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The Hunter–Saxton equation with noise [☆]

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Abstract

In this paper we develop an existence theory for the Cauchy problem to the stochastic Hunter–Saxton equation (1.1), and prove several properties of the blow-up of its solutions. An important part of the paper is the continuation of solutions to the stochastic equations beyond blow-up (wave-breaking). In the linear noise case, using the method of (stochastic) characteristics, we also study random wave-breaking and stochastic effects unobserved in the deterministic problem. Notably, we derive an explicit law for the random wave-breaking time.

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Contents

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1. Introduction

We consider the Hunter–Saxton equation [19] with noise:

$$\begin{aligned} \partial_t q + \partial_x(uq) + \partial_x(\sigma q) \circ \dot{W} &= \frac{1}{2}q^2, \\ \partial_x u &= q. \end{aligned} \tag{1.1}$$

Here evolution occurs on $[0,T] \times \mathbb{R}$, and over the stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, the process W is a standard one-dimensional Brownian motion and $\circ$ denotes Stratonovich multiplication. We also point out that in this paper we ultimately limit ourselves to the assumption that $\sigma = \sigma(x)$ is *linear*. This assumption simplifies the analysis considerably, but still allows the equation to manifest some stochastic effects. The Cauchy problem is posed with an initial condition $q|_{t=0} = q_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Other stochastic versions of the stochastic Hunter–Saxton equation exist, see [5,4], where the noise is introduced as a source term.

In the Itô formulation the stochastic Hunter–Saxton equation reads:

$$\partial_t q + \partial_x(uq) + \partial_x(\sigma q)\dot{W} - \frac{1}{2}\partial_x(\sigma \partial_x(\sigma q)) = \frac{1}{2}q^2. \tag{1.2}$$

The primary aim of this paper is to develop an existence theory for the stochastic Hunter–Saxton equation under the assumptions above. Our main theorem is Theorem 2.8, stating that the equation (1.1) has both conservative and dissipative global solutions when σ is linear. (The notions of conservative and dissipative solutions are discussed below.)

Our line of attack relies on the method of characteristics. Stochastic characteristics are used widely in the analysis of transport type equations in fluid dynamics and other applications (see [13] and [14, Ch. 4] and references there), where corresponding deterministic dynamics are perturbed by introducing noise to the characteristics. As explained in Appendix A, the physical relevance of this noise derives from its being a perturbation on the associated Hamiltonian of the system, following a discussion in [18] for stochastic soliton dynamics, so that the resulting equation follows from a variational principle applied to the stochastically perturbed Hamiltonian. Our formulation does *not* conform to the “Euler-Poincaré structure” specified in [18], however, except in the $\sigma'' = 0$ case, to which most of this paper nevertheless pertains. A fuller account of the diochotomy and similarities of these formulations is given in Appendix A.

The method of characteristics as applied to (1.1), departs from the regime treated by [13], however, as the transport term depends on the solution. This type of equation also falls outside the scope of the related investigation [15], which extended [13] in their use of the kinetic formulation. The non-locality of the dynamics of (1.1) means that the transport term depends not only on the values of the solution at a point, but on the integral thereof, precluding a "kinetic" treatment of well-posedness. A substantial part of this work will be devoted to showing that the characteristics can be extended beyond a blow-up that inevitably happens, also in the deterministic case. This blow-up, termed "wave-breaking", is explained in Section 1.1 below.

It turns out that on properly defined characteristics, it is possible to derive explicit solutions. As we are employing characteristics and solving equations on characteristics, it is also imperative that we reconcile "solutions-along-characteristics" with solutions as usually defined, and which reduces to the familiar weak solutions [20] in the deterministic case $\sigma = 0$. Relying on this explicit representation of solutions on characteristics, along the way we shall develop other aspects of the phenomenology for various solutions to these equations, including a connection between the distribution of blow-up times and exponential Brownian processes.

The organisation of this paper is as follows: In the remainder of this section, we describe the deterministic theory both to develop intuition about the dynamics of the stochastic Hunter–Saxton equation, and to give ourselves a template by which to understand corresponding features of the stochastic dynamics. Some pertinent calculations in the deterministic theory have been relegated to Appendix C. Physical arguments behind our particular choice of the noise, which suggest that the case we consider is of physical relevance, are contained in Appendix A.

In the next section we give precise definitions of solutions, and state a-priori bounds. These bounds are proven in Appendix B. In Section 3, we set up the method of characteristics framework used in subsequent sections. In particular, we show *how* the quantity q experiences finite-time blow-up in L^∞. We also describe how this blowup in q is reflected by the behaviour of the evolution of its antiderivative, u. In Section 4 we specialise to the case $\sigma'' \equiv 0$. We derive an explicit distribution for the wave-breaking stopping time in certain cases, and describe how characteristics behave up to the blow-up of q. In Section 5 we first describe strategies to continue characteristics and solutions beyond blow-up. We then prove global well-posedness of characteristics and well-posedness of solutions defined along characteristics, first on special initial data for clarity, before extending this to general data in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ in Section 5.3. Finally in Section 6, we reconcile various notions of solutions that we use in the article and show that the solutions defined along characteristics are included in more traditional partial differential equation-type (PDE-type) weak solutions. We postpone details of discussions on uniqueness and maximal dissipation that we shall mention in passing in Sections 2 and 6 to upcoming work.

1.1. Background and the deterministic setting

We shall provide here a rough sketch of the deterministic theory of the Hunter–Saxton equation by which our intuitions are driven and against which our results can be benchmarked. We will focus on the analysis of the characteristics following Dafermos [9]. Most of the material in this subsection can be found in classical papers by Hunter–Zheng [20,21], and also in [34].

Solutions in the weak sense to the equations

$$\begin{aligned} \partial_t q + u\partial_x q + \frac{1}{2}q^2 &= 0, \\ \partial_x u &= q, \end{aligned} \tag{1.3}$$

can be constructed quite explicitly by approximation with step functions. Approximating an initial function $q_0 \in L^2(\mathbb{R})$ by

$$q_0^n(x) = \sum_{-\infty}^{\infty} V_k^n \mathbb{1}_{[k/n,(k+1)/n)}(x), \qquad V_k^n = \fint_{k/n}^{(k+1)/n} q_0(x)\,\mathrm{d}x,$$

we can confine our discussion to the "box"-type initial condition $q_0 = V_0 \mathbb{1}_{[0,1)}$. This is true in spite of the equation being non-linear, see [20]. Here $\mathbb{1}_A$ denotes the characteristic, or indicator, function of a set A, and $\fint_A$ denotes the average over a set A, i.e., $\fint_A \psi(x)\,\mathrm{d}x = \frac{1}{|A|}\int_A \psi(x)\,\mathrm{d}x$.

The equation with initial data q_0 is solved uniquely for at least a finite time by

$$q(t,x) = \frac{2V_0}{2+V_0 t}\mathbb{1}_{\{2+V_0t>0\}}\mathbb{1}_{\{X(t,0)\le x<X(t,1)\}},$$

where $X(t,x)$ with $x \in [0,1)$ are the characteristics

$$\begin{aligned} X(t,x) = x + \int_0^t u(s, X(s,x))\,\mathrm{d}s &= x + \int_0^t \int_0^{X(s,x)} q(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ &= x + \frac{1}{4}(2+V_0t)^2, \end{aligned} \tag{1.4}$$

with u being the function almost everywhere satisfying $\partial_x u = q$, and the final equality established by solving the linear ordinary differential equation using the form of q postulated. A calculation gives

$$u(t,x) = \mathbb{1}_{\{2+V_0t>0\}} \begin{cases} 0, & x \le \frac{1}{4}(2+V_0t)^2, \\ \frac{2V_0x}{2+V_0t}, & \frac{1}{4}(2+V_0t)^2 < x \le 1 + \frac{1}{4}(2+V_0t)^2, \\ \frac{2V_0}{2+V_0t}\big(1+\frac{1}{4}(2+V_0t)^2\big), & x > 1+\frac{1}{4}(2+V_0t)^2. \end{cases}$$

The general solution to the nth approximation can be recovered by summing up these "boxes" defined on disjoint intervals at every t, see [20].

From the above we see that where $V_0 \ge 0$, this solution exists uniquely and globally. If $V_0 < 0$, however, there is a break-down time t^* at which u remains just absolutely continuous in the sense of the Lebesgue decomposition as it develops a steeper and steeper gradient over a smaller and smaller interval around $x = 0$, and $\|q\|_{L^\infty(\mathbb{R})}$ tends to infinity. This phenomenon, where $\|u\|_{L^\infty(\mathbb{R})}$ remains bounded but $\|q\|_{L^\infty(\mathbb{R})} = \|\partial_x u\|_{L^\infty(\mathbb{R})} \to \infty$ is known as *wave-breaking*.

Up to wave-breaking, the energy $\|q(t)\|_{L^2}$ is conserved. This means that the characteristics $X(t,x)$ starting between $x = 0$ and $x = 1$ contract to a point. The failure of $X(t)$ in remaining a homeomorphism on $\mathbb{R}$ at wave-breaking leads to uncountably many possible ways of continuing solutions past wave-breaking, even under the requirement that $\|q(t)\|_{H^{-1}_{\mathrm{loc}}}$ remains continuous in time.

At the point of wave-breaking $q^2(t)$ passes from $L^1(\mathbb{R})$ into a measure. We can think of this measure as a "defect" measure storing up the energy (or L^2_x-mass of q). It is possible to continue solutions in various ways past wave-breaking by releasing various amounts of this mass over various durations. The two extremes are generally termed "conservative" and "dissipative" solutions [20, p. 320]. Intermediates between these extremes when dissipation is not mandated everywhere, entirely, or eternally are also possible [16], as are more non-physical solutions exhibiting spontaneous energy generation. We relegate calculations showing this defect measure to Appendix C.

Conservative solutions are constructed by releasing all the mass stored in the defect measure instantaneously after wave-breaking. That is, noticing that the formula for q (less the characteristic function $\mathbb{1}_{\{2+V_0^1 t>0\}}$) returns to a bounded function of the same — conserved — $L^2(\mathbb{R})$-mass immediately post wave-breaking, and continues to satisfy the equation weakly, it is accepted that the formula defines a reasonable notion of solution. In particular:

$$\begin{aligned}
&q \in L^\infty([0,T]; L^2(\mathbb{R})) \cap \mathrm{Lip}([0,T]; H^{-1}_{\mathrm{loc}}(\mathbb{R})),\\
&u \in C([0,T]\times\mathbb{R}),\\
&0 = \partial_t(q^2) + \partial_x(uq) \text{ in the sense of distributions.}
\end{aligned} \tag{1.5}$$

Dissipative solutions arise when the "defect measure" stores up all mass eternally, and q is simply set to nought after the wave-breaking time t^*. In this case the equations remain satisfied, and the previous inclusions remain valid, but

$$0 \geq \partial_t(q^2) + \partial_x(uq) \text{ in the sense of distributions,}$$

reflecting the dissipation characterised by the defect measure.

These can be compared to continuation in the general stochastic setting, see Section 5.1.

We propose to approach the problem of well-posedness via the method of characteristics. As solutions are non-local, even though we have equations for characteristics $\mathrm{d}X(t,x)$ dependent on $u(t,X(t,x))$, and for $\mathrm{d}(q(t,X))$, there is no independent equation for $\mathrm{d}u(t,X(t,x))$. One of the aspects of this article is making sure that characteristics and functions constructed along them are defined without circularity, up to and beyond wave-breaking, where non-uniqueness is necessarily introduced into the problem. Whilst our approach reduces to that of [9] in the deterministic case, our analysis in the stochastic setting is complicated by the fact that at wave-breaking, where a choice must be made as to the way that characteristics should be continued beyond wave-breaking, the set of wave-breaking times are dependent on the spatial variable x and on the probability space. This means that wave-breaking occurs on a significantly more complicated set, and whereas in [6,9,10], for example, translating between a wave-breaking time and the set of initial points with characteristics leading up to a wave-breaking point at those times is a fairly straightforward affair, this operation is much more delicate in the stochastic setting. Even the measurability of wave-breaking times in the filtration of the stochastic basis needs to be established in order to start a characteristic at wave-breaking and match it up properly to characteristics leading up to that wave-breaking time (on those particular sample-paths). Moreover the characteristics themselves are rough, and it is standard that there are correction terms compensating for this roughness in evaluating functions on these characteristics. These issues compel us to set forth various notions of solutions to handle different aspects of the problem, and then later to reconcile them. We shall do this in the next section.

2. Solutions and a-priori estimates

2.1. Definition of solutions

In this subsection we give definitions of different types of solutions and state our main theorem. As in the deterministic setting, there are two extreme notions of solution on which we shall focus. Whereas we have discussed how these arise in the deterministic setting both in Section 1.1 above (supplemented by Appendix C below), we shall postpone the discussion regarding continuation beyond wave-breaking in the stochastic setting and the resultant non-uniqueness to Section 5.1, after we have developed the theory sufficiently before and up to wave-breaking, with their supporting calculations.

We are working on a fixed stochastic basis

$$(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P}) \tag{2.1}$$

to which the process W in (1.1) is adapted as a Brownian motion. Next we define weak solutions in the PDE sense in the usual way: Note that in Definition 2.1, we only consider time-independent test functions.

Definition 2.1 *(Weak Solution).* A weak solution to the stochastic Hunter–Saxton equation (1.1) with $\sigma \in (C^2 \cap \dot{W}^{1,\infty} \cap \dot{W}^{2,\infty})(\mathbb{R})$ and with initial condition (u_0, q_0) where $q_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and u_0 are related by

$$u_0(x) = \int_{-\infty}^{x} q_0(y)\,\mathrm{d}y,$$

is a pair (u, q) of $\{\mathscr{F}_t\}_{t\geq 0}$-adapted processes, with $u \in L^2(\Omega \times [0,T]; \dot{H}^1(\mathbb{R}))$ being absolutely continuous in x, and in $C([0,T] \times \mathbb{R}) \cap L^\infty([0,T]; \dot{H}^1(\mathbb{R}))$, $\mathbb{P}$-almost surely, and $q \in L^2(\Omega \times [0,T] \times \mathbb{R})$ and in $C([0,T]; H^{-1}_{\mathrm{loc}}(\mathbb{R})) \cap L^\infty([0,T]; L^2(\mathbb{R})))$, $\mathbb{P}$-almost surely. The solution (u, q) satisfies, for any $\varphi \in C_0^\infty(\mathbb{R})$ and for any $t \in [0,T]$, $\mathbb{P}$-almost surely,

$$0 = \int_{\mathbb{R}} \varphi q\,\mathrm{d}x\Big|_0^t - \int_0^t \int_{\mathbb{R}} \left(\partial_x \varphi\, uq + \frac{1}{2}\varphi q^2\right) \mathrm{d}x\,\mathrm{d}s - \int_0^t \int_{\mathbb{R}} \partial_x \varphi\, \sigma q\,\mathrm{d}x \circ \mathrm{d}W(s), \tag{2.2}$$
$$q = \partial_x u \quad \text{in } L^2([0,T] \times \mathbb{R}).$$

In addition, we require that $\mathbb{P}$-almost surely, $\lim_{r\to -\infty} u(r) = 0$.

Remark 2.2 *(The Itô formulation of the noise).* Using the definition of a weak solution (Definition 2.1), we have the temporal integrability to ensure that the stochastic integral of (2.2) is a martingale.

From the definition of the Stratonovich integral we have

$$\int_0^t \int_{\mathbb{R}} \partial_x \varphi\, \sigma q\,\mathrm{d}x \circ \mathrm{d}W = \int_0^t \int_{\mathbb{R}} \partial_x \varphi\, \sigma q\,\mathrm{d}x\,\mathrm{d}W + \frac{1}{2}\int_0^t \mathrm{d}\left\langle \int_{\mathbb{R}} \sigma q\, \partial_x \varphi\,\mathrm{d}x,\ W \right\rangle_s. \tag{2.3}$$

Consider now $\psi = \sigma\,\partial_x\varphi$ as a time-independent test function in (2.2) (σ is assumed to be at least once continuously differentiable), we find, $\mathbb{P}$-almost surely, that

$$\begin{aligned}\int_{\mathbb{R}} (\sigma\,\partial_x\varphi\, q)(t,\,\cdot)\,\mathrm{d}x &= \int_{\mathbb{R}} \psi q\,\mathrm{d}x\Big|_{t=0} + \int_0^t\int_{\mathbb{R}} \big(\partial_x\psi\, uq + \frac{1}{2}\psi q^2\big)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathbb{R}} \sigma q\,\partial_x\psi\,\mathrm{d}x\circ\mathrm{d}W \\ &= \int_{\mathbb{R}} \psi q\,\mathrm{d}x\Big|_{t=0} + \int_0^t\int_{\mathbb{R}} \big(\partial_x\psi\, uq + \frac{1}{2}\psi q^2\big)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathbb{R}} \sigma q\,\partial_x\big(\sigma\,\partial_x\varphi\big)\,\mathrm{d}x\,\mathrm{d}W + \frac{1}{2}\int_0^t \mathrm{d}\Big\langle \int_{\mathbb{R}} \sigma q\,\partial_x\psi\,\mathrm{d}x,\; W\Big\rangle_s .\end{aligned}$$

As all terms on the right-hand side except for the stochastic integral, are of finite variation, we also have

$$\begin{aligned}\int_0^t \mathrm{d}\Big\langle \int_{\mathbb{R}} \sigma q\,\partial_x\varphi\,\mathrm{d}x,\; W\Big\rangle_s &= \int_0^t \mathrm{d}\Big\langle \int_0^{(\cdot)}\int_{\mathbb{R}} \sigma q\,\partial_x\psi\,\mathrm{d}x\,\mathrm{d}W,\; W\Big\rangle_s \\ &= \int_0^t\int_{\mathbb{R}} \sigma q\,\partial_x\big(\sigma\,\partial_x\varphi\big)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Inserting this is in (2.3), we find

$$\int_0^t\int_{\mathbb{R}} \partial_x\varphi\,\sigma q\,\mathrm{d}x\circ\mathrm{d}W = \int_0^t\int_{\mathbb{R}} \partial_x\varphi\sigma q\,\mathrm{d}x\,\mathrm{d}W + \frac{1}{2}\int_0^t\int_{\mathbb{R}} \sigma q\,\partial_x\big(\sigma\,\partial_x\varphi\big)\,\mathrm{d}x\,\mathrm{d}s. \tag{2.4}$$

We can put this directly back into (2.2) and conclude that the weak solution as given can also be understood as a weak formulation of the Itô equation (1.2):

$$\partial_t q + \partial_x(uq) + \partial_x(\sigma q)\dot{W} - \frac{1}{2}\partial_x(\sigma\,\partial_x(\sigma q)) = \frac{1}{2}q^2.$$

Weak solutions are non-unique, a fact that shall be further expounded upon in Section 5.1. We can refine Definition 2.1 by concentrating on two types with additional properties as in the deterministic setting:

Definition 2.3 *(Conservative Weak Solutions).* A conservative weak solution is a weak solution of (1.1) satisfying the energy equality

$$\begin{aligned}\partial_t q^2 + \partial_x\Big(\big(u - \frac{1}{4}\partial_x\sigma^2\big)q^2\Big) + \partial_x\left(\sigma q^2\right)\dot{W} + \partial_x\sigma q^2\dot{W} - \frac{1}{2}\partial_{xx}^2\left(\sigma^2 q^2\right) \\ = q^2\Big(\big(\partial_x\sigma\big)^2 - \frac{1}{4}\partial_{xx}^2\sigma^2\Big),\end{aligned}\tag{2.5}$$

in the sense of distributions on $[0,\infty)\times\mathbb{R}$, $\mathbb{P}$-almost surely.

Remark 2.4. Equation (2.5) is derived in Appendix B, for $S \in W^{2,\infty}(\mathbb{R})$. Taking $S = S_\ell(q_\varepsilon) = q_\varepsilon^2 \wedge (2\ell|q| - \ell^2)$ for a mollified solution q_ε, and taking $\varepsilon \to 0$ before $\ell \to \infty$ (when $S_\infty(q) = q^2$), the conservation in the definition above follows from (B.23). The full calculation can be found in Lemma B.3 and the proof of Proposition 2.11 (also housed in Appendix B).

Remark 2.5 *(Energy conservation identity).* We shall prove in Theorem 5.6 that in the case $\sigma'' = 0$, conservative weak solutions that are also solutions-along-characteristics (Definition 2.9) also satisfy the energy identity that, $\mathbb{P}$-almost surely,

$$\int_{\mathbb{R}} q^2(t,x)\,\mathrm{d}x = \int_{\mathbb{R}} q_0^2(x)\exp(-\sigma' W(t))\,\mathrm{d}x.\tag{2.6}$$

In particular, for a deterministic initial value $q_0 \in L^2(\mathbb{R})$,

$$\begin{aligned}\mathbb{E}\int_{\mathbb{R}} q^2(t,x)\,\mathrm{d}x &= \mathbb{E}\int_{\mathbb{R}} q_0^2(x)\exp(-\sigma' W(t))\,\mathrm{d}x \\ &= \iint_{\mathbb{R}^2} q_0^2(x)\exp(-\sigma' y)\,\gamma_t(\mathrm{d}y)\,\mathrm{d}x = \|q_0\|_{L^2(\mathbb{R})} e^{(\sigma')^2 t/4},\end{aligned}\tag{2.7}$$

where γ_t is the one-dimensional Gaussian measure at t.

This shows both that $q \in L^\infty([0,T]; L^2(\mathbb{R}))$, $\mathbb{P}$-almost surely, and, in fact, also the additional integrability information in ω, namely that $q \in L^\infty([0,T]; L^2(\Omega\times\mathbb{R}))$. This inclusion holds for more general noise (see Proposition 2.11).

Definition 2.6 *(Dissipative Weak Solutions).* A dissipative weak solution is a weak solution of (1.1) satisfying the condition that $q(t,x)$ is almost surely bounded from above on every compact subset of $(0,\infty)\times\mathbb{R}$, i.e., on every compact $E \subseteq (0,\infty)\times\mathbb{R}$, for $\mathbb{P}$-almost every ω there exists $M_{\omega,E} < \infty$ such that $q(t,x) < M_{\omega,E}$ for any $(t,x) \in E$, in particular, M is allowed to depend on ω.

Remark 2.7 *(Energy dissipation identity and maximal energy dissipation).* We shall show in Proposition 2.11 that weak dissipative solutions also satisfy the energy inequality

$$\begin{aligned}\partial_t q^2 + \partial_x\Big(\big(u - \frac{1}{4}\partial_x\sigma^2\big)q^2\Big) + \partial_x\left(\sigma q^2\right)\dot{W} + \partial_x\sigma q^2\dot{W} - \frac{1}{2}\partial_{xx}^2\left(\sigma^2 q^2\right) \\ \le q^2\Big(\left(\partial_x\sigma\right)^2 - \frac{1}{4}\partial_{xx}^2\sigma^2\Big),\end{aligned}\tag{2.8}$$

in the sense of distributions (when integrated against non-negative test functions) on $[0,\infty)\times\mathbb{R}$, $\mathbb{P}$-almost surely.

Define the random variable t_x^*, parameterised by every $x\in\mathbb{R}$ that is a Lebesgue point of q_0, via the equation

$$-q_0(x)\int_0^{t_x^*}\exp\big(-\sigma' W(s)\big)\,\mathrm{d}s=2, \tag{2.9}$$

or set $t_x^*=\infty$ if this equality never holds. In the case $\sigma''=0$, we shall prove additionally in Theorem 5.7 that $\mathbb{P}$-almost surely, dissipative weak solutions that are also solutions-along-characteristics (Definition 2.9) satisfy the energy identity

$$\int_{\mathbb{R}} q^2(t,x)\,\mathrm{d}x=\int_{\mathbb{R}} q_0^2(x)\exp(-\sigma' W(t))\mathbb{1}_{\{t\le t_x^*\}}\,\mathrm{d}x. \tag{2.10}$$

This formula similarly shows that a dissipative weak solution in the $\sigma''=0$ case is in $L^\infty([0,T];L^2(\Omega\times\mathbb{R}))$ as the integrand on the right is non-negative and cannot be greater than (2.6) (again, see Proposition 2.11 for a more general statement).

