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Paul Trygsland

Topics in Applied Homotopy Theory

Doctoral thesis

NTNU
Norwegian University of Science and Technology
Thesis for the Degree of Philosophiae Doctor
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Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of Philosophiae Doctor (Ph.D.) at the Norwegian University of Science and Technology (NTNU) in Trondheim, Norway. It is the result of four years of work at the Department of Mathematical Sciences, for which one year is devoted to teaching. The research presented here was supervised by Professor Markus Szymik and Professor Sverre O. Smalø.

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Paul Trygsland
Trondheim, October 2021

To my mother.

Outline of the thesis

This thesis consists of an introductory chapter and the following three papers, two of which are joint work. Paper III is supplemented with Python code available at

`https://github.com/paultrygs/Section-Complex/`.

The introduction provides a context for the papers and gives a brief overview of their contents.

Paper I

Combinatorial models for topological Reeb spaces.
Submitted.

Paper II

Factorization, extensions and a theorem of Retakh for exact quasi-categories. Joint with Erlend Due Børve.
Preprint.

Paper III

Section complexes of height functions. Joint with Erik Hermansen and Melvin Vaupel.
Work in progress.

Introduction

Outline. Sections 1 and 2 provide a context for papers I and III, whereas Sections 3 and 4 do the same for paper II. In the last section, a summary of the three papers is presented. All work carried out in this thesis hinge on applications of simplicial techniques.

1 Simplicial sets and spaces

S. Eilenberg and J. Zilber introduced simplicial sets in 1950 [EZ50]. Their motivation was the need for a modification of simplicial complexes in (co)homology theory. For example, a singular simplex, a continuous map from a topological simplex into a space, is not determined by its faces [Eil44]. Hence singular simplices in a given space do not constitute a simplicial complex. Not long after, it was discovered that simplicial sets model all topological spaces, at least up to homotopy groups [Mil57]. Work of D. Kan and D. Quillen in the 50s and 60s marked the beginning of axiomatized homotopy theory via model categories [Kan57, Qui68, Qui06]. Nowadays it is common knowledge, among those acquainted with homotopy theory, that the Quillen model structure on simplicial sets is equivalent to the standard model structure on topological spaces. Simplicial sets thus provide a combinatorial framework for studying spaces.

Simplicial sets can be described in a very low-tech manner as they are all about simplices and their faces, but there is also a neat abstract definition. Indeed, if Δ is the simplex category, i.e. the full subcategory of small categories spanned by the total orders

$$0 \to 1 \to \cdots \to n,$$

one defines a simplicial set as a contravariant functors from Δ into the category of sets. The prefix ‘simplicial’ can thus be put in front of any category, defining the more general notion of simplicial objects. To demonstrate the usefulness of such a generalization, we can for instance consider simplicial abelian groups, which are equivalent to (bounded) chain complexes via the Dold–Kan correspondence [Dol58, Kan58]. Chain complexes are undoubtedly interesting from the theoretical point of view, and in recent years they have even been used to study the shape of data sets using the language of persistent homology [Car09]. The latter technology has among other things been used to identify a certain subgroup of breast cancers [NLC11].

Let us use the word ‘space’ as a substitute for both simplicial sets and topological spaces. Simplicial spaces can be presented as contravariant functors from $\Delta \times \Delta$ to sets, and hence they have two spatial directions. Similar to how double chain complexes C has an associated total complex $\mathrm{Tot}(C)$, simplicial spaces have an associated space. Explicitly, this amounts to mapping a simplicial space X to its geometrical realization $|X|$ [Mil57]. If you want to understand the homology of a chain complex $\mathrm{Tot}(C)$, which appears as the total complex of some double complex C, then there is a spectral sequence to approach it. In short, homological information in the horizontal and vertical chain complexes of C is structured to deduce the homology $\mathrm{H}_*\mathrm{Tot}(C)$, up to extension problems. G. Segal gives the cohomological analogue for simplicial spaces in [Seg68]: the vertical and horizontal spaces in X are used to compute $\mathrm{H}_*|X|$.

2 Simplicial spaces in applied topology and Morse theory

A simplicial space $X_\bullet$ is a collection of spaces X_n, $n = 0, 1, \ldots$, whose face and degeneracy maps d_i and s_j are maps of spaces. We now discuss how these can be relevant for real-world use, such as studying data sets.

Consider a numerable covering $U = (U_\alpha)_\Sigma$ of a space X. For any finite σ in Σ, we denote by U_σ the intersection $\cap_{\alpha\in\sigma} U_\alpha$. We define the simplicial space X_U whose space of n–simplices is $(X_U)_n = \coprod_{\sigma_0 \subset \cdots \subset \sigma_n} U_{\sigma_n}$. A 0–simplex in X_U is thus a point x contained in some finite intersection U_σ. A 1–simplex e in U_τ, labeled by $\sigma \subset \tau$, has faces $d_0 x = ix$ and $d_1 x = x$ in U_σ and U_τ, respectively, where $i\colon U_\tau \hookrightarrow U_\sigma$ is the inclusion.

Proposition 2.1 (Segal 1968)**.** If U is a numerable open covering of a space X, then the space $|X_U|$ is homotopy equivalent to X.

Many ideas in applied topology can be understood from this result. To understand how, we consider the spectral sequence coming from X_U. The first page is depicted:

$$
\begin{tikzcd}
q \\
\bigoplus_{\sigma_1} \mathrm{H}_2 U_{\sigma_1} & \bigoplus_{\sigma_1\subset\sigma_2} \mathrm{H}_2 U_{\sigma_2} \arrow[l, "\partial^1_{1,2}"'] & \bigoplus_{\sigma_1\subset\sigma_2\subset\sigma_3} \mathrm{H}_2 U_{\sigma_3} \arrow[l, "\partial^1_{2,2}"'] \\
\bigoplus_{\sigma_1} \mathrm{H}_1 U_{\sigma_1} & \bigoplus_{\sigma_1\subset\sigma_2} \mathrm{H}_1 U_{\sigma_2} \arrow[l, "\partial^1_{1,1}"'] & \bigoplus_{\sigma_1\subset\sigma_2\subset\sigma_3} \mathrm{H}_1 U_{\sigma_3} \arrow[l, "\partial^1_{2,1}"'] \\
\bigoplus_{\sigma_1} \mathrm{H}_0 U_{\sigma_1} & \bigoplus_{\sigma_1\subset\sigma_2} \mathrm{H}_0 U_{\sigma_2} \arrow[l, "\partial^1_{1,0}"'] & \bigoplus_{\sigma_1\subset\sigma_2\subset\sigma_3} \mathrm{H}_0 U_{\sigma_3} \arrow[l, "\partial^1_{2,0}"'] \\
 & & & p
\end{tikzcd}
$$

The differential $\partial^1_{p,q}$ is induced in H_q from the alternating sum of face maps, for $p = 1$ this is explicitly given by $\partial^1_{1,q} = \mathrm{H}_q d_0 - \mathrm{H}_q d_1$. The second page of the spectral sequence is computed by calculating homology of the rows. Proposition 2.1 tells us that this first page can be used to deduce the homology of X, up to extension problems. This is a Mayer–Vietoris type phenomenon: intersections in a cover are used to compute the homology groups of X.

We look at pre-images of an open cover in $\mathbb{R}$ under a real-valued function $f\colon X \to \mathbb{R}$. From that we will be able to recover level-set zigzag and Mapper [CdSM09, SMC+07].

Assume we are given a finite collection of open intervals I_k such that $U_k = f^{-1}I_k$ is an open cover of X. Moreover, we require $U_k \cap U_l \neq \emptyset$ only if $l = k \pm 1$. The latter assumption guarantees that all higher simplices in X_U are degenerate so that the spectral sequence collapses at the second page. Indeed, the chain complex

$$\bigoplus_{\sigma_1} \mathrm{H}_q U_{\sigma_1} \leftarrow \bigoplus_{\sigma_1 \subset \sigma_2} \mathrm{H}_q U_{\sigma_2} \leftarrow \bigoplus_{\sigma_1 \subset \sigma_2 \subset \sigma_3} \mathrm{H}_q U_{\sigma_3} \leftarrow \cdots$$

is homology equivalent to

$$\bigoplus_{\sigma_1} \mathrm{H}_q U_{\sigma_1} \leftarrow \bigoplus_{\sigma_1 \subsetneq \sigma_2} \mathrm{H}_q U_{\sigma_2}$$

where we only include non-trivial subset inclusions $\sigma_1 \subsetneq \sigma_2$ [GJ09, p.150]. Another simplification produces

$$\bigoplus_{k=1,\ldots,n} \mathrm{H}_q U_k \xleftarrow{\partial} \bigoplus_{k=1,\ldots,n-1} \mathrm{H}_q U_k \cap U_{k+1}$$

where ∂ sends a class α in $\mathrm{H}_q U_k \cap U_{k+1}$ to the difference $(j_k)_*\alpha - (i_k)_*\alpha$, where $(i_k)_*$ is the induced map on H_q coming from the inclusion $i_k\colon U_k \cap U_{k+1} \hookrightarrow U_k$, and similarly $(j_k)_*$ is obtained from the inclusion $j_k\colon U_k \cap U_{k+1} \hookrightarrow U_{k+1}$. Wrap out the direct sums to obtain the levelset zigzag

$$\mathrm{H}_q U_1 \leftarrow \mathrm{H}_q U_1 \cap U_2 \rightarrow \mathrm{H}_q U_2 \leftarrow \mathrm{H}_q U_2 \cap U_3 \rightarrow \cdots \leftarrow \mathrm{H}_q U_{n-1} \cap U_n \rightarrow \mathrm{H}_q U_n$$

as defined by G. Carlsson, V. de Silva and D. Morozov for real-valued functions of Morse type [CdSM09]. The spectral sequence thus incorporates information that is equivalent to levelset zigzag.

From a simplicial space $X_\bullet$, we can always produce the simplicial set $\pi_0 X_\bullet$ whose set of n–simplices is $\pi_0 X_n$. For the simplicial space X_U, defined from a continuous function $f\colon X \to \mathbb{R}$ as above, the simplicial set $\pi_0 X_U$ is a graph. This is because all of the higher simplices are degenerate. The vertices of $\pi_0 X_U$ correspond to the path components of opens U_i and $U_i \cap U_{i+1}$, whereas edges connect overlapping components of U_i, and U_{i+1}, with components in the intersection $U_i \cap U_{i+1}$. This construction is thus a subdivided version of the graph produced in the method Mapper [SMC+07].

We have described how the simplicial space associated to a covering relates to established methods in applied topology. Let us see how simplicial spaces can be relevant for a different application. In unpublished work of R. Cohen, J. Jones and G. Segal [CJS92], simplicial spaces are utilized to better understand the homotopical properties of Morse theory. Here I present a simplified version of their construction. Let $f\colon M \to \mathbb{R}$ be a Morse function on a closed Riemannian manifold M. The flow-lines $\gamma\colon [a,b] \to M$ of f are certain smooth curves such that $\gamma(a)$ and $\gamma(b)$ are critical points. Moreover, the flow-lines are parametrized as sections: $f(\gamma(c)) = c$. A piece-wise flow-line is a concatenation of flow-lines. We define a simplicial space $\mathcal{F}_f$ whose space of 0–simplices is the set of critical points, whereas the space of n–simplices consists of tuples $(\gamma_1, \ldots, \gamma_n)$ of piece-wise flow-lines γ_i that can be concatenated. The latter space is equipped with a non-trivial topology. It turns out that this simplicial space carries very interesting information. For one, it can be used to calculate the homology of M:

Theorem 2.2 (Cohen, Jones and Segal 1995)**.** Let $f\colon M \to \mathbb{R}$ be a Morse function on a closed Riemannian manifold M. The realization of $\mathcal{F}_f$ is homotopy equivalent to M.

If we apply π_0 level-wise to $\mathcal{F}_f$, then what we end up with is a simplicial set whose vertices are the critical points of f. Moreover, the edges are determined by classes of flow-lines between critical points. This thus defines a model for the Reeb graph, or topological Reeb space, as defined by G. Reeb [Ree46].

V. Nanda, D. Tamaki and K. Tanaka proves an analogous result to Theorem 2.2 for discrete Morse functions. Their idea is to look at certain combinatorial flow-lines, or flow paths, generalizing Forman's gradient paths [For98].

3 Quasi-categories

J. Boardman and R. Vogt discovered quasi-categories in the early 1970s [BV06]. Their motivation was to develop a homotopical machinery for recognizing loop spaces and infinite loop spaces, or "homotopy groups" and "homotopy abelian groups". A loop space naturally comes with a product via concatenation of loops, but there is no canonical choice of such. One could fix this by replacing a loop space with a strict topological group up to homotopy, as pointed out in [Ada78, p. 31]. A different idea, which quasi-categories incorporate, is to not require strict products, but rather only demand products up to homotopy. Another approach in this direction is the Γ–spaces of G. Segal [Seg74], paving the way for simplicial spaces known as Segal spaces [Rez01].

The idea of composition up to homotopy, as presented in [BV06], generalizes the notion of categories. Indeed, a quasi-category is a simplicial set in which two concatenated edges, or morphisms, f and g can be extended to a 2–simplex:

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, dashed] & & C \end{tikzcd}$$

The dashed arrow represents a *choice* of composition $g \circ f$. Moreover, the choice made is, of course, redundant up to homotopy. This framework truly generalizes strict category theory as most constructions, such as limits and colimits, carry over to a homotopy invariant version in quasi-categories [Joy02]. Moreover, any quasi-category $\mathcal{C}$ admits a homotopy category $\mathrm{h}\mathcal{C}$, which is defined by applying π_0 to mapping spaces. If $\mathcal{C}$ happens to be an ordinary category, then it most certainly agrees with its homotopy category; $\mathcal{C} = \mathrm{h}\mathcal{C}$.

The applications provided by quasi-categories go beyond (infinite) loop spaces. For one, they provide a model for a homotopy theory of homotopy theories, by interpreting a quasi-category itself as a ”homotopy theory”. This is made precise by the Joyal model structure on simplicial sets, whose fibrant, or ‘nice’, objects are quasi-categories [Lur09]. There are many other equivalent approaches in this direction such as Segal spaces [Rez01] or categories enriched in simplicial sets [DK80, Ber07].

4 Exact quasi–categories

Abelian categories are fundamental in algebra and topology. Some examples include the category of left (or right) modules over a ring and sheaves of abelian groups on a topological space. One advantage of abelian categories is that they allow for homological algebra, a framework for homology, exact sequences, diagram lemmas and derived functors to mention a few keywords. But there are many close-to-abelian categories in which homological algebra should be possible. For instance, filtered abelian groups and locally convex vector spaces are not abelian [Sch99, Pro00].

An exact category is an additive category together with a collection of short exact sequences subject to certain conditions [Hel58, Qui73]. Many constructions and results from the homological algebra of abelian categories carries over to exact categories. I refer to [Büh10] for a concise survey. An additive category $\mathcal{A}$ admits a minimal exact category by imposing that

$$A \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} A \oplus B \xrightarrow{\begin{bmatrix}0 & 1\end{bmatrix}} B$$

is exact for all objects A and B in $\mathcal{A}$. A kernel-cokernel pair in $\mathcal{A}$ is a sequence

$$A \xrightarrow{f} E \xrightarrow{g} B$$

such that f is the kernel of g and g is the cokernel of f. Short exact sequences in an exact category must constitute a kernel-cokernel pair. It is, however, not true that the collection of all kernel-cokernel pairs in an additive category defines a maximal exact structure [Rum08]. But there is a notion of quasi-abelian categories for which this is the case [Sch99]. Both filtered abelian groups and locally convex topological vector spaces are quasi-abelian, hence they admit this maximal exact structure.

The Hom–sets in an abelian category $\mathcal{A}$ are abelian groups. This means that $\mathrm{Hom}_{\mathcal{A}}(-,-)$ defines a bifunctor from $\mathcal{A}$ to abelian groups. Assume $\mathcal{A}$ to have enough injectives and projectives, and fix two objects A and B therein. The abelian group of n–extensions of B by A $\mathrm{Ext}^n_{\mathcal{A}}(B,A)$ is often defined as the nth (right) derived functor of $\mathrm{Hom}_{\mathcal{A}}(B,-)$ applied to A. Dually, it can be defined as the (right) derived functor of $\mathrm{Hom}_{\mathcal{A}}(-,A)$ applied to B. To compute Ext–groups, one calculates the homology of a chain complex $\mathrm{Ext}_{\mathcal{A}}(B,A)$, the total derived Hom–space. Unfortunately, this approach is rather restrictive and is not applicable to general exact categories.

We take a closer look at an alternative definition of Ext–groups. Let $\mathcal{A}$ be an abelian category, and fix two objects A and B therein. An n–extension of B by A is a long exact sequence

$$A \to E_1 \to \cdots E_n \to B \tag{1}$$

in $\mathcal{A}$. Note that (1) is equivalent to a concatenation of short exact sequences

$$\begin{tikzcd}[column sep=small] & E_1 \arrow[dr, two heads] & & E_2 \arrow[dr, two heads] & & E_n \arrow[dr, two heads] & \\ A \arrow[ur, tail] & & E_{1.5} \arrow[ur, tail] & & \cdots \arrow[ur, tail] & & B \end{tikzcd}$$

This means in particular that the notion of n–extensions of B by A makes perfect sense in the exact setting. The diagrams of shape (1) define a category $\mathscr{E}\mathrm{xt}^n_{\mathcal{A}}(B,A)$ whose morphisms are commutative ladders which restrict to identities at A and B. Let us interpret a category as a quasi-category in which compositions are unique. It is well-known that the set of Yoneda n–extensions $\pi_0\,\mathscr{E}\mathrm{xt}^n_{\mathcal{A}}(B,A)$ is an abelian group [Yon60]. Moreover, the abelian groups $\pi_0\,\mathscr{E}\mathrm{xt}^n_{\mathcal{A}}(B,A)$ and $\mathrm{Ext}^n_{\mathcal{A}}(B,A)$, as defined above, are isomorphic. One advantage of Yoneda Ext–groups is that they only depend on the exact structure in $\mathcal{A}$. V. Retakh shows that the extension categories assemble into a spectrum [Ret86].

Theorem 4.1 (Retakh 1986)**.** Let $\mathcal{A}$ be an abelian category, and fix two objects A and B therein. There is an Ω–spectrum $\mathscr{E}\mathrm{xt}_{\mathcal{A}}(B,A)$ whose nth entry is $\mathscr{E}\mathrm{xt}^n_{\mathcal{A}}(B,A)$.

This allows for ‘derived’ Hom–spaces in any exact category.

In recent work, C. Barwick has extended the concept of exact categories to exact quasi-categories [Bar15], utilizing the minimal axioms of B. Keller [**?**]. The name is perhaps best motivated by the fact that ordinary exact categories (interpreted as quasi-categories) are exact quasi-categories. Another extreme example is given by stable quasi-categories in the sense of J. Lurie [Lur17], which for instance includes the quasi-category of spectra. Many results in exact categories are translated to exact quasi-categories. This thesis extends Theroem 4.1 to exact quasi-categories.

I present a rather ironic observation here. The homotopy category of an exact category is obviously exact, but the homotopy category of a general exact quasi-category need not be. Indeed, the axioms imposed by Barwick require the existence of certain pushouts. But a pushout in an exact quasi-category need not descend to a pushout in the homotopy category. An explicit example is given by the quasi-category of spectra: homotopy pushouts are not pushouts in the homotopy category. More generally, any stable quasi-category provides a natural triangulated structure to its homotopy category. We can thus wonder: what kind of structure does an exact quasi-category induce on its homotopy category?

On the level of homotopy categories, or ordinary categories, H. Nakaoka and Y. Palu have introduced extriangulated categories as a generalization of both exact and triangulated categories [NP19]. Based on the above discussion, there is the question of whether the homotopy category of an exact quasi-category is extriangulated? This was recently answered in the positive [NP20]. But there are still many questions left unanswered. For instance, one might wonder just how much of the extriangulated data that is encoded in the higher structure inherent to an exact quasi-category.

5 Summary of papers

Paper I: Combinatorial models for topological Reeb spaces

For any continuous real-valued function $f\colon X \to \mathbb{R}$ on a topological space X, we naturally associate a topological category $\mathcal{S}_f$. Morphisms in $\mathcal{S}_f$ are the sections $\sigma\colon [a,b] \to X$ satisfying that $f \circ \sigma$ is the inclusion $[a,b] \hookrightarrow \mathbb{R}$. Our construction is inspired by the work of R. Cohen, J. Jones and G. Segal [CJS92]. The classifying space $|\mathrm{N}\mathcal{S}_f|$ does not have the homotopy type of X in general, but these homotopy types do agree for many examples. When X is a stratified space, for which the strata are C^1–manifolds, we introduce the class of Reeb functions. This class includes both smooth and combinatorial examples, e.g. Morse functions and piecewise linear functions. Moreover, we prove that $|\mathrm{N}\mathcal{S}_f| \simeq X$ whenever f is a Reeb function.

Two applications are discussed. First, the simplicial topological space $\mathrm{N}\mathcal{S}_f$ comes with a spectral sequence for computing $\mathrm{H}_*|\mathrm{N}\mathcal{S}_f|$ [Seg68]. We investigate its basic algebraic properties, especially when f is a Reeb function so that $\mathrm{H}_*X \simeq \mathrm{H}_*|\mathrm{N}\mathcal{S}_f|$. Secondly, we introduce the combinatorial Reeb space $\pi_0\mathrm{N}\mathcal{S}_f$ of any continuous function f, by applying the nerve followed by level-wise path components. It is proven that this simplicial set always has the homotopy type of a graph. But it does not always have the same homotopy type as the topological Reeb space, introduced by G. Reeb [Ree46]. However, the combinatorial and topological Reeb spaces agree when f is a Reeb function, and in particular a Morse function.

Paper II: Factorization, extensions and a theorem of Retakh for exact quasi-categories

We generalize a Theorem of V. Retakh [Ret86] to the framework of exact quasi-categories as defined by C. Barwick [Bar15].

Given an exact quasi-category $\mathcal{C}$, there is a notion of exact sequences therein. We define for every pair of objects A and B the quasi-category $\mathscr{E}\mathrm{xt}^n_{\mathcal{C}}(B,A)$, whose objects are the n–extensions of B by A. If $\mathcal{C}$ is an ordinary abelian category, then applying π_0 to $\mathscr{E}\mathrm{xt}^n_{\mathcal{C}}(B,A)$ produces the abelian group of Yoneda n–extensions.

We arrange all of the extension categories in a spectrum $\mathscr{E}\mathrm{xt}_{\mathcal{C}}(B,A)$ whose nth entry is $\mathscr{E}\mathrm{xt}^n_{\mathcal{C}}(B,A)$ for $n \geq 1$ and $\mathrm{map}(B,A)$ for $n = 0$. Inspired by the result of Retakh, this spectrum is proven to be an Ω–spectrum, and hence capture the $\mathbb{E}_\infty$–structure on $\mathrm{map}(B,A)$. In the case when $\mathcal{C}$ is an abelian category, this spectrum has the group of Yoneda n–extensions as its $(-n)$th homotopy group.

To prove our generalization of Retakh's theorem we identify a functorial Kan-fibrant replacement $\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathcal{C}}(B,A)$ of $\mathscr{E}\mathrm{xt}^n_{\mathcal{C}}(B,A)$, well suited for our needs. This is a modified version of Kan's well-known Ex–functor [Kan57]. But in contrast to Ex, we need only apply fEx a single time to obtain a Kan complex in the case of $\mathscr{E}\mathrm{xt}^n_{\mathcal{C}}(B,A)$. This observation heavily relies on lifting the factorization property of exact categories [VW20] to exact quasi-categories.

As an application we deduce that the homotopy category of an exact quasi-category is extriangulated, a result which was already proven by Nakaoka and Palu in 2020 [NP20]. Moreover, we show that the bifunctor $\mathscr{E}\mathrm{xt}_{\mathcal{C}}(-,-)$ into spectra not only determines a natural extriangulation on the homotopy category $\mathrm{h}\mathcal{C}$, but also descends to all of the (extriangulated) higher n–extension groups in $\mathrm{h}\mathcal{C}$.

Paper III: Section complexes of height functions

We investigate a discrete analogue of the theory developed in [Try21] by only considering piecewise linear functions on CW complexes. Since this class is purely combinatorial, we rather work with simplicial sets directly. Our model R for the real line is the nerve of the poset category $(\mathbb{R}, \leq)$, and a height function is a simplicial map $h\colon X \to \mathrm{R}$. To a height function h is associated naturally a bisimplicial set $\mathcal{S}_h$, the section complex of h. It is a combinatorial analogue of the (nerve of the) topological category introduced in [Try21]. We prove that the diagonal/realization of $\mathcal{S}_h$ always has the homotopy type of X.

Any bisimplicial set comes with a spectral sequence for computing the homology of its diagonal/realization [GJ09]. In particular, any height function h has an associated spectral sequence which computes the homology of $\mathrm{diag}\mathcal{S}_h \simeq X$. We call it the section spectral sequence. Note that it does not collapse at the second page in general. An explicit example in which there is a non-trivial differential on the second page is calculated. We extract the Reeb complexes from the chain complexes appearing on the first page of the section spectral sequence. These chain complexes carry information about how homology generators flow across height levels (or fibers). To better demonstrate computability of this discrete theory, Python code for computing section complexes and Reeb complexes is made available at `https://github.com/paultrygs/Section-Complex/`. Examples computed in the paper are also included in this repository.

If X is sufficiently subdivided, we prove that applying π_0 level-wise to $\mathcal{S}_h$ recovers the topological Reeb space [Ree46]. In particular, the zeroth Reeb complex computes the homology of the topological Reeb space. Moreover, Reeb complexes give rise to zigzag modules [CdS10]. For many examples these modules complement the level-set zigzag modules via the diamond principle [CdSM09, CdSM09]. In the special case when h is obtained from the iterated mapping cylinder of a filtration, our zigzag modules coincide with the standard persistence modules.

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Paper I

Combinatorial models for topological Reeb Spaces

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Combinatorial models for topological Reeb spaces

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Abstract. There are two rather distinct approaches to Morse theory nowadays: smooth and discrete. We propose to study a real-valued function by assembling all associated sections in a topological category. From this point of view, *Reeb functions* on stratified spaces are introduced, including both smooth and combinatorial examples. As a consequence of the simplicial approach taken, the theory comes with a spectral sequence for computing (generalized) homology. We also model the homotopy type of Reeb graphs/ topological Reeb spaces as simplicial sets, which are combinatorial in nature, as opposed to the typical description in terms of quotient spaces.

1 Introduction

Let X be a topological space and $f\colon X \to \mathbb{R}$ a continuous function on it. A *section* σ of f is a map $[a,b] \to X$, for some real numbers $a \leq b$, subject to $f \circ \sigma(c) = c$. Two sections $\sigma\colon [a,b] \to X$ and $\rho\colon [b,c] \to X$, such that $\sigma(b) = \rho(b)$, may be concatenated into a new section $\rho \circ \sigma\colon [a,c] \to X$. This data defines the *section category* $\mathcal{S}_f$ associated to f which is in fact a topological category. The nerve construction thus provides a simplicial topological space $\mathrm{N}\mathcal{S}_f$. We did not put any constraints on f as of yet. However, if the section category is to recover the homotopical information of X by realizing $\mathrm{N}\mathcal{S}_f$, some assumptions are necessary. This should be considered motivation for the concept of *Reeb functions* which requires f to be sufficiently 'nice'. Examples include Morse functions on smooth manifolds and piecewise linear functions on CW complexes. I refer to Definition 2.5 for a precise formulation.

Theorem 1.1. For any Reeb function $f\colon X \to \mathbb{R}$, the realization of the nerve of the section category of f is weakly equivalent to X, that is $X \simeq |\mathrm{N}\mathcal{S}_f|$.

Ralph L. Cohen, John D. S. Jones and Graeme B. Segal prove a similar result for Morse functions in [CJS92] as an attempt to better understand homotopical aspects of Morse theory [CJS95]. A purely combinatorial analogue can be found in [NTT18] which covers the discrete Morse theory of Robin Forman [For98]. Our work can thus be described as an attempt to find a common framework including both smooth and combinatorial examples.

Any simplicial topological space comes with a spectral sequence for computing the generalized homology of its classifying space [Seg68]. A shortcoming of the section category $\mathcal{S}_f$ is that its classifying space is huge, hence nowhere near computationally feasible. Reeb functions provide a way to extract the essential information in $\mathcal{S}_f$ into the much smaller *critical subcategory* $\mathcal{C}_f$ whose classifying space has unchanged homotopy type when compared to $\mathcal{S}_f$. Computing the homology of X via $\mathcal{C}_f$, as opposed to $\mathcal{S}_f$, is analogous to how Morse and CW homology reduces the complexity of singular homology. I refer to Section 4 for some basic algebraic properties together with a user-guide on how to carry out computations.

Consider a continuous function $f\colon X \to \mathbb{R}$ on a topological space X. The *topological Reeb space* R_f, often referred to as the Reeb graph, was introduced by Georges H. Reeb in [Ree46] to study singularities. Later on it was popularized in computer graphics due to the work of Y. Shinagawa, T. Kunii and Y. Kergosien [SKK91]. Since then there has been several applications in shape analysis [BGSF08]. This advertises the need to better understand combinatorial properties of the topological Reeb space R_f, commonly constructed as a certain quotient space of X depending on the extra data that is f. I refer to categorified Reeb graphs [dSMP16] and Mapper [SMC$^+$07] for related work. From the section category $\mathcal{S}_f$ we define the *combinatorial Reeb space* by first applying the nerve followed by taking path components level-wise $\pi_0\mathrm{N}\mathcal{S}_f$. It is important to note that this construction is no topological space, but rather a simplicial set. To compare topological and combinatorial Reeb spaces we make use of the fact that topological spaces and simplicial sets carry the same homotopical information: we identify the homotopy type of a simplicial set S with that of its geometric realization $|S|$. The combinatorial and topological Reeb spaces of f do not have the same homotopy type in general. But if we restrict ourselves to Reeb functions, then they do agree.

Theorem 1.2. For any Reeb function $f\colon X \to \mathbb{R}$, the simplicial set $\pi_0\mathrm{N}\mathcal{S}_f$ has the same homotopy type as the topological space R_f; there is a zigzag of weak homotopy equivalences between $|\pi_0\mathrm{N}\mathcal{S}_f|$ and R_f.

Topological Reeb spaces are not graphs in general (Example 5.2) and we might expect combinatorial Reeb spaces to have equally nasty homotopy types. But it turns out that combinatorial Reeb spaces are always weakly homotopic to graphs:

Theorem 1.3. The combinatorial Reeb space of any continuous function has the homotopy type of a 1–dimensional CW complex.

Outline. Section 2 is all about Reeb functions $f\colon X \to \mathbb{R}$. To better illustrate the theory we first restrict ourselves to functions on C^1–manifolds in Section 2.1 before handling more general stratified spaces in Section 2.2. Results that do not hinge upon any simplicial structure are proven along the way. In Section 3 we formally define the topological section category associated to a continuous function as well as the critical subcategory and other intermediate subcategories. Some simplicial background is then provided in

Section 3.2 before proving Theorem 3.9, which implies Theorem 1.1. The spectral sequence associated to section categories, as well as critical subcategories, is discussed in Section 4. General algebraic properties are deduced in Section 4.1, whereas Section 4.2 is concerned with how to use the critical subcategory for numerical computations. In the remaining Section 5 we introduce combinatorial Reeb spaces. More background on simplicial sets is presented in Section 5.2 before proving Theorems 1.2 and 1.3 in Sections 5.3 and 5.4, respectively.

Notation. Categories of familiar objects are put inside parentheses, for instance there is the category (topological spaces). The set of morphisms between objects x, y in a category is denoted $\mathrm{Map}(x,y)$. In the case of topological spaces $\mathrm{map}(X,Y)$ reads the topological space of continuous functions from X to Y. The standard n–simplex Δ^n is modeled as the convex hull of the standard basis vectors in $\mathbb{R}^{n+1}$. The 1–simplex will also be represented as the unit interval I. We denote by $[n]$ the category generated by the directed graph

$$0 \to 1 \to \cdots \to n$$

on n arrows. In particular, $[0]$ is the trivial one object category and $[1]$ is the category with two objects 0 and 1 connected by a non-trivial arrow $0 \to 1$.

2 Reeb functions

We shall clarify what it means for a function $f\colon X \to \mathbb{R}$ to be a Reeb function. In this paper, a stratified space is built out of C^1–manifolds, which will be covered more in depth later on. Hence we start out by restricting ourselves to the simplest spaces, namely the C^1–manifolds, in Section 2.1. Thereafter we move on to the more general stratified spaces in Section 2.2. The final Proposition 2.7 is utilized many times throughout the paper.

2.1 Reeb functions on C^1–manifolds

A continuous function f is said to be *proper* if the preimage of compact is compact.

Definition 2.1. Let M be a C^1–manifold and $f\colon M \to \mathbb{R}$ a C^1–function. Then f is a *Reeb function* if

i) the subspace of critical values of f is discrete inside $\mathbb{R}$ and

ii) the restriction of f to each component of M is proper.

Recall that a C^1–function $f\colon M \to \mathbb{R}$ has a differential $df\colon M \to \mathrm{T}^*M$ which is a section of the cotangent bundle; 1-form. Let us think of df in terms of its *gradient vec-*

tor field: Pick an inner product $\langle -,- \rangle$ on $\mathrm{T}M$, and characterize $\mathrm{grad}(f)\colon M \to \mathrm{T}M$ by $\langle \mathrm{grad}(f), \mathbf{v} \rangle = df(\mathbf{v})$ for all vector fields $\mathbf{v}\colon M \to \mathrm{T}M$. The integral curves of a vector field $\mathbf{v}\colon M \to \mathrm{T}M$ are the C^1–curves $l\colon (\alpha, \omega) \to M$, allowing $\pm\infty$, satisfying $\frac{dl}{dt} = \mathbf{v}_{l(t)}$. A *local flow* on M is a map $\Psi\colon U \to M$, defined on an open neighborhood U of $\{0\} \times M$ in $\mathbb{R} \times M$, such that $U \cap (\mathbb{R} \times \{p\})$ is an interval for which Ψ restricts to an integral curve. The maximal integral curves l_p of $\mathbf{v}\colon M \to \mathrm{T}M$ form the maximal flow $\Psi_{\mathbf{v}}(p,t) = l_p(t)$. It is maximal in the sense that there are no other local flows which contains the domain of $\Psi_{\mathbf{v}}$. For this maximal flow, let us write $(\alpha_p, \omega_p) = U \cap (\mathbb{R} \times \{p\})$, allowing $\pm\infty$ as endpoints. Then $l_p\colon (\alpha_p, \omega_p) \to M$ is the maximal integral curve subject to $l_p(0) = p$. If an integral curve passes through a point q with $\mathbf{v}(q) = 0$ then $l\colon \mathbb{R} \to M$, $t \longmapsto q$ is the obvious solution. This means, conversely, that all other integral curves are immersions. They do not have to be embeddings, in general. But it is the case whenever $\mathbf{v} = \mathrm{grad}(f)$ for a function f as above:

$$\frac{d(f \circ l)}{dt} = df_{l(t)}(\frac{dl}{dt}) = \langle \mathrm{grad}(f), \mathrm{grad}(f) \rangle_{l(t)}$$

which is greater than zero so that $f \circ l$ and hence l are both injective. The existence of integral curves follows by solving local differential equations. In fact, vector fields and maximal flows are in one-to-one correspondence [BJ82, p. 82-83]. I will refer to the maximal integral curves of $\mathrm{grad}(f)$ as the *flow-lines* of f.

Definition 2.2. Let $f\colon X \to \mathbb{R}$ be a continuous function. A *section* of f is a continuous function $\sigma\colon [a,b] \to \mathbb{R}$ such that $f \circ \sigma$ is the inclusion $[a,b] \hookrightarrow \mathbb{R}$.

The next assertion tells us how to continuously pick sections of Reeb functions, a property which will turn out to be extremely useful.

Proposition 2.3. Let $f\colon M \to \mathbb{R}$ be a Reeb function. For any pair $c < d$ of successive critical values, there is a continuous function $g\colon [c,d] \times f^{-1}(c,d) \to X$ such that for all x the curve $g_x = g(-,x)$

i) is a section; $f \circ g_x(t) = t$, and

ii) pass through x at $f(x)$; $g_x(f(x)) = x$.

Proof. The idea is simple: We would like to reparametrize the flow-lines of f. Restrict the maximal flow of f to define $\Psi\colon U \to f^{-1}(c,d)$. For every flow-line l_x, defined from (α_x, ω_x) to $f^{-1}(c,d)$, the preceding discussion implies that the composition $f \circ l_x\colon (\alpha_x, \omega_x) \to \mathbb{R}$ is injective. Corestriction thus defines a C^1–isomorphism which we will denote by $h_x\colon (\alpha_x, \omega_x) \to (f \circ l_x(\alpha_x), f \circ l_x(\omega_x))$. The target must necessarily equal (c,d), independently of x: there are no critical points in $f^{-1}(c,d)$ and so an integral curve must meet every fiber. If not, one could have extended it by solving a local differential equation, contradicting the maximality of Ψ. The reparametrized flow $h\colon U \to (c,d)\times f^{-1}(c,d)$, $(a,x)\longmapsto (h_x(a),x)$ is a C^1–diffeomorphism. Its inverse is explicitly given by $(a,x)\longmapsto (h_x^{-1}(a),x)$. Define

$$\tilde{g}\colon (c,d)\times f^{-1}(c,d) \xrightarrow{h^{-1}} U \xrightarrow{\Psi} f^{-1}(c,d),$$

then $l_x = \Psi(-,x)$ implies that the restriction $\tilde{g}_x = \tilde{g}(-,x)$ is equal to $\tilde{g}_x(t) = l_x(h_x^{-1}t)$ and thus

$$f\circ\tilde{g}_x(t) = (f\circ l_x)(h_x^{-1}(t)) = t.$$

Also, the equation $x = l_x(0)$ implies

$$\tilde{g}(f(x),x) = l_x(h_x^{-1}\circ f\circ l_x(0)) = l_x(0) = x.$$

Hence the map $\tilde{g}$ satisfies the asserted properties i) and ii).

The proof will be complete once we have extended the map $\tilde{g}$ to $[c,d]\times f^{-1}(c,d)$. One can alternatively view $\tilde{g}$ as a map $f^{-1}(c,d)\to \mathrm{map}((c,d),f^{-1}[c,d])$, utilizing the right adjoint. In fact, the two properties of $\tilde{g}$ above tells us that its adjoint factorizes through the subspace $\mathrm{Flow}_f(c,d)$, of $\mathrm{map}((c,d),f^{-1}[c,d])$, consisting of flow-lines reparametrized as sections $(c,d)\to f^{-1}[c,d]$. So the map $\tilde{g}$ might as well be interpreted as a map $f^{-1}(c,d)\to \mathrm{Flow}_f(c,d)$. Since f is Reeb, hence proper on connected components, the preimage $f^{-1}[c,d]$ is a disjoint union of compact topological spaces. Consequently any flow-line of the form $\tilde{g}_x\colon (c,d)\to f^{-1}[c,d]$ can be extended uniquely to a section $g_x\colon [c,d]\to f^{-1}[c,d]$. In other words, there is a function e from $\mathrm{Flow}_f(c,d)$ to $\mathcal{S}_f(c,d)$ that extends reparametrized low-lines on (c,d) to sections on $[c,d]$. The rather tedious task of demonstrating the continuity of e is all that remains. For then the composition

$$f^{-1}(c,d) \xrightarrow{\tilde{g}} \mathrm{Flow}_f(c,d) \xrightarrow{e} \mathcal{S}_f(c,d)$$

admits an adjoint $g\colon [c,d]\times f^{-1}(c,d)\to X$ satisfying the asserted properties.

For every $a\le b$ in $[c,d]$ and V open in M, denote by $\mathrm{C}([a,b],V)$ the subbasis element whose points are the maps $\rho\colon [a,b]\to M$ for which $\rho([a,b])\subset V$. Then the collection of all $\mathrm{C}([a,b],V)\cap\mathcal{S}_f(c,d)$ is a subbasis for $\mathcal{S}_f(c,d)$. Similarly, the collection of all $\mathrm{C}([a,b],V)\cap\mathrm{Flow}_f(c,d)$, with $c<a\le b<d$, is a subbasis for $\mathrm{Flow}_f(c,d)$. We need only verify that every preimage of the form $e^{-1}(\mathrm{C}([a,b],V)\cap\mathcal{S}_f(c,d))$ is open. This is trivial whenever $c<a$ and $b<d$, for then the preimage $e^{-1}(\mathrm{C}([a,b],V)\cap\mathcal{S}_f(c,d))$ is the set $\mathrm{C}([a,b],V)\cap\mathrm{Flow}_f(c,d)$ which is open. To complete the proof, we will assume $a=c$ and $b<d$ henceforth: The case $a>c$ and $b=d$ is similar, whereas $a=c$ and $b=d$ is a special case of the former.

Take an arbitrary flow-line $\tilde{g}$ in $e^{-1}(\mathrm{C}([c,b],V)\cap\mathcal{S}_f(c,d))$. Let $g=e\tilde{g}$ be the extension to $[c,d]$ so that $g(c)$ is the limit point of $\tilde{g}$ in $f^{-1}(c)$. We need only prove that there is an open neighborhood N, of $\tilde{g}$, inside $e^{-1}(\mathrm{C}([c,b],V)\cap\mathcal{S}_f(c,d))$. To construct such a neighborhood we first pick a monotone sequence (a_n) in $(c,b]$ converging to c. Ehresmann's fibration theorem [Ehr50] provides a C^1–diffeomorphism $E_{a'}$ over $\mathbb{R}$:

$$\begin{tikzcd} f^{-1}(c,d) \arrow[r, "E_{a'}"] \arrow[dr, "f"'] & f^{-1}(a') \times (c,d) \arrow[d, "\mathrm{pr}_2"] \\ & \mathbb{R} \end{tikzcd}$$

for every real number $b < a' < c$. The elementary opens in $(f^{-1}(a') \cap V) \times (c,d)$ are all of the form $B \times (c',d')$ where B is an open ball. Since every restriction $g|_{[a_n,b]}$ has compact image and g maps into V, there are cylinders $C_n = E_b^{-1}(B_n \times [a_n,b])$ contained in V with the property that $N_n = \mathrm{C}([a_n,b],C_n)$ is a neighborhood of $\tilde{g}$. Moreover, it is safe to assume that the radius of B_n tends to zero as n goes to infinity: If B' is a ball contained inside B, and $B \times (c',d')$ maps into V under Ehresmann's C^1–diffeomorphism, then surely so does $B' \times (c',d')$. I claim that we can choose $N = N_{n_0}$ for some n_0. Assume conversely that this is not the case. Then no N_n is contained in $e^{-1}(\mathrm{C}([c,a'],V) \cap \mathcal{S}_f(c,d))$. So for every n there is a flow-line ρ_n and a real number a'_n in $[c,a_n]$ such that $\rho_n(a'_n)$ is in the complement of V. But the sequence $(\rho_n(a'_n))$ converges to the point $g(c)$–inside V–by construction, a contradiction. $\square$

2.2 Extension to stratified spaces

There are several notions of 'stratified spaces' around. One of which is the locally cone-like spaces dating back to R. Thom's work in the late 60s [Tho69]. A more recent reference is [GM83]. For any topological space Z, there is the open cone $\mathrm{C}(Z)$ defined as $Z \times [0,1)/Z \times 0$. As an example the open cone on the $(n-1)$–sphere is the open n–disk. A *filtration-preserving map* between two filtrations $X_0 \subset X_1 \subset \cdots$ and $Y_0 \subset Y_1 \subset \cdots$ of topological spaces, consists of continuous functions $g_n \colon X_i \to Y_i$ which commute with the inclusions: $g_{n+1} \circ (X_n \subset X_{n+1}) = (Y_n \subset Y_{n+1}) \circ g_n$.

Definition 2.4. An n–dimensional *stratification* on a topological space X is a filtration

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$$

satisfying: i) every ith *stratum* $S_i = X_i \setminus X_{i-1}$ is an i–dimensional C^1–manifold and ii) for every point x in S_i there exists an open neighborhood U about x and an $(n-i-1)$–dimensional stratified space Z together with $h : U \simeq \mathbb{R}^i \times \mathrm{C}(Z)$, a filtration-preserving homeomorphism. The restriction which takes $U \cap S_{i+j+1}$ to $\mathbb{R}^i \times \mathrm{C}(Z_j - Z_{j-1})$, and its inverse, are both required to be C^1. We say that a topological space together with an n–dimensional stratification is a *stratified space* of dimension n.

Finite-dimensional stratified spaces and strata-wise C^1 filtration-preserving maps form a category. Include filtered colimits to get a more general notion of *stratified spaces*, allowing infinite filtrations. Every CW complex X fits into this larger category: The ith stratum of X is the disjoint union of its (open) i–cells. In particular, every weak homotopy type can be represented by such a space.

A continuous function $f : X \to \mathbb{R}$, from a stratified space X, is *strata-wise* C^1 if it is C^1 when restricted to each stratum. A point x in the ith stratum of X is *critical* if it is a critical point of the C^1–map $f|_{S^i}$.

We extend Definition 2.1 from differentiable manifolds to stratified spaces in the following way:

Definition 2.5. Let X be a stratified space and $f\colon X \to \mathbb{R}$ a strata-wise C^1–function. We say that f is a *Reeb function* if

i) the subspace of critical values of f is discrete inside $\mathbb{R}$ and

ii) for any connected component C of some stratum, the restriction of f to the closure of C, in X, is proper.

Example 2.6. For a given stratifiable space X, the definition of a strata-wise C^1–function depends on the choice of stratification. Because of this we can always assume a Reeb function to have critical values. Indeed, let $f\colon X \to \mathbb{R}$ be a Reeb function for which there are no critical values. We slightly modify the stratified structure on X: refine the already existing structure by dividing every stratum S into the three strata $f|_S^{-1}(-\infty,0)$, $f|_S^{-1}(0)$ and $f|_S^{-1}(0,\infty)$. Then f is still a Reeb function on X with this choice of stratification. Moreover, we now have a critical value 0.

For the purpose of proving Thoerem 1.1, this will turn out to be a satisfactory extension. In particular, there is the stratified version of Proposition 2.3.

Proposition 2.7. Let $f\colon X \to \mathbb{R}$ be a Reeb function. For any pair $c < d$ of successive critical values, there is a continuous function $g\colon [c,d] \times f^{-1}(c,d) \to X$ which satisfies

i) every $g_x\colon [c,d] \to X$, $g_x(t) = g(t,x)$ is a section and

ii) $g(f(x),x) = x$.

Proof. For a general stratified space X, and Reeb function $f\colon X \to \mathbb{R}$, let $i_1, i_2, \ldots$ denote the indices of the non-empty strata. The proof is by induction on i_n. To ease notation I will simply reindex $i_n \longmapsto n$. Define f_n to be the restriction of f to X_n. For $n = 0$ there is nothing to prove if X_0 is 0–dimensional, otherwise the base case follows by Proposition 2.3. Assume that a function $g_{n-1}\colon f_{n-1}^{-1}(c,d) \times [c,d] \to X_{n-1}$ is constructed to satisfy the assertion. We shall modify the gradient vector field on the nth stratum to take into account the flow on lower dimensional strata. Definition 2.4 tells us that a point x in $S_i \cap X_n \cap f^{-1}(c,d)$ admits a neighborhood N_x, contained in $f^{-1}(c,d)$, of the form $\mathbb{R}^i \times \mathrm{C}(Z)$ with Z an $(n-i-1)$–dimensional stratified space. We shall define a vector field on each $N_x \cap S_n$ to obtain a new vector field on all of S_n via a partition of unity.

If $i < n$, then the $(n-i-1)$st stratum of Z, which is locally C^1–diffeomorphic to $\mathbb{R}^{n-i-1}$, indicates the intersection between $\mathbb{R}^i \times \mathrm{C}(Z)$ and S_n. So the intersection of N_x and the nth stratum S_n may be covered by opens $N_{x,j_x} \simeq \mathbb{R}^i \times \mathrm{C}(\mathbb{R}^{n-i-1})$. Let us construct a vector field on one such N which meets S_n in $U \simeq \mathbb{R}^i \times \mathbb{R} \times \mathbb{R}^{n-i-1}$ and S_i in $V = N \cap S_i \simeq \mathbb{R}^i$. There is a C^1–map $U \to V$ which is given by the projection $\mathrm{pr}_1 : \mathbb{R}^i \times \mathbb{R} \times \mathbb{R}^{n-i-1} \to \mathbb{R}^i$ in coordinates. The induced map Tpr_1 on tangent spaces admits a right inverse $v \longmapsto (v,0,0)$. Hence a vector field on V defines a vector field on U. In particular, the vector field corresponding to an appropriate restriction of g_{n-1} defines a vector field $\mathbf{u}\colon U \to \mathrm{T}U$. Notice that an integral curve l of $\mathbf{u}$ cannot have a limit point in $X_{n-1} \cap f^{-1}(c,d)$ since g_{n-1} is a family of C^1–sections. For every x in X_{n-1}, also contained in the closure of S_n, we associate such a vector field $\mathbf{u}_x\colon U_x \to \mathrm{T}U_x$. Otherwise, if $i = n$ and x is not contained in any such U_x, then $N_x \simeq \mathbb{R}^n$ and we simply restrict the gradient vector field on S_n to N_x.

