Controllability analysis of planar snake robots influenced by viscous ground friction

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Abstract— **This paper investigates the controllability properties of planar snake robots influenced by viscous ground friction forces. The paper provides three contributions: 1) A partially feedback linearized model of a planar snake robot is developed. 2) A stabilizability analysis is presented proving that any asymptotically stabilizable control law for a planar snake robot to an equilibrium point must be *time-varying*. 3) A controllability analysis is presented proving that planar snake robots are *not controllable* when the viscous ground friction is *uniform*, but that a snake robot becomes *strongly accessible* when the viscous ground friction is *non-uniform*. The analysis also shows that the snake robot does *not* satisfy sufficient conditions for *small-time local controllability* (STLC).**

I. Introduction

Inspired by biological snakes, snake robots carry the potential of meeting the growing need for robotic mobility in challenging environments. Snake robots consist of serially connected modules capable of bending in one or more planes. The many degrees of freedom of snake robots make them difficult to control, but provides traversability in irregular environments that surpasses the mobility of the more conventional wheeled, tracked and legged forms of robotic mobility. Research on snake robots have been conducted for several decades. However, our understanding of snake locomotion so far is for the most part based on empirical studies of biological snakes and simulation-based synthesis of relationships between parameters of the snake robot. This paper is an attempt to contribute to the understanding of snake robots by employing well-established system analysis tools for investigating fundamental properties of their dynamics.

There are several reported works aimed at analysing and understanding snake locomotion. Gray [1] conducted empirical and analytical studies of snake locomotion already in the 1940s. Hirose [2] studied biological snakes and developed mathematical relationships characterizing their motion, such as the *serpenoid curve*. Ostrowski [3] considered a particular wheeled snake robot developed by Hirose and studied its controllability properties. The results are, however, not very relevant to the results of this paper since the snake robot is wheeled and since the analysis was performed on a pure kinematic level. Several models of wheel-less snake robots influenced by ground friction have been developed [4]–[10]. However, no formal controllability analysis of snake locomotion is reported in any of these works. This is also the case for the work by Nilsson [11], where energy arguments are employed to analyse planar snake locomotion influenced by Coulomb friction. The result is, however, restricted to one specific motion pattern of a snake robot. The work in [12] presents a feedback linearized model of the *joint* motion of a snake robot and studies the controllability of the *joints* under the assumption that one joint is passive. However, the analysis does not consider the controllability of the *position* of the snake robot. There are many reported works on control of robotic fish and eel-like mechanisms [13]–[15]. Research on these mechanisms is very relevant to land-based snake robots. The work in [13] is particularly interesting as it investigates the controllability of a robotic fish by employing mathematical tools also employed in this paper. The result is, unfortunately, not directly applicable to land-based snake robots due to some fundamental model differences.

This paper provides three contributions. The first contribution is a partially feedback linearized model of a planar snake robot that builds on a model previously presented in [10]. This approach resembles the work in [12]. However, the feedback linearized model in [12] does not include the position of the snake robot, which is a key ingredient in this paper. The second contribution is a stabilizability analysis for planar snake robots that proves that any asymptotically stabilizable control law for a planar snake robot to an equilibrium point must be *time-varying*, i.e. not of pure-state feedback type (see Theorem 1). This result is valid regardless of which type of friction the snake robot is subjected to. Finally, the third contribution is a controllability analysis for planar snake robots influenced by viscous ground friction forces. The analysis shows that a snake robot is *not* controllable when the viscous ground friction is *uniform* (see Theorem 5), but that a snake robot becomes *strongly accessible* when the viscous ground friction is *non-uniform* (see Theorem 6). The analysis also shows that the snake robot does *not* satisfy sufficient conditions for *small-time local controllability* (see Theorem 9). To the authors' best knowledge, no formal controllability analysis has previously been reported for the position and link angles of a locomoting snake robot influenced by ground friction. Note that the work in [12] studies the controllability of the joints of a snake robot under the assumption that one joint is passive. However, the analysis does not consider the position of the snake.

The paper is organized as follows. Section II gives a brief introduction to a snake robot model previously presented in [10]. Section III converts the model from [10] to a simpler form through partial feedback linearization. Section IV studies stabilizability properties of planar snake robots. Section V presents a controllability analysis of planar snake robots. Finally, Section VI presents concluding remarks.

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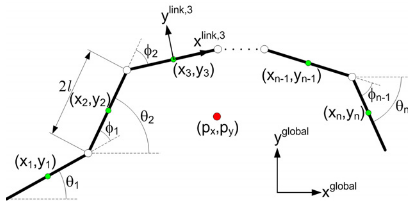

Fig. 1. Kinematic parameters for the snake robot.

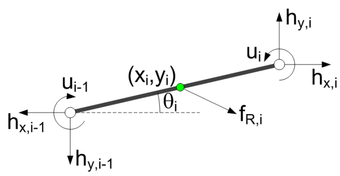

Fig. 2. Forces and torques acting on each link of the snake robot.

II. A MODEL OF THE SNAKE ROBOT

This section gives a brief introduction to a mathematical model of a planar snake robot previously presented in [10]. A feedback linearized form of this snake robot model is developed in Section III in order to simplify the controllability analysis presented in Section V.

A. Notations and defined identities

The snake robot consists of n links of length $2l$ interconnected by $n-1$ joints. The mathematical identities defined in order to describe the kinematics and dynamics of the snake robot are described in Table I and illustrated in Fig. 1 and Fig. 2. All n links have the same length, mass, and moment of inertia. The total mass of the snake robot is therefore nm. The mass of each link is uniformly distributed so that the link CM (center of mass) is located at its center point (at length l from the joint at each side).

The following vectors and matrices are used in the subsequent sections:

Symbol	Description	Associated vector
θ_i	Angle between link i and global x axis.	$\theta \in \mathbb{R}^n$
(x_i, y_i)	Global coordinates of CM of link i.	$x, y \in \mathbb{R}^n$
(p_x, p_y)	Global coordinates of CM of snake robot.	$p \in \mathbb{R}^2$
u_i	Actuator torque exerted on link i from link $i+1$.	$u \in \mathbb{R}^{n-1}$
u_{i-1}	Actuator torque exerted on link i from link $i-1$.	$u \in \mathbb{R}^{n-1}$
$f_{R,x,i}$	Friction force on link i in x direction.	$f_{R,x} \in \mathbb{R}^n$
$f_{R,y,i}$	Friction force on link i in y direction.	$f_{R,y} \in \mathbb{R}^n$
$h_{x,i}$	Joint constraint force in x direction on link i from link $i+1$.	$h_x \in \mathbb{R}^{n-1}$
$h_{y,i}$	Joint constraint force in y direction on link i from link $i+1$.	$h_y \in \mathbb{R}^{n-1}$
$h_{x,i-1}$	Joint constraint force in x direction on link i from link $i-1$.	$h_x \in \mathbb{R}^{n-1}$
$h_{y,i-1}$	Joint constraint force in y direction on link i from link $i-1$.	$h_y \in \mathbb{R}^{n-1}$