It was shown in Cieślak–Jamaróz [6] that this final requirement, in the deterministic setting, is implied by an Oleinik-type bound from above on q, and equivalent to a maximal energy dissipation admissibility criterion à la Dafermos [8–10]. The energy (in)equality is derived as part of the L^2-estimate worked out in the next subsection. As we also mention at the end of the paper, we shall show in an upcoming work that maximal energy dissipation is given by (2.10), as well as uniqueness of these (maximally) dissipative solutions.

Taking $\sigma\equiv 0$, we recover the well-known conservative and dissipative solutions, respectively, of [20].

The main aim of this paper is to establish the following theorem:

Theorem 2.8. *There exist conservative and dissipative weak solutions to the stochastic Hunter–Saxton equation* (1.1) *with σ for which $\sigma''=0$ and $q_0\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$.*

As we shall be working on characteristics, in Section 3.1 below we adopt yet another notion of solutions.

Definition 2.9 *(Solution-along-characteristics).* On the stochastic basis (2.1), an $\{\mathscr{F}_t\}$-adapted process $Q\in L^2(\Omega\times[0,T]\times\mathbb{R})$ and $Q\in C([0,T];H^{-1}_{\mathrm{loc}}(\mathbb{R}))\cap L^\infty([0,T];L^2(\mathbb{R}))$, $\mathbb{P}$-almost surely, is a solution-along-characteristics to the stochastic Hunter–Saxton equation (1.1) if there exists an $\{\mathscr{F}_t\}$-adapted process $U\in L^2(\Omega\times[0,T];\dot{H}^1(\mathbb{R}))$ and in $C([0,T]\times\mathbb{R})$, $\mathbb{P}$-almost surely, for which the following stochastic differential equations (SDEs) are satisfied strongly in the probabilistic sense and a.e. on $[0,T]\times\mathbb{R}$:

$$Q(t,x):=\partial_x U(t,x),$$

$$Q(t, X(t,x)) = q_0(x) - \frac{1}{2}\int_0^t Q^2(s, X(s,x))\,\mathrm{d}s - \int_0^t \sigma'(X(s,x))Q(s, X(s,x)) \circ \mathrm{d}W, \tag{2.11}$$

where $q_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and where,

$$X(t,x) = x + \int_0^t U(s, X(s,x))\,\mathrm{d}s + \int_0^t \sigma(X(s,x)) \circ \mathrm{d}W(s).$$

Remark 2.10 *(Conservative and dissipative solutions-along-characteristics).* The solutions so defined are individualised into conservative and dissipative solutions-along-characteristics according to how $U(t, X(t,x))$ (equivalently, X) are extended past a (unique) wave-breaking time t_x^* indexed by the initial point $x = X(0,x)$, cf. Theorems 5.6 and 5.7. We will in Section 6 provide theorems showing that solutions-along-characteristics are weak solutions.

As we shall see, the SDE (2.11) above is the Lagrangian formulation of the stochastic Hunter–Saxton equation (1.1). In the linear case $\sigma'' = 0$ (σ' is a constant) there is an explicit formula for the process $\mathfrak{Q} = \mathfrak{Q}(t,x)$ satisfying

$$\mathrm{d}\mathfrak{Q} = -\frac{1}{2}\mathfrak{Q}^2\,\mathrm{d}t - \sigma'(X(t,x))\mathfrak{Q} \circ \mathrm{d}W,$$

as we shall demonstrate in Section 3.1. Importantly, this SDE does not depend explicitly on t and x (cf. Remark 3.4).

This definition reflects our strategy of proof, which is to postulate a $U(t,x)$, and, using this function, define $Q(t,x) := \partial_x U(t,x)$ and the characteristics $X(t,x)$ for which

$$\mathrm{d}X(t,x) = U(t, X(t,x))\,\mathrm{d}t + \sigma(X(t,x)) \circ \mathrm{d}W,$$

and then show that $Q(t, X(t,x))$ coincides with the explicit formula for the process $\mathfrak{Q}(t,x)$. A schematic diagram for our construction is as follows:

$$\begin{tikzcd}
 & \text{construct } U(t,x) \arrow[dl] \arrow[dr] & \\
Q(t,x) := \partial_x U(t,x) \arrow[dr] & & \mathrm{d}X = U(t,X)\,\mathrm{d}t + \sigma(X) \circ \mathrm{d}W \arrow[dl] \\
 & Q(t, X(t,x)) \overset{?}{=} \mathfrak{Q}(t,x) &
\end{tikzcd}$$

2.2. A-priori bounds

In the deterministic setting [20, Section 4] (see also [34, Section 2.2.4], and references included there) it is known that weak conservative and dissipative solutions satisfy the following bounds:

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|q(t)\|_{L^2(\mathbb{R})} \le \|q_0\|_{L^2(\mathbb{R})},$$

$$\|q\|^{2+\alpha}_{L^{2+\alpha}([0,T]\times\mathbb{R})} \le C_{T,\alpha}\|q_0\|^2_{L^2(\mathbb{R})},$$

for $t \in [0, T]$ and $0 \le \alpha < 1$. In the stochastic setting, the same types of bounds are generally available only in expectation. In fact, we have the following result.

Proposition 2.11 *(A-priori bounds). Let q be a conservative or dissipative weak solution to the stochastic Hunter–Saxton equation (1.1), with $\sigma \in (C^2 \cap \dot{W}^{1,\infty} \cap \dot{W}^{2,\infty})(\mathbb{R})$, and initial condition $q(0) = q_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. The following bounds hold:*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \mathbb{E}\|q(t)\|^2_{L^2(\mathbb{R})} \le C_T\|q_0\|^2_{L^2(\mathbb{R})}, \tag{2.12}$$

$$\mathbb{E}\|q\|^{2+\alpha}_{L^{2+\alpha}([0,T]\times\mathbb{R})} \le C_{T,\alpha}\|q_0\|^2_{L^2(\mathbb{R})}, \tag{2.13}$$

for any $\alpha \in [0, 1)$.

Therefore we have

$$q \in L^\infty([0,T]; L^2(\Omega\times\mathbb{R})) \cap L^{2+\alpha}(\Omega\times[0,T]\times\mathbb{R})$$

for any $\alpha \in [0, 1)$. These bounds are not expected to hold for general weak solutions, because, as we shall see, spontaneous energy generation (spontaneous increase in L^2-mass even in expectation) in q is permissible under Definition 2.1.

We shall prove this proposition using renormalisation techniques. Calculations can be found in Appendix B. More precisely, we have the t-almost everywhere bounds:

$$\mathbb{E}\int_{\mathbb{R}} |q|^2\,\mathrm{d}x\Big|_0^t \le \mathbb{E}\int_0^t\int_{\mathbb{R}} q^2\Big((\partial_x\sigma)^2 - \frac{1}{4}\partial^2_{xx}\sigma^2\Big)\,\mathrm{d}x\,\mathrm{d}s \tag{2.14}$$

for $L^2_{\omega,x}$-control, and

$$\frac{1-\alpha}{2}\mathbb{E}\int_0^t\int_{\mathbb{R}} |q|^{2+\alpha}\,\mathrm{d}x\,\mathrm{d}s \le \mathbb{E}\int_{\mathbb{R}} q|q|^\alpha\,\mathrm{d}x\Big|_0^t - \frac{\alpha(\alpha+1)}{2}\mathbb{E}\int_0^t\int_{\mathbb{R}} q|q|^\alpha(\partial_x\sigma)^2\,\mathrm{d}x\,\mathrm{d}s$$
$$+\frac{\alpha}{4}\mathbb{E}\int_0^t\int_{\mathbb{R}} \partial^2_{xx}\sigma^2 q|q|^\alpha\,\mathrm{d}x\,\mathrm{d}s \tag{2.15}$$

for control in $L^{2+\alpha}_{\omega,t,x}$, by interpolation.

Because of the first term on the right-hand side of (2.15) and the use of interpolation/Hölder's inequality, and because we only have pointwise almost everywhere-in-time bounds for $\mathbb{E}\|q(t)\|_{L_x^p}$ with $p=2$, we cannot extend these estimates past $\alpha<1$ (but see Remark 5.5 regarding possible higher integrability as a manifestation of regularisation-by-noise).

Remark 2.12 *(Energy conservation).* With respect to (2.14), the equation $(\partial_x\sigma)^2=\partial_{xx}^2\sigma^2/4$, which implies energy conservation, can be solved explicitly by $\sigma(x)=Ae^{\pm x}$ or $\sigma(x)\equiv C$, the first of which does not satisfy our linearity assumption except with $A=0$. (See also Remark A.1 for the significance of this σ in a slightly different formulation of the stochastic Hunter–Saxton equation.)

3. The Lagrangian formulation and method of characteristics

3.1. Solving q on characteristics

Even though the Hunter–Saxton equation is not spatially local, in the deterministic setting, characteristics

$$\partial_t X(t,x)=u(t,X(t,x))$$

essentially fix the evolution of the equations because functions constant-in-space between two characteristics remain constant-in-space, and $\|q(t)\|_{L^2}$ is conserved up to wave-breaking (and also beyond — this being *one* way to characterise continuation of solutions past wave-breaking). In the stochastic setting the behaviour between characteristics is more complicated and there is no conserved quantity. Nevertheless, taking cue from the classical construction of characteristics, much can still be deduced for solutions to the stochastic equations.

The "characteristic equations" from which the stochastic Hunter–Saxton equation arises are written with Stratonovich noise, as pointed out by [1]:

$$X(t,x)=x+\int_0^t u(s,X(s,x))\,\mathrm{d}s+\int_0^t \sigma(X(s,x))\circ \mathrm{d}W(s). \tag{3.1}$$

Assuming that these characteristics are well-posed, via a general Itô–Wentzell formula [24], since $q(t;\omega)$ takes values in $L^2(\mathbb{R})$, one can derive from (1.1) the simpler (Lagrangian variables) equation:

$$\mathrm{d}q(t,X(t))=-\frac{1}{2}q^2(t,X(t))\,\mathrm{d}t-\sigma'(X(t))q(t,X(t))\circ \mathrm{d}W. \tag{3.2}$$

As mentioned after Definition 2.9 above, the SDE (3.2) satisfied by $q(t,X(t))$ (if suitably well-defined), can be written without reference to x or to compositions of solution with characteristics as:

$$\mathrm{d}\mathfrak{Q}=-\frac{1}{2}\mathfrak{Q}^2\,\mathrm{d}t-\sigma'\mathfrak{Q}\circ \mathrm{d}W, \tag{3.3}$$

and can in fact be solved explicitly without dependence on X, in the case $\sigma'' = 0$. We shall see this in (3.4) of Lemma 3.1.

As in the previous section, since we are working presently on the assumption of well-posedness, in this section we do not restrict ourselves to $\sigma'' = 0$. We shall do so starting in Section 4. We postpone resolving the issue of the well-posedness of the characteristics equation (3.2) to Section 5.1, but record here some properties of the composition $q(t, X(t,x))$ if it exists and is a strong solution of the SDE (3.2):

Lemma 3.1.

(i) *Assume that $X(t,x)$ is a collection of adapted processes with $\mathbb{P}$-almost surely continuous paths for each x in the collection of Lebesgue points of q_0. Suppose that the composition $q(t, X(t,x))$ is a strong solution to the SDE (3.2) with $\sigma \in C^2(\mathbb{R}) \cap \dot{W}^{2,\infty}(\mathbb{R})$ (i.e., u is C^2 with bounded second derivative), for each x in the same set. Then $q(t, X(t,x))$ can be expressed by the formula*

$$q(t, X(t,x)) = \frac{Z(t,x)}{\frac{1}{q_0(x)} + \frac{1}{2}\int_0^t Z(s,x)\,\mathrm{d}s}, \tag{3.4}$$

where $Z(t,x) = \exp\big(-\int_0^t \sigma'(X(s,x)) \circ \mathrm{d}W\big)$, up to the random time $t = t_x^*$ defined by

$$-\frac{1}{2}q_0(x)\int_0^{t_x^*} \exp\Big(-\int_0^s \sigma'(X(r,x)) \circ \mathrm{d}W(r)\Big)\,\mathrm{d}s = 1. \tag{3.5}$$

(ii) *For X as above assume further that $X(t)\colon \mathbb{R} \to \mathbb{R}$ is a homeomorphism of $\mathbb{R}$. If $q_0(x)$ can be written as a sum $q_1(0,x) + q_2(0,x)$ of functions of disjoint support, then*

$$q(t,x) = q_1(t, X(t, X(t)^{-1}(x))) + q_2(t, X(t, X(t)^{-1}(x))),$$

and $q_1(t)$ and $q_2(t)$ have $\mathbb{P}$-almost surely disjoint supports.

Remark 3.2 *(Non-associativity of the Stratonovich product).* Before we proceed to the proof we point out two obvious distinctions

(i) $(\mathrm{d}q)(t, X(t))$ is not $\mathrm{d}(q(t, X(t)))$; these are related by the Itô–Wentzell formula:

$$\mathrm{d}(q(t, X(t))) = (\mathrm{d}q)(t, X(t)) + (\partial_x q)(t, X(t)) \circ \mathrm{d}X + \frac{1}{2}\mathrm{d}\langle \partial_x q(t,y), X(t)\rangle\Big|_{y=X(t)};$$

to avoid the over-proliferation of parentheses, we take $\mathrm{d}q(t, X(t))$ always to mean $\mathrm{d}(q(t, X(t)))$.

(ii) Also, $(AB) \circ \mathrm{d}C$, for three processes A, B, and C with finite quadratic variation, is not $A(B \circ \mathrm{d}C)$. The difference is

$$(AB) \circ \mathrm{d}C - A(B \circ \mathrm{d}C) = \frac{1}{2}B\langle A, C\rangle.$$

For notational convenience $AB \circ \mathrm{d}C$ will always denote $(AB) \circ \mathrm{d}C$, which, as especially pointed out in [1, Lemma 3.1], is also equivalent to $A \circ (B \circ \mathrm{d}C)$.

Proof. No requirements on linearity need be made here, but we remark after the end of this proof how formulas derived simplify in an important way in this special case.

Using the change-of-variable $q(t, X(t)) \mapsto h(t) = 1/q(t, X(t))$ reduces the above to a linear SDE in $h(t)$:

$$\begin{aligned}\mathrm{d}h = \mathrm{d}\frac{1}{q(t,X(t))} &= \frac{-1}{q^2(t,X(t))} \circ \mathrm{d}q(t,X(t)) \\ &= \frac{-1}{q^2(t,X(t))} \circ \Big[-\frac{1}{2}q^2(t,X(t))\,\mathrm{d}t - \sigma'(X(t))q(t,X(t)) \circ \mathrm{d}W \Big] \\ &= \frac{1}{2}\,\mathrm{d}t + \sigma'(X(t))h(t) \circ \mathrm{d}W.\end{aligned}$$

From [23, Eq. IV.4.51], the equation for $h(t)$, and hence for $q(t, X(t))$, can be solved explicitly, being the solution of the *stochastic Verhulst equation*. Setting

$$Z(t) = Z(t,x) = \exp\Big(-\int_0^t \sigma'(X(s,x)) \circ \mathrm{d}W(s) \Big), \tag{3.6}$$

the linear equation for h and $q(t, X(t))$ can be solved explicitly:

$$h(t) = \frac{1}{Z(t)}\Big(h(0) + \frac{1}{2}\int_0^t Z(s)\,\mathrm{d}s \Big),$$

because

$$\begin{aligned}&\mathrm{d}\Big[\frac{1}{Z(t)}\Big(h(0) + \frac{1}{2}\int_0^t Z(s)\,\mathrm{d}s \Big) \Big] \\ &\quad= \frac{1}{Z(t)} \circ \frac{1}{2}Z(t)\,\mathrm{d}t - \Big(h(0) + \frac{1}{2}\int_0^t Z(s)\,\mathrm{d}s \Big) \circ \big(\frac{1}{Z^2(t)} \circ \mathrm{d}Z(t) \big) \\ &\quad= \frac{1}{2}\,\mathrm{d}t - \Big(h(0) + \frac{1}{2}\int_0^t Z(s)\,\mathrm{d}s \Big) \frac{1}{Z(t)} \circ (-\sigma'(X(t)) \circ \mathrm{d}W) \\ &\quad= \frac{1}{2}\,\mathrm{d}t + \sigma'(X(t))h \circ \mathrm{d}W\end{aligned}$$

as sought. Here we used the rule $A \circ (B \circ \mathrm{d}C) = (AB) \circ \mathrm{d}C$ repeatedly. And consequently,

$$q(t,X(t,x)) = \frac{Z(t,x)}{\frac{1}{q_0(x)} + \frac{1}{2}\int_0^t Z(s,x)\,\mathrm{d}s},$$

proving (3.4).

Since $Z > 0$ everywhere, and $X(0,x) = x$, blow-up of $q(t,X(t,x))$ occurs at $t = t_x^*$ at which

$$-\frac{1}{2}q_0(x)\int_0^{t_x^*} \exp\Big(-\int_0^s \sigma'(X(r,x))\circ \mathrm{d}W(r)\Big)\,\mathrm{d}s = 1. \tag{3.7}$$

It is immediate that if $q_0(x) = 0$, then $q(t,X(t,x)) = 0$. This implies that initial conditions with disjoint support give rise to solutions that have disjoint support, up to wave-breaking. $\square$

Remark 3.3 *(Pathwise formulation for constant σ).* It is similarly immediate that if $\sigma' = 0$ (σ constant), then the blow-up time coincides with that arising from deterministic dynamics. In fact, before we proceed to the next section, we point out that the case $\sigma' = 0$ is effectively the deterministic equations because in a “frame-of-reference” given via a path-wise transformation $x \mapsto x + \sigma W$, see [15, Prop. 2.6] and [13, Section 6.2], then modulo measurability concerns,

$$U(t,x) = u(t, x + \sigma W(t)), \quad V(t,x) = q(t, x+\sigma W(t))$$

solve the deterministic Hunter–Saxton equation

$$\begin{aligned} 0 &= \partial_t V + U\partial_x V + \frac{1}{2}V^2, \\ V &= \partial_x U, \end{aligned}$$

exactly when q and u solve (1.1) with constant σ. In fact, this is true for all equations of the form

$$0 = \partial_t u + \mathcal{B}[u] + \sigma\partial_x u \circ \dot{W},$$

in which $\mathcal{B}$ is an integro-differential functional in the spatial variable (but not directly dependent on the same) as these operations are invariant in x-translations. See also Remark 4.2.

Remark 3.4 *(The special case $\sigma'' = 0$).* Referring to (3.4), (3.5), and (3.7), consider the case of linear σ. Since then σ' is a constant, we conclude that $q(t,X)$ and the wave-breaking time depend on x only through q_0 — *and not also cyclically through $X(t,x)$*, and in (3.4), $Z(t,x) = \exp(-\sigma' W(t))$ is independent of x altogether.

The expression (3.4) can this case be written as

$$\mathfrak{Q}(t,x) = \frac{e^{-\sigma' W(t)}}{\frac{1}{q_0(x)} + \frac{1}{2}\int_0^t e^{-\sigma' W(s)}\,\mathrm{d}s}. \tag{3.8}$$

As mentioned after Definition 2.9, we shall define $\mathfrak{Q}(t,x)$ up to t_x^* in subsequent discussions where $\sigma'' = 0$, as a family of processes indexed by x by equation (3.8), and *not as the composition* of some yet unknown $q(t,x)$ with a yet unknown $X(t,x)$ (that is, for example, the expression $q(t,X(s,x))$ has no meaning for us yet where $s \neq t$).

Remark 3.5 *(An application of the theory of Bessel processes).* As an aside, we mention that it is possible to represent $\mathfrak{Q}$ as (a simple function of) a time-changed squared Bessel process of dimension 1 when $\sigma'' \equiv 0$ (that is, as the absolute value of some Brownian motion $\tilde{W}$).

A result of Lamperti [25], see also [31, XI.1.28], showed that there exists a Bessel process $R^{(\nu)}$ of index ν, i.e., of dimension $d = 2(\nu + 1)$, for which

$$\exp(W(t) + \nu t) = R^{(\nu)}\Big(\int_0^t \exp\big(2(W(s) + \nu s)\big)\,\mathrm{d}s\Big).$$

By a slight modification of Lamperti's result, it can be shown that there exists a squared Bessel process $\mathfrak{Z}^{(\delta)}(t)$ of dimension $d = 1 + 2c/(\sigma')^2$ for which

$$\frac{2}{(\sigma')^2}\exp(-\sigma' W + ct) = \mathfrak{Z}^{(\delta)}(\langle M, M\rangle(t)),$$
$$M(t) = -\int_0^t \frac{1}{\sqrt{2}}\exp\Big(\frac{1}{2}\big(-\sigma' W(s) + cs\big)\Big)\,\mathrm{d}W(s).$$

We can see this as follows. A squared Bessel process of dimension d (starting at λ) satisfies:

$$\mathfrak{Z}^{(\delta)}(t) = \lambda + 2\int_0^t \sqrt{\mathfrak{Z}^{(\delta)}}\,\mathrm{d}B + \delta t.$$

Letting B be the Brownian motion for which

$$B(\langle M, M\rangle(t)) = M(t)$$

under the Dambis–Dubins–Schwarz theorem,

$$\mathfrak{Z}^{(\delta)}(\langle M, M\rangle(t)) = \lambda + 2\int_0^t \sqrt{\mathfrak{Z}^{(\delta)}(\langle M, M\rangle(s))}\,\mathrm{d}M(s) + \delta\langle M, M\rangle(t). \tag{3.9}$$

Expanding $\langle M, M\rangle(t) = \frac{1}{2}\int_0^t \exp(-\sigma' W(s) + cs)\,\mathrm{d}s$, we find that with

$$\lambda = \frac{2}{(\sigma')^2}, \qquad \delta = \frac{2c}{(\sigma')^2} + 1,$$

the ansatz $Y(t) = \lambda\exp(-\sigma' W(t) + ct)$ satisfies the equation

$$\mathrm{d}Y(t) = \frac{-2}{\sqrt{2}}\sqrt{Y(t)}\exp\big(\frac{-\sigma' W(t) + ct}{2}\big)\,\mathrm{d}W(t) + \frac{\delta}{2}\exp(-\sigma' W(t) + ct)\,\mathrm{d}t,$$

which is (3.9) above with $Y(t) = \mathfrak{Z}^{(\delta)}(\langle M, M\rangle(t))$.

Therefore choosing $c = 0$ above, there exists a squared Bessel process $\mathfrak{Z}$ of dimension one (the absolute value of a Brownian motion) for which

$$\exp(-\sigma' W(t)) = \mathfrak{Z}(\frac{1}{2}\int_0^t \exp(-\sigma' W(s))\,\mathrm{d}s),$$

and hence,

$$q(t, X(t,x)) = \frac{\mathfrak{Z}(\frac{1}{2}\int_0^t \exp(-\sigma' W(s))\,\mathrm{d}s)}{\frac{1}{q_0(x)} + \frac{1}{2}\int_0^t \exp(-\sigma' W(s))\,\mathrm{d}s}.$$

Finally we prove our main technical lemma, which will be useful in establishing well-posedness later. This lemma is important because it describes the main feature of wave-breaking — that u gets steeper and steeper as q nears wave-breaking, but the jump is actually smaller and smaller, so that in the limit, around the point of wave-breaking, u remains absolutely continuous, but $(\partial_x u)^2 = q^2$ passes into a measure.