To define a vector field on all of S_n, we cover S_n with a family of opens (U_α) as described above and pick a partition of unity (ρ_{U_α}). The formula $\mathbf{v} = \sum_\alpha \rho_{U_\alpha} \mathbf{u}_\alpha$ defines a vector field $S_n \to \mathrm{T}S_n$. Notice how $df(\mathbf{v})$ is non-zero everywhere precisely because each $df(\mathbf{u}_\alpha)$ is non-zero everywhere. The corresponding maximal local flow thus results in a map $g_n\colon [c,d] \times f|_{S_n}^{-1}(c,d) \to X$. Combine g_{n-1} and g_n to define the parametrized family $g\colon [c,d] \times f_n^{-1}(c,d) \to X_n$

$$g(t,x) = \begin{cases} g_{n-1}(t,x) & \text{if } x \in X_{n-1} \\ g_n(t,x) & \text{if } x \in S_n \end{cases}$$

of sections. □

We end this entire section by proving a lemma. The result is analogous to two basic Morse lemmas that utilize flow-lines.

Lemma 2.8. Let $f\colon X \to \mathbb{R}$ be a Reeb function with at most one critical value. Then the inclusion $f^{-1}a \hookrightarrow X$ is a homotopy equivalence for all a if there is no critical value, otherwise it is a homotopy equivalence for a equal to the critical value.

Proof. Define a filtration $X_n = f^{-1}[a-n, a+n]$, $n \geq 0$, on X. Given that X is the homotopy colimit over X_n, it suffices to prove that the inclusion $i_n\colon f^{-1}a \hookrightarrow f^{-1}[a-n,a+n]$ is a weak homotopy equivalence. The inclusion certainly factorizes

$$f^{-1}a \overset{j_n}{\hookrightarrow} f^{-1}[a-n,a] \overset{k_n}{\hookrightarrow} f^{-1}[a-n,a+n]$$

and we will only argue that j_n is a weak homotopy equivalence. For the case of k_n is similar.

Utilize Proposition 2.7 to continuously map any point x, in $f^{-1}[a-n,a]$, to a section, or reparametrized flow-line, $g_x\colon [a-n,a] \to X$ through x. If x is in $f^{-1}a$, then $g_x\colon \{a\} \to X$ is the trivial section at x. Define a retract r_n of j_n by declaring $r_n(x) = g_x(a)$. This defines a homotopy equivalence. Indeed, a homotopy can e.g. be constructed

$$H(x,t) = g_x(ta + (1-t)f(x))$$

from $H(x,0) = x$ to $H(x,1) = j_n \circ r_n(x)$. □

3 The section category and its classifying space

In Section 3.1 we define the section category $\mathcal{S}_f$ of a continuous function f. Also, if $f\colon X \to \mathbb{R}$ is a Reeb function, then a subset A of $\mathbb{R}$, which contains the critical values of f, defines a subcategory $\mathcal{C}_f^A$ of $\mathcal{S}_f$. Section 3.2 is included for the reader that would like some background on simplicial sets. Thereafter Theorem 1.1 is deduced from the stronger Theorem 3.9 in Section 3.3.

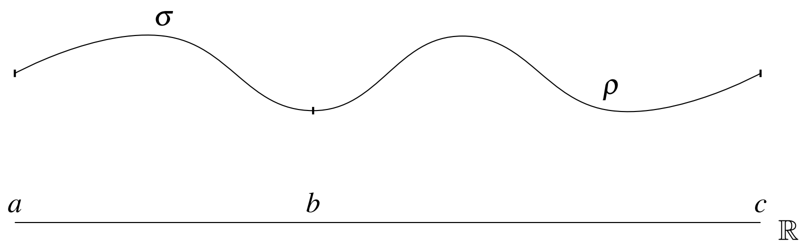
3.1 The section category

Let us first agree on the meaning of a 'topological category'. There are two different flavors: categories enriched in topological spaces and categories internal to topological spaces. In this paper a topological category is to be understood in the latter sense, following G. Segal [Seg68]. A category $\mathcal{C}$ can be described in terms of four structural maps: If ob$\mathcal{C}$ is the set of objects; mor$\mathcal{C}$ the set of morphisms; then they are source and target $s,t\colon \mathrm{mor}\mathcal{C} \to \mathrm{ob}\mathcal{C}$, injection of objects as identity morphisms $i\colon \mathrm{ob}\mathcal{C} \to \mathrm{mor}\mathcal{C}$ and composition $\circ\colon \mathrm{mor}\mathcal{C} \times_{\mathrm{ob}\mathcal{C}} \mathrm{mor}\mathcal{C} \to \mathrm{mor}\mathcal{C}$. The set $\mathrm{mor}\mathcal{C} \times_{\mathrm{ob}\mathcal{C}} \mathrm{mor}\mathcal{C}$ is the pullback obtained from the source and target; consists of pairs (m,m') of morphisms for which $s(m') = t(m)$ such that $m' \circ m$ is defined. A category $\mathcal{C}$ is a *topological category* if both ob$\mathcal{C}$ and mor$\mathcal{C}$ are equipped with topologies and the four structural maps s, t, i and $\circ$ are all continuous. Any topological space X defines a trivial topological category $\underline{X}$ whose object space and morphism space are both equal to X. The structural maps s,t,i all agree with the identity on X, whereas composition is the homeomorphism from the diagonal on X to X.

Assume that a continuous function $f\colon X \to \mathbb{R}$ from a topological space X is given. Recall that a *section* of f is a continuous function $\sigma\colon [a,b] \to X$ such that the composition $f \circ \sigma\colon [a,b] \to \mathbb{R}$ is the inclusion. Arrange all of the sections in the *space of* *all sections* $\mathrm{mor}\mathcal{S}_f = \coprod_{a \le b} \mathcal{S}_f[a,b]$, ranging over all pairs $a \le b$ in $\mathbb{R}$, equipped with the disjoint union topology. Notice how $f^{-1}a$ and $\mathcal{S}_f[a,a]$ are canonically homeomorphic. It follows that $\mathrm{ob}\mathcal{S}_f = \coprod_{a\in\mathbb{R}} f^{-1}a$ comes with an inclusion $i\colon \mathrm{ob}\mathcal{S}_f \to \mathrm{mor}\mathcal{S}_f$. Restricting the evaluation $\mathrm{eval}\colon \mathcal{S}_f[a,b] \times [a,b] \to X$ to a and b, provides source and target maps $s,t\colon \mathrm{mor}\mathcal{S}_f \to \mathrm{ob}\mathcal{S}_f$, respectively. If $\sigma\colon [a,b] \to X$ is a section, then applying source and target yields $s(\sigma) = \sigma(a)$ and $t(\sigma) = \sigma(b)$. Concatenation defines canonical maps $\mathcal{S}_f[b,c] \times_{f^{-1}b} \mathcal{S}_f[a,b] \to \mathcal{S}_f[a,c]$:

$$\rho \circ \sigma(r) = \begin{cases} \sigma(r) & \text{if } a \le r \le b \\ \rho(r) & \text{if } b \le r \le c \end{cases}$$

From which a composition $\circ\colon \mathrm{mor}\mathcal{S}_f \times_{\mathrm{ob}\mathcal{S}_f} \mathrm{mor}\mathcal{S}_f \to \mathrm{mor}\mathcal{S}_f$ is deduced.



It is straightforward to check that $\circ$ is associative: morphisms are canonically parametrized as a result of being sections. The inclusion is clearly unital. In other words, we have defined a topological category $\mathcal{S}_f$.

Definition 3.1. The *section category* of a continuous function $f\colon X \to \mathbb{R}$ is the topological category $\mathcal{S}_f$.

Two continuous functions $f\colon X \to \mathbb{R}$ and $f'\colon X' \to \mathbb{R}$, together with a continuous function $\phi\colon X \to X'$ over $\mathbb{R}$ in the sense that $f' \circ \phi = f$, induce a continuous functor between topological categories $\mathcal{S}_\phi\colon \mathcal{S}_f \to \mathcal{S}_{f'}$. So the assignment $f \longmapsto \mathcal{S}_f$ is functorial from the category of spaces over the real line.

Assume $f\colon X \to \mathbb{R}$ to be a Reeb function from here on. Every section $\sigma\colon [a,b] \to \mathbb{R}$ of f is decorated by two real numbers: $f(s\sigma) = a$ and $f(t\sigma) = b$. If A is a non-empty subset of $\mathbb{R}$ containing the critical values of f, we define the subcategory $\mathcal{C}_f^A$ of sections decorated only by real numbers in A:

Definition 3.2. Let $f\colon X \to \mathbb{R}$ be a Reeb function, and consider A a non-empty subset of $\mathbb{R}$ containing the critical values of f. Define $\mathcal{C}_f^A$ as the full subcategory of $\mathcal{S}_f$ with object space $\coprod_{a\subset A} f^{-1}a$.

If $A = \mathbb{R}$, then obviously $\mathcal{C}_f^A = \mathcal{S}_f$. And more is true: $\mathcal{C}_f^A$ and $\mathcal{S}_f$ carries the same homotopical information for any choice of A, as in the above definition. We shall make this precise in Section 3.3, after giving a brief recap on simplicial spaces.

3.2 Some background on simplicial spaces

A *simplicial set* is a family of sets X_n, $n \geq 0$, together with face maps $d_i \colon X_n \to X_{n-1}$ and degeneracy maps $s_j \colon X_n \to X_{n+1}$ satisfying certain relations [GJ09, p. 4]. It resembles a simplicial complex: the face map d_i applied to an n–simplex is the $(n-1)$–simplex to be interpreted as its ith face. The degeneracy maps, on the other hand, encode the number of ways in which one could consider an n–simplex as a degenerate $(n+1)$–simplex. The latter is not important to us, for all homotopy types in this paper are unaffected by simply omitting degeneracy maps. This can be made precise by verifying the goodness condition in [Seg74].

The *nerve* is a functor

$$\mathrm{N} \colon (\text{small categories}) \to (\text{simplicial sets}).$$

It maps a category $\mathcal{C}$ to the simplicial set $\mathrm{N}\mathcal{C}$ whose set of n–simplices is the n–fold pullback

$$(\mathrm{N}\mathcal{C})_n = \mathrm{mor}\mathcal{C} \times_{\mathrm{ob}\mathcal{C}} \cdots \times_{\mathrm{ob}\mathcal{C}} \mathrm{mor}\mathcal{C},$$

of composable n–tuples of morphisms. The outer face maps d_0 and d_n are given by omitting the first and last component, respectively, whereas the inner face maps d_i are given by composing the ith and $(i+1)$th component.

A *simplicial space* $X_\bullet$ is a simplicial set with the additional requirement that X_n is a topological space and the face and degeneracy maps are continuous. The nerve makes perfect sense as a functor

$$\mathrm{N} \colon (\text{topological categories}) \to (\text{simplicial spaces}).$$

Denote by $\Delta^\bullet$ the cosimplicial space with n–cosimplices the standard topological n–simplex Δ^n. The coface map $d^i \colon \Delta^n \to \Delta^{n+1}$ is the inclusion of Δ^{n+1}'s ith face, whereas the codegeneracy map $s^j \colon \Delta^n \to \Delta^{n-1}$ collapses Δ^n along its jth edge. Then the *geometrical realization*

$$|\cdot| \colon (\text{simplicial spaces}) \to (\text{topological spaces})$$

is defined by assigning to a simplicial space $X_\bullet$ the quotient space

$$|X_\bullet| = (\coprod_n X_n \times \Delta^n)/\sim$$

with relations $(d_i x, z) \sim (x, d^i z)$ and $(s_j x, z) \sim (x, s^j z)$. Compose the realization with the nerve to define the *classifying space*

$$\mathrm{B} = |\mathrm{N}(\cdot)| \colon (\text{topological categories}) \to (\text{topological spaces}).$$

Example 3.3. Let $f\colon X \to \mathbb{R}$ be a continuous function and consider the associated simplicial space $\mathrm{N}\mathcal{S}_f$ obtained from applying the nerve to the section category. For a sequence $a_0 \le \cdots \le a_n$ we introduce the topological space

$$\mathcal{S}_f[a_0,\ldots,a_n] = \mathcal{S}_f[a_0,a_1] \times_{f^{-1}a_1} \mathcal{S}_f[a_1,a_2] \times_{f^{-1}a_2} \cdots \times_{f^{-1}(a_{n-1})} \mathcal{S}_f[a_{n-1},a_n]$$

of sections $[a_0,a_n] \to X$ labeled by the given sequence. With this notation, we may identify the space of n–simplices

$$(\mathrm{N}\mathcal{S}_f) = \coprod_{a_0 \le \cdots \le a_n \subset \mathbb{R}} \mathcal{S}_f[a_0,\ldots,a_n].$$

If, in addition, the function $f\colon X \to \mathbb{R}$ is a Reeb function on a stratified space X, then the simplicial space $\mathrm{N}\mathcal{C}_f^A$ has an associated space of n–simplices

$$(\mathrm{N}\mathcal{C}_f^A) = \coprod_{a_0 \le \cdots \le a_n \subset A} \mathcal{S}_f[a_0,\ldots,a_n].$$

3.3 A proof of Theorem 1.1

Let $f\colon X \to \mathbb{R}$ be a continuous function from a topological space X to the real line. It is tempting to presume $X \simeq \mathrm{B}\mathcal{S}_f$ in general (Theorem 1.1). But that is not the case.

Example 3.4. There is a continuous function $f\colon I \to \mathbb{R}$, from the unit interval I, uniquely determined by the formula $f(x) = x\sin(\frac{1}{x})$. Proposition 4.5, to be proven, tells us that there cannot be a path from 1 to any other point in $\mathrm{B}\mathcal{S}_f$: such a path would have to meet an infinite number of 1–cells up to homotopy fixing endpoints. Hence $\mathrm{B}\mathcal{S}_f$ has at least two path components. In fact, $\mathrm{B}\mathcal{S}_f$ is weakly equivalent to the disjoint union of two points–see Example 4.6, to be computed.

Assume from here on that $f\colon X \to \mathbb{R}$ is a Reeb function and that A is a subset of the real line in concordance with Definition 3.2. Recall from Example 3.3 that the space of n–simplices in $\mathrm{N}\mathcal{C}_f^A$ is the disjoint union $\coprod \mathcal{S}_f[\bar{a}]$, indexed over non-decreasing sequences $\bar{a} = (a_0,\ldots,a_n)$ in A. Points in $\mathrm{B}\mathcal{C}_f^A$ are thus classes $[\bar{\sigma},\bar{t}]$ with $(\bar{\sigma},\bar{t})$ a tuple in $\mathcal{S}_f[\bar{a}] \times \Delta^n$. There is a map $\phi\colon \mathrm{B}\mathcal{C}_f^A \to X$ which is soon to be proven a weak homotopy equivalence. For a representative with $\bar{\sigma} = (\sigma_1,\ldots,\sigma_n)$, a sequence of composable sections $[a_{i-1},a_i] \to X$, and $\bar{t} = (t_0,\ldots,t_n)$, it is defined by $\phi[\bar{\sigma},\bar{t}] = \sigma_n \circ \cdots \circ \sigma_1(\bar{a}\bar{t})$. The notation $\bar{a}\bar{t}$ is short for the dot product $a_0t_0 + \cdots + a_nt_n$. It is a straightforward hassle to verify that this does in fact induce a well-defined map on $\mathrm{B}\mathcal{C}_f^A$. Post composition with f now defines a map $\bar{f} = f \circ \phi$ from $\mathrm{B}\mathcal{C}_f^A$ to the real line. Applying f to the above formula reveals that $\bar{f}[\bar{\sigma},\bar{t}] = \bar{a}\bar{t}$ on representatives–for the composition $\sigma_n \circ \cdots \circ \sigma_1$ in $\mathcal{C}_f^A$ is a section of f.

Let us establish some notation before proving some convenient lemmas. Given a finite non-decreasing sequence $\bar{a} = (a_0, \ldots, a_n)$ in A and a subspace K of Δ^n, we denote by $\langle \bar{a}, K \rangle$ the image of $\coprod \mathcal{S}_f[\bar{b}] \times K$, ranging over all subsequences $\bar{b}$ of $\bar{a}$, under the quotient map that defines $\mathrm{B}\mathcal{C}_f^A$. In particular, $\langle \bar{a}, \Delta^n \rangle$ is the subspace generated by classes whose representative is decorated by a subsequence of $\bar{a}$.

Lemma 3.5. If $m\colon K \to \mathrm{B}\mathcal{C}_f^A$ is a map from a compact space K, then there is an increasing sequence $\bar{a}$ in A of length n such that the image of K is contained in $\langle \bar{a}, \Delta^n \rangle$.

Proof. Let

$$\mathrm{sk}_k \mathrm{B}\mathcal{C}_f^A = (\coprod_{q \le k} (\mathrm{N}\mathcal{S}_f)_q \times \Delta^q)/\sim$$

be the k–skeleton of $\mathrm{B}\mathcal{C}_f^A$. It is well-known that the map m must factor through some k–skeleton of $\mathrm{B}\mathcal{C}_f^A$, because K is compact. In our notation, one may alternatively write the skeleton as a union $\mathrm{sk}_k \mathrm{B}\mathcal{C}_f^A = \cup \langle \bar{b}, \Delta^k \rangle$, ranging over all non-decreasing sequences $\bar{b}$ of length $\le k$ in A. Hence we can deduce even more: the image of m can only meet finitely many subspaces of the form $\langle \bar{b}, \Delta^k \rangle$, i.e. it factorizes through $\cup_{i=0,\ldots,q} \langle \bar{b}_i, \Delta^m \rangle$ for finitely many sequences $\bar{b}_i = (b_{i,0}, \ldots, b_{i,k})$. Include and order all the components $b_{i,j}$ to define the bigger increasing sequence $\bar{a} = (a_0, \ldots, a_n)$. A point $[\bar{\sigma}, \bar{t}]$ in the image of m comes with a subsequence of $\bar{a}$. □

Recall the spine $\mathrm{sp}\Delta^n$ of the topological n–simplex. It is the subspace parametrized by tuples $\bar{t} = (t_0, \ldots, t_n)$ satisfying that at most two successive entries are non-zero; there is an i such that $t_j = 0$ for all j except possibly $j = i-1, i$. Points, or classes, in $\langle \bar{a}, \mathrm{sp}\Delta^n \rangle$ have a particularly nice presentation: a point $[\bar{\sigma}, \bar{t}]$ in $\langle \bar{a}, \mathrm{sp}\Delta^n \rangle$ can be represented $[\sigma_i, (t_{i-1}, t_i)]$, because of how $\bar{t} = (0, \ldots, t_{i-1}, t_i, 0 \ldots, 0)$ for some i.

Lemma 3.6. Consider a Reeb function $f\colon X \to \mathbb{R}$ and $\bar{a}$ an increasing sequence in A. The subspace $\langle \bar{a}, \Delta^n \rangle$ deformation retracts onto $\langle \bar{a}, \mathrm{sp}\Delta^n \rangle$ in $\mathrm{B}\mathcal{C}_f^A$. Moreover, the deformation retract preserves fibers of $\bar{f}$.

Proof. A point $[\bar{\sigma}, \bar{t}] = [\sigma_i, (t_{i-1}, t_i)]$ in $\langle \bar{a}, \mathrm{sp}\Delta^n \rangle$ is mapped to $t_{i-1}a_{i-1} + t_i a_i$ under $\bar{f}$. For a fixed $\bar{\sigma}$ in $\mathcal{S}_f[\bar{a}]$ the map $\bar{f} \circ (\bar{\sigma}, \mathrm{id}_{\mathrm{sp}\Delta^n})\colon \mathrm{sp}\Delta^n \to \mathbb{R}$ is injective, because $\bar{a}$ is an increasing sequence. So for every $a_0 \le b \le a_n$ and $\bar{\sigma}$ there is a unique $\bar{s}_b$ in $\mathrm{sp}\Delta^n$ such that $\bar{f}[\bar{\sigma}, \bar{s}_b] = b$. Two points in $\langle \bar{a}, \Delta^n \rangle \cap \bar{f}^{-1} b$ both associate to the same $\bar{s}_b$. The homotopy

$$R([\bar{\sigma}, \bar{t}], t) = [\bar{\sigma}, (1-t)\bar{t} + t\bar{s}_{\bar{f}[\bar{\sigma}, \bar{t}]}]$$

is thus well-defined. And it satisfies $R(-, 0) = \mathrm{id}_{\langle \bar{a}, \Delta^n \rangle}$ whereas the image of $R(-, 1)$ is contained in $\langle \bar{a}, \mathrm{sp}\Delta^n \rangle$. It is a deformation retract because $\bar{s}_{\bar{f}[\bar{\sigma}, \bar{t}]} = \bar{t}$ whenever $\bar{t}$ is in the spine; the homotopy is trivial when restricted to the spine. □

The third lemma is analogous to Lemma 2.8.

Lemma 3.7. Consider a Reeb function $f\colon X \to \mathbb{R}$ and $\bar{a} = (a_0, \dots, a_n)$ an increasing sequence in A such that $[a_0, a_n]$ contains at most one critical value of f. Then the subspace $\langle \bar{a}, \mathrm{sp}\Delta^n \rangle$ of $\mathrm{B}\mathcal{C}_f^A$ deformation retracts onto

i) $f^{-1}a$ for any a in $\bar{a}$ if there is no critical value or

ii) $f^{-1}a$ for a equal to the critical value, otherwise.

Proof. We may assume $a = a_0$, much like we only consider the case $[a-n, a]$ in the proof of Lemma 2.8.

The deformation retract will be defined inductively in n. If $n = 0$, then there is nothing to prove since $\langle a, \Delta^0 \rangle$ is equal to $f^{-1}a$. Assume the existence of a deformation retract

$$R_{n-1}\colon \langle (a_0, \dots, a_{n-1}), \mathrm{sp}\Delta^{n-1} \rangle \times I \to \langle (a_0, \dots, a_{n-1}), \mathrm{sp}\Delta^{n-1} \rangle$$

onto $f^{-1}a$. A deformation retract of $\langle \bar{a}, \mathrm{sp}\Delta^n \rangle$ onto $\langle (a_0, \dots, a_{n-1}), \mathrm{sp}\Delta^{n-1} \rangle$ will be defined. Observe how $\langle \bar{a}, \mathrm{sp}\Delta^n \rangle$ is the union of $\langle (a_0, \dots, a_{n-1}), \mathrm{sp}\Delta^{n-1} \rangle$ and $\langle (a_{n-1}, a_n), \Delta^1 \rangle$, so surely it suffices to deformation retract $\langle (a_{n-1}, a_n), \Delta^1 \rangle$ onto $f^{-1}a_{n-1}$.

Choose a collection of reparametrized flow-lines g_x according to Proposition 2.7. A point $[x]$ in $f^{-1}(a_n)$, considered as a subspace of $\langle (a_{n-1}, a_n), \Delta^1 \rangle$, has a canonical choice of representative $[g_x, (0,1)]$ where $g_x\colon [a_{n-1}, a_n] \to X$ is a reparametrized flow-line. Hence every point in $\langle (a_{n-1}, a_n), \Delta^1 \rangle \setminus f^{-1}a_{n-1}$ is represented $[\sigma, (t_0, t_1)]$ with $\sigma\colon [a_{n-1}, a_n] \to X$ a section. To every section $\sigma\colon [a_{n-1}, a_n] \to X$ we associate the map $g_\sigma\colon [a_{n-1}, a_n]^2 \to X$ determined by $g_\sigma(b,c) = g_{\sigma(b)}(c)$, where $g_{\sigma(b)}$ is the reparametrized flow-line through the point $\sigma(b)$. The section σ can be identified with the composition

$$[a_{n-1}, a_n] \xrightarrow{d} [a_{n-1}, a_n]^2 \xrightarrow{g_\sigma} X$$

where d is the diagonal map $b \longmapsto (b,b)$. Similarly, for a fixed value b_0, we also recover $g_{\sigma(b_0)}$ as the composition

$$[a_{n-1}, a_n] \xrightarrow{(b_0, \mathrm{id})} [a_{n-1}, a_n]^2 \xrightarrow{g_\sigma} X.$$

The straight line homotopy $H_{b_0}\colon I \to \mathrm{map}([a_{n-1,a_n}], [a_{n-1,a_n}]^2)$,

$$H_{b_0}(t)(b) = ((1-t)b + tb_0, b)$$

between d an (b_0, id) will give us the desired deformation retract. Indeed, whenever the point $[\sigma, (t_0, t_1)]$ is contained in $\langle (a_{n-1}, a_n), \Delta^1 \rangle \setminus f^{-1}a_{n-1}$, we declare

$$R_n([\sigma, (t_0, t_1)], t) = [g_\sigma \circ H_{\bar{f}[\sigma,(t_0,t_1)]}(t), (1-t)(t_0, t_1) + t(1,0)].$$

When $t = 0$ the output is $[\sigma, (t_0, t_1)]$ whereas $t = 1$ yields $[g_{\phi[\sigma,(t_0,t_1)]}, (1,0)]$, or equivalently $[g_{\phi[\sigma,(t_0,t_1)]}(a_{n-1})]$, in $f^{-1}a_{n-1}$. We cannot guarantee the existence of a section through a_{n-1}, so we manually extend R_n to $\langle (a_{n-1}, a_n), \Delta^1 \rangle$: map a class $[x]$ in $f^{-1}a_{n-1}$ to $R_n([x], t) = [x]$ for all t. $\square$

The following lemma is easily proven directly.

Lemma 3.8. Assume that A' is constructed from A by adding a single real value a. Then the inclusion $\mathcal{C}_f^A \hookrightarrow \mathcal{C}_f^{A'}$ induces a weak homotopy equivalence on classifying spaces.

Proof. For any commutative diagram

$$\begin{tikzcd}
\partial\Delta^m \arrow[r] \arrow[d, "i"] & \mathcal{C}_f^A \arrow[d] \\
\Delta^m \arrow[r, "l"] & \mathcal{C}_f^{A'}
\end{tikzcd}$$

we find a lift $L\colon \Delta^m \to \mathcal{C}_f^A$ up to homotopy relative to $\partial\Delta^m$. Lemma 3.5 tells us that the map l factorizes through some subspace $\langle \bar{a}, \Delta^n \rangle$ of $\mathcal{C}_f^{A'}$. If a is not contained in $\bar{a}$, then l factorizes through $\mathcal{C}_f^A$, and we are done. Otherwise, let i be the index of a in $\bar{a}$. Every point $\bar{t}$ in the standard n–simplex has a closest point $\bar{t}_i$ in the ith face of Δ^n. If $l(\bar{s}) = [\bar{\sigma}, \bar{t}]$, we declare $L(\bar{s}) = [\bar{\sigma}, \bar{t}_i]$ to get the desired lift. □

We are prepared to prove that the homotopy types of $\mathrm{B}\mathcal{C}_f^A$ and X coincide for all eligible A. In particular, Theorem 1.1 follows.

Theorem 3.9. Let $f\colon X \to \mathbb{R}$ be a Reeb function. The map $\phi\colon \mathrm{B}\mathcal{C}_f^A \to X$ is a weak homotopy equivalence for all non-empty subsets $A \subset \mathbb{R}$ containing the critical values of f.

Proof. We can essentially reduce the problem to two special cases: 1. f has no critical values and 2. f has precisely one critical value. Indeed, assume that there is at least two critical values. Enumerate them (c_i) according to the standard ordering of the real line. Every critical value c_i is contained in $N_i = (c_{i-1}, c_{i+1})$, possibly interpreting $c_{i-1} = -\infty$ or $c_{i+1} = \infty$. Whence we deduce open covers of X and $\mathcal{C}_f^A$ by considering $U_i = f^{-1}N_i$ and $V_i = \bar{f}^{-1}N_i$, respectively. Do note that the only non-empty intersections are of the form $N_{i,j} = N_i \cap N_j$ for $j = i, i+1$. Hence it suffices to prove that ϕ restricts to weak homotopy equivalences $\phi|_{V_{i,j}}\colon V_{i,j} \to U_{i,j}$ with $V_{i,j} = V_i \cap V_j$ and $U_{i,j} = U_i \cap U_j$ subject to $j = i, i+1$. This essentially follows since X and $\mathrm{B}\mathcal{C}_f^A$ can be described as homotopy colimits over the given covers. See Theorem 6.7.9 in [Die08] for a precise reference.

The open cover N_i is constructed so that $N_{i,j} = N_i \cap N_j$ contains precisely one critical value when $i = j$, whereas it contains no critical values when $j = i+1$. So the reduction to the above two special cases is made precise.

Fix $N = N_{i,j}$, $U = U_{i,j}$ and $V = V_{i,j}$ for $j = i$ or $i+1$. Also, pick a value a in N. If $i = j$, then a must be chosen as the critical value. Otherwise, it can be any real value.

Lemma 3.8 allow us to assume that a is contained in A without loss of generality. We may thus assume the inclusion $f^{-1}a \hookrightarrow U$ to factorize through $\phi|_V$. By the two out of three property it only remains to see that $f^{-1}a \hookrightarrow U$ and $f^{-1}a \hookrightarrow V$ are weak homotopy equivalences. The first follows directly from Lemma 2.8.

To see that $f^{-1}a \hookrightarrow V$ is a weak homotopy equivalence we consider an arbitrary commutative diagram

$$\begin{tikzcd} \partial\Delta^m \arrow[r] \arrow[d, "i"] & f^{-1}a \arrow[d] \\ \Delta^m \arrow[r, "l"] & V \end{tikzcd}$$

and find a lift $L\colon \Delta^m \to f^{-1}a$ up to homotopy relative to $\partial\Delta^m$. Lemma 3.5 tells us that the image of l is contained in a subspace $\langle \bar{a}, \Delta^n \rangle$. The deformation retract onto $\langle \bar{a}, \mathrm{sp}\Delta^n \rangle$, provided by Lemma 3.6, fixes $f^{-1}a$ and hence l factorizes through $\langle \bar{a}, \mathrm{sp}\Delta^n \rangle$, up to homotopy relative $\partial\Delta^m$. A point in V must map into N under $\bar{f}$, hence we can assume $\bar{a}$ to be contained in N without loss of generality. Now Lemma 3.7 applies to give a deformation retract from $\langle \bar{a}, \mathrm{sp}\Delta^n \rangle$ to $f^{-1}a$. But then we can homotope l to a map which factorizes through $f^{-1}a$, up to homotopy relative to $\partial\Delta^m$. □

4 Reeb functions for homology computations

Section 4.1 introduces the *section spectral sequence* associated naturally to a continuous function $f\colon X \to \mathbb{R}$. A noteworthy property of this spectral sequence is that its second page only consists of two non-trivial columns (Proposition 4.3). In Section 4.2 we introduce the much smaller *critical spectral sequence*. Basic computational tools are deduced and illustrated.

4.1 The section spectral sequence

Let us fix a generalized homology theory k_*. Recall that a simplicial space $X_\bullet$ comes with a spectral sequence whose termination is $k_*|X_\bullet|$. The cohomological version is derived in [Seg68]. For every q, a simplicial abelian group $k_qX_\bullet\colon \Delta^{\mathrm{op}} \to$ (Abelian groups) is obtained by applying k_q level-wise. Entries on the first page are given by $\mathrm{E}^1_{p,q} = k_qX_p$, whereas the differential is induced by the face maps in $k_qX_\bullet$. Indeed, a simplicial abelian group $A_\bullet$ defines a chain complex by collapsing degeneracies entry-wise and defining the differential $\partial = \sum_i (-1)^i d_i$. Entries on the second page are given by $\mathrm{E}^2_{p,q} = \mathrm{H}_p k_q X_\bullet$. A map $F\colon X_\bullet \to Y_\bullet$ of simplicial spaces naturally induces morphisms $k_qF_p\colon \mathrm{E}^1_{p,q} \to \bar{\mathrm{E}}^1_{p,q}$ and hence morphisms $\mathrm{E}^2_{p,q} \to \bar{\mathrm{E}}^2_{p,q}$ on the second page. This does in fact define a map in the category of (homology) spectral sequences, but there is no need for further abstraction.

We have seen that a continuous function $f\colon X \to \mathbb{R}$, on a topological space X, defines a section category $\mathcal{S}_f$ whose morphisms are the sections $\sigma\colon [a,b] \to X$.

Definition 4.1. The *section spectral sequence* of a continuous function $f\colon X \to \mathbb{R}$ is the spectral sequence naturally associated to the simplicial topological space $\mathrm{N}\mathcal{S}_f$.

Theorem 1.1 tells us that for a Reeb function defined on a topological space X, the associated section spectral sequence converges to H_*X:

Proposition 4.2. For $f\colon X \to \mathbb{R}$ a Reeb function, the section spectral sequence converges to H_*X:

$$\mathrm{H}_p k_q \mathrm{N}\mathcal{S}_f \Rightarrow k_{p+q}X.$$

The first algebraic observation is concerned with computability: there are only two non-zero columns on the second page of the section spectral sequence. In particular, the sequence collapses on the second page; all differentials on the second page are zero.

Proposition 4.3. Let $f\colon X \to \mathbb{R}$ be a continuous function. The section spectral sequence of $\mathrm{N}\mathcal{S}_f$ satisfies $\mathrm{E}^2_{p,q} = 0$ for $p \geq 2$.

Proof. The additivity axiom of k_* implies

$$E^1_{p,q} = k_q\Big(\coprod_{a_0<\cdots<a_o} \mathcal{S}_f(a_0,\dots a_p)\Big) \simeq \bigoplus_{a_0<\cdots<a_p} k_q\mathcal{S}_f(a_0,\dots,a_p).$$

For an arbitrary q we fix a $p \geq 2$ and denote by $\partial_p\colon \mathrm{E}^1_{p,q} \to \mathrm{E}^1_{p-1,q}$ the differential. On the level of elements, an element α in $k_q\mathcal{S}_f[a_0,\dots,a_p]$ is mapped to the alternating sum $\sum(-1)^i d_i\alpha$ with $d_i\alpha$ an element in $k_q\mathcal{S}_f[a_0,\dots,\hat{a}_i,\dots,a_p]$. We shall see that the kernel of ∂_p is contained in the image of ∂_{p+1}, thus justifying the assertion.

An arbitrary element α in the kernel of ∂_p is a linear combination $c_1\alpha_1 + \cdots + c_n\alpha_n$ where each α_i is in some $k_q\mathcal{S}_f[a_{i,0},\dots,a_{i,p}]$. We proceed by defining a process which splits α into a sum $\alpha' + \partial_{p+1}\beta'$ and argue why finitely many iterations must eventually lead to $\alpha = \partial_{p+1}\beta$ for some β in $k_q(\mathrm{N}\mathcal{S}_f)_{p+1}$.

Start by picking the α_i with smallest possible $a_{i,0}$ and recursively smallest possible $a_{i,j+1}$ subject to smallest possible $a_{i,j}$. Since α is assumed to be in the kernel of ∂_p, there must be a j and n such that $d_n\alpha_j$ cancels $d_p\alpha_i$. But α_i was chosen so that n must necessarily equal p, otherwise we would have picked α_j in place of α_i. There are now two cases:

1. The class α_j restricts to α_i. That is, we can find a class β in $k_q\mathcal{S}_f[a_{i,0},\dots,a_{i,p},a_{j,p}]$ satisfying $d_{p+1}\beta = \alpha_i$ and $d_p\beta = \alpha_j$. In this case, we replace $(-1)^p(\alpha_i - \alpha_j)$ with the

linear combination $\partial_{p+1}\beta - \sum_{k\neq p,p+1} d_k\beta$ in the linear combination $c_1\alpha_1 + \cdots + c_n\alpha_n$ to rewrite $\alpha = c'_1\alpha'_1 + \cdots + c'_m\alpha'_m + \partial_{p+1}\beta$. Now there is one less summand α'_k with the minimal configuration of α_i.

2. Otherwise, if there is no such j, we rather apply d_{p-1} to α_i. This produces an element $d_{p-1}\alpha_i$ contained in $k_q\mathcal{S}_f[a_{i,0},\ldots,\hat{a}_{i,p-1},a_{i,p}]$. There must be a k and m such that $d_m\alpha_k$ cancels $d_{p-1}\alpha_i$. As in 1. the minimality of α_i implies that $m = p-1$. Under the assumption $p \geq 2$, it is possible to lift α_i to $k_q\mathcal{S}_f[a_{i,0},\ldots,a_{i,p-1},a_{k,p-1},a_{i,p}]$ by adding the distinct label from α_k. Denote such a lift β. This element satisfies $d_{p-1}\beta = \alpha_k$ and $d_p\beta = \alpha_i$. Moreover, among the $d_l\beta$ for $l \neq p-1,p$, only $d_{p+1}\beta$ has a smaller configuration than α_i. As in 1. we replace $(-1)^p(\alpha_i - \alpha_k)$ with $\partial_{p+1}\beta - \sum_{l\neq p-1,p} d_l\beta$, yielding a new linear combination $\alpha = c'_1\alpha'_1 + \cdots + c'_m\alpha'_m + \partial_{p+1}\beta$. We did not manage to reduce the number of minimal configurations, but rather replaced a minimal configuration by a smaller minimal configuration.

Repeating this process iteratively must terminate. Indeed, case 2. can only repeat finitely many times as there are only finitely many summands in α. So we have successfully constructed an iterative process that reduces the number of summands with a minimal configuration after finitely many steps. □

The second page of the section spectral sequence only consists of two adjacent non-trivial columns. This amounts to a collection of short exact sequences ([Wei95, p. 124])

$$\mathrm{H}_0 k_q(\mathrm{N}\mathcal{S}_f)_0 \hookrightarrow k_q\mathrm{B}\mathcal{S}_f \twoheadrightarrow \mathrm{H}_1 k_{q-1}(\mathrm{N}\mathcal{S}_f)_1$$

where the kernel map is induced from the evident inclusion of fibers. Moreover, source and target of sections induce $\partial^1_{1,q}\colon \mathrm{E}^1_{1,q} \to \mathrm{E}^1_{0,q}$, the differential on the first page. We recognize $\mathrm{H}_0 k_q(\mathrm{N}\mathcal{S}_f)_0$ and $\mathrm{H}_1 k_q(\mathrm{N}\mathcal{S}_f)_1$ as the cokernel and kernel of $\partial^1_{1,q}$, respectively. The sequences can thus be concatenated to obtain the following Mayer–Vietoris type result.

Corollary 4.4. Any continuous function $f\colon X \to \mathbb{R}$ determines a long exact sequence

$$\cdots \to \bigoplus_{a\in\mathbb{R}} k_q f^{-1}a \to k_q\mathrm{B}\mathcal{S}_f \to \bigoplus_{a\leq b} k_{q-1}\mathcal{S}_f[a,b] \to \bigoplus_{a\in\mathbb{R}} k_{q-1} f^{-1}a \to \cdots$$

for all $q \geq 1$.

An f–path is a path $p\colon I \to X$ in X which reparametrizes to a concatenation of paths contained in fibers of f and possibly reversed sections of f. Define the relation $\sim_f$ on X by declaring that points are equivalent if they can be connected by an f–path.

Proposition 4.5. For $f\colon X \to \mathbb{R}$ a continuous function, the abelian group $\mathrm{H}_0\mathrm{B}\mathcal{S}_f$ is the free abelian group on $X/\sim_f$.

Proof. The assertion is a direct consequence of $\mathrm{H}_0\mathrm{B}\mathcal{S}_f = \mathrm{E}^2_{0,0}$. Indeed, $\mathrm{E}^2_{0,0} = \mathrm{H}_0\mathrm{H}_0\mathrm{N}\mathcal{S}_f$ where the first application of H_0 identifies points in the same path components of fibers, whereas the second identifies points that can be connected via sections. $\square$

The following calculation justifies Example 3.4.

Example 4.6. Consider the continuous function $f\colon I \to \mathbb{R}$, $f(x) = x\sin(\frac{1}{x})$. We shall verify that the homology of $\mathrm{B}\mathcal{S}_f$ with coefficients in $\mathbb{Z}$ is

$$\mathrm{H}_n\mathrm{B}\mathcal{S}_f = \begin{cases} \mathbb{Z}^2 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

It follows from Proposition 4.3 that the associated spectral sequence has $\mathrm{E}^2_{p,q} = 0$ whenever $p \geq 2$. Moreover, all of the section spaces $\mathcal{S}_f[a_0,\dots,a_n]$ are discrete, except for the fiber $\mathcal{S}_f[0] = f^{-1}0$ which is homeomorphic to the subspace $\{\frac{1}{n} | n = 1,2,\dots\} \cup 0$ of $\mathbb{R}$. Nonetheless, the topological space $\mathcal{S}_f[0]$ is homotopically discrete as it is weakly equivalent to the natural numbers equipped with the discrete topology. We thus conclude that $\mathrm{E}^1_{p,q} = 0$ for $q \geq 1$. So in light of Proposition 4.3 it only remains to calculate $\mathrm{E}^2_{0,0}$ and $\mathrm{E}^2_{1,0}$.

Let $\partial_1\colon \mathrm{E}^1_{1,0} \to \mathrm{E}^1_{0,0}$ be the remaining non-zero differential on the first page and pick a linear combination $c_1\sigma_1 + \cdots + c_n\sigma_n$ in its kernel. Let m_i denote the minimum of σ_i, a section into I, and consider m the smallest number among the m_i. Observe how every σ_i maps into $I_m = [m,1]$. The inclusion $j\colon I_m \hookrightarrow I$ satisfies $f|_{I_m} = f \circ j$ and so there is an induced map $\bar{\mathrm{E}}^*_{p,q} \to \mathrm{E}^*_{p,q}$, between associated spectral sequences. Theorem 1.1 applies to f_{I_m} so that $\mathrm{H}_1 I_m = 0$ implies $\bar{\mathrm{E}}^2_{0,1} = 0$. Hence $c_1\sigma_1 + \cdots + c_n\sigma_n$ is in the image of $\bar{\partial}_1\colon \bar{\mathrm{E}}^1_{2,0} \to \bar{\mathrm{E}}^1_{1,0}$, but then it is also in the image of $\partial_2\colon \mathrm{E}^1_{2,0} \to \mathrm{E}^1_{1,0}$. The isomorphism $\mathrm{E}^2_{1,0} \simeq 0$ thus follows.

Proposition 4.5 implies $\mathrm{E}^2_{0,0} \simeq \mathbb{Z}^2$: All pairs in $(0,1]$ can be connected by a finite zigzag of sections, whereas no section starts nor terminates in 0. Hence $\mathrm{H}_0\mathrm{B}\mathcal{S}_f \simeq \mathbb{Z}^2$.

4.2 The critical spectral sequence

For a continuous function $f\colon X \to \mathbb{R}$, the associated section spectral sequence

$$\mathrm{H}_q k_p \mathrm{N}\mathcal{S}_f \Rightarrow \mathrm{H}_{p+q}\mathrm{B}\mathcal{S}_f$$

collapses on the second page which only consists of two non-zero columns. Alas, it has a huge problem when it comes to computability. Computing $\mathrm{H}_p k_q \mathrm{N}\mathcal{S}_f$ abouts to an uncountable number of homology computations. For we would have to determine $k_q\mathcal{S}_f[a,b]$ for all real numbers $a \leq b$. But if $f\colon X \to \mathbb{R}$ is a Reeb function, we shall see that the complexity is drastically reduced.

Recall from Section 3.1 that for every non-empty subset A of $\mathbb{R}$ which contains the critical values of f, there is the subcategory $\mathcal{C}_f^A$ of sections whose source and target are both contained in $f^{-1}A$. From here on we assume f to have at least one critical value and introduce the *critical category* $\mathcal{C}_f = \mathcal{C}_f^{\{\text{critical values}\}}$.

Definition 4.7. The *critical spectral sequence* of a Reeb function $f\colon X \to \mathbb{R}$ is the spectral sequence naturally associated to $\mathrm{N}\mathcal{C}_f$.

Theorem 3.9 tells us that the critical spectral sequence converges to k_*X.

Proposition 4.8. Let $f\colon X \to \mathbb{R}$ be a Reeb function. The critical spectral sequence converges to k_*X:

$$\mathrm{H}_q k_p \mathrm{N}\mathcal{C}_f \Rightarrow k_{p+q}X.$$

As opposed to the section spectral sequence, we need only compute the generalized homology groups of $\mathcal{S}_f[c_0, c_1]$ whenever $c_0 < c_1$ are critical values of f. If for example X is compact, this reduces the number of qth generalized homology groups needed to compute from uncountable to finite. We shall reduce the complexity even further: it suffices to compute the generalized homology groups of $\mathcal{S}_f[c_0, c_1]$ whenever $c_0 < c_1$ are *successive* critical values. Let us introduce some convenient notation before stating the formal result.

I remind that $\mathrm{E}^1_{p,q} \simeq \oplus k_q \mathcal{S}_f[c_0, c_1]$ ranging over all critical values $c_0 < c_1$. For every $q \geq 0$, the differential $\partial^1_{1,q}\colon \mathrm{E}^1_{1,q} \to \mathrm{E}^1_{0,q}$ restricts to a morphism

$$\partial^s_{1,q}\colon \bigoplus_{\text{successive } c_0 < c_1} k_q \mathcal{S}_f[c_0, c_1] \to \bigoplus k_q \mathcal{S}_f[c].$$

only ranging over successive pairs of critical values $c_0 < c_1$.

Proposition 4.9. For $f\colon X \to \mathbb{R}$ a Reeb function, the second page of the associated critical spectral sequence satisfies $\mathrm{E}^2_{1,q} \simeq \ker \partial^s_{1,q}$ for all $q \geq 2$.

Proof. Let β be an element in $\mathrm{E}^1_{2,q}$. The relation imposed by $\partial^1_{2,q}$ on $\mathrm{E}^2_{1,q}$ is determined by the equation $\partial^1_{2,q}\beta = d_0\beta - d_1\beta + d_2\beta$ and implies that $[d_1\beta] = [d_0\beta] + [d_2\beta]$. If β is in $k_q\mathcal{S}_f[c_0, c_1, c_2]$, then $d_0\beta$ is in $k_q\mathcal{S}_f[c_1, c_2]$ etc. So the map d_i simply forgets the ith label.

For a $[\alpha]$ in $\mathrm{E}^2_{1,q}$ represented by α in $k_q\mathcal{S}_f[c_0, c_1]$, we list all intermediate critical values $d_0 < \cdots < d_n$ with $d_0 = c_0$ and $d_n = c_1$. Now it is only a matter of applying the relation imposed by $\partial^1_{2,q}$ n times to rewrite $[\alpha]$ as a linear combination $[\alpha_1] + \cdots + [\alpha_n]$ with α_i in $k_q\mathcal{S}_f[d_{i-1}, d_i]$. Hence the elements in $\ker \partial^s_{1,q}$ generates $\mathrm{E}^2_{1,q}$. Moreover, the relation induced by $\partial^1_{2,q}$ is trivial on this set of generators, for they cannot be decomposed further. □

If $f\colon X \to \mathbb{R}$ is a Reeb function, then Proposition 4.9 implies

$$\mathrm{H}_q X \simeq (\mathrm{H}_q(\mathcal{C}_f)_0/\mathrm{im}\partial^s_{1,q}) \bigoplus \ker \partial^s_{1,q-1}$$

for $q \geq 1$ whenever we have chosen coefficients in a field.

As a last computational tool, we recognize the homotopy type of section spaces decorated by successive critical values. These section spaces have the homotopy type of any intermediate fiber. Hence the critical sequence recovers the homology of X using the homology type of certain fibers. I refer to Example 4.11 for a hands-on demonstration.

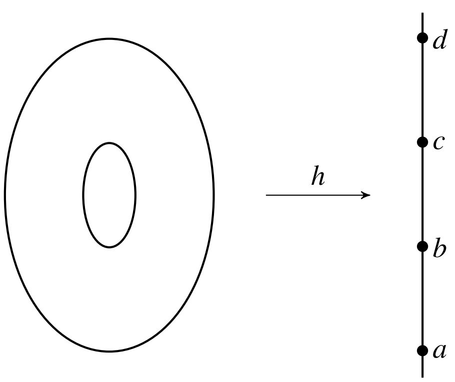
Proposition 4.10. Consider c and d two successive critical values of f as well as a real number a in (c,d). The evaluation map $\mathrm{eval}_a\colon \mathcal{S}_f[c,d] \to f^{-1}a$ is a homotopy equivalence.