TABLE I
DEFINED MATHEMATICAL IDENTITIES.

$$A := \begin{bmatrix} 1 & 1 & & \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ & & 1 & 1 \end{bmatrix}, D := \begin{bmatrix} 1 & -1 & & \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ & & 1 & -1 \end{bmatrix}$$

where $A \in \mathbb{R}^{(n-1)\times n}$ and $D \in \mathbb{R}^{(n-1)\times n}$. Furthermore,

$$e := \begin{bmatrix} 1 & . & . & 1 \end{bmatrix}^T \in \mathbb{R}^n \quad E = \begin{bmatrix} e & 0_{n\times 1} \\ 0_{n\times 1} & e \end{bmatrix} \in \mathbb{R}^{2n\times 2}$$

$$\sin\theta := \begin{bmatrix} \sin\theta_1 \,..\, \sin\theta_n \end{bmatrix}^T \in \mathbb{R}^n \ \cos\theta := \begin{bmatrix} \cos\theta_1 \,..\, \cos\theta_n \end{bmatrix}^T \in \mathbb{R}^n$$

$$S_\theta := \text{diag}(\sin\theta_1, .., \sin\theta_n) \quad C_\theta := \text{diag}(\cos\theta_1, .., \cos\theta_n)$$

B. Kinematics

The snake robot moves in the horizontal plane and has a total of $n+2$ degrees of freedom. The absolute angle, θ_i, of link i is expressed with respect to the global x axis with counterclockwise positive direction. As seen in Fig. 1, the relative angle between link i and link $i+1$ is given by $\phi_i = \theta_i - \theta_{i+1}$. The local coordinate system of each link is fixed in the CM (center of mass) of the link with x (tangential) and y (normal) axis oriented such that they are oriented in the directions of the global x and y axis, respectively, when the link angle is zero. The rotation matrix from the global frame to the frame of link i is given by

$$R^{\text{global}}_{\text{link},i} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix} \quad (1)$$

The position of the snake robot, p, is described through the coordinates of its CM (center of mass) and is given by

$$p := \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} \frac{1}{nm}\sum_{i=1}^{n} m x_i \\ \frac{1}{nm}\sum_{i=1}^{n} m y_i \end{bmatrix} = \frac{1}{n}\begin{bmatrix} e^T x \\ e^T y \end{bmatrix} \quad (2)$$

It is shown in [10] that the position of the CM of each link along the global x and y axis, respectively, is given by

$$\begin{aligned} x &= -lN^T \cos\theta + ep_x \\ y &= -lN^T \sin\theta + ep_y \\ N &:= A^T \left(DD^T\right)^{-1} D \in \mathbb{R}^{n\times n} \end{aligned} \quad (3)$$

The linear velocities of the links are derived by differentiating (3). This gives

$$\begin{aligned} \dot{x} &= lN^T S_\theta \dot{\theta} + e\dot{p}_x \\ \dot{y} &= -lN^T C_\theta \dot{\theta} + e\dot{p}_y \end{aligned} \quad (4)$$

An expression for the velocity of a single link may be found by investigating the structure of each row in (4). The derivation is not included here due to space restrictions, but it may be verified that the linear velocity of the CM of link i in the global x and y direction is given by

$$\begin{aligned} \dot{x}_i &= \dot{p}_x - \sigma_i S_\theta \dot{\theta} \\ \dot{y}_i &= \dot{p}_y + \sigma_i C_\theta \dot{\theta} \end{aligned} \quad (5)$$

where

$$\begin{aligned} \sigma_i &= \begin{bmatrix} a_1 \ a_2 \ \ldots \ a_{i-1} \ \frac{a_i+b_i}{2} \ b_{i+1} \ b_{i+2} \ \ldots \ b_n \end{bmatrix} \in \mathbb{R}^n \\ a_i &= \frac{l(2i-1)}{n} \\ b_i &= \frac{l(2i-1-2n)}{n} \end{aligned} \quad (6)$$

C. Viscous friction model

In this paper, we analyse the controllability properties of the snake robot when it is influenced by viscous ground friction forces. In this section, we present the viscous friction model, and in particular we present models for the different cases of uniform versus non-uniform viscous friction.

1) Uniform viscous friction: The friction forces are assumed to act on the CM of the links only. The uniform viscous friction force on link i in the global x and y direction is proportional to the global velocity of the link and is written

$$\begin{aligned} f_{R,x,i} &= -c\dot{x}_i = -c\dot{p}_x + c\sigma_i S_\theta \dot{\theta} \\ f_{R,y,i} &= -c\dot{y}_i = -c\dot{p}_y - c\sigma_i C_\theta \dot{\theta} \end{aligned} \tag{7}$$

where c is the viscous friction coefficient and the expression for the link velocity is given by (5). The friction forces on all links may be expressed in matrix form as

$$f_R = \begin{bmatrix} f_{R,x} \\ f_{R,y} \end{bmatrix} = -c \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = -c \begin{bmatrix} lN^T S_\theta \dot{\theta} + e\dot{p}_x \\ -lN^T C_\theta \dot{\theta} + e\dot{p}_y \end{bmatrix} \tag{8}$$

where the expression for the link velocities is given by (4). We disregard the friction torque caused by a link rotating with respect to the ground since this torque only has a minor impact on the motion of the snake robot.

2) Non-uniform viscous friction: Under non-uniform friction conditions, a link has two viscous friction coefficients, c_t and c_n, describing the friction force in the tangential (along link x axis) and normal (along link y axis) direction of the link, respectively. Using (1), the friction force on link i in the global frame as a function of the global link velocity, $\dot{x}_i$ and $\dot{y}_i$, is given by

$$\begin{aligned} f_{R,i}^{\text{global}} &= R_{\text{link},i}^{\text{global}} f_{R,i}^{\text{link},i} = -R_{\text{link},i}^{\text{global}} \begin{bmatrix} c_t & 0 \\ 0 & c_n \end{bmatrix} v_i^{\text{link},i} \\ &= -R_{\text{link},i}^{\text{global}} \begin{bmatrix} c_t & 0 \\ 0 & c_n \end{bmatrix} \left(R_{\text{link},i}^{\text{global}} \right)^T \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \end{bmatrix} \end{aligned} \tag{9}$$

where $f_{R,i}^{\text{link},i}$ and $v_i^{\text{link},i}$ are, respectively, the friction force and the link velocity expressed in the local link frame. Performing the matrix multiplication and assembling the friction forces on all links in matrix form gives

$$f_R = -\begin{bmatrix} c_t \left(C_\theta\right)^2 + c_n \left(S_\theta\right)^2 & \left(c_t - c_n\right) S_\theta C_\theta \\ \left(c_t - c_n\right) S_\theta C_\theta & c_t \left(S_\theta\right)^2 + c_n \left(C_\theta\right)^2 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \tag{10}$$

where $f_R = \begin{bmatrix} f_{R,x}^T & f_{R,y}^T \end{bmatrix}^T \in \mathbb{R}^{2n}$. Note that in the case of uniform friction ($c_t = c_n = c$), the expression for the friction forces is reduced to (8).