Lemma 3.6 *(Absolute continuity of u at wave-breaking). Let t_x^* be the wave-breaking time defined by (3.5) indexed by the Lebesgue points x of q_0. Assume that $X(t,x)$ is a collection of adapted processes with $\mathbb{P}$-almost surely continuous paths for each x in the collection of Lebesgue points of q_0. Suppose that the composition $q(t, X(t,x))$ is a strong solution to the SDE (3.2) for each x in the same collection. Set*

$$\begin{aligned}\mathfrak{u}(t,x;\omega) &= \mathfrak{u}(t,x)\\ &:= q(t,X(t,x))\exp\Big(\int_0^t q(s,X(s,x))\,\mathrm{d}s + \int_0^t \sigma'(X(s,x))\circ \mathrm{d}W(s)\Big).\end{aligned} \tag{3.10}$$

It holds that for such $x \in \mathbb{R}$ as aforementioned,

$$\mathbb{P} - a.s., \qquad \lim_{t \nearrow t_x^*} \mathfrak{u}(t,x) = 0.$$

Remark 3.7. The quantity (3.10) ought to be thought of heuristically as

$$q(t, X(t,x))\frac{\partial X}{\partial x},$$

and will be integrated in x to construct a function $U(t,x)$, defined on characteristics (cf. (5.18)). The exponential is a $\mathbb{P}$-almost surely finite quantity up to blow-up because we assume that σ' is bounded (and then constant in Section 4). Furthermore up to blow-up (if there is blow-up) there is always an upper bound on $q(t, X(t,x))$ depending on $q_0(x)$ and σ'. In the case $\sigma'' = 0$, we can define $\mathfrak{u}$ as a well-defined quantity with $\mathfrak{Q}(t,x)$ given by (3.8) in the place of $q(t, X(t,x))$, sans assumptions on q and X, so that $\mathfrak{u}$ is expressible as

$$\mathfrak{u}(t,x) := \mathfrak{Q}(t,x)\exp\Big(\int_0^t \mathfrak{Q}(s,x)\,\mathrm{d}s + \sigma' W(t)\Big), \tag{3.11}$$

which, as we shall see in the proof, cf. (3.13), reduces to

$$q_0(x)\Big(1 + \frac{1}{2}q_0(x)\int_0^t e^{-\sigma' W(s)}\,\mathrm{d}s\Big). \tag{3.12}$$

It is easily seen from the preceding formula that in the deterministic case, where the integral reduces further to $t/2$, we recover the linear term familiar in the deterministic theory.

Proof. Let $Z(t,x) = \exp(-\int_0^t \sigma'(X(s,x)) \circ \mathrm{d}W(s))$. Using the expression (3.4), we have

$$\begin{aligned}
\mathfrak{u}(t,x) &= q(t, X(t,x))\exp\Big(\int_0^t q(s, X(s,x))\,\mathrm{d}s + \int_0^t \sigma'(X(s,x)) \circ \mathrm{d}W(s)\Big) \\
&= \frac{Z(t,x)}{\frac{1}{q_0(x)} + \frac{1}{2}\int_0^t Z(s,x)\,\mathrm{d}s} \\
&\qquad \times \exp\Big(\int_0^t \frac{Z(s,x)}{\frac{1}{q_0(x)} + \frac{1}{2}\int_0^s Z(r,x)\,\mathrm{d}r}\,\mathrm{d}s + \int_0^t \sigma'(X(s,x)) \circ \mathrm{d}W(s)\Big) \\
&= \frac{Z(t,x)}{\frac{1}{q_0(x)} + \frac{1}{2}\int_0^t Z(s,x)\,\mathrm{d}s} \\
&\qquad \times \exp\Big(2\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\log\big(-\frac{1}{q_0(x)} - \frac{1}{2}\int_0^s Z(r,x)\,\mathrm{d}r\big)\,\mathrm{d}s + \int_0^t \sigma'(X(s,x)) \circ \mathrm{d}W(s)\Big) \\
&= \frac{Z(t,x)}{\frac{1}{q_0(x)} + \frac{1}{2}\int_0^t Z(s,x)\,\mathrm{d}s}\Big(-1 - \frac{1}{2}q_0(x)\int_0^t Z(s,x)\,\mathrm{d}s\Big)^2 e^{\int_0^t \sigma'(X(s,x))\circ \mathrm{d}W(s)} \\
&= Z(t,x)\exp\big(\int_0^t \sigma'(X(s,x)) \circ \mathrm{d}W(s)\big) q_0(x)\Big(1 + \frac{1}{2}q_0(x)\int_0^t Z(s,x)\,\mathrm{d}s\Big) \\
&= q_0(x)\Big(1 + \frac{1}{2}q_0(x)\int_0^t Z(s,x)\,\mathrm{d}s\Big).
\end{aligned} \tag{3.13}$$

By the definition of t_x^* given in (3.7), this quantity vanishes exactly at $t = t_x^*$. $\square$

Although the result derived above holds for general $\sigma \in W^{1,2}$, we emphasize again that whenever σ' is a constant, $Z(t,x)$ only depends on x through q_0. In the case σ' is constant, a closer look at (3.6) and (3.4) confirms that $Z(t,x)$ is independent of x, so if q_0 is constant over an interval $I \subseteq \mathbb{R}$, then for $x, y \in I$, until the blow-up time,

$$\mathfrak{Q}(t,x) = \mathfrak{Q}(t,y), \tag{3.14}$$

just as in the deterministic setting. Therefore the point of the Lemma 3.6 is that where we start with $q_0 = V_0 \mathbb{1}_{x\in[0,1]}$, we have $\mathfrak{Q}(t,x) = \mathfrak{Q}(t,\frac{1}{2})$ for $x \in [0,1]$, and $\mathfrak{u}(t,x)$ should be a constant multiple of the value of $u(t,x)$. We next explore finer properties concerning blow-up time.

4. Wave-breaking behaviour

4.1. Explicit calculation of the law of wave-breaking time using exponential Brownian motion

In this section we provide an expression for the distribution of the blow-up time t_x^* defined in (3.5), under the condition that $\sigma'' = 0$, from which we are also assured of its measurability. This is of independent interest as it describes the (random) time of wave-breaking precisely.

Where σ' is a constant, the blow-up condition (3.5) simplifies to

$$-\frac{1}{2} q_0(x) \int_0^{t_x^*} \exp\left(-\sigma' W(s)\right) \,\mathrm{d}s = 1.$$

Exponential Brownian functionals such as the one above have been studied in detail by Yor [33] and others (see also the surveys [27,28]). The distribution for the blow-up can be explicitly computed:

Let

$$A(t) := \frac{1}{2} \int_0^t \exp\left(-\sigma' W(s)\right) \,\mathrm{d}s,$$

$$A^{(\mu)}(t) := \int_0^t \exp(2\mu s + 2W(s)) \,\mathrm{d}s. \tag{4.1}$$

In [27, Theorem 4.1] (originally derived in another form in [32]) it was shown that

$$\mathbb{P}(A^{(\mu)}(t) \in \mathrm{d}\chi) = \frac{\mathrm{d}\chi}{\chi} \int_{\mathbb{R}} e^{\mu r - \mu^2 t/2} \exp\Big(-\frac{1+e^{2r}}{2\chi}\Big) \vartheta\Big(\frac{e^r}{\chi}, t\Big) \,\mathrm{d}r, \tag{4.2}$$

where the integral is taken against $\mathrm{d}r$, and

$$\vartheta(y,t) = \frac{y}{\sqrt{2\pi^3 t}} e^{\pi^2/(2t)} \int_0^\infty e^{-\xi^2/(2t)} e^{-y\cosh(\xi)} \sinh(\xi) \sin\left(\frac{\pi\xi}{t}\right) \mathrm{d}\xi.$$

We shall apply the explicit formula for the distribution of $A^{(\mu)}$ to give a similarly explicit formula for the distribution of the blow-up time t_x^*.

Proposition 4.1. *Let t_x^* be defined as in (3.5), and let $A^{(\mu)}$ be defined as in (4.1). Then*

$$\mathbb{P}(\{t_x^* \geq t\}) = \mathbb{P}\left(\left\{A^{(0)}\left(\frac{(\sigma')^2 t}{4}\right) \leq \frac{-(\sigma')^2}{2q_0(x)}\right\}\right). \tag{4.3}$$

Proof. In the following we use "$\sim$" to denote equality in law under $\mathbb{P}$.

We can use the scaling invariance of Brownian motion to show that

$$A(t) \sim \frac{2}{(\sigma')^2} A^{(0)}\left(\frac{(\sigma')^2 t}{4}\right), \tag{4.4}$$

which gives us the distribution of $A(t)$ explicitly:

$$\begin{aligned} A^{(0)}(t) = \int_0^t \exp(2W(s))\,\mathrm{d}s &= \int_0^{4t/(\sigma')^2} \exp\left(-\sigma' \frac{-2}{\sigma'} W\left(\frac{\tau}{(2/\sigma')^2}\right)\right) \mathrm{d}\frac{\tau}{4/(\sigma')^2} \\ &\sim \frac{(\sigma')^2}{4} \int_0^{4t/(\sigma')^2} \exp(-\sigma' \tilde{W}(\tau))\,\mathrm{d}\tau \\ &= \frac{(\sigma')^2}{2} A\big(4t/(\sigma')^2\big). \end{aligned}$$

Here $\tilde{W}$ is another standard Brownian motion, by the scaling invariance of the process.

We know that $A(0) = 0$ because it is an integral of a continuous process. It is also an increasing process because the integrand is positive. This implies that the supremum process $A^*(t) = \sup_{s \leq t} A(s)$ is simply $A(t)$. Finally, $-\frac{1}{q_0(x)} > 0$. Therefore,

$$\mathbb{P}(\{t_x^* \geq t\}) = \mathbb{P}(\{A(t) \leq -\frac{1}{q_0(x)}\}). \quad \square$$

Remark 4.2 *(Consistency in the limit $\sigma' \to 0$).* With regards to Remark 3.3, it is instructive to see that if $(\sigma')^2/4$ is treated as a parameter and taken to nought, then of course

$$A(t) = \frac{1}{2} t,$$

or alternatively,

$$\lim_{(\sigma')^2/4 \to 0} \frac{2}{(\sigma')^2} A^{(0)}\left(\frac{(\sigma')^2 t}{4}\right) = \frac{1}{2} \lim_{c \to 0} \frac{1}{c} \int_0^{ct} \exp(2W(s))\,\mathrm{d}s = \frac{1}{2} t,$$

by the Lebesgue differentiation theorem, and this matches the deterministic dynamics of wave-breaking exactly. This again verifies that the $\sigma' = 0$ setting cannot result in random blow-up.

4.2. Meeting time of characteristics

We turn our attention now to the characteristics themselves, described by (3.1) and reproduced below:

$$X(t,x) = x + \int_0^t u(s, X(s,x))\,\mathrm{d}s + \int_0^t \sigma(X(s,x)) \circ \mathrm{d}W(s).$$

Consider again the explicit "box" example with initial condition

$$q_0 = V_0\,\mathbb{1}_{[0,1]} \le 0, \qquad V_0 \in (-\infty, 0). \tag{4.5}$$

We seek to prove that in the case $\sigma'' = 0$, wave-breaking only occurs when characteristics meet, and when characteristics meet, wave-breaking occurs. This allows us later to use characteristics to capture precisely the behaviour of wave-breaking.

As mentioned after (3.14), in the case of "box" initial conditions (4.5), by (3.8) and reproduced here:

$$\mathfrak{Q}(t,x) = \frac{e^{-\sigma' W(t)}}{\frac{1}{q_0(x)} + \frac{1}{2}\int_0^t e^{-\sigma' W(s)}\,\mathrm{d}s},$$

we see from the dependence on x only via $q_0(x)$ that $\mathfrak{Q}$ is piecewise constant over x. In particular, this means $\mathfrak{Q}(t,x) = \mathfrak{Q}(t, \frac{1}{2})$ over $x \in (0,1)$.

We shall show that it is possible to construct a function $U(t,x)$ from this information, and characteristics from $U(t,x)$ in the next section. For now we assume that characteristics as defined by $\mathrm{d}X = U(t,X)\,\mathrm{d}t + \sigma(X) \circ \mathrm{d}W$ exist and that $(\partial_x U)(t, X(t))$ — the composition of $(\partial_x U)$ with a characteristic at the same time — is equal to the process $\mathfrak{Q}$ above. We shall establish this existence in Section 5.2 below.

Proposition 4.3 *(Characteristics meet at wave-breaking). Let $\sigma'' = 0$, and suppose $X(t,x)$ is a strong solution to the equation (3.1), for which $(\partial_x u)(s, X(t,x)) = \mathfrak{Q}(t,x)$ for each $x \in \mathbb{R}$, with $q_0 = V_0\mathbb{1}_{[0,1]}$. Then the first meeting time of any two characteristics $X(t,x)$ and $X(t,y)$,*

$$\tau_{x,y} := \inf\{t > 0 : X(t,x) = X(t,y)\}, \qquad x, y \in [0,1],$$

is $\mathbb{P}$-almost surely equal to the wave-breaking time $t^*_{1/2}$ defined by (3.5).

Remark 4.4. In particular, the explicit formula for the distribution of the meeting time of characteristics is also given by (4.3). In the case $\sigma'' \not\equiv 0$, we cannot immediately extract an explicit form for u and thereby one for X as in [1], because of nonlocality.

Proof. Recall that in the linear case, $\mathfrak{Q}$ is given via (3.8) as the process

$$\mathfrak{Q}(t,x) = \frac{e^{-\sigma' W(t)}}{\frac{1}{q_0(x)} + \frac{1}{2}\int_0^t e^{-\sigma' W(s)}\,\mathrm{d}s}.$$

If $x, y \in [0,1]$, then

$$\frac{u(s, X(s,x)) - u(s, X(s,y))}{X(s,x) - X(s,y)} = \mathfrak{Q}(s,x) = \mathfrak{Q}(s,y) = \mathfrak{Q}(s, \frac{1}{2}), \tag{4.6}$$

and similarly,

$$\frac{\sigma(X(s,x)) - \sigma(X(s,y))}{X(s,x) - X(s,y)} = \sigma',$$

as both q and σ' are constant in space over the interval $[X(s,0), X(s,1)]$. This leads us to

$$\begin{aligned} X(t,x) - X(t,y) = (x-y) + \int_0^t \mathfrak{Q}(s, \frac{1}{2})\,(X(s,x) - X(s,y))\,\mathrm{d}s \\ + \sigma' \int_0^t (X(s,x) - X(s,y)) \circ \mathrm{d}W(s), \end{aligned}$$

for $x, y \in [0,1]$. This is eminently solvable:

$$X(t,x) - X(t,y) = (x-y)\exp\Big(\int_0^t \mathfrak{Q}(s, \frac{1}{2})\,\mathrm{d}s + \sigma' W(t)\Big). \tag{4.7}$$

Since $\|q(t)\|_{L^2}^2$ is $\mathbb{P}$-almost surely bounded, the first meeting time $\tau_{0,1}$ cannot occur after the blow-up time t_x^* of $\mathfrak{Q}(t,x)$ on the characteristic $X(t,x)$ (which, again, by (3.14) is the same for any $x \in [0,1]$ — we have chosen $x = \frac{1}{2}$ for concreteness). The meeting time also cannot occur before the blow-up time, so that dissipation (instantaneous in the conservative case) cannot occur without wave-breaking.

To see this it suffices to ask how the exponential in (4.7) can possibly become nought — it cannot become so before $\mathfrak{Q}(t, \frac{1}{2})$ blows up to $-\infty$. □

The fact that the exponential *does* become nought when this happens gives us a rate in time at which $\mathfrak{Q}(s,x)$ blows up, which may otherwise have been difficult to extract from (3.8).

5. Existence of solutions

5.1. Solutions post wave-breaking: a discussion

This subsection consists solely of a discussion on different ways characteristics, and solutions defined along them, can be continued past wave-breaking. We shall not limit ourselves to

$\sigma'' = 0$. This is a question of cardinal importance because here as in the deterministic setting, non-uniqueness turns on there being various ways in which to continue solutions past wave-breaking. Accurately prescribing this continuation will allow us both to prove global existence of individual characteristics and thereby, on them, of q.

As noted following (3.7) in Lemma 3.1, if $q_0(x) = 0$, then along a characteristic starting at x, we expect $q(t, X(t,x)) = 0$. Therefore as in the deterministic setting, it should be possible to patch solutions together: That is, if $q_1(0)$, $q_2(0)$ are two $L^2(\mathbb{R})$-valued random variables (or simply $L^2(\mathbb{R})$ functions, if invariant over all but a measure zero set of Ω) of compact and disjoint support on $\mathbb{R}$, then the solution q with initial condition $q_0 = q_1(0) + q_2(0)$ is simply $q(t) = q_1(t) + q_2(t)$. Furthermore, from (3.5), the non-negativity of the exponential function also shows that there ought not to be blow-up along $X(t,x)$ if $q_0(x) \geq 0$. These heuristics imply that, as in the deterministic setting, "box"-type initial conditions given (4.5) should retain special interest in the stochastic setting.

As discussed in Section 1.1 there are two extreme ways by which solutions are continued past wave-breaking. They give rise to "conservative" and "dissipative" solutions.

In the deterministic setting, conservative solutions are constructed by simply extending the definition by explicit formulas to times $t > t_x^*$, as, in the example of the box, the explicit formula is undefined only at the point of wave-breaking, and reverts immediately to being well-defined thereafter. Seeing as $t \mapsto \int_0^t \exp(-\sigma'(X(s,x))W(s))\,\mathrm{d}s$ is $\mathbb{P}$-almost surely an increasing function in t for each fixed x, simply allowing $q(t, X(t))$ to be defined by (3.4) is similarly admissible in the stochastic setting (if the characteristics $X(t,x)$ are properly defined). Of course, continuity of $q(t)$ in suitable norms, and that of $X(t)$, requires proof. We also stress that there is no conservation of $L^2(\mathbb{R})$ even in expectation in the general stochastic setting — however, on taking $\sigma = 0$, we shall be able to recover the well-studied deterministic conservative solutions.

Alternatively, one can mandate dissipation by setting all concentrating $L^2(\mathbb{R})$-mass to nought at wave-breaking. This is the "dissipative solution". In the stochastic setting (complete) dissipation can also be replicated, though this is again predicated on proofs of continuity, for example, of the H^{-1}_{loc} norms of q. Suppose all characteristics $X(t,z)$ for $z \in [x,y]$ meet at the stopping time t_z^*. This is a stopping time by Proposition 4.1. Assuming σ' is locally bounded, as we always do, by the standard existence and uniqueness theorem for SDEs, these can be continued as

$$\mathrm{d}X(t_x^* + t, X(t_x^*, x)) = \sigma(X(t_x^* + t, X(t_x^*, x))) \circ \mathrm{d}\tilde{W}, \tag{5.1}$$

where $\tilde{W}$ is the Brownian motion starting at t_x^*, at the initial point $W(t_x^*)$.

5.2. *Well-posedness for box initial data*

We focus again on the $\sigma'' = 0$ case. Here we use the "box"-type initial condition (4.5) to illustrate the derivation of well-posedness, and the chief aspects of the general well-posedness theorem will appear here. We shall extend these results to the general data case in Section 5.3. In this subsection all solutions refer to conservative or dissipative solutions-along-characteristics.

Recall that by (3.14), for the case described by (4.5) the wave-breaking time t_x^* defined in (3.7) is uniform in $x \in [0,1]$. Thus, we denote this time simply by t^*:

$$-\frac{1}{2} q_0(\frac{1}{2}) \int_0^{t^*} \exp\Big(- \int_0^s \sigma' \circ \mathrm{d}W(r) \Big)\,\mathrm{d}s = 1. \tag{5.2}$$

The result of Lemma 3.6 then states that $\mathbb{P}$-almost surely, as $t \to t^*$ from below,

$$\mathfrak{u}(t;\omega) = \mathfrak{u}(t) := \mathfrak{Q}(t, \frac{1}{2}) \exp\Big(\int_0^t \mathfrak{Q}(s, \frac{1}{2})\, \mathrm{d}s + \sigma' W(t) \Big) \to 0, \tag{5.3}$$

where $\mathfrak{Q}(s, x)$, given explicitly by (3.8), is also uniform in $x \in [0, 1]$ because it only depends on x through the initial condition.

Next we proceed to the focus of this subsection — to resolve the primary questions of existence and uniqueness concerning the characteristics defined in (3.1), including the continuation of them past wave-breaking. This will in turn lead us to different ways of continuing $\mathfrak{Q}(t, x)$ (given by (3.8) in the "box"-type initial data case) past wave-breaking.

Our plan of attack is as follows (cf. diagram at the end of Section 2.1):

(i) Postulate a $U(t, x)$, and use it to find characteristics $X(t, x)$ satisfying

$$X(t, x) = x + \int_0^t U(s, X(s, x))\, \mathrm{d}s + \int_0^t \sigma(X(s, x)) \circ \mathrm{d}W(s). \tag{5.4}$$

(ii) Show that for $Q(t, x) = \partial_x U(t, x)$, the process $Q(t, X(t, x))$ agrees with $\mathfrak{Q}(t, x)$, $\mathbb{P}$-almost surely, up to $t = t^*$, and remains a strong solution to (3.2):

$$\mathrm{d}\tilde{Q}(t) = -\frac{1}{2}\tilde{Q}^2(t)\, \mathrm{d}t + \sigma' \tilde{Q}(t) \circ \mathrm{d}W.$$

(iii) Finally we extend U and Q past wave-breaking in ways that preserve their continuity pointwise and in $H^{-1}_{\mathrm{loc}}(\mathbb{R})$, respectively.

Our goal in this subsection is to prove the following two theorems:

Theorem 5.1 *(Conservative Solutions: Box Initial Data). Suppose $\sigma'' = 0$ and $q_0 = V_0 \mathbb{1}_{[0,1]}$, $V_0 \in \mathbb{R}$. There exists a $U \in C([0, \infty) \times \mathbb{R})$, $\mathbb{P}$-almost surely, absolutely continuous in x, such that for each $x \in \mathbb{R}$, the following SDE is globally well-posed:*

$$X(t, x) = x + \int_0^t U(s, X(s, x))\, \mathrm{d}s + \int_0^t \sigma(X(s, x)) \circ \mathrm{d}W(s).$$

For $Q(t, x) = \partial_x U(t, x)$, the process $Q(t, X(t, x))$ agrees $\mathbb{P}$-almost surely with $\mathfrak{Q}(t, x)$, defined in (3.8), up to $t = t^*$ and can be represented globally as

$$Q(t, X(t, x)) = \frac{\exp(-\sigma' W(t))}{\frac{1}{V_0} + \frac{1}{2}\int_0^t \exp(-\sigma' W(s))\, \mathrm{d}s} \mathbb{1}_{[0,1]}(x).$$

We have $Q(0, x) = q_0(x)$. In particular, $\tilde{Q}(t) = Q(t, X(t, x))$ satisfies (3.3) strongly and globally:

$$d\tilde{Q}(t) = -\frac{1}{2}\tilde{Q}^2(t)\,dt + \sigma'\tilde{Q}(t)\circ dW.$$

Similarly, for the dissipative solutions-along-characteristics, we have:

Theorem 5.2 *(Dissipative Solutions: Box Initial Data). Suppose $\sigma'' = 0$ and $q_0 = V_0\,\mathbb{1}_{[0,1]}$, $V_0 \in \mathbb{R}$. There exists a $U \in C((0,\infty)\times\mathbb{R})$, $\mathbb{P}$-almost surely, absolutely continuous in x, such that for each $x \in \mathbb{R}$, the SDE*

$$X(t,x) = x + \int_0^t U(s, X(s,x))\,ds + \int_0^t \sigma(X(s,x))\circ dW(s)$$

is globally well-posed.

For $Q(t,x) = \partial_x U(t,x)$, the process $Q(t, X(t,x))$ agrees $\mathbb{P}$-almost surely with $\mathfrak{Q}(t,x)$ as given by (3.8), up to $t = t^*$ and can be represented globally in time as

$$Q(t, X(t,x)) = \begin{cases} \frac{\exp(-\sigma' W(t))}{\frac{1}{V_0} + \frac{1}{2}\int_0^t \exp(-\sigma' W(s))\,ds}\mathbb{1}_{[0,1]}(x), & t < t^*, \\ 0, & t > t^*. \end{cases} \tag{5.5}$$

We have $Q(0,x) = q_0(x)$. In particular, $\tilde{Q}(t) = Q(t, X(t,x))$ satisfies (3.3) strongly and globally (in time):

$$d\tilde{Q}(t) = -\frac{1}{2}\tilde{Q}^2(t)\,dt + \sigma'\tilde{Q}(t)\circ dW.$$

We relegate the computation of H^{-1}_{loc} to Section 5.3 where it is done in the general context (see also Remark 5.10). Theorems 5.1 and 5.2 are proved in similar fashion and we shall present one in full and sketch out the other. In both of them the bulk of the work rests on a proper construction of U. Obviously in both proofs we shall be making heavy use of (3.8) and on our main technical result, Lemma 3.6.