Proof. Let $g\colon [c,d] \times f^{-1}(c,d) \to X$ be a family of reparametrized sections which exists per Proposition 2.7. Its adjoint $\bar{g}\colon f^{-1}(c,d) \to \mathcal{S}_f[c,d]$ restricts to a map from the fiber $g_a\colon f^{-1}a \to \mathcal{S}_f[c,d]$, mapping a point x in $f^{-1}a$ to the section, or reparametrized flow-line, g_x through x. It is clear that $\mathrm{eval}_a \circ g_a$ is the identity on $f^{-1}a$. Conversely, the composition $g_a \circ \mathrm{eval}_a$ maps a section $\sigma\colon [c,d] \to X$ to the section $g_{\sigma(a)}$ through $\sigma(a)$. Note that we may identify σ with the section $b \longmapsto g_{\sigma(b)}(b)$.

Denote by $H\colon\ [c,d] \times I \to [c,d]$ the straight line homotopy $H(b,t) = (1-t)b + ta$, from which we define a homotopy $G\colon \mathcal{S}_f[c,d] \times I \to \mathcal{S}_f[c,d]$. It maps a tuple (σ,t) to the section $b \longmapsto g_{\sigma\circ H(b,t)}(b)$. Notice that $G(-,0)$ is σ whereas $G(-,1)$ is $g_{\sigma(a)}$. We have thus constructed a homotopy from the identity on $\mathcal{S}_f[c,d]$ to $g_a \circ \mathrm{eval}_a$. $\square$

We calculate the torus' homology as means to illustrate the computational implication of Propositions 4.9 and 4.10.

Example 4.11. Let $h\colon T \to \mathbb{R}$ be the height function on the torus depicted



r	$f^{-1}r$
a	pt
$\frac{a+b}{2}$	$\partial\Delta^2$
b	$\partial\Delta^2 \vee \partial\Delta^2$
$\frac{b+c}{2}$	$\partial\Delta^2 \coprod \partial\Delta^2$
c	$\partial\Delta^2 \vee \partial\Delta^2$
$\frac{c+d}{2}$	$\partial\Delta^2$
d	pt

It has four critical values a, b, c and d. Proposition 4.10 tells us that the above table determines the homotopy type of all section spaces. We calculate the first page of the critical spectral sequence, knowing that we need only compute the homology of section spaces between successive critical values (Proposition 4.9):

$$
\begin{array}{c|cccc}
q & & & & \\
& 0 & & 0 & 0 \\
& 0 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus 0 & \xleftarrow{\ \partial_{1,1}\ } & \mathbb{Z} \oplus \mathbb{Z}^2 \oplus \mathbb{Z} & 0 \\
& \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{\ \partial_{1,0}\ } & \mathbb{Z} \oplus \mathbb{Z}^2 \oplus \mathbb{Z} & 0 \\
\hline
& & & & p
\end{array}
$$

Homology groups are split up according to the above table, e.g.

$$\mathrm{E}^1_{1,0} = \mathrm{H}_0 \partial \Delta^1 \oplus \mathrm{H}_0(\partial \Delta^1 \coprod \partial \Delta^1) \oplus \mathrm{H}_0 \partial \Delta^1 \simeq \mathbb{Z} \oplus \mathbb{Z}^2 \oplus \mathbb{Z}.$$

The differentials are induced by subtracting target from source: $\partial = d_0 - d_1 = t - s$. For instance, the induced map $\mathrm{H}_0 t \colon \mathrm{H}_0 f^{-1}(\frac{a+b}{2}) \to \mathrm{H}_0 f^{-1} b$ is the identity $1 \colon \mathbb{Z} \to \mathbb{Z}$ in coordinates. This is because the target of a flow-line through $f^{-1}\frac{a+b}{2}$ meets the path component of $f^{-1}b$. By such geometric reasoning we deduce

$$\partial_{1,0} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \partial_{1,1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Elementary linear algebra gives the second page:

$$
\begin{array}{c|ccc}
q & & & \\
& 0 & 0 & 0 \\
& \mathbb{Z} & \mathbb{Z} & 0 \\
& \mathbb{Z} & \mathbb{Z} & 0 \\
\hline
& & & p
\end{array}
$$

And so we read of that $\mathrm{H}_0 T \simeq \mathbb{Z}$, $\mathrm{H}_2 T \simeq \mathbb{Z}$ whereas $\mathrm{H}_1 T$ is an extension of $\mathbb{Z}$ by $\mathbb{Z}$ hence $\mathbb{Z}^2$. The remaining homology groups are trivial.

5 Reeb spaces

The combinatorial Reeb space is introduced in Section 5.1. Section 5.2 is merely a recap of Quillen's theorem A and the theory of collapsing schemes due to K. Brown. The last two Sections are dedicated to clarifying and proving Theorems 1.2 and 1.3.

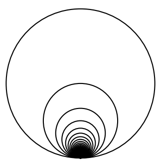
5.1 From topological to combinatorial Reeb spaces

The topological Reeb space is defined for any continuous function $f\colon X \to \mathbb{R}$. Given such a function one declares points to be equivalent if they are in the same path component of some fiber: $x \sim_f y$ if there is a real number a and $[x] = [y]$ in $f^{-1}(a)$.

Definition 5.1. The *topological Reeb space* associated to a continuous real-valued function $f\colon X \to \mathbb{R}$ is the quotient space $\mathrm{R}_f = X/\sim_f$.

The topological Reeb space is commonly referred to as the Reeb graph, which surely is accurate for e.g. Morse functions, albeit not accurate in general.

Example 5.2. Consider the Hawaiian earring $\mathbb{H}$ embedded as a subspace of $\mathbb{R}^2$:



The fibers of the horizontal projection $\mathrm{pr}_1\colon \mathbb{H} \to \mathbb{R}$ are all discrete. We thus conclude that the topological Reeb space $\mathrm{R}_{\mathrm{pr}_1}$ is homeomorphic to $\mathbb{H}$. But the fundamental group of $\mathbb{H}$ is not free [DS92].

I, for one, would very much like to define a Reeb space whose homotopy type is that of a graph. This could very well serve as motivation for our next definition. Recall that a continuous function $f\colon X \to \mathbb{R}$ has an associated section category $\mathcal{S}_f$. Applying the nerve and level-wise path components functor produce a simplicial set $\pi_0 \mathrm{N}\mathcal{S}_f$.

Definition 5.3. The *combinatorial Reeb space* of a continuous function $f\colon X \to \mathbb{R}$ is the simplicial set $\pi_0 \mathrm{N}\mathcal{S}_f$.

Simplicial sets carry a homotopy theory equivalent to the standard theory on topological spaces: The homotopy type of a simplicial set S is equivalent to that of the topological space $|S|$. In particular, the homotopy types of $\pi_0 \mathrm{N}\mathcal{S}_f$ and R_f can be compared by realizing $\pi_0 \mathrm{N}\mathcal{S}_f$.

5.2 More background on simplicial sets

I will give a brief reminder on Quillen's well-known theorem A [Qui73] as well as a theorem on collapsing schemes due to K. Brown [Bro92]. Both are useful to prove Theorems 1.2 and 1.3.

Let us first review Quillen's theorem A. For any functor $F\colon \mathcal{C} \to \mathcal{D}$ and object d in $\mathcal{D}$, we define the slice category $F \downarrow d$ as the pullback

$$\begin{tikzcd} F \downarrow y \arrow[r] \arrow[d] & \mathrm{Fun}([1],\mathcal{D}) \arrow[d, "(\mathrm{eval}_0,\mathrm{eval}_1)"] \\ \mathcal{C} \times [0] \arrow[r, "F\times d"] & \mathcal{D} \times \mathcal{D} \end{tikzcd}$$

where $\mathrm{Fun}([1],\mathcal{D})$ is the category of functors $[1] \to \mathcal{D}$. More explicitly, an object is a tuple (c,m) in the product $\mathrm{ob}\mathcal{C} \times \mathrm{mor}\mathcal{D}$ subject to $s(m) = F(c)$ and $t(m) = d$; a morphism $\alpha\colon (c,m) \to (c',m')$ is a morphism $\alpha\colon c \to c'$ such that $m = m' \circ F(\alpha)$. This data is commonly depicted

$$\begin{tikzcd} F(c) \arrow[d, "m"] \\ d \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F(c) \arrow[rr, "F(\alpha)"] \arrow[dr, "m"'] & & F(c') \arrow[dl, "m'"] \\ & d & \end{tikzcd}$$

Quillen's Theorem A gives a sufficient condition as to when F realizes to a weak homotopy equivalence: If $\mathrm{B}(F \downarrow d)$ is contractible for all d, then $\mathrm{B}F$ is a weak homotopy equivalence.

For S a simplicial set, the topological space $|S|$ is a CW complex whose cells are in bijection with the non-degenerate simplices in S. In particular, it makes sense to talk about $d_i e$, the ith face of a cell e in $|S|$. Moreover, we define the ith horn of e, which we will denote e_i, to be the union of all its faces except the ith one. We may safely deform $|S|$ onto a quotient space Y by collapsing e onto e_i without changing the homotopy type of S. Moreover, Y is clearly a CW complex again. Brown gives conditions for how to iterate this process of collapsing cells without changing the homotopy type of $|S|$, while making sure that Y is still a CW complex.

Partition the non-degenerate simplices of S into three classes: essential, redundant and collapsible. The cells corresponding to redundant simplices are to be deformed along the collapsible cells, hence they are truly "redundant". So a function c from redundant simplices to collapsible simplices that maps n–simplices to $(n+1)$–simplices is required. If s is redundant and cs admits another redundant face s', then we write $s' \leq s$. This data defines a *collapsing scheme* if i) c is a bijection from redundant n–simplices to collapsible $(n+1)$–simplices for all n and ii) there is no infinite descending chain of n–simplices $s \geq s' \geq s'' \geq \cdots$.

Proposition 1 in [Bro92] can then be formulated: For a collapsing scheme on S, the quotient map $|S| \to Y$ is a weak homotopy equivalence onto a CW complex Y whose n–cells are in bijection with the essential simplices in S.

5.3 Proof of Theorem 1.2

The nerve admits a left adjoint $\tau_1 :$ (simplicial sets) $\to$ (small categories) commonly referred to as the *fundamental category*. It agrees with the homotopy category when restricted to quasi-categories. A simplicial set S is sent to the category $\tau_1 S$ whose object set is S_0; morphism set is the *directed paths* $S_1 \coprod (S_1 \times_{S_0} S_1) \coprod \cdots$ modulo the relations $s_0 x \sim 1_x$ for all 0–simplices x and $d_1 s \sim d_0 s \circ d_2 s$ for all 2–simplices s. More explicitly, a directed path is a tuple $(e_1, \ldots, e_n)$ of edges/1–simplices such that the source of e_{i+1} is the target of e_i; $d_1 e_{i+1} = d_0 e_i$. The corresponding morphism in $\tau_1 S$ is denoted $e_1 \cdots e_n$, utilizing the word notation. We define the *length* of a word $e_1 \cdots e_n$ to be n if there is no equivalent word on fewer letters. An n–simplex in $\mathrm{N}\tau_1 S$ is a tuple $(w_1, \ldots, w_n)$ of composable words/morphisms. We define the length of $(w_1, \ldots, w_n)$ to be the length of the word $w_1 \cdots w_n$.

Since τ_1 is left adjoint to N, there is an associated unit map $\eta : S \to \mathrm{N}\tau_1 S$, which is natural in S. It is not a weak homotopy equivalence in general, as pointed out by Thomason in [Tho80]. But we shall see that the unit always induces a weak homotopy equivalence on combinatorial Reeb spaces.

Lemma 5.4. Let S be a simplicial set such that

i) S and $\mathrm{N}\tau_1 S$ only differ in cells of length ≥ 2 and

ii) any word of length n has a unique presentation on n letters.

Then the natural map $\eta : S \to \mathrm{N}\tau_1 S$ is a weak homotopy equivalence.

Proof. The theory of collapsing schemes, due to K. Brown [Bro92], is utilized to construct a homotopy inverse of the realized unit $|\eta|$. I refer to Section 5.2 for a quick summary of this theory. All morphisms in $\tau_1 S$ will be represented uniquely according to assumption ii).

To partition $\mathrm{N}\tau_1 S$ into redundant, collapsible and essential simplices we first declare every 1–simplex $e_1 \cdots e_n$ of length $n \geq 2$ redundant and define its associated collapsible 2–simplex

$$c(e_1 \cdots e_n) = (e_1 \cdots e_{n-1}, e_n).$$

This function is well-defined because such a presentation is unique. For $m \geq 2$ we declare an m–simplex of the form $(e_{1,1} \cdots e_{1,i_1}, \ldots, e_{m,1} \cdots e_{m,i_m})$, whose length is greater than or equal to 2, redundant if its not in the image of c. Its associated $(m+1)$–simplex is then determined by taking the biggest k such that $i_k \geq 2$ and factoring $e_{k,1} \cdots e_{k,i_k}$ as $(e_{k,1} \cdots e_{k,i_k-1}, e_{k,i_k})$. In other words, we factorize out the last letter not already factorized. The remaining simplices are declared essential. Do note that these are precisely the ones whose length is equal to 1.

The function c is constructed to be a bijection in the sense required by a collapsing scheme. Whereas the second demand follows since a chain associated to a redundant n–simplex s cannot exceed the length of s and is therefore necessarily bounded. We thus have a map $|\mathrm{N}\tau_1 S| \to Y$ with Y a CW complex whose n–cells correspond to essential n–simplices, i.e. those of length 1 in $|\mathrm{N}\tau_1 S|$. Hence Y is necessarily equal to $|S|$ per assumption i) and what we have constructed is a homotopy inverse to $|\eta|$. □

For a combinatorial Reeb space $\pi_0 \mathrm{N}\mathcal{S}_f$, a morphism in $\tau_1 \pi_0 \mathrm{N}\mathcal{S}_f$ is a word $[\sigma_1][\sigma_2] \cdots [\sigma_n]$ with representatives σ_i in $\mathcal{S}_f[a_{i-1}, a_i]$ subject to $[s\sigma_{i+1}] = [t\sigma_i]$. So the source and target of successive classes must agree up to path components in fibers.

Lemma 5.5. A word in $\tau_1 \pi_0 \mathrm{N}\mathcal{S}_f$ of length n has a unique presentation on n letters.

Proof. Assume that a word w is presented $[\sigma_1] \cdots [\sigma_n]$ and $[\rho_1] \cdots [\rho_n]$. We shall see that for all i the equality $[\sigma_i] = [\rho_i]$ holds. From the domains of σ_i and ρ_j we extract sequences $\bar{a} = (a_0, \ldots, a_n)$ and $\bar{b} = (b_0, \ldots, b_n)$ of real numbers.

These sequences must be equal. The statement is clear if $n = 1$ since $a_0 = b_0$ and $a_n = b_n$. If $n \geq 2$ we conversely assume that $\bar{a} \neq \bar{b}$. Consider i the smallest index such that $a_i \neq b_i$. We assume $a_i < b_i$ without loss of generality. Introduce the real number $c = \min(a_{i+1}, b_i)$ and define two letters $[\sigma'] = [\sigma_{i+1}|_{[a_i,c]}]$ and $[\rho'] = [\rho_i|_{[b_{i-1},c]}]$ to rewrite $[\sigma_i][\sigma'] = [\rho']$. We observe that $[\sigma_{i+1}]$ and $[\rho']$ overlap in $[\sigma']$ and so there must be a section τ such that $[\tau] = [\sigma_i][\sigma_{i+1}]$, contradicting the length of w.

Hence the equality $[\sigma_1] \cdots [\sigma_n] = [\rho_1] \cdots [\rho_n]$ can only be achieved if the letters are equal. For the only relation connecting them is to concatenate sections up to paths in fibers. □

The simplicial sets $\pi_0 \mathrm{N}\mathcal{S}_f$ and $\mathrm{N}\tau_1 \pi_0 \mathrm{N}\mathcal{S}_f$ clearly only differ by simplices of length ≥ 2:

a word $(w_1,\ldots,w_n)$ can only be of length 1 if $w_i = [\sigma_i]$ and there is a section ρ such that $[\rho] = [\sigma_1]\cdots[\sigma_n]$.

Lemma 5.6. Let $f\colon X \to \mathbb{R}$ be a continuous function. The unit $\eta : \pi_0 \mathrm{N}\mathcal{S}_f \to \mathrm{N}\tau_1\pi_0\mathrm{N}\mathcal{S}_f$ realizes to a weak homotopy equivalence.

Proof. A direct consequence of Lemmas 5.4 and 5.5. □

In the proof of Theorem 1.2 it will be convenient to utilize Theorem 1.1. But to do so, we must first verify that a Reeb function $f\colon X \to \mathbb{R}$ defines a Reeb function on the topological space R_f.

Lemma 5.7. If $f\colon X \to \mathbb{R}$ is a Reeb function, then R_f has the homeomorphism type of a 1–dimensional CW complex satisfying that the induced function $\bar{f}\colon \mathrm{R}_f \to \mathbb{R}$ is piecewise linear.

Proof. Let R_f be the topological Reeb space, presented as a quotient space according to Definition 5.1. A topological space Q, homeomorphic to R_f, will be constructed to satisfy the assertion. We assume, without loss of generality, that every point in X is either contained in some critical level or in between two critical levels. This can always be achieved by slightly modifying the stratification on X: one may e.g. present X as a filtered colimit of preimages of closed intervals under the map f.

The set of 0–cells is given by $\coprod \pi_0 f^{-1}c$ ranging over all critical values c, whereas the set of 1–cells is given by $\coprod \pi_0\mathcal{S}_f[c,d]$ ranging over successive critical values $c < d$. The attaching maps comes from the source and target: a 1–cell e labeled by a path component $[\sigma]$ in $\pi_0\mathcal{S}_f[c,d]$ admits a source in $\pi_0 f^{-1}c$; target in $\pi_0 f^{-1}d$. Denote the resulting CW complex Q.

Define the piecewise linear map $\bar{f}\colon Q \to \mathbb{R}$ as follows. On a closed 1–cell $e \simeq [0,1]$ labeled by a class in $\pi_0\mathcal{S}_f[c,d]$ it is the orientation-preserving linear map $[0,1] \to [c,d]$. Note that $\bar{f}\colon Q \to \mathbb{R}$ is constructed to be piecewise linear.

There is a rather evident surjective map $q\colon X \to Q$: If x is a point contained in some critical fiber $f^{-1}c$, then it is mapped to the 0–cell labeled by $[x]$ in $\pi_0 f^{-1}c$. Otherwise, consider the 1–cell e corresponding to g_x, the reparametrized flow-line provided by Proposition 2.7. We then send x to the point in e mapped to $f(x)$ under $\bar{f}$. Note that this map is constructed to be over $\mathbb{R}$ in the sense that $f = \bar{f}\circ q$.

It only remains to verify that Q has the universal quotient topology–for the topological space Q is clearly in bijection with R_f. We thus verify that a subset U of Q is open if $q^{-1}U$ is open in X. This is true if for any closed 1–cell e, the intersection $e\cap U$ is open in e. From the construction of Q it follows that there is a section $\sigma\colon [c,d] \to X$ satisfying that $q\circ\sigma\colon [c,d] \to Q$ corestricts to a homeomorphism $e \simeq [c,d]$. But $e\cap U$

corresponds to $\sigma([c,d]) \cap q^{-1}U$ under the given homeomorphism, and $\sigma([c,d]) \cap q^{-1}U$ is open in $\sigma([c,d])$ given that $q^{-1}U$ is open in X. $\square$

Let us end this section with a

proof of Theorem 1.2. Lemma 5.7 allows us to assume that $f\colon X \to \mathbb{R}$ induces a Reeb function $\bar{f}\colon \mathrm{R}_f \to \mathbb{R}$. Theorem 1.1 thus guarantees $\mathrm{R}_f \simeq \mathrm{B}\mathcal{S}_{\bar{f}}$. So it suffices to prove that the simplicial sets $\pi_0\mathrm{N}\mathcal{S}_f$ and $\mathrm{N}\mathcal{S}_{\bar{f}}$ are weakly homotopy equivalent.

Functoriality of $\mathcal{S}$ induces a functor $\mathcal{S}_f \to \mathcal{S}_{\bar{f}}$ from the quotient map $q\colon X \to \mathrm{R}_f$. It maps a section σ of f to the section $q \circ \sigma$ of $\bar{f}$. Sections in the same path components of $(\mathrm{N}\mathcal{S}_f)_1$ are obviously mapped to the same section of $\bar{f}$, so we have an induced simplicial map $F\colon \pi_0\mathrm{N}\mathcal{S}_f \to \mathrm{N}\mathcal{S}_{\bar{f}}$. A morphism in $\tau_1\pi_0\mathrm{N}\mathcal{S}_f$ is a word $[\sigma_1]\cdots[\sigma_n]$, represented by sections σ_i which are composable up to paths contained in fibers of f. The unit $\eta\colon \pi_0\mathrm{N}\mathcal{S}_f \to \mathrm{N}\tau_1\pi_0\mathrm{N}\mathcal{S}_f$ provides a factorization

$$\pi_0\mathrm{N}\mathcal{S}_f \xrightarrow{\eta} \mathrm{N}\tau_1\pi_0\mathrm{N}\mathcal{S}_f \xrightarrow{\mathrm{N}G} \mathrm{N}\mathcal{S}_{\bar{f}}$$

of F. We have already seen that η realizes to a weak homotopy equivalence in Lemma 5.6, so we need only verify that $\mathrm{N}G$ is a weak homotopy equivalence. On the level of objects, the functor $G\colon \tau_1\pi_0\mathrm{N}\mathcal{S}_f \to \mathcal{S}_{\bar{f}}$ sends a morphism/word $[\sigma_1]\cdots[\sigma_n]$ to the composition $(q\circ\sigma_n)\circ\cdots\circ(q\circ\sigma_1)$ of sections of $\bar{f}$. We shall construct an inverse functor G^{-1} from which we conclude that $\mathrm{B}G$ is in fact a homeomorphism.

Consider a section $\rho\colon [c,d] \to \mathrm{R}_f$ of $\bar{f}$ which passes no critical points, except possibly at the end points. Since $\bar{f}\colon \mathrm{R}_f \to \mathbb{R}$ is assumed to be piecewise linear on a 1-dimensional CW complex, the image of ρ must be contained in some edge e of R_f. Take any point x in X which maps to the interior of e and define $G^{-1}\rho$ to be $[g_x|_{[c,d]}]$, the reparametrized flow-line provided by Proposition 2.7. This class in $\pi_0\mathcal{S}_f[c,d]$ is independent of the choice of x. Indeed, assume that another point y is mapped to the interior of e. Any choice of path $p\colon I \to f^{-1}(c,d)$, between x and y, defines a path from $g_x|_{[c,d]}$ to $g_y|_{[c,d]}$ in $\mathcal{S}_f[c,d]$ via the composition

$$[c,d]\times I \xrightarrow{\mathrm{id}_{[c,d]}\times p} [c,d]\times f^{-1}(c,d) \xrightarrow{g} X.$$

A general section ρ of $\bar{f}$ can only pass finitely many critical values, because f is assumed a Reeb function. Whence we factorize it accordingly $\rho = \rho_n\circ\cdots\circ\rho_1$ and define $G^{-1}\rho$ to be the word $G^{-1}\rho_1\cdots G^{-1}\rho_n$.

Applying $G^{-1}\circ G$ to a word $[\sigma_1]\cdots[\sigma_n]$ returns $[g_{x_1}]\cdots[g_{x_n}]$, where x_i is chosen according to the above description of G^{-1}. We can verify the equality $[\sigma_i] = [g_{x_i}]$ by considering all reparametrized flow-lines through points in the image of $[\sigma_i]$. Hence $G^{-1}\circ G = \mathrm{id}_{\tau_1\pi_0\mathrm{N}\mathcal{S}_f}$. The remaining equality $G\circ G^{-1} = \mathrm{id}_{\mathcal{S}_{\bar{f}}}$ follows since the induced map $\bar{f}\colon \mathrm{R}_f \to \mathbb{R}$ is piecewise linear on a 1–dimensional CW complex; an edge in R_f uniquely determines a section that traverses it. $\square$

5.4 Combinatorial Reeb spaces are graphs

Before we prove the result, I must first elaborate on the meaning of a 'graph'. A simplicial set is a *graph* if it is aspherical–all higher homotopy groups are trivial–and the fundamental group is free for any choice of basepoint. The category (graphs) of graphs is then the evident full subcategory of (simplicial sets). Our definition of combinatorial Reeb spaces gives a functor

$$(\text{spaces over } \mathbb{R}) \to (\text{simplicial sets})$$

by mapping $f\colon X \to \mathbb{R}$ to $\pi_0 \mathrm{N}\mathcal{S}_f$, and we shall see that it does in fact define a functor

$$(\text{spaces over } \mathbb{R}) \to (\text{graphs}).$$

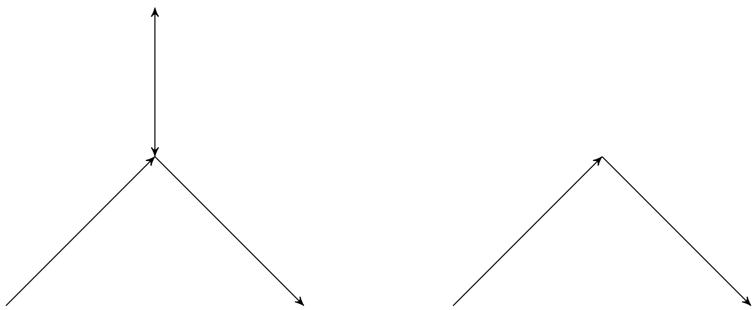
In light of Example 5.2, I would argue that this is one advantage over topological Reeb spaces.

Classifying spaces of groupoids are aspherical, a fact which is easily verified by using simplicial homotopy groups. There is a groupoidification functor from small categories to groupoids which assigns to a category $\mathcal{C}$ the groupoid $\mathcal{C}[\mathcal{C}^{-1}]$ in which all morphisms are formally inverted. It may abstractly be described as the left adjoint of the forgetful functor

$$(\text{groupoids}) \to (\text{small categories}).$$

For a given category $\mathcal{C}$, there is an evident functor $j\colon \mathcal{C} \to \mathcal{C}[\mathcal{C}^{-1}]$ which is not a weak homotopy equivalence in general: categories can represent all homotopy types, whereas groupoids cannot. I refer to [McD79] for details. But we shall verify that j does in fact realize to a weak homotopy equivalence for combinatorial Reeb spaces.

Recall that a morphism in $\tau_1\pi_0\mathrm{N}\mathcal{S}_f$ is a word $[\sigma_1]\cdots[\sigma_n]$, represented by sections that are composable up to paths in fibers. A morphism in the groupoid $\tau_1\pi_0\mathrm{N}\mathcal{S}_f[\tau_1\pi_0\mathrm{N}\mathcal{S}_f^{-1}]$ is thus a word $[\sigma_1]^{i_1}[\sigma_2]^{i_2}\cdots[\sigma_n]^{i_n}$ in which each $i_j = \pm 1$. The new relations are generated by $[\sigma][\rho]^{-1} = 1_{[s\sigma]}$ and $[\rho]^{-1}[\sigma] = 1_{[t\sigma]}$ whenever $[\sigma] = [\rho]$; they are in the same component of the section space $\mathcal{S}_f[f(s\sigma), f(t\sigma)]$. Geometrically you might want to interpret this as moving up and down, or down and up, along $[\sigma]$ cancels to the appropriate identity. A word $[\sigma][\rho]^{-1}$ is said to be *reducible* if there are factorizations $[\sigma] = [\sigma_2]\circ[\sigma_1]$ and $[\rho] = [\rho_1]\circ[\rho_2]$ such that $[\sigma_2] = [\rho_2]$. In particular, $[\sigma][\rho]^{-1} = [\sigma_1][\rho_1]^{-1}$. Dually, we declare what it means for $[\sigma]^{-1}[\rho]$ to be reducible. Intuitively, a part of $[\sigma]$ may overlap with $[\rho]$:



A morphism/word $[\sigma_1]^{i_1}[\sigma_2]^{i_2}\cdots[\sigma_n]^{i_n}$ in $\tau_1\pi_0\mathrm{N}\mathcal{S}_f[\tau_1\pi_0\mathrm{N}\mathcal{S}_f^{-1}]$ is then declared *irreducible* if the word has length equal to n and there is no reducible subword. Subject to this added requirement, we extend Lemma 5.5 to the groupoidification:

Lemma 5.8. Any morphism in $\tau_1\pi_0\mathrm{N}\mathcal{S}_f[\tau_1\pi_0\mathrm{N}\mathcal{S}_f^{-1}]$ is uniquely presentable as an irreducible word.

Proof. Assume w to be presented $[\sigma_1]^{i_1}\cdots[\sigma_n]^{i_n}$ and $[\rho_1]^{j_1}\cdots[\rho_n]^{j_n}$, both irreducible. Let $\bar{a}=(a_0,\dots,a_n)$ and $\bar{b}=(b_0,\dots,b_n)$ be the sequences obtained by successively considering the domains of sections that appear as representatives in the two words.

These sequences must be equal and so the letters must be equal. Indeed, all relations connecting them alters the associated sequences of real numbers. The statement is clear if both are of length 1 since $a_0=b_0$ and $a_n=b_n$. If the length is ≥ 2 we conversely assume that $\bar{a}\neq\bar{b}$. Consider $q\geq 1$ the smallest index such that $a_q\neq b_q$. We assume $a_q<b_q$ without loss of generality.

Case 1: $a_q<a_{q-1}$ and $b_q>b_{q-1}$. Apply $[\sigma_1]^{-i_1}\cdots[\sigma_{q-1}]^{-i_q}$ and $w'=[\rho]_n^{-j_n}\cdots[\rho_{q+1}]^{-j_{q+1}}$ to w. The result is an equality $[\sigma_q]^{i_q}\cdots[\sigma_n]^{i_n}w'=[\rho_q]^{j_q}$. For this particular case, we deduce $i_q=-1$ and $j_q=1$. But then the equality can only hold if something cancels $[\sigma_q]^{-1}$, contradicting the irreducibility of $[\sigma_1]^{i_1}\cdots[\sigma_n]^{i_n}$.

Case 2: $a_q>a_{q-1}$ and $b_q>b_{q-1}$, or $a_q<a_{q-1}$ and $b_q<b_{q-1}$. These are proved in a similar fashion as the previous case. $\square$

Before presenting the next lemma I remind that a natural transformation $F\Rightarrow G$ between two functors $\mathcal{C}\to\mathcal{D}$ is equivalent to a functor $\mathcal{C}\times[1]\to\mathcal{D}$ whose restriction to 0 and 1 in $[1]$ is F and G, respectively. A natural transformation $F\Rightarrow G$ thus defines a homotopy $\mathrm{B}F\sim\mathrm{B}G$. See e.g. Segal's paper [Seg68].

Lemma 5.9. The map $j\colon\ \tau_1\pi_0\mathrm{N}\mathcal{S}_f\to\tau_1\pi_0\mathrm{N}\mathcal{S}_f[\tau_1\pi_0\mathrm{N}\mathcal{S}_f^{-1}]$ realizes to a weak homotopy equivalence for any combinatorial Reeb space.

Proof. Consider an arbitrary object $[x]$ in $\tau_1\pi_0\mathrm{N}\mathcal{S}_f[\tau_1\pi_0\mathrm{N}\mathcal{S}_f^{-1}]$. Quilen's theorem A reduces the problem to proving that the comma category $j\downarrow[x]$ is contractible. An object in the comma category is a morphism/word w in $\tau_1\pi_0\mathrm{N}\mathcal{S}_f[\tau_1\pi_0\mathrm{N}\mathcal{S}_f^{-1}]$ terminating at $[x]$. All words are presented uniquely according to Lemmas 5.5 and 5.8. We shall define a homotopy from the identity on $\mathrm{B}(j\downarrow[x])$ to the trivial map $w\longmapsto 1_{[x]}$. There are two essential intermediate functors.

The first functor $\mathrm{pr}_+\colon j\downarrow[x]\to j\downarrow[x]$ reduces the length of words that start with a letter of the form $[\sigma_1]$. It maps a non-trivial word $w=[\sigma_1]^{i_1}\cdots[\sigma_n]^{i_n}$ to

$$\mathrm{pr}_+([\sigma_1]^{i_1}\cdots[\sigma_n]^{i_n})=\begin{cases}[\sigma_2]^{i_2}\cdots[\sigma_n]^{i_n} & \text{if } i_1=1\\ [\sigma_1]^{i_1}\cdots[\sigma_n]^{i_n} & \text{if } i_1=-1\end{cases}$$

A morphism $w''\colon w\to w'$ is a factorization $w=w'\circ j(w'')$ and hence there is a unique choice for pr_+w'' yielding a factorization $\mathrm{pr}_+w=\mathrm{pr}_+w'\circ(\mathrm{pr}_+w'')$. This data comes with a rather evident natural transformation η_+ from id to pr_+ since $[\sigma_1]$ defines a morphism from a word $[\sigma_1][\sigma_2]^{i_2}\cdots[\sigma_n]^{i_n}$ to $[\sigma_2]^{i_2}\cdots[\sigma_n]^{i_n}$. In other words, we have defined a functor $H_+\colon(j\downarrow[x])\times[1]\to(j\downarrow[x])$ whose restriction to $(j\downarrow[x])\times 0$ is id, whereas the restriction to $(j\downarrow[x])\times 1$ is pr_+.

The second functor $\mathrm{pr}_-\colon j\downarrow[x]\to j\downarrow[x]$ is complementary to pr_+. It maps a non-trivial word $w=[\sigma_1]^{i_1}\cdots[\sigma_n]^{i_n}$ to

$$\mathrm{pr}_-([\sigma_1]^{i_1}\cdots[\sigma_n]^{i_n})=\begin{cases}[\sigma_2]^{i_2}\cdots[\sigma_n]^{i_n} & \text{if } i_1=-1\\ [\sigma_1]^{i_1}\cdots[\sigma_n]^{i_n} & \text{if } i_1=1\end{cases}$$

Analogous to pr_+ this data comes with a homotopy $H_-\colon(j\downarrow[x])\times[1]\to(j\downarrow[x])$. But, in contrast to H_+, this homotopy starts at pr_- and terminates at id. This is because of how $[\sigma_1]$ defines a morphism from $[\sigma_2]^{i_2}\cdots[\sigma_n]^{i_n}$ to $[\sigma_1]^{-1}[\sigma_2]^{i_2}\cdots[\sigma_n]^{i_n}$.

For every object w, we need only alternate H_+ and H_- a finite number of times to obtain the trivial word $1_{[x]}$. We thus conclude that the identity on $(j\downarrow[x])_n$, generated by words of length $\leq n$, is homotopic to the trivial map for all n. It follows that the identity on $j\downarrow[x]$ must be homotopic to the trivial map. $\square$

We wrap up the discussion on Combinatorial Reeb spaces with a

proof of Theorem 1.3. We have seen in Lemma 5.9 that a Reeb space $\pi_0\mathrm{N}\mathcal{S}_f$ has the homotopy type of its groupoidification. In particular, it must be aspherical. It remains only to verify that the fundamental group is free, regardless of basepoint. So fix a basepoint $[x]$ and consider $\pi_1(\pi_0\mathrm{N}\mathcal{S}_f)$ which is isomorphic to the automorphism group at $[x]$, considered as an object in the groupoidification $\tau_1\pi_0\mathrm{N}\mathcal{S}_f[\tau_1\pi_0\mathrm{N}\mathcal{S}_f^{-1}]$. This group admits the irreducible words (Lemma 5.8) as a free generating set. For a non-trivial irreducible word cannot possibly be reduced further to the unit $1_{[x]}$. $\square$

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Paper II

Factorization, extensions and a theorem of Retakh for exact quasi-categories

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Preprint

Factorization, extensions and a theorem of Retakh for exact quasi-categories

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Abstract. We define extension quasi-categories for exact quasi-categories in an analogous way to the special case of ordinary exact categories. We show that these form an Omega-spectrum, generalizing a theorem of Retakh. In so doing, we give an explicit Kan fibrant resolution of these extension quasi-categories, which relies on a factorization property. Finally, we show that the homotopy groups of extension quasi-categories are naturally isomorphic to the higher extension groups of the extriangulated category given by the homotopy category.

1 Introduction

Roughly speaking, an exact category $\mathcal{C}$ is an additive category together with a collection of short exact sequences subject to certain constraints [Hel58], see [Kel90] for a modern approach. This allows for homological algebra to be performed in a more general context than abelian categories. Non-abelian examples include the category of vector bundles of a scheme and the category of Banach spaces. In addition to homological algebra, exact categories provide a natural framework for K–theory. Quillen first introduced the higher algebraic K–groups in this context [Qui73].

The definition has recently been extended by Barwick to quasi-categories in order to generalize definitions and results in K–theory [Bar15, Bar16, BR10]. This broadens the scope considerably; it captures both (nerves of) exact categories and stable quasi-categories. Moreover, every extension closed subcategory of a stable quasi-category has the structure of an exact quasi-category. In fact, all exact quasi-categories occur in this manner [Kle20]. One would thus expect more constructions and results about exact categories to generalize, providing potentially useful applications. In particular, one could study homological algebra from a quasi-categorical point of view.

In order to better understand the homological algebra of exact quasi-categories $\mathscr{C}$, we define quasi-categorical analogues of extension categories, denoted by $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$. These are well understood if $\mathscr{C}$ is the nerve of an ordinary exact category. For a start, it is well-known that the Yoneda Ext-groups $\mathrm{Ext}^n_{\mathscr{C}}(B, A)$ can be recovered as the zeroth homotopy groups of $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ [Mac12, Proposition XII.4.4].

For an exact quasi-category $\mathscr{C}$, we declare that $\mathscr{E}\mathrm{xt}^0_{\mathscr{C}}(B, A) := \hom_{\mathscr{C}}(B, A)$, the mapping space from B to A. We will define maps $R_n\colon\ \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A) \longrightarrow \Omega\,\mathscr{E}\mathrm{xt}^{n+1}_{\mathscr{C}}(B, A)$, resulting in a spectrum $\mathscr{E}\mathrm{xt}_{\mathscr{C}}(B, A)$. The main result of ours is the following.

Theorem 4.3. *Let $\mathscr{C}$ be an exact quasi-category and let A and B be objects in $\mathscr{C}$. Then the spectrum $\mathscr{E}\mathrm{xt}_{\mathscr{C}}(B, A)$ is an Ω–spectrum. In other words, all of the maps R_n are weak equivalences.*

This is a generalization of a theorem of Retakh, who proved the same result for abelian categories [Ret86, Theorem 2(b)]. It follows that we have isomorphisms

$$\pi_{-n}\,\mathscr{E}\mathrm{xt}_{\mathscr{C}}(B, A) \simeq \pi_0\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A) \longrightarrow \pi_1\,\mathscr{E}\mathrm{xt}^{n+1}_{\mathscr{C}}(B, A) \qquad (1.1)$$

of homotopy groups. This means, in particular, that the higher structure of $\mathscr{E}\mathrm{xt}_{\mathscr{C}}(B, A)$ descends to the abelian group structure on classical Ext–groups.

To prove Theorem 4.3 it is required of us to find a manageable model for $\Omega\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$. Similar to how you construct injective resolutions in homological algebra, this is achieved by finding Kan fibrant replacements of higher extension categories. We define a new variant fEx of the Ex–functor [Kan57], enjoying the property that $\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ is a Kan complex. This is more computationally feasible, as compared to the usual transfinite composite $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A) \longrightarrow \mathrm{Ex}^\infty\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$:

Theorem 6.1. *The simplicial set $\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ is a Kan complex (or ∞–groupoid).*

The proof of Theorem 6.1 relies on a factorization property of $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$. We refer to Proposition 3.6 for the details. In the case of an ordinary exact category $\mathcal{C}$, this factorization property was first addressed by Hermann [Her16, Definition 2.1.11] in order to explicitly describe the isomorphisms (1.1). See [Her16, Theorem 1] for details. Initially, it was not clear whether factorization was possible for all exact categories, but this has now been shown by Volkov–Witherspoon [VW20, Lemma 3.1]. Our Proposition 3.6 then generalizes their result to the quasi-categorical framework.

There is a notion extriangulated categories, defined by Nakaoka–Palu [NP19], which simultaneously generalizes exact and triangulated categories. Recently, Nakaoka–Palu have shown that the homotopy category of an exact quasi-category has a natural extriangulated structure [NP20, Theorem 4.22]. In particular, it encapsulates the now well-known result that the homotopy category of a stable quasi-category is canonically triangulated [Lur17, Theorem 1.1.2.14], and moreover the homotopy category of the nerve of an exact category is obviously exact. Note that the homotopy category of a stable quasi-category is hardly ever exact. Building on Nakaoka–Palu's work, we show that the Retakh spectra $\mathscr{E}\mathrm{xt}_{\mathscr{C}}(-, -)$ determine the extriangulation on $\mathrm{h}\,\mathscr{C}$.

For a non-negative integer n, Gorsky–Nakaoka–Palu [GNP21, Definition 3.1] define the n–extension groups for extriangulated categories, ultimately inspired by Yoneda's classic monograph on extension categories [Yon60]. We show that the higher structure contained in the Retakh spectra $\mathscr{E}\mathrm{xt}_{\mathscr{C}}(-,-)$ descends to the n–extension groups in the homotopy category.

Theorem 7.8. *Let $\mathscr{C}$ be an exact quasi-category, and consider $(\mathrm{h}\,\mathscr{C}, \pi_0\,\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(-,-))$ as an extriangulated category. The bifunctor $\pi_0\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(-,-)$ is naturally isomorphic to the nth extension functor of $\mathrm{h}\,\mathscr{C}$.*

Outline. Section 2 contains little to no original ideas. It is concerned with basics of exact quasi-categories and diagram lemmas. We define the extension categories for quasi-categories in Section 3. Moreover, the factorization property of exact quasi-categories is proven. We follow up with an overview of the Retakh spectrum in Section 4. Although Theorem 4.3 is stated here, technical details are postponed. The reader only interested in the application to extriangulated categories, can safely skip straight to Section 7 from here. Section 5 is all about the factorized fEx–functor and proofs of technical lemmas. Only the basic intuition on fEx is required to read and follow Section 6, where we prove Theorems 4.3 and 6.1. In the final Section 7 we discuss how Retakh spectra determine the extriangulation on homotopy categories. In particular, Theorem 7.8 is proven.

Notation and conventions. Throughout, we fix a Grothendieck universe U. Simplicial sets should thus be understood as simplicial U–sets unless otherwise is stated. We refer to Shulman [Shu08, Section 8] for a detailed treatment of Grothendieck universes.

A category J will be identified with its nerve $\mathrm{N}(J)$, so that it can be regarded as a quasi-category. An $(\infty,1)$-functor between quasi-categories $\mathscr{C}$ and $\mathscr{D}$ (i.e. a morphism of the underlying simplicial sets) will simply be referred to as a functor.

A *homotopy coherent diagram* of J in a quasi-category $\mathscr{C}$ is a functor $J \longrightarrow \mathscr{C}$. A *homotopy commutative diagram* is a functor from J into the homotopy category $\mathrm{h}\,\mathscr{C}$.

Recall that the limit (resp. colimit) of a diagram $D\colon J \longrightarrow \mathscr{C}$ arises as a left (resp. right) adjoint of the diagonal functor $\mathscr{C} \longrightarrow \mathscr{C}^J$. Suppose that $\mathscr{C}$ is a quasi-category in which all natural maps of the form

$$\coprod_{i=1}^{n} X_i \longrightarrow \prod_{i=1}^{n} X_i$$

are homotopy equivalences. In this case, we make the assumption that finite coproducts and finite product coincide. More precisely, the functors defining the limit $\prod_{i=1}^{n} X_i$ coincides strictly with the one defining the colimit $\coprod_{i=1}^{n} X_i$. The term *biproduct* will be used to emphasise that it is both a limit and a colimit, and it will be denoted by $\bigoplus_{i=1}^{n} X_i$.

Let $\bigoplus\limits_{i=1}^{n} X_i$ and $\bigoplus\limits_{j=1}^{m} Y_j$ be biproducts in $\mathscr{C}$. Then a map $f\colon \bigoplus\limits_{i=1}^{n} X_i \longrightarrow \bigoplus\limits_{j=1}^{m} Y_j$ is uniquely determined, up to homotopy, by its components $f_{j,i}\colon X_i \longrightarrow Y_j$. We will thus write f as a matrix $(f_{j,i})_{i,j}$.

The 0–simplices of a quasi-category $\mathscr{C}$ are often referred to as objects. Similarly, 1–simplices are referred to as maps or morphisms. We write $f \sim g$ when the 1-simplicies f and g are homotopic. A degenerate 1-simplex $s_0 X$ is denoted by $X =\!=\!= X$.

For two simplicial sets X and Y, we denote by $\hom(X, Y)$ the *function complex* (or internal hom) whose set of n–simplices is $\hom(X, Y)_n = \mathrm{Hom}_{\mathrm{sSet}}(X \times [n], Y)$. In the case of $\hom([1], \mathscr{C})$, where $\mathscr{C}$ is a quasi-category, there is the subcomplex $\hom_{\mathscr{C}}(B, A)$ whose 0–simplices are the 1–simplices $B \longrightarrow A$ in $\mathscr{C}$. Note that this space models the homotopy function complex, as defined in [DK80]. We refer to [Lur09, Rem 1.2.2.5] and [DS11, Cor 3.7] for details. For an ordinary category $\mathcal{C}$ we denote by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ the set of maps $X \to Y$.

2 Exact quasi-categories

We first review exact quasi-categories, a generalization of exact categories. Since the underlying category of an exact category is additive, it is only sensible to introduce the quasi-categorical generalization.

Definition 2.1. A quasi-category $\mathscr{C}$ is *additive* if the following hold.

(Add1) There is a zero object 0 in $\mathscr{C}_0$, which is to say that $\hom_{\mathscr{C}}(0, X)$ and $\hom_{\mathscr{C}}(X, 0)$ are contractible for all $X \in \mathscr{C}_0$.

(Add2) Finite products and coproducts exist in $\mathscr{C}$.

(Add3) For any finite family of objects $\{X_1, \ldots, X_n\}$, the natural map

$$\coprod_{i=1}^{n} X_i \longrightarrow \prod_{i=1}^{n} X_i,$$

induced by the identity maps $X_i \xrightarrow{1} X_i$, is a homotopy equivalence.

(Add4) For all $X, Y \in \mathscr{C}_0$, the Hom-set $\mathrm{Hom}_{h\mathscr{C}}(X, Y) := \pi_0 \hom_{\mathscr{C}}(X, Y)$ admits an abelian group structure: $f + g$ is the composite

$$X \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} X \coprod X \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}} Y \prod Y \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} Y$$

in $\mathrm{h}\mathscr{C}$.

A bicomplete quasi-category $\mathscr{C}$ is additive precisely when the homotopy category $\mathrm{h}\mathscr{C}$ is additive as an ordinary category. Analogously to how biproducts arise in additive categories, the axioms (Add2) and (Add3) permit the assumption that $\mathscr{C}$ has a finite biproducts, which we denote by $\bigoplus_{i=1}^{n}$. More precisely, we may choose the product and coproduct functors

$$\prod, \coprod : \mathscr{C}^n \longrightarrow \mathscr{C}$$

in such a way that they coincide strictly – for (co)limits in quasi-categories are homotopy invariant.

Barwick defines exact quasi-categories [Bar15, Definition 3.1] by adapting Keller's minimal set of axioms [Kel90] to the quasi-categorical setting.

Definition 2.2. Let $\mathscr{C}$ be an additive quasi-category and let $\mathscr{C}_\dagger$ and $\mathscr{C}^\dagger$ be subcategories that contain all objects in $\mathscr{C}$, as well as all homotopy equivalences. The maps in $\mathscr{C}_\dagger$ will be referred to as *cofibrations*, whereas morphisms in $\mathscr{C}^\dagger$ are *fibrations*. The triple $(\mathscr{C}, \mathscr{C}_\dagger, \mathscr{C}^\dagger)$ is called an *exact quasi-category* if the following axioms hold.

(Ex1) For any zero object 0 in $\mathscr{C}_0$, all morphisms of the form $0 \longrightarrow X$ are cofibrations and those of the form $X \longrightarrow 0$ are fibrations.

(Ex2) Pushouts of cofibrations exist and are cofibrations, and dually for pullbacks of fibrations.

(Ex3) The following are equivalent for a homotopy coherent square

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] & E \arrow[d, "p"] \\ C \arrow[r, "g"] & B \end{tikzcd}$$

(Ex3.1) The square is a pullback, the map g is a cofibration and p is a fibration.

(Ex3.2) The square is a pushout, the map i is a cofibration and f is a fibration.

If the subcategories $\mathscr{C}_\dagger$ and $\mathscr{C}^\dagger$ are implicitly specified, we simply say that $\mathscr{C}$ is exact. A triple $(\mathscr{C}, \mathscr{C}_\dagger, \mathscr{C}^\dagger)$ satisfying the axioms above is called an *exact structure* on $\mathscr{C}$.

Whenever a cofibration appears in a diagram, it will be drawn as follows: $\rightarrowtail$ (like a monomorphism in an ordinary category). A fibration will be drawn as a two-headed arrow $\twoheadrightarrow$.