D. Equations of motion

This section presents the equations of motion of the snake robot in terms of the acceleration of the link angles, $\ddot{\theta}$, and the acceleration of the CM of the snake robot, $\ddot{p}$. These coordinates describe all $n+2$ DOFs of the snake robot.

The forces and torques acting on link i are visualized in Fig. 2. The force balance for link i in global frame coordinates is given by

$$\begin{aligned} m\ddot{x}_i &= f_{R,x,i} + h_{x,i} - h_{x,i-1} \\ m\ddot{y}_i &= f_{R,y,i} + h_{y,i} - h_{y,i-1} \end{aligned} \tag{11}$$

while the torque balance for link i is given by

$$\begin{aligned} J\ddot{\theta}_i &= u_i - u_{i-1} \\ -l\sin\theta_i&(h_{x,i} + h_{x,i-1}) + l\cos\theta_i(h_{y,i} + h_{y,i-1}) \end{aligned} \tag{12}$$

Through straightforward calculations, it is shown in [10] that (11) and (12) may be rewritten for all links and combined into the following complete model of the snake robot:

$$\begin{aligned} M\ddot{\theta} + W\dot{\theta}^2 - lS_\theta N f_{R,x} + lC_\theta N f_{R,y} &= D^T u \\ nm\ddot{p} &= E^T f_R \end{aligned} \tag{13}$$

where θ and p represent the $n+2$ generalized coordinates of the system, $\dot{\theta}^2 = \text{diag}(\dot{\theta})\dot{\theta}$, and

$$\begin{aligned} M &:= JI_{n\times n} + ml^2 \left(S_\theta V S_\theta + C_\theta V C_\theta\right) \\ W &:= ml^2 \left(S_\theta V C_\theta - C_\theta V S_\theta\right) \\ N &:= A^T \left(DD^T\right)^{-1} D \\ V &:= A^T \left(DD^T\right)^{-1} A \end{aligned} \tag{14}$$

III. Partial feedback linearization of the model

This section transformes the model from [10], which was summarized in the previous section, to a simpler form through partial feedback linearization. This conversion greatly simplifies the controllability analysis presented in Section V. Partial feedback linearization of underactuated systems was introduced in [16] and consists of linearizing the dynamics corresponding to the actuated degrees of freedom of the system. In this section, we show how a change of coordinates makes it possible to employ this methodology by following the approach presented in [17].

A. Partitioning the model into an actuated and an unactuated part

Before partial feedback linearization can be carried out, the model of the snake robot in (13) must be partitioned into two parts representing the actuated and unactuated degrees of freedom, respectively [17]. The acceleration of the CM of the snake robot, $\ddot{p}$, belongs to the unactuated part since it is not directly influenced by the input, u. The acceleration of the link angles, $\ddot{\theta}$, represent one unactuated degree of freedom and $n-1$ actuated degrees of freedom since there are n link accelerations ($\theta \in \mathbb{R}^n$) and only $n-1$ control inputs ($u \in \mathbb{R}^{n-1}$). However, it is not possible to partition the equation for $\ddot{\theta}$ in (13) into an actuated and an unactuated part since the matrix D^T in front of the control input gives a direct influence between u and all the link accelerations. We therefore seek a form of the model where there is a direct influence between u and only $n-1$ link accelerations. This is achieved by modifying the choice of generalized coordinates from absolute link angles to relative joint angles. The generalized coordinates of the model in (13) are given by the absolute link angles, θ, and the CM position of the snake robot, p. We now replace these coordinates with

$$q = \begin{bmatrix} \phi \\ p \end{bmatrix} \in \mathbb{R}^{n+2} \tag{15}$$

where

$$\phi = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{n-1} & \theta_n \end{bmatrix}^T \in \mathbb{R}^n \tag{16}$$

contains the $n-1$ relative joint angles of the snake robot and the absolute link angle, $\theta_n \in \mathbb{R}$, of the head link. The relative joint angles are defined in Fig. 1. The coordinate transformation between absolute link angles and relative joint angles is easily shown to be given by

$$\theta = R\phi \tag{17}$$

$$R = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{18}$$

The dynamic model in the new coordinates is found by inserting (17) into (13). This gives

$$\begin{aligned} MR\ddot{\phi} + W\,\mathrm{diag}\left(R\dot{\phi}\right)R\dot{\phi} - lS_\theta N f_{R,x} \\ +lC_\theta N f_{R,y} = D^T u \\ nm\ddot{p} = E^T f_R \end{aligned} \tag{19}$$

where we have used that $\dot{\theta}^2 = \mathrm{diag}(\dot{\theta})\dot{\theta} = \mathrm{diag}\left(R\dot{\phi}\right)R\dot{\phi}$. Finally, we premultiply the first matrix equation in (19) with R^T in order to achieve the desired form of the input mapping matrix on the right-hand side by making the last of the n equations independent of the control input. This enables us to write the complete model of the snake robot as

$$\overline{M}(\phi)\ddot{q} + \overline{W}\left(\phi,\dot{\phi}\right) + \overline{G}(\phi) f_R\left(\phi,\dot{\phi},\dot{p}\right) = \overline{B}u \tag{20}$$

where

$$q = \begin{bmatrix}\phi \\ p\end{bmatrix} \in \mathbb{R}^{n+2} \tag{21}$$

$$\overline{M}(\phi) = \begin{bmatrix} R^T M(\phi) R & 0_{n\times 2} \\ 0_{2\times n} & nmI_2 \end{bmatrix} \tag{22}$$

$$\overline{W}\left(\phi,\dot{\phi}\right) = \begin{bmatrix} R^T W(\phi)\,\mathrm{diag}\left(R\dot{\phi}\right)R\dot{\phi} \\ 0_{2\times 1} \end{bmatrix} \tag{23}$$

$$\overline{G}(\phi) = \begin{bmatrix} -lR^T S_{R\phi} N & lR^T C_{R\phi} N \\ -e^T & 0_{1\times n} \\ 0_{1\times n} & -e^T \end{bmatrix} \tag{24}$$

$$\overline{B} = \begin{bmatrix} I_{n-1} \\ 0_{3\times n-1} \end{bmatrix} \tag{25}$$

and where $S_{R\phi} = S_\theta$ and $C_{R\phi} = C_\theta$. It is interesting to note that premultiplying the first matrix equation in (19) with R^T both causes the input mapping matrix to attain a desirable form and produces a symmetrical inertia matrix. Had we left the model in the form of (19), the inertia matrix would not have been symmetrical.