For dissipative solutions we can also show the one-sided Oleinik-type estimate (cf. discussion following Definition 2.6):

Corollary 5.3 *(One-Sided Estimate: Box Initial Data). Suppose $\sigma'' = 0$ and $q_0 = V_0\,\mathbb{1}_{[0,1]}$, $V_0 \in \mathbb{R}$. Then the dissipative solution $Q(t,x)$ with initial condition $Q(0) = q_0$ satisfies $\mathbb{P}$-almost surely the following one-sided bound:*

$$Q(t,x) \le \frac{\exp(-\sigma' W(t))}{\frac{1}{\max(V_0,0)} + \frac{1}{2}\int_0^t \exp(-\sigma' W(s))\,ds}.$$

Because of [27, Theorem 4.1], the law of the right-hand side is known.

We now present the proofs of the above theorems, starting with the conservative case.

Proof of Theorem 5.1. We divide the proof into two parts:

(1) We postulate U and construct globally (in time) extant characteristics $X(t,x)$.
(2) We show that $(\partial_x U)(t, X(t,x))$ satisfies (3.3).

1. Construction of U and global characteristics.

Using (3.8), $\mathfrak{Q}(t,x)$ is constant over $x \in [0,1]$ for time up to $t = t^*_{1/2}$ $(= t^*_0 = t^*_1$ by this constancy). Therefore we simply construct $U(t,\cdot)$ to be the piecewise linear function taking the value $U(t,x) = 0$ for $x < X(t,0)$ and $U(t,x) = \mathfrak{Q}(t,\frac{1}{2})(X(t,1) - X(t,0))$ for $x > X(t,1)$. (Because $U(t)$ is piecewise linear by construction, $Q(t)$ will be constant between $X(t,0)$ and $X(t,1)$.) This definition can be extended to all times $t \ge 0$ by taking $\mathfrak{Q}(s,\frac{1}{2})$ in the definition of $\mathfrak{u}$ (cf. (5.3)) to mean:

$$\mathfrak{Q}(t,\frac{1}{2}) = \frac{\exp(-\sigma' W(t))}{\frac{1}{V_0} + \frac{1}{2}\int_0^t \exp(-\sigma' W(s))\, \mathrm{d}s}.$$

The only difficulty is that U so defined depends on $X(t,0)$ and $X(t,1)$ in a circular fashion. To rectify this circularity, we take one more step back and define characteristics $X(t,0)$ and $X(t,1)$, which will later self-evidently be solutions to (5.4) at $x = 0$ and $x = 1$.

For $x \le 0$ or $x \ge 1$, set

$$X(t,x) = x + \mathbb{1}_{\{x \ge 1\}} \int_0^t \mathfrak{u}(s)\, \mathrm{d}s + \int_0^t \sigma(X(s,x)) \circ \mathrm{d}W(s), \tag{5.6}$$

which has a global unique strong solution in the space of adapted process with $\mathbb{P}$-almost surely continuous paths by the basic theorem on well-posedness of SDEs (see, e.g., [31, Thm. IX.II.2.4]), and by the boundedness of $\mathfrak{u}$ ensured by the formula (3.12). The function $\mathfrak{u}$ here has been defined explicitly in (3.12).

We now postulate the ansatz $U(t,x)$ for $u(t,x)$:

$$U(t,x) = \begin{cases} 0, & x \le X(t,0), \\ \frac{x - X(t,0)}{X(t,1) - X(t,0)} \mathfrak{u}(t), & x \in (X(t,0), X(t,1)), \\ \mathfrak{u}(t), & x \ge X(t,1), \end{cases} \tag{5.7}$$

where U is defined pointwise in (t,x), $\mathbb{P}$-almost surely.

In the $\sigma'' = 0$ case, $\mathfrak{u}(t)$ (given in (3.11)) does *not* depend on any characteristic.

Now we define $X(t,x)$ by the equation

$$X(t,x) = x + \int_0^t U(s, X(s,x))\, \mathrm{d}s + \int_0^t \sigma(X(s,x)) \circ \mathrm{d}W. \tag{5.8}$$

(We re-use the symbol X from above as this equation simply augments equation (5.6).) By taking a spatial derivative, we see that this SDE also has an explicit solution: for $x \in [0,1]$, $t < t^*$,

$$\frac{\partial X(t,x)}{\partial x} = e^{\sigma' W(t)}\Big(1 + \int_0^t \frac{e^{-\sigma' W(s)}}{X(s,1) - X(s,0)} \mathfrak{u}(s)\,\mathrm{d}s\Big),$$

and consequently,

$$X(t,x) = X(t,0) + x e^{\sigma' W(t)}\Big(1 + \int_0^t \frac{e^{-\sigma' W(s)}}{X(s,1) - X(s,0)} \mathfrak{u}(s)\,\mathrm{d}s\Big) \qquad x \in [0,1],\ t < t^*.$$

Again, by direct differentiation of the equation above, we can see that the derivative $\partial X/\partial x$ is independent of x,

$$\frac{\partial X(t,x)}{\partial x} = X(t,1) - X(t,0). \tag{5.9}$$

It is also signed, since alternatively to (3.13) we also have

$$\begin{aligned} \mathfrak{u}(s) &= \mathfrak{Q}(s, \tfrac{1}{2}) \exp\Big(\int_0^s \mathfrak{Q}(r, \tfrac{1}{2})\,\mathrm{d}r + \sigma' W(s)\Big) \\ &= e^{\sigma' W(s)} \frac{\mathrm{d}}{\mathrm{d}s} \exp\Big(\int_0^s \mathfrak{Q}(r, \tfrac{1}{2})\,\mathrm{d}r\Big), \end{aligned} \tag{5.10}$$

so solving the SDE for $X(t,1) - X(t,0)$,

$$\begin{aligned} X(t,1) - X(t,0) &= e^{\sigma' W(t)}\Big(1 + \int_0^t e^{-\sigma' W(s)} \mathfrak{u}(s)\,\mathrm{d}s\Big) \\ &= \exp\Big(\int_0^t \mathfrak{Q}(s, \tfrac{1}{2})\,\mathrm{d}s + \sigma' W(t)\Big) \ge 0, \end{aligned} \tag{5.11}$$

with strict inequality except at $t = t^*$.

We record the fact that characteristics do not cross except at wave-breaking as a lemma, see Lemma 5.4 after this proof.

The global well-posedness for the end-point characteristics $X(t,0)$ and $X(t,1)$, and (5.9), allow us to extend $X(t,x)$ globally (beyond t^*) via

$$\frac{X(t,x) - X(t,0)}{X(t,1) - X(t,0)} = x. \tag{5.12}$$

2. Verifying properties of $\partial_x U$.

Setting

$$Q(t,x) = \partial_x U(t,x),$$

we shall proceed to show that up to $t = t^*$, $\mathbb{P}$-almost surely,

$$Q(t, X(t, x)) = \mathfrak{Q}(t, x),$$

and that we have the (global) explicit formula:

$$Q(t, X(t, x)) = \frac{\exp(-\sigma' W(t))}{\frac{1}{V_0} + \frac{1}{2}\int_0^t \exp(-\sigma' W(s))\, \mathrm{d}s} \mathbb{1}_{x \in [0,1]}.$$

By construction U was built by integrating $\mathfrak{Q}(t, \frac{1}{2})$ in time. Using (5.7), (5.3), and (5.11) directly, it comes as no surprise that:

$$\begin{aligned}
\partial_x U(t, x) &= \begin{cases} 0, & x \leq X(t, 0), \\ \frac{\mathfrak{u}(t)}{X(t,1) - X(t,0)}, & x \in (X(t, 0), X(t, 1)), \\ 0, & x \geq X(t, 1), \end{cases} \\
&= \begin{cases} 0, & x \leq X(t, 0), \\ \mathfrak{Q}(t, \frac{1}{2}), & x \in (X(t, 0), X(t, 1)), \\ 0, & x \geq X(t, 1), \end{cases}
\end{aligned}$$

which, by differentiating directly, yields

$$\mathrm{d}Q(t, X(t, x)) = -\frac{1}{2} Q^2(t, X(t, x))\, \mathrm{d}t + \sigma' Q(t, X(t, x)) \circ \mathrm{d}W.$$

We emphasise once again that no conservation of any norms of Q is proven or even claimed. □

We state for clarity the following result, which simply re-establishes Proposition 4.3 without the unproven assumption concerning the existence of characteristics.

Lemma 5.4 *(Stochastic Flow of Diffeomorphisms before Wave-breaking). Let $q_0 = V_0 \mathbb{1}_{[0,1]}$ be a "box"-type initial data. Let $\sigma'' = 0$ and $\{X(t, x)\}_{x \in \mathbb{R}}$ be defined by (5.7), (5.8). Then up to t^* defined by (5.2), $\phi_t : x \mapsto X(t, x)$ is a flow (i.e., a one-parameter semi-group in t) of diffeomorphisms of $\mathbb{R}$.*

And for given (t, x), $t \neq t^*$, there is a unique random variable $y : \Omega \to \mathbb{R}$ for which $X(t, y) = x$.

We now turn to the proof in the dissipative case.

Proof of Theorem 5.2. First we notice that by construction and Lemma 3.6, at the wave-breaking time t^*, $U(t^*, \cdot) \equiv 0$, $\mathbb{P}$-almost surely. Since we have unique paths up to t^*, the pair of equations

$$\begin{cases} \mathrm{d}X(t, x) = U(t, X(t, x))\, \mathrm{d}t + \sigma(X(t, x)) \circ \mathrm{d}W(t), t^* > t \geq 0, \\ \mathrm{d}X(t^* + t, x) = \sigma(X(t^* + t, x)) \circ \mathrm{d}W(t^* + t), \quad t \geq 0, \end{cases} \tag{5.13}$$

gives unique global solutions $X(t,x)$ for each x that are continuous in t. These equations represent stopping the characteristic at the time t^*, and then starting it again where $U(X)$ becomes nought. Measurability is not an issue as W is strong Markov, and t^* was shown to be a stopping time in Section 4.1. Lemma 3.6 in fact guarantees that $U(X)$ tends continuously to zero at wave-breaking.

In effect we have postulated a truncated $\tilde{U}(t,x)$ in place of U in (5.7), to wit:

$$\tilde{U}(t,x) = \begin{cases} U(t,x), & t < t^*, \\ 0, & t \geq t^*, \end{cases}$$

and used the result of Lemma 3.6.

By defining

$$Q(t,x) = \begin{cases} \partial_x U(t,x), & t < t^*, \\ 0, & t \geq t^*. \end{cases}$$

It is clear that as in the previous proof, $Q(t,x)$ and $Q(t, X(t,x))$ still satisfy

$$\mathrm{d}Q(t,X(t,x)) = -\frac{1}{2}(Q(t,X(t,x)))^2\,\mathrm{d}t - \sigma' Q(t,X(t,x)) \circ \mathrm{d}W$$

over $t < t^*$, and that this holds trivially thereafter, as sought. $\square$

Proof of Corollary 5.3. This follows directly from (3.8), and from (5.5) in Theorem 5.2. $\square$

Remark 5.5 *(Optimality of higher integrability for the case $\sigma'' = 0$).* As we can extend solutions to and past wave-breaking, using (3.8), (3.13), and (4.7) it is possible to compute $\|q(t)\|_{L^2}$ explicitly for the "box"-type initial condition (4.5) *in the conservative case*, because $q(s)$, as in the deterministic case, does not vary over the interval $(X(s,0), X(s,1))$:

$$\begin{aligned}
\|q(t)\|_{L^2}^2 &= (X(t,0) - X(t,1))\mathfrak{Q}^2(t,\frac{1}{2}) \\
&= \mathfrak{Q}(t,\frac{1}{2})\,\mathfrak{u}(t,x) \\
&= \frac{Z(t,x)}{\frac{1}{q_0(x)} + \frac{1}{2}\int_0^s Z(r,x)\,\mathrm{d}r}\Big(\frac{1}{q_0(x)} + \frac{1}{2}\int_0^s Z(r,x)\,\mathrm{d}r\Big) \\
&= Z(t,x) = \exp(-\sigma' W(t)).
\end{aligned}$$

It may be hoped that if the distribution of t^* is sufficiently dispersed, then at any deterministic time t, only a $\mathbb{P}$ measure zero set of paths experience wave-breaking and higher integrability beyond $L^{3-\varepsilon}(\Omega \times [0,T] \times \mathbb{R})$ proven in Proposition 2.11 may be achieved. This hope proves false, however, as we shall now show:

By the preservation of boxes under the flow of the equations in the case $\sigma'' \equiv 0$,

$$\begin{aligned}\mathbb{E}\|q\|^p_{L^p([0,T]\times\mathbb{R})} &= \mathbb{E}\int_0^T |\mathfrak{Q}(t,\frac{1}{2})|^p (X(t,1)-X(t,0))\,\mathrm{d}t \\ &= \mathbb{E}\int_0^T |\mathfrak{Q}(t,\frac{1}{2})|^{p-1}|\mathfrak{u}(t)|\,\mathrm{d}t.\end{aligned}$$

With $\mathfrak{Q}(t,x)$ again given by (3.8) and (3.13), we can simplify the integrand as follows:

$$\begin{aligned}|\mathfrak{Q}(t,\frac{1}{2})|^{p-1}\mathfrak{u}(t) &= \frac{|Z(t)|^{p-1}}{|\frac{1}{q_0(x)}+\frac{1}{2}\int_0^t Z(s,x)\,\mathrm{d}s|^{p-1}}\Big(\frac{1}{q_0(x)}+\frac{1}{2}\int_0^s Z(r,x)\,\mathrm{d}r\Big) \\ &= \frac{|Z(t)|^{p-1}}{|\frac{1}{q_0(x)}+\frac{1}{2}\int_0^t Z(s,x)\,\mathrm{d}s|^{p-2}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{E}\big(|\mathfrak{Q}(t,\frac{1}{2})|^{p-1}|\mathfrak{u}(t)|\big) \\ &\quad = \int_{\xi\in[0,\infty)}\int_{r\in\mathbb{R}} \frac{|\exp(-\sigma' r)|^{p-1}}{|\frac{1}{V_0}+\chi|^{p-2}}\mathbb{P}\Bigg(\Bigg\{W(t)\in\mathrm{d}r,\ \frac{1}{2}\int_0^t \exp(-\sigma' W(s))\,\mathrm{d}s\in\mathrm{d}\chi\Bigg\}\Bigg).\end{aligned}$$

This law is almost given in [27, Theorem 4.1] (see also [32]), where using the notation established in Section 4.1, it was shown that for

$$A^{(\mu)}(t) = \int_0^t \exp(2\mu s + 2W(s))\,\mathrm{d}s,$$

one has

$$\mathbb{P}(\{A^{(\mu)}(t)\in\mathrm{d}\chi,\ W(t)\in\mathrm{d}r\}) = \frac{\mathrm{d}\chi}{\chi}e^{\mu r-\mu^2 t/2}\exp\Big(-\frac{1+e^{2r}}{2\chi}\Big)\vartheta\Big(\frac{e^r}{\chi},t\Big)\,\mathrm{d}r,$$

where again,

$$\vartheta(y,t) = \frac{y}{\sqrt{2\pi^3 t}}e^{\pi^2/(2t)}\int_0^\infty e^{-\xi^2/(2t)}e^{-y\cosh(\xi)}\sinh(\xi)\sin\Big(\frac{\pi\xi}{t}\Big)\,\mathrm{d}\xi.$$

It is possible simply to scale time in both $A^{(\mu)}(t)$ and $W(t)$ simultaneously as in (4.4):

$$\begin{aligned}&\mathbb{P}(\{A^{(0)}(t)\in \mathrm{d}\chi,\ W(t)\in \mathrm{d}r\})\\&\qquad=\mathbb{P}\Big(\Big\{\frac{(\sigma')^2}{4}\int_0^{4t/(\sigma')^2}\exp(-\sigma'\tilde{W}(\tau))\,\mathrm{d}\tau\in \mathrm{d}\chi,\ W(t)\in \mathrm{d}r\Big\}\Big),\end{aligned}$$

so that

$$\begin{aligned}&\mathbb{P}(\{A^{(0)}(t)\in \mathrm{d}\chi,\ W(t)\in \mathrm{d}r\})\\&\qquad=\mathbb{P}\Big(\Big\{\frac{(\sigma')^2}{4}\int_0^{4t/(\sigma')^2}\exp(-\sigma'\tilde{W}(\tau))\,\mathrm{d}\tau\in \mathrm{d}\chi,\ -\frac{\sigma'}{2}\tilde{W}\big(4t/(\sigma')^2\big)\in \mathrm{d}r\Big\}\Big),\end{aligned}$$

and

$$\begin{aligned}&\mathbb{P}\Big(\Big\{\int_0^t\exp(-\sigma'\tilde{W}(s))\,\mathrm{d}s\in \mathrm{d}\chi,\ \tilde{W}(t)\in \mathrm{d}r\Big\}\Big)\\&\qquad=\frac{-\sigma'}{2}\frac{\mathrm{d}\chi}{\chi}\exp\Big(-\frac{2(1+e^{-\sigma' r})}{(\sigma')^2\chi}\Big)\vartheta\Big(\frac{4e^{-\sigma' r/2}}{(\sigma')^2\chi},\frac{(\sigma')^2t}{4}\Big)\,\mathrm{d}r.\end{aligned}$$

Finally integrating in time we find

$$\begin{aligned}\int_0^T\mathbb{E}\big(|\mathfrak{Q}(t,\tfrac{1}{2})|^{p-1}|\mathfrak{u}(t)|\big)\,\mathrm{d}t&=\int_0^T\int_{\mathbb{R}}\int_0^\infty\frac{\exp(-(p-1)\sigma' r)}{|1/V_0+\chi|^{p-2}}\\&\quad\times\frac{-\sigma'}{2}\exp\Big(-\frac{2(1+e^{-\sigma' r})}{(\sigma')^2\chi}\Big)\vartheta\big(\frac{4e^{-\sigma' r/2}}{(\sigma')^2\chi},\frac{(\sigma')^2t}{4}\big)\,\frac{\mathrm{d}\chi}{\chi}\,\mathrm{d}r\,\mathrm{d}t.\end{aligned}$$

As can be seen, there is no bound for the blow-up of this quantity in the small ball $\chi\in B_\varepsilon(-1/V_0)$ except if $p-2<1$. However, it is still conceivable that there is higher integrability if $\sigma''\neq 0$). Under the principle that "boxes" are preserved under the flow, the spatial dimension is essentially lost in the triple integral (in space, time, and probability), but freeing up the spatial variable from this constraint gives us, effectively, an extra dimension to integrate, opening the possibility that the integral remains bounded at a higher exponent than $3-\varepsilon$. This can be understood as an effect of regularisation-by-*multiplicative* noise if indeed it holds [13].

5.3. Well-posedness for general data

Using the same procedure outlined after (5.3), we now extend our analysis to general data. We work directly with $L^2(\mathbb{R})\cap L^1(\mathbb{R})$-valued random variables. The following does not generalise easily beyond the linear σ case, again because in the $\sigma''=0$ case, there is no dependence of

$\mathfrak{Q}(t,x)$ on x through characteristics $X(t,x)$. In particular, as mentioned in Remark 3.4, $\mathfrak{Q}(t,x)$ is simply defined up to wave-breaking via (3.8):

$$\mathfrak{Q}(t,x) = \frac{e^{-\sigma W(t)}}{\frac{1}{q_0(x)} + \frac{1}{2}\int_0^t e^{-\sigma W(s)}\, \mathrm{d}s}. \tag{5.14}$$

In this subsection, all solutions refer exclusively to conservative or dissipative solutions-along-characteristics.

Theorem 5.6 *(Conservative Solutions: General Initial Data). Suppose $\sigma'' = 0$ and $q_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. There exists a $U \in C([0,\infty) \times \mathbb{R})$, absolutely continuous in x, $\mathbb{P}$-almost surely, such that for each $x \in \mathbb{R}$, the following SDE is globally well-posed:*

$$X(t,x) = x + \int_0^t U(s, X(s,x))\, \mathrm{d}s + \int_0^t \sigma(X(s,x)) \circ \mathrm{d}W(s).$$

For $Q(t,x) = \partial_x U(t,x)$, the process $Q(t, X(t,x))$ agrees $\mathbb{P}$-almost surely with $\mathfrak{Q}(t,x)$ as given by (3.8) up to $t = t^*$ and can be represented globally as

$$Q(t, X(t,x)) = \frac{\exp(-\sigma' W(t))}{\frac{1}{q_0(x)} + \frac{1}{2}\int_0^t \exp(-\sigma' W(s))\, \mathrm{d}s}.$$

In particular, $\tilde{Q}(t) = Q(t, X(t,x))$ satisfies (3.3):

$$\mathrm{d}\tilde{Q}(t) = -\frac{1}{2}\tilde{Q}^2(t)\, \mathrm{d}t + \sigma' \tilde{Q}(t) \circ \mathrm{d}W.$$

Furthermore, $Q \in L^2(\Omega \times [0,T] \times \mathbb{R})$ and in $C([0,T]; H^{-1}_{\mathrm{loc}}(\mathbb{R}))$, $\mathbb{P}$-almost surely, and the energy can be expressed $\mathbb{P}$-almost surely as

$$\int_{\mathbb{R}} Q^2(t,x)\, \mathrm{d}x = \int_{\mathbb{R}} Q^2(0,x) \exp(-\sigma' W(t))\, \mathrm{d}x. \tag{5.15}$$

Similarly, for the dissipative solutions-along-characteristics, we have:

Theorem 5.7 *(Dissipative Solutions: General Initial Data). Suppose $\sigma'' = 0$ and $q_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. There exists a $U \in C([0,\infty) \times \mathbb{R})$, absolutely continuous in x, $\mathbb{P}$-almost surely, such that for each $x \in \mathbb{R}$, the SDE*

$$X(t,x) = x + \int_0^t U(s, X(s,x))\, \mathrm{d}s + \int_0^t \sigma(X(s,x)) \circ \mathrm{d}W(s)$$

is globally well-posed.

For $Q(t,x) = \partial_x U(t,x)$, the process $Q(t, X(t,x))$ agrees $\mathbb{P}$-almost surely with $\mathfrak{Q}(t,x)$ up to $t = t_x^*$ and can be represented globally as

$$Q(t, X(t,x)) = \begin{cases} \frac{\exp(-\sigma' W(t))}{\frac{1}{q_0(x)} + \frac{1}{2}\int_0^t \exp(-\sigma' W(s))\,\mathrm{d}s}, & t < t_x^*, \\ 0, & t > t_x^*. \end{cases} \tag{5.16}$$

Here t_x^* is given by (2.9)*.*

In particular, $\tilde{Q}(t) = Q(t, X(t,x))$ satisfies (3.3)*:*

$$\mathrm{d}\tilde{Q}(t) = -\frac{1}{2}\tilde{Q}^2(t)\,\mathrm{d}t + \sigma'\tilde{Q}(t) \circ \mathrm{d}W.$$

Furthermore, $Q \in L^2(\Omega \times [0,T] \times \mathbb{R})$ and in $C([0,T]; H^{-1}_{\mathrm{loc}}(\mathbb{R}))$, $\mathbb{P}$-almost surely, and the energy can be expressed $\mathbb{P}$-almost surely as

$$\int_{\mathbb{R}} Q^2(t,x)\,\mathrm{d}x = \int_{\mathbb{R}} Q^2(0,x)\exp(-\sigma' W(t))\mathbb{1}_{\{t \le t_x^*\}}\,\mathrm{d}x. \tag{5.17}$$

This generalises the main theorem in [9, Thm. 4.1] to the stochastic setting.

Remark 5.8. The inclusions preceding (5.15) and (5.17) are implied by the respective equations. This was already shown in Remarks 2.5 and 2.7, respectively.

