^t \int_M F(\rho(s)) a_i(a_i(\psi)) \, dV_h \, ds \\ &- \int_0^t \int_M G_F(\rho(s)) \operatorname{div}_h u \, \psi \, dV_h \, ds - \sum_{i=1}^N \int_0^t \int_M G_F(\rho(s)) \operatorname{div}_h a_i \, \psi \, dV_h \, dW^i(s) \\ &- \frac{1}{2} \sum_{i=1}^N \int_0^t \int_M A_i(1) G_F(\rho(s)) \, \psi \, dV_h \, ds \\ &+ \frac{1}{2} \sum_{i=1}^N \int_0^t \int_M F''(\rho(s)) (\rho(s) \operatorname{div}_h a_i)^2 \, \psi \, dV_h \, ds \\ &- \sum_{i=1}^N \int_0^t \int_M G_F(\rho(s)) \bar{a}_i(\psi) \, dV_h \, ds. \end{aligned} \tag{3.6}$$

Remark 3.2. The quantity $\mathcal{J} := -\frac{1}{2} \sum_i A_i(1) G_F(\rho) \, dt + \sum_i \operatorname{div}_h (G_F(\rho) \bar{a}_i) \, dt$ in (3.3) takes the equivalent form

$$\mathcal{J} = \frac{1}{2} \sum_{i=1}^N A_i(1) G_F(\rho) \, dt + \sum_{i=1}^N \bar{a}_i(G_F(\rho)) \, dt,$$

if we apply the product rule to the divergence of the scalar $G_F(\rho)$ times the vector field \bar{a}_i , remembering that $A_i(1) = \operatorname{div}_h \bar{a}_i$, cf. (2.1) and (3.4). We will make use of this expression for \mathcal{J} in the upcoming computations.

We can now state the main result of this paper.

Theorem 3.2 (Renormalization Property). *Suppose conditions (2.6) and (2.9) hold. Consider a weak L^2 solution ρ of (1.1) with initial datum $\rho_0 \in L^2(M)$, according to Definition 3.2. Then ρ is renormalizable in the sense of Definition 3.3.*

As an application of this result, we obtain a uniqueness result for (1.1), if we further assume that $\operatorname{div}_h u \in L_t^1 L_x^\infty$, cf. (2.7). More precisely, we have

Corollary 3.3 (Uniqueness). *Suppose conditions (2.6), (2.7), and (2.9) hold. Then the initial-value problem for (1.1) possesses at most one weak L^2 solution ρ in the sense of Definition 3.2.*

According to Definition 3.2, a weak solution ρ belongs to the space $L_t^\infty L_{\omega,x}^2$. Combining the proof of Corollary 3.3 and a standard martingale argument, we can strengthen this through “shifting” esssup_t inside the expectation operator $\mathbb{E}[\cdot]$, so that $\rho \in L_\omega^2 L_t^\infty L_x^2$ and consequently, \mathbb{P} -a.s., $\rho \in L_t^\infty L_x^2$.

Corollary 3.4 (A Priori Estimate). *Suppose the assumptions of Corollary 3.3 are satisfied. Consider a weak L^2 solution ρ of (1.1) with initial datum $\rho_0 \in L^2(M)$. Then $\rho \in L^2(\Omega; L^\infty([0, T]; L^2(M)))$ and*

$$\mathbb{E} \text{esssup}_{t \in [0, T]} \|\rho(t)\|_{L^2(M)}^2 \leq \exp(Ct) \|\rho_0\|_{L^2(M)}^2, \tag{3.7}$$

where the constant C depends on $\|\text{div}_h u\|_{L_t^1 L_x^\infty}$ and $\max_i \|a_i\|_{C^2}$.

Remark 3.3. Throughout the paper, we assume that the vector fields driving the noise are smooth, $a_i \in C^\infty$. In the Euclidean setting [32], the renormalization property holds under appropriate Sobolev smoothness, say $a_i \in \overrightarrow{W^{1,p}}$ with $p \geq 4$. A conceivable but quite technical extension of our work would allow for $a_i \in W^{1,p}(M)$. We leave this extension for future work. We refer to [19] for proof of the existence of weak solutions. Beyond the existence result, in that paper, we identify a delicate “regularization by noise” effect for carefully chosen noise vector fields (these vector fields must be linked to the geometry of the underlying domain). Consequently, we obtain existence without an L^∞ assumption on the divergence of the velocity u .

4. Informal proof of Theorem 3.2

In this section, we give a motivational account of the proof of our main result, assuming simply that all considered functions have the necessary smoothness for the operations we perform on them. To this end, consider a solution ρ of (1.1), which in Itô form reads (1.4), cf. Lemma 3.1. An application of Itô’s formula with $F \in C^2(\mathbb{R})$ gives

$$\begin{aligned} dF(\rho) + F'(\rho) \text{div}_h(\rho u) dt + \sum_{i=1}^N F'(\rho) \text{div}_h(\rho a_i) dW^i(t) \\ = \frac{1}{2} \sum_{i=1}^N F'(\rho) A_i(\rho) dt + \frac{1}{2} \sum_{i=1}^N F''(\rho) (\text{div}_h(\rho a_i))^2 dt. \end{aligned} \tag{4.1}$$

By the product and chain rules,

$$F'(\rho) \text{div}_h(\rho V) = \text{div}_h(F(\rho)V) + G_F(\rho) \text{div}_h V, \quad V = u, a_i.$$

To take care of the term $F'(\rho)A_i(\rho)$, we need

Lemma 4.1. *Let S be a smooth symmetric (0, 2)-tensor field on M , $f \in C^1(M)$, and $F \in C^1(\mathbb{R})$. Then, as vector fields,*

$$\text{div}_h(F(f)S) = F(f) \text{div}_h(S) + F'(f) S(df, \cdot).$$

Proof. In any coordinates, by the product and chain rules,

$$\begin{aligned} \text{div}_h(F(f)S) &\stackrel{(2.2)}{=} \left[\partial_j (F(f)S^{ij}) + \Gamma_{lj}^i F(f)S^{lj} + \Gamma_{lj}^j F(f)S^{il} \right] \partial_i \\ &= F(f) \text{div}_h(S) + S^{ij} F'(f) \partial_j f \partial_i = F(f) \text{div}_h(S) + F'(f) S(df, \cdot). \quad \square \end{aligned}$$

In view of Lemmas 2.2 and 4.1,

$$\begin{aligned}
 \Lambda_i(F(\rho)) &= \operatorname{div}_h^2(F(\rho)\hat{a}_i) - \operatorname{div}_h(F(\rho)\nabla_{a_i}a_i) \\
 &= \operatorname{div}_h\left(F(\rho)\operatorname{div}_h(\hat{a}_i) + F'(\rho)\hat{a}_i(d\rho, \cdot)\right) - \operatorname{div}_h(F(\rho)\nabla_{a_i}a_i) \\
 &= F'(\rho)\operatorname{div}_h(\hat{a}_i)(\rho) + F(\rho)\operatorname{div}_h^2(\hat{a}_i) + F''(\rho)\hat{a}_i(d\rho, \cdot)(\rho) \\
 &\quad + F'(\rho)\operatorname{div}_h(\hat{a}_i(d\rho, \cdot)) - F'(\rho)\nabla_{a_i}a_i(\rho) - F(\rho)\operatorname{div}_h(\nabla_{a_i}a_i).
 \end{aligned}
 \tag{4.2}$$

where, cf. Remark 2.1, $\hat{a}_i(d\rho, \cdot)(\rho) = \hat{a}_i(d\rho, d\rho) = (a_i(\rho))^2$. On the other hand,

$$\begin{aligned}
 F'(\rho)\Lambda_i(\rho) &= F'(\rho)\operatorname{div}_h^2(\rho\hat{a}_i) - F'(\rho)\operatorname{div}_h(\rho\nabla_{a_i}a_i) \\
 &= F'(\rho)\operatorname{div}_h\left(\rho\operatorname{div}_h(\hat{a}_i) + \hat{a}_i(d\rho, \cdot)\right) - F'(\rho)\operatorname{div}_h(\rho\nabla_{a_i}a_i) \\
 &= F'(\rho)\operatorname{div}_h(\hat{a}_i)(\rho) + F'(\rho)\rho\operatorname{div}_h^2(\hat{a}_i) + F'(\rho)\operatorname{div}_h(\hat{a}_i(d\rho, \cdot)) \\
 &\quad - F'(\rho)\rho\operatorname{div}_h(\nabla_{a_i}a_i) - F'(\rho)\nabla_{a_i}a_i(\rho).
 \end{aligned}
 \tag{4.3}$$

Therefore, subtracting (4.2) from (4.3),

$$\begin{aligned}
 F'(\rho)\Lambda_i(\rho) - \Lambda_i(F(\rho)) &= G_F(\rho)\operatorname{div}_h^2(\hat{a}_i) - G_F(\rho)\operatorname{div}_h(\nabla_{a_i}a_i) - F''(\rho)(a_i(\rho))^2 \\
 &= G_F(\rho)\Lambda_i(1) - F''(\rho)(a_i(\rho))^2,
 \end{aligned}$$

where G_F is defined in (3.5) and $\Lambda_i(1)$ is defined in (3.4).

In view of the above computations, we can write (4.1) as

$$\begin{aligned}
 dF(\rho) + \operatorname{div}_h(F(\rho)u) dt + G_F(\rho)\operatorname{div}_h u dt \\
 + \sum_{i=1}^N \operatorname{div}_h(F(\rho)a_i) dW^i(t) + \sum_{i=1}^N G_F(\rho)\operatorname{div}_h a_i dW^i(t) \\
 = \frac{1}{2} \sum_{i=1}^N \Lambda_i(F(\rho)) dt + \frac{1}{2} \sum_{i=1}^N \Lambda_i(1)G_F(\rho) dt \\
 + \underbrace{\frac{1}{2} \sum_{i=1}^N F''(\rho)(\operatorname{div}_h(\rho a_i))^2 dt - \frac{1}{2} \sum_{i=1}^N F''(\rho)(a_i(\rho))^2 dt}_{=: \mathcal{Q}},
 \end{aligned}$$

where we need to take a closer look at the (potentially) problematic term \mathcal{Q} , which contains the difference between some quadratic terms linked to the covariation of the martingale part of Eq. (1.4) and the second order operators Λ_i .

We apply the product rule to write $\operatorname{div}_h(\rho a_i) = \rho\operatorname{div}_h a_i + a_i(\rho)$, and then expand the square $(\operatorname{div}_h(\rho a_i))^2$, yielding

$$\begin{aligned}
 (\operatorname{div}_h(\rho a_i))^2 - (a_i(\rho))^2 &= (\rho\operatorname{div}_h a_i)^2 + 2\rho a_i(\rho)\operatorname{div}_h a_i = (\rho\operatorname{div}_h a_i)^2 + 2\rho \bar{a}_i(\rho),
 \end{aligned}$$

where \bar{a} is defined in (3.4). As a result of this and the chain/product rules for \bar{a}_i ,

$$\begin{aligned}
 F''(\rho)\left((\operatorname{div}_h(\rho a_i))^2 - (a_i(\rho))^2\right) &= F''(\rho)(\rho\operatorname{div}_h a_i)^2 + 2\rho \bar{a}_i(F'(\rho))
 \end{aligned}$$

$$\begin{aligned} &= F''(\rho)(\rho \operatorname{div}_h a_i)^2 + 2\bar{a}_i(\rho F'(\rho)) - 2F'(\rho)\bar{a}_i(\rho) \\ &= F''(\rho)(\rho \operatorname{div}_h a_i)^2 + 2\bar{a}_i(G_F(\rho)), \end{aligned}$$

and so \mathcal{Q} becomes

$$\mathcal{Q} = \frac{1}{2} \sum_{i=1}^N F''(\rho)(\rho \operatorname{div}_h a_i)^2 dt + \sum_{i=1}^N \bar{a}_i(G_F(\rho)) dt.$$

We note here that the problematic term $(a(\rho))^2$ has cancelled out in the final expression for \mathcal{Q} . This is similar to what happens in the Euclidean setting [32]. On a curved manifold we must in addition exploit “cancellations” to control some error terms coming from the localization part of our regularization procedure, i.e., terms related to the geometry of the underlying domain (cf. Section 5 for details).

This concludes the informal argument for (3.3).

5. Rigorous proof of Theorem 3.2

The aim of this section is to develop a rigorous proof of Theorem 3.2. The proof will involve a series of long computations, which will be scattered over seven subsections. We begin with the procedure for regularizing tensor fields on a manifold, along with several commutator estimates for controlling the regularization error.

5.1. Pullback and extension of tensor fields

We first recall and extend some concepts from Section 2.1. Let V be an arbitrary (boundary-less) smooth manifold of dimension d . Consider an arbitrary chart $\kappa : X_\kappa \rightarrow \tilde{X}_\kappa$ for V , where X_κ and \tilde{X}_κ are open subsets of V and \mathbb{R}^d respectively. Let $\operatorname{RS}(\mathcal{T}_l^m(\tilde{X}_\kappa))$ denote the space of m covariant and l contravariant tensor fields on $\tilde{X}_\kappa \subset \mathbb{R}^d$, and define similarly $\operatorname{RS}(\mathcal{T}_l^m(X_\kappa))$ and $\operatorname{RS}(\mathcal{T}_l^m(V))$. Observe that we do not impose any assumptions on the regularity of the coefficients of the tensor fields; RS is an acronym for Rough Sections. Let $\operatorname{SS}(\mathcal{T}_l^m(\tilde{X}_\kappa))$ be the subspace of smooth sections, and define similarly $\operatorname{SS}(\mathcal{T}_l^m(X_\kappa))$, $\operatorname{SS}(\mathcal{T}_l^m(V))$.

We are going to define a procedure for pulling an element of $\operatorname{RS}(\mathcal{T}_l^m(\tilde{X}_\kappa))$ back to V . Indeed, given $\sigma \in \operatorname{RS}(\mathcal{T}_l^m(\tilde{X}_\kappa))$, we may transport it on $X_\kappa \subset V$ via the diffeomorphism κ , and we call the result $\kappa^*\sigma$, which will belong to $\operatorname{RS}(\mathcal{T}_l^m(X_\kappa))$. We refer to [25, Exercise 11-6] and Section 2.1 for details (the fact that κ is a diffeomorphism is crucial). Moreover, we may trivially extend it to the whole of V , by simply declaring that it is the null (m, l) -tensor field outside X_κ . Let us name the resulting object $(\kappa^*\sigma)^{\text{ext}}$. Let us give a name to the entire procedure:

$$\mathcal{L}_\kappa : \operatorname{RS}(\mathcal{T}_l^m(\tilde{X}_\kappa)) \ni \sigma \mapsto (\kappa^*\sigma)^{\text{ext}} \in \operatorname{RS}(\mathcal{T}_l^m(V)), \tag{5.1}$$

where \mathcal{L}_κ will be referred to as a “pullback-extension” operator.

Assuming in addition that

$$\operatorname{supp} \sigma \subset \tilde{X}_\kappa, \quad \sigma \in \operatorname{SS}(\mathcal{T}_l^m(\tilde{X}_\kappa)),$$

it is trivial to see that $\mathcal{L}_\kappa \sigma \in \operatorname{SS}(\mathcal{T}_l^m(V))$ and $\operatorname{supp} \mathcal{L}_\kappa \sigma \subset X_\kappa$.

In the following, starting from objects defined on open Euclidean subsets, we are going to use this procedure repeatedly to build global objects on M .

5.2. Regularization & commutator estimates

From now on we are going to use the atlas \mathcal{A} provided by Lemma 2.3. For fixed $\kappa \in \mathcal{A}$, the induced coordinates will be typically denoted by z or \bar{z} . We need a smooth partition of the unity $\{\mathcal{U}_\kappa\}_{\kappa \in \mathcal{A}}$ subordinate to \mathcal{A} , i.e.,

- (1) $\mathcal{U}_\kappa \geq 0, \sum_{\kappa \in \mathcal{A}} \mathcal{U}_\kappa = 1,$
- (2) $\mathcal{U}_\kappa \in C^\infty(M),$ and
- (3) $\text{supp } \mathcal{U}_\kappa \subset X_\kappa$ (and compact).

Let ρ be a weak L^2 -solution of (1.1) with initial datum $\rho_0 \in L^2(M)$. In what follows, we introduce a series of local objects that appear later in a localized version of (1.1), and establish their main properties. For $\kappa \in \mathcal{A}$, fix a standard mollifier ϕ on \mathbb{R}^d with support in $\overline{B_1(0)}$, and define the rescaled mollifier

$$\phi_\varepsilon(z) := \varepsilon^{-d} \phi\left(\frac{z}{\varepsilon}\right), \quad z \in \mathbb{R}^d, \tag{5.2}$$

whose support is contained in $\overline{B_\varepsilon(0)}, z = \kappa(x).$

5.2.1. Localization & smoothing of ρ, ρ_0

Set

$$\rho_\kappa(\omega, t, z) := \mathcal{U}_\kappa(z) \rho(\omega, t, z), \quad \rho_{0,\kappa}(\omega, z) := \mathcal{U}_\kappa(z) \rho_0(\omega, z), \tag{5.3}$$

for $\omega \in \Omega, t \in [0, T], z \in \tilde{X}_\kappa \subset \mathbb{R}^d.$

Remark 5.1. As is customary in differential geometry, we will use the convention of not explicitly writing the chart, in order to alleviate the notation. For example, if $f : M \rightarrow \mathbb{R}$, then we write $f(z)$ instead of $f(\kappa^{-1}(z)).$

We observe that for fixed $\omega \in \Omega, t \in [0, T],$

$$\begin{aligned} \text{supp } \rho_\kappa(\omega, t, \cdot) &\subset \kappa(\text{supp } \mathcal{U}_\kappa) \subset \subset \tilde{X}_\kappa \subset \mathbb{R}^d, \\ \text{supp } \rho_{0,\kappa}(\omega, \cdot) &\subset \kappa(\text{supp } \mathcal{U}_\kappa) \subset \subset \tilde{X}_\kappa \subset \mathbb{R}^d, \end{aligned}$$

and thus $\rho_\kappa(\omega, t, \cdot)$ and $\rho_{0,\kappa}(\omega, \cdot)$ may be viewed as global functions on $\mathbb{R}^d.$ Next we define spatial regularizations of ρ_κ and $\rho_{0,\kappa}.$ For $\omega \in \Omega, t \in [0, T], z \in \mathbb{R}^d,$

$$\begin{aligned} (\rho_\kappa)_\varepsilon(\omega, t, z) &:= \int_{\mathbb{R}^d} \rho_\kappa(\omega, t, \bar{z}) \phi_\varepsilon(z - \bar{z}) d\bar{z} = \int_{\mathbb{R}^d} \mathcal{U}_\kappa(\bar{z}) \rho(\omega, t, \bar{z}) \phi_\varepsilon(z - \bar{z}) d\bar{z}, \\ (\rho_{0,\kappa})_\varepsilon(\omega, z) &:= \int_{\mathbb{R}^d} \rho_{0,\kappa}(\omega, \bar{z}) \phi_\varepsilon(z - \bar{z}) d\bar{z} = \int_{\mathbb{R}^d} \mathcal{U}_\kappa(\bar{z}) \rho_0(\omega, \bar{z}) \phi_\varepsilon(z - \bar{z}) d\bar{z}. \end{aligned}$$

For later use, set

$$\varepsilon_\kappa := \text{dist}\left(\kappa(\text{supp } \mathcal{U}_\kappa), \partial \tilde{X}_\kappa\right) > 0, \quad \varepsilon_0 := \frac{1}{4} \min_{\kappa} \{\varepsilon_\kappa\} > 0. \tag{5.4}$$

The main properties of $(\rho_\kappa)_\varepsilon$ and $(\rho_{0,\kappa})_\varepsilon$ are collected in

Lemma 5.1. Fix $\kappa \in \mathcal{A}$, cf. Lemma 2.3. Then

- (1) $(\rho_\kappa)_\varepsilon(\omega, t, \cdot) \in C^\infty(\mathbb{R}^d)$, for all $(\omega, t) \in \Omega \times [0, T]$.
- (2) For $\varepsilon < \varepsilon_\kappa$ and for any $\omega \in \Omega, t \in [0, T]$,

$$\text{supp}(\rho_\kappa)_\varepsilon(\omega, t, \cdot) \subset \kappa (\text{supp} \mathcal{U}_\kappa + \overline{B_\varepsilon(0)}) \subset \subset \tilde{X}_\kappa.$$

This implies in particular that for any $(\omega, t) \in \Omega \times [0, T]$, the function $(\rho_\kappa)_\varepsilon(\omega, t, \cdot)$ can be seen as an element of $C^\infty(M)$, provided that we set it equal to zero outside of X_κ .