Nerves of exact categories are exact quasi-categories, where the the class of cofibrations consists of the admissible monomorphisms, and the fibrations are the admissible epimorphisms. Moreover, any stable quasi-category [Lur17, Definition 1.1.1.9] can be seen as an exact quasi-category where $\mathscr{C}_\dagger = \mathscr{C} = \mathscr{C}^\dagger$.

The axioms above provide a framework for exact sequences, in more or less the usual fashion.

Definition 2.3. Let $\mathscr{C}$ be an exact quasi-category. An *exact sequence* in $\mathscr{C}$ is a square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & E \arrow[d] \\ 0 \arrow[r, tail] & B \end{tikzcd}$$

where 0 is a zero object and the equivalent criteria in (Ex3) are met.

Equivalently, the exact sequences are the bicartesian squares

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[d, "p"] \\ 0 \arrow[r, tail] & B \end{tikzcd}$$

As for ordinary exact categories, exact sequences can be drawn horizontally

$$A \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} B,$$

omitting the zero object. There is not really any loss of information in such notation, for the choice of morphisms to/from zero is irrelevant up to homotopy.

Definition 2.4. A *map of exact sequences* is simply a map of bicartesian squares. Such maps will mostly be depicted as homotopy commutative diagrams of the form

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d] & E \arrow[r, two heads, "p"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, tail, "i'"] & E' \arrow[r, two heads, "p'"] & B' \end{tikzcd}$$

Exact functors also have a completely analogous definition.

Definition 2.5. Let $\mathscr{C}$ and $\mathscr{D}$ be exact quasi-categories. A functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ is *exact* if it preserves zero objects, cofibrations and pushouts of cofibrations.

When checking that a functor is exact, one may equivalently consider the dual statement for fibrations [Bar15, Proposition 4.8]. Exact sequences are obviously preserved by exact functors, and by considering split exact sequences it is seen that they preserve the additive structure as well.

Full subcategories $\mathscr{D}$ of a given exact quasi-category $\mathscr{C}$ inherit the exact structure if they are *closed under extensions*, i.e. for all exact sequences

$$A \rightarrowtail E \twoheadrightarrow B$$

with $A, B \in \mathscr{D}_0$, we have that $E \in \mathscr{D}_0$. The cofibrations $\mathscr{D}_\dagger$ are the maps in $\mathscr{C}_\dagger \cap \mathscr{D}$ whose cofiber is in $\mathscr{D}$, and dually for fibrations. The inclusion functor $\mathscr{D} \hookrightarrow \mathscr{C}$ is then exact.

It turns out that all exact quasi-categories occur as extension closed subcategories of stable quasi-categories.

Theorem 2.6 ([Kle20, Theorem 1]). *Given a small exact quasi-category $\mathscr{C}$, there exists a stable hull $\mathcal{H}^{\mathrm{st}}(\mathscr{C})$ into which $\mathscr{C}$ embeds exactly and universally.*

This provides a generalization of the Gabriel–Quillen embedding theorem for exact categories, which states that they embed exactly into abelian categories.

A number of results concerning exact quasi-categories generalize to results about exact quasi-categories. We will make frequent use of our two next lemmas, first generalized by Barwick.

Lemma 2.7 ([Bar15, Lemma 4.5]). *A homotopy coherent square*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & E \arrow[d, "p"] \\ C \arrow[r, "g"] & B \end{tikzcd}$$

in an exact quasi-category is bicartesian if either of the following conditions hold.

1. *The square is a pushout and i is a cofibration.*
2. *The square is a pullback and p is a fibration.*

Lemma 2.8 ([Bar15, Lemma 4.7]). *Let*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & E \arrow[d, "p"] \\ C \arrow[r, "g"] & B \end{tikzcd}$$

be a homotopy coherent square satisfying one of the properties 1. or 2. in Lemma 2.7. Then we have an exact sequence

$$A \xrightarrow{\begin{pmatrix} -i \\ f \end{pmatrix}} E \oplus C \xrightarrow{\begin{pmatrix} p & g \end{pmatrix}} B.$$

In particular, the maps $\begin{pmatrix} -i \\ f \end{pmatrix}$ *and* $\begin{pmatrix} i \\ -f \end{pmatrix}$ *are cofibrations and* $\begin{pmatrix} p & g \end{pmatrix}$ *is a fibration.*

Generalizations of celebrated diagram lemmas will also be helpful in later sections. The first relates pushouts to maps of exact sequences.

Lemma 2.9 ([Kle20, Proposition A.1])**.** *A pushout*

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d] & E \arrow[d] \\ C \arrow[r, tail] & F \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where i *is cofibration, can be extended to a map of exact sequences*

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d] & E \arrow[r, two heads, "p"] \arrow[d] & B \arrow[d, equal] \\ C \arrow[r, tail] & F \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, two heads] & B \end{tikzcd}$$

Conversely, the existence of such a map of exact sequences implies that the square is a pushout. A dual statement holds for pullbacks along fibrations.

Secondly, a map of exact sequences has a canonical factorization.

Lemma 2.10 ([Kle20, Proposition A.2])**.** *Any map of exact sequences*

$$\begin{tikzcd} A \arrow[r, tail, "i_A"] \arrow[d, "f'"] & E \arrow[r, two heads, "p_B"] \arrow[d, "f"] & B \arrow[d, "f''"] \\ C \arrow[r, tail, "i_C"] & F \arrow[r, two heads, "p_D"] & D \end{tikzcd}$$

can be factored

$$\begin{tikzcd} A \arrow[r, tail, "i_A"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & E \arrow[r, two heads, "p_B"] \arrow[d, "g"] & B \arrow[d, equal] \\ C \arrow[r, tail, "j"] \arrow[d, equal] & Z \arrow[r, two heads, "q"] \arrow[d, "h"] \arrow[dr, phantom, "\square"] & B \arrow[d, "f''"] \\ C \arrow[r, tail, "i_C"] & F \arrow[r, two heads, "p_D"] & D \end{tikzcd}$$

where the squares marked by $\square$ *are bicartesian.*

The renowned Five Lemma is a consequence of Lemma 2.10. It will be proved in Appendix A, since there is no significant difference from the special case of ordinary exact categories.

Lemma 2.11 (Five lemma)**.** *Consider a map of exact sequences*

$$\begin{tikzcd} A \arrow[r, tail, "i_A"] \arrow[d, "f'"] & E \arrow[r, two heads, "p_B"] \arrow[d, "f"] & B \arrow[d, "f''"] \\ C \arrow[r, tail, "i_C"] & F \arrow[r, two heads, "p_D"] & D \end{tikzcd}$$

If f' and f'' are homotopy equivalences (resp. cofibrations, resp. fibrations), so is f.

Lastly, the 3×3–lemma will come in handy. It is also proved in Appendix A.

Lemma 2.12 (3×3–lemma)**.** *Consider the homotopy coherent diagram with exact columns*

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, tail, "i_A"] & E' \arrow[r, "g'"] \arrow[d, tail, "i_E"] & B' \arrow[d, tail, "i_B"] \\ A \arrow[r, "f"] \arrow[d, two heads, "p_A"] & E \arrow[r, "g"] \arrow[d, two heads, "p_E"] & B \arrow[d, two heads, "p_B"] \\ A'' \arrow[r, "f''"] & E'' \arrow[r, "g''"] & B'' \end{tikzcd}$$

If the middle row and one of the other rows is exact, then the remaining row is exact.

3 Quasi-categories of extensions

Fix an exact quasi-category $\mathscr{C}$ and let $\mathscr{A}$ and $\mathscr{B}$ be arbitrary (not necessarily full) subcategories of $\mathscr{C}$. We define the quasi-categories of n–extensions $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(\mathscr{B}, \mathscr{A})$.

Recall that $[1] \times [1]$ consists of two 2–simplices glued along their 1–faces to obtain a square. A short exact sequence is thus a homotopy coherent diagram $\mathbb{E}\colon [1] \times [1] \to \mathscr{C}$ which defines a bicartesian square

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & E \arrow[d, two heads, "p"] \\ 0 \arrow[r, tail] & B \end{tikzcd}$$

The corners in $[1]\times[1]$ are commonly labeled (k,l) for $k,l=0,1$. Every inclusion $i_{l,k}\colon [0]\longrightarrow[1]\times[1]$ induces a map $\mathrm{eval}_{k,l}\colon \hom([1]\times[1],\mathscr{C})\longrightarrow\mathscr{C}$ which restricts to $\mathrm{eval}_{k,l}\colon \mathscr{E}\mathrm{xt}^1(\mathscr{C},\mathscr{C})\longrightarrow\mathscr{C}$. In particular a short exact sequence

$$\mathbb{E}\colon \qquad F' \rightarrowtail E \twoheadrightarrow F$$

satisfies $\mathrm{eval}_{0,0}\mathbb{E}=F'$, $\mathrm{eval}_{0,1}\mathbb{E}=0$, $\mathrm{eval}_{1,0}\mathbb{E}=E$ and $\mathrm{eval}_{1,1}\mathbb{E}=F$.

Definition 3.1. Formally, we define the 1–*extension* category

$$\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(\mathscr{B},\mathscr{A}) := e^{-1}(\mathscr{B}\times\mathscr{A}),$$

where e is the functor

$$e\colon\ \mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(\mathscr{C},\mathscr{C}) \xrightarrow{\mathrm{eval}_{0,0}\times\mathrm{eval}_{1,1}} \mathscr{C}\times\mathscr{C}.$$

In other words, $\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(\mathscr{B},\mathscr{A})$ is the subcategory of $\hom([1],\mathscr{C})$ generated exact sequences starting in $\mathscr{A}$ and ending in $\mathscr{B}$

Note that the 1–simplices in $\mathscr{E}\mathrm{xt}^1(\mathscr{C},\mathscr{C})$ are maps of exact sequences, as defined in Definition 2.4. If $\mathscr{A}$ and $\mathscr{B}$ consist of a single object, say A and B, respectively, and all of its morphisms are homotopic to the identity, we write $\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B,A)$. The Five Lemma has a very nice interpretation in this special case.

Proposition 3.2. *All maps in $\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B,A)$ are homotopy equivalences. Consequently, the quasi-category $\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B,A)$ is a Kan complex.*

Proof. The first assertion is a direct consequence of Lemma 2.11. Thus, the quasi-category $\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B,A)$ is a Kan complex [Joy02, Corollary 1.4]. □

Our next result implies that $\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B,A)$ does in fact define a bifunctor

$$\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}\colon \mathscr{C}^{\mathrm{op}}\times\mathscr{C}\longrightarrow\mathcal{S}$$

into the quasi-category of Kan complexes. We refer to [Lur09, p. 121] for generalities.

Proposition 3.3. *The map $e\colon\ \mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(\mathscr{C},\mathscr{C})\longrightarrow\mathscr{C}\times\mathscr{C}$ is a bifibration.*

Proof. It is required of us to find a lift in

$$\begin{tikzcd}
\Lambda^0[n] \arrow[r, "\gamma"] \arrow[d] & \mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(\mathscr{C},\mathscr{C}) \arrow[d] \\
{[n]} \arrow[r, "s"] & \mathscr{C}\times\mathscr{C}
\end{tikzcd}$$

whenever the first 1–simplex in s has a degenerate 1–simplex as its second component. There is also a dual horn filling requirement for $\Lambda^n[n]$ which is follows by replacing pushouts with pullbacks in the following arguments.

The case $n = 1$ reduces to filling out a diagram

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d] & E \arrow[r, two heads] & B \arrow[d, equal] \\
A & & B
\end{tikzcd}$$

to a map of exact sequences (Definition 2.4). Thus we need only compute the pushout along the leftmost vertical morphism and invoke Lemma 2.9 which results in

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[r, two heads] \arrow[d] & B \arrow[d, equal] \\
A' \arrow[r, tail] & P \arrow[r, two heads] & B
\end{tikzcd}$$

The case $n \geq 2$ is also a more or less direct consequence of Lemma 2.9. Indeed, the first 1–simplex in γ is of the form

$$\begin{tikzcd}
A_0 \arrow[r, tail] \arrow[d] & E_0 \arrow[r, two heads] \arrow[d] & B_0 \arrow[d, equal] \\
A_1 \arrow[r, tail] & E_1 \arrow[r, two heads] & B_0
\end{tikzcd}$$

so that the leftmost square/cocone is in fact a pushout square due to Lemma 2.9. A lift of s to $\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(\mathscr{C}, \mathscr{C})$ can now be deduced from the fact that pushout squares are initial in the category of cocones. $\square$

Before defining the higher extension categories, let us first explain how one may interpret the n–extensions when $\mathscr{C}$ is the nerve of an exact category. In this case, the *n–extensions* of B by A can be represented as diagrams

$$\begin{tikzcd}[column sep=small]
 & E_1 \arrow[rr, "d_1"] \arrow[dr, two heads, "p_1"] & & E_2 \arrow[rr, "d_2"] \arrow[dr, two heads, "p_2"] & & \cdots \arrow[rr, "d_{n-2}"] \arrow[dr, two heads, "p_{n-2}"] & & E_{n-1} \arrow[rr, "d_{n-1}"] \arrow[dr, two heads, "p_{n-1}"] & & E_n \arrow[dr, two heads, "d_n"] & \\
A \arrow[ur, tail, "d_0"] \arrow[dr, two heads] & \diamondsuit & E_{1.5} \arrow[ur, tail, "i_1"] \arrow[dr, two heads] & \diamondsuit & E_{2.5} \arrow[ur, tail, "i_2"] \arrow[dr, two heads] & \cdots & E_{(n-2).5} \arrow[ur, tail, "i_{n-2}"] \arrow[dr, two heads] & \diamondsuit & E_{(n-1).5} \arrow[ur, tail, "i_{n-1}"] \arrow[dr, two heads] & \diamondsuit & B \\
 & 0 \arrow[ur, tail] & & 0 \arrow[ur, tail] & & \cdots \arrow[ur, tail] & & 0 \arrow[ur, tail] & & 0 \arrow[ur, tail] &
\end{tikzcd}$$

Of course, it is common to simply think of

$$\begin{tikzcd} A \arrow[r, tail, "d_0"] & E_1 \arrow[r, "d_1"] & E_2 \arrow[r, "d_2"] & \cdots \arrow[r, "d_{n-1}"] & E_n \arrow[r, two heads, "d_n"] & B \end{tikzcd}$$

as the exact sequence. But such a view-point does not encode exactness, at least not for our purposes. An alternative approach is to rather look at the complementary diagram

$$\begin{tikzcd}[column sep=small]
 & E_1 \arrow[dr, "p_1"] & & E_2 \arrow[dr, "p_2"] & & \cdots \arrow[dr, "p_{n-2}"] & & E_{n-1} \arrow[dr, "p_{n-1}"] & & E_n \arrow[dr, "d_n"] & \\
A \arrow[ur, tail, "d_0"] \arrow[dr] & \diamondsuit & E_{1.5} \arrow[ur, tail, "i_1"] \arrow[dr] & \diamondsuit & E_{2.5} \arrow[ur, tail, "i_2"] \arrow[dr] & \cdots & E_{(n-2).5} \arrow[ur, tail, "i_{n-2}"] \arrow[dr] & \diamondsuit & E_{(n-1).5} \arrow[ur, tail, "i_{n-1}"] \arrow[dr] & \diamondsuit & B \\
 & 0 \arrow[ur, tail] & & 0 \arrow[ur, tail] & & \cdots \arrow[ur, tail] & & 0 \arrow[ur, tail] & & 0 \arrow[ur, tail] &
\end{tikzcd} \tag{3.1}$$

which does contain the information of exactness. Moreover, the boundary maps d_j are uniquely determined as the composition of p_j and i_j – so there is no loss of information. Do note that these diagrams are advantageous when working with quasi-categories due to the fact that the composites d_j of i_j and p_j have not been chosen.

The higher extension quasi-categories are defined recursively:

Definition 3.4. For $n \geq 2$ we define the *n–extension* quasi-category $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(\mathscr{B}, \mathscr{A})$ as the pullback of

$$\begin{tikzcd} & \mathscr{E}\mathrm{xt}^{n-1}(\mathscr{B}, \mathscr{C}) \arrow[d, "\mathrm{eval}_{0,0}"] \\ \mathscr{E}\mathrm{xt}^1(\mathscr{C}, \mathscr{A}) \arrow[r, "\mathrm{eval}_{1,1}"] & \mathscr{C} \end{tikzcd}$$

If $\mathscr{A}$ and $\mathscr{B}$ consist of a single object, say A and B, respectively, and all of its morphisms are homotopic to the identity, we write $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$. Note that $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ defines a functor into the quasi-category of quasi-categories. This essentially follows from the recursive definition combined with Proposition 3.3.

The objects in $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ are to be regarded as n conjoined exact sequences, as depicted in (3.1). As a matter of notation, we will omit the zeros from the elements and write a typical object $\mathbb{E}$ of $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ as

$$\begin{tikzcd}[column sep=small]
 & E_1 \arrow[dr, "p_1"] & & E_2 \arrow[dr, "p_2"] & & E_n \arrow[dr, "d_n"] & \\
A \arrow[ur, tail, "d_0"] & & E_{1.5} \arrow[ur, tail, "i_1"] & & \cdots \arrow[ur, tail, "i_{n-1}"] & & B
\end{tikzcd}$$

A map $f \colon \mathbb{E} \longrightarrow \mathbb{F}$ of such n–extensions, or 1–simplex in $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$, is then given by a homotopy coherent diagram

$$\begin{tikzcd}
 & E_1 \arrow[dr, two heads, "p_1^E"] \arrow[dd, "f_1"] & & E_2 \arrow[dr, "p_2^E"] \arrow[dd, "f_2"] & & E_n \arrow[dr, two heads, "d_n^E"] \arrow[dd, "f_n"] & \\
A \arrow[ur, "d_0^E"] \arrow[dd] & & E_{1.5} \arrow[ur, "i_1^E"] \arrow[dd, "f_{1.5}"] & & \cdots \arrow[ur, "i_{n-1}^E"] & & B \arrow[dd] \\
 & F_1 \arrow[dr, two heads, "p_1^F"] & & F_2 \arrow[dr, "p_2^F"] & & F_n \arrow[dr, two heads, "d_n^F"] & \\
A \arrow[ur, "d_0^F"] & & F_{1.5} \arrow[ur, "i_1^F"] & & \cdots \arrow[ur, "i_{n-1}^F"] & & B
\end{tikzcd}$$

At a first glance, our notion of maps between exact sequences may seem stronger than what is conventional. If $\mathscr{C}$ is the nerve of an exact category, a map of n–extensions is usually given as a commutative diagram

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & E_1 \arrow[r] \arrow[d, "f_1"] & E_2 \arrow[r] \arrow[d, "f_2"] & \cdots \arrow[r] & E_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & E_n \arrow[r, two heads] \arrow[d, "f_n"] & B \arrow[d, equal] \\
A' \arrow[r, tail] & F_1 \arrow[r] & F_2 \arrow[r] & \cdots \arrow[r] & F_{n-1} \arrow[r] & F_n \arrow[r, two heads] & B'
\end{tikzcd}$$

without mention of maps $f_{i.5}\colon E_{i.5} \longrightarrow F_{i.5}$. In our setting we must require the $f_{i.5}$s to exist, but they can be induced if $\mathscr{C}$ is an ordinary exact category. For example, we can induce $f_{1.5}\colon E_{1.5} \longrightarrow F_{1.5}$ since the left-most square in

$$\begin{tikzcd}
 & E_1 \arrow[rr] \arrow[dr, two heads, "p_1"] \arrow[dd, "f_1"] & & E_2 \arrow[r] \arrow[dd, "f_2"] & \cdots \\
A \arrow[ur, tail] \arrow[dd, equal] & & E_{1.5} \arrow[ur, tail] \arrow[dd, dashed, "f_{1.5}"] & & \\
 & F_1 \arrow[rr] \arrow[dr, two heads] & & F_2 \arrow[r] & \cdots \\
A \arrow[ur, tail] & & F_{1.5} \arrow[ur, tail] & &
\end{tikzcd}$$

commutes. Then using the fact that fibrations in nerves of exact categories are epimorphisms forces

$$\begin{tikzcd}
E_{1.5} \arrow[r] \arrow[d, "f_{1.5}"] & E_2 \arrow[d, "f_2"] \\
E_{1.5} \arrow[r] & F_2
\end{tikzcd}$$

to commute. Proceeding in such a manner yields $f_{i.5}$ for $1 \leq i \leq n-1$. But this process does not translate to exact quasi-categories; fibrations are not necessarily epimorphisms. The discussion thus far is summarized in

Proposition 3.5. *If $\mathscr{C}$ is the nerve of an exact category, then $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ is the nerve of the classical category of n–extensions.*

Extension categories enjoy a factorization property, which will be crucial later on.

Proposition 3.6. *Let $n \geq 1$. A map $f\colon \mathbb{E} \longrightarrow \mathbb{F}$ in $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$*

$$\begin{tikzcd}
 & E_1 \arrow[rr, "d_1^E"] \arrow[dr, "p_1^E"] \arrow[dd, "f_1"] & & E_2 \arrow[r, "d_2^E"] \arrow[dr, "p_2^E"] \arrow[dd, "f_2"] & \cdots \arrow[r, "d_{n-1}^E"] & E_n \arrow[dr, "d_n^E"] \arrow[dd, "f_n"] & \\
A \arrow[ur, "d_0^E"] \arrow[dd] & & E_{1.5} \arrow[ur, "i_1^E"] \arrow[dd, dashed, "f_{1.5}"] & & \cdots \arrow[ur, "i_{n-1}^E"] & & B \arrow[dd] \\
 & F_1 \arrow[rr] \arrow[dr, "p_1^F"] & & F_2 \arrow[r] \arrow[dr, "p_2^F"] & \cdots \arrow[r] & F_n \arrow[dr, "d_n^F"] & \\
A \arrow[ur, "d_0^F"] & & F_{1.5} \arrow[ur, "i_1^F"] & & \cdots \arrow[ur, "i_{n-1}^F"] & & B
\end{tikzcd}$$

can be factored

$$\mathbb{E} \xrightarrow{f'} \operatorname{cyl}(f) \xrightarrow{p} \mathbb{F}$$

where f' is a term-wise cofibration, and there exists a factorization $1_{\mathbb{F}} \sim pm$ where m is a term-wise cofibration.

Proof. The case $n = 1$ is easily deduced from Proposition 3.2; choose f' as the identity on $\mathbb{E}$ and $p = f$.

For $n \geq 2$, the exact sequence $\operatorname{cyl}(f)$ will be chosen as follows

$$\begin{tikzcd}
 & F_1 \oplus E_2 \arrow[dr, "\begin{pmatrix} p_1^F & 0 \\ 0 & 1 \end{pmatrix}"] & & F_2 \oplus E_3 \oplus E_2 \arrow[dr, "\begin{pmatrix} p_2^F & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}"] & & \cdots \\
A \arrow[ur, "\begin{pmatrix} d_0^F \\ 0 \end{pmatrix}"] & & F_{1.5} \oplus E_2 \arrow[ur, "\begin{pmatrix} i_1^F & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}"] & & F_{2.5} \oplus E_3 \arrow[ur, "\begin{pmatrix} i_2^F & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[dr, "\begin{pmatrix} p_{n-2}^F & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}"] & & F_{n-1} \oplus E_n \oplus E_{n-1} \arrow[dr, "\begin{pmatrix} p_{n-1}^F & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}"] & & F_n \oplus E_n \arrow[dr, "\begin{pmatrix} d_n^F & 0 \end{pmatrix}"] & \\
 & F_{(n-2).5} \oplus E_{n-1} \arrow[ur, "\begin{pmatrix} i_{n-2}^F & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}"] & & F_{(n-1).5} \oplus E_n \arrow[ur, "\begin{pmatrix} i_{n-1}^F & 0 \\ 0 & 1 \end{pmatrix}"] & & B
\end{tikzcd}$$

We deduce from Lemma 2.8 that $\mathrm{cyl}(f)$ is an n–extension. Since $\mathbb{F}$ is a direct summand of $\mathrm{cyl}(f)$, the map $p\colon \mathrm{cyl}(f) \longrightarrow \mathbb{F}$ is chosen as the projection, and $m\colon \mathbb{F} \longrightarrow \mathrm{cyl}(f)$ as the inclusion. For $f'\colon \mathbb{E} \longrightarrow \mathrm{cyl}(f)$ we choose the map in $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ with components

$$
\begin{aligned}
f'_1 &= \begin{pmatrix} f_1 \\ d^E_1 \end{pmatrix}, \\
f'_{j.5} &= \begin{pmatrix} f_{j.5} \\ i^E_j \end{pmatrix} \text{ for } 1 \leq j \leq n-1 \\
f'_j &= \begin{pmatrix} f_i \\ d^E_j \\ 1_{E_j} \end{pmatrix} \text{ for } 2 \leq j \leq n-1 \\
f'_n &= \begin{pmatrix} f_n \\ 1_{E_n} \end{pmatrix}.
\end{aligned}
$$

Clearly, we have that $f = pf'$, so it remains to show that f' has cofibrations in all components. For $j \neq 1$, we can use Lemma 2.8 (with the squares transposed). Only f'_1 remains. We have a map of exact sequences

$$
\begin{tikzcd}
A \arrow[r, "d^E_0"] \arrow[d, equal] & E_1 \arrow[r, two heads, "p^E_1"] \arrow[d, "f'_1"] \arrow[dr, phantom, "\square"] & E_{1.5} \arrow[d, "f'_{1.5}"] \\
A \arrow[r, "\begin{pmatrix} d^F_0 \\ 0 \end{pmatrix}"'] & F_1 \oplus E_2 \arrow[r, two heads, "\begin{pmatrix} p^F_1 & 0 \\ 0 & 1 \end{pmatrix}"'] & F_{1.5} \oplus E_2
\end{tikzcd}
$$

in which the right square, by Lemma 2.9, is bicartesian. As (Ex3) applies, the map f'_1 is a cofibration since $f'_{1.5}$ is. $\square$

In recent work of Volkow–Witherspoon [VW20, Lemma 3.1], this factorization property was shown to hold for ordinary exact categories. Our proof for exact quasi-categories can be specialized to this case. We will use it in Lemma 6.2, via Lemma 5.8, to prove that all loops in $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ are homotopic to a loop of length 1, as Hermann does for ordinary exact categories [Her16].

4 The Retakh spectrum

Inspired by Retakh [Ret86], we organize all of the extension categories in an Ω–spectrum object $\mathscr{E}\mathrm{xt}_{\mathscr{C}}(B, A)$, whose nth entry is $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$, called the *Retakh spectrum*. Such a construction comes with weak equivalences $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A) \longrightarrow \Omega\,\mathscr{E}\mathrm{xt}^{n+1}_{\mathscr{C}}(B, A)$. We thus start with a recap of loop spaces in simplicial sets.

Let X be any simplicial set. The *loop space* on X, based at x, is defined to be the homotopy pullback of

$$\begin{tikzcd} & {[0]} \arrow[d, "x"] \\ {[0]} \arrow[r, "x"] & X \end{tikzcd}$$

It can be computed by first replacing one $x\colon [0] \longrightarrow X$ with a Kan fibration $p\colon PX \twoheadrightarrow X$, where PX contractible, and then computing the ordinary pullback [BK72]. The problem of computing loop spaces can thus be solved by finding such a Kan fibration. This is analogous to how derived functors in homological algebra are calculated by first finding projective or injective resolutions.

There are two evaluation maps eval_0 and eval_1 on $\hom([1], X)$ induced by applying $\hom(-, X)$ to the inclusions $0\colon [0] \longrightarrow [1]$ and $1\colon [0] \longrightarrow [1]$ of 0 and 1 into $[1]$. These maps send a 1-simplex e in X, which is an object in $\hom([1], X)$, to its source $d_1 e$ and target $d_0 e$ in X. The reduced mapping space $\hom(([1], 0), (X, x))$, of 1–simplices starting at x, may then be defined as the pre-image $\mathrm{eval}_0^{-1} x$. Whenever X is a Kan complex, one may mimic the construction of loop spaces in topological spaces. For in this case, the evaluation eval_1 is a Kan fibration [GJ09, Corollary 5.3 on p. 21]. Hence the loop space of X based at x, denoted by $\Omega(X, x)$, fits into the pullback diagram

$$\begin{tikzcd} \Omega(X, x) \arrow[r] \arrow[d] & \hom(([1], 0), (X, x)) \arrow[d, "\mathrm{eval}_1"] \\ {[0]} \arrow[r, "x"] & X \end{tikzcd}$$

For future reference, we state this as a proposition:

Proposition 4.1. *If X is a Kan complex, then the loop space $\Omega(X, x)$ on X based at x is the pre-image of x under the first evaluation* $\mathrm{eval}_1\colon \hom(([1], 0), (X, x)) \longrightarrow X$.

There is a rather natural choice of basepoint in $\mathscr{E}\mathrm{xt}^n(B, A)$. It is the split exact sequence $\sigma_1(B, A)$:

$$A \rightarrowtail A \oplus B \twoheadrightarrow B$$

for $n = 1$ and $\sigma_n(B, A)$:

$$\begin{tikzcd}[column sep=small] & A \arrow[dr] & & \cdots \arrow[dr] & & B \arrow[dr, equal] & \\ A \arrow[ur, equal] & & 0 \arrow[ur] & & 0 \arrow[ur] & & B \end{tikzcd}$$

otherwise. To define the advertised Ω–spectrum object, we must first find a suitable model for $\Omega(\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A), \sigma_n(B, A))$, the loop space of $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ based at $\sigma_n(B, A)$, with the correct homotopy type.

Let us first consider loops on $\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B, A)$, as the corresponding homotopy type is easily calculated:

Lemma 4.2. *The pre-image* $\mathrm{eval}_1^{-1}(\sigma_1(B, A))$ *is the loop space on* $\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B, A)$*, based at* $\sigma_1(B, A)$.

Proof. Propositions 3.2 and 4.1 gives the result. $\square$

More explicitly, the objects of $\Omega(\mathscr{E}\mathrm{xt}^1_C(B, A), \sigma_1(B, A))$ are the maps

$$\sigma_1(B, A) \longrightarrow \sigma_1(B, A)$$

i.e. diagrams

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, equal] & A \oplus B \arrow[r, two heads] \arrow[d, "\phi"] & B \arrow[d, equal] \\ A \arrow[r, tail] & A \oplus B \arrow[r, two heads] & B \end{tikzcd}$$

whereas the maps are homotopy coherent cubes of such diagrams and so on for higher simplices.

The loop spaces of higher extension categories are not as easily determined: Although $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ is a quasi-category, it is not a Kan complex in general. And so it is required of us to do some work in order to model its loop space. Technicalities are postponed till section 5 and 6. But for the purpose of presenting the Ω–spectrum of extension categories we shall give a brief description of the objects in our model for $\Omega(\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A), \sigma_1(B, A))$ where $n \geq 2$. As we shall prove in Lemma 6.2, they are simple zigzags

$$\sigma_n(B, A) \longrightarrow \mathbb{E} \longleftarrow \sigma_n(B, A)$$

of length one in $\mathscr{E}\mathrm{xt}_{\mathscr{C}}(B, A)$, starting and terminating in $\sigma_n(B, A)$. All of the data contained in such a zigzag can be depicted

$$\begin{tikzcd}
 & A \arrow[dd, tail] \arrow[dr, two heads] & & \cdots \arrow[dr, two heads] & & B \arrow[dd] \arrow[dr, equal] & \\
A \arrow[ur, equal] \arrow[dd, equal] & & 0 \arrow[ur, tail] \arrow[dd] & & 0 \arrow[ur, tail] \arrow[dd] & & B \arrow[dd, equal] \\
 & E_1 \arrow[dr, two heads] & & \cdots \arrow[dr, two heads] & & E_n \arrow[dr, two heads] & \\
A \arrow[ur, tail] \arrow[dd, equal] & & E_{1.5} \arrow[ur, tail] & & E_{(n-1).5} \arrow[ur, tail] & & B \arrow[dd, equal] \\
 & A \arrow[uu, tail] \arrow[dr, two heads] & & \cdots \arrow[dr, two heads] & & B \arrow[uu] \arrow[dr, equal] & \\
A \arrow[ur, equal] & & 0 \arrow[ur, tail] \arrow[uu] & & 0 \arrow[ur, tail] \arrow[uu] & & B
\end{tikzcd}$$

in our exact category $\mathscr{C}$.

We now define the *zeroth Retakh functor*

$$R_0\colon \hom_{\mathscr{C}}(B, A) \longrightarrow \Omega(\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B, A), \sigma_1(B, A)),$$

which will serve as the first structure map in the Retakh spectrum. Every map $f\colon B \longrightarrow A$ induces a map

$$A \oplus B \xrightarrow{\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}} A \oplus B$$

which in turn can be seen as a endomap of $\sigma_1(B, A)$ in $\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B, A)$ (by Lemma 4.2)

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & A \oplus B \arrow[r, two heads] \arrow[d, "\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}"] & B \arrow[d, equal] \\
A \arrow[r, tail] & A \oplus B \arrow[r, two heads] & B
\end{tikzcd}$$

Functoriality follows since the construction hinges upon the universal property of the biproduct.

Recall that the homotopy category of an additive quasi-category is additive. In particular, the path components of $\hom$–spaces are abelian groups. Thus we can pick an additive inverse -1_B of 1_B, up to homotopy. For $n \geq 1$, we now describe the *nth Retakh functor* on objects

$$R_n\colon \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A) \longrightarrow \Omega(\mathscr{E}\mathrm{xt}^{n+1}_{\mathscr{C}}(B, A), \sigma_{n+1}(B, C)),$$

generalizing Hermann's constructive interpretation of Retakh's work [Her16]. It sends an n–extension $\mathbb{E}$:

$$\begin{tikzcd}
 & E_1 \arrow[dr, two heads, "p_1"] & & \cdots \arrow[dr, two heads, "p_{n-1}"] & & E_n \arrow[dr, two heads, "d_n"] & \\
A \arrow[ur, tail, "d_0"] & & E_{1.5} \arrow[ur, tail, "i_1"] & & E_{(n-1).5} \arrow[ur, tail, "i_{n-1}"] & & B
\end{tikzcd}$$

to the loop

$$\begin{tikzcd}
 & A \arrow[dd, tail, "d_0"] \arrow[dr] & & \cdots \arrow[dr] & & B \arrow[dd, tail, "\begin{pmatrix}1\\0\end{pmatrix}"'] \arrow[dr, equal] & \\
A \arrow[ur, equal] \arrow[dd, equal] & & 0 \arrow[ur, tail] \arrow[dd] & & 0 \arrow[ur, tail] \arrow[dd, "\begin{pmatrix}-1\\1\end{pmatrix}"'] & & B \arrow[dd, equal] \\
 & E_1 \arrow[dr, two heads, "p_1"] & & \cdots \arrow[dr, two heads, "d_n"] & & B \oplus B \arrow[dr, two heads, "\begin{pmatrix}1 & 1\end{pmatrix}"] & \\
A \arrow[ur, tail, "d_0"] \arrow[dd, equal] & & E_{1.5} \arrow[ur, tail, "i_1"] & & B \arrow[ur, tail] & & B \arrow[dd, equal] \\
 & A \arrow[uu, tail, "d_0"] \arrow[dr] & & \cdots \arrow[dr] & & B \arrow[uu, tail, "\begin{pmatrix}0\\1\end{pmatrix}"] \arrow[dr, equal] & \\
A \arrow[ur, equal] & & 0 \arrow[uu] \arrow[ur, tail] & & 0 \arrow[uu] \arrow[ur, tail] & & B
\end{tikzcd}$$

based at $\sigma_{n+1}(B, A)$. The middle row is the $(n+1)$–extension $(\mathbb{E}, \rho_B)$, the concatenation of $\mathbb{E}$ with the short exact sequence ρ_B:

$$B \xrightarrow{\begin{pmatrix}-1\\1\end{pmatrix}} B \oplus B \xrightarrow{\begin{pmatrix}1 & 1\end{pmatrix}} B$$

Altogether we have defined for every pair $(B, A) \in \mathscr{C}_0 \times \mathscr{C}_0$ a spectrum $\mathscr{E}\mathrm{xt}_{\mathscr{C}}(B, A)$ with structure maps $R_n\colon\ \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A) \longrightarrow \Omega(\mathscr{E}\mathrm{xt}^{n+1}_{\mathscr{C}}(B, A), \sigma_{n+1}(B, C))$. And, moreover, it is in fact an Ω–spectrum, to be proven in Propositons 6.3 and 6.5.

Theorem 4.3. *The spectrum $\mathscr{E}\mathrm{xt}_{\mathscr{C}}(B, A)$ is an Ω–spectrum. In other words, all of the Retakh maps R_n are weak equivalences.*

Corollary 4.4. *The $(-n)$th homotopy group of $\mathscr{E}\mathrm{xt}_{\mathscr{C}}(B, A)$ is $\pi_0\, \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$.*

Note that $\mathscr{E}\mathrm{xt}_{\mathscr{C}}(B, A)$ only adds to the already existing abelian group structure on the mapping space $\hom_{\mathscr{C}}(B, A)$. Indeed, the group structure on $\Omega(\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B, A), \sigma_1(B, A))$ is given by concatenating loops of the form

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & A \oplus B \arrow[r] \arrow[d, "\begin{pmatrix}1 & f\\0 & 1\end{pmatrix}"] & B \arrow[d, equal] \\
A \arrow[r, tail] & A \oplus B \arrow[r] & B
\end{tikzcd}$$

But a concatenation of such reduces to computing a matrix product

$$\begin{pmatrix}1 & f\\0 & 1\end{pmatrix}\begin{pmatrix}1 & g\\0 & 1\end{pmatrix} \sim \begin{pmatrix}1 & f+g\\0 & 1\end{pmatrix}$$

which incorporates the same algebraic structure as $\hom_{\mathscr{C}}(B, A)$.

If $\mathscr{C}$ is the nerve of a small abelian category $\mathcal{A}$, Theorem 4.3 specializes to Retakh's result that the extension categories $\mathscr{E}\mathrm{xt}^n_{\mathcal{A}}(B, A)$ form an Ω–spectrum [Ret86]. Moreover, $\pi_0\,\mathscr{E}\mathrm{xt}^n_{\mathcal{A}}(B, A)$ is easily seen to be the abelian group of Yoneda n–extensions. In particular, $\pi_0\,\mathscr{E}\mathrm{xt}^n(B, A)$ generalizes the standard Ext-groups from abelian categories to exact quasi-categories. On the other hand, if we rather consider $\mathrm{D}^b(\mathcal{A})$, the bounded derived category of $\mathcal{A}$, we see that the spectrum $\mathrm{Hom}_{\mathrm{D}^b(\mathcal{A})}(B, \Sigma^n A)$ coincides with the Retakh spectrum $\mathscr{E}\mathrm{xt}_{\mathcal{A}}(B, A)$. Indeed, it is well-known that Yoneda Ext-groups are isomorphic to Hom-groups in the bounded derived category [Ver96, Prop 3.2.2].

The construction of the Retakh spectrum determines a functor

$$\mathscr{E}\mathrm{xt}_{\mathscr{C}} \colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \mathrm{Sp}(\mathrm{Cat}_\infty)$$

where $\mathrm{Sp}(\mathrm{Cat}_\infty)$ is an appropriate category of Ω–spectra. Specifically, the quasi-category $\mathrm{Sp}(\mathrm{Cat}_\infty)$ is that of spectrum objects [Lur06, Section 8] in Cat_∞, the quasi-category of small quasi-categories. In this framework, a map $X^\bullet \longrightarrow Y^\bullet$ of Ω–spectra simply becomes a family of maps $X^n \longrightarrow Y^n$ between the corresponding components.

Proposition 4.5. *An exact functor*

$$F \colon \mathscr{C} \longrightarrow \mathscr{D}$$

induces a canonical natural transformation

$$\eta_F \colon\ \mathscr{E}\mathrm{xt}_{\mathscr{C}}(-,-) \Longrightarrow \mathscr{E}\mathrm{xt}_{\mathscr{D}}(F-, F-)$$

of functors $\mathscr{C}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \mathrm{Sp}(\mathrm{Cat}_\infty)$.

Proof. Let (B, A) be an object of $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$. Then F induces functors

$$\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A) \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{D}}(FA, FB)$$

sending an exact sequence of length n in $\mathscr{C}$ to its image in $\mathscr{D}$. Assembling these functors yields a map of spectra

$$\mathscr{E}\mathrm{xt}_{\mathscr{C}}(B, A) \longrightarrow \mathscr{E}\mathrm{xt}_{\mathscr{D}}(FB, FA).$$

More generally, given an n–simplex

$$(B_1, A_1) \longrightarrow (B_2, A_2) \longrightarrow \cdots \longrightarrow (B_n, A_n)$$

in $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$, one checks that the $[1] \times [n]$–shaped diagram

$$\begin{tikzcd}
\mathscr{E}\mathrm{xt}_{\mathscr{C}}(B_1, A_1) \arrow[r] \arrow[d] & \mathscr{E}\mathrm{xt}_{\mathscr{D}}(FB_1, FA_1) \arrow[d] \\
\mathscr{E}\mathrm{xt}_{\mathscr{C}}(B_2, A_2) \arrow[r] \arrow[d] & \mathscr{E}\mathrm{xt}_{\mathscr{D}}(FB_2, FA_2) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathscr{E}\mathrm{xt}_{\mathscr{C}}(B_n, A_n) \arrow[r] & \mathscr{E}\mathrm{xt}_{\mathscr{D}}(FB_n, FA_n)
\end{tikzcd} \tag{4.1}$$

is homotopy commutative.

A natural transformation of quasi-categories can be given by a simplicial map

$$\eta_F : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \hom([1], \mathrm{Sp}(\mathrm{Cat}_\infty)).$$

We choose this map as that sending an n–simplex to the diagram (4.1). It can be checked that the boundary and degeneracy maps behave as expected, yielding the desired map of simplicial sets. □

5 An adaption of the subdivision and Ex-functors

This section is all about introducing a version of Kan's Ex-functor which is tailored for the extension categories $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$. We call it the *factorized Ex-functor* and denote it by fEx. Only the basic intuition behind fEx is needed to follow Section 6, and in particular following the proofs of how $\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ is a Kan complex and that the nth Retakh functor is a weak equivalence.

We spend the next few paragraphs defining the *factorized subdivision functor*

$$\mathrm{fsd} : \Delta \longrightarrow \mathrm{Cat}.$$

The objects in $\mathrm{fsd}[m]$ will be generated recursively. Let $\mathrm{fsd}[0] = [0]$ be the discrete category with one object and one identity morphism. Suppose that $\mathrm{fsd}[m-1]$ has been defined, where $m \geq 1$, and that its objects are certain $m-1$ tuples

$$(M_1, \ldots, M_{m-1})$$

of pairwise disjoint subsets of $\{0, 1, \ldots, m-1\}$. We may also define $\mathrm{fsd}\, X$, for any totally ordered set X with m elements, simply by setting up a monotone bijection $X \longrightarrow [m-1]$. The objects of $\mathrm{fsd}[m]$ will then be certain m–tuples

$$(M_1, \ldots, M_{m-1}, M_m)$$

of pairwise disjoint subsets of

$$\{0, 1, \ldots, m\}.$$

We generate them from the objects of $\mathrm{fsd}[m-1]$ according to the procedure given by (O1)–(O3) below.

(O1) For each $0 \leq j \leq m$ and each object $(M_1, \ldots, M_{m-1})$ in $\mathrm{fsd}[0, \ldots, \widehat{j}, \ldots, m]$, we include $(M_1, \ldots, M_{m-1}, \emptyset)$. Although duplicates may occur, no element is counted twice.

(O2) For each object $(M_1, \dots, M_{m-1}, \emptyset)$ introduced in (O1), with the exception of those of the form $(j, \emptyset, \dots, \emptyset)$, we add $(M_1, \dots, M_{m-1}, M_m)$, where M_m is the set $\{0, \dots, m\} \setminus \left(\bigcup_{i=1}^{m-1} M_i \right)$ of remaining integers $\leq m$.

(O3) We add the object $(01 \cdots m, \emptyset, \dots, \emptyset)$.

The category $\mathrm{fsd}[m]$ will then be the preorder (i.e. category with at most one morphism connecting any pair of objects) generated by the following morphisms.

(M1) For each $0 \leq j \leq m$ and each morphism

$$f\colon (M_1, \dots, M_{m-1}) \longrightarrow (N_1, \dots, N_{m-1})$$

in $\mathrm{fsd}[0, \dots, \widehat{j}, \dots, m]$, we add a morphism

$$f^j\colon (M_1, \dots, M_{m-1}, \emptyset) \longrightarrow (N_1, \dots, N_{m-1}, \emptyset).$$

(M2) For each object $(M_1, \dots, M_{m-1}, M_m)$ introduced in (O2), we add morphisms

$$h_{(M_1, \dots, M_{m-1})}\colon (M_1, \dots, M_{m-1}, M_m) \longrightarrow (M_1, \dots, M_{m-1}, \emptyset).$$

(M3) Let $f^j\colon (M_1, \dots, M_{m-1}, \emptyset) \longrightarrow (N_1, \dots, N_{m-1}, \emptyset)$ be a morphism introduced in (M1). If $(M_1, \dots, M_{m-1}, \emptyset)$ is not of the form $(i, \emptyset, \dots, \emptyset)$, we include a morphism

$$\widehat{f^j}\colon (M_1, \dots, M_{m-1}, M_m) \longrightarrow (N_1, \dots, N_{m-1}, N_m).$$

If $(M_1, \dots, M_{m-1}, \emptyset) = (i, \emptyset, \dots, \emptyset)$, we add

$$\widehat{f^j}\colon (M_1, \dots, M_{m-1}, \emptyset) = (i, \emptyset, \dots, \emptyset) \longrightarrow (N_1, \dots, N_{m-1}, N_m).$$

Since $\mathrm{fsd}[m]$ is required to be a preorder, we add relations so that all resulting subdiagrams commute.

(M4) For each object $(M_1, M_2, \cdots, M_m)$ introduced in (O2), we have a morphism

$$i_{(M_1, M_2, \cdots, M_m)}\colon (M_1, M_2, \cdots, M_m) \longrightarrow (01 \cdots m, \emptyset, \dots, \emptyset)$$

We introduce the notation i_j for the composites

$$(j, \emptyset, \dots, \emptyset) \longrightarrow (01 \cdots m, \emptyset, \dots, \emptyset).$$

We note that all morphisms in $\mathrm{fsd}[m]$ fall into one of the cases (M1)–(M4).

From this point on, if an object $(M_1 \dots, M_m)$ has $M_i = \emptyset$ for all $k < i \leq m$, then we exclude the empty tail and simply write $(M_1, \dots, M_k)$.

The recursive procedure above produces the following category for $\mathrm{fsd}[1]$:

$$0 \xrightarrow{i_0} 01 \xleftarrow{i_1} 1,$$

whereas the category $\mathrm{fsd}[2]$ is graphically displayed as follows:

$$\begin{tikzcd}
 & & & 1 \arrow[dll, "i_1^2"'] \arrow[ddl, "\widehat{i}_1^2"'] \arrow[ddr, "\widehat{i}_1^0"] \arrow[drr, "i_1^0"] & & & \\
 & 01 & & & & 12 & \\
 & & (01,2) \arrow[ul, "h_{01}"'] \arrow[r, "i_{(01,2)}"] & 012 & (12,0) \arrow[ur, "h_{12}"] \arrow[l, "i_{(12,0)}"'] & & \\
 & & & (02,1) \arrow[u, "i_{(02,1)}"] \arrow[d, "h_{02}"] & & & \\
0 \arrow[uuur, "i_0^2"] \arrow[uurr, "\widehat{i}_0^2"] \arrow[urrr, "\widehat{i}_0^1"] \arrow[rrr, "i_0^1"] & & & 02 & & & 2 \arrow[uuul, "i_2^0"'] \arrow[uull, "\widehat{i}_2^0"'] \arrow[ulll, "\widehat{i}_2^1"'] \arrow[lll, "i_2^1"']
\end{tikzcd}$$

For this particular example, the usual barycentric subdivision $\mathrm{sd}[2]$ is clearly seen in the interior, although three of its 0–simplices have been given unusual indices. From the geometrical point of view we introduce the following terminology:

- Objects of the form j are the *singletons*.
- The full subcategory generated by objects (O1) is the *boundary* of $\mathrm{fsd}[m]$.
- The full subcategory generated by singletons and objects (O2) is the *thickening* of the boundary in $\mathrm{fsd}[m]$.
- The object $01 \cdots m$ is the *central point*.

With this in mind, the morphisms (M2) are the irreducible morphisms from the thickening to the boundary, and morphisms (M4) are the irreducible morphisms from the thickening to the central point.

To complete the definition of the functor $\mathrm{fsd}\colon \Delta \longrightarrow \mathrm{Cat}$, we need to define its action on morphisms. Recall how the simplex category Δ is generated by the coface maps $\delta_i^m\colon [m-1] \longrightarrow [m]$ and the codegeneracy maps $\sigma_i^m\colon [m+1] \longrightarrow [m]$, subject to the cosimplicial relations [GJ09, p. 4]. We proceed to define fsd on the coface and codegeneracy maps.