5.4. Conservative solutions

In the case $\sigma'' = 0$, $\mathfrak{Q}(t,x)$ in (3.8) is independent of $X(t,x)$, and only depends on x via $q_0(x)$. It becomes possible, if $q_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, to define $U(t, X(t,x))$ as the spatial integral of u. However, in order to avoid cyclic dependencies when U is used to define X via an SDE analogous to (3.1), we define first an auxiliary function which should be thought of as $U(t, X(t,y))$:

$$\Psi(t,y) = \int_{-\infty}^{y} \mathfrak{u}(t,x)\,\mathrm{d}x. \tag{5.18}$$

Recall that $\mathfrak{u}$ is explicitly given in (3.12) and depends on x only via q_0. In the conservative construction we extend this definition by the same formula to $t > t_x^*$ as we did in the specific cases of "box"-type data.

Define the characteristics via the equation:

$$X(t,x) = x + \int_0^t \Psi(s,x)\,\mathrm{d}s + \int_0^t \sigma(X(s,x)) \circ \mathrm{d}W(s), \tag{5.19}$$

which is straightforward as σ is linear and $\Psi(t,y)$ is a well-defined process, being dependent only on $\mathfrak{u}$, which in turn is defined explicitly in (3.12), as, analogous to (4.7), the derivative

$$\frac{\partial X(t,x)}{\partial x} = \exp\Big(\int_0^t \mathfrak{Q}(s,x)\,\mathrm{d}s + \sigma' W(t)\Big) \tag{5.20}$$

is well-defined and non-negative, the right-hand side again being dependent on x only through q_0.

This allows us to define

$$U(t,x) := \Psi(t,y), \qquad X(t,y;\omega) = x,$$

as long as $t \neq t_y^*$ (cf. (3.7)). Such a y exists because $\partial X/\partial x$ is $\mathbb{P}$-almost surely bounded, and strictly positive. The function U is well-defined even though y as a random variable may not be unique because U only depends on y via $X(t,y)$. The variable y is therefore a device for shifting stochasticity back-and-forth between x and $X(t,y)$, and depends on the Jacobian $\partial X(y)/\partial y$ being non-singular. To expand on this point we record a general version of Lemma 5.4:

Lemma 5.9 *("Stochastic Flow of Diffeomorphism" before Wave-breaking for General Data). Given t and x deterministic, there is a random variable $y\colon \Omega \to \mathbb{R}$ such that $X(t,y) = x$, $\mathbb{P}$-almost surely. If there are two such random variables y_1 and y_2 that satisfy this equation, then $y_1 - y_2$ is supported on the set $\{\omega : t_{y_1}^* = t\} \cap \{\omega : t_{y_2}^* = t\}$ in the sense that on the full $\mathbb{P}$-measure of the complement, the difference is nought.*

We emphasize here the hierarchy of dependencies, being that U depends on X, which depends on Ψ in the above. The function Ψ in turn depends on $\mathfrak{u}$, which in the $\sigma'' = 0$ case, is given explicitly by formula (3.12), derived using the similarly explicit formula (3.8) for the process $\mathfrak{Q}(t,x)$.

The definition of U ensures that

$$U(t, X(t,x)) = \Psi(t,x)$$

and consequently

$$X(t,x) = x + \int_0^t U(s, X(s,x))\,\mathrm{d}s + \int_0^t \sigma(X(s,x)) \circ \mathrm{d}W(s).$$

It remains for us to check that, $\mathbb{P}$-almost surely,

(i) $Q(t, X(t,x)) = (\partial_x U)(t, X(t,x))$ satisfies (3.3), and
(ii) $Q \in C([0,T]; H^{-1}_{\mathrm{loc}}(\mathbb{R}))$.

By continuity in H^{-1}_{loc} we mean that for every pre-compact $B \in \mathbb{R}$, $\|Q(t)\|_{H^{-1}(B)}$ is continuous. In turn, the space H^{-1} is defined as the dual space of compactly supported H^1 functions. It is norm-equivalent to L^2 of the anti-derivative on compact sets.

Proof of Theorem 5.6. By construction, (i) is already satisfied. We can take the spatial derivative easily enough:

$$
\begin{aligned}
(\partial_x U)(t,x) &= \partial_x \int_0^y \mathfrak{Q}(t,z)\frac{\partial X}{\partial z}\,\mathrm{d}z \\
&= \mathfrak{Q}(t,y)\frac{\partial X}{\partial y}\frac{\partial y}{\partial x} \\
&= \mathfrak{Q}(t,y).
\end{aligned}
$$

Putting $X(t,x)$ in the place of x, we can put x in the place of y, giving us:

$$(\partial_x U)(t, X(t,x)) = \mathfrak{Q}(t,x).$$

To prove (5.15) we again invoke Lemma 3.6 (in particular, (3.13)) and (5.20):

$$
\begin{aligned}
\int |Q(t,x)|^2\,\mathrm{d}x &= \int |Q(t,X(t,y))|^2 \frac{\partial X(y)}{\partial y}\,\mathrm{d}y \\
&= \int \frac{Z(t,y)}{\frac{1}{Q(0,y)} + \frac{1}{2}\int_0^t Z(s,y)\,\mathrm{d}s} Q(t,X(t,y)) \\
&\qquad\qquad \times \exp\left(\int_0^t Q(s,X(s,y))\,\mathrm{d}s + \sigma' W(t)\right)\mathrm{d}y \\
&= \int Q(0,y)^2 Z(t,y)\,\mathrm{d}y \\
&= \int Q(0,y)^2 \exp(-\sigma' W(t))\,\mathrm{d}y,
\end{aligned}
$$

where again we have used the notation $Z(t,y) = \exp\left(\int_0^t \sigma'\, dW\right) = \exp(\sigma' W(t))$.

Finally to see (ii), we consider the almost sure continuity of $\|U(t)\|^2_{L^2(B)}$ over a pre-compact set $B \subseteq \mathbb{R}$:

$$\int_B |U(t,x)|^2\,\mathrm{d}x = \int_B \left| \int_{-\infty}^x \mathfrak{u}(t,y)\,\mathrm{d}y \right|^2 \mathrm{d}x. \tag{5.21}$$

As was shown in (3.12), (3.13) in Lemma 3.6,

$$\mathfrak{u}(t,y) = q_0(y)\left(1 + \frac{q_0(y)}{2}\int_0^t \exp(-\sigma' W(s))\,\mathrm{d}s\right), \tag{5.22}$$

which is path-by-path continuous in time for each fixed y that is a Lebesgue point of u. The boundedness of the integral on the right in (5.21) is then a result of the assumption $q_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Therefore,

$$\begin{aligned}
&\|U(t)-U(s)\|^2_{L^2(B)}\\
&\quad=\int_B\left|\int_{-\infty}^{x}\mathfrak{u}(t,y)\,\mathrm{d}y-\int_{-\infty}^{x}\mathfrak{u}(s,y)\,\mathrm{d}y\right|^2\mathrm{d}x\\
&\quad=\int_B\left|\int_{-\infty}^{x}\frac{q_0^2(y)}{2}\right|^2\mathrm{d}x\times\left(\int_s^t\exp(-\sigma' W(\tau))\,\mathrm{d}\tau\right)^2. \qquad (5.23)
\end{aligned}$$

The same boundedness of integral of $\mathfrak{u}$, and continuity of $\mathfrak{u}(\cdot,y)$ in time means that the limit as $s\to t$ is almost surely 0. This shows the continuity of $\|Q(t)\|_{H^{-1}_{\mathrm{loc}}}$ in time. □

Remark 5.10 *(Temporal continuity of $\|Q(t)\|^2_{H^{-1}(B)}$).* From second factor in the integral with respect to x of the foregoing calculation, (5.23), upon comparison with (5.22), it can be seen that in fact $\|Q(t)\|^2_{H^{-1}(B)}$ is $\mathbb{P}$-almost surely in $C^{1/2-0}$, and not simply continuous. Even though on taking the square root $\|Q(t)\|_{H^{-1}(B)}$ possesses strictly higher regularity-in-time than simply $\mathbb{P}$-almost sure inclusion in $C(\mathbb{R})$, this still contrasts with the local Lipschitz continuity of $\|q(t)\|_{H^{-1}_{\mathrm{loc}}}$ that deterministic solutions q possess (cf. (1.5)).

5.5. Dissipative solutions

We proceed directly to the proof of Theorem 5.7.

Proof of Theorem 5.7. By dissipative we mean solutions for which

$$Q(t,X(t,x))=\begin{cases}\exp(-\sigma' W(t))\left(\frac{1}{q_0(x)}+\frac{1}{2}\int_0^t\exp(-\sigma' W(s))\,\mathrm{d}s\right)^{-1}, & t<t_x^*,\\ 0, & t>t_x^*.\end{cases}$$

Again, as $\sigma''=0$, the right-hand side only depends on x via q_0.

Defining $U(t,X(t,x))$ as before, we can write

$$\begin{aligned}
U(t,X(t,x))&=\int_{-\infty}^{x}Q(t,X(t,y))\frac{\partial X}{\partial x}\,\mathrm{d}y\\
&=\int_{-\infty}^{x}q_0(y)\left(1+\frac{q_0(y)}{2}\int_0^t\exp(-\sigma' W(s))\,\mathrm{d}s\right)\mathbb{1}_{\{t<t_y^*\}}\,\mathrm{d}y. \qquad (5.24)
\end{aligned}$$

Therefore, again, there is no dependence of $U(t,X(t,x))$ on $X(t,x)$, and $U(t,X(t,x))$ is explicitly known. From this and the boundedness of U it is clear that we can find a global solution to

$$X(t,x)=x+\int_0^t U(s,X(s,x))\,\mathrm{d}s+\int_0^t\sigma(X(s,x))\circ\mathrm{d}W.$$

It follows as in the conservative solution that

(i) $X(t,x)$ also satisfies

$$\mathrm{d}X(t,x) = U(t,X(t,x))\,\mathrm{d}t + \sigma(X(t,x)) \circ \mathrm{d}W$$

up to $t = t_x^*$, and remains well-defined beyond this time, and

(ii) $Q(t,X(t,x)) = (\partial_x U)(t,X(t,x))$ satisfies (3.3).

To prove (5.17) we invoke Lemma 3.6 (in particular, (3.13)) and (5.20) exactly as in the proof immediately foregoing:

$$\begin{aligned}
\int |Q(t,x)|^2\,\mathrm{d}x &= \int |Q(t,X(t,y))|^2 \frac{\partial X(y)}{\partial y}\,\mathrm{d}y \\
&= \int \frac{\exp(-\sigma' W(t))}{\frac{1}{Q(0,y)} + \frac{1}{2}\int_0^t \exp(-\sigma' W(t))\,\mathrm{d}s} \\
&\qquad \times \mathbb{1}_{\{t \le t_y^*\}} Q(t,X(t,y)) \exp\Big(\int_0^t Q(s,X(s,y))\,\mathrm{d}s + \sigma' W(t)\Big)\,\mathrm{d}y \\
&= \int Q(0,y)^2 \exp(-\sigma' W(t)) \mathbb{1}_{\{t \le t_y^*\}}\,\mathrm{d}y.
\end{aligned}$$

We also show $Q \in C([0,T]; H^{-1}_{\mathrm{loc}}(\mathbb{R}))$ by showing that $\|U(t)\|_{L^2(B)}$ is continuous in time. As before we have

$$\int_B |U(t,x)|^2\,\mathrm{d}x = \int_B \Bigg| \int_{-\infty}^x q_0(y)\Big(1 + \frac{q_0(y)}{2} \int_0^t \exp(-\sigma' W(s))\,\mathrm{d}s\Big) \mathbb{1}_{\{t<t_y^*\}}\,\mathrm{d}y \Bigg|^2 \mathrm{d}x.$$

Continuity follows as in part (ii) of the proof of Theorem 5.6. The only difference is continuity at wave-breaking. This in turn follows from Lemma 3.6, where this time we invoke its main conclusion that at t_y^*, the integrand of the inner integral, $\mathfrak{u}(t,y)$, tends $\mathbb{P}$-almost surely to nought. In dissipative solutions, we continue U past wave-breaking by simply setting $\partial_x U(t,x)$ to be nought after $t = t_y^*$ for a y where $X(t,y) = x$. □

Finally, as in the case of "box"-type initial data, we retain the Oleinik-type one-sided estimate:

Corollary 5.11. *Suppose $\sigma'' = 0$ and $q_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then the dissipative solution Q with initial condition $Q(0) = q_0$ in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ satisfies $\mathbb{P}$-almost surely the following one-sided bound:*

$$Q(t,X(t,y)) \le \frac{\exp(-\sigma' W(t))}{\frac{1}{\max(q_0(y),0^+)} + \frac{1}{2}\int_0^t \exp(-\sigma' W(s))\,\mathrm{d}s}.$$

Remark 5.12 *(Discrete approximations).* From Lemma 3.1 (ii), it may be possible first to consider well-posedness in the space of step functions, and thereafter to extend this by a limiting procedure to more general compactly supported $L^2(\mathbb{R})$ functions. As in the deterministic setting, see e.g., [34], it is enough to add the boxes together:

Let $P = (x_0, \dots, x_n)$ be a partition of $[x_0, x_n] \subset \mathbb{R}$, and q_0 be the function

$$q_0(x) = \sum_{i=1}^{n} V_0^i \mathbb{1}_{(x_{i-1}, x_{i+1})}(x), \qquad V_0^i \in \mathbb{R}.$$

For $i = 1, \dots, n$, let t_i^* be the wave-breaking time for the ith box. These are obviously not dependent on one another. Where $V_0^i \geq 0$, we put $t_i^* = \infty$, $\mathbb{P}$-everywhere.

As neighbouring intervals are almost disjoint on $\mathbb{R}$ the analysis on any one box can be extended to show that where $U_i(t, x)$ is counterpart of (5.7),

$$U_i(t,x) = \begin{cases} 0, & x \leq X(t, x_{i-1}), \\ \frac{x - X(t, x_{i-1})}{X(t,x_i) - X(t, x_{i-1})} \mathfrak{u}_i(t), & x \in (X(t, x_{i-1}), X(t, x_i)), \\ \mathfrak{u}_i(t), & x \geq X(t, x_i), \end{cases}$$

with (Recall that the left-hand side does not actually depend on some $X(s, \frac{1}{2}(x_{i-1} + x_i))$, but only on the value $q_0(\frac{1}{2}(x_{i-1} + x_i))$.)

$$\mathfrak{u}_i(t) := \mathfrak{Q}(t, \frac{1}{2}(x_{i-1} + x_i)) \exp\Big(\int_0^t \mathfrak{Q}(s, \frac{1}{2}(x_{i-1} + x_i))\, \mathrm{d}s + \sigma' W(t) \Big),$$

we can write the solution $u(t, x)$ as the sum

$$u(t,x) = \sum_{i=1}^{n} U_i(t,x).$$

This can be extended to an $L^2(\mathbb{R})$ initial condition q_0 by setting

$$V_0^i = \fint_{x_{i-1}}^{x_i} q_0(x)\, \mathrm{d}x,$$

so that the approximation with the partition P is

$$q_0^P(x) = \sum_{i=1}^{n} V_0^i \mathbb{1}_{(x_{i-1}, x_{i+1})}(x).$$

Next, suppose one can find spaces on which the set $\{u^P, q^P\}_{\|P\|>0}$ is weakly compact, and on which the associated collection of laws $\{\mu^P\}_{\|P\|>0}$ is correspondingly tight (see Ondreját [29] for conditions giving compact embeddings into spaces of functions weakly continuous in time, and $W^{k,p}_{\mathrm{loc}}(\mathbb{R})$ in space). Invoking the Jakubowski–Skorohod theorem [22] in taking the limit of a subsequence as $\|P\| \to 0$, one obtains a limit process whose law on a new stochastic basis is the same as that of the weak-star limit μ of the tight sequence $\{\mu^P\}$ on the original stochastic basis, that is, the same conclusions as for the conventional Skorohod theorem, but applied to function spaces without the requisite separability.

It then only behooves one to conclude the argument by showing that the stochastic integrals against $\mathrm{d}\tilde{W}$, where $\tilde{W}$ is the representation of the original Brownian motion in the new stochastic basis, remain martingales, in the manner of [2,11].

6. Reconciling different notions of solutions

Finally we complement the results concerning conservative and dissipative solutions-along-characteristics by reconciling them with conservative and dissipative weak solutions, respectively, which are more traditional to the subject of partial differential equations. These notions of solutions are all defined in Section 2.1.

Proposition 6.1 *(Existence of Conservative Weak Solutions). Suppose $q_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\sigma'' = 0$. For processes given by*

$$U(t, X(t,x)) = \int_{-\infty}^{x} q_0(y)\left(1 + \frac{q_0(y)}{2}\int_0^t \exp(-\sigma' W(s))\,\mathrm{d}s\right)\mathrm{d}y,$$

$$X(t,x) = x + \int_0^t U(s, X(s,x))\,\mathrm{d}s + \int_0^t \sigma(X(s,x)) \circ \mathrm{d}W,$$

$$Q(t, X(t,x)) = \exp(-\sigma' W(t))\left[\frac{1}{q_0(x)} + \frac{1}{2}\int_0^t \exp(-\sigma' W(s))\,\mathrm{d}s\right]^{-1},$$

the function defined by

$$q(t,x) = Q(t, X(t,y)),$$

where $y \in \mathbb{R}$ satisfies $x = X(t,y)$, is a conservative weak solution.

Proposition 6.2 *(Existence of Dissipative Weak Solutions). Suppose $q_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\sigma'' = 0$. For a collection $\{t^*_x\}$ of random variables defined by*

$$-q_0(x)\int_0^{t^*_x} \exp\big(-\sigma' W(s)\big)\,\mathrm{d}s = 2,$$

indexed by the Lebesgue points x of $q_0(x)$, and processes given by

$$U(t, X(t,x)) = \int_{-\infty}^{x} q_0(y)\Big(1 + \frac{q_0(y)}{2}\int_0^t \exp(-\sigma' W(s))\,\mathrm{d}s\Big)\mathbb{1}_{\{t<t_y^*\}}\,\mathrm{d}y,$$

$$X(t,x) = x + \int_0^t U(s, X(s,x))\,\mathrm{d}s + \int_0^t \sigma(X(s,x)) \circ \mathrm{d}W,$$

$$Q(t, X(t,x)) = \begin{cases} \exp(-\sigma' W(t))\Big[\frac{1}{q_0(x)} + \frac{1}{2}\int_0^t \exp(-\sigma' W(s))\,\mathrm{d}s\Big]^{-1}, & t < t_x^*, \\ 0, & t > t_x^*, \end{cases}$$

the function defined by

$$q(t,x) = Q(t, X(t,y)),$$

where $y \in \mathbb{R}$ satisfies $x = X(t,y)$, is a dissipative weak solution.

Proof of Proposition 6.1. Since the process $\tilde{Q}(t) = Q(t, X(t,y))$ satisfies (3.3),

$$\mathrm{d}\tilde{Q}(t) = -\frac{1}{2}\tilde{Q}^2(t)\,\mathrm{d}t - \sigma'\tilde{Q}(t) \circ \mathrm{d}W,$$

up to $t < t_y^*$, pointwise for y in the set of Lebesgue points of q_0, by the Itô formula it manifestly holds that up to the same stopping time,

$$\mathrm{d}\tilde{Q}^2(t) = -\tilde{Q}^3(t)\,\mathrm{d}t - 2\sigma'\tilde{Q}^2(t) \circ \mathrm{d}W. \tag{6.1}$$

On $\mathbb{P}$-almost every path, except at the time $t = t_y^*$, we have shown that these equations remain valid. This is possible because we are only concerned with the Lebesgue points of q_0, which is a deterministic, time independent object. Let $\varphi \in C_0^\infty(\mathbb{R})$. First we observe that since $\partial X(t,y)/\partial y > 0$ for almost every $(t,y) \in [0,T] \times \mathbb{R}$, $\mathbb{P}$-almost surely, it holds that for almost every t, $\mathbb{P}$-almost surely,

$$\int_{\mathbb{R}} Q^2(t, X(t,y))\varphi(X(t,y))\,X(\mathrm{d}y) = \int_{\mathbb{R}} Q^2(t,x)\varphi(x)\,\mathrm{d}x, \tag{6.2}$$

where we have used $X(\mathrm{d}y)$ instead of $\mathrm{d}X(y)$ to denote the *deterministic* differential to emphasise integration in the *spatial*, and not the temporal variable. We can disregard the measure zero set in t (wave-breaking only occurs once along each characteristic) as we shall be integrating over t.

By (5.20), in the sense of Itô, we have the $\mathbb{P}$-almost sure equality

$$\mathrm{d}\Big(\int_{\mathbb{R}} Q^2(t, X(t,y))\varphi(X(t,y))\,X(\mathrm{d}y)\Big)$$
$$= \mathrm{d}\Big(\int_{\mathbb{R}} Q^2(t, X(t,y))\varphi(X(t,y))\frac{\partial X(t,y)}{\partial y}\,\mathrm{d}y\Big)$$

$$= \int_{\mathbb{R}} \mathrm{d}Q^2(t, X(t,y)) \circ \Big(\varphi(X(t,y)) \frac{\partial X(t,y)}{\partial y}\Big)\,\mathrm{d}y$$

$$+ \int_{\mathbb{R}} Q^2(t, X(t,y)) \circ \mathrm{d}\Big(\varphi(X(t,y)) \frac{\partial X(t,y)}{\partial y}\Big)\,\mathrm{d}y. \tag{6.3}$$

We already know how to expand $\mathrm{d}Q^2(t, X(t,y))$ from (6.1). Therefore we inspect the second summand in the final line of the foregoing calculation. Since φ is a smooth, deterministic function, by the regular chain rule,

$$\begin{aligned}
&\mathrm{d}\Big(\varphi(X(t,y)) \frac{\partial X(t,y)}{\partial y}\Big) \\
&\quad = \mathrm{d}\varphi(X(t,y)) \circ \frac{\partial X(t,y)}{\partial y} + \varphi(X(t,y)) \circ \mathrm{d}\frac{\partial X(t,y)}{\partial y} \\
&\quad = \frac{\partial X(t,y)}{\partial y} \circ \Big(\partial_x \varphi(X(t,y)) \circ \mathrm{d}X(t,y)\Big) + \varphi(X(t,y)) \circ \mathrm{d}\frac{\partial X(t,y)}{\partial y}.
\end{aligned}$$

Recalling Remark 3.2, and using (5.20) and the equation for the characteristics in the theorem statement,

$$\begin{aligned}
&\mathrm{d}\Big(\varphi(X(t,y)) \frac{\partial X(t,y)}{\partial y}\Big) \\
&\quad = \frac{\partial X(t,y)}{\partial y} \partial_x \varphi(X(t,y)) \circ \mathrm{d}X(t,y) + \varphi(X(t,y)) \circ \mathrm{d}\frac{\partial X(t,y)}{\partial y} \\
&\quad = \partial_x \varphi(X(t,y)) U(t, X(t,y)) \frac{\partial X(t,y)}{\partial y}\,\mathrm{d}t \\
&\qquad + \partial_x \varphi(X(t,y)) \sigma(X(t,y)) \frac{\partial X(t,y)}{\partial y} \circ \mathrm{d}W \\
&\qquad + \varphi(X(t,y)) \circ \Big(Q(t, X(t,y)) \frac{\partial X(t,y)}{\partial y}\,\mathrm{d}t + \sigma' \frac{\partial X(t,y)}{\partial y} \circ \mathrm{d}W\Big).
\end{aligned}$$

Inserting this into (6.3) and using (6.1) and (6.2), we recover the weak energy balance (2.5), where $\partial^2_{xx}\sigma = 0$ in the linear σ case. $\square$

For dissipative solutions, we shall be multiplying by an extra factor of $\mathbb{1}_{\{t<t^*_y\}}$ in the proof below. The selection of y for times $t > t^*_y$ has in fact been dealt with in Section 5.2, where we have shown how to extend characteristics globally through a wave-breaking point.

Remark 6.3. If it can be shown that any conservative weak solution (u, q) can be used to construct characteristics

$$\mathrm{d}X(t,y) = u(t, X(t,y))\,\mathrm{d}t + \sigma(X(t,y)) \circ \mathrm{d}W$$

that are for almost every $t \in [0, T]$ and $\mathbb{P}$-almost surely a C^1 surjection of $\mathbb{R}$ for which $\partial X/\partial x \geq 0$, then the calculations of the foregoing proof can be done in reverse to attain the reverse implication that conservative weak solutions are necessarily conservative solutions-along-characteristics. This would imply uniqueness of solutions. We relegate this proof to an upcoming work.