- (3) For any $p \in [1, 2]$ and $(\omega, t) \in \Omega \times [0, T]$,

$$(\rho_\kappa)_\varepsilon(\omega, t, \cdot) \xrightarrow{\varepsilon \downarrow 0} \mathcal{U}_\kappa(\cdot)\rho(\omega, t, \cdot) \text{ in } L^p(M).$$

Therefore, for any $q \in [1, \infty)$, $(\rho_\kappa)_\varepsilon \xrightarrow{\varepsilon \downarrow 0} \mathcal{U}_\kappa \rho$ in $L^q([0, T]; L^2(\Omega \times M))$.

The listed properties hold true for $(\rho_{0,\kappa})_\varepsilon$ as well.

Proof. Claims (1) and (2) follow from standard properties of convolution. To prove claim (3), recall Lemma 2.3. Indeed, on X_κ we use the coordinates given by κ , for which $|h_\kappa(z)|^{1/2} = 1$, to compute as follows:

$$\begin{aligned} & \int_M |(\rho_\kappa)_\varepsilon(t, x) - \mathcal{U}_\kappa(x)\rho(t, x)|^p dV_h(x) \\ &= \int_{X_\kappa} |(\rho_\kappa)_\varepsilon(t, x) - \mathcal{U}_\kappa(x)\rho(t, x)|^p dV_h(x) \\ &= \int_{\tilde{X}_\kappa} |(\rho_\kappa)_\varepsilon(t, z) - \mathcal{U}_\kappa(z)\rho(t, z)|^p dz \\ &= \int_{\tilde{X}_\kappa} |(\rho_\kappa)_\varepsilon(t, z) - \rho_\kappa(t, z)|^p dz. \end{aligned}$$

The last integral converges to zero as ε goes to zero by standard properties of mollifiers, since $\rho_\kappa(\omega, t, \cdot)$ is in $L^2(\mathbb{R}^d)$ and has compact support. This follows easily from our assumption $\rho \in L^\infty([0, T]; L^2(\Omega \times M))$.

Claims (1), (2), and (3) can be proved in a similar way for $(\rho_{0,\kappa})_\varepsilon$. \square

We define the global counterparts of $(\rho_\kappa)_\varepsilon$ and $(\rho_{0,\kappa})_\varepsilon$ as follows:

$$\rho_\varepsilon(\omega, t, x) := \sum_\kappa (\rho_\kappa)_\varepsilon(\omega, t, x), \quad \rho_{0,\varepsilon}(x) := \sum_\kappa (\rho_{0,\kappa})_\varepsilon(x),$$

for $\omega \in \Omega, t \in [0, T], x \in M$, and $\varepsilon < \varepsilon_0$.

5.2.2. Localization & smoothing of ρa_i

Consider for $\omega \in \Omega, t \in [0, T]$, and $z \in \tilde{X}_\kappa$, the object $\{\rho_\kappa(\omega, t, z)a_i^l(z)\}_l$, which belongs to $\text{RS}(\mathcal{T}_1^0(\tilde{X}_\kappa))$ and is compactly supported in $\kappa (\text{supp} \mathcal{U}_\kappa) \subset \tilde{X}_\kappa$, uniformly in ω, t .

Remark 5.2. By writing $a_i(z)$ we mean the vector field evaluated at the point z , not differentiation. It is a minor abuse of notation that should not provoke too much confusion. We will use this convention in the following also for other objects.

We regularize $\{\rho_\kappa(\omega, t, z)a_i^l(z)\}_l$ componentwise via the mollifier ϕ_ε (as above). The result is an object in $\text{SS}(\mathcal{T}_1^0(\tilde{X}_\kappa))$ that is compactly supported, uniformly in ε . We denote this regularized object by

$$(\rho_\kappa(t)a_i)_\varepsilon = (\rho_\kappa(t)a_i)_\varepsilon(x) = (\rho_\kappa(\omega, t)a_i)_\varepsilon(x).$$

We apply the pullback-extension operator \mathcal{L}_κ defined in (5.1), yielding

$$\mathcal{L}_\kappa (\rho_\kappa(t)a_i)_\varepsilon \in \text{SS} (\mathcal{T}_1^0(M)).$$

Finally, we define the companion vector field $(\rho_\kappa)_\varepsilon(t)a_i \in \text{SS} (\mathcal{T}_1^0(M))$.

We need the following version of a well-known result found in [10].

Lemma 5.2 (DiPerna–Lions Commutator; “ $\text{div}_h(\rho a_i)$ ”). Fix $\kappa \in \mathcal{A}$, cf. Lemma 2.3, and define for $(\omega, t, x) \in \Omega \times [0, T] \times M$ the smooth (in x) functions

$$r_{\kappa,\varepsilon,i}(\omega, t, x) := \text{div}_h \mathcal{L}_\kappa (\rho_\kappa(t)a_i)_\varepsilon(x) - \text{div}_h((\rho_\kappa)_\varepsilon(t)a_i)(x), \tag{5.5}$$

for $i = 1, \dots, N$. Then $r_{\kappa,\varepsilon,i} \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^q([0, T]; L^2(\Omega \times M))$, for any $q \in [1, \infty)$.

Furthermore, for $x \in X_\kappa$ (the only relevant case), in the coordinates induced by κ , we have the representations

$$\begin{aligned} a_i (r_{\kappa,\varepsilon,i}) &= a_i^m \partial_{ml} (\rho_\kappa a_i^l)_\varepsilon - a_i^m a_i^l \partial_{ml}(\rho_\kappa)_\varepsilon - a_i^m \partial_m a_i^l \partial_l(\rho_\kappa)_\varepsilon \\ &\quad - a_i^m \partial_l a_i^l \partial_m(\rho_\kappa)_\varepsilon - (\rho_\kappa)_\varepsilon a_i^m \partial_{ml} a_i^l, \end{aligned}$$

and

$$\text{div}_h \mathcal{L}_\kappa (\rho_\kappa(t)a_i)_\varepsilon = \partial_l (\rho_\kappa(t)a_i^l)_\varepsilon.$$

Proof. Let $x \in X_\kappa$. Then, in the coordinates induced by κ , we have by definition that $((\rho_\kappa a_i)_\varepsilon)^l = (\rho_\kappa a_i^l)_\varepsilon$ ($l = 1, \dots, d$) and $\Gamma_{ab}^a = 0$; therefore, div_h coincides with the Euclidean divergence and thus

$$\text{div}_h \mathcal{L}_\kappa (\rho_\kappa(s)a_i)_\varepsilon = \partial_l (\rho_\kappa(s)a_i^l)_\varepsilon.$$

Moreover,

$$r_{\kappa,\varepsilon,i} = \partial_l (\rho_\kappa a_i^l)_\varepsilon - \partial_l(\rho_\kappa)_\varepsilon a_i^l - (\rho_\kappa)_\varepsilon \partial_l a_i^l.$$

Differentiating this expression according to a_i leads to the claimed representation for $a_i (r_{\kappa,\varepsilon,i})$. The convergence claim is also clear. Indeed, with “ $\rho a_i \in L_t^\infty L_x^2$ ”, repeated applications of Lemma A.1 lead to $r_{\kappa,\varepsilon,i} \rightarrow 0$ in $L^q([0, T]; L^2(\Omega \times \tilde{X}_\kappa))$, for any $q \in [1, \infty)$. Because the support of $r_{\kappa,\varepsilon,i}$ is compactly contained in X_κ and $|h_\kappa|^{\frac{1}{2}} = 1$ therein, we immediately deduce the result. \square

For $(\omega, t, x) \in \Omega \times [0, T] \times M$, we define the function

$$r_{\varepsilon,i}(\omega, t, x) := \sum_{\kappa \in \mathcal{A}} r_{\kappa,\varepsilon,i}(\omega, t, x) \tag{5.6}$$

and the vector field

$$(\rho(t)a_i)_\varepsilon(x) := \sum_{\kappa} \mathcal{L}_\kappa(\rho_\kappa(t)a_i)_\varepsilon(x),$$

which both are smooth in x , $i = 1, \dots, N$. Clearly, as $\varepsilon \rightarrow 0$, the global remainder function $r_{\varepsilon,i}$ converges to zero in $L^q([0, T]; L^2(\Omega \times M))$, for any $q \in [1, \infty)$.

5.2.3. Localization & smoothing of $\rho \nabla_{a_i} a_i$.

Consider for $\omega \in \Omega$, $t \in [0, T]$, and $z \in \tilde{X}_\kappa$, the vector field

$$\{\rho_\kappa(\omega, t, z)(\nabla_{a_i} a_i)^l(z)\}_l,$$

which is an object in $\text{RS} \left(\mathcal{T}_1^0(\tilde{X}_\kappa) \right)$ that is compactly supported in $\kappa \left(\text{supp} \mathcal{U}_\kappa \right) \subset \tilde{X}_\kappa$, uniformly in ω, t . Observe that

$$\rho_\kappa(\nabla_{a_i} a_i)^l = \rho_\kappa a_i^m \partial_m a_i^l + \rho_\kappa \Gamma_{mb}^l a_i^m a_i^b, \quad l = 1, \dots, d.$$

We regularize $\rho_\kappa \nabla_{a_i} a_i$ componentwise using the mollifier ϕ_ε , denoting the result by $(\rho_\kappa(t) \nabla_{a_i} a_i)_\varepsilon$. By definition, $\left((\rho_\kappa(t) \nabla_{a_i} a_i)_\varepsilon \right)^l = \left((\rho_\kappa(t) \nabla_{a_i} a_i)^l \right)_\varepsilon$. We apply the pullback-extension operator \mathcal{L}_κ , arriving at the compactly supported vector field

$$\mathcal{L}_\kappa \left(\rho_\kappa(s) \nabla_{a_i} a_i \right)_\varepsilon \in \text{SS} \left(\mathcal{T}_1^0(M) \right).$$

Also in this case, we define the companion vector field $(\rho_\kappa)_\varepsilon(t) \nabla_{a_i} a_i \in \text{SS} \left(\mathcal{T}_1^0(M) \right)$.

Lemma 5.3 (DiPerna–Lions Commutator; “ $\text{div}_h(\rho \nabla_{a_i} a_i)$ ”). Fix $\kappa \in \mathcal{A}$, cf. Lemma 2.3, and define for $(\omega, t, x) \in \Omega \times [0, T] \times M$ the smooth (in x) functions

$$\tilde{r}_{\kappa, \varepsilon, i}(\omega, t, x) := \text{div}_h \mathcal{L}_\kappa \left(\rho_\kappa(t) \nabla_{a_i} a_i \right)_\varepsilon(x) - \text{div}_h \left((\rho_\kappa)_\varepsilon(t) \nabla_{a_i} a_i \right)(x),$$

for $i = 1, \dots, N$. Then $\tilde{r}_{\kappa, \varepsilon, i} \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^q([0, T]; L^2(\Omega \times M))$, for any $q \in [1, \infty)$.

Furthermore, for $x \in X_\kappa$ (the only relevant case), in the coordinates induced by κ , we have the representation

$$\text{div}_h \mathcal{L}_\kappa \left(\rho_\kappa(t) \nabla_{a_i} a_i \right)_\varepsilon = \partial_l \left(a_i^m \partial_m a_i^l \rho_\kappa(t) \right)_\varepsilon(z) + \partial_l \left(a_i^m a_i^r \Gamma_{mr}^l \rho_\kappa(t) \right)_\varepsilon(z).$$

Proof. The proof is identical to the one of Lemma 5.2, since the vector fields $\nabla_{a_1} a_1, \dots, \nabla_{a_N} a_N$ are smooth. \square

We also introduce the global function

$$\tilde{r}_{\varepsilon, i}(\omega, t, x) := \sum_{\kappa \in \mathcal{A}} \tilde{r}_{\kappa, \varepsilon, i}(\omega, t, x) \tag{5.7}$$

and the global vector field

$$\left(\rho(t) \nabla_{a_i} a_i \right)_\varepsilon(x) := \sum_{\kappa \in \mathcal{A}} \mathcal{L}_\kappa \left(\rho_\kappa(t) \nabla_{a_i} a_i \right)_\varepsilon(x),$$

which both are smooth in x and defined for $(\omega, t, x) \in \Omega \times [0, T] \times M, i = 1, \dots, N$. Clearly, as $\varepsilon \rightarrow 0$, we have $\tilde{r}_{\varepsilon, i} \rightarrow 0$ in $L^q([0, T]; L^2(\Omega \times M))$, for any $q \in [1, \infty)$.

5.2.4. Localization & smoothing of $\rho \hat{a}_i$

Recalling Remark 2.1 ($\hat{a}_i^{ml} = a_i^m a_i^l$), let us consider $\{ \rho_\kappa(\omega, t, z) \hat{a}_i^{ml}(z) \}_{m, l}$, for $\omega \in \Omega, t \in [0, T]$, and $z \in \tilde{X}_\kappa$, which defines a symmetric object in $\text{RS} \left(\mathcal{T}_2^0(\tilde{X}_\kappa) \right)$ that is compactly supported in $\kappa \left(\text{supp} \mathcal{U}_\kappa \right) \subset \tilde{X}_\kappa$, uniformly in ω, t . We regularize this object componentwise using the mollifier ϕ_ε , thereby obtaining a symmetric element in $\text{SS} \left(\mathcal{T}_2^0(\tilde{X}_\kappa) \right)$, whose support is contained in \tilde{X}_κ , uniformly in ε . We denote this smooth (0, 2)-tensor field by $(\rho_\kappa(t) \hat{a}_i)_\varepsilon$; clearly, by definition, $\left((\rho_\kappa(t) \hat{a}_i)_\varepsilon \right)^{ml} = \left(\rho_\kappa(s) \hat{a}_i^{ml} \right)_\varepsilon$. Applying the pullback-extension operator \mathcal{L}_κ , we obtain

$$\mathcal{L}_\kappa \left((\rho_\kappa(t) \hat{a}_i)_\varepsilon \right) \in \text{SS} \left(\mathcal{T}_2^0(M) \right),$$

with support in X_κ , uniformly in ε , and symmetric. We also need the globally defined object

$$(\rho(t)\hat{a}_i)_\varepsilon(x) := \sum_\kappa \mathcal{L}_\kappa (\rho_\kappa(t)\hat{a}_i)_\varepsilon(x),$$

for $(\omega, t, x) \in \Omega \times [0, T] \times M$ and, cf. (5.4), $\varepsilon < \varepsilon_0$.

Let us compute $\operatorname{div}_h^2 \mathcal{L}_\kappa ((\rho_\kappa(t)\hat{a}_i)_\varepsilon)$ in the local coordinates given by Lemma 2.3. The only relevant case is $x \in X_\kappa$, where we use the coordinates induced by κ . Recall that in these coordinates we have $\Gamma_{j\alpha}^\alpha = 0$ for all j . Hence,

$$\begin{aligned} \operatorname{div}_h^2 \mathcal{L}_\kappa ((\rho_\kappa(t)\hat{a}_i)_\varepsilon) &= \partial_{ml} ((\rho_\kappa(t)\hat{a}_i)_\varepsilon)^{ml}(z) + \partial_l \left[\Gamma_{mj}^l ((\rho_\kappa(s)\hat{a}_i)_\varepsilon)^{mj} \right](z) \\ &= \partial_{ml} (\rho_\kappa(t)\hat{a}_i^{ml})_\varepsilon + \partial_l \left[\Gamma_{mj}^l (\rho_\kappa(t)\hat{a}_i^{mj})_\varepsilon \right](z). \end{aligned} \tag{5.8}$$

To be able to control the regularization error linked to $\operatorname{div}_h^2(\rho\hat{a}_i)$, we need first to consider some additional terms appearing in the definition of div_h^2 that is related to the Christoffel symbols Γ of the Levi Civita connection. To this end, consider

$$V_{\kappa,i} := \left\{ \rho_\kappa(\omega, t, z) \Gamma_{mj}^l(z) \hat{a}_i^{mj}(z) \right\}_l, \tag{5.9}$$

which belongs to $\operatorname{RS}(\mathcal{T}_1^0(\tilde{X}_\kappa))$ and is compactly supported in $\kappa(\operatorname{supp} \mathcal{U}_\kappa) \subset \tilde{X}_\kappa$, uniformly in ω, t . The regularized counterpart of $V_{\kappa,i}$ is denoted by $V_{\kappa,i,\varepsilon}$. Clearly, $(V_{\kappa,i,\varepsilon})^l = (\rho_\kappa(t) \Gamma_{mj}^l \hat{a}_i^{mj})_\varepsilon$. Applying the pullback-extension operator \mathcal{L}_κ yields

$$\mathcal{L}_\kappa V_{\kappa,i,\varepsilon} \in \operatorname{SS}(\mathcal{T}_1^0(M)).$$

We multiply the components of $(\rho_\kappa(t)\hat{a}_i)_\varepsilon$ by the Christoffel symbols Γ (written in the coordinates induced by κ) and then add them together. The result is

$$\bar{V}_{\kappa,i,\varepsilon} := \left\{ \Gamma_{mj}^l (\rho_\kappa(t)\hat{a}_i^{mj})_\varepsilon \right\}_l, \tag{5.10}$$

an object in $\operatorname{SS}(\mathcal{T}_1^0(\tilde{X}_\kappa))$ that is compactly supported, uniformly in ε . Pushing forward $\bar{V}_{\kappa,i,\varepsilon}$ to M via \mathcal{L}_κ , we obtain

$$\mathcal{L}_\kappa \bar{V}_{\kappa,i,\varepsilon} \in \operatorname{SS}(\mathcal{T}_1^0(M)).$$

For $x \in X_\kappa$, in the coordinates given by κ ($z = \kappa(x)$), we have

$$\begin{aligned} \operatorname{div}_h \mathcal{L}_\kappa V_{\kappa,i,\varepsilon} - \operatorname{div}_h \mathcal{L}_\kappa \bar{V}_{\kappa,i,\varepsilon} &= \partial_l \left[(\rho_\kappa(t) \Gamma_{mj}^l \hat{a}_i^{mj})_\varepsilon(z) - \Gamma_{mj}^l (\rho_\kappa(t)\hat{a}_i^{mj})_\varepsilon(z) \right]. \end{aligned} \tag{5.11}$$

We also need the globally defined objects

$$V_{i,\varepsilon}(\omega, t, x) := \sum_{\kappa \in \mathcal{A}} \mathcal{L}_\kappa V_{\kappa,i,\varepsilon}(\omega, t, x), \quad \bar{V}_{i,\varepsilon}(\omega, t, x) := \sum_{\kappa \in \mathcal{A}} \mathcal{L}_\kappa \bar{V}_{\kappa,i,\varepsilon}(\omega, t, x),$$

for $(\omega, t, x) \in \Omega \times [0, T] \times M$ and $\varepsilon < \varepsilon_0$.

Finally, consider the smooth (in x) and compactly supported (in \tilde{X}_κ) vector field $\{(\rho_\kappa)_\varepsilon(t) \Gamma_{mj}^l \hat{a}_i^{mj}\}_l$, denoted by $(\rho_\kappa)_\varepsilon \gamma_i$. Applying \mathcal{L}_κ , we obtain

$$\mathcal{L}_\kappa [(\rho_\kappa)_\varepsilon \gamma_i] \in \operatorname{SS}(\mathcal{T}_1^0(M)),$$

which is compactly supported in X_κ , uniformly in ε .