The first $n-1$ equations of (20) represent the dynamics of the relative joint angles of the snake robot, i.e. the *actuated* degrees of freedom of the snake robot. The last three equations represent the dynamics of the absolute orientation and position of the snake robot, i.e. the *unactuated* degrees of freedom. The model may therefore be partitioned as

$$\overline{M}_{11}\ddot{q}_a + \overline{M}_{12}\ddot{q}_u + \overline{W}_1 + \overline{G}_1 f_R = u \tag{26}$$

$$\overline{M}_{21}\ddot{q}_a + \overline{M}_{22}\ddot{q}_u + \overline{W}_2 + \overline{G}_2 f_R = 0_{3\times 1} \tag{27}$$

where $q_a = \begin{bmatrix}\phi_1 & \cdots & \phi_{n-1}\end{bmatrix}^T \in \mathbb{R}^{n-1}$ represents the actuated degrees of freedom, $q_u = \begin{bmatrix}\theta_n & p_x & p_y\end{bmatrix}^T \in \mathbb{R}^3$ represents the unactuated degrees of freedom, $\overline{M}_{11} \in \mathbb{R}^{n-1\times n-1}$, $\overline{M}_{12} \in \mathbb{R}^{n-1\times 3}$, $\overline{M}_{21} \in \mathbb{R}^{3\times n-1}$, $\overline{M}_{22} \in \mathbb{R}^{3\times 3}$, $\overline{W}_1 \in \mathbb{R}^{n-1}$, $\overline{W}_2 \in \mathbb{R}^3$, $\overline{G}_1 \in \mathbb{R}^{n-1\times 2n}$, and $\overline{G}_2 \in \mathbb{R}^{3\times 2n}$. Note that $\overline{M}(\phi)$ only depends on the relative joint angles of the snake robot and not on the absolute orientation of the head link, θ_n. Formally, this is a result of the fact that θ_n is a *cyclic* coordinate [18]. Less formally, this is quite obvious since it would not be reasonable that the inertial properties of a planar snake robot is dependent on how the snake robot is oriented in the plane. We therefore have that $\overline{M} = \overline{M}(q_a)$.

B. Partial feedback linearization

We are now ready to present an input transformation that linearizes the dynamics of the actuated degrees of freedom in (26). $\overline{M}_{22}$ is an invertible 3×3 matrix as a consequence of the uniform positive definiteness of the complete system inertia matrix, $\overline{M}(q_a)$. We may therefore solve (27) for $\ddot{q}_u$ as

$$\ddot{q}_u = -\overline{M}_{22}^{-1}\left(\overline{M}_{21}\ddot{q}_a + \overline{W}_2 + \overline{G}_2 f_R\right) \tag{28}$$

Inserting (28) into (26) gives

$$\begin{aligned} \left(\overline{M}_{11} - \overline{M}_{12}\overline{M}_{22}^{-1}\overline{M}_{21}\right)\ddot{q}_a + \overline{W}_1 + \overline{G}_1 f_R \\ -\overline{M}_{12}\overline{M}_{22}^{-1}\left(\overline{W}_2 + \overline{G}_2 f_R\right) = u \end{aligned} \tag{29}$$

Consequently, the following linearizing controller

$$\begin{aligned} u = \left(\overline{M}_{11} - \overline{M}_{12}\overline{M}_{22}^{-1}\overline{M}_{21}\right)v + \overline{W}_1 \\ +\overline{G}_1 f_R - \overline{M}_{12}\overline{M}_{22}^{-1}\left(\overline{W}_2 + \overline{G}_2 f_R\right) \end{aligned} \tag{30}$$

enables us to rewrite (26) and (27) as

$$\ddot{q}_a = v \tag{31}$$

$$\ddot{q}_u = \mathcal{A}(q,\dot{q}) + \mathcal{B}(q_a)v \tag{32}$$

where

$$\mathcal{A}(q,\dot{q}) = -\overline{M}_{22}^{-1}\left(\overline{W}_2 + \overline{G}_2 f_R\right) \in \mathbb{R}^3 \tag{33}$$

$$\mathcal{B}(q_a) = -\overline{M}_{22}^{-1}\overline{M}_{21} \in \mathbb{R}^{3\times n-1} \tag{34}$$

This model may be written in the standard form of a control-affine system by defining $x_1 = q_a$, $x_2 = q_u$, $x_3 = \dot{q}_a$, $x_4 = \dot{q}_u$, and $x = \begin{bmatrix}x_1^T & x_2^T & x_3^T & x_4^T\end{bmatrix}^T \in \mathbb{R}^{2n+4}$. This gives

$$\dot{x} = \begin{bmatrix} x_3 \\ x_4 \\ 0_{n-1\times 1} \\ \mathcal{A}(x) \end{bmatrix} + \begin{bmatrix} 0_{n-1\times n-1} \\ 0_{3\times n-1} \\ I_{n-1} \\ \mathcal{B}(x_1) \end{bmatrix} v = f(x) + \sum_{j=1}^{n-1} g_j(x_1) v_j \tag{35}$$

where

$$f(x) = \begin{bmatrix} x_3 \\ x_4 \\ 0_{n-1\times 1} \\ \mathcal{A}(x) \end{bmatrix}, g_j(x_1) = \begin{bmatrix} 0_{n-1\times 1} \\ 0_{3\times 1} \\ e_j \\ \mathcal{B}_j(x_1) \end{bmatrix} \tag{36}$$

and where e_j denotes the jth standard basis vector in $\mathbb{R}^{n-1}$ (the jth column of I_{n-1}) and $\mathcal{B}_j(x_1)$ denotes the jth column of $\mathcal{B}(x_1)$.

IV. Stabilizability properties of planar snake robots

This section presents and proves a fundamental theorem concerning the properties of an asymptotically stabilizable control law for snake robots to any equilibrium point. From (35), the model of the snake robot is given by

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ v \\ \mathcal{A}(x) + \mathcal{B}(x_1)v \end{bmatrix} \tag{37}$$

The equation (37) maps the state, x, and the controller input, v, of the snake robot into the resulting derivative of the state vector, $\dot{x}$. For any equilibrium point $(x_1 = x_1^e, x_2 = x_2^e, x_3 = 0, x_4 = 0)$, where (x_1^e, x_2^e) is the configuration of the system at the equilibrium point, we have that $\dot{x} = 0$.