Proof of Proposition 6.2. The proof here essentially follows the one for Proposition 6.1 with the exception that there is a defect measure arising from the temporal derivative, and we employ (6.1) in evaluating the quantity:

$$\mathrm{d}(Q^2(t, X(t, y))\mathbb{1}_{\{t \leq t_y^*\}}) = \mathbb{1}_{\{t \leq t_y^*\}}\, \mathrm{d}Q^2(t, X(t, y)) - Q^2(t, X(t, y))\delta(t - t_y^*)\, \mathrm{d}t,$$

understood in the weak sense. (See Appendix C for the deterministic analogue, along with a discussion of this “defect measure”.)

Since $Q^2\delta \geq 0$, the inequality replaces the equal sign when this measure is suppressed. This is the weak energy inequality (2.8). See also (C.10) for the deterministic analogue.

Almost sure boundedness from above is given by Lemma 5.11. Except on the set $\{\omega : t_x^* \geq t\}$, for $\mathbb{P}$-almost every ω there exists a unique y such that $X(t, y) = x$. On that set we know that $Q(t, x)$ can be bounded by 0. Since every $x = X(t, y)$ can be reached from some y at $t = 0$ on a characteristic, the one sided estimate holds for $Q(t, x)$ in the general case. □

Remark 6.4 *(Maximal dissipation of energy).* With regards to comments following Definition 2.6, we intend to show in an upcoming work that maximal energy dissipation is given by (2.10), as well as the uniqueness of dissipative weak solutions.

Appendix A. Lagrangian and Hamiltonian approaches to the Hunter–Saxton equation

Here we motivate the stochastic Hunter–Saxton equation (1.1) that we study in this paper.

From Hunter–Zheng [20] we know that the evolution part of the Hunter–Saxton equation is given by

$$\partial_t u = D^{-1}\frac{\delta H(u)}{\delta u}, \tag{A.1}$$

where the Hamiltonian reads

$$H(u) = \frac{1}{2}\int u(\partial_x u)^2\, \mathrm{d}x,$$

and $D^{-1} = \int^x$. We find that

$$\frac{\delta H(u)}{\delta u} = \frac{1}{2}(\partial_x u)^2 - \partial_x(u\partial_x u),$$

which yields

$$\partial_x(\partial_t u + u\partial_x u) = \frac{1}{2}(\partial_x u)^2.$$

Note that we can write (A.1) as

$$\partial_t q = \frac{\delta H(u)}{\delta u}.$$

If we perturb the Hamiltonian as in, e.g., [18], by

$$\tilde{H}(u) = H(u) + \frac{1}{2}\int \sigma(\partial_x u)^2 \circ \dot{W}\, \mathrm{d}x, \tag{A.2}$$

we find

$$\frac{\delta \tilde{H}(u)}{\delta u} = \frac{1}{2}(\partial_x u)^2 - \partial_x(u\partial_x u) - \partial_x(\sigma\partial_x u) \circ \dot{W},$$

which yields

$$\partial_x(\partial_t u + u\partial_x u) = \frac{1}{2}(\partial_x u)^2 - \partial_x(\sigma\partial_x u) \circ \dot{W}, \tag{A.3}$$

and this is the stochastic Hunter–Saxton equation.

An alternative approach is based on a Lagrangian formulation. Let $L = L(u, \partial_t u, \partial_x u)$ denote the Lagrangian. If we take the first variation

$$\delta \iint L(u, \partial_t u, \partial_x u)\, \mathrm{d}x\, \mathrm{d}t,$$

we find that the Euler–Lagrange equation reads

$$\frac{\partial}{\partial x}\frac{\partial L}{\partial(\partial_x u)} + \frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial_t u)} - \frac{\partial L}{\partial u} = 0.$$

Introduce [20]

$$L(u, \partial_t u, \partial_x u) = \partial_x u \partial_t u + u(\partial_x u)^2 + \sigma(\partial_x u)^2 \circ \dot{W}.$$

Then we find again that

$$\partial_x(\partial_t u + u\partial_x u) = \frac{1}{2}(\partial_x u)^2 - \partial_x(\sigma\partial_x u) \circ \dot{W}. \tag{A.4}$$

Remark A.1 *(The Euler-Poincaré structure).* The perturbation (A.2) can be compared with the way that [7,18] treated Camassa–Holm and Hunter–Saxton equations, emphasising the geometric “Euler-Poincaré structure” to which these related equations conform, manifested in the deterministic setting as

$$0 = \partial_t m + \partial_x(mu) + m\partial_x u,$$

with $m = u - \partial_{xx}^2 u$ for the Camassa–Holm equation and $m = \partial_{xx}^2 u$ for the Hunter–Saxton equation. To "stochasticise" these equations whilst respecting the Euler-Poincaré structure, [7,17,18]replaced u by $u + \sigma \dot{W}$, and replaced all simple products by the Stratonovich product.

This amounted to perturbing the Hamiltonian by

$$\tilde{H}[m] = H[m] + \int (m\sigma) \circ \dot{W} \, \mathrm{d}x$$

and leads to the equation

$$0 = \partial_t m + \partial_x (mu) + m\partial_x u + \big(\partial_x (m\sigma) + m\partial_x \sigma\big) \circ \dot{W}.$$

In the Hunter–Saxton case, by setting $m = \partial_{xx}^2 u = \partial_x q$, one formally arrives at

$$0 = \partial_x \left(\partial_t q + \partial_x (uq) - \frac{1}{2} q^2 \right) + (\partial_x (\partial_x q\sigma) + \partial_x q \, \partial_x \sigma) \circ \dot{W}.$$

Taking an antiderivative, we see that the noise agrees with our equation (1.1) in the $\sigma'' = 0$ case. The analytic properties do not at first look as easy to exploit in the general case where the noise formulated as above. Since $\partial_x q \partial_x \sigma$ is not a full derivative, one may be forced to work on the "level" of $\partial_t (\partial_x q)$ for noises of this form with general σ.

Nevertheless, Crisan-Holm [7] showed heuristically that the stochastic Camassa–Holm equation thus derived, respecting the Euler-Poincaré structure, can be understood as a compatibility condition for the deterministic Camassa–Holm isospectral problem and a stochastic evolution equation for its eigenvalue if one had $\sigma(x) = Ae^x + Be^{-x} + C$ for $A, B, C \in \mathbb{R}$. This σ is of some heuristic interest for our formulation as well, as noted in Remark 2.12. (Note that there is a calculation error in (2.13) of [7] that invalidates Theorem 16 there — see also Remark 3.3 and Section 4.1 for genuinely stochastic wave-breaking.)

Appendix B. A-priori bounds

In this appendix we shall establish the a-priori estimates of Proposition 2.11, which we re-state below:

Proposition B.1 *(A-priori bounds). Let q be a conservative or dissipative weak solution to the stochastic Hunter–Saxton equation (1.1), with $\sigma \in (C^2 \cap \dot{W}^{1,\infty} \cap \dot{W}^{2,\infty})(\mathbb{R})$, and initial condition $q(0) = q_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. The following bounds hold:*

$$\operatorname*{ess\,sup}_{t \in [0,T]} \mathbb{E} \|q(t)\|_{L^2(\mathbb{R})}^2 \le C_T \|q_0\|_{L^2(\mathbb{R})}^2, \tag{B.1}$$

$$\mathbb{E} \|q\|_{L^{2+\alpha}([0,T] \times \mathbb{R})}^{2+\alpha} \le C_{T,\alpha} \|q_0\|_{L^2(\mathbb{R})}^2, \tag{B.2}$$

for any $\alpha \in [0, 1)$.

Consider a standard mollifier defined by $J_\varepsilon(x) = \frac{1}{\varepsilon} J\left(\frac{x}{\varepsilon}\right)$ where

$$0 \le J \in C_c^\infty(\mathbb{R}), \quad \operatorname{supp}(J) \subseteq [-1,1], \quad J(-x) = J(x), \quad \int_{\mathbb{R}} J(x)\,\mathrm{d}x = 1.$$

We write

$$h_\varepsilon := J_\varepsilon \star h$$

for the (spatial) convolution of a function h. We prove the following technical lemma on mollifiers.

Lemma B.2 *(Regularisation Lemma). Let q be a weak solution to the stochastic Hunter–Saxton equation (1.1) with $\sigma \in (C^2 \cap \dot{W}^{1,\infty} \cap \dot{W}^{2,\infty})(\mathbb{R})$. The mollified equation holds pointwise in $\mathbb{R}$ over $t < T$, in the sense of Itô that:*

$$\mathrm{d}q_\varepsilon = -J_\varepsilon \star \partial_x(uq)\,\mathrm{d}t + \frac{1}{2} J_\varepsilon \star q^2\,\mathrm{d}t - J_\varepsilon \star \partial_x(\sigma q)\,\mathrm{d}W + \frac{1}{2} J_\varepsilon \star \partial_x(\sigma \partial_x(\sigma q))\,\mathrm{d}t. \quad \text{(B.3)}$$

In particular, for fixed ε, there is a representative of q_ε (also called q_ε) such that for each $\omega \in \Omega$,

$$q_\varepsilon(\omega) \in C([0,T]; C^\infty(\mathbb{R})).$$

Proof. The main point is to check that there q_ε is $\mathbb{P}$-almost surely pointwise continuous in time, so that there are no dissipative effects when an entropy is applied to it, and so that Itô's formula can be applied pointwise in x.

By the Burkholder–Davis–Gundy inequality, for $\beta, \theta > 0$ and deterministic times $s, t \in [0,T]$,

$$\begin{aligned}
\mathbb{E}\left\| \int_s^t J_\varepsilon \star \partial_x(\sigma q)\,\mathrm{d}W \right\|_{H^\beta_{\mathrm{loc}}}^\theta &\le C\mathbb{E}\left(\int_s^t \left\| J_\varepsilon \star \partial_x(\sigma q)\right\|_{H^\beta_{\mathrm{loc}}}^2 \mathrm{d}r \right)^{\theta/2} \\
&= C\mathbb{E}\left(\int_s^t \left\| (\partial_x^{\beta+1} J_\varepsilon) \star (\sigma q)\right\|_{L^2_{\mathrm{loc}}}^2 \mathrm{d}r \right)^{\theta/2} \\
&\le C\mathbb{E}\left(\int_s^t \left\| (\partial_x^{\beta+1} J_\varepsilon) \star (\sigma q)\right\|_{L^\infty_{\mathrm{loc}}}^2 \mathrm{d}r \right)^{\theta/2} \\
&\le C_{\beta,\varepsilon,\theta,T,\sigma} |t-s|^{\theta/2},
\end{aligned}$$

By Kolmogorov's continuity theorem, for fixed $\varepsilon > 0$, we have a $C^{1/2-0}([0,T]; H^\beta_{\mathrm{loc}}(\mathbb{R}))$ representative of the martingale

$$J_\varepsilon \star \int_0^t \partial_x(\sigma q)\,\mathrm{d}W = \int_0^t J_\varepsilon \star \partial_x(\sigma q)\,\mathrm{d}W.$$

By the Sobolev embedding theorem on $\mathbb{R}$, for $\beta \geq 1$, we have spatial continuity to arbitrary order.

All the other temporal integrals are integrals of finite variation, and hence continuous in t, with integrands that are convolutions against a fixed, smooth function, and hence smooth in x. This means that

$$q_\varepsilon(t) - q_\varepsilon(0) = -\int_0^t J_\varepsilon \star \partial_x(uq)\,\mathrm{d}s - \frac{1}{2} J_\varepsilon \star q^2\,\mathrm{d}s - J_\varepsilon \star \int_0^t \partial_x(\sigma q) \circ \mathrm{d}W$$

is also pointwise continuous. This means there is no dissipation arising from the mollified equation for fixed $\varepsilon > 0$.

Moreover, since

$$J_\varepsilon \star \int_0^t \partial_x(\sigma q)\,\mathrm{d}W$$

has a $C^{1/2-0}([0,T]; H^\beta_{\mathrm{loc}}(\mathbb{R}))$ continuous representative, we can write its cross-variation with W as

$$\left\langle J_\varepsilon \star \int_0^{(\cdot)} \partial_x(\sigma q)\,\mathrm{d}W, W \right\rangle_t = J_\varepsilon \star \int_0^t \partial_x(\sigma q)\,\mathrm{d}s.$$

Therefore the normal Itô formula is sufficient to establish equivalence of the Stratonovich and Itô formulations. □

Lemma B.3 *(Mollification error bounds). On the same assumptions as Lemma B.2, the mollified equation (B.3) can be re-written as*

$$dq_\varepsilon + \partial_x(u_\varepsilon q_\varepsilon)\,\mathrm{d}t + \partial_x(\sigma q_\varepsilon)\mathrm{d}W - \frac{1}{2}\partial_x(\sigma\partial_x(\sigma q_\varepsilon)) = \frac{1}{2}q_\varepsilon^2\,\mathrm{d}t + (r_\varepsilon + \rho_\varepsilon)\,\mathrm{d}t + \tilde{r}_\varepsilon\,\mathrm{d}W, \tag{B.4}$$

where the mollification error

$$r_\varepsilon := \Big(\partial_x(u_\varepsilon q_\varepsilon) - \partial_x\big(uq\big) \star J_\varepsilon\Big) + \frac{1}{2}\Big(q^2 \star J_\varepsilon - q_\varepsilon^2\Big) \tag{B.5}$$

tends to zero in $L^1([0,T]; L^1(\mathbb{R}))$ as $\varepsilon \to 0$, $\mathbb{P}$-almost surely,

$$\tilde{r}_\varepsilon := \Big(\partial_x(\sigma q_\varepsilon) - \partial_x\big(\sigma q\big) \star J_\varepsilon\Big) \tag{B.6}$$

tends to zero in $L^2([0,T]; L^2(\mathbb{R}))$ as $\varepsilon \to 0$, $\mathbb{P}$-almost surely, and, for any $S \in C^{1,1}(\mathbb{R})$ with $\sup_{r\in\mathbb{R}}(|S'(r)| + |S''(r)|) < \infty$,

$$\int_0^T \int_{\mathbb{R}} \Big(- S'(q_\varepsilon)\rho_\varepsilon + \frac{1}{2} S''(q_\varepsilon)\Big((\partial_x(\sigma q_\varepsilon))^2 - \big(\partial_x(\sigma q) \star J_\varepsilon\big)^2 \Big)\Big) \,\mathrm{d}x\,\mathrm{d}t$$

tends to zero as $\varepsilon \to 0$, $\mathbb{P}$-almost surely, where

$$\rho_\varepsilon := \frac{1}{2}\Big(\partial_x\big(\sigma \partial_x(\sigma q)\big) \star J_\varepsilon - \partial_x(\sigma \partial_x(\sigma q_\varepsilon))\Big). \tag{B.7}$$

Proof. By Definition 2.1, for a weak solution (u, q) to the stochastic Hunter–Saxton equation (1.1), $q \in L^\infty([0,T]; L^2(\mathbb{R}))$, and $u \in L^\infty([0,T]; \dot{H}^1(\mathbb{R}))$ with unit $\mathbb{P}$-probability. This fact will be implicitly invoked along with the dominated convergence theorem, among other instances, in the following tripartite calculations.

1. Estimate of r_ε.

We can decompose $\partial_x(u_\varepsilon q_\varepsilon) - \partial_x\big(uq\big) \star J_\varepsilon$ as follows:

$$\begin{aligned}\partial_x(u_\varepsilon q_\varepsilon) &- \partial_x\big(uq\big) \star J_\varepsilon \\ &= \big(\partial_x(u_\varepsilon q_\varepsilon) - \partial_x(u q_\varepsilon)\big) + \big(\partial_x(u q_\varepsilon) - \partial_x\big(uq\big) \star J_\varepsilon\big).\end{aligned}$$

We estimate the above in $L^1(\mathbb{R})$ term-by-term.

Treating the $L^1_{\mathrm{loc}}(\mathbb{R})$ integral as an $\dot{H}^1(\mathbb{R})$ – $H^{-1}(\mathbb{R})$ pairing between $|u - u_\varepsilon|$ and $|\partial_x q_\varepsilon|$, we have

$$\|\partial_x\big((u - u_\varepsilon)q_\varepsilon\big)\|_{L^1(\mathbb{R})} \le C\big(\|u - u_\varepsilon\|_{L^\infty(B_R)} + \|q - q_\varepsilon\|_{L^2(\mathbb{R})}\big)\|q_\varepsilon\|_{L^2(B_R)}$$

as $\varepsilon \to 0$, by standard results on convolutions, $\mathbb{P}$-almost surely.

The second term tends to nought in $L^1(\mathbb{R})$ for almost every $t \in [0,T]$, $\mathbb{P}$-almost surely by [26, Lemma 2.3] (also see [12, Lemma II.1]).

The final part of r_ε is

$$q^2 \star J_\varepsilon - q_\varepsilon^2 = q^2 \star J_\varepsilon - q^2 + q(q - q_\varepsilon) + (q - q_\varepsilon)q_\varepsilon.$$

It tends to nought in $L^1(\mathbb{R})$ for almost every $t \in [0,T]$, $\mathbb{P}$-almost surely by standard theorems on convolutions. By the $\mathbb{P}$-almost sure inclusion $q \in L^\infty([0,T]; L^2(\mathbb{R}))$ for weak solutions, $\mathbb{P}$-almost surely the $L^1(\mathbb{R})$ norm of the expression above can be uniformly bounded by $C \sup_{t\in[0,T]} \|q(t)\|^2_{L^2(\mathbb{R})}$. This expression is of course integrable over $[0,T]$. Therefore, by the dominated convergence theorem, $\mathbb{P}$-almost surely, the $L^1([0,T]; L^1(\mathbb{R}))$ convergence follows from the pointwise-in-t convergence to zero of

$$\|q^2(t) \star J_\varepsilon - q^2(t)\|_{L^1(\mathbb{R})} + \|q(q - q_\varepsilon)\|_{L^1(\mathbb{R})} + \|(q - q_\varepsilon)q_\varepsilon\|_{L^1(\mathbb{R})},$$

as $\varepsilon \to 0$.

By the estimate

$$\|\partial_x(u q_\varepsilon)(t) - \partial_x(uq)(t) \star J_\varepsilon\|_{L^1_{\mathrm{loc}}(\mathbb{R})} \le C\|q(t)\|^2_{L^2(\mathbb{R})},$$

also established in [26, Lemma 2.3], it again follows from the $\mathbb{P}$-almost sure inclusion $\|q(t)\|_{L^2(\mathbb{R})} \in L^\infty([0,T])$ via the dominated convergence theorem that

$$\int_0^T \|r_\varepsilon(t)\|_{L^1(\mathbb{R})} \,\mathrm{d}t \le C_{T,\varepsilon} \int_0^T \|q(t)\|^2_{L^2(\mathbb{R})} \,\mathrm{d}t, \tag{B.8}$$

where $C_{T,\varepsilon}$ is a quantity independent of t, vanishing as $\varepsilon \to 0$, and therefore $r_\varepsilon \to 0$ in $L^1([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$, $\mathbb{P}$-almost surely.

2. Estimate of $\tilde{r}_\varepsilon$.

This is treated similarly to the second term above, with σ in place of u.

Since $\sigma \in \dot{W}^{1,\infty}$ and $q \in L^\infty([0,T]; L^2(\mathbb{R}))$, it holds that $\partial_x(\sigma q) \in L^\infty([0,T]; L^2_{\mathrm{loc}}(\mathbb{R}))$ with unit $\mathbb{P}$ probability. Therefore this time we have slightly higher spatial integrability, allowing us to conclude via [26, Lemma 2.3] that

$$\tilde{r}_\varepsilon(t) = \partial_x(\sigma q_\varepsilon)(t) - \partial_x(\sigma q)(t) \star J_\varepsilon \to 0$$

in $L^2(\mathbb{R})$ for almost every $t \in [0,T]$, $\mathbb{P}$-almost surely, as $\varepsilon \to 0$.

Next, by an application of the dominated convergence theorem in a manner previously demonstrated, we can conclude that

$$\int_0^T \|\tilde{r}_\varepsilon(t)\|^2_{L^2(\mathbb{R})} \,\mathrm{d}t \le C_{T,\varepsilon} \int_0^T \|q(t)\|^2_{L^2(\mathbb{R})} \,\mathrm{d}t, \tag{B.9}$$

where $C_{T,\varepsilon}$ depends on the continuity properties of σ and its derivatives, in additional to ε, for which we have the limit $C_{T,\varepsilon} \to 0$ as $\varepsilon \to 0$, $\mathbb{P}$-almost surely. Hence $\tilde{r}_\varepsilon \to 0$ in $L^2([0,T]; L^2(\mathbb{R}))$, $\mathbb{P}$-almost surely.

3. Estimate of ρ_ε.

The estimate of ρ_ε takes inspiration from the proof of [30, Prop. 3.4]. However, whereas they considered the commutator between the operators $\tilde{\boldsymbol{\sigma}} f := \sigma \partial_x f$ and $\mathbf{j}^\varepsilon f := f \star J_\varepsilon$, we shall have to consider the analogous question for $\boldsymbol{\sigma} f := \partial_x(\sigma f)$ and $\mathbf{j}^\varepsilon$.

Recall that here, we seek not to show that ρ_ε vanishes but that the following quantity does:

$$\int_0^T \int_{\mathbb{R}} \Big(- S'(q_\varepsilon)\rho_\varepsilon + \frac{1}{2} S''(q_\varepsilon)\Big((\partial_x(\sigma q_\varepsilon))^2 - \big(\partial_x(\sigma q) \star J_\varepsilon\big)^2 \Big)\Big) \,\mathrm{d}x\,\mathrm{d}t.$$

We can write ρ_ε as

$$\begin{aligned}
\rho_\varepsilon &:= \frac{1}{2}\Big(\partial_x\big(\sigma \partial_x(\sigma q)\big) \star J_\varepsilon - \partial_x(\sigma \partial_x(\sigma q_\varepsilon)) \Big) \\
&= \frac{1}{2}\big(\mathbf{j}^\varepsilon \boldsymbol{\sigma}\boldsymbol{\sigma} q - \boldsymbol{\sigma}\boldsymbol{\sigma}\mathbf{j}^\varepsilon q \big) \\
&= \frac{1}{2}\big(\mathbf{j}^\varepsilon \boldsymbol{\sigma}\boldsymbol{\sigma} q - \boldsymbol{\sigma}\mathbf{j}^\varepsilon \boldsymbol{\sigma} q + \boldsymbol{\sigma}\mathbf{j}^\varepsilon \boldsymbol{\sigma} q - \boldsymbol{\sigma}\boldsymbol{\sigma}\mathbf{j}^\varepsilon q \big)
\end{aligned}$$

$$= \frac{1}{2}\big(\big[\mathbf{j}^{\varepsilon}, \boldsymbol{\sigma}\big](\boldsymbol{\sigma} q) + \boldsymbol{\sigma}\big[\mathbf{j}^{\varepsilon}, \boldsymbol{\sigma}\big](q)\big), \tag{B.10}$$

where

$$\big[\mathbf{j}^{\varepsilon}, \boldsymbol{\sigma}\big](q) = \mathbf{j}^{\varepsilon}\boldsymbol{\sigma} q - \boldsymbol{\sigma}\mathbf{j}^{\varepsilon} q.$$

Similarly, we can write the remaining part of the integrand as

$$(\partial_x(\sigma q_\varepsilon))^2 - \big(\partial_x(\sigma q) \star J_\varepsilon\big)^2 = \big((\boldsymbol{\sigma}\mathbf{j}^{\varepsilon} q)^2 - (\mathbf{j}^{\varepsilon}\boldsymbol{\sigma} q)^2\big). \tag{B.11}$$

Therefore, following the calculations in [30, p. 655], we find that

$$\begin{aligned}
\frac{1}{2} S''(q_\varepsilon) \cdot (\text{B.11}) &= \frac{1}{2} S''(q_\varepsilon)\big(\boldsymbol{\sigma}\mathbf{j}^{\varepsilon} q - \mathbf{j}^{\varepsilon}\boldsymbol{\sigma} q\big)\big(\boldsymbol{\sigma}\mathbf{j}^{\varepsilon} q + \mathbf{j}^{\varepsilon}\boldsymbol{\sigma} q\big) \\
&= -\frac{1}{2} S''(q_\varepsilon)\big(\big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q)\big)^2 + S''(q_\varepsilon)(\boldsymbol{\sigma}\mathbf{j}^{\varepsilon} q)\big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q) \\
&= -\frac{1}{2} S''(q_\varepsilon)\big(\big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q)\big)^2 + S''(q_\varepsilon)\partial_x \sigma q_\varepsilon \big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q) \\
&\quad + \sigma \partial_x (S'(q_\varepsilon))\big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q) \\
&= -\frac{1}{2} S''(q_\varepsilon)\big(\big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q)\big)^2 + S''(q_\varepsilon)\partial_x \sigma q_\varepsilon \big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q) \\
&\quad + \partial_x\big(\sigma S'(q_\varepsilon)\big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q)\big) - S'(q_\varepsilon)\partial_x\big(\sigma\big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q)\big) \\
&= -\frac{1}{2} S''(q_\varepsilon)\big(\big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q)\big)^2 + S''(q_\varepsilon)\partial_x \sigma q_\varepsilon \big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q) \\
&\quad + \partial_x\big(\sigma S'(q_\varepsilon)\big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q)\big) - S'(q_\varepsilon)\boldsymbol{\sigma}\big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q),
\end{aligned}$$

by invoking the definition of $\boldsymbol{\sigma}$.