Lemma 5.4 (DiPerna–Lions Commutator; “ $\operatorname{div}_h(\rho\Gamma\hat{a})$ ”). Fix $\kappa \in \mathcal{A}$, cf. Lemma 2.3, and define for $(\omega, t, x) \in \Omega \times [0, T] \times M$ the smooth (in x) functions

$$\bar{r}_{\kappa,\varepsilon,i}(\omega, t, x) := \operatorname{div}_h \mathcal{L}_\kappa V_{\kappa,i,\varepsilon}(x) - \operatorname{div}_h \mathcal{L}_\kappa [(\rho_\kappa(t))_\varepsilon \gamma_i](x) \tag{5.12}$$

for $i = 1, \dots, N$. Then, $\bar{r}_{\kappa,\varepsilon,i} \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^q([0, T]; L^2(\Omega \times M))$, for any $q \in [1, \infty)$. Moreover, in X_κ (the only relevant case) in the coordinates induced by κ ($z = \kappa(x)$), $\bar{r}_{\kappa,\varepsilon,i}$ takes the form

$$\bar{r}_{\kappa,\varepsilon,i}(\omega, t, z) = \partial_l \left(\rho_\kappa(t) \Gamma_{mj}^l \hat{a}_i^{mj} \right)_\varepsilon(z) - \partial_l \left((\rho_\kappa(t))_\varepsilon \Gamma_{mj}^l \hat{a}_i^{mj} \right)(z).$$

Proof. See the proofs of Lemmas 5.2 and 5.3. \square

As before, we introduce the global function

$$\bar{r}_{\varepsilon,i}(\omega, t, x) := \sum_{\kappa \in \mathcal{A}} \bar{r}_{\kappa,\varepsilon,i}(\omega, t, x), \tag{5.13}$$

with $(\omega, t, x) \in \Omega \times [0, T] \times M$ and, cf. (5.4), $\varepsilon < \varepsilon_0$, $i = 1, \dots, N$. Obviously, as $\varepsilon \rightarrow 0$, we have $\bar{r}_{\varepsilon,i} \rightarrow 0$ in $L^q([0, T]; L^2(\Omega \times M))$, for any $q \in [1, \infty)$.

Now we are now going to analyze the key terms ($i = 1, \dots, N$)

$$\begin{aligned} R_{\varepsilon,i}(\omega, t, x) &:= \operatorname{div}_h^2 (\rho(t)\hat{a}_i)_\varepsilon(x) - \operatorname{div}_h^2 (\rho_\varepsilon(t)\hat{a}_i)(x) \\ &\quad + \operatorname{div}_h V_{i,\varepsilon}(t, x) - \operatorname{div}_h \bar{V}_{i,\varepsilon}(t, x), \end{aligned} \tag{5.14}$$

where $\rho_\varepsilon(t)\hat{a}_i := \sum_{\kappa \in \mathcal{A}} (\rho_\kappa)_\varepsilon(t)\hat{a}_i \in \operatorname{SS}(\mathcal{T}_2^0(M))$. By definition,

$$\begin{aligned} R_{\varepsilon,i}(\omega, t, x) &= \sum_{\kappa \in \mathcal{A}} \left\{ \operatorname{div}_h^2 \mathcal{L}_\kappa (\rho_\kappa(t)\hat{a}_i)_\varepsilon(x) - \operatorname{div}_h^2 ((\rho_\kappa)_\varepsilon(t)\hat{a}_i)(x) \right. \\ &\quad \left. + \operatorname{div}_h \mathcal{L}_\kappa V_{\kappa,i,\varepsilon}(t, x) - \operatorname{div}_h \mathcal{L}_\kappa \bar{V}_{\kappa,i,\varepsilon}(t, x) \right\} \\ &=: \sum_{\kappa \in \mathcal{A}} R_{\kappa,\varepsilon,i}(\omega, t, x). \end{aligned}$$

Fix $\kappa \in \mathcal{A}$, cf. Lemma 2.3. The quantity $R_{\kappa,\varepsilon,i}(t, \cdot)$ is supported in $X_\kappa \subset M$, and there we are going to use the coordinates induced by κ , $z = \kappa(x)$. From the definition of div_h^2 , cf. (2.3), and by means of formulas (5.8) and (5.11), we deduce

$$\begin{aligned} R_{\kappa,\varepsilon,i}(t, z) &= \partial_{ml} (\rho_\kappa(t)\hat{a}_i^{ml})_\varepsilon(z) + \partial_l \left[\Gamma_{mj}^l (\rho_\kappa(t)\hat{a}_i^{mj})_\varepsilon \right](z) \\ &\quad - \partial_{ml} ((\rho_\kappa)_\varepsilon(t)\hat{a}_i^{ml})(z) - \partial_l \left[\Gamma_{mj}^l (\rho_\kappa)_\varepsilon(t)\hat{a}_i^{mj} \right](z) \\ &\quad + \partial_l \left[(\rho_\kappa(t)\Gamma_{mj}^l \hat{a}_i^{mj})_\varepsilon \right](z) - \partial_l \left[\Gamma_{mj}^l (\rho_\kappa(t)\hat{a}_i^{mj})_\varepsilon \right](z) \\ &= \partial_{ml} (\rho_\kappa(t)\hat{a}_i^{ml})_\varepsilon(z) - \partial_{ml} ((\rho_\kappa)_\varepsilon(t)\hat{a}_i^{ml})(z) + \bar{r}_{\kappa,\varepsilon,i}(t, z), \end{aligned}$$

where the reminder $\bar{r}_{\kappa,\varepsilon,i}$ is defined in (5.12)

Remark 5.3. Be mindful of the fact that we have computed $R_{\varepsilon,i}$ in the (convenient) coordinates provided by Lemma 2.3. With a different choice of coordinates, we would need to handle some additional terms involving the Christoffel symbols Γ_{ab}^b , further complicating the analysis.

By expanding $-\partial_{ml} ((\rho_\kappa)_\varepsilon(t)\hat{a}_i^{ml})(z)$,

$$\begin{aligned} \partial_{ml} ((\rho_\kappa)_\varepsilon \hat{a}_i^{ml}) &= \partial_{ml} (\rho_\kappa)_\varepsilon \hat{a}_i^{ml} + 2\partial_l (\rho_\kappa)_\varepsilon \partial_m a_i^m a_i^l \\ &\quad + 2\partial_l (\rho_\kappa)_\varepsilon \partial_m a_i^l a_i^m + (\rho_\kappa)_\varepsilon \partial_{ml} (a_i^m a_i^l), \end{aligned}$$

and making use of Lemma 5.2, we arrive at

$$R_{\kappa,\varepsilon,i}(t, z) = 2C_\varepsilon [\rho_\kappa(t), a_i](z) + 2a_i(r_{\kappa,\varepsilon,i}(t, z)) + 2(\rho_\kappa)_\varepsilon(t)a_i^m \partial_{ml} a_i^l(z) - (\rho_\kappa)_\varepsilon(t) \partial_{ml} (a_i^m a_i^l)(z) + \bar{r}_{\kappa,\varepsilon,i}(t, z),$$

where

$$C_\varepsilon [\rho_\kappa(t), a_i] := \frac{1}{2} \partial_{ml} (\rho_\kappa \hat{a}_i^{ml})_\varepsilon - a_i^m \partial_{ml} (\rho_\kappa a_i^l)_\varepsilon + \frac{1}{2} \partial_{ml} (\rho_\kappa)_\varepsilon \hat{a}_i^{ml}. \tag{5.15}$$

We recognize $C_\varepsilon [\rho_\kappa(t), a_i]$ as a “second order” commutator, first identified in [33] (cf. the appendix herein for more details). Using again the product rule,

$$R_{\kappa,\varepsilon,i}(t, z) = 2 \left\{ C_\varepsilon [\rho_\kappa(t), a_i](z) - \frac{1}{2} (\rho_\kappa)_\varepsilon (\partial_m a_i^m)^2 - \frac{1}{2} (\rho_\kappa)_\varepsilon \partial_l a_i^m \partial_m a_i^l \right\} + 2a_i (r_{\kappa,\varepsilon,i}(t, z)) + \bar{r}_{\kappa,\varepsilon,i}(t, z), \tag{5.16}$$

where $r_{\kappa,\varepsilon,i}$ is defined in (5.5). For convenience, let us set $g_{\kappa,i} := \partial_l a_i^m \partial_m a_i^l$. In our coordinates (cf. Lemma 2.3), $\text{div}_h a_i = \partial_m a_i^m$. By compactness of the supports of the involved functions, the quantities $(\rho_\kappa)_\varepsilon \partial_l a_i^m \partial_m a_i^l$ and $C_\varepsilon [\rho_\kappa(t), a_i]$ can be thought of as globally defined (on M). In other words, (5.16) may be seen as a global identity on M , that is, for $x \in M$,

$$R_{\kappa,\varepsilon,i}(t, x) = 2G_{\kappa,\varepsilon,i}(t, x) + 2a_i (r_{\kappa,\varepsilon,i}(t, x)) + \bar{r}_{\kappa,\varepsilon,i}(t, x), \tag{5.17}$$

where

$$G_{\kappa,\varepsilon,i}(\omega, t, x) := C_\varepsilon [\rho_\kappa(t), a_i](x) - \frac{1}{2} (\rho_\kappa)_\varepsilon (\text{div}_h a_i(x))^2 - \frac{1}{2} (\rho_\kappa)_\varepsilon g_{\kappa,i}(x). \tag{5.18}$$

If $x \notin X_\kappa$ for some κ , then (5.17) reduces to the trivial statement “ $0 = 0$ ”. Referring to the Appendix, a simple application of Lemma A.3 shows that $G_{\kappa,\varepsilon,i} \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^q([0, T]; L^2(\Omega \times M))$, for any $q \in [1, \infty)$.

Let us summarize our findings.

Lemma 5.5 (Second Order Commutator; “ $\text{div}_h^2(\rho \hat{a})$ ”). For $(\omega, t, x) \in \Omega \times [0, T] \times M$ and, cf. (5.4), $\varepsilon < \varepsilon_0$, the remainder $R_{\varepsilon,i}$ defined in (5.14) takes the form

$$R_{\varepsilon,i}(\omega, t, x) = 2a_i (r_{\varepsilon,i}(\omega, t, x)) + \bar{r}_{\varepsilon,i}(\omega, t, x) + 2G_{\varepsilon,i}(\omega, t, x), \tag{5.19}$$

where $\bar{r}_{\varepsilon,i}$ is defined in (5.12) and $G_{\varepsilon,i} := \sum_{\kappa \in \mathcal{A}} G_{\kappa,\varepsilon,i}$ with $G_{\kappa,\varepsilon,i}$ defined in (5.18).

Furthermore,

$$\bar{r}_{\varepsilon,i}, G_{\varepsilon,i} \xrightarrow{\varepsilon \downarrow 0} 0 \text{ in } L^q([0, T]; L^2(\Omega \times M)), \text{ for any } q \in [1, \infty).$$

Remark 5.4. Regarding the error term $G_{\varepsilon,i}$, in general we cannot “sum away” κ due to the nonlinearity of the domain M , which is manifested in the local nature of the commutator $C_\varepsilon [\rho_\kappa(t), a_i]$, cf. (5.15), and its dependence on different mollifiers!

Remark 5.5. In view of (5.19), we do not expect the remainder term

$$R_{\varepsilon,i} = \text{div}_h^2 \left((\rho(t) \hat{a}_i)_\varepsilon - \rho_\varepsilon(t) \hat{a}_i \right) + \text{div}_h \left(V_{i,\varepsilon}(t) - \bar{V}_{i,\varepsilon}(t) \right)$$

to converge to zero as $\varepsilon \rightarrow 0$, although $\bar{r}_{\varepsilon,i}$ and $G_{\varepsilon,i}$ do! Indeed, there is no reason to expect $a_i (r_{\varepsilon,i})$ to have a limit as $\varepsilon \rightarrow 0$. By good fortune, it turns out that this quantity is going to

cancel out with a term that appears when applying the Itô formula, see the upcoming equation (5.38). This cancellation is the reason why the renormalization property holds for weak L^2 solutions without having to assume some kind of “parabolic” regularity (cf. the discussion in Section 1).

5.2.5. Localization & smoothing of the vector field u

For $\omega \in \Omega, t \in [0, T]$, and $z \in \tilde{X}_\kappa$, consider the object $\{\rho_\kappa(\omega, t, z)u^l(t, z)\}_l \in \text{RS}(\mathcal{T}_1^0(\tilde{X}_\kappa))$, that is compactly supported in κ ($\text{supp} \mathcal{U}_\kappa \subset \tilde{X}_\kappa$, uniformly in $\omega \in \Omega, s \in [0, T]$). As before, we regularize $\{\rho_\kappa(\omega, t, z)u^l(t, z)\}_l$ componentwise via the mollifier ϕ_ε . The result is an object in $\text{SS}(\mathcal{T}_1^0(\tilde{X}_\kappa))$ that is compactly supported, uniformly in ε . We denote the regularized object by $(\rho_\kappa(t)u(t))_\varepsilon$. Applying the pullback-extension operator \mathcal{L}_κ ,

$$\mathcal{L}_\kappa(\rho_\kappa(t)u(t))_\varepsilon \in \text{SS}(\mathcal{T}_1^0(M)).$$

Finally, we define the companion vector field $(\rho_\kappa)_\varepsilon(t)u(t)$, which for a.e. t belongs to $W^{1,2}(\mathcal{T}_1^0(M))$ since by assumption $u \in L^1([0, T]; W^{1,2}(\mathcal{T}_1^0(M)))$.

Lemma 5.6 (DiPerna–Lions Commutator; “ $\text{div}_h(\rho u)$ ”). Fix $\kappa \in \mathcal{A}$, cf. Lemma 2.3, and define for $(\omega, t, x) \in \Omega \times [0, T] \times M$ the smooth (in x) function

$$r_{\kappa,\varepsilon,u}(\omega, t, x) := \text{div}_h \mathcal{L}_\kappa(\rho_\kappa(t)u(t))_\varepsilon(x) - \text{div}_h((\rho_\kappa)_\varepsilon(t)u(t))(x).$$

Then $r_{\kappa,\varepsilon,u} \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^1([0, T]; L^2(\Omega; L^1(M)))$.

Besides, in X_κ (the only relevant case) in the coordinates induced by κ ,

$$\text{div}_h \mathcal{L}_\kappa(\rho_\kappa(t)u(t))_\varepsilon = \partial_l(\rho_\kappa(t)u^l(t))_\varepsilon.$$

Proof. At this point, only the convergence claim needs some explanation. Since u is deterministic, cf. (2.6),

$$u \in L^1([0, T]; L^\infty(\Omega; W^{1,2}(\mathcal{T}_1^0(M)))).$$

Moreover, $\rho \in L^\infty([0, T]; L^2(\Omega \times M))$. Fixing t , we apply Lemma A.1 with $Z = \Omega, G = \tilde{X}_\kappa, p_1 = p_2 = q_1 = 2, q_2 = \infty$ to obtain $r_{\kappa,\varepsilon,u}(t) \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^2(\Omega; L^1(M))$. Utilizing the dominated convergence theorem in t and the bounds provided by Lemma A.1, we conclude $r_{\kappa,\varepsilon,u} \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^1([0, T]; L^2(\Omega; L^1(M)))$. \square

For $(\omega, t, x) \in \Omega \times [0, T] \times M$, we define the global remainder function

$$r_{\varepsilon,u}(\omega, t, x) := \sum_{\kappa \in \mathcal{A}} r_{\kappa,\varepsilon,u}(\omega, t, x), \tag{5.20}$$

which belongs to $L^1([0, T]; L^2(\Omega; L^1(M)))$ and is smooth in x . Likewise, we define the global smooth vector field

$$(\rho(t)u(t))_\varepsilon(x) := \sum_{\kappa} \mathcal{L}_\kappa(\rho_\kappa(t)u(t))_\varepsilon(x).$$

Clearly, $r_{\varepsilon,u} \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^1([0, T]; L^2(\Omega; L^1(M)))$.

5.2.6. Localization & smoothing of partition of unity terms

For reasons that will become apparent later, we need to apply the machinery developed so far to some additional terms that are related to the partition of unity $\{\mathcal{U}_\kappa\}_\kappa$ and its derivatives. These terms are linked to the nonlinear geometry of the manifold M .

For $\omega \in \Omega$, $t \in [0, T]$, and $z \in \tilde{X}_\kappa$, we introduce the functions

$$\begin{aligned} A_{\kappa,i}^1(\omega, t, z) &:= \rho(\omega, t, z) a_i(\mathcal{U}_\kappa)(z), \\ A_{\kappa,i}^2(\omega, t, z) &:= \rho(\omega, t, z) \nabla^2 \mathcal{U}_\kappa(a_i, a_i)(z), \\ A_{\kappa,i}^3(\omega, t, z) &:= \rho(\omega, t, z) (\nabla_{a_i} a_i)(\mathcal{U}_\kappa)(z), \end{aligned} \tag{5.21}$$

cautioning the reader that the superscripts do not mean exponentiation. Observe that these functions have their supports contained in $\text{supp } \mathcal{U}_\kappa$, uniformly in ω, t . Besides, recalling the local expressions for $\nabla^2 \mathcal{U}_\kappa$ and $(\nabla_{a_i} a_i)$,

$$\begin{aligned} A_{\kappa,i}^2(t, z) &= \rho(t, z) \left[\partial_{ml} \mathcal{U}_\kappa(z) a_i^m(z) a_i^l(z) - \Gamma_{ml}^j(z) \partial_j \mathcal{U}_\kappa(z) a_i^m(z) a_i^l(z) \right], \\ A_{\kappa,i}^3(t, z) &= \rho(t, z) \left[\partial_l \mathcal{U}_\kappa(z) \partial_m a_i^l(z) a_i^m(z) + \Gamma_{ml}^j(z) \partial_j \mathcal{U}_\kappa(z) a_i^m(z) a_i^l(z) \right]. \end{aligned}$$

We regularize these functions using the mollifier ϕ_ε and then apply the pullback-extension operator (5.1). The next lemma is analogous to Lemma 5.1, with the proof being evident at this stage.

Lemma 5.7. Fix $\kappa \in \mathcal{A}$, cf. Lemma 2.3. Then

- (1) $(A_{\kappa,i}^1)_\varepsilon(\omega, t, \cdot)$, $(A_{\kappa,i}^2)_\varepsilon(\omega, t, \cdot)$, and $(A_{\kappa,i}^3)_\varepsilon(\omega, t, \cdot)$ belong to $C^\infty(\mathbb{R}^d)$, for each fixed $(\omega, t) \in \Omega \times [0, T]$.
- (2) For $\varepsilon < \varepsilon_\kappa$, cf. (5.4), and for any $(\omega, t) \in \Omega \times [0, T]$, the supports of the functions in (1) are contained in

$$\kappa(\text{supp } \mathcal{U}_\kappa) + \overline{B_\varepsilon(0)} \subset \subset \tilde{X}_\kappa.$$

This implies in particular that the functions $(A_{\kappa,i}^1)_\varepsilon(\omega, t, \cdot)$, $(A_{\kappa,i}^2)_\varepsilon(\omega, t, \cdot)$, and $(A_{\kappa,i}^3)_\varepsilon(\omega, t, \cdot)$ can be seen as elements of $C^\infty(M)$, provided that we set them equal to zero outside of X_κ , for each fixed $(\omega, t) \in \Omega \times [0, T]$.

- (3) For any $p \in [1, 2]$ and $(\omega, t) \in \Omega \times [0, T]$,

$$\begin{aligned} (A_{\kappa,i}^1)_\varepsilon(\omega, t, \cdot) &\xrightarrow{\varepsilon \downarrow 0} \rho(\omega, t, \cdot) a_i(\mathcal{U}_\kappa)(\cdot), \\ (A_{\kappa,i}^2)_\varepsilon(\omega, t, \cdot) &\xrightarrow{\varepsilon \downarrow 0} \rho(\omega, t, \cdot) \nabla^2 \mathcal{U}_\kappa(a_i, a_i)(\cdot), \\ (A_{\kappa,i}^3)_\varepsilon(\omega, t, \cdot) &\xrightarrow{\varepsilon \downarrow 0} \rho(\omega, t, \cdot) (\nabla_{a_i} a_i)(\mathcal{U}_\kappa)(\cdot), \end{aligned}$$

where the convergences taking place in $L^p(M)$, and

$$\begin{aligned} (A_{\kappa,i}^1)_\varepsilon &\xrightarrow{\varepsilon \downarrow 0} \rho a_i(\mathcal{U}_\kappa), & (A_{\kappa,i}^2)_\varepsilon &\xrightarrow{\varepsilon \downarrow 0} \rho \nabla^2 \mathcal{U}_\kappa(a_i, a_i), \\ (A_{\kappa,i}^3)_\varepsilon &\xrightarrow{\varepsilon \downarrow 0} \rho (\nabla_{a_i} a_i)(\mathcal{U}_\kappa), \end{aligned}$$

in $L^q([0, T]; L^2(\Omega \times M))$, for any $q \in [1, \infty)$.