As a map of objects, we set $\operatorname{fsd}\delta_i^m$ to be

$$\begin{aligned} \operatorname{ob}\operatorname{fsd}[m-1] &\xrightarrow{\operatorname{fsd}\delta_i^m} \operatorname{ob}\operatorname{fsd}[m], \\ (M_1,\dots,M_{m-1}) &\longmapsto (\delta_i^m M_1,\dots,\delta_i^m M_{m-1},\emptyset). \end{aligned} \tag{5.1}$$

A morphism f in $\operatorname{fsd}[m-1]$ will be sent to f^i, introduced in (M1). By considering the way in which (M1) generates morphisms in $\operatorname{fsd}[m]$ from those in $\operatorname{fsd}[m-1]$, it is easy to see that $\operatorname{fsd}\delta_i^m$ is a fully faithful functor for each $i \leq m$.

The action on the degeneracy maps will be chosen as follows on the level of objects:

$$\begin{aligned} \operatorname{ob}\operatorname{fsd}[m+1] &\xrightarrow{\operatorname{fsd}\sigma_j^m} \operatorname{ob}\operatorname{fsd}[m], \\ (M_1,\dots,M_m,M_{m+1}) &\longmapsto (\sigma_j^m M_1,\dots,\sigma_j^m M_m). \end{aligned} \tag{5.2}$$

Let us proceed with the tedious task of checking that these are indeed functors:

Lemma 5.1. *The formulas* (5.1) *and* (5.2) *define functors* $\operatorname{fsd}\delta_i^m$ *and* $\operatorname{fsd}\sigma_j^m$*, respectively.*

Proof. For case of $\operatorname{fsd}\delta_i^m$, we refer to the above discussion. We can claim that $\operatorname{fsd}\sigma_j^m$ are functors by showing the following: if there exists a morphism $\mathbf{M}\longrightarrow\mathbf{N}$ in $\operatorname{fsd}[m+1]$, then there exists a morphism $\operatorname{fsd}\sigma_j^m(\mathbf{M})\longrightarrow\operatorname{fsd}\sigma_j^m(\mathbf{N})$ in $\operatorname{fsd}[m]$. After all, we are mapping a preorder into a preorder, so there is at most one morphism connecting any pair of objects.

We proceed by induction on m. The base case $m=0$ concerns functors into the category $[0]$, where the claim trivially holds. Let $m>0$ and suppose that the claim holds for $m-1$. Given a morphism $g\colon \mathbf{M}\longrightarrow\mathbf{N}$ in $\operatorname{fsd}[m+1]$, we check that there exists a morphism $\operatorname{fsd}\sigma_j^m g\colon \operatorname{fsd}\sigma_j^m(\mathbf{M})\longrightarrow\operatorname{fsd}\sigma_j^m(\mathbf{N})$ in $\operatorname{fsd}[m]$. This is achieved by considering the cases (M1)–(M4).

If g is in (M1), it is of the form f^i for some f in $\operatorname{fsd}[0,\dots,\widehat{i},\dots,m+1]$. Since the functors of the form $\operatorname{fsd}\sigma_k^{m-1}$ have been assumed to exist, they can be exploited to prove the existence of a morphism $\operatorname{fsd}\sigma_j^m g\colon \operatorname{fsd}\sigma_j^m(\mathbf{M})\longrightarrow\operatorname{fsd}\sigma_j^m(\mathbf{N})$. Indeed, consider the cosimplicial relation

$$\sigma_j^m\circ\delta_i^{m+1} = \begin{cases} \delta_i^m\circ\sigma_{j-1}^{m-1} & \text{if } 0\leq i<j\leq m, \\ \mathrm{id} & \text{if } i=j,j+1, \\ \delta_{i-1}^m\circ\sigma_j^{m-1} & \text{if } 0\leq j+1<i\leq m. \end{cases}$$

Thus, if $\mathbf{O}$ is any object in $\mathrm{fsd}[m]$, we have that

$$\mathrm{fsd}\,\sigma_j^m(\delta_i^{m+1}\mathbf{O}) := \begin{cases} (\mathrm{fsd}\,\delta_i^m \circ \mathrm{fsd}\,\sigma_{j-1}^{m-1})(\mathbf{O}) & \text{if } 0 \le i < j \le m \\ \mathbf{O} & \text{if } i = j, j+1 \\ (\mathrm{fsd}\,\delta_{i-1}^m \circ \mathrm{fsd}\,\sigma_j^{m-1})(\mathbf{O}) & \text{if } 0 \le j+1 < i \le m. \end{cases}$$

Since the induction hypothesis guarantees that both $\mathrm{fsd}\,\sigma_{j-1}^{m-1}$ and $\mathrm{fsd}\,\sigma_j^{m-1}$ induce morphisms from g, it follows that $\mathrm{fsd}\,\sigma_j^m$ does. This is because g is contained in the image of δ_i^{m+1}.

Should g be in (M2), the domain and codomain are sent to the same object. The identity then works as the image of g.

The case of (M3) can be reduced to (M1). Since the last component is removed in (5.2), the domain and codomain of the morphism $\widehat{f^j}$ in (M3) are sent to the same as the domain and codomain, respectively, of f^j.

The last case (M4) can be addressed by the explicit formula $\mathrm{fsd}\,\sigma_j^m(i_{\mathbf{M}}) = i_{\sigma_j^m\mathbf{M}}$, whenever $\sigma_j^m\mathbf{M}$ does not agree with the central point in $\mathrm{fsd}[m]$. Otherwise we define $\mathrm{fsd}\,\sigma_j^m$ as the identity at $0\cdots m$. □

Since the cosimplicial identities hold in Δ, it immediately follows that the functors of the form $\mathrm{fsd}\,\delta_j^m$ and $\mathrm{fsd}\,\sigma_k^m$ satisfy the same relations when considered as maps of objects. Since the categories $\mathrm{fsd}[m]$ are preorders, these relations hold when applied to morphisms as well. We are therefore in a position to conclude:

Lemma 5.2. *We have defined a functor* $\mathrm{fsd}\colon \Delta \longrightarrow \mathrm{Cat}$. *In other words, it is a cosimplicial category.*

We will compare the factorizing subdivision $\mathrm{fsd}[m]$ with the standard simplex $[m]$ via the subdivision $\mathrm{sd}[m]$ as introduced by Kan in [Kan57]. To do so it will be beneficial to define $\mathrm{sd}[m]$ in a recursive way that resembles the description of $\mathrm{fsd}[m]$. This is achieved by removing the thickening of faces that occur in (O2), (M2) and (M3). So we simply omit these constraints and alter (M4) to rather require maps from the boundary to the central point $01\cdots m$. With such a description there is an evident map $r\colon\ \mathrm{fsd}[n] \longrightarrow \mathrm{sd}[n]$ obtained by collapsing the information in (O2), (M2) and (M3) onto (O1) and (M1). The result is in fact a map $\ell\colon\ \mathrm{fsd} \longrightarrow \mathrm{sd}$ of cosimplicial categories.

Lemma 5.3. *For every* $m \ge 0$ *the factorizing subdivision* $\mathrm{fsd}[m]$ *is contractible. In particular, the map* $\ell_m\colon\ \mathrm{fsd}[m] \longrightarrow \mathrm{sd}[m]$ *is a weak equivalence.*

Proof. Consider the full subcategory $\mathrm{c}[m]$ of $\mathrm{fsd}[m]$ generated by singletons, objects in the thickening and the central point. This category is contractible since the object $01\cdots m$

is terminal. So it suffices to prove that $\operatorname{fsd}[m]$ is weakly equivalent to $\mathrm{c}[m]$ in the Quillen model structure on simplicial sets. It is well-known that taking the opposite of a category preserves homotopy types. Hence we rather consider the opposites of $\mathrm{c}[m]$ and $\operatorname{fsd}[m]$. This allows us to deform along opposites of (M2) from the boundary to the thickening.

We define a retract of the inclusion $i\colon \mathrm{c}[m]^{\mathrm{op}} \hookrightarrow \operatorname{fsd}[m]^{\mathrm{op}}$. Recall that the boundary of $\operatorname{fsd}[m]$, the union of its faces, is the full subcategory generated by the objects (O1). When we consider the opposite of $\operatorname{fsd}[m]$ every object (O1), excluding the singletons, admit a unique irreducible morphism $h^{\mathrm{op}}_{M_1,\ldots,M_{m-1}}$ to the thickening, defined by (M2). To define a retract $r\colon \operatorname{fsd}[m]^{\mathrm{op}} \longrightarrow \mathrm{c}[m]^{\mathrm{op}}$ we simply send every $h^{\mathrm{op}}_{M_1,\ldots,M_{m-1}}$ to the identity at $(M_1,\ldots,M_{m-1},M_m)$. The morphisms in the boundary, which are all of the form $(M_1,\ldots,M_{m-1},\emptyset) \longrightarrow (N_1,\ldots,N_{m-1},\emptyset)$, are then forced to be sent to their respective morphisms in the thickening (M3). Geometrically, the functor r collapses the boundary of $\operatorname{fsd}[m]^{\mathrm{op}}$ onto $c[m]^{\mathrm{op}}$. Since r fixes singletons as well as objects in (O2) and (O3), it is clear that $r \circ i$ is the identity on $\mathrm{c}[m]^{\mathrm{op}}$.

There is a rather evident natural transformation η from the identity on $\operatorname{fsd}[m]^{\mathrm{op}}$ to $i \circ r$ defined by utilizing the irreducible morphisms $h^{\mathrm{op}}_{M_1,\ldots,M_{m-1}}$ from the boundary to the thickening: an object $\mathbf{M} = (M_1,\ldots,M_{m-1},\emptyset)$ in (O1), except the singleton, associates uniquely to the morphism

$$\eta_{\mathbf{M}} = (M_1,\ldots,M_{m-1},\emptyset) \longrightarrow (M_1,\cdots,M_{m-1},12\ldots m \setminus \cup M_i).$$

Such a natural transformation is equivalent to a homotopy $\operatorname{fsd}[m]^{\mathrm{op}} \times [1] \longrightarrow \operatorname{fsd}[m]^{\mathrm{op}}$ from the identity to $i \circ r$. □

Any simplicial set X is canonically presentable as a colimit of m–simplices [GJ09, p. 7]. More precisely, the simplex category Δ, defines, for a fixed simplicial set X, a slice category $\Delta \downarrow X$ and $X = \operatorname*{colim}_{[m] \longrightarrow X} [m]$. Hence we can lift the factorizing subdivision from the simplex category to all simplicial sets by declaring $\operatorname{fsd} X = \operatorname*{colim}_{[m] \longrightarrow X} \operatorname{fsd}[m]$.

Lemma 5.4. *Every horn inclusion* $\operatorname{fsd}\Lambda^k[m] \hookrightarrow \operatorname{fsd}[m]$ *is an acyclic cofibration.*

Proof. The inclusion is clearly a cofibration, so we need only verify that it is in fact a weak equivalence. The k–horn $\Lambda^k[m]$ is the union of all faces in $[m]$ except the kth one. It can for example be presented as the coequalizer of

$$\coprod_{0 \le i < j \le m} [m-2] \rightrightarrows \coprod_{i \ne k} [m-1]$$

where both of the arrows are coproducts of face inclusions, whence they are cofibrations. Since both sd and fsd preserve colimits, we can apply them to the coequalizer and obtain $\operatorname{sd}\Lambda^k[m]$ and $\operatorname{fsd}\Lambda^k[m]$ as coequalizers. By Lemma 5.3, the natural map $\ell_m\colon \operatorname{fsd}[m] \longrightarrow \operatorname{sd}[m]$ then provides a level-wise weak equivalence of cofibrant coequlizer diagrams. Hence we conclude that the induced map $\operatorname{fsd}\Lambda^k[m] \longrightarrow \operatorname{sd}\Lambda^k[m]$ is a weak equivalence, and that $\operatorname{fsd}\Lambda^k[m]$ is contractible. This means that the inclusion $\operatorname{fsd}\Lambda^k[m] \hookrightarrow \operatorname{fsd}[m]$ is a map of contractible spaces; a weak equivalence. $\square$

Analogous to how Kan defines the well-known functor Ex in terms of the subdivision sd, we define a variant which is tailored for extension categories.

Definition 5.5. The factorizing Ex–functor is defined by $\operatorname{fEx} X_m = \operatorname{Hom}_{\mathrm{sSet}}(\operatorname{fsd}[m], X)$ with face and degeneracy maps induced from the cosimplicial structure of fsd.

The next observation is immediate from the definition.

Proposition 5.6. *The factorizing* Ex*–functor* fEx *is right adjoint to the factorizing subdivision* fsd.

We turn our attention to proving that the factorizing Ex–functor preserves homotopy types. This is done in a very similar fashion as to how you would prove the analogous result for Kan's Ex–functor. First, we derive a map $\eta\colon X \longrightarrow \operatorname{fEx} X$ from the natural map $\ell_m\colon \operatorname{fsd}[m] \longrightarrow \operatorname{sd}[m]$, as described in the discussion preceding Lemma 5.3, and the *last vertex map* $v_m\colon \operatorname{sd}[m] \longrightarrow [m]$ [GJ09, p. 183]. On the level of simplices, this is achieved by pre-composing an m–simplex $\operatorname{fsd}[m] \longrightarrow X$ with ℓ_m and v_m. Taking Lemmas 5.3 and 5.4 into account we more or less follow the acyclic models argument outlined by Goerss–Jardine [GJ09, Chapter 3.4] directly.

Theorem 5.7. *The map $\eta\colon X \longrightarrow \operatorname{fEx} X$ is a weak equivalence for any simplicial set X.*

We end this section with three important diagram lemmas, the two first are vital when proving the last.

Lemma 5.8. *Let J be a finite partially ordered set (considered as a category), and let $D\colon J \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ be a be homotopy coherent diagram. Then D extends to a diagram*

$$H\colon J \times [1] \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$$

with the following properties:

1. *$H_1 := H|_{J\times 1}$ is D,*
2. *all arrows in $H_0 := H|_{J\times 0}$ are term-wise cofibrations and*
3. *if i is an object in J which does not occur as the target of any morphism, then H is the identity when restricted to $i \times [1]$.*

Proof. Pick an arbitrary morphism $f\colon i \longrightarrow j$ in J. We shall first describe a procedure to define $H\colon J \times [1] \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ satisfying that $H(f, 0)$ is a term-wise cofibration. Moreover, if $D(g)\colon D(k) \longrightarrow D(j)$ is a term-wise cofibration, then also $H(g, 0)$ should be a term-wise cofibration. The same goes for morphisms $D(g)\colon D(k) \longrightarrow D(i)$. Applying the factorization in Proposition 3.6 to $D(f)\colon D(i) \longrightarrow D(j)$ provides a commutative diagram

$$\begin{tikzcd} D(i) \arrow[r, tail, "D(f)'"] \arrow[d, equal] & \mathrm{cyl}D(f) \arrow[d, "p"] \\ D(i) \arrow[r, "D(f)"] & D(j) \end{tikzcd}$$

which is equivalent to a functor $G\colon [1] \times [1] \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ satisfying properties 1.–3. above. We extend G to a functor $H\colon J \times [1] \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$. An object $k \neq i, j$ in J is mapped to $H(k, 0) = H(k, 1) = D(k)$. On the level of morphisms $g\colon k \longrightarrow \ell$ we can thus automatically define $H(g, 0) = H(g, 1) = D(g)$ whenever k and ℓ both differ from j. If g starts at $k = j$, then we choose a pre-composition $H(g, 0) = D(g) \circ p$. And lastly, if g terminates at $\ell = j$, then we post-compose with the right inverse m of p resulting in $H(g, 0) = m \circ D(g)$. The procedure is functorial: the only non-trivial case to check is $g\colon k \longrightarrow j$ and $h\colon j \longrightarrow \ell$, and here we have that $H(g, 0) = D(g) \circ p$ and $H(h, 0) = m \circ D(h)$, whence $H(h, 0) \circ H(g, 0) \sim D(g) \circ D(h)$. Observe that if a morphism of the form $D(g)\colon D(k) \longrightarrow D(j)$ is a term-wise cofibration, then also $H(g, 0)$ is a term-wise cofibration, since the class of cofibrations is closed under composition.

To finish the proof we shall iterate the above construction to turn every morphism into a term-wise cofibration. Since J is a finite partially ordered set, there is a finite set of irreducible morphisms f_I which uniquely factorizes all morphisms in J. Since J is a finite partially ordered set, any object k admits a *height* h. It is defined as the length of the longest chain of irreducible morphisms terminating at k. We define the height of an irreducible morphism f_I as the height of its target.

Iterate the above process on the set f_I, of irreducible morphisms, by first picking those whose height is equal to 1, then proceeding to 2 and so on. This is to guarantee that whenever we apply the procedure to an irreducible morphism $f\colon i \longrightarrow j$, no morphism out of j has already been replaced by a term-wise cofibration. Moreover, irreducible morphisms with smaller height will still be replaced by term-wise cofibrations. Indeed, this follows by the last sentence in the first paragraph. In the end, every irreducible morphism is replaced by a term-wise cofibration. If we denote by $[1] \amalg_{[0]} [1]$ the concatenation of $[1]$ by $[1]$, head to tail. What we have produced is a simplicial map

$$\widehat{H}\colon J \times \big([1] \amalg_{[0]} \cdots \amalg_{[0]} [1]\big) \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$$

where finitely many copies of $[1]$ are concatenated. But $\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ is a quasi-category, so choosing finitely many compositions determine a functor $H\colon J \times [1] \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$. The functor is constructed to satisfy the three asserted properties 1.–3. $\square$

Lemma 5.9. *Let J be a finite connected partially ordered set (considered as a category) with at least three objects. Let $D\colon J \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ be a homotopy coherent diagram. Assume that all morphisms of the form $D(f)$, where f is a morphism in J, are term-wise cofibrations. Then the diagram D has a cocone.*

Proof. Recall that D has a cocone if it can be extended to a homotopy coherent diagram $D^{\triangleright}\colon J^{\triangleright} \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$, where the category $J^{\triangleright}$ is obtained from J by adding a terminal object $*$. We refer to $D^{\triangleright}(*)$ as the cocone of D when its incident morphisms are obvious from context.

We prove the claim by induction on the number ℓ of minimal objects in J. The base case $\ell = 1$ amounts to the existence of an initial object i in J. We prove the base case by induction on the number of objects in J, denoted by m.

The base case $m = 3$ only pertains to the diagrams depicted as the solid part of the diagrams below.

$$\begin{tikzcd} & D(i) \arrow[dl] \arrow[dr] & \\ D(a) \arrow[dr, dashed] & & D(b) \arrow[dl, dashed] \\ & \mathbb{P} & \end{tikzcd} \qquad \begin{tikzcd} & D(i) \arrow[dl] \arrow[dr] & & \\ D(a) \arrow[rr, "D(f)"] \arrow[dr, dashed, "\alpha"'] & & D(b) \arrow[dl, dashed] \arrow[dr, dotted] & \\ & \mathbb{P} \arrow[rr, dotted] & & \mathbb{Q} \end{tikzcd}$$

In both cases, we construct an object $\mathbb{P}$ by taking the pushout of the arrows out of i, adding new dashed arrows above. The category on the left then obtains a cocone. One more pushout is needed to identify a cocone in the right-hand diagram, which is that of $D(f)$ along α.

Suppose that the claim has been proven for $m - 1$. If J has m objects, let a be a maximal one. The induction hypothesis then gives a cocone $\mathbb{P}$ of the restricted homotopy coherent diagram

$$D|_{J \setminus a}\colon J \setminus a \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A),$$

obtained by removing a and all its incident arrows from J. A cocone of D can be found by constructing an object $\mathbb{Q}$ as well as morphisms $\mathbb{P} \longrightarrow \mathbb{Q}$ and $D(a) \longrightarrow \mathbb{Q}$, such that for all immediate predecessors a_1 of a, the square

$$\begin{tikzcd} D(a_1) \arrow[r] \arrow[d] & D(a) \arrow[d] \\ \mathbb{P} \arrow[r] & \mathbb{Q} \end{tikzcd}$$

commutes. Suppose that a has k immediate predecessors a_1, a_2, …, a_k. The solid part of the following diagram depicts this situation.

$$\begin{tikzcd}
 & & & D(a) \arrow[r, dashed] & \mathbb{Q}_1 \arrow[r, dashed] & \mathbb{Q}_2 \arrow[r, dashed] & \cdots \arrow[r, dashed] & \mathbb{Q}_k \\
D(a_1) \arrow[urrr, "D(f_1)"] \arrow[drrrr, "\alpha_1"'] & D(a_2) \arrow[urr] \arrow[drrr] & \cdots & D(a_k) \arrow[u, "D(f_k)"] \arrow[dr, "\alpha_k"] & & & & \\
 & & & & \mathbb{P} \arrow[uu, dashed] \arrow[uur, dashed] \arrow[uurrr, dashed] & & &
\end{tikzcd}$$

The pushout of α_1 along $D(f_1)$ gives an object $\mathbb{Q}_1$ and the dashed morphisms into it. Next, we take the pushout of α_2 along the composite $D(a_2) \longrightarrow D(a) \longrightarrow \mathbb{Q}_1$, producing $\mathbb{Q}_2$. This procedure terminates after k steps, producing a cocone $\mathbb{Q} = \mathbb{Q}_k$ of D.

Having just proved the claim in the case where J has a unique minimal object, we now deal with the inductive step. Suppose that the claim holds when J has fewer than ℓ minimal objects, for some $\ell \geq 2$. If J contains exactly ℓ minimal objects $a_1, \dots, a_\ell$, we let J_1 be the subposet of J containing the objects x such that $x \geq a_1$. The base case (shown above) provides a cocone $\mathbb{P}_1$ of the subdiagram given by J_1.

Now, let J' be the full subcategory of J spanned by the objects of $\bigcup_{i \geq 2} J_{\geq a_i}$. We extend J' to J'' by adding an object p_1 and morphisms $a \longrightarrow p_1$ for all a in $J_1 \cap J'$, such that all resulting triangles commute. The construction of $\mathbb{P}_1$ in the last paragraph paves the way for a homotopy coherent diagram $D'' \colon J'' \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ such that $D''(p_1) = \mathbb{P}_1$. Since J'' is a partially ordered set with $\ell - 1$ minimal objects, it follows from the induction hypothesis that this diagram has a cocone $\mathbb{P}$. We claim that $\mathbb{P}$ serves as a cocone for J. We have suitable morphisms from objects in $D(J')$ to $\mathbb{P}$ by definition, and we may choose composites via $\mathbb{P}_1$ to get suitable morphisms out of objects in $D(J_1)$. $\square$

In a Kan complex, all horns can be filled. Two subcategories of $\mathrm{fsd}[m]$ are therefore of particular interest. Consider this statement in (O1), and add the extra assumption that $j > 0$. In full:

($\Lambda 1$) For each $0 < j \leq m$ and each object $(M_1, \dots, M_{m-1})$ in $\mathrm{fsd}[0, \dots, \widehat{j}, \dots, m]$, we include $(M_1, \dots, M_{m-1}, \emptyset)$. Although duplicates may arise, no element is counted twice.

The objects given by this refining of (O1) describes $\mathrm{fsd}\,\Lambda^0[m]$, the factorizing subdivision applied to the 0–horn. We depict $\mathrm{fsd}\,\Lambda^0[m]$ as the solid part of:

$$\begin{tikzcd}
 & & & 1 \arrow[dll, "i_1^2"'] \arrow[ddl, dashed, "\widehat{i}_1^2"] & \\
 & 01 & & & \\
 & & (01,2) \arrow[ul, dashed, "h_{01}"'] \arrow[r, dashed, "i_{(01,2)}"] & 012 & \\
 & & & (02,1) \arrow[u, dashed, "i_{(02,1)}"] \arrow[d, dashed, "h_{02}"] & \\
0 \arrow[uuur, "i_0^2"] \arrow[uurr, dashed, "\widehat{i}_0^2"] \arrow[urrr, dashed, "\widehat{i}_0^1"] \arrow[rrr, "i_0^1"] & & & 02 & 2 \arrow[l, "i_2^1"'] \arrow[ul, dashed, "\widehat{i}_2^1"']
\end{tikzcd}$$

The procedures put forward in (O2) and (O3) can then be applied to the objects in $\mathrm{fsd}\,\Lambda^0[m]$ (i.e. those described in (Λ1)), to generate a larger subcategory $\mathrm{fsd}_0^+[m]$ of $\mathrm{fsd}[m]$. This subcategory is a "fat" 0–horn because it also contains the thickening of j–faces, $j > 0$, as well as the central point. The dashed arrows above illustrate $n = 2$.

Lemma 5.10. *Any homotopy coherent diagram of the form*

$$\mathbb{E}\colon\ \mathrm{fsd}\,\Lambda^0[m] \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$$

fills out to

$$\mathbb{E}^+\colon\ \mathrm{fsd}_0^+[m] \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A).$$

Proof. We are given the objects generated by (Λ1). The first step is to fill out to the thickening. Finally, we add the central point of (O3), and make sure that all morphisms in (M4) are included. These two steps are achieved by invoking Lemmas 5.8 and 5.9, respectively.

To address (O2), (M2) and (M3) we ought to provide a filler to the thickening. This amounts to a homotopy

$$H\colon\ \mathrm{fsd}\,\Lambda^0[m] \times [1] \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$$

such that H is the identity when restricted to $j \times [1]$, for all singletons j. Lemma 5.8 thus provides a filler. The restriction

$$\mathrm{fsd}\,\Lambda^0[m] \xrightarrow{(\mathrm{id},0)} \mathrm{fsd}\,\Lambda^0[m] \times [1] \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$$

of H to the thickening satisfy that all entries are term-wise cofibrations. Hence it admits a cocone $\mathbb{P}$ by Lemma 5.9. Altogether we have defined a filler $\mathbb{E}^+$ which is equal to $H(-, 0)$ on the thickening and maps the central point to $\mathbb{P}$. $\square$

6 Loop spaces of extension categories

We have constructed a modified fEx–functor to Kan's Ex–functor. This was achieved by modifying the subdivision functor sd, cf. the beginning of Section 5. The technical Lemma 5.10 will now be used to prove that $\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ is a Kan complex and that the nth Retakh functor is a weak equivalence.

Theorem 6.1. *Let $\mathscr{C}$ be an exact quasi-category in which A and B are objects. Then the simplicial set $\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ is a Kan complex.*

Proof. Let $\mathbb{E}\colon \Lambda^k[m] \longrightarrow \mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$ be a k–horn in $\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$. We devise a horn-filling algorithm extending $\mathbb{E}$ to an m–simplex $\mathbb{F}\colon [m] \longrightarrow \mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A)$. Our algorithm does not depend on the choice of k, up to a re-indexing of 0–simplices. So we will only consider $k = 0$. Moreover, utilizing the left adjoint fsd we may rather extend a diagram $\mathbb{E}$ defined on $\mathrm{fsd}\,\Lambda^0[m]$ to an m–simplex $\mathbb{F}\colon\ \mathrm{fsd}[m] \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}$ (Definition 5.5).

We treat the special cases $m = 1$ and $m = 2$ to give the reader a geometric understanding of the horn-filling procedure. If $m = 1$, the 0–horn is simply the discrete category with a single object 1. Imposing that

$$\mathbb{F}(0) = \mathbb{F}(01) = \mathbb{F}(1)$$

gives an adequate horn filler. For $m = 2$, we consider a diagram indexed by the 0–horn $\mathrm{fsd}\,\Lambda^0[2]$, which lifts to the solid part of

$$\begin{tikzcd}
 & & & 1 \arrow[dll] \arrow[ddl] \arrow[ddr, dashed] \arrow[drr, dashed] & & & \\
 & 01 & & & & 12 \arrow[dl, equal] & \\
 & & (01,2) \arrow[ul] \arrow[r] & 012 \arrow[r, equal] & (12,0) & & \\
 & & & (02,1) \arrow[u] \arrow[d] & & & \\
0 \arrow[uuur] \arrow[uurr] \arrow[urrr] \arrow[rrr] & & & 02 & & & 2 \arrow[lll] \arrow[ulll] \arrow[uull, dashed] \arrow[uuul, dashed]
\end{tikzcd}$$

utilizing Lemma 5.10. Note that the solid part is equivalent to a homotopy coherent diagram in $\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B,A)$ indexed by $\mathrm{fsd}_0^+[2]$, the thick 0–horn. Proceed as indicated by the dashed arrows to complete a diagram $\mathbb{F}\colon\ \mathrm{fsd}[2] \longrightarrow \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B,A)$.

For a general $m \geq 2$ we do the same procedure, but some extra care is required when adding identities in the end. Lemma 5.10 gives a diagram $\mathbb{E}^+$ defined on the thick 0–horn $\mathrm{fsd}_0^+[m]$, which restricts to $\mathbb{E}$ on $\mathrm{fsd}\,\Lambda^0[m]$. Comparing $\mathrm{fsd}_0^+[m]$ and $\mathrm{fsd}[m]$, we see that the objects in the interior of $\mathrm{fsd}[m]$'s zeroth face $\mathrm{fsd}[1,\dots,m]$ and their duplicates in the thickening are missing. The boundary of $\mathrm{fsd}[1,\dots,m]$, i.e. the union over $\mathrm{fsd}[1,\dots,\widehat{k},\dots,m]$, its thickening in $\mathrm{fsd}[m]$ and the central point $0\cdots m$ defines a subcategory $I[m-1]$ equivalent to $\mathrm{fsd}[m-1]$. Similarly, the thickening $T[m-1]$ of $\mathrm{fsd}[1,\dots,m]$ in $\mathrm{fsd}[m]$, is of course equivalent to $\mathrm{fsd}[m-1]$. So we extend $\mathbb{E}^+$ to include $T[m-1]$ by mapping irreducible morphisms from $T[m-1]$ onto $I[m-1]$ to identities. Lastly, the m–simplex $\mathbb{F}$ is obtained by also sending irreducible morphisms from the thickening $T[m-1]$ onto $\mathrm{fsd}[1,\dots,m]$ to identities. These two steps of adding identities specializes to the above picture for $m=2$. $\square$

Recall that $\mathrm{eval}_1\colon\ \hom(([1],0),(\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B,A),\sigma_n(B,A))) \longrightarrow \mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B,A)$ is the evaltuation at 1, which is induced by the inclusion of the terminal object 1 into $[1]$.

Lemma 6.2. *The pre-image* $\mathrm{eval}_1^{-1}(\sigma(B,A))$ *models the loop space on* $\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B,A)$ *based at* $\sigma(B,A)$.

Proof. This is a direct consequence of Proposition 4.1 and Theorem 6.1. $\square$

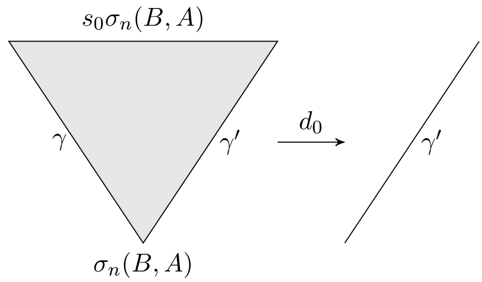
This is a fairly explicit description of the loop space. In particular, the objects are simple zigzags

$$\sigma_n(B,A) \longrightarrow \mathbb{E} \longleftarrow \sigma_n(B,A),$$

i.e. diagrams

$$\begin{tikzcd}
 & A \arrow[dl, equals] \arrow[dd, tail] \arrow[dr, two heads] & & \cdots \arrow[dr, two heads] & & B \arrow[dr, equals] \arrow[dd] & \\
A \arrow[dd, equals] & & 0 \arrow[ur, tail] \arrow[dd] & & 0 \arrow[ur, tail] \arrow[dd] & & B \arrow[dd, equals] \\
 & E_1 \arrow[dr, two heads] & & \cdots \arrow[dr, two heads] & & E_n \arrow[dr, two heads] & \\
A \arrow[ur, tail] & & E_{1.5} \arrow[ur, tail] & & E_{(n-1).5} \arrow[ur, tail] & & B \\
 & A \arrow[uu, tail] \arrow[dl, equals] \arrow[dr, two heads] & & \cdots \arrow[dr, two heads] & & B \arrow[uu] \arrow[dr, equals] & \\
A \arrow[uu, equals] & & 0 \arrow[uu] \arrow[ur, tail] & & 0 \arrow[uu] \arrow[ur, tail] & & B \arrow[uu, equals]
\end{tikzcd}$$

Notice how $\mathrm{eval}_1^{-1}(\sigma_n(B,A))$ is the space of automaps on $\sigma_n(B,A)$ inside the Kan complex $\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B,A)$. One model for this space has a particularly neat description in terms of simplices in $\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B,A)$. See [Lur09, Remark 1.2.2.5] for generalities. An m–simplex in $\mathrm{eval}_1^{-1}(\sigma_n(B,A))$ is an $(m+1)$–simplex S in $\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B,A)$ with the following properties. The zeroth 0–simplex in S is equal to $\sigma_n(B,A)$, also the zeroth face d_0S is equal to $s_0^m\sigma_n(B,A)$, the m–dimensional copy of $\sigma_n(B,A)$. The ith face of S, considered as an m–simplex in $\mathrm{eval}_1^{-1}(\sigma_n(B,A))$, is the m–simplex $d_{i+1}S$ in $\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B,A)$. Here is a simple illustration of a 1–simplex between two loops γ and γ' together with the zeroth face map:



Let us proceed with a proof of Theorem 4.3. The technique used to show that the Retakh functor R_n is a weak equivalence depends on n. The case $n=1$ is significantly simpler than $n \geq 2$: the quasi-category $\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B,A)$ is a Kan complex, whereas we need to consider $\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B,A)$ for higher n. Nonetheless, all Retakh maps are proven to be weak equivalences via their homotopy fiber. If $F\colon X \longrightarrow Y$ is a simplicial map between Kan complexes, then for all y in Y the comma category $F \downarrow y$ models the homotopy fiber at y. We refer to either [BK72, p. 298] or [Lur09, Theorem 4.1.3.1] for details. Its description is very similar to the above mapping space. An m–simplex in $F \downarrow y$ is an $(m+1)$–simplex S in Y so that the $(m+1)$st 0–simplex is equal to y, whereas $d_{m+1}S$ is in the image of F. A 1–simplex and its first face is typically depicted

$$\begin{tikzcd} F(x) \arrow[d] & & F(x) \arrow[rr, "F(e)"] \arrow[dr] & & F(x') \arrow[dl] \\ y & & & y & \end{tikzcd}$$

Recall the zeroth Retakh functor

$$R_0\colon \hom_{\mathscr{C}}(B,A) \longrightarrow \Omega(\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B,A), \sigma_1(B,A))$$

mapping $f\colon B \longrightarrow A$ to the endomap

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & A \oplus B \arrow[r] \arrow[d, "\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}"] & B \arrow[d, equal] \\
A \arrow[r, tail] & A \oplus B \arrow[r] & B
\end{tikzcd}$$

of $\sigma_1(B, A)$

Proposition 6.3. *The zeroth Retakh functor*

$$R_0\colon \hom_{\mathscr{C}}(B, A) \longrightarrow \Omega(\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B, A), \sigma_1(B, A))$$

is a weak homotopy equivalence.

Proof. It suffices to show that the homotopy fiber, modeled as the comma category $R_0 \downarrow \gamma$, is contractible for all loops $\gamma \in \Omega(\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B, A), \sigma_1(B, A))$. Any loop is a endomap of $\sigma_1(B, A)$, and all endomaps of $\sigma_1(B, A)$ are homotopic to a loop in the image of R_0. Hence the homotopy fiber $R_0 \downarrow \gamma$ is easily seen to deformation retract onto the identity at γ. This shows that the homotopy fiber is contractible at any loop γ. $\square$

In Section 4 we defined the nth Retakh functor

$$R_n\colon \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B, A) \longrightarrow \Omega(\mathscr{E}\mathrm{xt}^{n+1}_{\mathscr{C}}(B, A), \sigma_{n+1}(B, A)).$$

In short, the loop $R_n\mathbb{E}$ is given by

$$\sigma_{n+1}(B, A) \longrightarrow (\mathbb{E}, \rho_B) \longleftarrow \sigma_{n+1}(B, A).$$

where $(\mathbb{E}, \rho_B)$ is the concatenation of $\mathbb{E}$ and ρ_B:

$$B \xrightarrow{\begin{pmatrix} -1 \\ 1 \end{pmatrix}} B \oplus B \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} B.$$

Fix a loop γ in $\Omega(\mathscr{E}\mathrm{xt}^{n+1}_{\mathscr{C}}(B, A), \sigma_{n+1}(B, A))$, which is a diagram in $\mathscr{E}\mathrm{xt}^{n+1}_{\mathscr{C}}(B, A)$ of the form

$$\sigma_{n+1}(B, A) \longrightarrow \mathbb{F} \longleftarrow \sigma_{n+1}(B, A).$$

Now we aim to construct a universal map of loops $U_\gamma\colon R_n\mathbb{E} \longrightarrow \gamma$ from the image of the Retakh functor. To better see how such a map can be constructed, we depict γ in full:

$$
\begin{tikzcd}[column sep=small]
 & A \arrow[dr, two heads] \arrow[dd, tail, "i_1"] & & \cdots \arrow[dr, two heads] & & B \arrow[dr, equal] \arrow[dd, "\gamma_1"] & \\
A \arrow[ur, equal] \arrow[dd, equal] & & 0 \arrow[ur, tail] \arrow[dd] & & 0 \arrow[ur, tail] \arrow[dd] & & B \arrow[dd, equal] \\
 & F_1 \arrow[dr, two heads] & & \cdots \arrow[dr, two heads] & & F_{n+1} \arrow[dr, two heads, "p_{n+1}"] & \\
A \arrow[ur, tail, "i_1"] \arrow[dd, equal] & & F_{1.5} \arrow[ur, tail] & & F_{n.5} \arrow[ur, tail, "i_{n+1}"] & & B \arrow[dd, equal] \\
 & A \arrow[uu, tail, "i_1"] \arrow[dr, two heads] & & \cdots \arrow[dr, two heads] & & B \arrow[uu, "\gamma_2"] \arrow[dr, equal] & \\
A \arrow[ur, equal] & & 0 \arrow[ur, tail] \arrow[uu] & & 0 \arrow[ur, tail] \arrow[uu] & & B
\end{tikzcd}
$$

It is determined by $\mathbb{F}$ together with two right inverses γ_1 and γ_2 of p_{n+1}, i.e. $p_{n+1}\gamma_i \sim 1_B$. Consequently, a composite

$$
-\gamma_1 + \gamma_2 \coloneqq \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} : B \longrightarrow F_{n-1}
$$

factors through i_{n+1}, which is to say that we have 2-simplex

$$
\begin{tikzcd}
B \arrow[rr, "-\gamma_1+\gamma_2"] \arrow[dr, "b"'] & & F_{n-1} \\
 & F_{n.5} \arrow[ur, "i_{n+1}"'] &
\end{tikzcd}
$$

in $\mathscr{C}$. Note that, up to homotopy, the map b is the unique map such that $-\gamma_1+\gamma_2 \sim i_{n+1} \circ b$. This is because of how it is defined from the universal mapping property of the short exact sequence

$$
F_{n.5} \overset{i_{n+1}}{\rightarrowtail} F_{n+1} \overset{p_{n+1}}{\twoheadrightarrow} B.
$$

Lemma 2.9 provides a map of n–extensions

$$
\begin{tikzcd}[column sep=small]
 & F_1 \arrow[dr, two heads] \arrow[dd, equal] & & \cdots \arrow[dr, two heads] & & E_n \arrow[dr, two heads] \arrow[dd, "u_n"] \arrow[ddr, phantom, "\square"] & \\
A \arrow[ur, tail] \arrow[dd, equal] & & F_{1.5} \arrow[ur, tail] \arrow[dd, equal] & & F_{(n-1).5} \arrow[ur, tail] \arrow[dd, equal] & & B \arrow[dd, "b"] \\
 & F_1 \arrow[dr, two heads] & & \cdots \arrow[dr, two heads] & & F_n \arrow[dr, two heads] & \\
A \arrow[ur, tail] & & F_{1.5} \arrow[ur, tail] & & F_{(n-1).5} \arrow[ur, tail] & & F_{n.5}
\end{tikzcd}
$$

in which E_n is the pullback. Let $\mathbb{E}$ be the top row, then the above diagram extends to a map of loops $U_\gamma \colon R_n\mathbb{E} \longrightarrow \gamma$ given by

$$
\begin{tikzcd}
\sigma_{n+1}(B,A) \arrow[r, equal] \arrow[d] & \sigma_{n+1}(B,A) \arrow[d] \\
(\mathbb{E}, \rho_B) \arrow[r, "u_\gamma"] & \mathbb{F} \\
\sigma_{n+1}(B,A) \arrow[r, equal] \arrow[u] & \sigma_{n+1}(B,A) \arrow[u]
\end{tikzcd}
$$

where u_γ is the map

$$
\begin{tikzcd}
& \cdots \arrow[dr, two heads] & & E_n \arrow[dr, two heads] \arrow[dd, "u_n"] & & B \oplus B \arrow[dr, "\begin{pmatrix} 1 & 1 \end{pmatrix}"] \arrow[dd, "\begin{pmatrix} \gamma_1 & \gamma_2 \end{pmatrix}"] & \\
A \arrow[ur, tail] \arrow[dd, equal] & & F_{(n-1).5} \arrow[ur, tail] \arrow[dd, equal] & & B \arrow[ur, "\begin{pmatrix} -1 \\ 1 \end{pmatrix}"] \arrow[dd, "b"] \arrow[dl, phantom, "\square"] \arrow[dr, phantom, "\square"] & & B \arrow[dd, equal] \\
& \cdots \arrow[dr, two heads] & & F_n \arrow[dr, two heads] & & F_{n+1} \arrow[dr, "p_{n+1}"] & \\
A \arrow[ur, tail] & & F_{(n-1).5} \arrow[ur, tail] & & F_{n.5} \arrow[ur, "i_{n+1}"] & & B
\end{tikzcd}
$$

If $\gamma = R_n\mathbb{E}$, then both $-\gamma_1 + \gamma_2$ and i_{n+1} agree with $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ so that $b \sim 1_B$. The Five Lemma (Lemma 2.11) thus implies that u_n is a homotopy equivalence so that U_γ is the identity on $R_n\mathbb{E}$.

Let J be an indexing category, and consider $D \colon J \longrightarrow \Omega\,\mathscr{E}\mathrm{xt}^{n+1}_{\mathscr{C}}(B,A)$ a homotopy coherent diagram. We can apply the above steps to all of D, resulting in a diagram contained in the image of R_n. Indeed, no matter how complicated D is when translated to a diagram in $\mathscr{C}$, the above process only hinges upon the universal mapping property into short exact sequences as well as functoriality of pullbacks. We summarize the discussion thus far.

Lemma 6.4. *A homotopy coherent diagram $D \colon J \longrightarrow \Omega\,\mathscr{E}\mathrm{xt}^{n+1}_{\mathscr{C}}(B,A)$, indexed by some category J, extends to a diagram*

$$H \colon J \times [1] \longrightarrow \Omega\,\mathscr{E}\mathrm{xt}^{n+1}_{\mathscr{C}}(B,A)$$

by declaring $H|_{\gamma\times[1]} = U_\gamma$ for all $\gamma = D(j)$. Moreover, the extended diagram H enjoys the following properties:

1. *$H_1 \coloneqq H|_{J\times 1}$ is D,*

2. *$H_0 \coloneqq H|_{J\times 0}$ is in the image of R_n and*

3. *if i is an object in J such that $D(i) = R_n\mathbb{E}$, then H is the identity when restricted to $i \times [1]$.*

The zeroth Retakh functor was proven a weak equivalence in Proposition 6.3, so in order to finish a proof of Theorem 4.3 we need only verify that the nth Retakh functor is a weak equivalence.

Proposition 6.5. *The nth Retakh functor*

$$R_n\colon \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(B,A) \longrightarrow \Omega(\mathscr{E}\mathrm{xt}^{n+1}_{\mathscr{C}}(B,A), \sigma_n(B,A))$$

is a weak equivalence for all $n \geq 1$.

Proof. The objects A and B are fixed within $\mathscr{C}$, so we simply write $\sigma = \sigma_{n+1}(B,A)$ and $\mathscr{E}\mathrm{xt}^{n+1} = \mathscr{E}\mathrm{xt}^{n+1}_{\mathscr{C}}(B,A)$ throughout the proof. As in the proof of Proposition 6.3, we show that the homotopy fiber $R_n \downarrow \gamma$ is contractible for a fixed loop γ:

$$\sigma \xrightarrow{\gamma_1} \mathbb{F} \xleftarrow{\gamma_2} \sigma.$$

To do so, we prove that the universal map of loops $U_\gamma\colon R_n\mathbb{E}_1 \longrightarrow \gamma$ is initial in $R_n \downarrow \gamma$. The definition of an initial object [Joy02, Definition 4.1] requires that every simplicial sphere

$$\partial[m] \longrightarrow R_n \downarrow \gamma$$

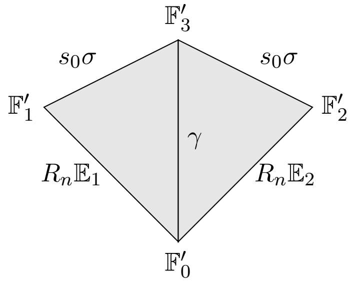
whose restriction to 0 is U_γ can be filled. We first treat the case $m = 1$ to get a better geometric grasp of the algorithm for general m.

Step 1: The 0–dimensional simplicial sphere $\partial[1] \longrightarrow R_n \downarrow \gamma$ consists of two loops $R_n\mathbb{E}_1$ and $R_n\mathbb{E}_2$ which constitute a 2–horn

$$\begin{tikzcd} R_n\mathbb{E}_1 \arrow[dr, "U_\gamma"] & & R_n\mathbb{E}_2 \arrow[dl] \\ & \gamma & \end{tikzcd}$$

whose 0–simplices are $R_n\mathbb{E}_1, R_n\mathbb{E}_2, \gamma$. Notice how the left-most map is U_γ.

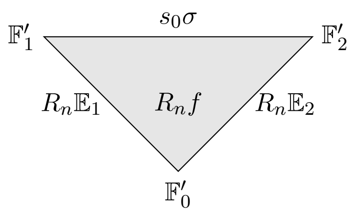
Step 2: A 1–simplex in $\Omega\,\mathscr{E}\mathrm{xt}^{n+1}$ is a 2–simplex in $\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^{n+1}$. The data in Step 1 thus defines a diagram $\mathbb{F}'$ in $\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^{n+1}$ depicted



Every 0–simplex $\mathbb{F}_i'$ is equal to σ. Define $\mathbb{F}(123) = s_0^2\sigma$, where $s_0^2\sigma$ is the degenerate 2–dimensional copy of σ, to obtain a 3–horn $\mathbb{F}$ from $\mathbb{F}'$.

Step 3: We fill out $\mathbb{F}$ to $\mathbb{F}^+$ according to Lemma 5.10. Define a diagram D by restricting $\mathbb{F}^+$ to the boundary spanned by $\mathbb{F}_0', \mathbb{F}_1', \mathbb{F}_2'$, its thickening as well as the central point. This translates to a diagram $D\colon [1] \longrightarrow \Omega\,\mathscr{E}\mathrm{xt}^{n+1}$ from $R_n\mathbb{E}_1$ to $R_n\mathbb{E}_2$. Although $D(01)$ need not be in the image of R_n, this is easily fixed by applying Lemma 6.4: we have that D extends to a homotopy H from a diagram in the image of R_n to D. In particular, $H(-,0)$ defines a filling of $\mathbb{F}^+$ to the thickening of the third face. We further extend $\mathbb{F}^+$ to all of [3] by adding equalities from the zeroth face to its thickening.

This completes the case $m = 1$. Indeed, the constructed 3–simplex $[3] \longrightarrow \mathrm{fEx}\,\mathscr{E}\mathrm{xt}^{n+1}$ has a third face of the form



which encodes an arrow $R_nf\colon R_n\mathbb{E}_1 \longrightarrow R_n\mathbb{E}_2$ and fills the horn in step 1:

$$\begin{tikzcd} R_n\mathbb{E}_1 \arrow[rr, "R_nf"] \arrow[dr, "U_\gamma"'] & & R_n\mathbb{E}_2 \arrow[dl] \\ & \gamma & \end{tikzcd}$$

There is nothing special about $m = 1$, except for the possibility of drawing pictures. Here is the general algorithm outlined above:

Step 1: The simplicial sphere $\partial[m] \longrightarrow R_n \downarrow \gamma$ has 0–simplices $R_n\mathbb{E}_i \longrightarrow \gamma$, which we index by $i = 1, \ldots, m+1$, and the first map agrees with U_γ. This data defines an $(m+1)$–horn in $\Omega\,\mathscr{E}\mathrm{xt}^{n+1}$ since all arrows terminate in γ. We order the 0–simplices $R_n\mathbb{E}_1, \dots, R_n\mathbb{E}_{m+1}, \gamma$.