Adding this to (B.10), we find that

$$\begin{aligned}
&-S'(q_\varepsilon) \cdot (\text{B.10}) + \frac{1}{2} S''(q_\varepsilon) \cdot (\text{B.11}) \\
&\quad = -\frac{1}{2} S''(q_\varepsilon)\big([\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}](q)\big)^2 + S''(q_\varepsilon)\partial_x \sigma q_\varepsilon \big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q) + \partial_x\big(\sigma S'(q_\varepsilon)\big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q)\big) \\
&\qquad - S'(q_\varepsilon)\boldsymbol{\sigma}\big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q) - \frac{1}{2} S'(q_\varepsilon)\big([\mathbf{j}^{\varepsilon}, \boldsymbol{\sigma}](\boldsymbol{\sigma} q) + \boldsymbol{\sigma}\big[\mathbf{j}^{\varepsilon}, \boldsymbol{\sigma}\big](q)\big) \\
&\quad = -\frac{1}{2} S''(q_\varepsilon)\big([\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}](q)\big)^2 + S''(q_\varepsilon)\partial_x \sigma q_\varepsilon \big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q) \\
&\qquad + \partial_x\big(\sigma S'(q_\varepsilon)\big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q)\big) + \frac{1}{2} S'(q_\varepsilon)\big([\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}](\boldsymbol{\sigma} q) - \boldsymbol{\sigma}[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}](q)\big) \\
&\quad = -\frac{1}{2} S''(q_\varepsilon)\big([\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}](q)\big)^2 + S''(q_\varepsilon)\partial_x \sigma q_\varepsilon \big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q) \\
&\qquad + \partial_x\big(\sigma S'(q_\varepsilon)\big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big](q)\big) + \frac{1}{2} S'(q_\varepsilon)\Big[\big[\boldsymbol{\sigma}, \mathbf{j}^{\varepsilon}\big], \boldsymbol{\sigma}\Big](q).
\end{aligned} \tag{B.12}$$

We have already established that $\big[\boldsymbol{\sigma}, \mathbf{j}^\varepsilon\big](q) = \tilde{r}_\varepsilon \to 0$ in $L^2([0,T]; L^2(\mathbb{R}))$ as $\varepsilon \to 0$.

Therefore, we focus on the double commutator, which, for clarity, is

$$\Big[\big[\boldsymbol{\sigma}, \mathbf{j}^\varepsilon\big], \boldsymbol{\sigma}\Big](q) = \big[\boldsymbol{\sigma}, \mathbf{j}^\varepsilon\big](\boldsymbol{\sigma} q) - \boldsymbol{\sigma}\big[\boldsymbol{\sigma}, \mathbf{j}^\varepsilon\big](q)$$
$$= 2\boldsymbol{\sigma}\mathbf{j}^\varepsilon\boldsymbol{\sigma} q - \mathbf{j}^\varepsilon\boldsymbol{\sigma}\boldsymbol{\sigma} q - \boldsymbol{\sigma}\boldsymbol{\sigma}\mathbf{j}^\varepsilon q. \tag{B.13}$$

Term-by-term in this commutator we have

$$2\boldsymbol{\sigma}\mathbf{j}^\varepsilon\boldsymbol{\sigma} q(x) = 2\int_{\mathbb{R}} \partial^2_{xx} J_\varepsilon(x-y)\sigma(x)\sigma(y)q(y)\,\mathrm{d}y \tag{B.14}$$
$$+ 2\int_{\mathbb{R}} \partial_x J_\varepsilon(x-y)\partial_x\sigma(x)\sigma(y)q(y)\,\mathrm{d}y, \tag{B.15}$$

$$\mathbf{j}^\varepsilon\boldsymbol{\sigma}\boldsymbol{\sigma} q(x) = \int_{\mathbb{R}} \partial^2_{xx} J_\varepsilon(x-y)\sigma^2(y)q(y)\,\mathrm{d}y \tag{B.16}$$
$$- \int_{\mathbb{R}} \partial_x J_\varepsilon(x-y)\sigma(y)\partial_y\sigma(y)q(y)\,\mathrm{d}y, \tag{B.17}$$

and

$$\boldsymbol{\sigma}\boldsymbol{\sigma}\mathbf{j}^\varepsilon q(x) = \int_{\mathbb{R}} J_\varepsilon(x-y)\partial_x\big(\sigma(x)\partial_x\sigma(x)\big)q(y)\,\mathrm{d}y \tag{B.18}$$
$$+ 3\int_{\mathbb{R}} \partial_x J_\varepsilon(x-y)\sigma(x)\partial_x\sigma(x)q(y)\,\mathrm{d}y \tag{B.19}$$
$$+ \int_{\mathbb{R}} \partial^2_{xx} J_\varepsilon(x-y)\sigma^2(x)q(y)\,\mathrm{d}y. \tag{B.20}$$

There are more terms here than in [30] because we do not necessarily have the divergence-free condition $\partial_x\sigma = 0$.

Now we can estimate (B.14) to (B.20) above by considering the sums

$$\mathfrak{I}_1 := \text{(B.15)} - \text{(B.17)} - \text{(B.19)},$$
$$\mathfrak{I}_2 := \text{(B.14)} - \text{(B.16)} - \text{(B.20)},$$

and finally the stand-alone integral (B.18), where from (B.13), we see that

$$\Big[\big[\boldsymbol{\sigma}, \mathbf{j}^\varepsilon\big], \boldsymbol{\sigma}\Big](q) = \mathfrak{I}_1 + \mathfrak{I}_2 - \text{(B.18)}. \tag{B.21}$$

We shall use [12, Lemma II.1] to establish that this sum above tends to nought in an appropriate topology. Estimating these integrals separately, we have

$$
\begin{aligned}
\|\mathfrak{I}_1\|_{L^2(\mathbb{R})} &= \Big\| \int_{\mathbb{R}} \partial_x J_\varepsilon(\cdot - y)\Big(2\sigma(y)\partial_x\sigma(\cdot) + \sigma(y)\partial_y\sigma(y) - 3\sigma(\cdot)\partial_x\sigma(\cdot)\Big)q(y)\,\mathrm{d}y\Big\|_{L^2(\mathbb{R})} \\
&= \Big\| \int_{\mathbb{R}} \partial_x J_\varepsilon(\cdot - y) \\
&\quad \times \Big(2\big(\sigma(y)-\sigma(\cdot)\big)\partial_x\sigma(\cdot) + \big(\sigma(y)\partial_y\sigma(y) - \sigma(\cdot)\partial_x\sigma(\cdot)\big)\Big)q(y)\,\mathrm{d}y\Big\|_{L^2(\mathbb{R})} \\
&\le \Big\| \int_{\mathbb{R}} |\partial_x J_\varepsilon(\cdot - y)| \\
&\quad \times \Big(2|\sigma(y)-\sigma(\cdot)|\,|\partial_x\sigma(\cdot)| + |\sigma(y)\partial_y\sigma(y) - \sigma(\cdot)\partial_x\sigma(\cdot)|\Big)|q(y)|\,\mathrm{d}y\Big\|_{L^2(\mathbb{R})} \\
&\le C\Big\| \int_{\mathbb{R}} \frac{1}{\varepsilon}|\cdot - y| J_\varepsilon(\cdot - y) \\
&\quad \times \Big(2\Big|\frac{\sigma(y)-\sigma(\cdot)}{y-\cdot}\partial_x\sigma(\cdot)\Big| + \Big|\frac{\sigma(y)\partial_y\sigma(y) - \sigma(\cdot)\partial_x\sigma(\cdot)}{y-\cdot}\Big|\Big)|q(y)|\,\mathrm{d}y\Big\|_{L^2(\mathbb{R})} \\
&\le C\Big(\|\partial_x\sigma\|^2_{L^\infty(\mathbb{R})} + \|\sigma\partial_x\sigma\|_{L^\infty(\mathbb{R})}\Big)\Big\| \int_{\mathbb{R}} \frac{1}{\varepsilon}|\cdot - y| J_\varepsilon(\cdot - y)|q(y)|\,\mathrm{d}y\Big\|_{L^2(\mathbb{R})} \\
&\le C\Big(\|\partial_x\sigma\|^2_{L^\infty(\mathbb{R})} + \|\sigma\partial_x\sigma\|_{L^\infty(\mathbb{R})}\Big)\Big\|\frac{1}{\varepsilon}|\cdot| J_\varepsilon(\cdot)\Big\|_{L^1(\mathbb{R})}\|q\|_{L^2(\mathbb{R})} \\
&\le C\big(\|\partial_x\sigma\|^2_{L^\infty(\mathbb{R})} + \|\sigma\partial_x\sigma\|_{L^\infty(\mathbb{R})}\big)\|q\|_{L^2(\mathbb{R})}.
\end{aligned}
$$

Here we used that $|\partial_x J_\varepsilon| \lesssim \varepsilon^{-1} J_\varepsilon$ and Young's inequality for convolutions. Similarly we find that

$$
\begin{aligned}
\|\mathfrak{I}_2\|_{L^2(\mathbb{R})} &= \Big\| \int_{\mathbb{R}} \partial^2_{xx} J_\varepsilon(\cdot - y)\big(2\sigma(\cdot)\sigma(y) - \sigma^2(\cdot) - \sigma^2(y)\big)q(y)\,\mathrm{d}y\|_{L^2(\mathbb{R})} \\
&= \Big\| \int_{\mathbb{R}} \partial^2_{xx} J_\varepsilon(\cdot - y)\big(\sigma(\cdot) - \sigma(y)\big)^2 q(y)\,\mathrm{d}y\Big\|_{L^2(\mathbb{R})} \\
&\le \Big\| \int_{\mathbb{R}} |\partial^2_{xx} J_\varepsilon(\cdot - y)|\big|2\sigma(\cdot)\sigma(y) - \sigma^2(\cdot) - \sigma^2(y)\big|\,|q(y)|\,\mathrm{d}y\|_{L^2(\mathbb{R})} \\
&\le C\Big\| \int_{\mathbb{R}} \frac{1}{\varepsilon^2}(\cdot - y)^2 J_\varepsilon(\cdot - y)\Big|\frac{\sigma(\cdot)-\sigma(y)}{y-\cdot}\Big|^2 |q(y)|\,\mathrm{d}y\Big\|_{L^2(\mathbb{R})} \\
&\le C\|\partial_x\sigma\|^2_{L^\infty(\mathbb{R})}\Big\| \int_{\mathbb{R}} \frac{1}{\varepsilon^2}(\cdot - y)^2 J_\varepsilon(\cdot - y)|q(y)|\,\mathrm{d}y\Big\|_{L^2(\mathbb{R})}
\end{aligned}
$$

$$\leq C\|\partial_x\sigma\|^2_{L^\infty(\mathbb{R})}\left\|\frac{1}{\varepsilon^2}(\,\cdot\,)^2 J_\varepsilon(\,\cdot\,)\right\|_{L^1(\mathbb{R})}\|q\|_{L^2(\mathbb{R})}$$
$$\leq C\|\partial_x\sigma\|^2_{L^\infty(\mathbb{R})}\|q\|_{L^2(\mathbb{R})}.$$

We also have

$$\|(\text{B.18})\|_{L^2(\mathbb{R})} \leq C\|J_\varepsilon\|_{L^1(\mathbb{R})}\|\partial_x(\sigma\partial_x\sigma)\|_{L^\infty(\mathbb{R})}\|q(t)\|_{L^2(\mathbb{R})}.$$

Now for smooth functions q,

$$\begin{aligned}
\mathfrak{I}_2 &= \int_{\mathbb{R}} \partial^2_{xx} J_\varepsilon(x-y)\big(2\sigma(x)\sigma(y)-\sigma^2(x)-\sigma^2(y)\big)q(y)\,\mathrm{d}y \\
&= -2\int_{\mathbb{R}} \partial^2_{xx} J_\varepsilon(x-y)\frac{(x-y)^2}{2}\left(\frac{\sigma(y)-\sigma(x)}{(y-x)}\right)^2 q(y)\,\mathrm{d}y \\
&= -2(\partial_x\sigma)^2 q(x)\int_{\mathbb{R}} \frac{z^2}{2}\partial^2_{zz}J_\varepsilon(z)\,\mathrm{d}z + o(\varepsilon).
\end{aligned}$$

A similar calculation can be done for $\mathfrak{I}_1$, where there is only one derivative on the mollifier, and which can be found directly in the proof of [12, Lemma II.1].

The limit of (B.18) as $\varepsilon \to 0$ for smooth q is standard.

Reasoning then as in the proof of [12, Lemma II.1], we find that

$$\mathfrak{I}_1 \to \partial_x(\sigma\partial_x\sigma)q + 2(\partial_x\sigma)^2 q, \qquad \mathfrak{I}_2 \to -2(\partial_x\sigma)^2 q, \qquad -(\text{B.18}) \to -\partial_x(\sigma\partial_x\sigma)q,$$

in $L^2(\mathbb{R})$ almost everywhere in time, $\mathbb{P}$-almost surely as $\varepsilon \to 0$. Adding these together, with reference to (B.21), we can conclude that $\Big[\big[\boldsymbol{\sigma}, \mathbf{j}^\varepsilon\big], \boldsymbol{\sigma}\Big](q) \to 0$ in $L^2([0,T]; L^2(\mathbb{R}))$ $\mathbb{P}$-almost surely as $\varepsilon \to 0$.

Recall (B.12). We have the $\mathbb{P}$-almost sure bound,

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{R}}\Big[-S'(q_\varepsilon)\rho_\varepsilon + \frac{1}{2}S''(q_\varepsilon)\Big((\partial_x(\sigma q_\varepsilon))^2 - \big(\partial_x(\sigma q)\star J_\varepsilon\big)^2\Big)\Big]\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad = \left|\int_0^T\int_{\mathbb{R}}\Big(-S'(q_\varepsilon)\cdot(\text{B.10}) + \frac{1}{2}S''(q_\varepsilon)\cdot(\text{B.11})\Big)\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad \leq C_{T,\sigma,\varepsilon}\|q\|_{L^2([0,T]\times\mathbb{R})},
\end{aligned} \tag{B.22}$$

where $C_{T,\sigma,\varepsilon} \to 0$ as $\varepsilon \to 0$. □

Next we prove Proposition 2.11:

Proof of Proposition 2.11. We carry out this proof in three steps:

(1) We first renormalise the mollified equation, finding an equation for $S(q_\varepsilon)$ with $S \in C^{1,1}$.
(2) Using the renormalisation in (1) prove the explicit L^2-bound (2.12).
(3) Exploiting the explicit L^2-bound, we demonstrate the $L^{2+\alpha}$-bound (2.13).

1. Renormalisation.
Since convolution commutes with differentiation in x,

$$\partial_x u_\varepsilon = q_\varepsilon.$$

For any non-negative $S \in C^2(\mathbb{R})$, we can use Itô's formula to write

$$\begin{aligned}
0 = \mathrm{d}q_\varepsilon &+ \big(\partial_x(u_\varepsilon q_\varepsilon) - \frac{1}{2}q_\varepsilon^2\big)\,\mathrm{d}t + \partial_x(\sigma q_\varepsilon)\,\mathrm{d}W \\
&- \frac{1}{2}\partial_x\big(\sigma\,\partial_x(\sigma q_\varepsilon)\big)\,\mathrm{d}t - (r_\varepsilon + \rho_\varepsilon)\,\mathrm{d}t - \tilde{r}_\varepsilon\,\mathrm{d}W.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
0 &= \mathrm{d}S(q_\varepsilon) + S'(q_\varepsilon)\bigg(\partial_x(u_\varepsilon q_\varepsilon) - \frac{1}{2}q_\varepsilon^2 - \frac{1}{2}\partial_x(\sigma\,\partial_x(\sigma q_\varepsilon))\bigg)\,\mathrm{d}t + S'(q_\varepsilon)\partial_x(\sigma q_\varepsilon)\,\mathrm{d}W \\
&\quad - \frac{1}{2}S''(q_\varepsilon)\big(\partial_x(\sigma q_\varepsilon) - \tilde{r}_\varepsilon\big)^2\,\mathrm{d}t - S'(q_\varepsilon)(r_\varepsilon\,\mathrm{d}t + \rho_\varepsilon\,\mathrm{d}t + \tilde{r}_\varepsilon\,\mathrm{d}W) \\
&= \mathfrak{L} - \frac{1}{2}S''(q_\varepsilon)\big(\partial_x(\sigma q_\varepsilon) - \tilde{r}_\varepsilon\big)^2\,\mathrm{d}t - S'(q_\varepsilon)(r_\varepsilon\,\mathrm{d}t + \rho_\varepsilon\,\mathrm{d}t + \tilde{r}_\varepsilon\,\mathrm{d}W).
\end{aligned}$$

For the first term $\mathfrak{L}$ we find

$$\begin{aligned}
\mathfrak{L} &= \mathrm{d}S(q_\varepsilon) + S'(q_\varepsilon)\big(\partial_x(u_\varepsilon q_\varepsilon) - \frac{1}{2}q_\varepsilon^2\big)\,\mathrm{d}t + S'(q_\varepsilon)\partial_x(\sigma q_\varepsilon)\,\mathrm{d}W - \frac{1}{2}S'(q_\varepsilon)\partial_x(\sigma\,\partial_x(\sigma q_\varepsilon))\,\mathrm{d}t \\
&= \mathrm{d}S(q_\varepsilon) + \big(\partial_x(u_\varepsilon S(q_\varepsilon)) - q_\varepsilon S(q_\varepsilon) + \frac{1}{2}S'(q_\varepsilon)q_\varepsilon^2\big)\,\mathrm{d}t - \frac{1}{2}S'(q_\varepsilon)\partial_x(\sigma\,\partial_x(\sigma q_\varepsilon))\,\mathrm{d}t \\
&\quad + \partial_x(\sigma S(q_\varepsilon))\,\mathrm{d}W + \partial_x\sigma\big(q_\varepsilon S'(q_\varepsilon) - S(q_\varepsilon)\big)\,\mathrm{d}W,
\end{aligned}$$

and the last term on the first line can be further expanded in order to maximise the number of terms in divergence form:

$$\begin{aligned}
\mathfrak{L} = \mathrm{d}S(q_\varepsilon) &+ \big(\partial_x(u_\varepsilon S(q_\varepsilon)) - q_\varepsilon S(q_\varepsilon) + \frac{1}{2}S'(q_\varepsilon)q_\varepsilon^2\big)\,\mathrm{d}t \\
&+ \partial_x(\sigma S(q_\varepsilon))\,\mathrm{d}W + \partial_x\sigma\big(q_\varepsilon S'(q_\varepsilon) - S(q_\varepsilon)\big)\,\mathrm{d}W \\
&- \frac{1}{2}\partial_{xx}^2(S(q_\varepsilon)\sigma^2)\,\mathrm{d}t + \frac{1}{4}\partial_x(S(q_\varepsilon)\partial_x\sigma^2)\,\mathrm{d}t + \frac{1}{4}\partial_x\big((S(q_\varepsilon) - S'(q_\varepsilon)q_\varepsilon)\partial_x\sigma^2\big)\,\mathrm{d}t \\
&+ \frac{1}{2}S''(q_\varepsilon)(\sigma\,\partial_x q_\varepsilon)^2\,\mathrm{d}t + \frac{1}{4}S''(q_\varepsilon)\partial_x\sigma^2 q_\varepsilon\partial_x q_\varepsilon\,\mathrm{d}t.
\end{aligned}$$

Re-arranging the terms, one arrives at:

$$
\begin{aligned}
\mathfrak{L} &= \mathrm{d}S(q_\varepsilon) + \partial_x\Big(u_\varepsilon S(q_\varepsilon) + \frac{1}{4}\partial_x\sigma^2 S(q_\varepsilon)\Big)\,\mathrm{d}t + \partial_x(\sigma S(q_\varepsilon))\,\mathrm{d}W \\
&\quad + \partial_x\sigma\big(q_\varepsilon S'(q_\varepsilon) - S(q_\varepsilon)\big)\,\mathrm{d}W - \frac{1}{2}\partial_{xx}^2\big(\sigma^2 S(q_\varepsilon)\big)\,\mathrm{d}t - \Big(q_\varepsilon S(q_\varepsilon) - \frac{1}{2}S'(q_\varepsilon)q_\varepsilon^2\Big)\,\mathrm{d}t \\
&\quad - \frac{1}{4}\partial_{xx}^2\sigma^2\big(q_\varepsilon S'(q_\varepsilon) - S(q_\varepsilon)\big)\,\mathrm{d}t - \frac{1}{2}S''(q_\varepsilon)\big((\partial_x(\sigma q_\varepsilon))^2 \\
&\quad - (\sigma\,\partial_x q_\varepsilon)^2\big)\,\mathrm{d}t + \frac{1}{2}S''(q_\varepsilon)(\partial_x(\sigma q_\varepsilon))^2\,\mathrm{d}t.
\end{aligned}
$$

Introducing $G_S(v) = vS'(v) - S(v)$, we can simplify the above as:

$$
\begin{aligned}
\mathfrak{L} &= \mathrm{d}S(q_\varepsilon) + \partial_x\left(u_\varepsilon S(q_\varepsilon) + \frac{1}{4}\partial_x\sigma^2 S(q_\varepsilon) - \frac{1}{2}\partial_x\sigma^2 G_S(q_\varepsilon)\right)\mathrm{d}t - \Big(q_\varepsilon S(q_\varepsilon) - \frac{1}{2}q_\varepsilon^2 S'(q_\varepsilon)\Big)\,\mathrm{d}t \\
&\quad - \left[\frac{1}{2}\partial_{xx}^2\big(\sigma^2 S(q_\varepsilon)\big) + \frac{1}{2}q_\varepsilon G_S'(q_\varepsilon)(\partial_x\sigma)^2 - \frac{1}{4}\partial_{xx}^2\sigma^2 G_S(q_\varepsilon)\right]\mathrm{d}t \\
&\quad + \partial_x(\sigma S(q_\varepsilon))\,\mathrm{d}W + \partial_x\sigma G_S(q_\varepsilon)\,\mathrm{d}W + \frac{1}{2}S''(q_\varepsilon)(\partial_x(\sigma q_\varepsilon))^2\,\mathrm{d}t.
\end{aligned}
$$

There is no pathwise energy estimate in the stochastic setting because of the term $\partial_x\sigma G_S(q_\varepsilon)\,\mathrm{d}W$, which is not an exact spatial derivative.