Define for $(\omega, t, x) \in \Omega \times [0, T] \times M$, the global functions

$$A_{i,\varepsilon}^j(\omega, t, x) := \sum_{\kappa} \mathcal{L}_{\kappa} \left(A_{\kappa,i}^j \right)_{\varepsilon} (\omega, t, x), \quad j = 1, \dots, 3, \tag{5.22}$$

where the pullback-extension operator \mathcal{L}_{κ} is defined in (5.1).

Finally, for $\omega \in \Omega, t \in [0, T]$, and $z \in \tilde{X}_{\kappa}$, consider the vector field

$$A_{\kappa,i}^4(\omega, t, z) := \left\{ \rho(\omega, t, z) a_i(\mathcal{U}_{\kappa}) a_i(z)^l \right\}_l, \tag{5.23}$$

which belongs to $\text{RS} \left(\mathcal{T}_1^0(\tilde{X}_{\kappa}) \right)$ and is compactly supported in $\kappa(\text{supp} \mathcal{U}_{\kappa}) \subset \tilde{X}_{\kappa}$, uniformly in ω, t . Following our (by now) standard procedure, we regularize $A_{\kappa,i}^4$ componentwise via the mollifier ϕ_{ε} , cf. (5.2). We denote the resulting object by $(A_{\kappa,i}^4)_{\varepsilon}$, and observe that, by definition, $\left((A_{\kappa,i}^4)_{\varepsilon} \right)^l = (\rho a_i(\mathcal{U}_{\kappa}) a_i^l)_{\varepsilon}$. We apply the pullback-extension operator \mathcal{L}_{κ} to obtain the compactly supported vector field

$$\mathcal{L}_{\kappa} \left(A_{\kappa,i}^4 \right)_{\varepsilon} \in \text{SS} \left(\mathcal{T}_1^0(M) \right).$$

Summing over $\kappa \in \mathcal{A}$, we obtain the global object

$$A_{i,\varepsilon}^4(\omega, t, x) := \sum_{\kappa \in \mathcal{A}} \mathcal{L}_{\kappa} \left(A_{\kappa,i}^4 \right)_{\varepsilon} (\omega, t, x).$$

From the very definitions of $A_{i,\varepsilon}^1$ and $\mathcal{L}_{\kappa}(A_{\kappa,i}^4)_{\varepsilon}$, we have, for $x \in M$,

$$A_{i,\varepsilon}^1(t, x) a_i(x) = \sum_{\kappa \in \mathcal{A}} \left(A_{\kappa,i}^1 \right)_{\varepsilon} (t, x) a_i(x) = \sum_{\kappa \in \mathcal{A}} (\rho(t) a_i(\mathcal{U}_{\kappa}))_{\varepsilon} (x) a_i(x),$$

and

$$\mathcal{L}_{\kappa} \left(A_{\kappa,i}^4 \right)_{\varepsilon} (t, x) = \mathcal{L}_{\kappa} \left(A_{\kappa,i}^1(t) a_i \right)_{\varepsilon} (x).$$

We define for $(\omega, t, x) \in \Omega \times [0, T] \times M$ the smooth (in x) remainder function

$$\begin{aligned} r_{\kappa,\varepsilon,i}^* (\omega, t, x) &:= \text{div}_h \mathcal{L}_{\kappa} \left(A_{\kappa,i}^4 \right)_{\varepsilon} (t, x) - \text{div}_h \left((\rho(t) a_i(\mathcal{U}_{\kappa}))_{\varepsilon} a_i \right) (x) \\ &= \text{div}_h \mathcal{L}_{\kappa} \left(A_{\kappa,i}^1(t) a_i \right)_{\varepsilon} (x) - \text{div}_h \left((A_{\kappa,i}^1)_{\varepsilon} (t, x) a_i(x) \right). \end{aligned}$$

Observe that for $x \in X_{\kappa}$ in the coordinates given by $\kappa, z = \kappa(x)$, we have the representation

$$r_{\kappa,\varepsilon,i}^* (t, z) = \partial_l \left(\rho(t) a_i(\mathcal{U}_{\kappa}) a_i^l \right)_{\varepsilon} (z) - \partial_l \left((\rho(t) a_i(\mathcal{U}_{\kappa}))_{\varepsilon} a_i^l \right) (z).$$

By now the following lemma should be easy to prove.

Lemma 5.8 (DiPerna–Lions Commutator; “ $\text{div}_h((\rho a_i(\mathcal{U}_{\kappa})) a_i)$ ”). For any $q \in [1, \infty)$, $r_{\kappa,\varepsilon,i}^* \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^q([0, T]; L^2(\Omega \times M))$.

We define a global remainder function by summing over κ :

$$r_{\varepsilon,i}^* (\omega, t, x) := \sum_{\kappa \in \mathcal{A}} r_{\kappa,\varepsilon,i}^* (\omega, t, x), \quad (\omega, t, x) \in \Omega \times [0, T] \times M. \tag{5.24}$$

Clearly, $r_{\varepsilon,i}^* \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^q([0, T]; L^2(\Omega \times M))$, for all $q \in [1, \infty)$.

Finally, we introduce the function

$$A_{\kappa,u}(\omega, t, z) := \rho(\omega, t, z) u(t, z) (\mathcal{U}_{\kappa}), \quad (\omega, t, z) \in \Omega \times [0, T] \times \tilde{X}_{\kappa}, \tag{5.25}$$

which has support contained in $\text{supp } \mathcal{U}_\kappa$, uniformly in ω, t . We regularize $A_{\kappa,u}$ in z using the mollifier ϕ_ε , and then apply the pullback-extension operator (5.1) to produce a function defined on M . We state the following lemma (without proof), noting that it is related to Lemmas 5.1 and 5.7.

Lemma 5.9. Fix $\kappa \in \mathcal{A}$, cf. Lemma 2.3. Then

- (1) $(A_{\kappa,u})_\varepsilon(\omega, t, \cdot) \in C^\infty(\mathbb{R}^d)$, for $(\omega, t) \in \Omega \times [0, T]$.
- (2) For $\varepsilon < \varepsilon_\kappa$, cf. (5.4), and $(\omega, t) \in \Omega \times [0, T]$, the support of $(A_{\kappa,u})_\varepsilon(\omega, t, \cdot)$ is contained in

$$\kappa(\text{supp } \mathcal{U}_\kappa) + \overline{B_\varepsilon(0)} \subset \subset \tilde{X}_\kappa.$$

This implies in particular that $(A_{\kappa,u})_\varepsilon(\omega, t, \cdot)$ can be seen as a function in $C^\infty(M)$, provided that we set it equal to zero outside of X_κ .

- (3) $(A_{\kappa,u})_\varepsilon \xrightarrow{\varepsilon \downarrow 0} \rho u(\mathcal{U}_\kappa)$ in $L^1([0, T]; L^1(\Omega \times M))$.

We also introduce the globally defined function

$$A_{u,\varepsilon}(\omega, t, x) := \sum_\kappa (A_{\kappa,u})_\varepsilon(\omega, t, x), \quad (\omega, t, x) \in \Omega \times [0, T] \times M.$$

Clearly, $A_{u,\varepsilon} \xrightarrow{\varepsilon \downarrow 0} \rho u(1) = 0$ in $L^1([0, T]; L^1(\Omega \times M))$.

5.3. Localized SPDEs and regularization

Fix $\kappa \in \mathcal{A}$, $\kappa : X_\kappa \subset M \rightarrow \tilde{X}$, $z = \kappa(x)$, cf. Lemma 2.3, and recall that $\{\mathcal{U}_\kappa\}_{\kappa \in \mathcal{A}}$ denotes the partition of the unity introduced at the beginning of Section 5.2. By inserting into (3.2) the test function $\psi \mathcal{U}_\kappa$, $\psi \in C^\infty(M)$, we obtain, \mathbb{P} -a.s., for any $t \in [0, T]$,

$$\begin{aligned} \int_M \rho(t) \mathcal{U}_\kappa \psi \, dV_h &= \int_M \rho_0 \mathcal{U}_\kappa \psi \, dV_h + \sum_{i=1}^N \int_0^t \int_M \rho(s) \mathcal{U}_\kappa a_i(\psi) \, dV_h \, dW^i(s) \\ &+ \frac{1}{2} \sum_{i=1}^N \int_0^t \int_M \rho(s) \mathcal{U}_\kappa \left[(\nabla^2 \psi)(a_i, a_i) + (\nabla_{a_i} a_i)(\psi) \right] \, dV_h \, ds \\ &+ \sum_{i=1}^N \int_0^t \int_M \rho(s) \psi a_i(\mathcal{U}_\kappa) \, dV_h \, dW^i(s) \\ &+ \frac{1}{2} \sum_{i=1}^N \int_0^t \int_M \rho(s) \left[\psi (\nabla^2 \mathcal{U}_\kappa)(a_i, a_i) \right. \\ &\quad \left. + \psi (\nabla_{a_i} a_i)(\mathcal{U}_\kappa) + 2d\psi \otimes d\mathcal{U}_\kappa(a_i, a_i) \right] \, dV_h \, ds \\ &+ \int_0^t \int_M \left[\rho(s) \mathcal{U}_\kappa u(s)(\psi) + \rho(s) \psi u(s)(\mathcal{U}_\kappa) \right] \, dV_h \, ds, \end{aligned} \tag{5.26}$$

where we have used the tensorial identity

$$\nabla^2(fg) = g(\nabla^2 f) + f(\nabla^2 g) + df \otimes dg + dg \otimes df, \quad f, g \in C^2(M).$$

By making use of the coordinates induced by κ , and some of quantities introduced previously, Eq. (5.26) amounts to writing

$$\begin{aligned}
 \int_{\tilde{X}_\kappa} \rho_\kappa(t) \psi \, d\bar{z} &= \int_{\tilde{X}_\kappa} \rho_{0,\kappa} \psi \, d\bar{z} + \sum_{i=1}^N \int_0^t \int_{\tilde{X}_\kappa} \rho_\kappa(s) a_i(\psi) \, d\bar{z} \, dW^i(s) \\
 &+ \frac{1}{2} \sum_{i=1}^N \int_0^t \int_{\tilde{X}_\kappa} \rho_\kappa(s) \left[(\nabla^2 \psi)(a_i, a_i) + (\nabla_{a_i} a_i)(\psi) \right] \, d\bar{z} \, ds \\
 &+ \sum_{i=1}^N \int_0^t \int_{\tilde{X}_\kappa} A_{\kappa,i}^1(s) \psi \, d\bar{z} \, dW^i(s) \\
 &+ \frac{1}{2} \sum_{i=1}^N \int_0^t \int_{\tilde{X}_\kappa} \left[\psi A_{\kappa,i}^2(s) + \psi A_{\kappa,i}^3(s) + 2A_{\kappa,i}^4(s)(\psi) \right] \, d\bar{z} \, ds \\
 &+ \int_0^t \int_{\tilde{X}_\kappa} \left[\rho_\kappa(s) u(s)(\psi) + A_{\kappa,u}(s)\psi \right] \, d\bar{z} \, ds,
 \end{aligned} \tag{5.27}$$

where $\rho_\kappa, \rho_{0,\kappa}, A_{\kappa,i}^1, A_{\kappa,i}^2, A_{\kappa,i}^3, A_{\kappa,i}^4, A_{\kappa,u}$ are defined in (5.3), (5.21), (5.23), (5.25).

For convenience, set $\tilde{B}_\kappa := \kappa(\text{supp } \mathcal{U}_\kappa) + B_{\varepsilon_\kappa/2}(0)$. From now on, we consider only $\varepsilon < \varepsilon_\kappa/4$, cf. (5.4). Let us introduce the following family of test functions $\psi_{z,\varepsilon}$ (parametrized by $z \in \tilde{X}_\kappa$ and $\varepsilon < \varepsilon_\kappa/4$):

$$\psi_{z,\varepsilon}(\cdot) := \begin{cases} \phi_\varepsilon(z - \cdot), & \text{if } B_\varepsilon(z) \cap \partial \tilde{X}_\kappa = \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

We observe that these functions may be seen as elements of $C_c^\infty(\tilde{X}_\kappa)$ and that, for fixed $(\omega, t) \in \Omega \times [0, T]$, $\text{supp } ((\cdot \cdot)_\varepsilon(\omega, t, \cdot)) \subset \kappa(\text{supp } \mathcal{U}_\kappa) + \overline{B_\varepsilon(0)} \subset \tilde{B}_\kappa$, where $(\cdot \cdot)_\varepsilon$ denotes any one of the objects defined previously. Moreover, for $\omega \in \Omega, t \in [0, T]$, and $z \in \tilde{X}_\kappa \setminus \tilde{B}_\kappa$, we have that $(\cdot \cdot)_\varepsilon(\omega, t, z) = 0$ and $(\cdot \cdot)_\varepsilon(\omega, t, z)$ coincides with the action of $(\cdot \cdot)_\varepsilon(\omega, t)$ on the function $\psi_{z,\varepsilon}$.

We make use of $\psi_{z,\varepsilon}$ as test function in (5.27), which results in

$$\begin{aligned}
 (\rho_\kappa)_\varepsilon(t)(z) - (\rho_{0,\kappa})_\varepsilon(z) &= - \sum_{i=1}^N \int_0^t \int_{\tilde{X}_\kappa} \rho_\kappa(s) a_i^l(\bar{z}) (\partial_l \phi_\varepsilon)(z - \bar{z}) \, d\bar{z} \, dW^i(s) \\
 &+ \frac{1}{2} \sum_{i=1}^N \int_0^t \int_{\tilde{X}_\kappa} \rho_\kappa(s) (\partial_{lm} \phi_\varepsilon)(z - \bar{z}) a_i^m(\bar{z}) a_i^l(\bar{z}) \, d\bar{z} \, ds \\
 &- \frac{1}{2} \sum_{i=1}^N \int_0^t \int_{\tilde{X}_\kappa} \rho_\kappa(s) a_i^m(\bar{z}) \partial_m a_i^l(\bar{z}) (\partial_l \phi_\varepsilon)(z - \bar{z}) \, d\bar{z} \, ds \\
 &+ \sum_{i=1}^N \int_0^t \int_{\tilde{X}_\kappa} \rho(s) \phi_\varepsilon(z - \bar{z}) a_i^l(\bar{z}) \partial_l \mathcal{U}_\kappa(\bar{z}) \, d\bar{z} \, dW^i(s) \\
 &+ \frac{1}{2} \sum_{i=1}^N \int_0^t \int_{\tilde{X}_\kappa} \rho(s) \phi_\varepsilon(z - \bar{z}) \partial_{ml} \mathcal{U}_\kappa(\bar{z}) a_i^m(\bar{z}) a_i^l(\bar{z}) \, d\bar{z} \, ds \\
 &+ \frac{1}{2} \sum_{i=1}^N \int_0^t \int_{\tilde{X}_\kappa} \rho(s) \phi_\varepsilon(z - \bar{z}) a_i^l(\bar{z}) \partial_l a_i^m(\bar{z}) \partial_m \mathcal{U}_\kappa(\bar{z}) \, d\bar{z} \, ds
 \end{aligned}$$

$$\begin{aligned}
 & - \sum_{i=1}^N \int_0^t \int_{\tilde{X}_\kappa} \rho(s) (\partial_l \phi_\varepsilon)(z - \bar{z}) \partial_m \mathcal{U}_\kappa(\bar{z}) a_i^l(\bar{z}) a_i^m(\bar{z}) d\bar{z} ds \\
 & - \int_0^t \int_{\tilde{X}_\kappa} \rho_\kappa(s) u^l(s, \bar{z}) (\partial_l \phi_\varepsilon)(z - \bar{z}) d\bar{z} ds \\
 & + \int_0^t \int_{\tilde{X}_\kappa} \rho(s) \phi_\varepsilon(z - \bar{z}) u^l(s, \bar{z}) \partial_l \mathcal{U}_\kappa(\bar{z}) d\bar{z} ds,
 \end{aligned} \tag{5.28}$$

valid for $z \in \tilde{X}_\kappa$ and $\varepsilon < \varepsilon_\kappa/4$. Note that for the terms involving the covariant Hessian we have used the geometric identities appearing in Lemma 2.1. We can rewrite (5.28) in the following (pointwise) form:

$$\begin{aligned}
 (\rho_\kappa)_\varepsilon(t, z) - (\rho_{0,\kappa})_\varepsilon(z) &= - \sum_{i=1}^N \int_0^t \operatorname{div}_h \mathcal{L}_\kappa(\rho_\kappa(s) a_i)_\varepsilon(z) dW^i(s) \\
 &+ \frac{1}{2} \sum_{i=1}^N \int_0^t \left[\operatorname{div}_h^2 \mathcal{L}_\kappa(\rho_\kappa(s) \hat{a}_i)_\varepsilon(z) - \operatorname{div}_h \mathcal{L}_\kappa(\rho_\kappa(s) \nabla_{a_i} a_i)_\varepsilon(z) \right] ds \\
 &+ \frac{1}{2} \sum_{i=1}^N \int_0^t \left[\operatorname{div}_h \mathcal{L}_\kappa V_{\kappa,i,\varepsilon}(s, z) - \operatorname{div}_h \mathcal{L}_\kappa \bar{V}_{\kappa,i,\varepsilon}(s, z) \right] ds \\
 &+ \sum_{i=1}^N \int_0^t (A_{\kappa,i}^1)_\varepsilon(s, z) dW^i(s) + \frac{1}{2} \sum_{i=1}^N \int_0^t \left[(A_{\kappa,i}^2)_\varepsilon(s, z) + (A_{\kappa,i}^3)_\varepsilon(s, z) \right] ds \\
 &- \sum_{i=1}^N \int_0^t \operatorname{div}_h \mathcal{L}_\kappa (A_{\kappa,i}^4)_\varepsilon(s, z) ds - \int_0^t \operatorname{div}_h \mathcal{L}_\kappa(\rho_\kappa(s) u(s))_\varepsilon(z) ds \\
 &+ \int_0^t (A_{\kappa,u})_\varepsilon(s, z) ds,
 \end{aligned} \tag{5.29}$$

where $V_{\kappa,i,\varepsilon}, \bar{V}_{\kappa,i,\varepsilon}$ are respectively the regularized versions of $V_{\kappa,i}$, cf. (5.9), and $\bar{V}_{\kappa,i}$, cf. (5.10). Next, we turn these regularized local SPDEs, one equation for each chart $\kappa \in \mathcal{A}$, into a globally defined SPDE.

5.4. Global SPDE for ρ_ε

Summing the local equation (5.29) over $\kappa \in \mathcal{A}$, we arrive at the global equation

$$\begin{aligned}
 \rho_\varepsilon(t, x) - \rho_{0,\varepsilon}(x) &= - \sum_{i=1}^N \int_0^t \operatorname{div}_h (\rho(s) a_i)_\varepsilon(x) dW^i(s) \\
 &+ \frac{1}{2} \sum_{i=1}^N \int_0^t \left[\operatorname{div}_h^2 (\rho(s) \hat{a}_i)_\varepsilon(x) - \operatorname{div}_h (\rho(s) \nabla_{a_i} a_i)_\varepsilon(x) \right] ds \\
 &+ \frac{1}{2} \sum_{i=1}^N \int_0^t \left[\operatorname{div}_h V_{i,\varepsilon}(s, x) - \operatorname{div}_h \bar{V}_{i,\varepsilon}(s, x) \right] ds
 \end{aligned} \tag{5.30}$$

$$\begin{aligned}
 & + \sum_{i=1}^N \int_0^t A_{i,\varepsilon}^1(s, x) dW^i(s) + \frac{1}{2} \sum_{i=1}^N \int_0^t [A_{i,\varepsilon}^2(s, x) + A_{i,\varepsilon}^3(s, x)] ds \\
 & - \sum_{i=1}^N \int_0^t \operatorname{div}_h A_{i,\varepsilon}^4(s, x) ds - \int_0^t \operatorname{div}_h (\rho(s)u(s))_\varepsilon(x) ds + \int_0^t A_{u,\varepsilon}(s, x) ds,
 \end{aligned}$$

which holds \mathbb{P} -a.s., for all $(t, x) \in [0, T] \times M$, and any $\varepsilon < \varepsilon_0$, cf. (5.4).