Step 2: For every $i = 1, \dots, m+1$ we now have an m–simplex Γ_i in $\Omega\,\mathscr{E}\mathrm{xt}^{n+1}$ spanned by

$$R_n\mathbb{E}_1, \dots, \widehat{R_n\mathbb{E}_i}, R_n\mathbb{E}_{m+1}, \gamma.$$

Let $S_1, \dots, S_{m+1}$ be the corresponding $(m+1)$–simplices in $\mathrm{fEx}\,\mathscr{E}\mathrm{xt}^{n+1}$. We also introduce $S_0 = s_0^{m+1}\sigma$. These simplices assemble into an $(m+2)$–dimensional horn $\mathbb{F}\colon \Lambda^{m+2}[m+2] \longrightarrow \mathrm{fEx}\,\mathscr{E}\mathrm{xt}^{n+1}$. Indeed, it suffices to check that $d_iS_j = d_{j-1}S_i$ for $i < j$ and $j \neq m+1$ [GJ09, p. 10]. If $j > 0$, then applying d_0 to S_j always produces $s_0^m\sigma = d_{j-1}s_0^{m+1}$. So we need only consider $i > 0$. The m–simplex d_iS_j corresponds to $d_{i-1}\Gamma_i$ spanned by

$$R_n\mathbb{E}_1, \dots, \widehat{R_n\mathbb{E}_i}, \dots, \widehat{R_n\mathbb{E}_j}, \dots R_n\mathbb{E}_{m+1}, \gamma$$

where the jth term was removed first. If we apply d_{j-1} to Γ_i, we get the same output: we first remove $R_n\mathbb{E}_i$ in the construction of Γ_i so that reindexing to $j-1$ removes $R_n\mathbb{E}_j$ thereafter.

Step 3: Fill out $\mathbb{F}$ to $\mathbb{F}^+$ according to Lemma 5.10. To further extend it to an $(m+2)$–simplex we proceed as in the proof of Theorem 6.1. But instead of adding equalities to include the thickening of the $(m+2)$nd face, we rather invoke Lemma 6.4. Thereafter, we include the $(m+2)$nd face by sending irreducible morphisms from its thickening to identities. If we interpret the $(m+2)$nd face as an $(m+1)$–simplex of loops, then this simplex is necessarily in the image of R_n. The simplicial sphere $\partial[m] \longrightarrow R_n \downarrow \gamma$ is thus filled out. $\square$

7 Extriangulated homotopy categories and higher extensions

It has recently been shown that the homotopy category $\mathrm{h}\,\mathscr{C}$ of an exact quasi-category $\mathscr{C}$ has a natural extriangulated structure [NP20, Theorem 4.22], when equipped with the bifunctor $\pi_0\,\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B, A)$ as well as a readily accessible additive realization. In this section, we use of the Retakh Ext-Ω–spectrum to shed new light on Nakaoka–Palu's result. More specifically, our claim is that the Retakh spectrum determines the extriangulation of the homotopy category as well as extriangulated functors between them. We will also prove in Theorem 7.8 that higher extension categories induce the higher extension groups when passing to the homotopy category.

We need a few preliminary definitions to review the relatively new theory of extriangulated categories.

Definition 7.1. Let $\mathcal{C}$ be an additive category and let $\mathbf{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathrm{Ab}$ be an additive bifunctor into the category of abelian groups. We refer to group elements $\xi \in \mathbf{E}(B, A)$

as $\mathbf{E}$–*extensions*. A *morphism of* $\mathbf{E}$–*extensions* $\xi \in \mathbf{E}(B_1, A_1)$ and $\xi' \in \mathbf{E}(B_2, A_2)$ is a pair of morphisms $(a\colon A_1 \longrightarrow A_2, b\colon B_1 \longrightarrow B_2)$ such that $\mathbf{E}(b, A_2)(\xi') = \mathbf{E}(B_1, a)(\xi)$.

Having defined a notion of a morphism of $\mathbf{E}$–extensions, it is readily checked that there is a category of such. We denote this by $\mathbf{E}\text{-}\mathrm{Ext}^1_{\mathcal{C}}$.

The role played by $\mathbf{E}$–extensions is that of equivalence classes of exact sequences. For example, if $\mathscr{C}$ is an exact quasi-category we have an additive bifunctor

$$\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(-,-)\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \mathrm{Sp}$$

(recall that Sp is the quasi-category of spectrum objects). This is the first component in the Retakh spectrum functor $\mathscr{E}\mathrm{xt}_{\mathscr{C}}(-,-)$. It induces a biadditive functor

$$\mathbf{E} := \pi_0\, \mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(-,-)\colon (\mathrm{h}\mathscr{C})^{\mathrm{op}} \times \mathrm{h}\mathscr{C} \longrightarrow \mathrm{Ab}\,.$$

which sends a tuple (B, A) to the set of equivalence classes of exact sequences in $\mathscr{C}$ of the form $A \rightarrowtail E \twoheadrightarrow B$ (i.e. to a connected component of $\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B, A)$). An $\mathbf{E}$–extension is simply an equivalence class of exact sequences. Since the homomorphism $\mathbf{E}(B, a)$ sends an exact sequence $A \rightarrowtail E \twoheadrightarrow B$ to the bottom of the following commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "a"] \arrow[dr, phantom, "\square"] & E \arrow[r] \arrow[d] & B \arrow[d, equal] \\
A' \arrow[r] & Z \arrow[r] & B
\end{tikzcd}$$

and dually for $\mathbf{E}(b, A)$, it follows by Lemma 2.10 that a morphism of $\mathbf{E}$–extensions is an equivalence class of morphisms of complexes

$$(a, e, b)\colon \quad \begin{tikzcd}
A \arrow[r] \arrow[d, "a"] & E \arrow[r] \arrow[d, "e"] & B \arrow[d, "b"] \\
A' \arrow[r] & E' \arrow[r] & B'
\end{tikzcd}$$

which is uniquely determined by the pair (a, b).

Lemma 7.2. *Let* $\mathbf{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathrm{Ab}$ *and* $\mathbf{E}'\colon \mathcal{C}'^{\mathrm{op}} \times \mathcal{C}' \longrightarrow \mathrm{Ab}$ *be additive bifunctors. If* $F\colon \mathcal{C} \longrightarrow \mathcal{C}'$ *is an additive functor, then a natural transformation*

$$\eta\colon \mathbf{E}(-,-) \longrightarrow \mathbf{E}'(F^{\mathrm{op}}-, F-)$$

induces a functor $F_\eta\colon \mathbf{E}\text{-}\mathrm{Ext}_{\mathcal{C}} \longrightarrow \mathbf{E}'\text{-}\mathrm{Ext}_{\mathcal{C}'}$. *If* F *is an equivalence and* η *is a natural isomorphism, then* F_η *is an equivalence.*

Proof. The maps $\eta_{B,A}\colon \mathbf{E}(B,A) \longrightarrow \mathbf{E}'(FB,FA)$ establish a suitable map of objects. If $(a,b)\colon \xi_1 \longrightarrow \xi_2$ is a morphism of $\mathbf{E}$–extensions $\xi_1 \in \mathbf{E}(B_1,A_1)$ and $\xi_2 \in \mathbf{E}(B_2,A_2)$, the commutativity of the diagram

$$\begin{tikzcd}
\mathbf{E}(B_1,A_1) \arrow[r, "\eta_{B_1,A_1}"] \arrow[d, "\mathbf{E}(B_1,a)"'] & \mathbf{E}'(FB_1,FA_1) \arrow[d, "\mathbf{E}'(FB_1,a)"] \\
\mathbf{E}(B_1,A_2) \arrow[r, "\eta_{B_1,A_2}"] & \mathbf{E}'(FB_1,FA_2) \\
\mathbf{E}(B_2,A_2) \arrow[r, "\eta_{B_2,A_2}"] \arrow[u, "\mathbf{E}(b,A_2)"] & \mathbf{E}'(FB_2,FA_2) \arrow[u, "\mathbf{E}'(b,FA_2)"']
\end{tikzcd}$$

shows that (Fa,Fb) is a morphism $\eta_{B_1,A_1}\xi_1 \longrightarrow \eta_{B_2,A_2}\xi_2$ of $\mathbf{E}'$–extensions.

If F is an equivalence and η is a natural isomorphism, it is clear that F_η is essentially surjective. Full fidelity follows from the full fidelity of F. $\square$

Since we have a natural isomorphism $\pi_{-1}\,\mathscr{E}\mathrm{xt}_{\mathscr{C}} \longrightarrow \pi_0\,\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}$, between two bifunctors $(\mathrm{h}\,\mathscr{C})^{\mathrm{op}} \times \mathrm{h}\,\mathscr{C} \longrightarrow \mathrm{Ab}$, Lemma 7.2 provides an equivalence

$$\mathbf{E}'\text{-}\mathrm{Ext}_{\mathrm{h}\,\mathscr{C}} \longrightarrow \mathbf{E}\text{-}\mathrm{Ext}_{\mathrm{h}\,\mathscr{C}} \tag{7.1}$$

where $\mathbf{E} = \pi_0\,\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}$ and $\mathbf{E}' = \pi_{-1}\,\mathscr{E}\mathrm{xt}_{\mathscr{C}}$.

Unlike $\pi_0\,\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}$, an arbitrary additive bifunctor $\mathbf{E}$ need not be directly tied to exact sequences. In the general case, it necessary to manually link the elements of the abstract group $\mathbf{E}(B,A)$ to concrete diagrams of the form $A \longrightarrow E \longrightarrow B$, or rather equivalence classes of such.

Definition 7.3. Let $\mathcal{C}$ be an additive category. Two diagrams of the form $A \longrightarrow E \longrightarrow B$ and $A \longrightarrow E' \longrightarrow B$ are said to be *equivalent* if there exists an isomorphism $e\colon E \longrightarrow E'$ rendering the following diagram commutative.

$$\begin{tikzcd}
 & E \arrow[dr] \arrow[dd, "e"] & \\
A \arrow[ur] \arrow[dr] & & B \\
 & E' \arrow[ur] &
\end{tikzcd}$$

The equivalence class containing $A \longrightarrow E \longrightarrow B$ is denoted $[A \longrightarrow E \longrightarrow B]$.

It will be convenient to regard the equivalence classes of sequences as objects in a category $\mathbb{S}_{\mathcal{C}}$. A morphism in this category from $[A \longrightarrow E \longrightarrow B]$ to $[A' \longrightarrow E' \longrightarrow B']$ is pair of morphisms $(a\colon A \longrightarrow A', b\colon B \longrightarrow B')$ for which there exists a commutative diagram as follows:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"] & E \arrow[r] \arrow[d, "e"] & B \arrow[d, "b"] \\ A' \arrow[r] & E' \arrow[r] & B' \end{tikzcd}$$

Composition in this category is clearly well defined and associative.

The attentive reader might have had a *dejà vu* experience when reading the last paragraph. Indeed, the category of $\pi_0\,\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}$–extensions is remarkably similar to $\mathbb{S}_{\mathrm{h}\mathscr{C}}$. The difference is that $\mathbb{S}_{\mathrm{h}\mathscr{C}}$ contains more objects, giving rise to an embedding $\iota_{\mathscr{C}}\colon \mathbf{E}\text{-}\mathrm{Ext}^1_{\mathrm{h}\mathscr{C}} \hookrightarrow \mathbb{S}_{\mathrm{h}\mathscr{C}}$, where $\mathbf{E} = \pi_0\,\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}$. This is an example of the general notion of realization.

Definition 7.4. Let $\mathbf{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathrm{Ab}$ be an additive bifunctor. A *realization* is a functor

$$\mathfrak{s}\colon \mathbf{E}\text{-}\mathrm{Ext}^1_{\mathcal{C}} \longrightarrow \mathbb{S}_{\mathcal{C}}$$

such that an $\mathbf{E}$–extension $\xi \in \mathbf{E}(B, A)$ is sent to an equivalence class $[A \longrightarrow E \longrightarrow B]$ with appropriate endpoints. Equivalently, a *realization* is a correspondence $\mathfrak{s}$ that sends an $\mathbf{E}$–extension $\xi \in \mathbf{E}(B, A)$ to an equivalence class $[A \longrightarrow E \longrightarrow B]$, satisfying the following property: If (a, b) is a morphism of $\mathbf{E}$–extensions that are realized by $[A \longrightarrow E \longrightarrow B]$ and $[A' \longrightarrow E' \longrightarrow B']$, there exists a morphism $e\colon E \longrightarrow E'$ such that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"] & E \arrow[r] \arrow[d, "e"] & B \arrow[d, "b"] \\ A' \arrow[r] & E' \arrow[r] & B' \end{tikzcd}$$

is commutative.

A realization $\mathfrak{s}$ is *additive* if $\mathfrak{s}({}_B0_A) = [\sigma_1(B, A)]$, where ${}_B0_A$ is the zero element in $\mathbf{E}(B, A)$ and $\sigma_1(B, A)$ is the split exact sequence, and $\mathfrak{s}(\xi_1 \oplus \xi_2) = \mathfrak{s}(\xi_1) \oplus \mathfrak{s}(\xi_2)$.

A realization restricts to a map from $\mathbf{E}(B, A)$ to a subset of the set of equivalence classes of the form $[A \longrightarrow E \longrightarrow B]$. Since $\mathbf{E}(B, A)$ is an abelian group, this map gives a group structure to this subset. Moreover, an additive realization gives rise to a group structure that behaves similarly to Yoneda Ext-groups in exact categories.

The embedding $\iota_{\mathscr{C}} \colon \mathbf{E}\text{-}\mathrm{Ext}^1_{\mathrm{h}\mathscr{C}} \hookrightarrow \mathbb{S}_{\mathrm{h}\mathscr{C}}$, where $\mathbf{E} = \pi_0\, \mathscr{E}\mathrm{xt}^1_{\mathscr{C}}$, was our motivating example of a realization. It is easy to see that this is additive. Composing with the above equivalence (7.1) yields an additive realization

$$\mathbf{E}'\text{-}\mathrm{Ext}^1_{\mathrm{h}\mathscr{C}} \longrightarrow \mathbf{E}\text{-}\mathrm{Ext}^1_{\mathrm{h}\mathscr{C}} \hookrightarrow \mathbb{S}_{\mathrm{h}\mathscr{C}}$$

of $\mathbf{E}' = \pi_{-1}\, \mathscr{E}\mathrm{xt}_{\mathscr{C}}$.

We now have all ingredients to define what is meant by an extriangulated category and extriangulated functors.

Definition 7.5 ([NP19, Definition 2.12])**.** Let $\mathcal{C}$ be an additive category, and consider a bifunctor $\mathbf{E} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathrm{Ab}$ as well as an additive realization $\mathfrak{s}$ of $\mathbf{E}$. The triple $(\mathcal{C}, \mathbf{E}, \mathfrak{s})$ is an *extriangulated category* if the follows axioms hold.

(ET3) Let $\xi_1 \in \mathbf{E}(B_1, A_1)$ and $\xi_2 \in \mathbf{E}(B_2, A_2)$ be $\mathbf{E}$–extensions with realizations

$$\mathfrak{s}(\xi_1) = [A_1 \longrightarrow E_1 \longrightarrow B_1] \qquad \mathfrak{s}(\xi_2) = [A_2 \longrightarrow E_2 \longrightarrow B_2]$$

If the solid part of the following diagram commutes

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & B_1 \arrow[d, dashed] \\
A_2 \arrow[r] & E_2 \arrow[r] & B_2
\end{tikzcd}$$

the dashed morphism exists and the resulting diagram is commutative.

(ET3$^{\mathrm{op}}$) Dual of 7.5 [NP19, (ET3$^{\mathrm{op}}$) in Definition 2.12].

(ET4) Let $\xi_1 \in \mathbf{E}(B_1, A_1)$ and $\xi_2 \in \mathbf{E}(C, B)$ be $\mathbf{E}$–extensions with realizations

$$\mathfrak{s}(\xi_1) = [A \longrightarrow E \longrightarrow B], \qquad \mathfrak{s}(\xi_2) = [E \longrightarrow F \longrightarrow C].$$

There exists a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "p"] \arrow[d] & B \arrow[d, "j"] \\
A \arrow[r] & F \arrow[r] \arrow[d] & G \arrow[d, "q"] \\
& C \arrow[r, equal] & C
\end{tikzcd}$$

and an $\mathbf{E}$–extension $\xi_3 \in \mathbf{E}(G, A)$ which is realized by the middle row. Moreover, we require that

(ET4.1) $\mathfrak{s}(\mathbf{E}(C, p)(\xi_2)) = [B \longrightarrow G \longrightarrow C]$,

(ET4.2) $\mathbf{E}(j, A)(\xi_3) = \xi_1$,

(ET4.3) $\mathbf{E}(G, i)(\xi_3) = \mathbf{E}(q, E)(\xi_2)$.

(ET4$^{\mathrm{op}}$) Dual of 7.5 [NP19, Remark 2.22].

Definition 7.6 ([BTS21, Definition 2.32]). Let $(\mathcal{C}, \mathbf{E}, \mathfrak{s})$ and $(\mathcal{C}', \mathbf{E}', \mathfrak{s}')$ be extriangulated categories. An *extriangulated functor* consists of an additive functor $F\colon \mathcal{C} \longrightarrow \mathcal{C}'$ for which there exists a natural transformation $\eta\colon \mathbf{E}(-,-) \longrightarrow \mathbf{E}'(F^{\mathrm{op}}-, F-)$ such that diagram of functors

$$
\begin{tikzcd}
\mathbf{E}\text{-}\mathrm{Ext}_{\mathcal{C}} \arrow[r, "F_\eta"] \arrow[d, "\mathfrak{s}"] & \mathbf{E}'\text{-}\mathrm{Ext}_{\mathcal{C}'} \arrow[d, "\mathfrak{s}'"] \\
\mathbb{S}_{\mathcal{C}} \arrow[r, "\mathbb{S}_F"] & \mathbb{S}_{\mathcal{C}'}
\end{tikzcd}
$$

commutes, where F_η is as in Lemma 7.2 and $\mathbb{S}_F$ sends the class $[A \longrightarrow E \longrightarrow B]$ to the class $[FA \longrightarrow FE \longrightarrow FB]$. An extriangulated functor is an *equivalence* if F is an equivalence of categories.

As stated above, the homotopy category of an exact quasi-category admits a natural extriangulation. More precisely:

Theorem 7.7 ([NP20, Theorem 4.22 and Proposition 4.28]). *Let $\mathscr{C}$ be an exact quasi-category, and consider the additive bifunctor*

$$\mathbf{E} = \pi_0\, \mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(-,-)\colon (\mathrm{h}\,\mathscr{C})^{\mathrm{op}} \times \mathrm{h}\,\mathscr{C} \longrightarrow \mathrm{Ab},$$

as well as the additive realization

$$\mathbf{E} = \pi_0\, \mathscr{E}\mathrm{xt}^1_{\mathscr{C}} \xhookrightarrow{\iota_{\mathscr{C}}} \mathbb{S}_{\mathrm{h}\,\mathscr{C}}\,.$$

Then the triple $(\mathrm{h}\,\mathscr{C}, \mathbf{E}, \iota_{\mathscr{C}})$ is an extriagulated category. Furthermore, if $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ is an exact functor, then the induced functor $\mathrm{h}\,F\colon\ \mathrm{h}\,\mathscr{C} \longrightarrow \mathrm{h}\,\mathscr{D}$ between homotopy categories is an extriangulated functor.

An extriangulated category is *topological* if it is equivalent to the homotopy category of some exact quasi-category.

We noted above that we have a natural isomorphism of bifunctors

$$\varphi\colon \pi_{-1}\,\mathscr{E}\mathrm{xt}_{\mathscr{C}}(-,-) \Longrightarrow \pi_0\,\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(-,-).$$

A realization of $\pi_{-1}\,\mathscr{E}\mathrm{xt}_{\mathscr{C}}(-,-)$ is achieved by precomposing the realization $\iota_{\mathscr{C}}$ with the equivalence (7.1). It now follows from Lemma 7.2 that the identity functor $1_{\mathrm{h}\,\mathscr{C}}$ on $\mathrm{h}\,\mathscr{C}$ yields an extriangulated equivalence

$$(\mathrm{h}\,\mathscr{C}, \pi_{-1}\,\mathscr{E}\mathrm{xt}_{\mathscr{C}}(-,-), \iota_{\mathscr{C}} \circ (7.1)) \longrightarrow (\mathrm{h}\,\mathscr{C}, \pi_0\,\mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(-,-), \iota_{\mathscr{C}}).$$

In other words, we can present the extriangulation on the topological extriangulated category $\mathrm{h}\,\mathscr{C}$ using the Retakh spectrum.

Let $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ be an exact functor of exact quasi-categories. By Proposition 4.5 we have a natural transformation

$$\mathscr{E}\mathrm{xt}_{\mathscr{C}}(-,-) \Longrightarrow \mathscr{E}\mathrm{xt}_{\mathscr{D}}(F-,F-)$$

of functors $\mathscr{C}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \mathrm{Sp}$. We can thus induce a natural transformation

$$\eta\colon \pi_{-1}\,\mathscr{E}\mathrm{xt}_{\mathscr{C}}(-,-) \Longrightarrow \pi_0\,\mathscr{E}\mathrm{xt}^1_{\mathscr{D}}(F-,F-)$$

of functors $(\mathrm{h}\,\mathscr{C})^{\mathrm{op}} \times \mathrm{h}\,\mathscr{C} \longrightarrow \mathrm{Ab}$. Pushing this natural transformation down to the homotopy category ensures that the functor

$$\mathrm{h}\,F\colon\ \mathrm{h}\,\mathscr{C} \longrightarrow \mathrm{h}\,\mathscr{D}$$

is extriangulated. Indeed, by Lemma 7.2, we have a functor

$$\mathrm{h}\,F_\eta\colon \pi_{-1}\,\mathscr{E}\mathrm{xt}_{\mathscr{C}}\text{-}\mathrm{Ext} \longrightarrow \pi_{-1}\,\mathscr{E}\mathrm{xt}_{\mathscr{C}}\text{-}\mathrm{Ext},$$

since the vertical maps in the diagram

$$\begin{tikzcd}
\pi_{-1}\,\mathscr{E}\mathrm{xt}_{\mathscr{C}}\text{-}\mathrm{Ext} \arrow[r, "\mathrm{h}\,F_\eta"] \arrow[d] & \pi_{-1}\,\mathscr{E}\mathrm{xt}_{\mathscr{D}}\text{-}\mathrm{Ext} \arrow[d] \\
\mathbb{S}_{\mathrm{h}\,\mathscr{C}} \arrow[r, "\mathbb{S}_F"] & \mathbb{S}_{\mathrm{h}\,\mathscr{D}}
\end{tikzcd}$$

sends an equivalence class of extensions $[A \rightarrowtail E \twoheadrightarrow B]$ to itself, it is easy to see that the diagram commutes, and consequently that $\mathrm{h}\,F$ is an extriangulated functor.

Klemenc' embedding theorem (restated here as Theorem 2.6), as well as the discussion above, proves that any small topological extriangulated category embeds into a triangulated category.

As a new contribution, we generalise Theorem 7.7. The spectrum object $\mathscr{E}\mathrm{xt}_{\mathscr{C}}$ does not only determine a natural extriangulation on $\mathrm{h}\mathscr{C}$. It also gives the higher extension functors, recently defined by Gorsky–Nakaoka–Palu [GNP21, Definition 3.1]. We review their definition first.

Let F and G be bifunctors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathrm{Ab}$. For a fixed pair of objects (X, Y), consider the bifunctor $G(-, Y) \underset{\mathbb{Z}}{\otimes} F(X, -)$. The coend of this defines a product of G and F.

$$(G \diamond F)(X, Y) := \int^{C \in \mathcal{C}} G(C, Y) \underset{\mathbb{Z}}{\otimes} F(X, C).$$

Since the category of abelian groups is cocomplete, the coend of a bifunctor S on $\mathcal{C}$ into Ab can be obtained as the coequalizer of the diagram

$$\coprod_{f\colon c_1 \to c_2} S(c_2, c_1) \underset{S(f,c_1)}{\overset{S(c_2,f)}{\rightrightarrows}} \coprod_{c} S(c, c), \tag{7.2}$$

where the first coproduct is indexed over all morphisms in $\mathcal{C}$, and the second over all its objects.

Let $(\mathscr{C}, \mathbf{E}, \mathfrak{s})$ be an extriangulated category. We set the zeroth extension functor to be the Hom-functor $\mathcal{C}(-, -)$, and the first to be $\mathbf{E}$. The higher extension functors $\mathbf{E}^{\diamond n}$ are then inductively defined inductively by

$$\mathbf{E}^{\diamond n} := \mathbf{E}^{\diamond n-1} \diamond \mathbf{E}.$$

Theorem 7.8. *Let $\mathscr{C}$ be an exact quasi-category. The bifunctor $\pi_0\, \mathscr{E}\mathrm{xt}^n_{\mathscr{C}}(-, -)$ is naturally isomorphic to the nth extension functor $\pi_0\, \mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(-, -)^{\diamond n}$.*

Proof. We proceed by induction. The base cases will be $\ell = 0$ and $\ell = 1$, both of which holding by definition.

Suppose that we have natural isomorphisms

$$\varphi^{\ell}_{X,Y} \colon \pi_0\, \mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(X, Y)^{\diamond \ell} \longrightarrow \pi_0\, \mathscr{E}\mathrm{xt}^{\ell}_{\mathscr{C}}(X, Y),$$

for some $\ell \geq 1$. Applying the tensor functor gives another natural isomorphism

$$\pi_0\, \mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(X, Y)^{\diamond \ell} \underset{\mathbb{Z}}{\otimes} \pi_0\, \mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B, A) \xrightarrow{\varphi^{\ell}_{X,Y} \otimes 1} \pi_0\, \mathscr{E}\mathrm{xt}^{\ell}_{\mathscr{C}}(X, Y) \underset{\mathbb{Z}}{\otimes} \pi_0\, \mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B, A).$$

Consider also the homomorphism

$$c_{B,A} \colon \pi_0\, \mathscr{E}\mathrm{xt}^{\ell}_{\mathscr{C}}(C, A) \underset{\mathbb{Z}}{\otimes} \pi_0\, \mathscr{E}\mathrm{xt}^1_{\mathscr{C}}(B, C) \longrightarrow \pi_0\, \mathscr{E}\mathrm{xt}^{\ell+1}_{\mathscr{C}}(B, A)$$

that concatenates exact sequences. Our strategy to complete the inductive step will be to show that $c_{B,A} \circ (\varphi^{\ell}_{C,A} \otimes 1)$ is a coequalizer of the diagram (7.2), with the tensor product $\pi_0 \,\mathscr{E}\mathrm{xt}^{\ell}_{\mathscr{C}}(-, A) \underset{\mathbb{Z}}{\otimes} \pi_0 \,\mathscr{E}\mathrm{xt}^{1}_{\mathscr{C}}(B, -)$ in place of S. We simplify our notation by setting $S = \pi_0 \,\mathscr{E}\mathrm{xt}^{\ell}_{\mathscr{C}}(-, A) \underset{\mathbb{Z}}{\otimes} \pi_0 \,\mathscr{E}\mathrm{xt}^{1}_{\mathscr{C}}(B, -)$ for the remainder of this proof.

In order to have a coequalizer, it must be the case that

$$c_{B,A} \circ (\varphi^{\ell}_{C,A} \otimes 1) \circ S(c_2, f) = c_{B,A} \circ (\varphi^{\ell}_{C,A} \otimes 1) \circ S(f, c_1). \tag{7.3}$$

In other words, it should be shown that

$$c_{B,A} \circ (\varphi^{\ell}_{C,A} \otimes 1)$$

is a cofork of the diagram (7.2). The situation is illustrated by the solid part of the diagram

$$\begin{tikzcd}
A \arrow[r, dashed] \arrow[d, dashed, "1"] & \cdots \arrow[r, dashed] & E_{(\ell-1).5} \arrow[r, dashed] \arrow[d, dashed, "1"] & C' \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & C_2 \arrow[r] \arrow[d, "f"'] & F \arrow[r] \arrow[d, dashed] & B \arrow[d, dashed, "1"] \\
A \arrow[r] & \cdots \arrow[r] & E_{(\ell-1).5} \arrow[r] & E_\ell \arrow[r] & C_1 \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner", very near end] & F' \arrow[r, dashed] & B
\end{tikzcd} \tag{7.4}$$

The diagram is completed by taking a pullback and pushout of f along the morphisms $E_\ell \longrightarrow C_1$ and $C_2 \longrightarrow F$, respectively, and then adding identity morphisms so that we end up with a morphism of exact sequences. The top row of the entire diagram then displays a representative of an equivalence class in the left hand side of (7.3), and the bottom row displays one the right hand side. Since the diagram (7.4) is a morphism between them in $\mathscr{E}\mathrm{xt}^{\ell+1}_{\mathscr{C}}(B, A)$, these representatives belong to the same equivalence class, as desired.

Lastly, we give reasons as to why $c_{B,A} \circ (\varphi^{\ell}_{C,A} \otimes 1)$ is a universal cofork of (7.2). Given an abelian group M and a group homomorphism

$$\coprod_{c} S(c, c) \xrightarrow{\ m\ } M$$

satisfying $m \circ S(c_2, f) = m \circ S(f, c_1)$, our task is to find a homomorphism

$$\overline{m} \colon \pi_0 \,\mathscr{E}\mathrm{xt}^{\ell+1}_{\mathscr{C}}(B, A) \longrightarrow M$$

such that $m = \overline{m} \circ c_{B,A} \circ (\varphi^{\ell}_{C,A} \otimes 1)$, and then show that $\overline{m}$ is unique. Let $\overline{m}(\mathbb{E}) := m(\widetilde{\mathbb{E}})$, where $\widetilde{\mathbb{E}}$ is a choice of preimage of $\mathbb{E}$ in $\coprod_{c} S(c, c)$. This is a well-defined homomorphism; a different choice of preimage would yields the same result. The uniqueness of $\overline{m}$ is a direct consequence of the fact that $c_{B,A} \circ (\varphi^{\ell}_{C,A} \otimes 1)$ is an epimorphism. $\square$

A Proofs of diagram lemmas

We ended Section 2 with a series of diagram lemmas for exact categories, claiming that they generalize to exact quasi-categories. These results are found in Bühler's monograph [Büh10], as well as countless other texts. The proofs presented here are in a quasi-categorical context. Unlike the special case of ordinary exact categories where cofibrations (resp. fibrations) are monomorphisms (resp. epimorphisms), which we may not assume in a general quasi-categorical context.

Lemma 2.11. *Consider a map of exact sequences*

$$\begin{tikzcd} A \arrow[r, tail, "i_A"] \arrow[d, "f'"] & E \arrow[r, two heads, "p_B"] \arrow[d, "f"] & B \arrow[d, "f''"] \\ C \arrow[r, tail, "i_C"] & F \arrow[r, two heads, "p_D"] & D \end{tikzcd}$$

If f' and f'' are homotopy equivalences (resp. cofibrations, resp fibrations), so is f.

Proof. If f' and f'' are homotopy equivalences, we use Lemma 2.10 to construct a diagram

$$\begin{tikzcd} A \arrow[r, tail, "i_A"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & E \arrow[r, two heads, "p_B"] \arrow[d, "g"] & B \arrow[d, equal] \\ C \arrow[r, tail, "j"] \arrow[d, equal] & Z \arrow[r, two heads, "q"] \arrow[d, "h"] \arrow[dr, phantom, "\square"] & D \arrow[d, "f''"] \\ C \arrow[r, tail, "i_C"] & F \arrow[r, two heads, "p_D"] & D \end{tikzcd}$$

with exact rows. Since pushouts and pullbacks of homotopy equivalences are homotopy equivalences, the maps g and h are homotopy equivalences, whence $f = hg$ is.

If f' and f'' are cofibrations, then g is a cofibration since it is a pushout of f', and h is a cofibration as a direct consequence of (Ex3). Thus, the composite $f = hg$ is a cofibration.

If f' and f'' are fibrations, it is shown dually that f is a fibration. □

Lemma 2.12. *Consider the commutative diagram with exact columns*

$$
\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, tail, "i_A"] & E' \arrow[r, "g'"] \arrow[d, tail, "i_E"] & B' \arrow[d, tail, "i_B"] \\
A \arrow[r, "f"] \arrow[d, "p_A"] & E \arrow[r, "g"] \arrow[d, "p_E"] & B \arrow[d, "p_B"] \\
A'' \arrow[r, "f''"] & E'' \arrow[r, "g''"] & B''
\end{tikzcd}
$$

If the middle row and one of the other rows is exact, then the remaining row is exact.

Proof. We assume that the top and middle rows are exact, and show that the bottom row is. The other case is dual.

The map of exact sequences connecting the top and middle rows can be factored

$$
\begin{tikzcd}
A' \arrow[r, tail, "f'"] \arrow[d, tail, "i_A"] \arrow[dr, phantom, "\square"] & E' \arrow[r, two heads, "g'"] \arrow[d, tail, "\overline{i_A}"] & B' \arrow[d, equal] \\
A \arrow[r, tail, "\overline{f}"] \arrow[d, equal] & Z \arrow[r, two heads, "\overline{g}"] \arrow[d, tail, "\overline{i_B}"] \arrow[dr, phantom, "\square"] & B' \arrow[d, tail, "i_B"] \\
A \arrow[r, tail, "f"] & E \arrow[r, two heads, "g"] & B
\end{tikzcd}
$$

as asserted by Lemma 2.10. Our aim is to prove that f'' is a cofibration and that the square

$$
\begin{tikzcd}
A'' \arrow[r, "f''"] \arrow[d] & E'' \arrow[d, "g''"] \\
0 \arrow[r, tail] & B''
\end{tikzcd}
$$

is bicartesian. It suffices to show that the top square and outer rectangle in the diagram

$$
\begin{tikzcd}
Z \arrow[r, tail, "\overline{i_B}"] \arrow[d, "q"] & E \arrow[d, "p_E"] \\
A'' \arrow[r, "f''"] \arrow[d] & E'' \arrow[d, "g''"] \\
0 \arrow[r, tail] & B''
\end{tikzcd}
$$

are bicartesian, where q is a cofiber of $\overline{i_A}$ (by Lemma 2.9, the target of q can indeed be chosen as A''). Here, the outer rectangle is obtained by pasting two known bicartesian squares

$$\begin{tikzcd}
Z \arrow[r, tail, "\overline{i_B}"] \arrow[d, "\overline{g}"] \arrow[dr, phantom, "\square"] & E \arrow[d, "g"] \\
B' \arrow[r, "i_B"] \arrow[d] \arrow[dr, phantom, "\square"] & B \arrow[d, "p_B"] \\
0 \arrow[r, tail] & B''
\end{tikzcd}$$

and the upper square appears on the bottom right when applying Lemma 2.10 to the map

$$\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, tail, "i_A"] & E' \arrow[d, tail, "i_E"] \\
A \arrow[r, "f"] \arrow[d, "p_A"] & E \arrow[d, "p_E"] \\
A'' \arrow[r, "f''"] & E''
\end{tikzcd}$$

of exact sequences. □

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Paper III

Section complexes of height functions

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Manuscript

Section complexes of height functions

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Abstract. Piecewise linear functions appear everywhere in mathematics and nature. For instance, these maps are intrinsic to surface triangulations and data sets often come with a real-valued parametrization such as time or density. We model piecewise linear functions as certain simplicial maps, which we refer to as height functions. This allows us to introduce the associated section complex. To do so, higher section spaces are defined, which encode flow-lines along the height in a combinatorial way. Our construction comes with a spectral sequence which computes homology of the height function's domain. We extract Reeb complexes from the spectral sequence, which provide a first order approximation of how homology generators flow along height levels. In this way, Reeb complexes can be thought of as higher variants of Reeb graphs. Moreover, section complexes give rise to zigzag modules. These are different, but related, to the ones obtained by level-set zigzag persistence. In particular, barcodes of filtrations can be calculated using section complexes.

1 Introduction

A data set often comes with a natural real-valued function, such as time or density. Existing methods studying this kind of structure, for example level-set zigzag and Mapper [CdSM09, SMC$^+$07], often build on the idea of looking at preimages of an open cover in $\mathbb{R}$. We instead consider combinatorial flow-lines in order to directly capture how homology generators flow across height levels. This is achieved by developing a discrete variant of the section spaces from [Try21].

Consider a real-valued piecewise linear function f on a CW complex T (e.g. a geometric simplicial complex). This information is combinatorial in nature: all CW-complexes can be realized from simplicial sets. Hence we rather work with a simplicial map $h\colon X \to \mathrm{R}$ on a simplicial set X directly. Here R is our preferred simplicial model for the real line (see Definition 2.3). We refer to h as a *height function* (Definition 2.6).

The fibers of h assemble into a simplicial set $\coprod_{a\in\mathbb{R}} h^{-1}a$ that we call the *space of* 0–*sections*. To capture how these fibers are connected across p levels, we introduce the p–sections. A 1–section from height a to height b is a 1–simplex in X starting in $h^{-1}a$

and ending in $h^{-1}b$, a 2–section is a 2–simplex labeled by three height values and so on. There is a simplicial set, $(\mathcal{S}_h)_p$, containing the p–sections as its vertices, which we refer to as the *space of p–sections*. It turns out that the section spaces are also simplicial in p. Thus, we may define a bisimplicial set, $\mathcal{S}_h$, associated with the height function h, called the *section complex* of h.

The section complex essentially splits up the homology information of X into two directions: the horizontal direction along h and the vertical one transversal to it. This is encapsulated in the following main result:

Theorem 1.1. Let $h\colon X \to \mathrm{R}$ be a height function. The diagonal of the bisimplicial set $\mathcal{S}_h$ is homotopy equivalent to X:

$$\operatorname{diag}\mathcal{S}_h \simeq X.$$

The computational implications of this result comes from the existence of a spectral sequence which computes the homology of the diagonal, $\operatorname{diag}\mathcal{S}_h$, from homological features of the section spaces $(\mathcal{S}_h)_p$ (see e.g. [Seg68, GJ09]). We refer to the spectral sequence associated to $\mathcal{S}_h$ as the *section spectral sequence*. A similar result was proved for smooth Morse functions in unpublished work of Cohen, Jones and Segal [CJS92]. In their case, the associated spectral sequence reduces to the widely used Morse homology. Nanda, Tamaki and Tanaka also have an analogous result for discrete Morse functions [NTT18].

From the first page of the section spectral sequence we extract for $q = 0,1,2,\dots$ chain complexes $\mathcal{G}_q$. We refer to $\mathcal{G}_q$ as the qth *Reeb complex* of h. It is a chain complex which reveals how homology generators in H_q flow across height levels. Moreover, for a finite simplicial set X, the Reeb complexes can be computed in finite time.

Proposition 1.2. Let $h\colon X \to \mathrm{R}$ be a height function and $\mathcal{G}_q$ its qth Reeb complex. The associated section spectral sequence has $\mathrm{H}_p\mathcal{G}_q$ appearing as the (p,q)th entry on the second page $\mathrm{E}^2_{p,q} \simeq \mathrm{H}_p\mathcal{G}_q$ and converges to the homology of X:

$$\mathrm{H}_p\mathcal{G}_q \Rightarrow \mathrm{H}_{p+q}X.$$

This means that the spectral sequence uses the Reeb complexes as a first order approximation of the homology of X, and subsequently uses $\mathrm{H}_p\mathcal{G}_q$ in an iterative process to recover H_*X. The complexity of the section spectral sequence depends on how *subdivided* X is relative to h. We say that X is *subdivided according to h* if every 1–section traverses only successive height levels. In this case, the section spectral sequence collapses at the second page. This means that the Reeb complexes $\mathcal{G}_q$ recover the homology of X directly:

$$\mathrm{H}_nX = \bigoplus_{p+q=n} \mathrm{H}_p\mathcal{G}_q.$$

As the name suggests, the Reeb complexes are closely related to Reeb graphs [Ree46]. Indeed, if X is subdivided according to h, then the Reeb graph is obtained by applying π_0 level-wise to the section spaces $(\mathcal{S}_h)_p$ in $\mathcal{S}_h$ (Proposition 3.6). In particular, the zeroth Reeb complex $\mathcal{G}_0$ calculates the homology of the Reeb graph. Contrast this to how Mapper produces a graph from applying π_0 to preimages of intervals under h [SMC+07].

Given a filtration $\mathbb{X} = X_0 \overset{i_1}{\hookrightarrow} X_1 \overset{i_2}{\hookrightarrow} \ldots \overset{i_n}{\hookrightarrow} X_n$ we can construct its iterated mapping cylinder $C_{\mathbb{X}}$. The indices of $\mathbb{X}$ provide a natural height function $h_{\mathbb{X}}$ on $C_{\mathbb{X}}$. In this case the associated section spectral sequence amounts to a series of zigzag modules $\mathbb{G}_q$, with which we recover the barcode of $\mathbb{X}$:

Theorem 1.3. The barcodes of $\mathrm{H}_q\mathbb{X}$ and $\mathbb{G}_q$ are the same.

More generally, for any X subdivided according to a height function h, the Reeb complex $\mathcal{G}_q$ is equivalent to a zigzag module $\mathbb{G}_q$:

$$\begin{tikzcd}[column sep=small]
\mathrm{H}_q h^{-1}a_0 & & \mathrm{H}_q h^{-1}a_1 & \cdots & \mathrm{H}_q h^{-1}a_{n-1} & & \mathrm{H}_q h^{-1}a_n \\
& \mathrm{H}_q\mathcal{S}_h[a_0,a_1] \arrow[ul, "\mathrm{H}_q d_1^h"] \arrow[ur, "\mathrm{H}_q d_0^h"'] & & & & \mathrm{H}_q\mathcal{S}_h[a_{n-1},a_n] \arrow[ul, "\mathrm{H}_q d_1^h"] \arrow[ur, "\mathrm{H}_q d_0^h"'] &
\end{tikzcd}$$

where $\mathcal{S}_h[a_i,a_{i+1}]$ is the simplicial set of 1–sections from a_i to a_{i+1}. Our final result relates these to level-set zigzag modules [CdSM09]

$$\begin{tikzcd}[column sep=small]
& \mathrm{H}_q h^{-1}[a_0,a_1] & & & & \mathrm{H}_q h^{-1}[a_{n-1},a_n] & \\
\mathrm{H}_q h^{-1}a_0 \arrow[ur] & & \mathrm{H}_q h^{-1}a_1 \arrow[ul] & \cdots & \mathrm{H}_q h^{-1}a_{n-1} \arrow[ur] & & \mathrm{H}_q h^{-1}a_n \arrow[ul]
\end{tikzcd}$$

via the diamond principle [CdS10].

Theorem 1.4 (Diamond Principle)**.** Consider a height function $h\colon X \to \mathrm{R}$ for which

i) X is subdivided according to h and

ii) the image of 0–simplices, $h(X_0)$, is discrete as a subset of the real numbers.

Then for every pair of successive critical values $a < b$ the sequence

$$\mathrm{H}_q\mathcal{S}_h[a,b] \to \mathrm{H}_q h^{-1}a \oplus \mathrm{H}_q h^{-1}b \to \mathrm{H}_q h^{-1}[a,b]$$

is exact at the middle term.

Outline. In section 2, we give a brief introduction of simplicial sets, followed by an in-depth description of the *section complex*. Thereafter, Theorem 1.1 is proved in Section 2.4. Section 3 starts by introducing the Reeb complexes, followed by the more general section spectral sequence in Section 3.3. Numerous examples are computed to illustrate the theory. We end the discussion in Section 3.4 with a brief comparison of our discrete theory and the topological theory in [Try21]. Finally, Section 4 starts by explaining how zigzag modules are produced from Reeb complexes. Sections 4.2 and 4.3 are dedicated to clarifying and proving Theorems 1.3 and 1.4, respectively.

Computer code. A Python implementation for computing section complexes, as well as Reeb complexes, is found at `https://github.com/paultrygs/Section-Complex/`.

Notation. Categories of familiar objects are put inside parenthesis, e.g. (Simplicial Sets). The hom-set of maps $X \to Y$ is denoted $\mathrm{Map}(X,Y)$, while $\mathrm{map}(X,Y)$ refers to the simplicial set of maps. We will in general consider chain complexes over coefficients in some field k.

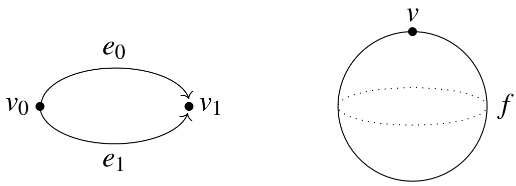
2 The section complex

We introduce higher section spaces $(\mathcal{S}_h)_p$ and explain how they assemble into the bisimplicial set $\mathcal{S}_h$. In the end, we prove our main Theorem 1.1.

2.1 Background on simplicial sets

A *simplicial set* X is a sequence X_n of sets, ranging over $n = 0,1,2,\dots$, together with face maps $d_i \colon X_n \to X_{n-1}$ and degeneracy maps $s_j \colon X_n \to X_{n+1}$ satisfying certain relations [GJ09, p.4]. An element x in X_n is interpreted as an n–simplex whose ith face is $d_i x$, whereas $s_j x$ incorporates ways to consider x as an $(n+1)$–simplex. In contrast to simplicial complexes, this for example implies that an $(n+1)$–simplex y can have an $(n-1)$–simplex x as its face; $d_i y = s_j x$. Moreover, two distinct n–simplices x and y can have equal faces $d_i x = d_i y$ for all i. Equivalently, the data of a simplicial set X can be organized into a functor $X \colon \Delta^{\mathrm{op}} \to$ (Sets), where Δ is the *simplex category*.

Example 2.1. We construct a circle from 0–simplices v_0 and v_1 and 1–simplices e_0 and e_1, not counting degeneracies, by declaring $d_0 e_i = v_1$ and $d_1 e_i = v_0$, for $i = 0,1$. A sphere can be obtained from a single 0–simplex v and 2–simplex f. In this case, all faces of f must be equal to $s_0 v$, a degenerate 0–simplex. This means that the boundary of f is equal to the point v.

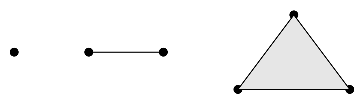


A *simplicial map* $f\colon X \to Y$ is a series of maps $f_n\colon X_n \to Y_n$ which commute with face and degeneracy maps. Pictures as above are produced by labeling each n–simplex in X with the topological n–simplex Δ_t^n and identifying appropriate simplices via the *geometric realization* $|X| = (\coprod_n X_n \times \Delta_t^n)/\sim$. The quotient glues simplices along faces and collapses degenerate simplices. Note that the realization is a functor from (Simplicial Sets) to (topological spaces).

Any small category $\mathcal{C}$ defines a simplicial set $\mathrm{N}\mathcal{C}$, called the *nerve* of $\mathcal{C}$. The set of 0–simplices, $\mathrm{N}\mathcal{C}_0$, consists of objects in $\mathcal{C}$, and $\mathrm{N}\mathcal{C}_n$ consists of tuples $(m_1,\ldots,m_n)$, of composable morphisms within $\mathcal{C}$. The ith face of $(m_1,\ldots,m_n)$ is determined by composition $(m_1,\ldots,m_{i+1}\circ m_i,\ldots,m_n)$ for $i \neq 0,n$, whereas d_0 and d_n omit m_0 and m_n, respectively. We depict a 2–simplex (f,g):

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "g\circ f"] & & C. \end{tikzcd}$$

Example 2.2. Let $[n]$ be the category generated by the directed graph $0 \to 1 \to \cdots \to n$. Applying the nerve yields the *standard simplicial* n–simplex $\Delta^n = \mathrm{N}[n]$. It consists of a unique n–simplex coming from the tuple $(0 \to 1, 1 \to 2, \ldots, n-1 \to n)$ with $n+1$ distinct faces. We recover the topological n–simplex Δ_t^n as $|\Delta^n|$.



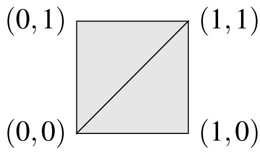
There are *simplicial inclusions* $\delta_i\colon \Delta^n \to \Delta^{n+1}$ which identify Δ^n with the ith face of Δ^{n+1}. Observe that $\delta^i(q)$ equals q if $q < i$ and $q+1$ otherwise. Conversely, there are *simplicial collapses* $\sigma^j\colon \Delta^{n+1} \to \Delta^n$ for which $\sigma^i(q)$ equals q if $n \leq i$ and $q-1$ otherwise.

Definition 2.3. Let $(\mathbb{R},\leq)$ be the real line equipped with its usual ordering. We define the *simplicial real line* $\mathrm{R} = \mathrm{N}(\mathbb{R},\leq)$.

An n–simplex in R is uniquely determined by a non-decreasing sequence $\bar{a} = (a_0,\ldots,a_n)$ of real numbers.

Two simplicial sets X and Y define a product $X \times Y$ with n–simplices $X_n \times Y_n$, whose face and degeneracy maps are computed component-wise.