A well-known result by Brockett [19] states that a necessary condition for the existence of a *time-invariant* (i.e. not explicitly dependent on time) *continuous* state feedback law, $v = v(x)$, that makes $(x_1^e, x_2^e, 0, 0)$ asymptotically stable, is that the image of the mapping $(x, v) \mapsto \dot{x}$ contains some neighbourhood of $\dot{x} = 0$. A result by Coron and Rosier [20] states that a control system that can be asymptotically stabilized (in the Filippov sense [20]) by a *time-invariant discontinuous* state feedback law can be asymptotically stabilized by a *continuous time-varying* state feedback law. If, moreover, the control system is *affine* (i.e. linear with respect to the control input), then it can be asymptotically stabilized by a *time-invariant continuous* state feedback law. We now employ these results to prove the following fundamental theorem for planar snake robots:

Theorem 1: An asymptotically stabilizable control law for a planar snake robot described by (37) to any equilibrium point must be time-varying, i.e. of the form $v = v(x, t)$.

Proof: The result by Brockett [19] states that the mapping $(x_1, x_2, x_3, x_4, v) \mapsto (x_3, x_4, v, \mathcal{A}(x) + \mathcal{B}(x_1) v)$ must map an arbitrary neighbourhood of $(x_1 = x_1^e, x_2 = x_2^e, x_3 = 0, x_4 = 0, v = 0)$ onto a neighbourhood of $(x_3 = 0, x_4 = 0, v = 0, \mathcal{A}(x) + \mathcal{B}(x_1) v = 0)$. For this to be true, points of the form $(x_3 = 0, x_4 = 0, v = 0, \mathcal{A}(x) + \mathcal{B}(x_1) v = \epsilon)$ must be contained in this mapping for some arbitrary $\epsilon \neq 0$ because points of this form are contained in every neighbourhood of $\dot{x} = 0$. However, these points do not exist for the system in (37) because $x_3 = 0$, $x_4 = 0$, and $v = 0$ means that $\mathcal{A}(x) + \mathcal{B}(x_1) v = 0 \neq \epsilon$. Hence, the snake robot *cannot* be asymptotically stabilized to $(x_1 = x_1^e, x_2 = x_2^e, x_3 = 0, x_4 = 0)$ by a *time-invariant continuous* state feedback law. Moreover, since the system in (37) is affine and *cannot* be asymptotically stabilized by a *time-invariant continuous* state feedback law, the result by Coron and Rosier [20] proves that the system can neither be asymptotically stabilized by a *time-invariant discontinuous* state feedback law. We can therefore conclude that an asymptotically stabilizable control law for a planar snake robot to any equilibrium point must be time-varying, i.e. of the form $v = v(x, t)$. ∎

Remark 2: Theorem 1 is independent of the choice of friction model and applies to any planar snake robot described by a friction model with the property that $\mathcal{A}(x^e) = 0$ for any equilibrium point x^e.

V. Controllability analysis of planar snake robots

This section studies the controllability of planar snake robots influenced by viscous ground friction forces. The first subsection presents a brief summary of selected tools for analyzing controllability of nonlinear systems. Most literary sources of this theory present the theory in the context of complex mathematical notations and terminologies. The summary given below is therefore formulated in an intuitive form that aims to be easily understandable, and thus hopefully can be a valuable contribution to readers that want an introduction to nonlinear controllability analysis.

A. Controllability concepts for nonlinear systems

Analyzing the controllability of a *linear* system is easy and involves a simple test (the *Kalman rank condition* [21]) on the constant system matrices. However, studying the controllability of a *nonlinear* system is far more complex and constitutes an active area of research. In the following, we summarize important controllability concepts for control-affine nonlinear systems, i.e. systems of the form

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x) v_j \ , \ x \in \mathbb{R}^n \ , \ v \in \mathbb{R}^m \qquad (38)$$

where the vector fields of the system are the drift vector field, $f(x)$, and the control vector fields, $g_j(x)$, $j \in [1, .., m]$.

A nonlinear system is said to be *controllable* if there exist admissible control inputs that will bring the system between two arbitrary states in finite time. However, conditions for this kind of controllability that are both necessary and sufficient do not exist. Nonlinear controllability is instead typically analyzed by investigating the local behaviour of the system near equilibrium points.

The simplest approach to studying controllability of a nonlinear system is to linearize the system about an equilibrium point, x^e. If the linearized system satisfies the Kalman rank condition at x^e, the nonlinear system is controllable in the sense that the set of states that can be reached from x^e contains a neighborhood of x^e [21]. Unfortunately, many underactuated systems do not have a controllable linearization. Moreover, a nonlinear system can be controllable even though its linearization is not.

A necessary (but not sufficient) condition for controllability from a state x_0 (not necessarily an equilibrium) is that the nonlinear system satisfies the *Lie algebra rank condition* (LARC), also called the *accessibility rank condition* [21]. If this is the case, the system is said to be *locally accessible* from x_0. This property means that the space that the system can reach within any time $T > 0$ is fully n-dimensional, i.e. the reachable space from x_0 has a dimension equal to the dimension of the state space. A slightly stronger property is *strong accessibility*, which means that the space that the system can reach in *exactly* time T for any $T > 0$ is fully n-dimensional.

Accessibility of a nonlinear system is investigated by computing the *accessibility algebra*, here denoted Δ, of the system. Computation of Δ requires knowledge of the *Lie bracket* [21], which is now briefly explained. The drift and control vector fields of the nonlinear system (38) indicate directions in which the state x can move. These directions will generally only span a subset of the complete state space. However, through combined motion along two or more of these vector fields, it is possible for the system to move in directions not spanned by the original system vector fields. The Lie bracket between two vector fields Y and Z produces a new vector field defined as $[Y, Z] = \frac{\partial Z}{\partial x} Y - \frac{\partial Y}{\partial x} Z$. When Y and Z are any of the system vector fields, the Lie bracket $[Y, Z]$ *approximates* the net motion produced when the system follows these two vector fields in an alternating fashion. The classical example is parallel parking with a car, where sideways motion of the car may be achieved through an alternating turning and forward/backward motion. Note that Lie brackets can be computed from other Lie brackets, thereby producing nested Lie brackets. The *accessibility algebra*, Δ, is a set of vector fields composed of the system vector fields, f and g_j, the Lie brackets between the system vector fields, and also higher order Lie brackets generated by nested Lie brackets. The LARC is satisfied at x_0 if the vector fields in $\Delta(x_0)$ span the entire n-dimensional state space ($\text{span}(\Delta) = n$). The following result is proved in [21]:

Theorem 3: The system (38) is *locally accessible* from x_0 if and only if the LARC is satisfied at x_0. The system is *locally strongly accessible* if the drift field f by itself (i.e. unbracketed) is not included in the *accessibility algebra*.