5.5. Global SPDE for $F(\rho_\varepsilon)$; $u \equiv 0$

For simplicity of presentation, let us assume that the vector field $u(t, \cdot)$ is the zero section for all $t \in [0, T]$, and then derive the equation satisfied by the stochastic process $(\omega, t) \mapsto F(\rho_\varepsilon(\cdot, \cdot, x))$, $x \in M$ fixed. The general case ($u \not\equiv 0$) will be handled later.

Let us apply Itô’s formula to $F(\rho_\varepsilon(\cdot, x))$, where $F \in C^2(\mathbb{R})$ with F, F', F'' bounded on \mathbb{R} . In view of (5.30), we obtain

$$\begin{aligned}
 F(\rho_\varepsilon(t, x)) - F(\rho_{0,\varepsilon}(x)) &= - \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \operatorname{div}_h (\rho(s)a_i)_\varepsilon(x) dW^i(s) \\
 & + \frac{1}{2} \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \left[\operatorname{div}_h^2 (\rho(s)\hat{a}_i)_\varepsilon(x) - \operatorname{div}_h (\rho(s)\nabla_{a_i} a_i)_\varepsilon(x) \right] ds \\
 & + \frac{1}{2} \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \left[\operatorname{div}_h V_{i,\varepsilon}(s, x) - \operatorname{div}_h \bar{V}_{i,\varepsilon}(s, x) \right] ds \\
 & + \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) A_{i,\varepsilon}^1(s, x) dW^i(s) \\
 & + \frac{1}{2} \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \left[A_{i,\varepsilon}^2(s, x) + A_{i,\varepsilon}^3(s, x) \right] ds \tag{5.31} \\
 & - \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \operatorname{div}_h A_{i,\varepsilon}^4(s, x) ds \\
 & + \frac{1}{2} \sum_{i=1}^N \int_0^t F''(\rho_\varepsilon(s, x)) (\operatorname{div}_h (\rho(s)a_i)_\varepsilon(x))^2 ds \\
 & + \frac{1}{2} \sum_{i=1}^N \int_0^t F''(\rho_\varepsilon(s, x)) (A_{i,\varepsilon}^1(s, x))^2 ds \\
 & - \sum_{i=1}^N \int_0^t F''(\rho_\varepsilon(s, x)) \operatorname{div}_h (\rho(s)a_i)_\varepsilon(x) A_{i,\varepsilon}^1(s, x) ds,
 \end{aligned}$$

valid \mathbb{P} -a.s., for $(t, x) \in [0, T] \times M$, and any $\varepsilon < \varepsilon_0$.

In a nutshell, to prove Theorem 3.2, i.e., the renormalized equation (3.6), we need to send $\varepsilon \rightarrow 0$ (after integrating in x). The task is rather nontrivial, and, before we can accomplish that, we need several intermediate results, which will involve crucial cancellations among some of the terms in (5.31).

First of all, to bring (5.31) into the form of a stochastic continuity equation for $F(\rho_\varepsilon(\cdot, x))$, we “add and subtract” the two terms $\frac{1}{2} \sum_i \int_0^t \Lambda_i(F(\rho_\varepsilon)) ds$ and $\sum_i \int_0^t \operatorname{div}_h (F(\rho_\varepsilon) a_i) dW^i(s)$.

The resulting equation is

$$F(\rho_\varepsilon(t, x)) - F(\rho_{0,\varepsilon}(x)) = \sum_{\ell=1}^{13} \mathcal{I}_\ell(\omega, t, x; \varepsilon), \tag{5.32}$$

where $\mathcal{I}_\ell = \mathcal{I}_\ell(\omega, t, x; \varepsilon)$, $\ell = 1, \dots, 13$, are defined by

$$\begin{aligned} \mathcal{I}_1 &= - \sum_{i=1}^N \int_0^t \operatorname{div}_h (F(\rho_\varepsilon(s, x)) a_i) dW^i(s), \\ \mathcal{I}_2 &= \frac{1}{2} \sum_{i=1}^N \int_0^t A_i (F(\rho_\varepsilon(s, x))) ds, \\ \mathcal{I}_3 &= - \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \operatorname{div}_h (\rho(s) a_i)_\varepsilon(x) dW^i(s), \\ \mathcal{I}_4 &= \sum_{i=1}^N \int_0^t \operatorname{div}_h (F(\rho_\varepsilon(s, x)) a_i) dW^i(s), \\ \mathcal{I}_5 &= \frac{1}{2} \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \left[\operatorname{div}_h^2 (\rho(s) \hat{a}_i)_\varepsilon(x) - \operatorname{div}_h (\rho(s) \nabla_{a_i} a_i)_\varepsilon(x) \right] ds, \\ \mathcal{I}_6 &= - \frac{1}{2} \sum_{i=1}^N \int_0^t A_i (F(\rho_\varepsilon(s, x))) ds, \\ \mathcal{I}_7 &= \frac{1}{2} \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \left[\operatorname{div}_h V_{i,\varepsilon}(s, x) - \operatorname{div}_h \bar{V}_{i,\varepsilon}(s, x) \right] ds, \\ \mathcal{I}_8 &= \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) A_{i,\varepsilon}^1(s, x) dW^i(s), \\ \mathcal{I}_9 &= \frac{1}{2} \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \left[A_{i,\varepsilon}^2(s, x) + A_{i,\varepsilon}^3(s, x) \right] ds, \\ \mathcal{I}_{10} &= - \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \operatorname{div}_h A_{i,\varepsilon}^4(s, x) ds, \\ \mathcal{I}_{11} &= \frac{1}{2} \sum_{i=1}^N \int_0^t F''(\rho_\varepsilon(s, x)) (\operatorname{div}_h (\rho(s) a_i)_\varepsilon(x))^2 ds, \\ \mathcal{I}_{12} &= \frac{1}{2} \sum_{i=1}^N \int_0^t F''(\rho_\varepsilon(s, x)) (A_{i,\varepsilon}^1(s, x))^2 ds, \\ \mathcal{I}_{13} &= - \sum_{i=1}^N \int_0^t F''(\rho_\varepsilon(s, x)) \operatorname{div}_h (\rho(s) a_i)_\varepsilon(x) A_{i,\varepsilon}^1(s, x) ds. \end{aligned} \tag{5.33}$$

Lemma 5.10 (“ $\mathcal{I}_3 + \mathcal{I}_4$ ”). With \mathcal{I}_3 and \mathcal{I}_4 defined in (5.33),

$$\mathcal{I}_3 + \mathcal{I}_4 = \mathcal{I}_{3+4,1} + \mathcal{I}_{3+4,2},$$

$$\begin{aligned} \mathcal{I}_{3+4,1} &:= - \sum_{i=1}^N \int_0^t G_F(\rho_\varepsilon(s, x)) \operatorname{div}_h a_i(x) dW^i(s), \\ \mathcal{I}_{3+4,2} &:= - \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) r_{\varepsilon,i}(s, x) dW^i(s), \end{aligned}$$

where G_F is defined in (3.5) and $r_{\varepsilon,i}$ is defined in (5.6).

Proof. By definition, in a coordinate-free notation,

$$\begin{aligned} \operatorname{div}_h (\rho(s)a_i)_\varepsilon(x) &= \sum_{\kappa \in \mathcal{A}} \operatorname{div}_h \mathcal{L}_\kappa (\rho_\kappa(s)a_i)_\varepsilon(x) \\ &= \sum_{\kappa \in \mathcal{A}} \left\{ \operatorname{div}_h ((\rho_\kappa)_\varepsilon(s)a_i)(x) + r_{\kappa,\varepsilon,i}(s, x) \right\} = \operatorname{div}_h (\rho_\varepsilon(s)a_i)(x) + r_{\varepsilon,i}(s, x). \end{aligned}$$

Therefore, by the product and chain rules, recalling (3.5), the lemma follows. \square

Lemma 5.11 (“ $\mathcal{I}_5 + \mathcal{I}_6$ ”). With $\mathcal{I}_5, \mathcal{I}_6$ defined in (5.33), $\mathcal{I}_5 + \mathcal{I}_6 = \sum_{\ell=1}^4 \mathcal{I}_{5+6,\ell}$,

$$\begin{aligned} \mathcal{I}_{5+6,1} &:= \frac{1}{2} \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \left[\operatorname{div}_h^2 (\rho(s)\hat{a}_i)_\varepsilon(x) - \tilde{r}_{\varepsilon,i} \right] ds, \\ \mathcal{I}_{5+6,2} &:= -\frac{1}{2} \sum_{i=1}^N \int_0^t F''(\rho_\varepsilon(s, x)) (a_i(\rho_\varepsilon(s, x)))^2 ds, \\ \mathcal{I}_{5+6,3} &:= \frac{1}{2} \sum_{i=1}^N \int_0^t G_F(\rho_\varepsilon(s, x)) \Lambda_i(1) ds, \\ \mathcal{I}_{5+6,4} &:= -\frac{1}{2} \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \operatorname{div}_h^2 (\rho_\varepsilon(s)\hat{a}_i)(x) ds, \end{aligned}$$

where $\tilde{r}_{\varepsilon,i}, G_F, \Lambda_i(1)$ are defined respectively in (5.7), (3.5), (3.4).

Proof. By (4.2) and arguing exactly as we have done several times before (i.e., expanding the sum over κ that defines the global objects, working locally relative to a chart κ , and in the end reaggregate), we arrive at $\mathcal{I}_5 + \mathcal{I}_6 = \sum_{\ell=1}^8 \mathcal{J}_\ell$, where $\mathcal{J}_\ell = \mathcal{J}_\ell(\omega, t, x, \varepsilon)$, $\ell = 1, \dots, 8$, are defined by

$$\begin{aligned} \mathcal{J}_1 &= \mathcal{I}_{5+6,1}, \quad \mathcal{J}_2 = -\frac{1}{2} \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \operatorname{div}_h (\rho_\varepsilon(s)\nabla_{a_i} a_i)(x) ds, \\ \mathcal{J}_3 &= -\frac{1}{2} \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) (\operatorname{div}_h(\hat{a}_i))(\rho_\varepsilon(s, x)) ds, \\ \mathcal{J}_4 &= -\frac{1}{2} \sum_{i=1}^N \int_0^t F(\rho_\varepsilon(s, x)) \operatorname{div}_h^2 (\hat{a}_i)(x) ds, \\ \mathcal{J}_5 &= -\frac{1}{2} \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \operatorname{div}_h (\hat{a}_i(d\rho_\varepsilon(s), \cdot))(x) ds, \end{aligned}$$

$$\mathcal{J}_6 = \mathcal{I}_{5+6,2}, \quad \mathcal{J}_7 = \frac{1}{2} \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) (\nabla_{a_i} a_i) (\rho_\varepsilon(s, x)) ds,$$

$$\mathcal{J}_8 = \frac{1}{2} \sum_{i=1}^N \int_0^t F(\rho_\varepsilon(s, x)) \operatorname{div}_h (\nabla_{a_i} a_i) (x) ds.$$

By expanding $\operatorname{div}_h (\rho_\varepsilon(s) \nabla_{a_i} a_i) = \rho_\varepsilon(s) \operatorname{div}_h (\nabla_{a_i} a_i) + (\nabla_{a_i} a_i) (\rho_\varepsilon(s))$ and recalling the definition of G_F , we obtain

$$\mathcal{I}_5 + \mathcal{I}_6 = \mathcal{J}_1 + \mathcal{J}_3 + \mathcal{J}_4 + \mathcal{J}_5 + \mathcal{J}_6 - \frac{1}{2} \sum_{i=1}^N \int_0^t G_F(\rho_\varepsilon(s, x)) \operatorname{div}_h (\nabla_{a_i} a_i) (x) ds.$$

Considering (4.2) with $F(u) = u$ and $u \in C^2(M)$,

$$\operatorname{div}_h^2 (u \hat{a}_i) = \operatorname{div}_h (\hat{a}_i) (u) + u \operatorname{div}_h^2 (\hat{a}_i) + \operatorname{div}_h (\hat{a}_i (du, \cdot)).$$

Using this identity and recalling once more (3.5),

$$\mathcal{I}_5 + \mathcal{I}_6 = \mathcal{J}_1 + \mathcal{J}_6 + \frac{1}{2} \sum_{i=1}^N \int_0^t G_F(\rho_\varepsilon(s, x)) [\operatorname{div}_h^2 (\hat{a}_i) - \operatorname{div}_h (\nabla_{a_i} a_i)] ds + \mathcal{I}_{5+6,4}.$$

Recalling (3.4), the third term on the right-hand side of the equality sign is $\mathcal{I}_{5+6,3}$. Since $\mathcal{J}_1 = \mathcal{I}_{5+6,1}$ and $\mathcal{J}_6 = \mathcal{I}_{5+6,2}$, this concludes the proof. \square

Lemma 5.12 (“ \mathcal{I}_{11} ”). With \mathcal{I}_{11} defined in (5.33), $\mathcal{I}_{11} = \sum_{\ell=1}^4 \mathcal{I}_{11,\ell}$,

$$\mathcal{I}_{11,1} := \frac{1}{2} \sum_{i=1}^N \int_0^t F''(\rho_\varepsilon(s, x)) (a_i(\rho_\varepsilon(s, x)))^2 ds,$$

$$\mathcal{I}_{11,2} := \frac{1}{2} \sum_{i=1}^N \int_0^t F''(\rho_\varepsilon(s, x)) (\rho_\varepsilon(s, x) \operatorname{div}_h a_i)^2 ds,$$

$$\mathcal{I}_{11,3} := \frac{1}{2} \sum_{i=1}^N \int_0^t F''(\rho_\varepsilon(s, x)) (r_{\varepsilon,i}(s, x))^2 ds,$$

$$\mathcal{I}_{11,4} := \sum_{i=1}^N \int_0^t F''(\rho_\varepsilon(s, x)) \operatorname{div}_h (\rho_\varepsilon(s, x) a_i) r_{\varepsilon,i}(s, x) ds,$$

$$\mathcal{I}_{11,5} := \sum_{i=1}^N \int_0^t \bar{a}_i (G_F(\rho_\varepsilon(s, x))) ds,$$

where $r_{\varepsilon,i}$ is defined in (5.6), $\bar{a}_i = (\operatorname{div}_h a_i) a_i$, and G_F is defined in (3.5).

Proof. Writing

$$\operatorname{div}_h (\rho(s) a_i)_\varepsilon (x) = \operatorname{div}_h (\rho_\varepsilon(s) a_i) (x) + r_{\varepsilon,i}(s, x) \tag{5.34}$$

and expanding $(\operatorname{div}_h (\rho_\varepsilon(s) a_i) (x) + r_{\varepsilon,i}(s, x))^2$, we obtain $\mathcal{I}_{11} = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3$, where

$$\mathcal{J}_1 = \frac{1}{2} \sum_{i=1}^N \int_0^t F''(\rho_\varepsilon(s, x)) (\operatorname{div}_h (\rho_\varepsilon(s) a_i) (x))^2 ds, \quad \mathcal{J}_2 = \mathcal{I}_{11,2}, \quad \mathcal{J}_3 = \mathcal{I}_{11,3}.$$

By applying the product and chain rules to $\operatorname{div}_h(\rho_\varepsilon a_i)$ and then expanding the square $(\operatorname{div}_h(\rho_\varepsilon(s)a_i))^2$ into $(a_i(\rho_\varepsilon(s)))^2 + (\rho_\varepsilon(s)\operatorname{div}_h a_i)^2 + 2\rho_\varepsilon(s)\bar{a}_i(\rho_\varepsilon(s))$,

$$\mathcal{J}_1 = \mathcal{I}_{11,1} + \mathcal{I}_{11,2} + \sum_{i=1}^N \int_0^t \rho_\varepsilon(s, x) F''(\rho_\varepsilon(s, x)) \bar{a}_i(\rho_\varepsilon(s, x)) ds.$$

Because $G'_F(\xi) = \xi F''(\xi)$ and $G'_F(\rho_\varepsilon)\bar{a}_i(\rho_\varepsilon) = \bar{a}_i(G_F(\rho_\varepsilon))$, the last term becomes $\mathcal{I}_{11,4}$. This concludes the proof of the lemma. \square

In view of Lemmas 5.10, 5.11, 5.12 and noting the cancellation $\mathcal{I}_{5+6,2} + \mathcal{I}_{11,1} = 0$, Eq. (5.32) becomes

$$\begin{aligned} F(\rho_\varepsilon(t, x)) - F(\rho_{0,\varepsilon}(x)) &= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_{3+4,1} + \mathcal{I}_{3+4,2} + \mathcal{I}_{5+6,1} + \mathcal{I}_{5+6,3} + \mathcal{I}_{5+6,4} \\ &\quad + \mathcal{I}_{11,2} + \mathcal{I}_{11,3} + \mathcal{I}_{11,4} + \mathcal{I}_{11,5} + \mathcal{I}_7 + \mathcal{I}_8 + \mathcal{I}_9 + \mathcal{I}_{10} + \mathcal{I}_{12} + \mathcal{I}_{13}. \end{aligned} \tag{5.35}$$

Keeping an eye on the end result (3.3) ($u \equiv 0$) while inspecting (5.35) as well as reorganizing and relabeling some of the terms, we rewrite (5.35) in the form

$$\begin{aligned} F(\rho_\varepsilon(t, x)) - F(\rho_{0,\varepsilon}(x)) &+ \sum_{i=1}^N \int_0^t \operatorname{div}_h(F(\rho_\varepsilon(s, x)) a_i) dW^i(s) \\ &+ \sum_{i=1}^N \int_0^t G_F(\rho_\varepsilon(s, x)) \operatorname{div}_h a_i dW^i(s) = \frac{1}{2} \sum_{i=1}^N \int_0^t \Lambda_i(F(\rho_\varepsilon(s, x))) ds \\ &+ \frac{1}{2} \sum_{i=1}^N \int_0^t G_F(\rho_\varepsilon(s, x)) \Lambda_i(1) ds + \frac{1}{2} \sum_{i=1}^N \int_0^t F''(\rho_\varepsilon(s, x)) (\rho_\varepsilon(s, x) \operatorname{div}_h a_i)^2 ds \\ &+ \sum_{i=1}^N \int_0^t \bar{a}_i(G_F(\rho_\varepsilon(s, x))) ds + \mathcal{R}_\varepsilon(\omega, t, x), \end{aligned} \tag{5.36}$$

where the third, fourth, fifth, sixth, seventh, and eighth terms correspond to $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_{3+4,1}, \mathcal{I}_{5+6,3}, \mathcal{I}_{11,2}, \mathcal{I}_{11,5}$, respectively. The remaining terms from (5.35) have been transferred to the remainder \mathcal{R}_ε . Modulo the “ ε -subscripts” and the remainder term, we recognize (5.36) as the sought-after renormalized equation (3.3), cf. also Remark 3.2. One of the remaining tasks is to demonstrate that $\mathcal{R}_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, weakly in x and strongly in (ω, t) . The remainder term $\mathcal{R}_\varepsilon = \mathcal{R}_\varepsilon(\omega, t, x)$ is

$$\begin{aligned} \mathcal{R}_\varepsilon &= \sum_{\ell=1}^{11} \mathcal{H}_\ell, \quad \text{where } \mathcal{H}_1 := \mathcal{I}_{3+4,2}, \\ \mathcal{H}_2 &:= \frac{1}{2} \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \left[\operatorname{div}_h^2(\rho(s)\hat{a}_i)_\varepsilon(x) - \operatorname{div}_h^2(\rho_\varepsilon(s)\hat{a}_i)(x) \right] ds, \\ \mathcal{H}_3 &:= -\frac{1}{2} \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \tilde{r}_{\varepsilon,i}(s, x) ds, \\ \mathcal{H}_4 &:= \mathcal{I}_{11,3}, \quad \mathcal{H}_5 := \mathcal{I}_{11,4}, \quad \mathcal{H}_6 := \mathcal{I}_7, \quad \mathcal{H}_7 := \mathcal{I}_8, \\ \mathcal{H}_8 &:= \mathcal{I}_9, \quad \mathcal{H}_9 := \mathcal{I}_{10}, \quad \mathcal{H}_{10} := \mathcal{I}_{12}, \quad \mathcal{H}_{11} := \mathcal{I}_{13}. \end{aligned} \tag{5.37}$$

Some crucial cancellations will occur also in the remainder term \mathcal{R}_ε . This is the subject matter of the remaining lemmas in this subsection, involving the terms \mathcal{H}_2 , \mathcal{H}_5 , \mathcal{H}_6 , and \mathcal{H}_9 .