Example 2.4. Consider the product of two copies of the standard 1–simplex: $\Delta^1 \times \Delta^1$. Decomposing $0 \to 1$ in components $(0 \to 1, 0 \to 1) = (0 \to 1, 1 \to 1) \circ (0 \to 0, 0 \to 1)$ yields the top 2–simplex in its realization:

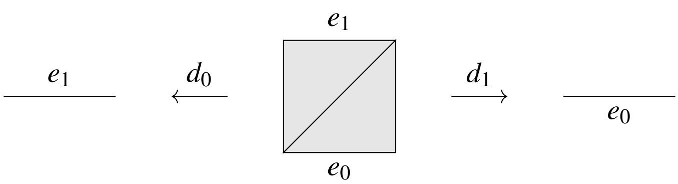


The bottom one is obtained as $(0 \to 1, 0 \to 1) = (1 \to 1, 0 \to 1) \circ (0 \to 1, 0 \to 0)$.

A *simplicial homotopy* is a simplicial map $H\colon X \times I \to Y$ such that I realizes to the standard unit interval. Note that a simplicial homotopy realizes to an ordinary homotopy in topological spaces [Seg68].

The *simplicial mapping space* $\mathrm{map}(X,Y)$ have the simplicial maps $f\colon X \to Y$ as its 0–simplices . An n–simplex in $\mathrm{map}(X,Y)_n$ is a simplicial map $X \times \Delta^n \to Y$. Since Δ^1 is a model of the interval, the 1–simplices are homotopies. Face and degeneracy maps, d_i and s_j, are obtained by pre-composing with component-wise maps $\mathrm{id}_X \times \delta^i$ and $\mathrm{id}_X \times \sigma^j$, respectively (Example 2.2). Applying d_i to $f\colon X \times \Delta^n \to Y$ thus restricts Δ^n to its ith face, whereas s_j adds appropriate identities.

Example 2.5. A 0–simplex in $\mathrm{map}(\Delta^1, X)$ is a simplicial map $e\colon \Delta^1 \to X$, uniquely determined by a 1–simplex e in X. Homotopies $H\colon \Delta^1 \times \Delta^1 \to X$, or 1–simplices, are determined by squares in X connecting two 1–simplices e_0 and e_1.



2.2 Sections of height functions

Let $f\colon T \to \mathbb{R}$ be a continuous function on a topological space. In [Try21], a *section* of f is defined as a continuous function $\rho\colon [a,b] \to T$, such that the composition $f \circ \rho$ is the inclusion $[a,b] \hookrightarrow \mathbb{R}$. Denote by $\mathrm{map}([a,b],T)$ the topological space of maps $[a,b] \to T$, with the compact-open topology, and define $\mathrm{Sect}_f[a,b]$ as the subspace whose points are the sections of f. These are then arranged in the space of all sections $\mathrm{Sect}_f = \coprod \mathrm{Sect}_f[a,b]$, ranging over all real numbers $a \leq b$. Note that two sections, one in $\mathrm{Sect}_f[a,b]$ and the other in $\mathrm{Sect}_f[b,c]$, with compatible ending and starting points can be concatenated to a section in $\mathrm{Sect}_f[a,c]$. This makes it possible to define the *section category* of f, a category internal to topological spaces with Sect_f as its space of morphisms. In [Try21] it is then shown that under fairly mild assumptions, the classifying space of this category is homotopy equivalent to T. These assumptions are for example met by piecewise linear functions.

We will now describe how to obtain such piecewise linear functions from height functions on simplicial sets. To that end, recall the definition of the simplicial real line R (Definition 2.3).

Definition 2.6. Let X be a simplicial set. A height function h on X is defined as a simplicial map $h\colon X \to \mathrm{R}$.

We can equivalently characterize a height function, h, as a map between sets $h\colon X_0 \to \mathbb{R}$. Indeed, h associates to each 0–simplex v in X_0 a height $h(v)$ in $\mathbb{R}$. Conversely, assigning to every v in X_0 a height $h(v)$ such that the orientation of the 1–simplices in X is respected, defines a unique height function $h\colon X \to \mathrm{R}$. We call the image $h(X_0)$ the *height levels* of h.

We observe from the previous section that a point in the realization $|\mathrm{R}|$ is a class $[\bar{a},\bar{t}]$ where $\bar{a} = (a_0,\dots,a_n)$ is a non-decreasing sequence of real numbers and $\bar{t} = (t_0,\dots,t_n)$ a point in the topological n–simplex. The dot product $\bar{a}\bar{t}$ defines a continuous function $c\colon |\mathrm{R}| \to \mathbb{R}$ from the realization of R to the real line. Any height function $h\colon X \to \mathrm{R}$ thus associates to a piecewise linear function $f\colon |X| \to \mathbb{R}$ by composing $|h|$ and c. It is shown in [Try21] that no homotopical information is lost if we only consider those sections that start and end at the height levels of h.

We now ask the following question: is it possible to construct a *simplicial version* of the section category directly from the simplicial height function $h\colon X \to \mathrm{R}$ rather than from the associated piecewise linear function f? This would render all the information involved combinatorial, thus accessible for the methods in computational topology.

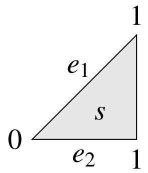
There is a natural choice for replacing the topological space of sections between two heights a_0 and a_1 with a simplicial set, $\mathcal{S}_h[a_0,a_1]$. Namely, the subspace of the mapping space, $\mathrm{map}(\Delta^1,X)$, carved out by the pullback:

$$\begin{tikzcd}
\mathcal{S}_h[a_0,a_1] \arrow[r] \arrow[d] & \mathrm{map}(\Delta^1,X) \arrow[d, "\mathrm{map}(\Delta^1,h)"] \\
\Delta^0 \arrow[r, "(a_0,a_1)"] & \mathrm{map}(\Delta^1,\mathrm{R}),
\end{tikzcd}$$

where we interpret the 1–simplex (a_0, a_1) in R as the simplicial map $\Delta^1 \to$ R, with $0 \mapsto a_0$ and $1 \mapsto a_1$. We may then define $(\mathcal{S}_h)_1$, *the space of* 1*–sections* of h as the disjoint union

$$(\mathcal{S}_h)_1 = \coprod_{a_0 \leq a_1} \mathcal{S}_h[a_0, a_1].$$

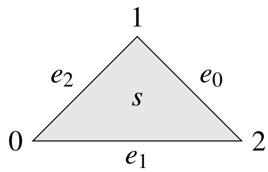
Example 2.7. We take as our simplicial set the standard 2–simplex Δ^2 and define a height function by the labels of the figure:



The two horizontal 1–simplices e_1 and e_2 in Δ^2 are 0–simplices in the section space $\mathcal{S}_h[0,1]$. The 2–simplex s corresponds to a 1–simplex connecting e_1 and e_2; $d_0 s = e_1$ and $d_1 s = e_2$.

The following example illustrates why we cannot proceed as in the construction of the topological section category.

Example 2.8. We define a height function on the standard 2–simplex Δ^2 as follows:



There is a unique 1–simplex between each distinct pair of heights. This means that the 1–section spaces are: $\mathcal{S}_h[0,1] = \{e_2\}$, $\mathcal{S}_h[1,2] = \{e_0\}$ and $\mathcal{S}_h[0,2] = \{e_1\}$. Hence the space of 1–sections cannot be utilized to recover the 2–simplex s in Δ^2 connecting e_0, e_1 and e_2. Contrast this with the corresponding topological situation. In that case, the sections $|e_2|$ and $|e_0|$ may be concatenated into $|e_2| * |e_0|$, a section from height 0 to height 2 running through $0 \to 1 \to 2$. We would also find a continuous path from $|e_2| * |e_0|$ to $|e_1|$ moving along the bottom of the triangle, given by continuous deformation through the realization of the 2–simplex. In particular, this process encodes the topological information provided by $|s|$.

To recover higher simplices across more than two height levels, we introduce *higher sections*.

2.3 Higher sections and the section complex

Definition 2.9. Given a height function $h\colon X \to \mathrm{R}$ on a simplicial set X, we may construct for every p–simplex $\bar{a}\colon \Delta^p \to \mathrm{R}$, a simplicial set $\mathcal{S}_h[\bar{a}]$ as the pullback

$$\begin{tikzcd} \mathcal{S}_h[\bar{a}] \arrow[r] \arrow[d] & \mathrm{map}(\Delta^p, X) \arrow[d, "\mathrm{map}(\Delta^p, h)"] \\ \Delta^0 \arrow[r, "\bar{a}"'] & \mathrm{map}(\Delta^p, R). \end{tikzcd}$$

Take the disjoint union over all p–simplices in R to obtain

$$(\mathcal{S}_h)_p := \coprod_{\bar{a} \in \mathrm{R}_p} \mathcal{S}_h[\bar{a}],$$

which we refer to as the *space of p–sections of h.*

By definition, the space of p–sections is a simplicial set. This is guaranteed by the fact that pullbacks and coproducts always exist in the category of simplicial sets. The 0–simplices are the *p–sections of h*, i.e. the p–simplices in X spanning p height levels. The q–simplices are then the (p,q)–sections in $(\mathcal{S}_h)_p$ corresponding to a simplicial map $\rho\colon \Delta^p \times \Delta^q \to X$ such that there is some p–simplex $\bar{a} = (a_0, \ldots, a_p)$ in R_p for which

$$\begin{tikzcd} \Delta^p \times \Delta^q \arrow[r, "\mathrm{pr}_0"] \arrow[d, "\rho"] & \Delta^p \arrow[r, "\bar{a}"] & \mathrm{R} \\ X \arrow[urr, "h"'] & & \end{tikzcd}$$

commutes. For instance, the 2-simplex, s, of Example 2.7 is a $(1,1)$–section in $(S_h)_1$. In Example 2.8, on the other hand, s is a $(2,0)$–section in $(S_h)_2$.

In Section 2.1, we explained how the face and degeneracy maps in $\mathrm{map}(\Delta^p, X)$ are obtained by pre-composing with $\mathrm{id}_{\Delta^p} \times \delta^i$ and $\mathrm{id}_{\Delta^p} \times \sigma^j$:

$$\begin{tikzcd} \Delta^p \times \Delta^{q-1} \arrow[r, "\mathrm{pr}_0"] \arrow[d, "\mathrm{id}_{\Delta^p} \times \delta^i"] \arrow[dd, bend right=60, "d_i^v(\rho)"'] & \Delta^p \arrow[r, "\bar{a}"] & \mathrm{R} \\ \Delta^p \times \Delta^q \arrow[d, "\rho"] & & \\ X \arrow[uurr, "h"'] & & \end{tikzcd} \qquad \begin{tikzcd} \Delta^p \times \Delta^{q+1} \arrow[r, "\mathrm{pr}_0"] \arrow[d, "\mathrm{id}_{\Delta^p} \times \sigma^j"] \arrow[dd, bend right=60, "s_j^v(\rho)"'] & \Delta^p \arrow[r, "\bar{a}"] & \mathrm{R} \\ \Delta^p \times \Delta^q \arrow[d, "\rho"] & & \\ X \arrow[uurr, "h"'] & & \end{tikzcd}$$

These diagrams characterize the face and degeneracy maps

$$d_i^v\colon (\mathcal{S}_h)_{p,q} \to (\mathcal{S}_h)_{p,q-1} \quad \text{and} \quad s_j^v\colon (\mathcal{S}_h)_{p,q} \to (\mathcal{S}_h)_{p,q+1}$$

in the space of p–sections. As the superscript v indicates, we call these the *vertical face and degeneracy maps*.

The above use of 'vertical' hints to the fact that there is a second, *horizontal*, simplicial structure. Indeed, we can pre-compose a (p,q)–section with a simplicial inclusion or collapse applied to the first component in $\Delta^p \times \Delta^q$ to obtain commutative diagrams

$$\begin{tikzcd}
\Delta^{p-1} \times \Delta^q \arrow[r, "\mathrm{pr}_0"] \arrow[d, "\delta^i \times \mathrm{id}_{\Delta^q}"] \arrow[dd, bend right=60, "d_i^h(\rho)"'] & \Delta^{p-1} \arrow[r, "\delta^i"] & \Delta^p \arrow[r, "\bar{a}"] & \mathrm{R} \\
\Delta^p \times \Delta^q \arrow[d, "\rho"] & & & \\
X \arrow[urrr, "h"] & & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Delta^{p+1} \times \Delta^q \arrow[r, "\mathrm{pr}_0"] \arrow[d, "\sigma^j \times \mathrm{id}_{\Delta^q}"] \arrow[dd, bend right=60, "s_j^h(\rho)"'] & \Delta^{p+1} \arrow[r, "\sigma^j"] & \Delta^p \arrow[r, "\bar{a}"] & \mathrm{R} \\
\Delta^p \times \Delta^q \arrow[d, "\rho"] & & & \\
X \arrow[urrr, "h"] & & &
\end{tikzcd}$$

These characterize maps of sets

$$d_i^h \colon (\mathcal{S}_h)_{p,q} \to (\mathcal{S}_h)_{p-1,q} \quad \text{and} \quad s_j^h \colon (\mathcal{S}_h)_{p,q} \to (\mathcal{S}_h)_{p+1,q}$$

which we refer to as the *horizontal face and degeneracy maps*. Alternatively, the horizontal face maps can be induced from the universal property of the pullback via:

$$\begin{tikzcd}
 & & \mathrm{map}(\Delta^p, X) \arrow[dr, "\mathrm{map}(\delta^i, X)"] \arrow[dd] & \\
 & & & \mathrm{map}(\Delta^{p-1}, X) \arrow[dd] \\
\Delta^0 \arrow[rr, "\bar{a}"] \arrow[dr, equal] & & \mathrm{map}(\Delta^p, \mathrm{R}) \arrow[dr, "\mathrm{map}(\delta^i, \mathrm{R})"] & \\
 & \Delta^0 \arrow[rr, "\bar{a} \circ \delta^i"] & & \mathrm{map}(\Delta^{p-1}, \mathrm{R}),
\end{tikzcd}$$

and similarly for horizontal degeneracy maps. This shows that the horizontal face maps are in fact simplicial maps

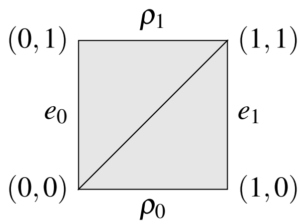
$$d_i^h \colon (\mathcal{S}_h)_p \to (\mathcal{S}_h)_{p-1} \quad \text{and} \quad s_j^h \colon (\mathcal{S}_h)_p \to (\mathcal{S}_h)_{p+1}$$

going from p–sections to $(p-1)$ and $(p+1)$–sections, respectively. The intuition is that d_i^h restricts a p–section in $\mathcal{S}_h[a_0,\dots,a_p]$ to a $(p-1)$–section in $\mathcal{S}_h[a_0,\dots,\hat{a}_i,\dots,a_p]$, whereas s_j^h adds a degenerate label $\mathcal{S}_h[a_0,\dots,a_j,a_j,\dots,a_p]$. The set $(\mathcal{S}_h)_{p,q}$ is therefore simplicial in both p and q, defining a *bisimplicial set.*

Definition 2.10. The *section complex* of a height function $h\colon X \to \mathrm{R}$ is the bisimplicial set $\mathcal{S}_h$ with (p,q)–simplices given by $(\mathcal{S}_h)_{p,q}$, i.e. the (p,q)–sections $\rho\colon \Delta^p \times \Delta^q \to X$. It has horizontal and vertical face and degeneracy maps as defined above.

Example 2.11. Consider once more the standard 2–simplex with height function like in Example 2.8. The simplicial set $(\mathcal{S}_h)_0$ is the disjoint union $\mathcal{S}_h[0]\coprod\mathcal{S}_h[1]\coprod\mathcal{S}_h[2]$. All these components consist of a single point determined by the 0–simplices at the corresponding heights. If we don't count degeneracies, the simplicial set $(\mathcal{S}_h)_1$ is the disjoint union $\mathcal{S}_h[0,1]\coprod\mathcal{S}_h[1,2]\coprod\mathcal{S}_h[0,2]$. Again, all of the components are singletons corresponding to the 1–simplices e_2, e_0 and e_1, respectively. Lastly, $(\mathcal{S}_h)_2 = \mathcal{S}_h[0,1,2]$, containing the 2–section corresponding to s. In this example, the horizontal face maps of s correspond to the ordinary face maps of the standard 2–simplex; $d_0^h s = e_0$, $d_1^h s = e_1$ and $d_2^h s = e_2$. Notice how the higher section space $(S_h)_2$ makes it possible to recover the topology of the 2–simplex.

Example 2.12. Consider the product of two standard 1–simplices as in Example 2.4.



We obtain a height function by projecting the labels of the vertices to their first component $h\colon (i,j) \mapsto i$. The space of 0–sections is $(\mathcal{S}_h)_0 = \mathcal{S}_h[0]\coprod\mathcal{S}_h[1] = h^{-1}0\coprod h^{-1}1$, with two components corresponding to the two 1–simplices e_0 and e_1. The space of 1–simplices is $(\mathcal{S}_h)_1 = \mathcal{S}_h[0,1]$. Consider the $(1,1)$–section defined in terms of the identity $\mathrm{id}_{\Delta^1\times\Delta^1}\colon \Delta^1\times\Delta^1 \to \Delta^1\times\Delta^1$. It has two horizontal faces e_0 and e_1 and two vertical faces given by the two 1–sections ρ_0 and ρ_1. We interpret this as $\mathrm{id}_{\Delta^1\times\Delta^1}$ being a homotopy from ρ_0 to ρ_1.

2.4 Proof of Theorem 1.1

We will now prove that the diagonal of the section complex $\mathcal{S}_h$, associated to a height function $h : X \to \mathrm{R}$, is homotopy equivalent to X (Theorem 1.1). This makes it possible to use the spectral sequence of $\mathcal{S}_h$ to extract homological features of X, which will be discussed in Section 3.

The *diagonal* of $\mathcal{S}_h$, $(\operatorname{diag}\mathcal{S}_h)_n$, is defined to have (n,n)–sections $\rho\colon \Delta^n \times \Delta^n \to X$ as n–simplices. Since horizontal and vertical face maps are independent, we can safely define $d_i = d_i^h d_i^v$ which is equal to $d_i^v d_i^h$. Similarly, $s_j = s_j^h s_j^v$.

Let us understand how to relate $\operatorname{diag}\mathcal{S}_h$ and X: the n–simplices in $\operatorname{diag}\mathcal{S}_h$ define a subset of $\operatorname{map}(\Delta^n, X)_n = \operatorname{Map}(\Delta^n \times \Delta^n, X)$, while the n–simplices of X are given by the set $\operatorname{Map}(\Delta^n, X)$. There are maps $(\mathrm{id}_{\Delta^n}, \mathrm{id}_{\Delta^n})\colon \Delta^n \to \Delta^n \times \Delta^n$, $i \mapsto (i,i)$ and, conversely, the projection onto the "section component" is defined: $\mathrm{pr}_0\colon \Delta^n \times \Delta^n \to \Delta^n$, $(i,j) \mapsto i$. Pre-composition defines maps

$$(\mathrm{id},\mathrm{id})^*\colon \operatorname{diag}\mathcal{S}_h \to X \quad \text{and} \quad \mathrm{pr}_0^*\colon X \to \operatorname{diag}\mathcal{S}_h$$

which will be proven mutual homotopy inverses. The latter map is well-defined. Indeed, if $\tau\colon \Delta^n \to X$ is any n–simplex in X, then the composition

$$\Delta^n \times \Delta^n \xrightarrow{\mathrm{pr}_0} \Delta^n \xrightarrow{\tau} X \xrightarrow{h} R$$

is in $\mathcal{S}_h(\bar{a})$, where $\bar{a}$ is defined by the n–simplex $h \circ \tau$. Furthermore, these maps are clearly simplicial and the composition $(\mathrm{id},\mathrm{id})^* \circ \mathrm{pr}_0^*$ is the identity. The proof of the theorem is thus reduced to finding a simplicial homotopy

$$H\colon \mathrm{id}_{\operatorname{diag}\mathcal{S}_h} \Rightarrow \mathrm{pr}_0^* \circ (\mathrm{id},\mathrm{id})^*.$$

To do so we first introduce for every $n \geq 0$, two families of simplicial maps

$$\{\phi_{n,s}\colon \Delta^n \times \Delta^n \to \Delta^n \times \Delta^n\}_{0 \leq s \leq n+1} \tag{1}$$

and

$$\{\psi_{n,s}\colon \Delta^n \times \Delta^n \to \Delta^n \times \Delta^n\}_{0 \leq s \leq n+1}. \tag{2}$$

Pulling these maps back along sections in $(\operatorname{diag}\mathcal{S}_h)_n$ will then provide us with the components of our homotopy. Note that the parameter s will be necessary to make these components fit together into a simplicial map.

We specify how the maps (1) and (2) act on 0–simplices:

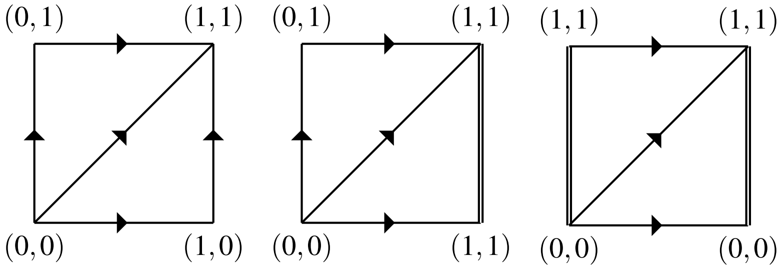
$$\phi_{n,s}(i,j) = \begin{cases} (i,i) & \text{if } i > n-s \quad \text{and } j \leq i \\ (i,j) & \text{else} \end{cases} \tag{3}$$

and

$$\psi_{n,s}(i,j) = \begin{cases} (i,i) & \text{if } j \leq i \\ (i,i) & \text{if } i < s \quad \text{and } j \geq i \\ (i,j) & \text{else} \end{cases} \tag{4}$$

Note that the so defined assignments preserve the preorder on 0–simplices in $\Delta^n \times \Delta^n$. Because $\Delta^n \times \Delta^n$ is the nerve of the category $[n] \times [n]$, and the nerve functor preserves products, (3) and (4) uniquely determine the families (1) and (2), respectively.

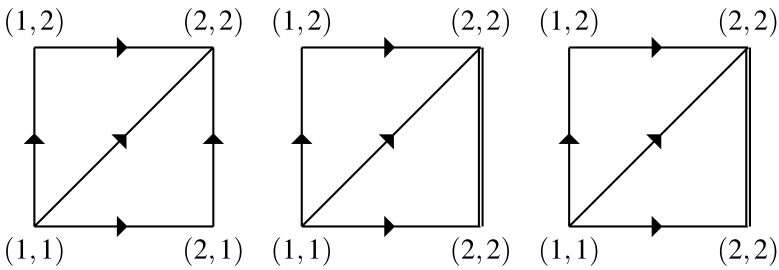
The following figure depicts the maps $\phi_{1,s}$ and $\psi_{1,s}$ in terms of their image.



$\phi_{1,0}$ $\qquad$ $\phi_{1,1} = \phi_{1,2} = \psi_{1,0}$ $\qquad$ $\psi_{1,1} = \psi_{1,2}$

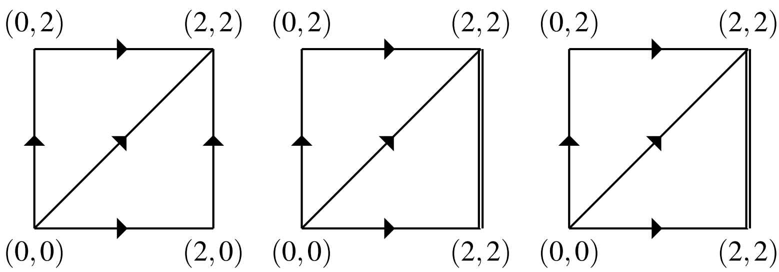
In general, the map $\phi_{n,0}$ is always the identity and $\psi_{n,n} = \psi_{n,n+1} = (\mathrm{id}, \mathrm{id}) \circ \mathrm{pr}_0$. These maps will respectively correspond to the start and end of our final homotopy.

We proceed by looking at how the maps $\phi_{2,s}$ act on the three diagonal faces of $\Delta^2 \times \Delta^2$.
$\mathrm{Im}(\delta^0 \times \delta^0(\Delta^1 \times \Delta^1))$:



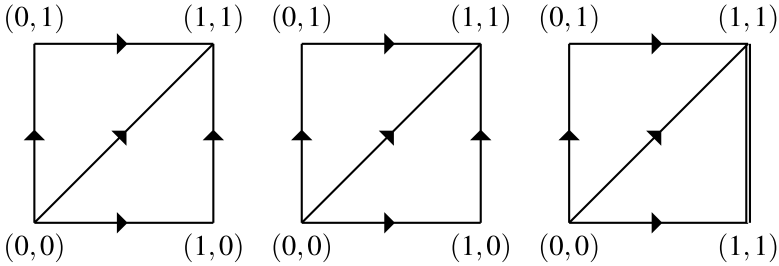
$\phi_{2,0}$ $\qquad$ $\phi_{2,1}$ $\qquad$ $\phi_{2,2} = \phi_{2,3}$

$\mathrm{Im}(\delta^1 \times \delta^1(\Delta^1 \times \Delta^1))$:



$\phi_{2,0}$ $\qquad$ $\phi_{2,1}$ $\qquad$ $\phi_{2,2} = \phi_{2,3}$

$\mathrm{Im}(\delta^2 \times \delta^2(\Delta^1 \times \Delta^1))$:



$\phi_{2,0}$ $\phi_{2,1}$ $\phi_{2,2} = \phi_{2,3}$

The key observation to be made from looking at these pictures is that on the diagonal face $(\delta^l \times \delta^l)(\Delta^1 \times \Delta^1)$ the map $\phi_{2,s}$ is determined by $\phi_{1,s}$ if $l \leq 2-s$ and by $\phi_{1,s-1}$ if $l > 2-s$. This pattern generalizes to all dimensions n, also for the maps $\psi_{n,s}$. With this insight in mind we prove the following lemma.

Lemma 2.13. The family of simplicial maps $\{\phi\}_{n,s}$ satisfies for every $n \geq 1$

$$\begin{tikzcd}
\Delta^{n-1} \times \Delta^{n-1} \arrow[r, "\delta^l \times \delta^l"] \arrow[d, "\phi_{n-1,s}"'] & \Delta^n \times \Delta^n \arrow[d, "\phi_{n,s}"] \\
\Delta^{n-1} \times \Delta^{n-1} \arrow[r, "\delta^l \times \delta^l"] & \Delta^n \times \Delta^n,
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta^n \times \Delta^n \arrow[r, "\sigma^l \times \sigma^l"] \arrow[d, "\phi_{n,s}"'] & \Delta^{n-1} \times \Delta^{n-1} \arrow[d, "\phi_{n-1,s}"] \\
\Delta^n \times \Delta^n \arrow[r, "\sigma^l \times \sigma^l"] & \Delta^{n-1} \times \Delta^{n-1}
\end{tikzcd}
\quad \text{for } l \leq n-s \tag{5}$$

and

$$\begin{tikzcd}
\Delta^{n-1} \times \Delta^{n-1} \arrow[r, "\delta^l \times \delta^l"] \arrow[d, "\phi_{n-1,s-1}"'] & \Delta^n \times \Delta^n \arrow[d, "\phi_{n,s}"] \\
\Delta^{n-1} \times \Delta^{n-1} \arrow[r, "\delta^l \times \delta^l"] & \Delta^n \times \Delta^n
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta^n \times \Delta^n \arrow[r, "\sigma^l \times \sigma^l"] \arrow[d, "\phi_{n,s}"'] & \Delta^{n-1} \times \Delta^{n-1} \arrow[d, "\phi_{n-1,s-1}"] \\
\Delta^n \times \Delta^n \arrow[r, "\sigma^l \times \sigma^l"] & \Delta^{n-1} \times \Delta^{n-1}
\end{tikzcd}
\quad \text{for } l > n-s \tag{6}$$

and likewise for the family $\{\psi\}_{n,s}$.

Proof. The proof is straightforward: We compute the images of 0–simplices along both sides of the asserted diagrams. As previously mentioned, this suffices because morphisms between products of standard simplices are uniquely determined by the image of 0–simplices. We first consider the left-hand diagram in (5). It should commute whenever $l \leq n-s$. An arbitrary 0–simplex (i,j) is mapped to

$$(\delta^l \times \delta^l) \circ \phi_{n-1,s} = \begin{cases} (\delta^l \times \delta^l)(i,i) & \text{if } i > n-1-s \text{ and } j \leq i \\ (\delta^l \times \delta^l)(i,j) & \text{else} \end{cases}$$

along the lower left composition and

$$\phi_{n,s} \circ (\delta^l \times \delta^l) = \begin{cases} (\delta^l \times \delta^l)(i,i) & \text{if } \delta^l(i) > n-s \text{ and } \delta^l(j) \leq \delta^l(i) \\ (\delta^l \times \delta^l)(i,j) & \text{else} \end{cases}$$

along the upper right composition. Observe how the inequalities $j \leq i$ and $\delta^l(j) \leq \delta^l(i)$ are equivalent. Moreover, the inequality $i > n-1-s$ is equivalent to $\delta^l(i) > n-s$. Indeed, if $i > n-1-s$, then $l \leq n-s \leq i$ so that $\delta^l(i) > n-s$. Conversely, if $\delta^l(i) > n-s$, then clearly $i > n-s-1$. Hence the first diagram commutes.

The commutativity of all the other diagrams is shown in the same manner. No complications arise in the corresponding computations and we will thus not spell them out here. $\square$

With this recursive description of the families $\{\phi_{n,s}\}$ and $\{\psi_{n,s}\}$ we can now give the homotopy

$$H\colon \mathrm{id}_{\mathrm{diag}\,\mathcal{S}_h} \Rightarrow \mathrm{pr}_1^* \circ (\mathrm{id}, \mathrm{id})^*,$$

which finishes the proof.

Proof of Theorem 1.1. Our model for the interval will be the $(2,2)$–horn $\Lambda^2_2\colon 0 \to 2 \leftarrow 1$. Note that an n–simplex in Λ^2_2 is equivalent to a map m from $\{0,1,\ldots,n\}$ to either $\{0,2\}$ or $\{1,2\}$, respecting the ordering. In the first case, we use the notation $(0:n-s+1, 2:s)$, counting the number of times m meets 0 and 2. Dually, $(2:n-s+1, 1:s)$ is used in the second case.

The components of the asserted homotopy are given by

$$H_n\colon (\mathrm{diag}\,\mathcal{S}_h)_n \times (\Lambda^2_2)_n \to (\mathrm{diag}\,\mathcal{S}_h)_n$$

where

$$H_n(\rho, t) = \begin{cases} \rho \circ \phi_{n,s} & t = (0:n-s+1, 2:s) \\ \rho \circ \psi_{n,s} & t = (2:n-s+1, 1:s) \end{cases}$$

For this to constitute a simplicial map, there must be commutative diagrams

$$\begin{tikzcd}
(\mathrm{diag}\,\mathcal{S}_h)_n \times (\Lambda^2_2)_n \arrow[r, "H_n"] \arrow[d, "d_l"] & (\mathrm{diag}\,\mathcal{S}_h)_n \arrow[d, "d_l"] \\
(\mathrm{diag}\,\mathcal{S}_h)_{n-1} \times (\Lambda^2_2)_{n-1} \arrow[r, "H_{n-1}"] & (\mathrm{diag}\,\mathcal{S}_h)_{n-1}
\end{tikzcd}$$

whenever $0 \leq l \leq n$ and

$$\begin{tikzcd}
(\mathrm{diag}\,\mathcal{S}_h)_{n-1} \times (\Lambda^2_2)_{n-1} \arrow[r, "H_{n-1}"] \arrow[d, "s_l"] & (\mathrm{diag}\,\mathcal{S}_h)_{n-1} \arrow[d, "s_l"] \\
(\mathrm{diag}\,\mathcal{S}_h)_n \times (\Lambda^2_2)_n \arrow[r, "H_n"] & (\mathrm{diag}\,\mathcal{S}_h)_n
\end{tikzcd}$$

whenever $0 \leq l < n$. We will only verify the case $t = (0 : n-p+1, 2 : p)$. This is because of how $t = (2 : n-p+1, 1 : p)$ is completely analogous. The upper right composition in the first diagram is:

$$\begin{aligned} d_l H_n(\rho, (0 : n-s+1, 2 : s)) =& d_l(\rho \circ \phi_{n,s}) \\ =& \rho \circ \phi_{n,s} \circ (\delta^l \times \delta^l) \end{aligned}$$

Hence, we deduce

$$d_l H_n(\rho, t) = \begin{cases} \rho \circ (\delta^l \times \delta^l) \circ \phi_{n-1,s} & l \leq n-s \\ \rho \circ (\delta^l \times \delta^l) \circ \phi_{n-1,s-1} & l > n-s \end{cases}$$

due to the left-hand diagrams (5) and (6) given in Lemma 2.13. Since

$$d_l(\rho, t) = \begin{cases} (d_l\rho, (0 : n-s, 2 : s)) & l \leq n-s \\ (d_l\rho, (0 : n-s+1, 2 : s-1)) & l > n-s, \end{cases}$$

we have $d_l H_n(\rho, t) = H_{n-1} d_l(\rho, t)$. This establishes the commutativity of the first diagram.

Using the right-hand diagrams in (5) and (6) we can show compatibility with the degeneracy maps in the same way. This concludes the construction of the homotopy and thus the proof. □

3 Reeb complexes and the section spectral sequence

We first introduce the *Reeb complexes* and show how these encode the flow of homology generators across height levels. Thereafter, we will see that the Reeb complexes are constructed from the first page of the more general *section spectral sequence*. This will enable us to recover the homology of X. Finally, we compare the combinatorial section space with the topological section space.

3.1 Reeb Complexes

Recall that the section complex, $\mathcal{S}_h$, consists of all section spaces $(\mathcal{S}_h)_p$. We can thus apply any homology functor H_q to $(\mathcal{S}_h)_p$ and induce $\mathrm{H}_q d_i^h \colon \mathrm{H}_q(\mathcal{S}_h)_p \to \mathrm{H}_q(\mathcal{S}_h)_{p-1}$. This defines a simplicial vector space $\mathrm{H}_q\mathcal{S}_h$, because every set of p–simplices, $\mathrm{H}_q(\mathcal{S}_h)_p$, is a vector space. Furthermore, for any simplicial vector space V, there is an associated chain complex $\mathrm{C}V$, called the *Moore Complex*. Its pth entry $\mathrm{C}V_p$ is equal to the vector space V_p, and its differentials are induced by the alternating sum of face maps, $\partial = \sum(-1)^i d_i$. Denote by $\mathrm{D}V$ the subcomplex of $\mathrm{C}V$ whose pth entry only consists of the degenerate p–simplices in V_p. The differential induces a well-defined differential $\mathrm{C}V_p/\mathrm{D}V_p \to \mathrm{C}V_{p-1}/\mathrm{D}V_{p-1}$ from which we define the *non-degenerate complex* $\mathrm{C}V/\mathrm{D}V$.

Definition 3.1. For a height function $h\colon X \to \mathrm{R}$ and integer $q \geq 1$, we define the qth *Reeb complex* $\mathcal{G}_q$ as the chain complex $\mathrm{C}(\mathrm{H}_q\mathcal{S}_h)/\mathrm{D}(\mathrm{H}_q\mathcal{S}_h)$.

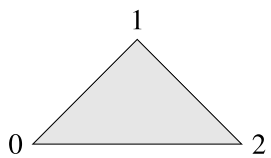
Example 3.2. The qth Reeb complex associated to a height function $h\colon X \to \mathrm{R}$ has

$$(\mathcal{G}_q)_p = \mathrm{H}_q\left(\bigsqcup \mathcal{S}_h(\bar{a})\right) \simeq \bigoplus \mathrm{H}_q\mathcal{S}_h[\bar{a}]$$

as its pth entry, ranging over all *increasing* sequences $\bar{a} = (a_0, \ldots, a_p)$ in R_p.

Reeb complexes provide an approximate tool to better understand homological features of the underlying space X. This is achieved by understanding how homology generators flow along height levels.

Example 3.3. Recall the standard 2–simplex Δ^2, with heights as indicated by the labels:



We glue two copies of Δ^2 together along their boundary $\partial\Delta^2$ to obtain $\Delta^2 \coprod_{\partial\Delta^2} \Delta^2$, a simplicial model for the 2–sphere. Label 0 and 1–simplices by their integers, e.g. 01 is the 1–simplex from 0 to 1. The two 2–simplices sharing a common boundary, are denoted a and b. To determine a basis for $\mathrm{H}_q\mathcal{S}_h$, we identify the homotopy types of all section spaces, indexed by increasing sequences:

$\bar{a}$	(0)	(1)	(2)	$(0,1)$	$(0,2)$	$(1,2)$	$(0,1,2)$
$\mathcal{S}_h[\bar{a}]$	$\{0\}$	$\{1\}$	$\{2\}$	$\{01\}$	$\{02\}$	$\{12\}$	$\{a,b\}$
Homotopy type	pt	pt	pt	pt	pt	pt	$\mathrm{pt}\coprod\mathrm{pt}$

Hence, all Reeb complexes with $q \geq 1$ are trivial. For $q = 0$, however, we determine the boundary maps

$$\partial_1\colon \oplus \mathrm{H}_0(\mathcal{S}_h)_1 \to \oplus \mathrm{H}_0(\mathcal{S}_h)_0 \text{ and } \partial_2\colon \mathrm{H}_0(\mathcal{S}_h)_2 \to \oplus \mathrm{H}_0(\mathcal{S}_h)_1.$$

Picking the evident bases from the above table yields

$$\partial_1 = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } \partial_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix}$$

in coordinates. As an example, the first column of ∂_1 is obtained by applying the target d_0^h and source d_1^h to generators in $\mathrm{H}\mathcal{S}_h[0,1]$: $\partial_1[01] = [1] - [0]$. Hence, we can present

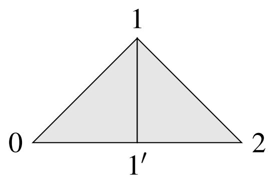
$$\mathcal{G}_0\colon\ k^3 \xleftarrow{\partial_1} k^3 \xleftarrow{\partial_2} k^2.$$

Elementary linear algebra gives $\mathrm{H}_0\mathcal{G}_0 = k$ and $\mathrm{H}_2\mathcal{G}_0 = k$, whereas other homology groups are trivial. In this particular example, the zeroth Reeb complex carries the homology of the underlying space $\Delta^2 \coprod_{\partial\Delta^2} \Delta^2$.

Definition 3.4. We say that a simplicial set X is *subdivided* according to a height function $h\colon X \to \mathrm{R}$ if all section spaces $\mathcal{S}_h[a,b]$ are empty whenever there is an intermediate height level $a < c < b$.

Whenever X is subdivided with respect to a height function $h\colon X \to \mathrm{R}$, the Reeb complexes only have two non-zero entries $(\mathcal{G}_q)_p$. Indeed, the pth entry, $\oplus\mathcal{S}_h[a_0,\dots,a_p]$, of the formula in Example 3.2, is zero for $p \geq 2$. Thus, choosing coordinates in this case reduces the information contained in $\mathcal{G}_q$ to a single matrix. Interpreting this matrix as an *incidence matrix* provides a graph which gives insight as to how homology generators flow across height levels. This is illustrated with an example.

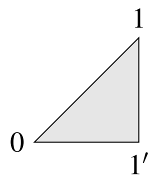
Example 3.5. We subdivide Δ^2 according to the heights given in Example 3.3:



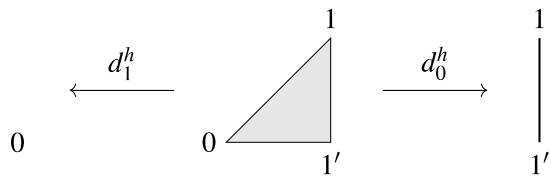
The subdivided 2–simplex still maps to R by also sending $1'$ to 1 in $\mathrm{R}_0 = \mathbb{R}$. We construct a space, much like in the previous example, by gluing two copies of the subdivided 2–simplex together along the boundary defined by the cycle $0 \to 1 \to 2 \leftarrow 1' \leftarrow 0$. It is not difficult to determine the homotopy types of associated section spaces:

$\bar{a}$	(0)	(1)	(2)	$(0,1)$	$(0,2)$	$(1,2)$	$(0,1,2)$
$\mathcal{S}_h[\bar{a}]$	pt	S^1	pt	S^1	$\emptyset$	S^1	$\emptyset$

For instance, the homotopy type of $\mathcal{S}_h[0,1]$ is deduced as follows. There are two 0–sections represented by the edges 01 and $01'$. Each copy of the subdivided 2–simplex provides a $(1,1)$-section, i.e. a homotopy



between the two sections 01 and $01'$, but no higher simplices connect them. Thus, $\mathcal{S}_h[0,1]$ is isomorphic to two 1–simplices glued tail to tail and head to head. The horizontal face maps used to calculate Reeb complexes can be depicted:



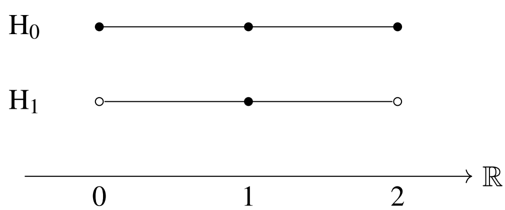
This translates to $\mathrm{H}_1\mathcal{S}_h[0,1] \xrightarrow{0} \mathrm{H}_1\mathcal{S}_h[0]$ and $\mathrm{H}_1\mathcal{S}_h[0,1] \xrightarrow{1} \mathrm{H}_1\mathcal{S}_h[1]$ on H_1. We calculate the two non-trivial Reeb complexes in coordinates

$$\mathcal{G}_0\colon\ k^3 \xleftarrow{\partial} k^2, \text{ with } \partial = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

and

$$\mathcal{G}_1\colon\ k \xleftarrow{\partial'} k^2, \text{ with } \partial' = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

To draw the associated graphs, we think of the basis elements in $(\mathcal{G}_0)_0$ as vertices whereas the basis elements in $(\mathcal{G}_0)_1$ define edges. Then the first column in ∂ tells us that the edge given by the first basis vector in k^2 connects the first and second vertices (basis elements) in k^3. For $\mathcal{G}_1$ we have to take a bit care as we have two edges and one vertex. One edge starts in the vertex, the other ends in it. This is due to face maps being sent to zero maps in H_1, a phenomenon that does not occur for H_0. We can assemble this information in a barcode-like diagram:



In the previous example, the reader familiar with Reeb graphs may have observed that the graph determined by introducing coordinates to $\mathcal{G}_0$ is the Reeb graph of the given height function. This observation is true in general, if X is subdivided according to $h\colon X \to \mathrm{R}$.

Proposition 3.6. If X is subdivided according to a height function $h\colon X \to \mathrm{R}$. Then the simplicial set $\pi_0\mathcal{S}_h$ is the Reeb graph of h. In particular, the zeroth Reeb complex $\mathcal{G}_0$ computes the homology of the associated Reeb graph.

Proof. The result is an immediate consequence of Proposition 3.14, to be proved in Section 3.4, and Theorem 1.2 in [Try21]. □

3.2 Background on spectral sequences

A double chain complex C is a collection $C_{p,q}$ of vector spaces together with horizontal and vertical boundary maps $\partial_h\colon C_{p,q} \to C_{p-1,q}$ and $\partial_v\colon C_{p,q} \to C_{p,q-1}$. The maps are further required to satisfy $\partial_h^2 = 0$, $\partial_v^2 = 0$ and $\partial_v\partial_h = \partial_h\partial_v$. We always assume a double chain complex to be contained within the first quadrant, so that all entries with p or q negative are zero. To a double complex C, we can functorially associate a chain complex $\mathrm{Tot}C$, the *total complex* of C with $\mathrm{Tot}C_n = \oplus_{p+q=n}C_{p,q}$.

There is a functor $\mathrm{F}\colon$ (Bisimplicial Sets) $\to$ (Double Complexes). It sends a bisimplicial set X to the double complex $\mathrm{F}X$ with $(\mathrm{F}X)_{p,q} = \mathrm{F}X_{p,q}$, the free vector space on $X_{p,q}$. The horizontal and vertical boundary maps are induced by the horizontal and vertical face maps: $\partial_h = \sum(-1)^i d_i^h$ and $\partial_v = \sum(-1)^i d_i^v$. Total complexes thus define a functor (Bisimplicial Sets) $\to$ (Chain Complexes), by mapping a bisimplicial set X to the total complex $\mathrm{TotF}X$. A theorem of Dold and Puppe [DP61, GJ09] tells us that $\mathrm{TotF}X$ is homology equivalent to $\mathrm{diag}X$, the diagonal on X:

$$\mathrm{H}_*\mathrm{TotF}X \simeq \mathrm{H}_*\,\mathrm{diag}X.$$

Therefore in order to understand the homology of $\mathrm{diag}X$, one may rather consider the homology of $\mathrm{TotF}X$. One advantage of the total complex, is that it comes with a *spectral sequence* for computing its homology. The following is a brief recap of how this computational tool works. We refer to [McC01] for a more in-depth introduction.

Given a double complex C, we define the zeroth page of the spectral sequence $\mathrm{E}^0_{p,q} = C_{p,q}$ and remember only the vertical boundary maps $\partial_v = \partial^0$:

$$\begin{tikzcd}
C_{0,2} \arrow[d, "\partial^0_{0,2}"] & C_{1,2} \arrow[d, "\partial^0_{1,2}"] & C_{2,2} \arrow[d, "\partial^0_{2,2}"] \\
C_{0,1} \arrow[d, "\partial^0_{0,1}"] & C_{1,1} \arrow[d, "\partial^0_{1,1}"] & C_{2,1} \arrow[d, "\partial^0_{2,1}"] \\
C_{0,0} & C_{1,0} & C_{2,0}
\end{tikzcd}$$

Applying homology produces the first page $\mathrm{E}^1_{p,q} = \mathrm{H}_q C_{p,q}$ with induced differentials ∂^1 from the horizontal differentials of C.

$$\begin{tikzcd}
\mathrm{H}_2 C_{0,2} & \mathrm{H}_2 C_{1,2} \arrow[l, "\partial^1_{1,2}"'] & \mathrm{H}_2 C_{2,2} \arrow[l, "\partial^1_{2,2}"'] \\
\mathrm{H}_1 C_{0,1} & \mathrm{H}_1 C_{1,1} \arrow[l, "\partial^1_{1,1}"'] & \mathrm{H}_1 C_{2,1} \arrow[l, "\partial^1_{2,1}"'] \\
\mathrm{H}_0 C_{0,0} & \mathrm{H}_0 C_{1,0} \arrow[l, "\partial^1_{1,0}"'] & \mathrm{H}_0 C_{2,0} \arrow[l, "\partial^1_{2,0}"']
\end{tikzcd}$$

Computing homology yet again gives the second page $\mathrm{E}^2_{p,q} = \mathrm{H}_p \mathrm{H}_q C_{p,q}$. There are also induced maps on the second page $\partial^2_{p,q} \colon \mathrm{E}^2_{p,q} \to \mathrm{E}^2_{p-2,q+1}$. One can show that the following description on the level of representatives is well-defined. If $[c]$ in $\mathrm{E}^1_{p,q} = \mathrm{H}_q C_{p,q}$ represents an element α in $\mathrm{E}^2_{p,q}$, then it is mapped to zero under $\partial^1_{p,q}[c] = [\partial_h c]$. This in turn means that $\partial_h c$ is in the image of $\partial_v = \partial^0_{p-1,q+1}$. Hence there is a b in $C_{p-1,q+1}$ such that $\partial_v b = \partial_h c$ and applying ∂_h then produces an element $\partial_h b$ which can be verified to represent an element in $\mathrm{E}^1_{p-2,q+1}$. Denote by β the element in $\mathrm{E}^2_{p-2,q+1}$ represented by $[\partial_h b]$, and define $\partial^2_{p,q}\alpha = \beta$. This is, of course, difficult to compute in general.

$$
\begin{tikzcd}
\mathrm{H}_0\mathrm{H}_2C_{0,2} & \mathrm{H}_1\mathrm{H}_2C_{1,2} & \mathrm{H}_2\mathrm{H}_2C_{2,2} \\
\mathrm{H}_0\mathrm{H}_1C_{0,1} & \mathrm{H}_1\mathrm{H}_1C_{1,1} & \mathrm{H}_2\mathrm{H}_1C_{2,1} \arrow[llu, "\partial^2_{2,1}"'] \\
\mathrm{H}_0\mathrm{H}_0C_{0,0} & \mathrm{H}_1\mathrm{H}_0C_{1,0} & \mathrm{H}_2\mathrm{H}_0C_{2,0} \arrow[llu, "\partial^2_{2,0}"']
\end{tikzcd}
$$

The process now iterates: $\mathrm{E}^3_{p,q}$ is defined as the homology at $\mathrm{E}^2_{p,q}$. There are induced differentials $\partial^3_{p,q}\colon \mathrm{E}^3_{p,q} \to \mathrm{E}^3_{p-3,q+2}$, much like in the case of E^2. What we end up with is a collection $\mathrm{E}^r_{p,q}$ of vector spaces together with differentials $\partial^r_{p,q}\colon \mathrm{E}^r_{p,q} \to \mathrm{E}^r_{p-r,q-1+r}$ satisfying that $\mathrm{E}^{r+1}_{p,q}$ is obtained from $\mathrm{E}^r_{p,q}$ by computing homology. Note that the process terminates; at some point $\mathrm{E}^{r+n}_{p,q} \simeq \mathrm{E}^r_{p,q}$ for all $n \geq 0$. This is because of how differentials must eventually be zero when they leave the first quadrant in the (p,q)–plane. Let $\mathrm{E}^\infty_{p,q}$ be the stable value of $\mathrm{E}^r_{p,q}$. It is well-known that

$$\mathrm{H}_n \mathrm{Tot} C \simeq \oplus_{p+q=n} \mathrm{E}^\infty_{p,q}.$$

Thus, if $C = \mathrm{F}X$ for some bisimplicial set X, then we have described a procedure to compute $\mathrm{H}_* \operatorname{diag} X$ from $\mathrm{Tot}\mathrm{F}X$.