Accessibility does *not* imply controllability since it only infers conclusions on the dimension of the reachable space from x_0. Accessibility is, however, a necessary (but not sufficient) condition for *small-time local controllability* (STLC) [22]. STLC is desirable since it is in fact a stronger property than controllability. If a system is STLC, then the control input can steer the system in any direction in an arbitrarily small amount of time. For second-order systems, STLC is therefore only possible from equilibrium states since it is not possible for a second-order system to instantly move in one direction if it already has a velocity in the opposite direction. Only sufficient (not necessary) conditions for STLC exist.

Sussmann presented sufficient conditions for STLC in [22]. These results were later extended by Bianchini and Stefani [23]. We now summarize these conditions. For any Lie bracket term B generated from the system vector fields, define the θ-degree of B, denoted $\delta_\theta(B)$, and the l-degree of B, denoted $\delta_l(B)$, as

$$\delta_\theta(B) = \frac{1}{\theta}\delta^0(B) + \sum_{j=1}^{m}\delta^j(B) \ , \ \delta_l(B) = \sum_{j=0}^{m} l_j \delta^j(B) \quad (39)$$

respectively, where $\delta^0(B)$ is the number of times the drift vector field f appears in the bracket B, $\delta^j(B)$ is the number of times the control vector field g_j appears in the bracket B, θ is an arbitrary number satisfying $\theta \in [1, \infty)$, and l_j is an arbitrary number satisfying $l_j \geq l_0 \geq 0, \ \forall \ j \in \{0, .., m\}$. The bracket B is said to be *bad* if $\delta^0(B)$ is odd and $\delta^1(B), ..., \delta^m(B)$ are all even. A bracket is *good* if it is not bad. As an example, we have that the bracket $[g_j, [f, g_k]]$ is *bad* for $j = k$ and *good* for $j \neq k$. This classification is motivated by the fact that a bad bracket *may* have directional constraints. E.g. the drift vector f is *bad* because it only allows motion in its positive direction and not in its negative direction, $-f$. A bad bracket is said to be *θ-neutralized* (resp. *l-neutralized*) if it can be written as a linear combination of brackets of lower θ-degree (resp. l-degree). The *Sussmann condition* and the *Bianchini and Stefani condition* for STLC are now combined in the following theorem:

Theorem 4: The system (38) is *small-time locally controllable* (STLC) from an equilibrium point x^e ($f(x^e) = 0$) if the LARC is satisfied at x^e and either all *bad* brackets are *θ-neutralized* (Sussmann [22]) or all *bad* brackets are *l-neutralized* (Bianchini and Stefani [23]).

B. Controllability with uniform viscous friction

We begin the controllability analysis of the snake robot by first assuming that the viscous ground friction is uniform. In this case, it turns out that the equations of motion take on a particularly simple form that enables us to study controllability through pure inspection of the equations of motion. From (13) we have that the acceleration of the CM of the snake robot is given by

$$\begin{bmatrix} \ddot{p}_x \\ \ddot{p}_y \end{bmatrix} = \begin{bmatrix} \frac{1}{nm} e^T f_{R,x} \\ \frac{1}{nm} e^T f_{R,y} \end{bmatrix} = \frac{1}{nm} \begin{bmatrix} \sum\limits_{i=1}^{n} f_{R,x,i} \\ \sum\limits_{i=1}^{n} f_{R,y,i} \end{bmatrix} \quad (40)$$

Inserting (7) into (40) gives

$$\begin{bmatrix} \ddot{p}_x \\ \ddot{p}_y \end{bmatrix} = \frac{c}{nm} \begin{bmatrix} -n\dot{p}_x + \left(\sum\limits_{i=1}^{n}\sigma_i\right) S_\theta \dot{\theta} \\ -n\dot{p}_y - \left(\sum\limits_{i=1}^{n}\sigma_i\right) C_\theta \dot{\theta} \end{bmatrix} = -\frac{c}{m} \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \end{bmatrix} \quad (41)$$

because it may be shown that $\sum\limits_{i=1}^{n} \sigma_i = 0$. This enables us to state the following theorem:

Theorem 5: A planar snake robot influenced by *uniform* viscous ground friction is *not controllable*.

Proof: In order to control the position, the snake robot must accelerate its CM (center of mass). From (41) it is clear that the CM acceleration is proportional to the CM velocity. If the snake robot starts from rest ($\dot{p} = 0$), it is therefore impossible to achieve acceleration of the CM. The position of the snake robot is in other words completely uncontrollable in this case, which renders the snake robot uncontrollable. ■

C. Controllability with non-uniform viscous friction

The equations of motion of the snake robot in (35) become far more complex under non-uniform friction conditions. We therefore employ the controllability concepts presented in Section V-A and begin by computing the Lie brackets of the system vector fields. The drift and control vector fields of the snake robot are given in (36). As explained in Section V-A, Lie bracket computation involves partial derivatives of the components of the vector fields. These computations can be carried out without dealing with the complex expressions contained in $\mathcal{A}(x)$ and $\mathcal{B}(x_1)$ given by (33) and (34), respectively, since we only need to know which variables each row of the vector fields depend on. As an example, consider column j of $\mathcal{B}(x_1)$. Since we know that it only depends on x_1, we may immediately write $\frac{\partial \mathcal{B}_j(x_1)}{\partial x} = \begin{bmatrix} \frac{\partial \mathcal{B}_j(x_1)}{\partial x_1} & 0_{3\times n+5} \end{bmatrix}$. This methodology enables us to compute the following Lie brackets of the system vector fields (evaluated at an equilibrium point):

$$[f, g_j]^{\dot{q}=0} = \begin{bmatrix} -e_j \\ -\mathcal{B}_j \\ 0_{n-1\times 1} \\ -\mathcal{C}_j \end{bmatrix}, [f, [f, g_j]]^{\dot{q}=0} = \begin{bmatrix} 0_{n-1\times 1} \\ \mathcal{C}_j \\ 0_{n-1\times 1} \\ \frac{\partial \mathcal{A}}{\partial x_4}\mathcal{C}_j \end{bmatrix}$$
$$[[f, g_j], [f, g_k]]^{\dot{q}=0} = \begin{bmatrix} 0_{n-1\times 1} \\ \mathcal{D}_{jk} \\ 0_{n-1\times 1} \\ \mathcal{E}_{jk} \end{bmatrix} \quad (42)$$