Putting back in r_ε, ρ_ε, and $\tilde{r}_\varepsilon$, we arrive at

$$
\begin{aligned}
0 &= \mathrm{d}S(q_\varepsilon) + \partial_x\left(u_\varepsilon S(q_\varepsilon) + \frac{1}{4}\partial_x\sigma^2 S(q_\varepsilon) - \frac{1}{2}\partial_x\sigma^2 G_S(q_\varepsilon)\right)\mathrm{d}t - \Big(q_\varepsilon S(q_\varepsilon) - \frac{1}{2}q_\varepsilon^2 S'(q_\varepsilon)\Big)\,\mathrm{d}t \\
&\quad - \left[\frac{1}{2}\partial_{xx}^2\big(\sigma^2 S(q_\varepsilon)\big) + \frac{1}{2}q_\varepsilon G_S'(q_\varepsilon)(\partial_x\sigma)^2 - \frac{1}{4}\partial_{xx}^2\sigma^2 G_S(q_\varepsilon)\right]\mathrm{d}t \\
&\quad + \partial_x(\sigma S(q_\varepsilon))\,\mathrm{d}W + \partial_x\sigma G_S(q_\varepsilon)\,\mathrm{d}W \\
&\quad + \frac{1}{2}S''(q_\varepsilon)\Big((\partial_x(\sigma q_\varepsilon))^2 - \big(\partial_x(\sigma q)\star J_\varepsilon\big)^2\Big)\,\mathrm{d}t - S'(q_\varepsilon)(r_\varepsilon\,\mathrm{d}t + \rho_\varepsilon\,\mathrm{d}t + \tilde{r}_\varepsilon\,\mathrm{d}W),
\end{aligned}
\tag{B.23}
$$

where we have used

$$
\partial_x(\sigma q)\star J_\varepsilon = \partial_x(\sigma q_\varepsilon) - \tilde{r}_\varepsilon.
$$

This puts most terms of the equation in divergence form and also sets up the mollification term ready for an application of Lemma B.3.

2. The L^2-bound.

The L^2-bound follows directly from the requirement (2.5) of Definition 2.3 for conservative weak solutions.

We show that the weak energy balance (2.8) holds for weak dissipative solutions, from which shall follow the L^2-bound (2.12).

We can estimate $\|q(t)\|_{L_x^2}^2$ using the entropies:

$$
S(v) = S_\ell(v) := \begin{cases} v^2, & |v| \le \ell, \\ 2\ell|v| - \ell^2, & |v| > \ell. \end{cases}
$$

This ensures that S_ℓ has bounded first and second derivatives for $\ell < \infty$, and allows us to exploit the convergences in $\varepsilon \to 0$ of r_ε, ρ_ε, and $\tilde{r}_\varepsilon$ proven in Lemma B.3. In particular,

$$S'_\ell(v) = \begin{cases} 2v, & |v| \le \ell, \\ 2\ell\,\mathrm{sgn}(v), & |v| > \ell, \end{cases} \qquad S''(v) = 2\mathbb{1}_{\{|v|\le\ell\}}.$$

Furthermore, we have

$$G_S(v) = vS'_\ell(v) - S_\ell(v) = v^2 \wedge \ell^2, \qquad G'_S(v) = vS''_\ell(v) = 2v\mathbb{1}_{\{|v|\le\ell\}},$$

and

$$q_\varepsilon S_\ell(q_\varepsilon) - \frac{1}{2}q_\varepsilon^2 S'(q_\varepsilon) = \ell\, q_\varepsilon(|q_\varepsilon| - \ell)\mathbb{1}_{\{|q_\varepsilon|>\ell\}}.$$

Inserting these into (B.23) and integrating in x and s, we are left with

$$\begin{aligned} 0 = &\int_{\mathbb{R}} S(q_\varepsilon)\,\mathrm{d}x\Big|_0^t - \int_0^t\int_{\mathbb{R}} \ell\, q_\varepsilon(|q_\varepsilon|-\ell)\mathbb{1}_{\{|q_\varepsilon|>\ell\}}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{R}} \partial_x\sigma\, G_S(q_\varepsilon)\,\mathrm{d}x\,\mathrm{d}W \\ &- \int_0^t\int_{\mathbb{R}} q_\varepsilon^2\big((\partial_x\sigma)^2 - \frac{1}{4}\partial_{xx}^2\sigma^2\big)\mathbb{1}_{\{|q_\varepsilon|\le\ell\}}\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{4}\int_0^t\int_{\mathbb{R}} \ell^2\partial_{xx}^2\sigma^2\mathbb{1}_{\{|q_\varepsilon|>\ell\}}\,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_0^t\int_{\mathbb{R}} \mathbb{1}_{\{|q_\varepsilon|\le\ell\}}\Big((\partial_x(\sigma q_\varepsilon))^2 - \big(\partial_x(\sigma q)\star J_\varepsilon\big)^2\Big)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_{\mathbb{R}} S'_\ell(q_\varepsilon)\rho_\varepsilon\,\mathrm{d}x\,\mathrm{d}s \\ &- \int_{\mathbb{R}}\int_0^t S'_\ell(q_\varepsilon)(r_\varepsilon\,\mathrm{d}t + \tilde{r}_\varepsilon\,\mathrm{d}W)\,\mathrm{d}x. \end{aligned} \tag{B.24}$$

We provide further bounds for the terms

$$\int_0^t\int_{\mathbb{R}} \ell\, q_\varepsilon(|q_\varepsilon|-\ell)\mathbb{1}_{\{|q_\varepsilon|>\ell\}}\,\mathrm{d}x\,\mathrm{d}s, \tag{B.25}$$

and

$$\frac{1}{4}\int_0^t\int_{\mathbb{R}} \ell^2\partial_{xx}^2\sigma^2\mathbb{1}_{\{|q_\varepsilon|>\ell\}}\,\mathrm{d}x\,\mathrm{d}s, \tag{B.26}$$

which cannot immediately be dealt with by Gronwall's inequality.

By splitting q_ε into positive and negative parts of essentially disjoint support, i.e., $q_\varepsilon = q_\varepsilon^+ + q_\varepsilon^-$ so that $q_\varepsilon^- \le 0 \le q_\varepsilon^+$, we see that

$$(\text{B.25}) = \int_0^t \int_{\mathbb{R}} \ell(q_\varepsilon^+ + q_\varepsilon^-)(|q_\varepsilon| - \ell)\mathbb{1}_{\{|q_\varepsilon|>\ell\}}\,\mathrm{d}x\,\mathrm{d}s \le \int_0^t \int_{\mathbb{R}} \ell q_\varepsilon^+(|q_\varepsilon| - \ell)\mathbb{1}_{\{|q_\varepsilon|>\ell\}}\,\mathrm{d}x\,\mathrm{d}s.$$

We shall be taking the limits in the order $\varepsilon \to 0$ first and then $\ell \to \infty$ later. Using the upper-boundedness of weak dissipative equations mandated in Definition 2.6, we can take $\varepsilon \to 0$ and conclude that there is always a sufficiently large ℓ beyond which the term $(|q_\varepsilon| - \ell)\mathbb{1}_{\{|q_\varepsilon|>\ell\}}$ simply vanishes.

Secondly, by Markov's inequality,

$$|(\text{B.26})| \le \frac{\|\partial_{xx}^2 \sigma^2\|_{L^\infty}}{4} \int_0^t \int_{\mathbb{R}} |q_\varepsilon|^2 \mathbb{1}_{\{|q_\varepsilon|>\ell\}}\,\mathrm{d}x\,\mathrm{d}s.$$

Finally, by Lemma B.3, equations (B.8), (B.9), and (B.22), the last two lines of (B.24) are bounded by $C_{\varepsilon,T}$, where $C_{\varepsilon,T} \to 0$ as $\varepsilon \to 0$. This means all terms can either be handled by Gronwall's inequality or are bounded. First integrating against $\mathrm{d}\mathbb{P}$, we then take the limits $\varepsilon \to 0$ and $\ell \to \infty$ and use Fatou's lemma in order to get the limit energy inequality

$$\mathbb{E} \int_{\mathbb{R}} |q|^2\,\mathrm{d}x \bigg|_0^t \le \mathbb{E} \int_0^t \int_{\mathbb{R}} q^2 \Big((\partial_x \sigma)^2 - \frac{1}{4} \partial_{xx}^2 \sigma^2 \Big)\,\mathrm{d}x\,\mathrm{d}s \tag{B.27}$$

for almost every $t \in [0, T]$.

3. The $L^{2+\alpha}$-bound.

For the $L^{2+\alpha}$-bound, with $\alpha \in [0, 1)$, we use the entropies S_ℓ defined by

$$S(v) = S_\ell(v) := \begin{cases} \frac{1}{2}\alpha \ell^{2-\alpha} v^3 + \frac{1}{2}(2-\alpha)\ell^{-\alpha} v, & |v| \le \ell^{-1}, \\ v|v|^\alpha, & \ell^{-1} < |v| \le \ell, \\ (1+\alpha) v \ell^\alpha - \alpha \ell^{1+\alpha} \operatorname{sgn}(v), & |v| > \ell. \end{cases}$$

In this way,

$$S_\ell'(v) = \begin{cases} \frac{3}{2}\alpha \ell^{2-\alpha} v^2 + \frac{1}{2}(2-\alpha)\ell^{-\alpha}, & |v| \le \ell^{-1}, \\ (1+\alpha)|v|^\alpha, & \ell^{-1} < |v| \le \ell, \\ (1+\alpha)\ell^\alpha, & |v| > \ell, \end{cases}$$

and

$$S_\ell''(v) = \begin{cases} 3\alpha \ell^{2-\alpha} v, & |v| \le \ell^{-1}, \\ \alpha(1+\alpha)|v|^{\alpha-1} \operatorname{sgn}(v), & \ell^{-1} < |v| \le \ell, \\ 0, & |v| > \ell. \end{cases}$$

The values for $S_\ell(v)$ in the interval $[-\ell^{-1}, \ell^{-1}]$ are the Hermite interpolation polynomial, matching the values and first derivatives of $v|v|^\alpha$ at the end-points $v = \pm\ell^{-1}$, so that S_ℓ' and S_ℓ'' stay bounded for fixed ℓ, as we require them to do.

Using these to compute $G_S(v) := vS'_\ell(v) - S_\ell(v)$ and its derivatives, we find

$$G_S(v) = \begin{cases} \alpha\ell^{2-\alpha}v^3, & |v| \le \ell^{-1}, \\ \alpha v|v|^\alpha, & \ell^{-1} < |v| \le \ell, \\ \alpha\ell^{1+\alpha}\operatorname{sgn}(v), & |v| > \ell, \end{cases}$$

and

$$\begin{aligned} G'_S(v) := vS''_\ell(v) &= \begin{cases} 3\alpha\ell^{2-\alpha}v^2, & |v| \le \ell^{-1}, \\ \alpha(1+\alpha)|v|^\alpha, & \ell^{-1} < |v| \le \ell, \\ 0, & |v| > \ell, \end{cases} \\ &\le 3\alpha|v|^\alpha. \end{aligned}$$

Moreover,

$$vS_\ell(v) - \frac{1}{2}v^2S'_\ell(v) = \begin{cases} -\frac{1}{4}\alpha\ell^{2-\alpha}v^4 + \frac{1}{4}(2-\alpha)\ell^{-\alpha}v^2, & |v| \le \ell^{-1}, \\ \frac{1}{2}(1-\alpha)|v|^{\alpha+2}, & \ell^{-1} < |v| \le \ell, \\ \frac{1}{2}(1+\alpha)v^2\ell^\alpha - \alpha\ell^{1+\alpha}|v|, & |v| > \ell. \end{cases}$$

Clearly, $S_\ell(v) \to v|v|^\alpha$ and $G_S(v) \to \alpha v|v|^\alpha$ as $\ell \to \infty$.

We can re-arrange (B.23), and integrate in x and s to get

$$\begin{aligned} &\mathbb{E}\int_0^t\int_{\mathbb{R}} \left(q_\varepsilon S(q_\varepsilon) - \frac{1}{2}q_\varepsilon^2 S'(q_\varepsilon)\right) \mathrm{d}x\,\mathrm{d}s \\ &\quad\le \mathbb{E}\int_{\mathbb{R}} S(q_\varepsilon)\,\mathrm{d}x\Big|_0^t - \mathbb{E}\int_0^t\int_{\mathbb{R}} \left[\frac{1}{2}q_\varepsilon G'_S(q_\varepsilon)(\partial_x\sigma)^2 - \frac{1}{4}\partial_{xx}^2\sigma^2 G_S(q_\varepsilon)\right]\mathrm{d}x\,\mathrm{d}s \\ &\qquad + \frac{1}{2}\int_0^t\int_{\mathbb{R}} S''(q_\varepsilon)\Big((\partial_x(\sigma q_\varepsilon))^2 - \big(\partial_x(\sigma q)\star J_\varepsilon\big)^2\Big)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_{\mathbb{R}} S'(q_\varepsilon)\rho_\varepsilon\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad - \int_{\mathbb{R}}\int_0^t S'(q_\varepsilon)(r_\varepsilon\,\mathrm{d}s + \tilde{r}_\varepsilon\,\mathrm{d}W)\,\mathrm{d}x, \end{aligned}$$

and insert the definitions of S_ℓ and G_S, and their derivatives.

By inspection, S'_ℓ and S''_ℓ are uniformly bounded on $\mathbb{R}$ for fixed ℓ, so again by Lemma B.3, and Eqs. (B.8), (B.9), and (B.22), the last two lines of (B.24) are bounded by $C_{\varepsilon,T}$, where $C_{\varepsilon,T} \to 0$ as $\varepsilon \to 0$. We then take the limits $\varepsilon \to 0$ and $\ell \to \infty$ and use Fatou's lemma in order to get the limit energy inequality

$$\frac{1}{2}(1-\alpha)\mathbb{E}\int_0^t\int_{\mathbb{R}}|q|^{2+\alpha}\,\mathrm{d}x\,\mathrm{d}s \le \mathbb{E}\int q|q|^\alpha\,\mathrm{d}x\Big|_0^t - \frac{1}{2}\alpha(\alpha+1)\,\mathbb{E}\int_0^t\int_{\mathbb{R}} q|q|^\alpha(\partial_x\sigma)^2\,\mathrm{d}x\,\mathrm{d}s$$
$$+\frac{\alpha}{4}\,\mathbb{E}\int_0^t\int_{\mathbb{R}}\partial_{xx}^2\sigma^2 q|q|^\alpha\,\mathrm{d}x\,\mathrm{d}s \tag{B.28}$$

for almost every $t\in[0,T]$. From the second part of this proof, we have the inclusions $q\in L^2(\Omega\times[0,T]\times\mathbb{R})$ and $q\in L^\infty([0,T];L^2(\Omega\times\mathbb{R}))$. This allows us to interpolate between $L^2_{t,x}$ and $L^1_{t,x}$ or between L^2_x and L^1_x to bound the integrals on the right, thereby allowing us to control $\mathbb{E}\|q\|^{2+\alpha}_{L^{2+\alpha}_{x,t}}$ as well.

This concludes the proof. $\square$

Appendix C. The defect measure in the deterministic setting

Here we construct explicit and easily verifiable solutions in the manner of [34] to a problem with step functions as the initial distribution, and show explicitly how blow-up and a defect measure recording that blow-up, arise. This is to complement the discussion on the defect measure in Section 1.1.

Let $[a,b]$ be evenly partitioned into n intervals, with endpoints $x_i=a+i(b-a)/n$ for $i=0,\dots,n$.

First we approximate q_0 by defining

$$V^n_{0,i}=\fint_{x_{i-1}}^{x_i} q_0(x)\,\mathrm{d}x,$$

and setting

$$q^n_0(x)=\sum_{i=1}^n V^n_{0,i}\mathbb{1}_{(x_{i-1},x_i)},\qquad q^n_0(b)=V^n_{0,n}.$$

Next we postulate the following characteristics:

$$X^n_i(t)=a+\frac{(b-a)}{4n}\sum_{j=1}^i(2+V^n_{0,j}t)^2\mathbb{1}_{\{t\ge 0:2+V^n_{0,j}t>0\}}. \tag{C.1}$$

Notice that two characteristics X^n_{i-1} and X^n_i coincide and remain coincident after $t=-2/V^n_{0,i}$ if $V^n_{0,i}<0$.

Setting

$$q^n(t,x)=\sum_{i=1}^n\frac{2V^n_{0,i}}{2+V^n_{0,i}t}\mathbb{1}_{\{X^n_{i-1}(t)<x<X^n_i(t),\,2+V^n_{0,i}t>0\}}, \tag{C.2}$$

$$u^n(t,x) = \int_{-\infty}^{x} q^n(t,y)\,\mathrm{d}y = \int_{X_0^n(t)}^{x} q^n(t,y)\,\mathrm{d}y, \tag{C.3}$$

we have by direct substitution of (C.3) in (C.1),

$$\mathrm{d}X_i^n(t) = u^n(t, X_i^n(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W.$$

For simplicity we set $q^n(t, X_i^n(t)) = 0$ on the (finitely many) characteristics X_i^n, thereby defining $q^n(t)$ pointwise, and so that from the definition, if and when two characteristics eventually meet, there is no mass concentrated along their coincident path. This is the defining feature of a dissipative solution — that L_x^2-mass is completely and eternally annihilated at wave-breaking — on which we shall expound further below.

From the definition of q^n in (C.2), we have

$$\partial_t q^n + u^n \partial_x(q^n) = -\frac{1}{2}(q^n)^2 \tag{C.4}$$

in the sense of distributions — to wit, from (C.2):

$$\begin{aligned}
\partial_t(q^n)(t,x) = \sum_{i=1}^{n} \Bigg[& \frac{-2(V_{0,i}^n)^2}{(2+V_{0,i}^n t)^2} \mathbb{1}_{\{X_{i-1}^n < x < X_i^n,\, 2+V_{0,i}^n t > 0\}} \\
& - \frac{2V_{0,i}^n}{2+V_{0,i}^n t} \mathbb{1}_{\{X_{i-1}^n(t) < x < X_i^n(t)\}} \delta(2+V_{0,i}^n t) \\
& - \frac{2V_{0,i}^n}{2+V_{0,i}^n t} \mathbb{1}_{\{2+V_{0,i}^n t > 0\}} \big(\delta(x - X_{i-1}^n) u^n(t, X_{i-1}^n) - \delta(x - X_i^n) u^n(t, X_i^n)\big) \Bigg],
\end{aligned} \tag{C.5}$$

$$\partial_x(q^n)(t,x) = \sum_{i=1}^{n} \frac{2V_{0,i}^n}{2+V_{0,i}^n t} \big(\delta(x - X_{i-1}^n) - \delta(x - X_i^n)\big) \mathbb{1}_{\{2+V_{0,i}^n t > 0\}}, \tag{C.6}$$

$$(q^n)^2(t,x) = \sum_{i=1}^{n} \left| \frac{2V_{0,i}^n}{2+V_{0,i}^n t} \right|^2 \mathbb{1}_{\{X_{i-1}^n < x < X_i^n,\, 2+V_{0,i}^n t > 0\}}. \tag{C.7}$$

The quantity

$$\frac{2V_{0,i}^n}{2+V_{0,i}^n t} \mathbb{1}_{\{X_{i-1}^n(t) < x < X_i^n(t)\}} \delta(2+V_{0,i}^n t)$$

in the equation for $\partial_t(q^n)(t,x)$ means

$$\lim_{\varepsilon \to 0} \frac{2V_{0,i}^n}{2+V_{0,i}^n t} \mathbb{1}_{\{X_{i-1}^n(t) < x < X_i^n(t)\}} \frac{1}{\varepsilon} \eta\Big(\frac{1}{\varepsilon}(t + 2/V_{0,i}^n + \varepsilon)\Big), \tag{C.8}$$

where η is a symmetric smooth bump of unit L^1-mass, supported on $[0,1]$. The limit is taken in the topology of distributions on $[0,T]\times\mathbb{R}$. We can interpret the expression thus, as differentiation is continuous in the topology of distributions. The limit evaluates to nought in the sense of distributions because $X_i^n(t)-X_{i-1}^n(t)$ is proportional to $(2+V_{0,i}^n t)^2$. Nevertheless a similar term is enormously important in the equation for $\partial_t(q^n)^2$ because dissipation arises from this term, which characterises dissipative solutions.

From the expression for the difference $X_i^n(t)-X_{i-1}^n(t)$ in (C.1), and as mentioned there, we see that the difference is zero for $2+V_{0,i}^n t\le 0$. Therefore by the expression for $(q^n)^2$, (C.7), we can compute that, $\mathbb{P}$-almost surely,

$$\int_{\mathbb{R}}(q^n)^2(t,x)\,\mathrm{d}x=\frac{b-a}{n}\sum_{i=1}^n \mathbb{1}_{\{t\ge 0:2+V_{0,i}^n t>0\}}(V_{0,i}^n)^2\le\frac{b-a}{n}\sum_{i=1}^n(V_{0,i}^n)^2=\int_{\mathbb{R}}(q_0^n(x))^2\,\mathrm{d}x, \tag{C.9}$$

a constant.

We can record the dissipation of $\|q^n(t)\|_{L^2(\mathbb{R})}^2$ as a defect measure:

$$\mathrm{m}^n(\mathrm{d}t,\mathrm{d}x)=\sum_{i=0}^n\frac{b-a}{n}(V_{0,i}^n)^2\delta(x-X_i^n(t))\delta(t+V_{0,i}^n/2)\,\mathrm{d}x\,\mathrm{d}t. \tag{C.10}$$

From this measure we see that dissipation gives rise to the admissibility condition in [20, Definition 2.2],

$$\partial_t(q^n)^2+\partial_x(u^n(q^n)^2)=-\frac{\mathrm{m}^n(\mathrm{d}t,\mathrm{d}x)}{\mathrm{d}t\,\mathrm{d}x}\le 0.$$

We carry out this computation explicitly below:

$$\begin{aligned}\partial_t(q^n)^2=\sum_{i=1}^n\Big[&\Big|\frac{2V_{0,i}^n}{2+V_{0,i}^n t}\Big|^2\big(\delta(x-X_i^n)u(X_i^n)-\delta(x-X_{i-1}^n)u(X_{i-1}^n)\big)\mathbb{1}_{\{2+V_{0,i}^n t>0\}}\\&-\left(\frac{2V_{0,i}^n}{2+V_{0,i}^n t}\right)^3\mathbb{1}_{\{X_{i-1}^n<x<X_i^n,\,2+V_{0,i}^n t>0\}}\\&-\frac{(2V_{0,i}^n)^2}{(2+V_{0,i}^n t)^2}\mathbb{1}_{\{X_{i-1}^n(t)<x<X_i^n(t)\}}\delta(2+V_{0,i}^n t)\Big],\end{aligned}$$

$$\begin{aligned}\partial_x(u^n(q^n)^2)&=u^n\partial_x(q^n)^2+(q^n)^3\\&=u^n(x)\sum_{i=1}^n\Big[\Big|\frac{2V_{0,i}^n}{2+V_{0,i}^n t}\Big|^2\big(\delta(x-X_{i-1}^n)-\delta(x-X_i^n)\big)\\&\qquad+\left(\frac{2V_{0,i}^n}{2+V_{0,i}^n t}\right)^3\mathbb{1}_{\{X_{i-1}^n<x<X_i^n,\,2+V_{0,i}^n t>0\}}\Big].\end{aligned}$$

Therefore again with due consideration for the difference $X_i^n(t) - X_{i-1}^n(t) = (2 + V_{i,0}^n t)^2(b - a)/4n$,

$$\partial_t (q^n)^2 + \partial_x (u^n (q^n)^2) = -\sum_{i=1}^{n} \frac{b-a}{n} (V_{0,i}^n)^2 \delta(x - X_i^n(t)) \delta(2 + V_{0,i}^n t),$$

where we understand the expression

$$\frac{(2V_{0,i}^n)^2}{(2 + V_{0,i}^n t)^2} \mathbb{1}_{\{X_{i-1}^n(t) < x < X_i^n(t)\}} \delta(2 + V_{0,i}^n t)$$

as in (C.8) above.

The times at which (L_x^2-)mass is released from this defect measure and returned to the solution, with a necessary corresponding determination of how characteristics $X_i^n(t)$ are to be continued past $\{t : 2 + V_{0,i}^n t > 0\}$, determines the types of solution one seeks. When it is never returned (when the indicator function in (C.10) attains unity for all t sufficiently great), the solutions are "dissipative"; when the measure only retains mass instantaneously, as in [3] in for the similar Camassa–Holm equation, solutions are "conservative". There are uncountably many possibilities between these extremes.

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