3.3 The section spectral sequence

The previous Section implies the existence of a spectral sequence associated to $\mathcal{S}_h$ which calculates the homology of $\operatorname{diag} \mathcal{S}_h$.

Definition 3.7. The *section spectral sequence* of a height function $h\colon X \to \mathrm{R}$ is the spectral sequence naturally associated to $\mathcal{S}_h$.

Entries on the zeroth page are determined by the free double complex $\mathrm{F}\mathcal{S}_h$. Explicitly, $\mathrm{E}^0_{p,q} = (\mathrm{F}\mathcal{S}_h)_{p,q}$ is the free vector space on $\coprod_{\bar{a}\in \mathrm{R}_p} \mathcal{S}_h[\bar{a}]_q$, ranging over all non-decreasing real-valued sequences $\bar{a} = (a_0, \ldots, a_p)$. Differentials $\partial^0_{p,q}\colon \mathrm{E}^0_{p,q} \to \mathrm{E}^0_{p,q-1}$ are induced from the alternating sum of vertical face maps $\sum_i (-1)^i d^v_i$ in the spatial q–direction. Computing homology vertically (in the q–direction) thus produces the entries of the first page $\mathrm{E}^1_{p,q} = \oplus_{\bar{a}\in \mathrm{R}_p} \mathrm{H}_q \mathcal{S}_h[\bar{a}]$. Differentials on the first page are then induced in homology from the alternating sum of the horizontal face maps in the section p–direction $\sum_i (-1)^i \mathrm{H}_q d^h_i$. Proceeding as in Section 3.2, the section spectral sequence tells us how to recover the homology of $\operatorname{diag} \mathcal{S}_h$, and thus of X.

Proposition 3.8 (Proposition 1.2)**.** Let $h\colon X \to \mathrm{R}$ be a height function. The associated section spectral sequence satisfies $\mathrm{E}^2_{p,q} \simeq \mathrm{H}_p\mathcal{G}_q$ and converges to the homology of X:

$$\mathrm{H}_p\mathcal{G}_q \Rightarrow \mathrm{H}_{p+q}X$$

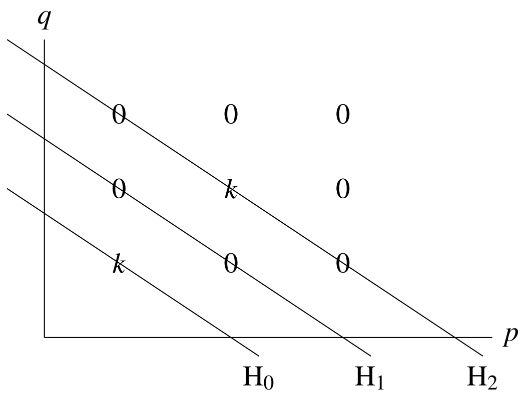
Proof. Theorem 1.1 tells us that $\operatorname{diag}\mathcal{S}_h$ is homotopy equivalent, hence homology equivalent, to X. So it only remains to verify that $\mathrm{E}^2_{p,q} \simeq \mathrm{H}_p\mathcal{G}_q$. The pth entry of the qth Reeb complex $\mathcal{G}_q$ is

$$(\mathcal{G}_q)_p = \bigoplus \mathrm{H}_p\mathcal{S}_h[\bar{a}]$$

ranging over all increasing sequences in R_p, whereas $\mathrm{E}^1_{p,q}$ ranges over all non-decreasing sequences. Hence we observe that $\mathcal{G}_q$ is the non-degenerate complex $\mathrm{E}^1_{-,q}/\mathrm{DE}^1_{-,q}$. It is well-known that $\mathrm{E}^1_{-,q}$ is chain homotopic to $\mathrm{E}^1_{-,q}/\mathrm{DE}^1_{-,q}$, see e.g. [GJ09, p.150]. In particular, $\mathrm{E}^2_{p,q} = \mathrm{H}_p\mathrm{E}^1_{-,q}$ is isomorphic to $\mathrm{H}_p\mathcal{G}_q = \mathrm{H}_p(\mathrm{E}^1_{-,q}/\mathrm{DE}^1_{-,q})$. $\square$

To summarize Proposition 3.8: It does not matter if we exchange the qth row $\mathrm{E}^1_{-,q}$, including all non-decreasing sequences, with the Reeb complex $\mathcal{G}_q$ (including only increasing sequences). Note the importance of this fact for making computations with section complexes feasible within finite time. This is perhaps best illustrated through some simple examples:

Example 3.9. In Example 3.3 we identified the first page of the section spectral sequence of $X = \Delta^2 \coprod_{\partial\Delta^2} \Delta^2$ with a single row – the Reeb complex $\mathcal{G}_0$. Hence, the differentials on the second page must be zero and we conclude that the homology of X and $\mathcal{G}_0$ coincide. In Example 3.5, where the 2-simplices were subdivided prior to gluing, we are left with two non-zero rows on the first page: $\mathcal{G}_0$ and $\mathcal{G}_1$. Moreover, the only non-trivial entries of both $\mathcal{G}_0$ and $\mathcal{G}_1$ are in $p = 0, 1$, implying that the differentials on the second page must be equal to zero. We calculate $\mathrm{H}_p\mathcal{G}_q$ and thus the second page in coordinates:



Since the sequence has converged, we extract $\mathrm{H}_0X = k$, $\mathrm{H}_2X = k$ and $\mathrm{H}_nX = 0$ otherwise.

It was not a coincidence that the section spectral sequence from Example 3.5 converged on the second page, as we shall make precise.

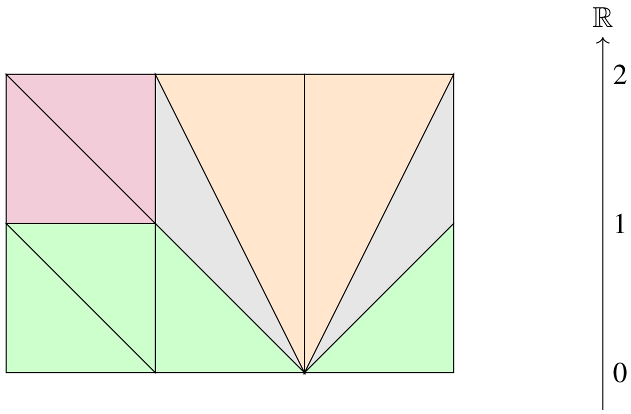
Definition 3.10. For a height function $h\colon X \to \mathrm{R}$, we introduce the *subdivision number* as the biggest n for which there is an increasing sequence $\bar{a} = (a_0, \dots, a_n)$ such that the section space $\mathcal{S}_h[\bar{a}]$ is non-empty.

Proposition 3.11. Let $h\colon X \to \mathrm{R}$ be a height function with subdivision number s. The section spectral sequence collapses at the $(s+1)$st page: $\mathrm{E}^{n+s}_{p,q} \simeq \mathrm{E}^{s}_{p,q}$ for all $n \geq 0$.

Proof. From the assumption, it follows that every Reeb complex $\mathcal{G}_q$ (Example 3.2) has trivial entries above s. Hence, the first page consists of zeros for $p \geq s+1$ and the differentials on the $(s+1)$st page must all terminate outside of the first quadrant; be equal to zero. □

Thus, the number of pages we need to compute is bounded by how subdivided X is relative to $h\colon X \to \mathrm{R}$.

Example 3.12. Construct a simplicial cylinder X by gluing together the leftmost and rightmost vertical 1–simplices in:



A height function $h\colon X \to \mathrm{R}$ is indicated by the right-hand values. Each section space of the form $\mathcal{S}_h[a_0]$, equal to $h^{-1}a_0$ for $a_0 = 0,1,2$, has one connected component. There are three 1–section spaces $\mathcal{S}_h[a_0,a_1]$, all of which have a single connected component indicated by the simplices colored in green, orange and purple above. The section space $\mathcal{S}_h[0,1,2]$ has two connected components represented by the gray 2–simplices. Only $\mathcal{S}_h[0]$ and $\mathcal{S}_h[2]$ have generators in H_1, obtained by following the horizontal edges at the bottom and top of the cylinder. We mimic the calculations in Example 3.5 to deduce

$$\partial^1_{1,0} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \partial^1_{2,0} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix}$$

which gives the first page:

$$
\begin{tikzcd}
q & & & & \\
& k^2 & 0 & 0 & \\
& k^3 & k^3 \arrow[l, "\partial^1_{1,0}"'] & k^2 \arrow[l, "\partial^1_{2,0}"'] & \\
& & & & p
\end{tikzcd}
$$

By computing homology again we obtain the second page:

$$
\begin{tikzcd}
q & & & & \\
& k^2 & 0 & 0 & \\
& k & 0 & k \arrow[ull, "\partial^2_{2,0}"'] & \\
& & & & p
\end{tikzcd}
$$

The sequence must collapse on the next page, and, as the homology of X is not calculated yet, the differential cannot be zero. The representative in $\mathrm{E}^2_{2,0} = k$ is given by the difference of the two gray 2–simplices. The alternating sum of the surrounding 1–simplices is in the image of ∂_v. Geometrically, this happens by applying ∂_v to the sum of all 2–simplices (i.e. 1–simplices in the section spaces $\mathcal{S}_h[a_0, a_1]$) not colored gray. Applying ∂_h to this sum gives the difference of the generators in $\mathrm{E}^2_{0,1} \simeq k^2$. As an example, the top generator is obtained from applying the target to purple and orange 2–simplices. We can thus conclude that $\partial^2_{2,0}$ is the transpose of $\begin{bmatrix} 1 & -1 \end{bmatrix}$. The third page only has two non-zero entries: $\mathrm{E}^3_{0,0} \simeq k$ and $\mathrm{E}^3_{0,1} \simeq k$. In particular, we calculate $\mathrm{H}_0 X = \mathrm{H}_1 X = k$ and $\mathrm{H}_n X = 0$ otherwise.

3.4 Comparison with topological spaces

In Section 2.2, we saw that a height function $h\colon X \to \mathrm{R}$ always associates to a piecewise linear function $f\colon |X| \to \mathbb{R}$. Example 2.8 illustrated that, in general, the topological space of sections Sect_f from [Try21] and the simplicial space of 1–sections $(\mathcal{S}_h)_1$ are significantly different. A topological section of the form $[a,b] \to T$ factorizes into smaller sections defined on $[a,c]$ and $[c,b]$ for any real number $a \leq c \leq b$. Conversely, two sections $\rho\colon [a,c] \to T$ and $\tau\colon [c,b] \to T$ compose to a section on $[a,b]$ via a canonical concatenation. This means that in contrast to the simplicial sections in $(\mathcal{S}_h)_1$, the topological sections are automatically subdivided. In particular, the spectral sequence obtained from Sect_f terminates on the second page, reflecting the fact that all information about the homology of T is contained in Sect_f. This is not true for $(\mathcal{S}_h)_1$ in general, which led us to introduce higher sections. Consider now the case where the simplicial set X is subdivided according to h (Definition 3.4). Then we observed, in Section 3.3, that the spectral sequence associated to $\mathcal{S}_h$ terminates on the second page as well. Whenever X is subdivided according to h, we can thus expect the space of 1–sections $(\mathcal{S}_h)_1$ to contain the same homological information as the topological section space Sect_f.

Generally, for fixed real values $a \leq b$ we can define a map from the realization $|\mathcal{S}_h[a,b]|$ to the space $\mathrm{Sect}_f[a,b]$ as follows. A point in $|\mathcal{S}_h[a,b]|$ is a class $[\rho,\bar{t}]$ with $\rho\colon \Delta^1 \times \Delta^n \to X$ an n–simplex in $\mathcal{S}_h[a,b]$ and $\bar{t}$ a point in the standard topological n–simplex $|\Delta^n|$. If we realize ρ, then we obtain a continuous function $|\rho|\colon |\Delta^1| \times |\Delta^n| \to |X|$ which hinges upon the existence of a homeomorphism $|\Delta^1 \times \Delta^n| \simeq |\Delta^1| \times |\Delta^n|$. For a fixed $\bar{t}$, the restriction of $|\rho|$ to $|\Delta^1| \times \bar{t}$ is a section of f up to the linear orientation-preserving homeomorphism $L_{a,b}\colon [a,b] \to |\Delta^1|$. Indeed, the composition $h \circ \rho$ maps the unique non-degenerate 1–simplex in Δ^1 to $a \leq b$ in R regardless of its second component. See Definition 2.9. It follows that $|h| \circ |\rho||_{|\Delta^1| \times \bar{t}}$ identifies $|\Delta^1|$ with the 1–cell labeled by $a \leq b$ in $|X|$. Whence we define a continuous function $\Phi_h\colon |\mathcal{S}_h[a,b]| \to \mathrm{Sect}_f[a,b]$ from the formula $\Phi_h[\rho,\bar{t}] = |\rho| \circ (L_{a,b},\bar{t})$.

Example 3.13. Consider the height function $h\colon \Delta^2 \coprod_{\partial\Delta^2} \Delta^2 \to \mathrm{R}$ from Example 3.3. The section space $\mathcal{S}_h[0,1]$ only consists of a single point represented by the 1–simplex $0 \to 1$, whereas the topological version $\mathrm{Sect}_f[0,1]$ is a circle. Hence, the map Φ_h cannot be a weak equivalence. While, if we subdivide $\Delta^2 \coprod_{\partial\Delta^2} \Delta^2$ as in Example 3.5, then $\mathcal{S}_h[0,1]$ is a circle and Φ_h is a weak equivalence.

Proposition 3.14. Assume that X is subdivided according to $h\colon X \to \mathrm{R}$. For every pair of successive height levels $a \leq b$, the continuous function $\Phi_h\colon |\mathcal{S}_h[a,b]| \to \mathrm{Sect}_f[a,b]$ is a homology equivalence.

We can assume without loss of generality that the only non-empty height levels of h are 0 and 1. The strategy for the proof is then to shift all homological information of X into the space $\mathcal{S}_h[0,1]$. This can be done by filling out all the simplices in the fibers of h by by means of the following pushout

$$\begin{tikzcd}
h^{-1}(0)\coprod h^{-1}(1) \arrow[r, "{(0,\mathrm{id})}"] \arrow[d, hook] & \coprod_{a=0,1} (\Delta^1 \times h^{-1}(a))/((1,x) \sim (1,y)) \arrow[d] \arrow[ddr, bend left, "a"] & \\
X \arrow[r] \arrow[drr, bend right, "h"'] & \tilde{X} \arrow[dr, "\tilde{h}"] & \\
& & \mathrm{R}
\end{tikzcd}$$

from that we in particular get an induced height function $\tilde{h}\colon \tilde{X} \to \mathrm{R}$.

Lemma 3.15. Let $h\colon X \to \mathrm{R}$ be a height function that only meets $a = 0$ and $b = 1$ and let $\tilde{h}\colon \tilde{X} \to \mathrm{R}$ be the replacement constructed above. Then

$$\mathcal{S}_h[0,1] = \mathcal{S}_{\tilde{h}}[0,1].$$

Proof. Postcomposing a section ρ in $\mathcal{S}_h[0,1]$ with the inclusion

$$\begin{tikzcd} \Delta^1 \times \Delta^n \arrow[rr, "\tilde{\rho}"] \arrow[dr, "\rho"'] & & \tilde{X} \\ & X \arrow[ur, hook] & \end{tikzcd}$$

yields a section $\tilde{\rho}$ in $\mathcal{S}_{\tilde{h}}[0,1]$. Moreover, starting with any $\tilde{\rho}$ in $\mathcal{S}_{\tilde{h}}[0,1]$ it always factorizes like that. Indeed, if we assume that for a given section $\tilde{\rho}$ in $\mathcal{S}_{\tilde{h}}[0,1]$ such a factorization does not exist. Then the image of this $\tilde{\rho}$ contains a simplex that is not in X. This simplex must then lie either in $f^{-1}(0)$ or in $f^{-1}(1)$ and thus be a horizontal face of $\tilde{\rho}$. Furthermore it contain one of the two vertices in $\tilde{X}$ which are not in X. But as this vertex is clearly no horizontal face of any section we get a contradiction. Thus every section $\tilde{\rho}$ in $\mathcal{S}_{\tilde{h}}[0,1]$ factors through X giving us the desired isomorphism. $\square$

We can now proof Proposition 3.14 by reducing to the case of contractible fibers.

Proof of Proposition 3.14. Let $h\colon X \to \mathrm{R}$ be a height function that only meets the height levels 0 and 1 and for which the fibers are contractible. Denote by $\mathrm{T}\mathcal{S}_h$ the simplicial topological space obtained by realizing $\mathcal{S}_h$ level-wise; whose space of p–simplices is the topological space of p–sections $(\mathrm{T}\mathcal{S}_h)_p = |(\mathcal{S}_h)_p|$. We can then extend Φ_h to a morphism of simplicial spaces

$$\Phi_h\colon |\mathrm{T}\mathcal{S}_h| \to \mathrm{NSect}_f$$

that acts as the identity on zero-simplices. It follows from standard theory that the realization $|\mathrm{T}\mathcal{S}_h|$ is isomorphic to $|\operatorname{diag}\mathcal{S}_h|$. Combining this fact with the homotopy equivalences from Theorem 1.1 of this paper and from Theorem 1.1 of [Try21] yields a commutative diagram

$$\begin{tikzcd} |\mathrm{T}\mathcal{S}_h| \arrow[r, "\cong"] \arrow[d, "|\Phi_h|"] & |\operatorname{diag}\mathcal{S}_h| \arrow[d, "\sim"] \\ |\mathrm{NSect}_f| \arrow[r, "\sim"] & |X| \end{tikzcd}$$

and exhibits $|\Phi_h|$ to be a homotopy equivalence as well. Using the result by Dold and Puppe [DP61, GJ09], that was already mentioned in Section 3.2 gives us the commutative square

$$
\begin{tikzcd}
\mathrm{H}_*\mathrm{TotF}\mathcal{S}_h \arrow[r] \arrow[d] & \mathrm{H}_*|\mathrm{T}\mathcal{S}_h| \arrow[d, "\mathrm{H}_*|\Phi_h|"] \\
\mathrm{H}_*\mathrm{TotNSect}_f \arrow[r] & \mathrm{H}_*|\mathrm{NSect}_f|
\end{tikzcd}
$$

where all the arrows are isomorphisms. Consider now the two spectral sequences associated to S_h and NSect_f respectively. Because X is subdivided according to h these both converge on the second page. Combine this with the contractability of the fibers of h to obtain for $q \geq 1$ the following extension of the above diagram

$$
\begin{tikzcd}
\mathrm{H}_q(\mathcal{S}_h)_1 \arrow[r] \arrow[d, "\mathrm{H}_q\Phi_h"] & \mathrm{H}_{q+1}\mathrm{TotF}\mathcal{S}_h \arrow[d] \\
\mathrm{H}_q\mathrm{Sect}_f \arrow[r] & \mathrm{H}_{q+1}\mathrm{TotNSect}_f
\end{tikzcd}
$$

from which we can conclude that $\mathrm{H}_q\Phi_h$ is an isomorphism for all $q \geq 1$.

For $q = 0$ we have to do some extra work. This is because the horizontal differential

$$\partial^1_{1,0}\colon E^1_{1,0} \to E^1_{0,0}$$

is non-trivial in both spectral sequences. It's kernel-cokernel pair however induces the diagram

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_1\mathrm{H}_0(\mathcal{S}_h)_1 \arrow[r] \arrow[d, "\mathrm{H}_1\mathrm{H}_0\Phi_h"] & \mathrm{H}_0(\mathcal{S}_h)_1 \arrow[r, "\partial^1_{1,0}"] \arrow[d, "\mathrm{H}_0\Phi_h"'] & \mathrm{H}_0(h^{-1}(0) \sqcup h^{-1}(1)) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_0\mathrm{H}_0(h^{-1}(0) \sqcup h^{-1}(1)) \arrow[d, "\mathrm{H}_0\mathrm{H}_0\Phi_h"] \\
0 \arrow[r] & \mathrm{H}_1\mathrm{H}_0\mathrm{Sect}_f \arrow[r] & \mathrm{H}_0\mathrm{Sect}_f \arrow[r, "\partial^1_{1,0}"] & \mathrm{H}_0(f^{-1}(0) \sqcup f^{-1}(1)) \arrow[r] & \mathrm{H}_0\mathrm{H}_0(f^{-1}(0) \sqcup f^{-1}(1))
\end{tikzcd}
$$

where the horizontal rows are exact. Using a similar argument as for $\mathrm{H}_q\Phi_h$ above we see that $\mathrm{H}_1\mathrm{H}_0\Phi_h$ and $\mathrm{H}_0\mathrm{H}_0\Phi_h$ are isomorphisms. An application of the five lemma exhibits the morphism $\mathrm{H}_0\Phi_h$ as an isomorphism as well and thus concludes the proof for the case of contractible fibers. The more general case follows from the above, combined with Lemma 3.15 □

4 Persistence theory

We start out by illustrating how the Reeb complexes of a height function $h\colon X \to \mathrm{R}$ may be interpreted as representations. In particular, they reduce to zigzag modules whenever X is subdivided according to h. In the end, we discuss persistent homology of filtrations and level-set zigzag.

4.1 From Reeb complexes to zigzag modules

A *zigzag module* $\mathbb{V}$ is a finite diagram

$$V_0 \overset{p_1}{\longleftrightarrow} V_1 \overset{p_2}{\longleftrightarrow} V_2 \overset{p_3}{\longleftrightarrow} \cdots \overset{p_n}{\longleftrightarrow} V_n$$

of vector spaces V_i and linear maps p_i of the form $V_{i-1} \to V_i$ or $V_i \to V_{i-1}$ [CdS10]. Representation theorists often refer to such diagrams as representations of quivers of type A_n. A celebrated theorem of Gabriel [Gab72] classifies all such representations. In particular, it shows that there is a unique decomposition of zigzag modules into *interval modules* of the form $\mathbb{I}(a,b)$:

$$I_0 \overset{p_1}{\longleftrightarrow} I_1 \overset{p_2}{\longleftrightarrow} I_2 \overset{p_2}{\longleftrightarrow} \cdots \overset{p_n}{\longleftrightarrow} I_n$$

where

$$I_i = \begin{cases} k & \text{if } a \leq i \leq b \\ 0 & \text{if } i < a \text{ or } i > b \end{cases}$$

and p_i is the identity on k for $i = a, \dots, b-1$. Decomposing in this way allows for the definition of *barcodes* as an invariant of the persistence module, which has proven useful in studying data sets [Car09]. The barcode $\mathrm{Pers}(\mathbb{V}) = \{[a_i, b_i]\}_i$ of a zigzag module $\mathbb{V}$ is obtained from the intervals appearing in its decomposition into interval modules. We emphasize that a zigzag module $\mathbb{V}$ is called a *persistence module* when all linear maps point in the same direction.

Let us consider a simplicial set X together with a height function $h \colon X \to \mathrm{R}$. The qth Reeb complex $\mathcal{G}_q$ has a differential

$$\partial \colon \bigoplus_{\bar{a}=a_0<\cdots<a_p} \mathrm{H}_q\mathcal{S}_h[\bar{a}] \to \bigoplus_{\bar{b}=b_0<\cdots<b_{p-1}} \mathrm{H}_q\mathcal{S}_h[\bar{b}]$$

for every $p \geq 1$, determined by $\mathrm{H}_q d_i^h \colon \mathrm{H}_q\mathcal{S}_h[\bar{a}] \to \mathrm{H}_q\mathcal{S}_h[a_0, \dots, \hat{a}_i, \dots, a_n]$. Restricting ∂ to a single summand may be depicted as a pyramid

$$\begin{tikzcd}[column sep=tiny]
 & & \mathrm{H}_q\mathcal{S}_h[a_0,\dots,a_p] \arrow[lld, "\mathrm{H}_q d_p^h"'] \arrow[ld, "\mathrm{H}_q d_{p-1}^h"] \arrow[rd, "\mathrm{H}_q d_1^h"'] \arrow[rrd, "\mathrm{H}_q d_0^h"] & & \\
\mathrm{H}_q\mathcal{S}_h[a_0,\dots,a_{p-1}] & \mathrm{H}_q\mathcal{S}_h[a_0,\dots,a_{p-2},a_p] & \cdots & \mathrm{H}_q\mathcal{S}_h[a_0,a_2\dots,a_p] & \mathrm{H}_q\mathcal{S}_h[a_1,\dots,a_p]
\end{tikzcd}$$

Iterating this recursively over all simplices in R yields a representation $\mathbb{G}_q$ over some directed graph. There is no hiding the fact that the underlying graph of $\mathbb{G}_q$ is rather nasty, in general. Thus, we cannot expect to classify representations arising from height functions in coordinates. For instance, assuming that h only meets 0, 1 and 2, $\mathbb{G}_q$ is drawn:

$$
\begin{tikzcd}
 & \mathrm{H}_q\mathcal{S}_h[0,1,2] \arrow[dl, "\mathrm{H}_q d_2^h"'] \arrow[d, "\mathrm{H}_q d_1^h"] \arrow[dr, "\mathrm{H}_q d_0^h"] & \\
\mathrm{H}_q\mathcal{S}_h[0,1] \arrow[d, "\mathrm{H}_q d_1^h"'] \arrow[dr, "\mathrm{H}_q d_0^h" near start] & \mathrm{H}_q\mathcal{S}_h[0,2] \arrow[dl, "\mathrm{H}_q d_1^h" near end] \arrow[dr, "\mathrm{H}_q d_0^h" near start] & \mathrm{H}_q\mathcal{S}_h[1,2] \arrow[dl, "\mathrm{H}_q d_1^h" near end] \arrow[d, "\mathrm{H}_q d_0^h"] \\
\mathrm{H}_q\mathcal{S}_h[0] & \mathrm{H}_q\mathcal{S}_h[1] & \mathrm{H}_q\mathcal{S}_h[2]
\end{tikzcd}
$$

Let us make two assumptions henceforth:

i) X is subdivided according to h, and

ii) the image of h only meets finitely many 0–simplices $a_0 < \cdots < a_n$ in R_0.

In this simplified setting, all non-empty section spaces are of the form $\mathcal{S}_h[a_i] = h^{-1}a_i$ or $\mathcal{S}_h[a_i, a_{i+1}]$. Hence, the representation $\mathbb{G}_q$ is a zigzag module:

$$
\begin{tikzcd}
\mathrm{H}_q\mathcal{S}_h[a_0] & & \mathrm{H}_q\mathcal{S}_h[a_1] & \cdots & \mathrm{H}_q\mathcal{S}_h[a_{n-1}] & & \mathrm{H}_q\mathcal{S}_h[a_n] \\
 & \mathrm{H}_q\mathcal{S}_h[a_0,a_1] \arrow[ul, "\mathrm{H}_q d_1^h"] \arrow[ur, "\mathrm{H}_q d_0^h"'] & & & & \mathrm{H}_q\mathcal{S}_h[a_{n-1},a_n] \arrow[ul, "\mathrm{H}_q d_1^h"] \arrow[ur, "\mathrm{H}_q d_0^h"'] &
\end{tikzcd}
$$

In addition to capturing persistence of homology generators, such a diagram is closely related to the Reeb graph of h. Indeed, the Reeb graph is obtained by applying π_0 instead of H_q (Propostion 3.6).

4.2 Persistent homology from section complexes

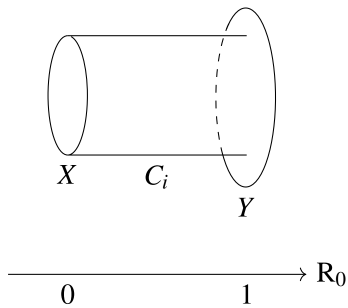
From an inclusion $i\colon X \to Y$ we form the *simplicial mapping cylinder* $C_i = (X \times \Delta^1) \coprod_X Y$ as the pushout of

$$
\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "(\mathrm{id}_X{,}1)"] & Y \\
X \times \Delta^1 &
\end{tikzcd}
$$

where id_X is the identity on X and $1\colon X \to \Delta^1$ is the trivial map $x \mapsto 1$ for all x in X. The pushout C_i is thus obtained by gluing the "top" $X \times 1$ of the cylinder $X \times \Delta^1$ to Y via i, and with $X \times 0 \cong X$ the "bottom" of C_i. The evident inclusion $j\colon \Delta^1 \hookrightarrow \mathrm{R}$ then induces a height function $h_i\colon C_i \to \mathrm{R}$ from the commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "(\mathrm{id},1)"] & Y \arrow[d, "1"] \\ X \times \Delta^1 \arrow[r, "j \circ \mathrm{pr}_1"'] & \mathrm{R}. \end{tikzcd}$$

The height function h_i thus sends the bottom of the cylinder to 0 and the top of the cylinder to 1 as depicted below.



The associated section space then has $\mathcal{S}_{h_i}[0] = X$ and $\mathcal{S}_{h_i}[1] = Y$.

Lemma 4.1. Let $i\colon X \to Y$ be some simplicial inclusion and $h_i\colon C_i \to \mathrm{R}$ the height function of the associated mapping cylinder, like defined above. The first horizontal face map, or source, $d_1^h\colon \mathcal{S}_{h_i}[0,1] \to X$ is a homology equivalence.

Proof. Recall from Section 3.4 that there is a continuous function $c : |R| \to \mathbb{R}$ and an induced height function of topological spaces $f_i = c \circ |h_i|\colon |C_i| \to \mathbb{R}$. The realization induces $|d_1^h|\colon |\mathcal{S}_{h_i}[0,1]| \to |X|$, a continuous function which fits into a commutative diagram

$$\begin{tikzcd} {|\mathcal{S}_{h_i}[0,1]|} \arrow[rr, "\Phi_{h_i}"] \arrow[dr, "|d_1^h|"'] & & \mathrm{Sect}_{f_i}[0,1] \arrow[dl, "\mathrm{eval}_0"] \\ & {|X|} & \end{tikzcd}$$

where eval_0 evaluates a section at its source. Indeed, recall that a class $[\rho,\bar{t}]$ in $|\mathcal{S}_{h_i}[0,1]|$ consists of an n–simplex $\rho\colon \Delta^1 \times \Delta^n \to X$ together with a point $\bar{t}$ in the topological n–simplex. Applying $\mathrm{eval}_0 \circ \Phi_{h_i}$ to $[\rho,\bar{t}]$ amounts to evaluating $|\rho|\colon |\Delta^1| \times |\Delta^n| \to |X|$ at $(0,\bar{t})$. This is achieved by taking the source of ρ, $d_1^h\rho$, which is an n–simplex in X

and labeling it with $\bar{t}$. In other words, $\mathrm{eval}_0 \circ \Phi_{h_i}[\rho, \bar{t}] = |d_1^h|[\rho, \bar{t}]$. Proposition 3.14 tells us that Φ_{h_i} is a homology equivalence, whereas eval_0 is a homotopy equivalence due to Proposition 4.10 in [Try21]. $\square$

Lemma 4.2. If $i\colon X \to Y$ is an inclusion of simplicial sets with associated height function $h\colon C_i \to \mathrm{R}$, then the diagram

$$\begin{tikzcd} & S_h(0,1) \arrow[dl, "d_1^h"'] \arrow[dr, "d_0^h"] & \\ X \arrow[rr, "i"] & & Y \end{tikzcd}$$

commutes up to homotopy.

Proof. It suffices to define a simplicial homotopy

$$\eta : \mathcal{S}_h(0,1) \times \Delta^1 \to Y$$

from $i \circ d_1^h$ to d_0^h. Because $\mathcal{S}_h(0,1) \subset \mathrm{map}(\Delta^1, C_i)$, it will be convenient to define η in terms of its adjoint

$$\tilde{\eta} : \mathcal{S}_h(0,1) \to \mathrm{map}(\Delta^1, Y).$$

Post-composing a section $\rho : \Delta^1 \times \Delta^n \to C_i$ with $\phi : C_i \to Y$, induced from the universal property of the pushout

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & Y \arrow[d] \arrow[ddr, "\mathrm{id}"] & \\ X \times \Delta^1 \arrow[r] \arrow[drr, "i \circ \mathrm{pr}_0"'] & C_i \arrow[dr, "\phi"] & \\ & & Y, \end{tikzcd}$$

sends ρ to a simplicial map $\phi \circ \rho : \Delta^1 \times \Delta^n \to Y$. So we define $\tilde{\eta}(\rho) = \phi \circ \rho$. $\square$

Lemmas 4.1 and 4.2 imply that

$$\begin{tikzcd} & S_h(0,1) \arrow[dl, "d_1^h"'] \arrow[dr, "d_0^h"] & \\ X \arrow[rr, "i"] & & Y \end{tikzcd}$$

maps to a commutative square

$$
\begin{tikzcd}
\mathrm{H}_q S_h(0,1) \arrow[r, "\mathrm{H}_q d_0^h"] \arrow[d, "\simeq"'] & \mathrm{H}_q Y \arrow[d, equal] \\
\mathrm{H}_q X \arrow[r, "\mathrm{H}_q i"] & \mathrm{H}_q Y
\end{tikzcd}
$$

under the qth homology functor.

Let $\mathbb{X}\colon X_0 \overset{i_1}{\hookrightarrow} X_1 \overset{i_2}{\hookrightarrow} \cdots \overset{i_n}{\hookrightarrow} X_n$ be a finite filtration of simplicial sets. It is common practice to define the persistence module

$$\mathrm{H}_q\mathbb{X}\colon \mathrm{H}_qX_0 \xrightarrow{\mathrm{H}_q i_1} \mathrm{H}_qX_1 \xrightarrow{\mathrm{H}_q i_2} \cdots \xrightarrow{\mathrm{H}_q i_n} \mathrm{H}_qX_n$$

for all $q \geq 0$ in the theory of persistent homology. Surely, there is no topological significance in adding equalities. Hence, we prefer to work with the associated degenerate zigzag module

$$\mathrm{H}_q\mathbb{X}\colon \mathrm{H}_qX_0 \xleftarrow{=} \mathrm{H}_qX_0 \xrightarrow{\mathrm{H}_q i_1} \mathrm{H}_qX_1 \xleftarrow{=} \mathrm{H}_qX_1 \xrightarrow{\mathrm{H}_q i_2} \cdots \mathrm{H}_qX_{n-1} \xleftarrow{=} \mathrm{H}_qX_{n-1} \xrightarrow{\mathrm{H}_q i_n} \mathrm{H}_qX_n$$

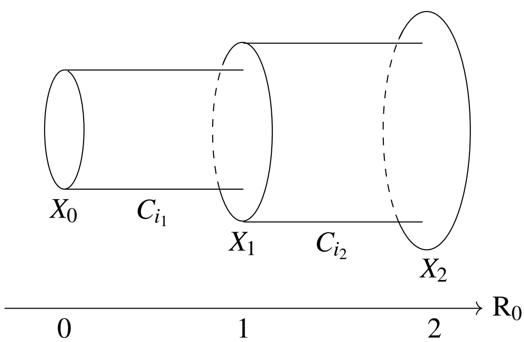
From $\mathbb{X}$ we may glue together mapping cylinders to obtain the iterated mapping cylinder:

$$C_{\mathbb{X}} = C_{i_1} \coprod_{X_1} C_{i_2} \coprod_{X_2} \cdots \coprod_{X_{n-1}} C_{i_n}.$$

It is the pushout of

$$
\begin{tikzcd}
 & C_{i_1} & & & C_{i_n} & \\
X_0 \arrow[ur, hook] & & X_1 \arrow[ul, hook'] & \cdots\; X_{n-1} \arrow[ur, hook] & & X_n. \arrow[ul, hook']
\end{tikzcd}
$$

If we modify the height function $h_{i_j}\colon C_{i_j} \to \mathrm{R}$ to meet the heights $j-1 < j$, we obtain an evident height function $h_{\mathbb{X}}$ on $C_{\mathbb{X}}$. Here is an illustration for $n = 2$:



Observe how $C_{\mathbb{X}}$ is automatically subdivided according to $h_{\mathbb{X}}$. Hence, the associated qth Reeb complex $\mathcal{G}_q$ defines a zigzag module $\mathbb{G}_q$:

$$\begin{tikzcd}
\mathrm{H}_q X_0 & & \mathrm{H}_q X_1 & \cdots & \mathrm{H}_q X_{n-1} & & \mathrm{H}_q X_n \\
& \mathrm{H}_q \mathcal{S}_{h_{\mathbb{X}}}[0,1] \arrow[ul, "\mathrm{H}_q d_1^h"] \arrow[ur, "\mathrm{H}_q d_0^h"'] & & & & \mathrm{H}_q \mathcal{S}_{h_{\mathbb{X}}}[n-1,n] \arrow[ul, "\mathrm{H}_q d_1^h"] \arrow[ur, "\mathrm{H}_q d_0^h"'] &
\end{tikzcd}$$

whose leftward pointing linear maps are all isomorphisms (Lemma 4.1).

Proposition 4.3 (Theorem 1.3)**.** For any finite filtration $\mathbb{X}$ and $q \geq 0$, the associated zigzag modules $\mathbb{G}_q$ and $\mathrm{H}_q\mathbb{X}$ are isomorphic.

Proof. Lemma 4.2 provides a zigzag module map from $\mathbb{G}_q$ to $\mathrm{H}_q\mathbb{X}$ via the commutative ladder

$$\begin{tikzcd}
\mathrm{H}_q X_0 \arrow[d, equal] & \mathrm{H}_q \mathcal{S}_{h_{\mathbb{X}}}[0,1] \arrow[l, "\mathrm{H}_q d_1^h"'] \arrow[r, "\mathrm{H}_q d_0^h"] \arrow[d, "\mathrm{H}_q d_1^h"] & \mathrm{H}_q X_1 \arrow[d, equal] & \mathrm{H}_q \mathcal{S}_{h_{\mathbb{X}}}[1,2] \arrow[l, "\mathrm{H}_q d_1^h"'] \arrow[r, "\mathrm{H}_q d_0^h"] \arrow[d, "\mathrm{H}_q d_1^h"] & \cdots \arrow[r, "\mathrm{H}_q d_0^h"] & \mathrm{H}_q X_n \arrow[d, equal] \\
\mathrm{H}_q X_0 \arrow[r, equal] & \mathrm{H}_q X_0 \arrow[r, "\mathrm{H}_q i_1"] & \mathrm{H}_q X_1 \arrow[r, equal] & \mathrm{H}_q X_1 \arrow[r, "\mathrm{H}_q i_2"] & \cdots \arrow[r, "\mathrm{H}_q i_n"] & \mathrm{H}_q X_n
\end{tikzcd}$$

Every vertical linear map is an isomorphism due to Lemma 4.1. □

Corollary 4.4. The barcodes of $\mathbb{G}_q$ and $\mathrm{H}_q\mathbb{X}$ are the same.

Proof. In the classification of representations of A_n–type quivers from [Gab72], isomorphic representations have the same decomposition into interval modules and thus the same barcodes. The result now follows from Proposition 4.3. □

4.3 Connection to level-set zigzag

Consider a real-valued function $f\colon T \to \mathbb{R}$ on a topological space T together with a sequence of real numbers $a_0 < \cdots < a_n$. Considering preimages under f and using inclusions gives a diagram $\mathbb{T}$ of topological spaces

$$\begin{tikzcd}
& f^{-1}[a_0,a_1] & & & & f^{-1}[a_{n-1},a_n] & \\
f^{-1}a_0 \arrow[ur] & & f^{-1}a_1 \arrow[ul] & \cdots & f^{-1}a_{n-1} \arrow[ur] & & f^{-1}a_n \arrow[ul]
\end{tikzcd}$$

Applying homology produces zigzag modules $\mathrm{H}_q\mathbb{T}$, called *level-set zigzags* [CdSM09].

Given a simplicial set X with a height function $h : X \to R$, adhering to the assumptions at the end of Section 4.1, we can define an analogous diagram, $\mathbb{X}$, of simplicial inclusions and compute level-set zigzags $\mathrm{H}_q\mathbb{X}$:

$$\begin{tikzcd}[column sep=small]
 & \mathrm{H}_q h^{-1}[a_0,a_1] & & & & \mathrm{H}_q h^{-1}[a_{n-1},a_n] & \\
\mathrm{H}_q h^{-1}a_0 \arrow[ur] & & \mathrm{H}_q h^{-1}a_1 \arrow[ul] & \cdots & \mathrm{H}_q h^{-1}a_{n-1} \arrow[ur] & & \mathrm{H}_q h^{-1}a_n \arrow[ul]
\end{tikzcd}$$

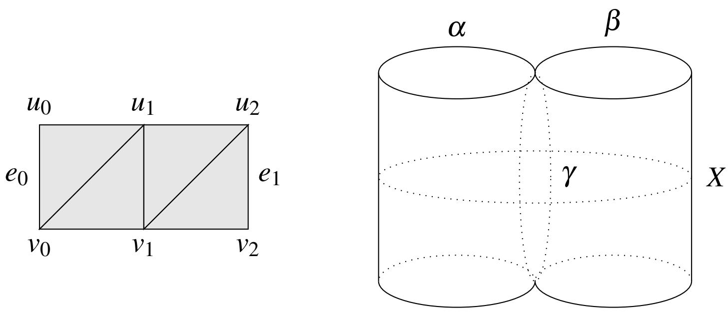
Compare $\mathrm{H}_q\mathbb{X}$ with the diagram $\mathbb{G}_q$ given at the end in Section 4.1. There are two immediate differences between the zigzag modules:

1) $\mathcal{S}_h[a_{i-1},a_i]$ and $h^{-1}[a_{i-1},a_i]$ are different spaces in general and

2) the arrows in $\mathbb{G}_q$ are reversed compared to $\mathrm{H}_q\mathbb{X}$.

The following example illuminates some of these differences:

Example 4.5. Identify e_0 and e_1 in the left-hand simplicial set



to obtain a cylinder. Pinch together the top circle by identifying $u_0 = u_2$ with u_1. Similarly, we pinch together the bottom circle. From this, we obtain the simplicial set X depicted on the right-hand side above. Let h be the height function mapping the bottom two circles to zero and the top two circles to one. Proceed as in Examples 3.3 and 3.5 to compute $\mathbb{G}_0$ and $\mathbb{G}_1$ in coordinates:

$$\begin{tikzcd}
k & & k & & k^2 & & k^2 \\
 & k \arrow[ul, "-1"'] \arrow[ur, "1"] & & & & k \arrow[ul, "\begin{bmatrix}-1\\-1\end{bmatrix}"'] \arrow[ur, "\begin{bmatrix}1\\1\end{bmatrix}"] &
\end{tikzcd}$$

The preimage $h^{-1}(0,1) = X$ deformation retracts onto the two horizontal circles α, β and the vertical circle γ depicted above. Pick these three circles as generators in H_1 to calculate $\mathrm{H}_0\mathbb{X}$ and $\mathrm{H}_1\mathbb{X}$ in coordinates:

$$\begin{tikzcd} & k & \\ k \arrow[ur, "1"] & & k \arrow[ul, "1"'] \end{tikzcd} \qquad \begin{tikzcd} & k^3 & \\ k^2 \arrow[ur, "{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}}"] & & k^2 \arrow[ul, "{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}"'] \end{tikzcd}$$

Note the difference between the zigzag modules $\mathbb{G}_1$ and $\mathrm{H}_1\mathbb{X}$. However, taking direct sums across the middle rows in concatenated diamonds

$$\begin{tikzcd} & k & \\ k \arrow[ur] & & k \arrow[ul] \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} & k^3 & \\ k^2 \arrow[ur] & & k^2 \arrow[ul] \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$

results in short exact sequences

$$k \xrightarrow{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} k^2 \xrightarrow{\begin{bmatrix} 1 & 1 \end{bmatrix}} k \qquad k \xrightarrow{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}} k^4 \xrightarrow{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix}} k^3$$

This means that, in this particular example, we can translate between the barcodes of $\mathrm{H}_q\mathbb{X}$ and $\mathbb{G}_q$ via the diamond principle in [CdS10].

We do not get a short exact sequence in general, but the sequence formed in the above example is always exact in the middle term.

Proposition 4.6 (Theorem 1.4)**.** Consider a height function $h\colon X \to \mathrm{R}$ for which

i) X is subdivided according to h and

ii) the image of 0–simplices, $h(X_0)$, is discrete as a subset of the real numbers.

Then for every pair of successive critical values $a < b$ the sequence

$$\mathrm{H}_q\mathcal{S}_h[a,b] \to \mathrm{H}_q h^{-1}a \oplus \mathrm{H}_q h^{-1}b \to \mathrm{H}_q h^{-1}[a,b]$$

is exact at the middle term.

Proof. Proposition 3.14 implies the existence of a commutative ladder

$$\begin{array}{ccccc} \mathrm{H}_q\mathcal{S}_h[a,b] & \longrightarrow & \mathrm{H}_q h^{-1}a \oplus \mathrm{H}_q h^{-1}b & \longrightarrow & \mathrm{H}_q h^{-1}[a,b] \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}_q\mathrm{Sect}_f[a,b] & \longrightarrow & \mathrm{H}_q f^{-1}a \oplus \mathrm{H}_q f^{-1}b & \longrightarrow & \mathrm{H}_q f^{-1}[a,b] \end{array}$$

where $f\colon |X| \to \mathrm{R}$ is induced from h and all vertical arrows are isomorphisms. Evaluation at $\frac{a+b}{2}$ defines a homotopy equivalence $\mathrm{Sect}_f[a,b] \to f^{-1}(\frac{a+b}{2})$, see Proposition 3.10 in [Try21]. The homotopy inverse is given by associating canonical sections/flow-lines to points in the intermediate fiber $f^{-1}(\frac{a+b}{2})$ and defines a map $f^{-1}(\frac{a+b}{2}) \to f^{-1}a \coprod f^{-1}b$ which results in a commutative ladder

$$\begin{array}{ccccc} \mathrm{H}_q\mathrm{Sect}_f[a,b] & \longrightarrow & \mathrm{H}_q f^{-1}a \oplus \mathrm{H}_q f^{-1}b & \longrightarrow & \mathrm{H}_q f^{-1}[a,b] \\ \downarrow & & \| & & \| \\ \mathrm{H}_q f^{-1}(\frac{a+b}{2}) & \longrightarrow & \mathrm{H}_q f^{-1}a \oplus \mathrm{H}_q f^{-1}b & \longrightarrow & \mathrm{H}_q f^{-1}[a,b] \end{array}$$

in which the vertical arrows are all isomorphisms. Let U_a and U_b be open neighborhoods of a and b, respectively, in $\mathbb{R}$. We can safely assume that U_a only contains a from $h(X_0)$, similarly U_b only contains b from $h(X_0)$, and that the union $U_a \cup U_b$ contains $f^{-1}[a,b]$. Inclusions define

$$\begin{array}{ccccc} \mathrm{H}_q f^{-1}(\frac{a+b}{2}) & \longrightarrow & \mathrm{H}_q f^{-1}a \oplus \mathrm{H}_q f^{-1}b & \longrightarrow & \mathrm{H}_q f^{-1}[a,b] \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}_q f^{-1}(f^{-1}U_a \cap f^{-1}U_b) & \longrightarrow & \mathrm{H}_q f^{-1}U_a \oplus \mathrm{H}_q f^{-1}u_b b & \longrightarrow & \mathrm{H}_q f^{-1}U_a \cup f^{-1}U_b \end{array}$$

The vertical arrows are isomorphisms due to Lemma 2.8 in [Try21]. We recognize the final row as part of the well-known Mayer-Vietoris sequence which is exact. Compose the given ladders to finish the proof. □

This final result allows for a method to translate between the barcodes of $\mathrm{H}_q\mathbb{X}$ (level-set zigzag) and $\mathbb{G}_q$ (coming from the q–th Reeb complex) via Theorem 5.6 in [Car09].

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