where $j, k \in \{1, ..., n-1\}$ and

$$\mathcal{C}_j = \frac{\partial \mathcal{A}}{\partial x_3} e_j + \frac{\partial \mathcal{A}}{\partial x_4}\mathcal{B}_j \ , \ \mathcal{D}_{jk} = \frac{\partial \mathcal{B}_k}{\partial x_1} e_j - \frac{\partial \mathcal{B}_j}{\partial x_1} e_k$$
$$\mathcal{E}_{jk} = \frac{\partial \mathcal{C}_k}{\partial x_1} e_j - \frac{\partial \mathcal{C}_j}{\partial x_1} e_k + \frac{\partial \mathcal{C}_k}{\partial x_2}\mathcal{B}_j - \frac{\partial \mathcal{C}_j}{\partial x_2}\mathcal{B}_k + \frac{\partial \mathcal{C}_k}{\partial x_4}\mathcal{C}_j - \frac{\partial \mathcal{C}_j}{\partial x_4}\mathcal{C}_k \quad (43)$$

The Lie brackets have been evaluated at zero velocity ($\dot{q} = 0$) since we are interested in controllability from an equilibrium point. The above vector fields represent our choice of vector fields to be contained in the *accessibility algebra*, Δ, of the system. To construct Δ of full rank, we need to find $(2n+4)$ independent vector fields since the snake robot has a $(2n+4)$-dimensional state space. Each of the four types of vector fields above represent $(n-1)$ vector fields. Solving $4(n-1) \geq 2n+4$ gives that our analysis is only valid if the snake robot has $n \geq 4$ links. This is a mild requirement, however, since a snake robot generally will have more than four links. In the remainder of this section, we assume that the snake robot consists of exactly $n = 4$ links (and thereby $n-1 = 3$ active joints) and argue that the following controllability results will also be valid for snake robots with more links. In particular, a snake robot with $n > 4$ links can behave as a snake robot with $n = 4$ links by fixing $(n-4)$ joint angles at zero degrees and allowing the remaining three joint angles to move. This means that controllability of the snake robot with $n = 4$ is a sufficient although not necessary condition for controllability of snake robots with $n > 4$.

With $n = 4$ links, the system has a $(2n+4) = 12$-dimensional state space. The system satisfies the *Lie algebra rank condition* (LARC) if the above vector fields span a 12-dimensional space. We therefore assemble the 12 vector fields into the following matrix, which represents the *accessibility algebra* of the system evaluated at an equilibrium point x^e:

$$\begin{aligned}\Delta(x^e) &= [g_1, g_2, g_3, [f, g_1], [f, g_2], [f, g_3], \\ &[f,[f,g_1]], [f,[f,g_2]], [f,[f,g_3]], \\ &[[f,g_1],[f,g_2]], [[f,g_1],[f,g_3]], [[f,g_2],[f,g_3]]] \\ &= \begin{bmatrix} 0_{3\times3} & -I_3 & 0_{3\times3} & 0_{3\times1} \\ 0_{3\times3} & -\mathcal{B} & \mathcal{C} & \mathcal{D} \\ I_3 & 0_{3\times3} & 0_{3\times3} & 0_{3\times1} \\ \mathcal{B} & -\mathcal{C} & \frac{\partial\mathcal{A}}{\partial x_4}\mathcal{C} & \mathcal{E} \end{bmatrix} \in \mathbb{R}^{12\times12}\end{aligned} \tag{44}$$

where

$$\begin{aligned}\mathcal{C} &= \frac{\partial\mathcal{A}}{\partial x_3} + \frac{\partial\mathcal{A}}{\partial x_4}\mathcal{B} \in \mathbb{R}^{3\times3} \\ \mathcal{D} &= \begin{bmatrix}\mathcal{D}_{12} & \mathcal{D}_{13} & \mathcal{D}_{23}\end{bmatrix}, \mathcal{E} = \begin{bmatrix}\mathcal{E}_{12} & \mathcal{E}_{13} & \mathcal{E}_{23}\end{bmatrix}\end{aligned} \tag{45}$$

We now state the following theorem regarding the accessibility of the snake robot:

Theorem 6: A planar snake robot with $n \geq 4$ links influenced by non-uniform viscous ground friction ($c_t \neq c_n$) is *locally strongly accessible* from any equilibrium point x^e ($\dot{q} = 0$) satisfying $\det(\mathcal{C}) \neq 0$ and $\det\left(\mathcal{E} - \frac{\partial\mathcal{A}}{\partial x_4}\mathcal{D}\right) \neq 0$, where $\det(*)$ denotes the determinant evaluated at x^e.

Proof: By Theorem 3, the system is *locally strongly accessible* from x^e if $\Delta(x^e)$, given by (44), has full rank, i.e. spans a 12-dimensional space. The proof is complete if we can show that this is the case as long as $\det(\mathcal{C}) \neq 0$ and $\det\left(\mathcal{E} - \frac{\partial\mathcal{A}}{\partial x_4}\mathcal{D}\right) \neq 0$ at x^e. The matrix $\Delta(x^e)$ has full rank when all its columns are linearly independent. By investigating the particular structure of $\Delta(x^e)$, we see that the first and third row contains an identity matrix and then zeros in the remaining elements of these rows. It is therefore impossible to write the columns containing the two identity matrices as linear combinations of other columns. We can therefore conclude that any linear dependence between the columns of $\Delta(x^e)$ must be caused by linear dependence between the columns of the following submatrix of $\Delta(x^e)$:

$$\widetilde{\Delta}(x^e) = \begin{bmatrix} \mathcal{C} & \mathcal{D} \\ \frac{\partial\mathcal{A}}{\partial x_4}\mathcal{C} & \mathcal{E}\end{bmatrix} \in \mathbb{R}^{6\times6} \tag{46}$$

Linear dependence between columns of a square matrix causes its determinant to become zero. We therefore calculate the determinant of $\widetilde{\Delta}(x^e)$ by employing the following well-known mathematical relationship concerning the determinant of a block matrix (see e.g. [24]):

$$\det\left(\begin{bmatrix} A & B \\ C & D\end{bmatrix}\right) = \det(A)\det\left(D - CA^{-1}B\right) \tag{47}$$

where A and D are any square matrices and A is non-singular. This gives

$$\det\left(\widetilde{\Delta}(x^e)\right) = \det(\mathcal{C})\det\left(\mathcal{E} - \frac{\partial\mathcal{A}}{\partial x_4}\mathcal{D}\right) \tag{48}$$