Lemma 5.13 (“ \mathcal{H}_5 ”). With $\mathcal{H}_5 = \mathcal{I}_{11,4}$ defined in Lemma 5.12,

$$\begin{aligned} \mathcal{H}_5 &= \mathcal{H}_{5,1} + \mathcal{H}_{5,2}, \quad \text{where} \\ \mathcal{H}_{5,1} &:= \sum_{i=1}^N \int_0^t G'_F(\rho_\varepsilon(s, x)) \operatorname{div}_h a_i r_{\varepsilon,i}(s, x) ds, \\ \mathcal{H}_{5,2} &:= - \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) a_i(r_{\varepsilon,i}(s, x)) ds, \\ \mathcal{H}_{5,3} &:= \sum_{i=1}^N \int_0^t a_i(F'(\rho_\varepsilon(s)) r_{\varepsilon,i}(s))(x) ds, \end{aligned}$$

where $r_{\varepsilon,i}$ is defined in (5.6) and G_F is defined in (3.5).

Proof. In $\mathcal{I}_{11,4}$, expand $\operatorname{div}_h(\rho_\varepsilon(s) a_i)$ and then use the product rule for a_i , finally noting that $\xi F''(\xi) = G'_F(\xi)$. \square

Lemma 5.14 (“ $\mathcal{H}_2 + \mathcal{H}_6$ ”). Consider \mathcal{H}_2 defined in (5.37) and $\mathcal{H}_6 = \mathcal{I}_7$ with \mathcal{I}_7 defined in (5.33). Then

$$\begin{aligned} \mathcal{H}_2 + \mathcal{H}_6 &= \sum_{\ell=1}^3 \mathcal{H}_{2+6,\ell}, \quad \text{where} \\ \mathcal{H}_{2+6,1} &:= \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) a_i(r_{\varepsilon,i}(s, x)) ds, \\ \mathcal{H}_{2+6,2} &:= \frac{1}{2} \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \bar{r}_{\varepsilon,i}(s, x) ds, \\ \mathcal{H}_{2+6,3} &:= \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \sum_{\kappa \in \mathcal{A}} G_{\kappa,\varepsilon,i}(s, x) ds, \end{aligned}$$

where $r_{\varepsilon,i}$, $\bar{r}_{\varepsilon,i}$, $G_{\kappa,\varepsilon,i}$ are defined respectively in (5.6), (5.13), (5.18).

Proof. By inspecting the integrands of \mathcal{H}_2 and \mathcal{H}_6 , we recognize that the claim follows from the “second order” commutator Lemma 5.5. \square

Lemma 5.15 (“ \mathcal{H}_9 ”). With $\mathcal{H}_9 = \mathcal{I}_{10}$ defined in (5.33),

$$\begin{aligned} \mathcal{H}_9 &= \mathcal{H}_{9,1} + \mathcal{H}_{9,2}, \quad \text{where} \\ \mathcal{H}_{9,1} &:= - \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) r_{\varepsilon,i}^*(s, x) ds, \\ \mathcal{H}_{9,2} &:= - \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) \operatorname{div}_h (A_{i,\varepsilon}^1(s, x) a_i) ds, \end{aligned}$$

where $r_{\varepsilon,i}^*$, $A_{i,\varepsilon}^1$ are defined respectively in (5.24), (5.22).

Proof. This follows from the very definition of $r_{\varepsilon,i}^*$. \square

Lemma 5.16 (“ \mathcal{H}_{11} ”). With $\mathcal{H}_9 = \mathcal{I}_{13}$ defined in (5.33)

$$\begin{aligned} \mathcal{H}_{11} &= \mathcal{H}_{11,1} + \mathcal{H}_{11,2}, \quad \text{where} \\ \mathcal{H}_{11,1} &:= - \sum_{i=1}^N \int_0^t A_{i,\varepsilon}^1(s, x) a_i(F'(\rho_\varepsilon(s, x))) ds, \\ \mathcal{H}_{11,2} &:= - \sum_{i=1}^N \int_0^t F''(\rho_\varepsilon(s, x)) A_{i,\varepsilon}^1(s, x) (r_{\varepsilon,i}(s, x) + \rho_\varepsilon(s, x) \operatorname{div}_h a_i) ds, \end{aligned}$$

where $r_{\varepsilon,i}$, $A_{i,\varepsilon}^1$ are defined respectively in (5.6), (5.22).

Proof. Making use of (5.34) and the product rule,

$$\operatorname{div}_h(\rho(s) a_i)_\varepsilon = r_{\varepsilon,i}(s) + \rho_\varepsilon(s) \operatorname{div}_h a_i + a_i(\rho_\varepsilon(s)).$$

Thus the claim follows by noting that $F''(\rho_\varepsilon(s)) a_i(\rho_\varepsilon(s)) = a_i(F'(\rho_\varepsilon(s)))$. \square

Lemma 5.17 (“ $\mathcal{H}_9 + \mathcal{H}_{11}$ ”). Consider $\mathcal{H}_9 = \mathcal{I}_{10}$ and $\mathcal{H}_{11} = \mathcal{I}_{13}$ with \mathcal{I}_{10} and \mathcal{I}_{13} defined in (5.33). Then

$$\begin{aligned} \mathcal{H}_9 + \mathcal{H}_{11} &= \sum_{\ell=1}^4 \mathcal{H}_{9+11,\ell}, \quad \text{where} \\ \mathcal{H}_{9+11,1} &:= - \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) r_{\varepsilon,i}^*(s, x) ds, \\ \mathcal{H}_{9+11,2} &:= - \sum_{i=1}^N \int_0^t F'(\rho_\varepsilon(s, x)) A_{i,\varepsilon}^1(s, x) \operatorname{div}_h a_i ds, \\ \mathcal{H}_{9+11,3} &:= - \sum_{i=1}^N \int_0^t a_i(F'(\rho_\varepsilon(s, x)) A_{i,\varepsilon}^1(s, x)) ds, \\ \mathcal{H}_{9+11,4} &:= - \sum_{i=1}^N \int_0^t F''(\rho_\varepsilon(s, x)) A_{i,\varepsilon}^1(s, x) (r_{\varepsilon,i}(s, x) + \rho_\varepsilon(s, x) \operatorname{div}_h a_i) ds, \end{aligned}$$

where $r_{\varepsilon,i}^*$, $A_{i,\varepsilon}^1$, $r_{\varepsilon,i}$ are defined respectively in (5.24), (5.22), (5.6).

Proof. Note that $\mathcal{H}_{9+11,1} = \mathcal{H}_{9,1}$, cf. Lemma 5.15, and $\mathcal{H}_{9+11,4} = \mathcal{H}_{11,2}$, cf. Lemma 5.16. Moreover, applying the product rule to $\operatorname{div}_h(A_{i,\varepsilon}^1(s, x) a_i)$, we find that $\mathcal{H}_{9+11,2} + \mathcal{H}_{9+11,3} = \mathcal{H}_{9,2} + \mathcal{H}_{11,1}$. \square

Combining (5.37) and Lemmas 5.13, 5.14, and 5.17, we arrive at the following expression for the remainder term \mathcal{R}_ε , which is a function of $(\omega, t, x) \in \Omega \times [0, T] \times M$:

$$\begin{aligned} \mathcal{R}_\varepsilon &= \mathcal{H}_1 + \mathcal{H}_3 + \mathcal{H}_4 + \mathcal{H}_7 + \mathcal{H}_8 + \mathcal{H}_{10} + \mathcal{H}_{5,1} + \mathcal{H}_{5,3} \\ &\quad + \mathcal{H}_{2+6,2} + \mathcal{H}_{2+6,3} + \mathcal{H}_{9+11,1} + \mathcal{H}_{9+11,2} + \mathcal{H}_{9+11,3} + \mathcal{H}_{9+11,4}, \end{aligned} \tag{5.38}$$

where the difficult terms involving $a_i(r_{\varepsilon,i})$ cancel out: $\mathcal{H}_{2+6,1} + \mathcal{H}_{5,2} = 0$.

5.6. Passing to the limit in SPDE for $F(\rho_\varepsilon)$; $u \equiv 0$

We wish to send $\varepsilon \rightarrow 0$ in the x -weak form of (5.36), analyzing the limiting behavior of the remainder \mathcal{R}_ε , which is going to vanish, separately from the other terms in (5.36), which are going to converge to their respective terms in (3.6). Denote by $\langle \cdot, \cdot \rangle$ the canonical pairing between functions on M . In the following, we will make repeated (unannounced) use of Lemma A.6 and the (stochastic) Fubini theorem.

We begin with the convergence analysis of the remainder term.

Proposition 5.18 (Convergence of the Remainder \mathcal{R}_ε). *For any $\psi \in C^\infty(M)$, $\langle \mathcal{R}_\varepsilon, \psi \rangle \rightarrow 0$ in $L^1(\Omega \times [0, T])$ as $\varepsilon \rightarrow 0$.*

Proof. We recall the following convergences, which are consequences of Lemmas 5.1, 5.2, 5.3, 5.4, 5.5, 5.7, and 5.8:

$$\begin{aligned} \rho_\varepsilon &\xrightarrow{\varepsilon \downarrow 0} \rho, & \rho_{0,\varepsilon} &\xrightarrow{\varepsilon \downarrow 0} \rho_0, & r_{\varepsilon,i} &\xrightarrow{\varepsilon \downarrow 0} 0, & \tilde{r}_{\varepsilon,i} &\xrightarrow{\varepsilon \downarrow 0} 0, \\ \tilde{r}_{\varepsilon,i} &\xrightarrow{\varepsilon \downarrow 0} 0, & r_{\varepsilon,i}^* &\xrightarrow{\varepsilon \downarrow 0} 0, & G_{\kappa,\varepsilon,i} &\xrightarrow{\varepsilon \downarrow 0} 0, & A_{i,\varepsilon}^1 &\xrightarrow{\varepsilon \downarrow 0} \rho a_i(1) = 0, \\ A_{i,\varepsilon}^2 &\xrightarrow{\varepsilon \downarrow 0} \rho \nabla^2 1(a_i, a_i) = 0, & A_{i,\varepsilon}^3 &\xrightarrow{\varepsilon \downarrow 0} \rho (\nabla_{a_i} a_i)(1) = 0, \end{aligned} \tag{5.39}$$

which take place in $L^q([0, T]; L^2(\Omega \times M))$, for any $q \in [1, \infty)$.

We multiply the remainder identity (5.38) by ψ and integrate over M , and then analyze the convergence (in ω, t) of the resulting terms separately. Consider first the term $\langle \mathcal{H}_1(\omega, t), \psi \rangle$, $\mathcal{H}_1 = -\sum_i \int_0^t F'(\rho_\varepsilon) r_{\varepsilon,i} dW^i(s)$. Since F' is bounded,

$$|\langle F'(\rho_\varepsilon(\omega, s)) r_{\varepsilon,i}(\omega, s), \psi \rangle| \leq \|\psi\|_{L^2(M)} \|F'\|_\infty \|r_{\varepsilon,i}(\omega, s)\|_{L^2(M)}.$$

Therefore, in view of (5.39) with $q = 2$, $\langle F'(\rho_\varepsilon) r_{\varepsilon,i}, \psi \rangle \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^2(\Omega \times [0, T])$. Consequently, by the Itô isometry, $\sup_{t \in [0, T]} (\mathbb{E} \langle \mathcal{H}_1, \psi \rangle^2)^{1/2} \xrightarrow{\varepsilon \downarrow 0} 0$; whence

$$\langle \mathcal{H}_1, \psi \rangle \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^2(\Omega \times [0, T]).$$

The term $\langle \mathcal{H}_7(\omega, t), \psi \rangle$, where $\mathcal{H}_7 = \sum_i \int_0^t F'(\rho_\varepsilon) A_{i,\varepsilon}^1 dW^i(s)$, is treated in the same way, yielding the convergence

$$\langle \mathcal{H}_7, \psi \rangle \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^2(\Omega \times [0, T]).$$

Again thanks to (5.39), $\langle F'(\rho_\varepsilon) \tilde{r}_{\varepsilon,i}, \psi \rangle \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^2(\Omega \times [0, T])$. It is therefore straightforward to deduce

$$\langle \mathcal{H}_3, \psi \rangle \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^2(\Omega \times [0, T]),$$

recalling that $\mathcal{H}_3 := -\frac{1}{2} \sum_i \int_0^t F'(\rho_\varepsilon) \tilde{r}_{\varepsilon,i} ds$.

The two terms $\langle \mathcal{H}_{2+6,2}(\omega, t), \psi \rangle$, and $\langle \mathcal{H}_{9+11,1}(\omega, t), \psi \rangle$, where we recall that $\mathcal{H}_{2+6,2} = \frac{1}{2} \sum_i \int_0^t F'(\rho_\varepsilon) \tilde{r}_{\varepsilon,i} ds$ and $\mathcal{H}_{9+11,1} = -\sum_i \int_0^t F'(\rho_\varepsilon) r_{\varepsilon,i}^* ds$, are dealt with exactly in the same way, supplying

$$\langle \mathcal{H}_{2+6,2}, \psi \rangle, \langle \mathcal{H}_{9+11,1}, \psi \rangle \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^2(\Omega \times [0, T]),$$

We continue with $\langle \mathcal{H}_4(\omega, t), \psi \rangle$, $\mathcal{H}_4 = \frac{1}{2} \sum_i \int_0^t F''(\rho_\varepsilon) r_{\varepsilon,i}^2 ds$. Since F'' is bounded,

$$\left| \langle F''(\rho_\varepsilon(\omega, s)) (r_{\varepsilon,i}(\omega, s))^2, \psi \rangle \right| \leq \|\psi\|_{L^\infty(M)} \|F''\|_\infty \|r_{\varepsilon,i}(\omega, s)\|_{L^2(M)}^2,$$

and so, once again using (5.39), $\langle F''(\rho_\varepsilon) r_{\varepsilon,i}^2, \psi \rangle \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^1(\Omega \times [0, T])$. As a result

$$\langle \mathcal{H}_4, \psi \rangle \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^1(\Omega \times [0, T]).$$

Similarly, with $\mathcal{H}_{10} = \frac{1}{2} \sum_i \int_0^t F''(\rho_\varepsilon) (A_{i,\varepsilon}^1)^2 ds$ and (5.39),

$$\langle \mathcal{H}_{10}, \psi \rangle \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^1(\Omega \times [0, T]).$$

Let us analyze $\langle \mathcal{H}_{5,1}(\omega, t), \psi \rangle$, $\mathcal{H}_{5,1} = \sum_i \int_0^t \rho_\varepsilon F''(\rho_\varepsilon) \operatorname{div}_h a_i r_{\varepsilon,i} ds$. We have

$$\begin{aligned} & \left| \langle \rho_\varepsilon(\omega, s) F''(\rho_\varepsilon(\omega, s)) \operatorname{div}_h a_i r_{\varepsilon,i}(\omega, s), \psi \rangle \right| \\ & \leq \| \psi \|_{L^\infty(M)} \| F'' \|_\infty \| \operatorname{div} a_i \|_{L^\infty(M)} \| \rho_\varepsilon(\omega, s) \|_{L^2(M)} \| r_{\varepsilon,i}(\omega, s) \|_{L^2(M)}. \end{aligned}$$

Therefore, by the Cauchy–Schwarz inequality,

$$\begin{aligned} & \iint_{\Omega \times [0, T]} \left| \langle \rho_\varepsilon F''(\rho_\varepsilon) \operatorname{div}_h a_i r_{\varepsilon,i}, \psi \rangle \right| ds d\mathbb{P} \\ & \lesssim_\psi \| \rho_\varepsilon \|_{L^2(\Omega \times [0, T] \times M)} \| r_{\varepsilon,i} \|_{L^2(\Omega \times [0, T] \times M)}. \end{aligned}$$

Accordingly, again making use of (5.39) with $q = 2$,

$$\langle \mathcal{H}_{5,1}, \psi \rangle \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^1(\Omega \times [0, T]).$$

Likewise, for $\mathcal{H}_{9+11,4} = - \sum_i \int_0^t F''(\rho_\varepsilon) A_{i,\varepsilon}^1 (r_{\varepsilon,i} + \rho_\varepsilon \operatorname{div}_h a_i) ds$, we find that

$$\langle \mathcal{H}_{9+11,4}, \psi \rangle \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^1(\Omega \times [0, T]).$$

Next we deal with the term $\langle \mathcal{H}_{5,3}(\omega, t), \psi \rangle$, where $\mathcal{H}_{5,3} = \sum_i \int_0^t a_i (F'(\rho_\varepsilon) r_{\varepsilon,i}) ds$. Integration by parts yields

$$\langle \mathcal{H}_{5,3}(\omega, t), \psi \rangle = - \sum_{i=1}^N \int_0^t \int_M F'(\rho_\varepsilon(s, x)) r_{\varepsilon,i}(s, x) \operatorname{div}(\psi a_i) dV_h(x) ds.$$

Since $\operatorname{div}(\psi a_i) \in L^\infty(M)$, we conclude as before that

$$\langle \mathcal{H}_{5,3}, \psi \rangle \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^2(\Omega \times [0, T]).$$

We treat $\langle \mathcal{H}_{9+11,3}(\omega, t), \psi \rangle$, where $\mathcal{H}_{9+11,3} = - \sum_i \int_0^t a_i (F'(\rho_\varepsilon) A_{i,\varepsilon}^1) ds$, in a similar way, writing

$$\langle \mathcal{H}_{9+11,3}(\omega, t), \psi \rangle = \sum_{i=1}^N \int_0^t \int_M F'(\rho_\varepsilon(s, x)) A_{i,\varepsilon}^1(s, x) \operatorname{div}(\psi a_i) dV_h(x) ds$$

and also in this case hammering out the convergence

$$\langle \mathcal{H}_{9+11,3}, \psi \rangle \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^2(\Omega \times [0, T]).$$

Putting into service once more the boundedness of F' and the convergences (5.39), we infer

$$\langle \mathcal{H}_8, \psi \rangle, \langle \mathcal{H}_{9+11,2}, \psi \rangle \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^2(\Omega \times [0, T]),$$

where $\mathcal{H}_8 = \frac{1}{2} \sum_i \int_0^t F'(\rho_\varepsilon) [A_{i,\varepsilon}^2 + A_{i,\varepsilon}^3] ds$, $\mathcal{H}_{9+11,2} = - \sum_i \int_0^t F'(\rho_\varepsilon) A_{i,\varepsilon}^1 \operatorname{div}_h a_i ds$.

Finally, we deal with $\langle \mathcal{H}_{9+11,3}(\omega, t), \psi \rangle$, where $\mathcal{H}_{9+11,3} = \sum_i \int_0^t F'(\rho_\varepsilon) \sum_\kappa G_{\kappa,\varepsilon,i} ds$. We clearly have

$$\langle \mathcal{H}_{9+11,3}(\omega, t), \psi \rangle = \sum_{i=1}^N \sum_{\kappa \in \mathcal{A}} \int_0^t \langle F'(\rho_\varepsilon(s)) G_{\kappa,\varepsilon,i}(s), \psi \rangle ds,$$

along with the following bounds on the integrands:

$$|\langle F'(\rho_\varepsilon(\omega, s)) G_{\kappa,\varepsilon,i}(\omega, s), \psi \rangle| \leq \|F'\|_\infty \|\psi\|_{L^2(M)} \|G_{\kappa,\varepsilon,i}(\omega, s)\|_{L^2(M)}.$$

Recalling that $G_{\kappa,\varepsilon,i} \rightarrow 0$ in L^2 , cf. (5.39), we obtain

$$\langle \mathcal{H}_{9+11,3}, \psi \rangle \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^2(\Omega \times [0, T]).$$

Summarizing our findings, $\langle \mathcal{R}_\varepsilon, \psi \rangle \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^1(\Omega \times [0, T])$. \square

Proposition 5.19 (Limit SPDE, $u \equiv 0$). *The function $F(\rho)$ satisfies the weak (in x) formulation (3.6), \mathbb{P} -a.s., for all $t \in [0, T]$, for each $\psi \in C^\infty(M)$.*

Proof. We multiply (5.36) by $\psi \in C^\infty(M)$ and integrate over M . Let us write the resulting identity symbolically as

$$\mathcal{I}_{1,\varepsilon} - \mathcal{I}_{2,\varepsilon} + \mathcal{I}_{3,\varepsilon} + \mathcal{I}_{4,\varepsilon} = \mathcal{I}_{5,\varepsilon} + \mathcal{I}_{6,\varepsilon} + \mathcal{I}_{7,\varepsilon} + \mathcal{I}_{8,\varepsilon} + \langle \mathcal{R}_\varepsilon, \psi \rangle.$$

In what follows, we analyze separately the terms $\mathcal{I}_{\ell,\varepsilon} = \mathcal{I}_{\ell,\varepsilon}(\omega, t)$, for $\ell = 1, \dots, 7$, referring to Proposition 5.18 for $\langle \mathcal{R}_\varepsilon, \psi \rangle$.