The determinant of $\widetilde{\Delta}(x^e)$ is zero when $\det(\mathcal{C}) = 0$ or when $\det\left(\mathcal{E} - \frac{\partial\mathcal{A}}{\partial x_4}\mathcal{D}\right) = 0$. This means that $\widetilde{\Delta}(x^e)$, and thereby also $\Delta(x^e)$, has full rank whenever $\det(\mathcal{C}) \neq 0$ and $\det\left(\mathcal{E} - \frac{\partial\mathcal{A}}{\partial x_4}\mathcal{D}\right) \neq 0$. This completes the proof. ■

The requirement regarding the two determinants in Theorem 6 is not very restrictive, but implies that the snake robot can attain configurations that are singular, i.e. certain shapes of the snake robot are obstructive from a control perspective since the dimension of the reachable space from these configurations is not full-dimensional. Note that we only consider joint angles satisfying $\phi_i < |90°|$ since larger joint angles are not common for snake robots. Our investigations so far indicate that the only singular configurations of a planar snake robot are configurations where all joint angles are equal ($\phi_1 = \phi_2 = ... = \phi_{n-1}$). These are postures where the snake robot is either lying straight or forming an arc of a circle. Such postures are easily avoided during snake locomotion. Unfortunately, the expressions for $\det(\mathcal{C})$ and $\det\left(\mathcal{E} - \frac{\partial\mathcal{A}}{\partial x_4}\mathcal{D}\right)$ are extremely complex and hard to analyze, even when employing a mathematical software tool such as *Matlab Symbolic Math Toolbox*. At this point, we therefore cannot rule out the possibility that other singular postures also exist, but we consider it unlikely. We may anyhow state the following corollary:

Corollary 7: Theorem 6 is not satisfied at equilibrium points where all relative joint angles are equal ($\phi_1 = \phi_2 = ... = \phi_{n-1}$).

Proof: It is straightforward to verify with a mathematical software tool such as *Matlab Symbolic Math Toolbox* that $\det(\mathcal{C})\big|_{\phi_1=\phi_2=...=\phi_{n-1}} = 0$, thus violating the conditions in Theorem 6. ■

Remark 8: Corollary 7 implies that *the joint angles of a snake robot should be out of phase during snake locomotion*. This claim has been stated in several previous works [1], [2], [4], [7], but no formal mathematical proof was given.

We now show that the snake robot does *not* satisfy sufficient conditions for *small-time local controllability* (STLC):

Theorem 9: A planar snake robot with $n \geq 4$ links influenced by viscous ground friction does *not* satisfy the sufficient conditions for *small-time local controllability* (STLC) stated in Theorem 4.

Proof: The proof is complete if we can show that there are *bad* brackets of the system vector fields that cannot be neither θ-*neutralized* nor l-*neutralized* (see Theorem 4). The *bad* brackets with the lowest θ-degree and the lowest l-degree (except for f, which vanishes at any equilibrium point) are $[g_j, [f, g_j]]$, $j \in \{1, 2, 3\}$. Theorem 4 requires these vectors to be written as linear combinations of *good* brackets with either lower θ-degree or lower l-degree. The only such *good* brackets are $g_j, [f, g_j], [f, [f, g_j]], .., [f, [\cdots [f, g_j]] \cdots], j \in \{1, 2, 3\}$. Brackets of the form $[g_k, g_j]$ are not considered because $[g_k, g_j] = 0$, $j, k \in \{1, 2, 3\}$. For a proper choice of θ and l_j, $j \in \{0, 1, 2, 3\}$, these brackets have both lower θ-degree and lower l-degree. It is straightforward to verify that $[g_j, [f, g_j]] \in \mathbb{R}^{2n+4=12}$ is a vector of all zeros except for element number $2n+2 = 10$. The only way to write this vector as a linear combination of the *good* brackets listed above is if these *good* brackets span the entire 12-dimensional state space. This is not the case, however, because the vectors $[f, [f, g_j]], .., [f, [\cdots [f, g_j]] \cdots]$ are linearly dependent, as can be seen by assembling the following matrix:

$$[[f, [f, g_j]], [f, [f, [f, g_j]]], [f, [f, [f, [f, g_j]]]], \cdots]$$
$$= \begin{bmatrix} 0_{3\times3} & 0_{3\times1} & 0_{3\times1} & \cdots \\ \mathcal{C} & -\frac{\partial \mathcal{A}}{\partial x_4}\mathcal{C} & \left(\frac{\partial \mathcal{A}}{\partial x_4}\right)^2 \mathcal{C} & \cdots \\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1} & \cdots \\ \frac{\partial \mathcal{A}}{\partial x_4}\mathcal{C} & -\left(\frac{\partial \mathcal{A}}{\partial x_4}\right)^2 \mathcal{C} & \left(\frac{\partial \mathcal{A}}{\partial x_4}\right)^3 \mathcal{C} & \cdots \end{bmatrix} \quad (49)$$

and noting that the fourth row is a multiple of the second row. For the system in (35), it is therefore not possible to neither θ-*neutralize* nor l-*neutralize* the *bad* brackets of the system. This completes the proof. ∎

Note that *necessary* conditions for STLC do not exist. The snake robot may therefore be STLC even though the *sufficient* conditions in Theorem 4 are violated.

We end this section with a note on Theorem 6. This theorem clearly shows that *non-uniform* friction is an important property for a snake robot. In the snake robot literature, it is common for snake robots to possess the property $c_n \gg c_t$. The extreme case of this property is realized by installing passive wheels along the snake body since this ideally means that $c_t = 0$ and $c_n = \infty$. However, from Theorem 6 it is clear that the only requirement for *strong accessibility* is that the friction coefficients are *not equal*. The property $c_t > c_n$ is therefore also valid. This means that the passive wheels commonly mounted tangential to the snake body may equally well be mounted transversal to the snake body. The resulting motion will off course be different, but the *strong accessibility* property is still preserved.

VI. Conclusion

This paper has investigated the controllability properties of planar snake robots influenced by viscous ground friction forces. The first contribution of the paper is a *partially feedback linearized model* of a planar snake robot. The second contribution is a *stabilizability analysis* proving that any asymptotically stabilizable control law for a planar snake robot to an equilibrium point must be *time-varying*. This result is valid regardless of which type of friction the snake robot is subjected to. The third and final contribution is a *controllability analysis* proving that planar snake robots are *not controllable* when the viscous ground friction is *uniform*, but that a snake robot becomes *strongly accessible* when the viscous ground friction is *non-uniform*. This analysis showed that the joint angles of a snake robot should be *out of phase* during snake locomotion. The analysis also showed that the snake robot does *not* satisfy sufficient conditions for *small-time local controllability* (STLC).

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