The term $\mathcal{I}_{1,\varepsilon}(\omega, t) = \langle F(\rho_\varepsilon(t)), \psi \rangle$ is easily handled. Indeed, we have

$$|\langle F(\rho_\varepsilon(\omega, t)) - F(\rho(\omega, t)), \psi \rangle| \leq \|\psi\|_{L^\infty(M)} \|F'\|_\infty \|\rho_\varepsilon(\omega, t) - \rho(\omega, t)\|_{L^2(M)},$$

and hence, by (5.39), $\mathcal{I}_{1,\varepsilon} \xrightarrow{\varepsilon \downarrow 0} \langle F(\rho), \psi \rangle$ in $L^2(\Omega \times [0, T])$. Similarly, we have $\mathcal{I}_{2,\varepsilon}(\omega) = \langle F(\rho_{0,\varepsilon}), \psi \rangle \xrightarrow{\varepsilon \downarrow 0} \langle F(\rho_0), \psi \rangle$ in $L^2(\Omega \times [0, T])$.

Integration by parts yields

$$\mathcal{I}_{5,\varepsilon}(\omega, t) = \left\langle \frac{1}{2} \sum_{i=1}^N \int_0^t \Lambda_i(F(\rho_\varepsilon(s))) ds, \psi \right\rangle = \frac{1}{2} \sum_{i=1}^N \int_0^t \langle F(\rho_\varepsilon(s)), a_i(a_i(\psi)) \rangle ds.$$

Noting that

$$\begin{aligned} & |\langle F(\rho_\varepsilon(\omega, s)) - F(\rho(\omega, s)), a_i(a_i(\psi)) \rangle| \\ & \leq \|a_i(a_i(\psi))\|_{L^2(M)} \|F'\|_\infty \|\rho_\varepsilon(\omega, s) - \rho(\omega, s)\|_{L^2(M)}, \end{aligned}$$

we use again (5.39) to infer

$$\mathcal{I}_{5,\varepsilon} \xrightarrow{\varepsilon \downarrow 0} \frac{1}{2} \sum_{i=1}^N \int_0^t \langle F(\rho(s)), a_i(a_i(\psi)) \rangle ds \quad \text{in } L^2(\Omega \times [0, T]).$$

Let us analyze the stochastic integral

$$\mathcal{I}_{3,\varepsilon}(\omega, t) = - \sum_{i=1}^N \int_0^t \langle F(\rho_\varepsilon(s)), a_i(\psi) \rangle dW^i(s),$$

where integration by parts was used to obtain the right-hand side. Making use of the estimates ($I = 1, \dots, N$)

$$\begin{aligned} & \left| \langle F(\rho_\varepsilon(\omega, s)) - F(\rho(\omega, s)), a_i(\psi) \rangle \right| \\ & \leq \|a_i(\psi)\|_{L^2(M)} \|F'\|_\infty \|\rho_\varepsilon(\omega, s) - \rho(\omega, s)\|_{L^2(M)}, \end{aligned}$$

the Itô isometry, and (5.39), we obtain

$$\mathcal{J}_{3,\varepsilon} \xrightarrow{\varepsilon \downarrow 0} \sum_{i=1}^N \int_0^\cdot \langle F(\rho(s)), a_i(\psi) \rangle dW^i(s) \quad \text{in } L^2(\Omega \times [0, T]).$$

The other term involving a stochastic integral is dealt with in a similar fashion. Indeed, recalling (3.5), $\mathcal{J}_{4,\varepsilon} = \mathcal{J}_{4_1,\varepsilon} + \mathcal{J}_{4_2,\varepsilon}$, where

$$\begin{aligned} \mathcal{J}_{4_1,\varepsilon}(\omega, t) & := \sum_{i=1}^N \int_0^t \langle \rho_\varepsilon(s) F'(\rho_\varepsilon(s)) \operatorname{div}_h a_i, \psi \rangle dW^i(s), \\ \mathcal{J}_{4_2,\varepsilon}(\omega, t) & := - \sum_{i=1}^N \int_0^t \langle F(\rho_\varepsilon(s)) \operatorname{div}_h a_i, \psi \rangle dW^i(s). \end{aligned}$$

As before,

$$\mathcal{J}_{6_4,\varepsilon} \xrightarrow{\varepsilon \downarrow 0} \sum_{i=1}^N \int_0^\cdot \langle F(\rho(s)) \operatorname{div}_h a_i, \psi \rangle dW^i(s) \quad \text{in } L^2(\Omega \times [0, T]).$$

Regarding $\mathcal{J}_{4_1,\varepsilon}$, note that

$$\begin{aligned} & \left| \langle \rho_\varepsilon(\omega, s) F'(\rho_\varepsilon(\omega, s)) - \rho(\omega, s) F'(\rho(\omega, s)), \operatorname{div}_h a_i \psi \rangle \right| \\ & \leq \|\psi \operatorname{div} a_i\|_{L^2(M)} \|\rho_\varepsilon(\omega, s) F'(\rho_\varepsilon(\omega, s)) - \rho(\omega, s) F'(\rho(\omega, s))\|_{L^2(M)} \end{aligned}$$

and, invoking Lemma A.4 (Appendix),

$$\|\rho_\varepsilon F'(\rho_\varepsilon) - \rho F'(\rho)\|_{L^2(M)} \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^2(\Omega \times [0, T]).$$

Hence, appealing once more to the Itô isometry,

$$\mathcal{J}_{4_1,\varepsilon} \xrightarrow{\varepsilon \downarrow 0} \sum_{i=1}^N \int_0^\cdot \langle \rho(s) F'(\rho(s)) \operatorname{div}_h a_i, \psi \rangle dW^i(s) \quad \text{in } L^2(\Omega \times [0, T]).$$

By a similar reasoning process, we compute easily the limits

$$\begin{aligned} \mathcal{J}_{6,\varepsilon}(\omega, t) & = \left\langle \frac{1}{2} \sum_{i=1}^N \int_0^\cdot G_F(\rho_\varepsilon(s)) A_i(1) ds, \psi \right\rangle \\ & \xrightarrow{\varepsilon \downarrow 0} \frac{1}{2} \sum_{i=1}^N \int_0^\cdot \langle G_F(\rho(s)) A_i(1), \psi \rangle ds \quad \text{in } L^2(\Omega \times [0, T]), \end{aligned}$$

and, after an integration by parts,

$$\begin{aligned} \mathcal{J}_{8,\varepsilon}(\omega, t) & = \left\langle \sum_{i=1}^N \int_0^t \bar{a}_i(G_F(\rho_\varepsilon(s))) ds, \psi \right\rangle \\ & \xrightarrow{\varepsilon \downarrow 0} - \sum_{i=1}^N \int_0^\cdot \langle G_F(\rho(s)), \operatorname{div}_h(\psi \bar{a}_i) \rangle ds \quad \text{in } L^2(\Omega \times [0, T]). \end{aligned}$$

It remains to deal with $\mathcal{J}_{7,\varepsilon}(\omega, t) = \left\langle \frac{1}{2} \sum_i \int_0^t F''(\rho_\varepsilon) (\rho_\varepsilon \operatorname{div}_h a_i)^2 ds, \psi \right\rangle$. Paying attention to the estimates ($i = 1, \dots, N$)

$$\begin{aligned} & \left| (\rho(\omega, s))^2 F''(\rho_\varepsilon(\omega, s)) - (\rho(\omega, s))^2 F''(\rho(\omega, s)), (\operatorname{div}_h a_i)^2 \psi \right| \\ & \leq \|\psi (\operatorname{div}_h a_i)^2\|_{L^\infty(M)} \left\| (\rho_\varepsilon(\omega, s))^2 F''(\rho_\varepsilon(\omega, s)) - (\rho^2(\omega, s))^2 F''(\rho(\omega, s)) \right\|_{L^1(M)}. \end{aligned}$$

Since $\rho_\varepsilon^2 \xrightarrow{\varepsilon \downarrow 0} \rho^2$ in $L^1(\Omega \times [0, T] \times M)$, cf. (5.39), and $F'' \in C_b(\mathbb{R})$, we can again invoke Lemma A.4 to arrive at

$$\mathcal{J}_{7,\varepsilon} \xrightarrow{\varepsilon \downarrow 0} \frac{1}{2} \sum_{i=1}^N \int_0^t \left\langle F''(\rho(s)) (\rho(s) \operatorname{div}_h a_i)^2, \psi \right\rangle ds \quad \text{in } L^1(\Omega \times [0, T]).$$

In view of the established convergences, it is clear that $F(\rho)$ satisfies the weak formulation (3.6) (with $u \equiv 0$), \mathbb{P} -a.s., for a.e. $t \in [0, T]$. To improve this to all times $t \in [0, T]$ note that the right-hand side of (3.6) defines a continuous stochastic process. Therefore, $(\omega, t) \mapsto \int_M F(\rho(t)) \psi dV_h$ admits a continuous modification. This concludes the proof of the proposition. \square

As of now, we have proved our main result (Theorem 3.2) under the additional assumption that $u \equiv 0$.

5.7. The general case $u \neq 0$

Let us adapt the prior proof to the general case. First, (5.31) continues to hold provided we add to the right-hand side the terms

$$- \int_0^t F'(\rho_\varepsilon(s, x)) \operatorname{div}_h (\rho(s) u(s))_\varepsilon(x) ds, \quad \widetilde{\mathcal{J}}_A := \int_0^t F'(\rho_\varepsilon(s, x)) A_{u,\varepsilon}(s, x) ds,$$

where the first term is (by now) easily seen to be equal to

$$\begin{aligned} \widetilde{\mathcal{J}}_u & := - \int_0^t \operatorname{div}_h (F(\rho_\varepsilon(s, x)) u(s)) ds - \int_0^t G_F(\rho_\varepsilon(s, x)) \operatorname{div}_h u(s) ds \\ & \quad - \int_0^t F'(\rho_\varepsilon(s, x)) r_{\varepsilon,u}(s, x) ds, \end{aligned}$$

where G_F is defined in (3.5) and the remainder $r_{\varepsilon,u}$ is defined in (5.20). In other words, Eq. (5.36) for $F(\rho_\varepsilon)$ continues to hold with $-\widetilde{\mathcal{J}}_u$ and $-\widetilde{\mathcal{J}}_A$ added to the left-hand side of the equality sign.

To conclude proof of Theorem 3.2 (in the general case, $u \neq 0$), we need strong convergence results for the following terms related to $\widetilde{\mathcal{J}}_u$ and $\widetilde{\mathcal{J}}_A$:

$$\begin{aligned} \mathcal{J}_{1,\varepsilon}(\omega, t) & := \left\langle \int_0^t \operatorname{div}_h (F(\rho_\varepsilon(s)) u(s)) ds, \psi \right\rangle, \\ \mathcal{J}_{2,\varepsilon}(\omega, t) & := \left\langle \int_0^t G_F(\rho_\varepsilon(s)) \operatorname{div}_h u(s) ds, \psi \right\rangle, \\ \mathcal{J}_{3,\varepsilon}(\omega, t) & := \left\langle \int_0^t F'(\rho_\varepsilon(s)) A_{u,\varepsilon}(s) ds, \psi \right\rangle, \\ \mathcal{J}_{4,\varepsilon}(\omega, t) & := \left\langle \int_0^t F'(\rho_\varepsilon(s)) r_{\varepsilon,u}(s) ds, \psi \right\rangle, \end{aligned}$$

for any $\psi \in C^\infty(M)$ and $(\omega, t) \in \Omega \times [0, T]$.

Lemma 5.20 (Convergence of Terms Related to u). Fix $\psi \in C^\infty(M)$. Then

$$\mathcal{J}_{1,\varepsilon} \xrightarrow{\varepsilon \downarrow 0} - \int_0^\cdot \langle F(\rho(s)), u(s)(\psi) \rangle ds \quad \text{in } L^1(\Omega \times [0, T]), \tag{5.40}$$

$$\mathcal{J}_{2,\varepsilon} \xrightarrow{\varepsilon \downarrow 0} \int_0^\cdot \langle G_F(\rho(s)) \operatorname{div}_h u(s), \psi \rangle ds \quad \text{in } L^1(\Omega \times [0, T]), \tag{5.41}$$

$$\mathcal{J}_{3,\varepsilon} \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^1(\Omega \times [0, T]), \quad \mathcal{J}_{4,\varepsilon} \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^1(\Omega \times [0, T]). \tag{5.42}$$

Proof. Clearly, $\mathcal{J}_{1,\varepsilon} = - \int_0^t \langle F(\rho_\varepsilon(s)), u(s)(\psi) \rangle ds$ and

$$\begin{aligned} & \left| \langle F(\rho_\varepsilon(\omega, s)) - F(\rho(\omega, s)), u(s)(\psi) \rangle \right| \\ & \lesssim_\psi \|F'\|_\infty \|u(s)\|_{L^2(\mathcal{T}_1^0(M))} \|\rho_\varepsilon(\omega, s) - \rho(\omega, s)\|_{L^2(M)}. \end{aligned}$$

Recalling (5.39), the latter estimate implies that $\left| \langle F(\rho_\varepsilon) - F(\rho), u(\psi) \rangle \right| \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^1(\Omega \times [0, T])$. Thus, invoking Lemma A.6, the claim (5.40) follows.

Similarly, looking back on (3.5),

$$\begin{aligned} I_1(\omega, s) & := \left| \langle \rho_\varepsilon(\omega, s)F'(\rho_\varepsilon(\omega, s)) - \rho(\omega, s)F'(\rho(\omega, s)), \psi \operatorname{div}_h u(s) \rangle \right| \\ & \lesssim_\psi \|\operatorname{div}_h u(s)\|_{L^2(M)} \|\rho_\varepsilon(\omega, s)F'(\rho_\varepsilon(\omega, s)) - \rho(\omega, s)F'(\rho(\omega, s))\|_{L^2(M)}, \\ I_2(\omega, s) & := \left| \langle F(\rho_\varepsilon(\omega, s)) - F(\rho(\omega, s)), \psi \operatorname{div}_h u(s) \rangle \right| \\ & \lesssim_\psi \|\operatorname{div}_h u(s)\|_{L^2(M)} \|\rho_\varepsilon(\omega, s) - \rho(\omega, s)\|_{L^2(M)}. \end{aligned}$$

Recalling that $\rho_\varepsilon \in L_t^\infty L_{\omega,x}^2$ and $\operatorname{div}_h u \in L_t^1 L_x^2$, cf. (2.6), we thus obtain

$$\begin{aligned} & \iint_{\Omega \times [0, T]} I_1(\omega, s) ds d\mathbb{P} \lesssim_\psi \int_0^T \tilde{I}_1(s) ds, \\ \tilde{I}_1(s) & := \|\rho_\varepsilon(s)F'(\rho_\varepsilon(s)) - \rho(s)F'(\rho(s))\|_{L^2(\Omega \times M)} \|\operatorname{div}_h u(s)\|_{L^2(M)}, \end{aligned}$$

and

$$\begin{aligned} & \iint_{\Omega \times [0, T]} I_2(\omega, s) ds d\mathbb{P} \lesssim_\psi \int_0^T \tilde{I}_2(s) ds, \\ \tilde{I}_2(s) & := \|F(\rho_\varepsilon(s)) - F(\rho(s))\|_{L^2(\Omega \times M)} \|\operatorname{div}_h u(s)\|_{L^2(M)}. \end{aligned}$$

The functions $\|\rho_\varepsilon F'(\rho_\varepsilon) - \rho F'(\rho)\|_{L^2(\Omega \times M)}$ and $\|F(\rho_\varepsilon) - F(\rho)\|_{L^2(\Omega \times M)}$ converge to zero in $L^q([0, T])$ for any $q \in [1, \infty)$, by Lemma A.5, and also a.e. on $[0, T]$ (up to subsequences). Moreover, for a.e. $s \in [0, T]$,

$$\tilde{I}_1(s), \tilde{I}_2(s) \lesssim \|\rho\|_{L^2(\Omega \times M)} \|\operatorname{div}_h u\|_{L^2(M)} \in L^1([0, T]).$$

Hence, by the dominated convergence theorem, $\int_0^T \int_\Omega I_1(\omega, s) ds d\mathbb{P} \xrightarrow{\varepsilon \downarrow 0} 0$ and $\int_0^T \int_\Omega I_2(\omega, s) ds d\mathbb{P} \xrightarrow{\varepsilon \downarrow 0} 0$. In combination with Lemma A.6, this gives (5.41).

Finally, since $A_{u,\varepsilon} \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^1(\Omega \times [0, T] \times M)$, cf. Lemma 5.9, and $r_{\varepsilon,u} \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^1([0, T]; L^2(\Omega; L^1(M)))$, cf. Lemma 5.6, we easily arrive at (5.42). \square

6. Uniqueness and a priori estimate

6.1. Uniqueness, proof of Corollary 3.3

The aim is to prove Corollary 3.3, relying on the renormalization property of L^2 weak solutions (Theorem 3.2). The renormalization property holds for bounded nonlinearities $F : \mathbb{R} \rightarrow \mathbb{R}$. To handle $F(\xi) = \xi^2$ we employ an approximation (truncation) procedure.

To this end, pick any increasing function $\chi \in C^\infty([0, \infty))$ such that $\chi(\xi) = \xi$ for $\xi \in [0, 1]$, $\chi(\xi) = 2$ for $\xi \geq 2$, $\chi(\xi) \in (1, 2)$ for $\xi \in (1, 2)$, and $A_0 := \sup_{\xi \geq 0} \chi'(\xi) > 1$. Set $A_1 := \sup_{\xi \geq 0} |\chi''(\xi)|$. We need the rescaled function $\chi_\mu(\xi) = \mu \chi(\xi/\mu)$, for $\mu > 0$. The relevant approximation of $F(\xi) = \xi^2$ is

$$F_\mu(\xi) := \chi_\mu(\xi^2), \quad \xi \in \mathbb{R}, \quad \mu > 0.$$

Some tedious computations will reveal that

$$\begin{aligned} F_\mu &\in C^\infty(\mathbb{R}), \quad \lim_{\mu \rightarrow \infty} F_\mu(\xi) = \xi^2, \quad \sup_{\xi \in \mathbb{R}} F_\mu(\xi) \leq 2\mu, \quad \sup_{\mu > 0} F_\mu(\xi) \leq 2\xi^2, \\ \sup_{\xi \in \mathbb{R}} |F'_\mu(\xi)| &\leq 2\sqrt{2}A_0\sqrt{\mu}, \quad \sup_{\mu > 0} |F'_\mu(\xi)| \leq 2\sqrt{2}A_0|\xi|, \quad \lim_{\mu \rightarrow \infty} F'_\mu(\xi) = 2\xi, \\ \lim_{\mu \rightarrow \infty} F''_\mu(\xi) &= 2, \quad |F''_\mu(\xi)| \leq 8A_1 + 2A_0, \quad \text{for } \xi \in \mathbb{R}, \mu > 0. \end{aligned} \tag{6.1}$$

Furthermore, the function $G_{F_\mu}(\xi) = \xi F'_\mu(\xi) - F_\mu(\xi)$, cf. (3.5), satisfies

$$\begin{aligned} \sup_{\xi \in \mathbb{R}} |G_{F_\mu}(\xi)| &\leq (4A_0 + 2)\mu, \quad \sup_{\mu > 0} |G_{F_\mu}(\xi)| \leq 2(\sqrt{2}A_0 + 1)\xi^2, \\ \text{and } \lim_{\mu \rightarrow \infty} G_{F_\mu}(\xi) &= \xi^2, \quad \text{for } \xi \in \mathbb{R}, \mu > 0. \end{aligned} \tag{6.2}$$

Finally, to prove Corollary 3.4, we will also make use of the bounds

$$|G_{F_\mu}(\xi)| \leq C_\chi F_\mu(\xi), \quad |\xi^2 F''_\mu(\xi)| \leq C_\chi \begin{cases} F_\mu(\xi), & |\xi| \leq \mu \\ \xi^2, & |\xi| \in [\sqrt{\mu}, \sqrt{2\mu}] \\ 0, & |\xi| > \sqrt{2\mu}, \end{cases} \tag{6.3}$$

for some constant C_χ that does not depend on μ .

Consider weak L^2 -solution ρ of (1.1) with initial datum $\rho_0 \in L^2(M)$. Referring to Theorem 3.2, taking $F = F_\mu$ and $\psi \equiv 1$ in (3.6) supplies the equation

$$\begin{aligned} \int_M F_\mu(\rho(t)) dV_h &= \int_M F_\mu(\rho_0) dV_h - \int_0^t \int_M G_{F_\mu}(\rho(s)) \operatorname{div}_h u(s) dV_h ds \\ &\quad - \sum_{i=1}^N \int_0^t \int_M G_{F_\mu}(\rho(s)) \operatorname{div}_h a_i dV_h dW^i(s) \\ &\quad - \frac{1}{2} \sum_{i=1}^N \int_0^t \int_M \Lambda_i(1) G_{F_\mu}(\rho(s)) dV_h ds \\ &\quad + \frac{1}{2} \sum_{i=1}^N \int_0^t \int_M F''_\mu(\rho(s)) (\rho(s) \operatorname{div}_h a_i)^2 dV_h ds, \end{aligned} \tag{6.4}$$

which holds \mathbb{P} -a.s., for all $t \in [0, T]$, and for any finite $\mu > 0$. Recall that $\Lambda_i(1)$ equals $\operatorname{div}_h \bar{a}_i$ and $\bar{a}_i = (\operatorname{div}_h a_i) a_i$.

In view of the bounds on G_{F_μ} , cf. (6.2), it is clear that the stochastic integral in (6.4) is a zero-mean martingale, and taking the expectation leads then to

$$\begin{aligned} \mathbb{E} \int_M F_\mu(\rho(t)) dV_h &= \mathbb{E} \int_M F_\mu(\rho_0) dV_h - \mathbb{E} \int_0^t \int_M G_{F_\mu}(\rho(s)) \operatorname{div}_h u(s) dV_h ds \\ &\quad - \frac{1}{2} \sum_{i=1}^N \mathbb{E} \int_0^t \int_M \Lambda_i(1) G_{F_\mu}(\rho(s)) dV_h ds \\ &\quad + \frac{1}{2} \sum_{i=1}^N \mathbb{E} \int_0^t \int_M F''_\mu(\rho(s)) (\rho(s) \operatorname{div}_h a_i)^2 dV_h ds, \end{aligned} \tag{6.5}$$

for all $t \in [0, T]$ and any $\mu > 0$. In view of the properties of F_μ and G_{F_μ} , cf. (6.1) and (6.2), the assumption $\operatorname{div}_h u \in L^1([0, T]; L^\infty(M))$, cf. (2.7), and $\rho \in L^\infty([0, T]; L^2(\Omega \times M))$, it is straightforward to use the dominated convergence theorem to send $\mu \rightarrow \infty$ in (6.5), eventually concluding that

$$\begin{aligned} \mathbb{E} \|\rho(t)\|_{L^2(M)}^2 &= \mathbb{E} \|\rho_0\|_{L^2(M)}^2 - \int_0^t \mathbb{E} \int_M (\rho(s))^2 \operatorname{div}_h u(s) dV_h ds \\ &\quad - \frac{1}{2} \sum_{i=1}^N \int_0^t \mathbb{E} \int_M \Lambda_i(1) (\rho(s))^2 dV_h ds + \sum_{i=1}^N \int_0^t \mathbb{E} \int_M (\rho(s))^2 (\operatorname{div}_h a_i)^2 dV_h ds. \end{aligned}$$

Setting $f(t) := \mathbb{E} \|\rho(t)\|_{L^2(M)}^2$ for $t > 0$ and $f(0) := \|\rho_0\|_{L^2(M)}^2$, this identity implies

$$f(t) \leq f(0) + \int_0^t \|\operatorname{div}_h u(s)\|_{L^\infty(M)} f(s) ds + \bar{C} \int_0^t f(s) ds, \quad t \in [0, T],$$

where $\bar{C} = \sum_{i=1}^N \left(\frac{1}{2} \|\Lambda_i(1)\|_{L^\infty(M)} + \left\| (\operatorname{div}_h a_i)^2 \right\|_{L^\infty(M)} \right)$. By Grönwall’s inequality, there is a constant C depending on \bar{C} , T , and $\|\operatorname{div}_h u\|_{L^1_t L^\infty_x}$ such that

$$\mathbb{E} \|\rho(t)\|_{L^2(M)}^2 \leq C \mathbb{E} \|\rho_0\|_{L^2(M)}^2, \quad t \in [0, T].$$

This, in combination with the linearity of the SPDE (1.1), implies Corollary 3.3.

6.2. A priori estimate, proof of Corollary 3.4

Define $f_\mu : [0, T] \rightarrow \mathbb{R}$ by

$$f_\mu(t) = \mathbb{E} \operatorname{esssup}_{r \in [0, t]} \int_M F_\mu(\rho(r)) dV_h, \quad \text{for } t > 0,$$

and $f_\mu(0) = \int_M F_\mu(\rho_0) dV_h$. By the boundedness of F_μ , note that $f_\mu \in L^\infty$ (for a forthcoming application of Grönwall’s inequality, we simply need $f_\mu \in L^1$). Set

$$M_\mu(t) := \sum_{i=1}^N \int_0^t \int_M G_{F_\mu}(\rho(s)) \operatorname{div}_h a_i dV_h dW^i(s), \quad t \in [0, T].$$

Kicking off from (6.4) and utilizing (6.3), it is not difficult to deduce

$$f_\mu(t) \lesssim f_\mu(0) + \int_0^t \|\operatorname{div}_h u(s)\|_{L^\infty(M)} \mathbb{E} \int_M F_\mu(\rho(s)) dV_h, ds + \mathbb{E} \sup_{r \in [0,t]} |M_\mu(r)| + \int_0^t \mathbb{E} \int_M F_\mu(\rho(\tau)) dV_h, ds + o(1/\mu), \quad t \in [0, T],$$

where we have taken advantage of the assumption $\rho \in L_t^\infty L_{\omega,x}^2$ to conclude that $\rho^2 \in L^1(\Omega \times [0, T] \times M)$ and thus

$$\iiint_{\{\rho^2 > \mu\}} \rho^2 dV_h ds d\mathbb{P} = o(1/\mu) \xrightarrow{\mu \uparrow \infty} 0.$$

The constant hidden in “ \lesssim ” depends on $\max_i \|a_i\|_{C^2}$ and χ .

By the Burkholder–Davis–Gundy inequality (2.8),

$$\begin{aligned} \mathbb{E} \sup_{r \in [0,t]} |M_\mu(r)| &\leq 3\mathbb{E} \left(\sum_{i=1}^N \int_0^t \left(\int_M G_{F_\mu}(\rho(s)) \operatorname{div}_h a_i dV_h \right)^2 ds \right)^{\frac{1}{2}} \\ &\stackrel{(6.3)}{\leq} C_1 \mathbb{E} \left(\int_0^t \left(\int_M F_\mu(\rho(s)) dV_h \right)^2 ds \right)^{\frac{1}{2}} \\ &\leq C_1 \mathbb{E} \left(\operatorname{esssup}_{\tau \in [0,t]} \int_M F_\mu(\rho(\tau)) dV_h \int_0^t \int_M F_\mu(\rho(s)) dV_h ds \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} \mathbb{E} \operatorname{esssup}_{r \in [0,t]} \int_M F_\mu(\rho(r)) dV_h + \frac{C_1}{2} \int_0^t \mathbb{E} \int_M F_\mu(\rho(s)) dV_h ds, \end{aligned}$$

where the constant C_1 is independent of μ, t (but it depends on $\max_i \|a_i\|_{C^1}$). On that account, we obtain

$$f_\mu(t) \lesssim f_\mu(0) + \int_0^t (1 + \|\operatorname{div}_h u(s)\|_{L^\infty(M)}) f_\mu(s) ds + o(1/\mu), \quad t \in [0, T],$$

which, in combination with the Grönwall inequality, implies

$$f_\mu(t) \leq \exp(Ct) f_\mu(0) + o(1/\mu) \stackrel{(6.3)}{\leq} 2 \exp(Ct) \|\rho_0\|_{L^2(M)}^2 + o(1/\mu),$$

for some μ -independent constant C . Relying on the Fatou lemma, the a priori estimate (3.7) emerges after sending $\mu \rightarrow \infty$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix. Some technical results

We collect here several results that have been used throughout the paper (often unannounced), starting with a minor generalization of a well-known commutator estimate, see [10, Lemma II.1] or [26, Lemma 2.3].

We fix a standard Friedrichs mollifier $\phi_\varepsilon (= \varepsilon^{-d}\phi(x/\varepsilon))$ on \mathbb{R}^d . In what follows, we will consider functions and vector fields defined on an open (bounded or unbounded) subset of the Euclidean space \mathbb{R}^d .

We say that a triple $(\alpha_1, \alpha_2, \beta)$ is (1)-admissible if $\alpha_1, \alpha_2 \in [1, \infty]$, $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \leq 1$, $\frac{1}{\beta} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2}$ if $\alpha_1 < \infty$ or $\alpha_2 < \infty$, and $\beta \in [1, \infty)$ is arbitrary if $\alpha_1 = \alpha_2 = \infty$.

Lemma A.1 (DiPerna–Lions). *Let (Z, μ) be a finite measure space. Suppose*

$$g \in L^{q_1}(Z; L^{p_1}_{\text{loc}}(G)), \quad V \in L^{q_2}\left(Z; W^{1,p_2}_{\text{loc}}(G; \mathbb{R}^d)\right),$$

for some (1)-admissible triples $(p_1, p_2, p), (q_1, q_2, q)$. Then, for any compact $K \subset G$,

$$\begin{aligned} & \left\| \operatorname{div}(gV)_\varepsilon - \operatorname{div}(g_\varepsilon V) \right\|_{L^q(Z; L^p(K))} \\ & \leq C \|g\|_{L^{q_1}(Z; L^{p_1}(K))} \|V\|_{L^{q_2}(Z; W^{1,p_2}(K; \mathbb{R}^d))}, \end{aligned} \tag{A.1}$$

for some constant C that does not depend on ε, p, g, V . Furthermore,

$$\left\| \operatorname{div}(gV)_\varepsilon - \operatorname{div}(g_\varepsilon V) \right\|_{L^q(Z; L^p(K))} \xrightarrow{\varepsilon \downarrow 0} 0. \tag{A.2}$$

Proof. For brevity, let us write $c_\varepsilon(z, x) := \operatorname{div}(gV)_\varepsilon(z, x) - \operatorname{div}(g_\varepsilon V)(z, x)$, for $z \in Z$ and $x \in G$. By the classical DiPerna–Lions theory (cf. [10, Lemma II.1] or [26, Lemma 2.3]), $c_\varepsilon(z, \cdot) \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^p(K)$ for μ -a.e. $z \in Z$. Besides,

$$\|c_\varepsilon(z, \cdot)\|_{L^p(K)} \lesssim \|g(z, \cdot)\|_{L^{p_1}(K)} \|V(z, \cdot)\|_{W^{1,p_2}(K; \mathbb{R}^d)},$$

for μ -a.e. $z \in Z$. Suppose $q_1 < \infty$ or $q_2 < \infty$. We raise to the power q this inequality and apply the generalized Hölder inequality to demonstrate that the resulting right-hand side is an integrable function (i.e., a μ -dominant integrable function). Therefore, by the dominated convergence theorem, we obtain the desired convergence result (A.2) as well the bound (A.1). The case $q_1 = q_2 = \infty$ is treated analogously. Indeed, for any $q \in [1, \infty)$,

$$\|c_\varepsilon(z, \cdot)\|_{L^p(K)}^q \lesssim \|g\|_{L^\infty(Z; L^{p_1}(K))}^q \|V\|_{L^\infty(Z; W^{1,p_2}(K; \mathbb{R}^d))}^q,$$

for μ -a.e. $z \in Z$, and once again we have obtained a μ -dominant integrable function and conclude by dominated convergence. \square

Remark A.1. In this paper, we apply Lemma A.1 with the finite measure space (Z, μ) equal to $(\Omega, \mathbb{P}), ([0, T], dt)$, or $(\Omega \times [0, T], \mathbb{P} \otimes dt)$.

Our next lemma is about the convergence of a “second order” commutator. The lemma is taken from Punshon-Smith’s preprint [32].

We say that a triple $(\alpha_1, \alpha_2, \beta)$ is (2)-admissible if $\alpha_1, \alpha_2 \in [1, \infty]$, $\frac{1}{\alpha_1} + \frac{2}{\alpha_2} \leq 1$, $\frac{1}{\beta} = \frac{1}{\alpha_1} + \frac{2}{\alpha_2}$ if $\alpha_1 < \infty$ or $\alpha_2 < \infty$, and $\beta \in [1, \infty)$ is arbitrary if $\alpha_1 = \alpha_2 = \infty$.

Lemma A.2 (Punshon-Smith). *Suppose*

$$g \in L^{p_1}_{\text{loc}}(G), \quad V \in W^{1,p_2}_{\text{loc}}(G; \mathbb{R}^d),$$

for some (2)-admissible triple (p_1, p_2, p) , and define

$$C_\varepsilon[g, V] := \frac{1}{2} \partial_{ij} (V^i V^j g)_\varepsilon - V^i \partial_{ij} (V^j g)_\varepsilon + \frac{1}{2} V^i V^j \partial_{ij} g_\varepsilon.$$

For any compact subset $K \subset G$,

$$\mathcal{C}_\varepsilon [g, V] - \frac{1}{2} \left((\operatorname{div} V)^2 + \partial_i V^j \partial_j V^i \right) g_\varepsilon \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^p(K).$$

Furthermore, there is a constant C independent of ε, p, g, V such that

$$\begin{aligned} & \left\| \mathcal{C}_\varepsilon [g, V] - \frac{1}{2} \left((\operatorname{div} V)^2 + \partial_i V^j \partial_j V^i \right) g_\varepsilon \right\|_{L^p(K)} \\ & \leq C \|V\|_{W^{1,p_2}(K;\mathbb{R}^d)}^2 \|g\|_{L^{p_1}(K)}. \end{aligned}$$

Proof. By [32, Lemma 3.2],

$$\mathcal{C}_\varepsilon [g, V] - \frac{1}{2} \left((\operatorname{div} V)^2 + \partial_i V^j \partial_j V^i \right) g \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^p(K),$$

and

$$\begin{aligned} & \left\| \mathcal{C}_\varepsilon [g, V] - \frac{1}{2} \left((\operatorname{div} V)^2 + \partial_i V^j \partial_j V^i \right) g \right\|_{L^p(K)} \\ & \leq C \|V\|_{W^{1,p_2}(K;\mathbb{R}^d)}^2 \|g\|_{L^{p_1}(K)}, \end{aligned}$$

for some constant C independent of ε, p, g, V . The lemma follows from this, the triangle inequality, and the bound $(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2/2})$

$$\begin{aligned} & \left\| \left((\operatorname{div} V)^2 + \partial_i V^j \partial_j V^i \right) (g - g_\varepsilon) \right\|_{L^p(K)} \\ & \leq \left\| (\operatorname{div} V)^2 + \partial_i V^j \partial_j V^i \right\|_{L^{p_2/2}(K)} \|g - g_\varepsilon\|_{L^{p_1}(K)} \lesssim \|V\|_{W^{1,p_2}(K)}^2 \|g\|_{L^{p_1}(K)}. \quad \square \end{aligned}$$

Let us also state the following generalization of Lemma A.2, which is analogous to Lemma A.1 (the proof is also the same).

Lemma A.3. *Let (Z, μ) be a finite measure space. Suppose*

$$g \in L^{q_1} \left(Z; L_{\text{loc}}^{p_1}(G) \right), \quad V \in L^{q_2} \left(Z; W_{\text{loc}}^{1,p_2}(G; \mathbb{R}^d) \right),$$

for some (2)-admissible triples $(p_1, p_2, p), (q_1, q_2, q)$. Then, for any compact $K \subset G$,

$$\begin{aligned} & \left\| 2\mathcal{C}_\varepsilon [g, V] - g_\varepsilon (\operatorname{div} V)^2 - g_\varepsilon \partial_i V^j \partial_j V^i \right\|_{L^q(Z; L^p(K))} \\ & \leq C \|g\|_{L^{q_1}(Z; L^{p_1}(K))} \|V\|_{L^{q_2}(Z; W^{1,p_2}(K;\mathbb{R}^d))}^2, \end{aligned}$$

for some constant C that does not depend on ε, p, g, V . Furthermore,

$$\left\| 2\mathcal{C}_\varepsilon [g, V] - g_\varepsilon (\operatorname{div} V)^2 - g_\varepsilon \partial_i V^j \partial_j V^i \right\|_{L^q(Z; L^p(K))} \xrightarrow{\varepsilon \downarrow 0} 0.$$

On several occasions we use the following basic convergence lemma:

Lemma A.4. *Fix $r \in [1, \infty]$ and $H \in C_b(\mathbb{R})$. Let $\{f_j\}_{j \geq 1}$ be a sequence in $L^r(\Omega \times [0, T] \times M)$ converging to f in $L^r(\Omega \times [0, T] \times M)$. Then, as $j \rightarrow \infty$,*

$$H(f_j) f_j \rightarrow H(f) f \quad \text{in } L^r(\Omega \times [0, T] \times M).$$

Proof. We can assume $r < \infty$, as the result is trivial for $r = \infty$. Fix an arbitrary subsequence $\{f_{j_n}\}_{n \geq 1} \subset \{f_j\}_{j \geq 1}$. Then, by the “inverse dominated convergence” theorem, there exists a

sub-subsequence $\{f_{j_{n_k}}\}_{k \geq 1} \subset \{f_{j_n}\}_{n \geq 1}$ which converges a.e. to f , and there exists a function $g \in L^r$ that dominates $\{f_{j_{n_k}}\}_{k \geq 1}$, see [8, Theorem 4.9]. Clearly, $H(f_{j_{n_k}}) \rightarrow H(f)$ a.e. as $k \rightarrow \infty$, and

$$\sup_k \left\| H\left(f_{j_{n_k}}\right) \right\|_{L^\infty} < \infty, \quad H(f) \in L^\infty(\Omega \times [0, T] \times M).$$

Therefore, by the dominated convergence theorem,

$$H\left(f_{j_{n_k}}\right) f_{j_{n_k}} \xrightarrow{k \uparrow \infty} H(f) f \quad \text{in } L^r(\Omega \times [0, T] \times M).$$

By the arbitrariness of the fixed subsequence and the uniqueness of the limit, the original sequence must converge as well. \square

We will also need an easy variant of the previous lemma.

Lemma A.5. Fix $q \in [1, \infty)$. *Lemma A.4 continues to hold with $L^r(\Omega \times [0, T] \times M)$ replaced by $L^q([0, T]; L^2(\Omega \times M))$.*

Finally, we recall (without proof) a simple result that has been utilized several times when passing to the limit in the space-weak formulation of the SPDE.

Lemma A.6. Fix $r \in [1, \infty]$. Let $\{f_j\}_{j \geq 1}$ be a sequence in $L^r(\Omega \times [0, T])$ converging to f in $L^r(\Omega \times [0, T])$. Consider the functions

$$F_j(\omega, t) := \int_0^t f_j(\omega, s) ds, \quad F(\omega, t) := \int_0^t f(\omega, s) ds.$$

Then $F_j \rightarrow F$ in $L^r(\Omega \times [0, T])$ as $j \rightarrow \infty$.

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