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Project Thesis

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Abstract

In this paper we first discuss concepts of efficiency for uncertain multi-objective optimization problems by using different set order relations. In all we discuss four different relations: the upper set less order relation introduced by Kuroiwa [1], the lower set less order relation introduced by Kuroiwa [1], the set less order relation introduced by Young [2] and the strict set less order relation which was introduced as alternative set less order relation by Ide et al. [3]. We then discuss the different characteristics of these order relations and the differences between them. Secondly, we look at two methods that can be used to solve multi-objective optimization problems where the feasible set is nonconvex and even disconnected, Weighted-constraint method introduced by Burachik et al. [4] and Pascoletti-Serafini method introduced by Pascoletti et al. [5]. In further work, we want to see if these two methods can be used to find all efficient solutions associated with the different set order relations we have introduced in this paper even when the feasible sets is set in such a way that methods such as Weighted sum scalarization and ϵ-constraint method, approaches discussed in this paper, fails to obtain them all.

Contents

Chapter 1

Introduction

The main topic we are setting our focus on in this paper is uncertain multi-objective optimization. Uncertainty is something that is common in the real-world when we have optimization problems. When we are making decision between the different solutions we are not always certain about how the future will unfold, but it still has influence on the decision we are making. Other reasons for uncertainty in the problem can be faulty data due to errors. When we are faced with uncertainty, the literature have suggested two main approaches - either stochastic optimization where the uncertain parameter(s) is assumed to posses a probability distribution. We then optimize the expected value of our cost functions which are defined by the objectives while we have other solutions which are still possible with some probability. The other approach is the one we are going to focus on, namely robust optimization. In this scenario, we consider the case where no stochastic information about the uncertain parameter(s) is given. There is different variations of what a robust solution is. One of the concepts that have been discussed is minmax robustness which was firstly introduced by Soyster [6]. With this approach, the goal is to find solutions which are feasible for every future scenario. Hence, the objective becomes to optimize on the worst-case scenario for each solution. What we want to accomplish with finding robust solutions is to have solutions that are less sensitive to perturbations in the data. If we for instance as a company choose a solution which earns the company a lot of money if everything goes according to plan but makes the company go bankrupt if something suddenly goes a bit wrong, then this is a very sensitive solution and is hence not a very robust solution as it becomes non-feasible for some scenarios. This type of choices are the ones we try to eliminate with optimizing on the worst-case scenario.

This becomes a set-based method because we then assume that the uncertain parameter(s) belong to an uncertainty set that is known prior to the solving of the optimization problem.

In this paper we are going to discuss this further, namely to use a set order relation to define the robust solutions of uncertain multi-objective optimization problems. In the example when we optimize on the worst-case scenario it can be said that this is a set order relation, named in the literature as upper set less order relation [1], with a pessimistic approach as we hedge against the scenarios with the worst outcomes. In addition to this one, we are going to discuss other set order relations to better define different approaches a decision-maker can have, for instance an optimistic approach. Throughout the discussions on these set order relations, we are going to look at uncertain multi-objective problems where only the objective functions are affected by uncertain data which is given by an arbitrary uncertainty set $\mathcal{U}$. Within this set, all possible scenarios of the uncertain input data is represented. With this, we want to show that by using the different definitions of set order relations we introduce it is able to end up with different set of efficient solutions for the same uncertain multi-objective optimization problem. For each set order relation, we will discuss how it affects the set of efficient solutions and also show methods that can be used to find the different efficient solutions.

It is however easy to construct uncertain multi-objective optimization problems that has characteristics which can make it impossible for methods such as Weighted sum scalarization and ϵ-constraint method to obtain all the different efficient solutions given the different set order relations. Examples of this is non-convexity or discontinuity in the feasible sets which we are going to look at in Chapter 3. To deal with this, we are going to look at two other methods which have been used in multi-objective optimization to solve problems where the sets have these characteristics, namely Weighted-constraint method [4] and Pascoletti-Serafini method [5].

Chapter 2

Preliminaries

Firstly we need to define some notation on multi-objective optimization. Given a feasible set $\mathcal{X} \subseteq \mathbb{R}^n$ defined by some constraints, we want to minimize a function $f : \mathcal{X} \to \mathbb{R}^k$. We can write it more formally as

$$\begin{aligned} \textbf{min} \quad & f(x) \\ \textbf{s.t.} \quad & x \in \mathcal{X}. \end{aligned} \tag{$\mathcal{P}$}$$

Due to the fact that we are comparing solutions in $\mathbb{R}^k$, it is necessary to define relations to compare them as we lack total order. To do this, we use the relations $\{\leqq, \leq, <\}$, referred to in Ehrgott [7]. Let $\{y_1, y_2\} \in \mathbb{R}^k$, then we say that

$$\begin{aligned} y_1 \leqq y_2 &\iff y_2^i \in [y_1^i, \infty) \; \forall \; i \in 1, \ldots, k \\ y_1 \leq y_2 &\iff y_1 \leqq y_2, y_1 \neq y_2 \\ y_1 < y_2 &\iff y_2^i \in (y_1^i, \infty) \; \forall \; i \in 1, \ldots, k. \end{aligned}$$

Furthermore, we define the ordering cones $\left\{\mathbb{R}^k_{\geqq}, \mathbb{R}^k_{\geq}, \mathbb{R}^k_{>}\right\}$ as

$$\mathbb{R}^k_{[\geqq, \geq, >]} := \left\{ x \in \mathbb{R}^k : x[\geqq, \geq, >]0 \right\}.$$

With this ordering, we want to find all feasible solutions $x \in \mathcal{X}$ to ($\mathcal{P}$) that are [*strictly*/ $\cdot$ / *weakly*] *efficient*, which means that its function value, $f(x)$, is not dominated by any other function value, $f(\hat{x})$, from a point $\hat{x} \in \mathcal{X} \setminus \{x\}$. We can write this as

$$x \text{ is [strictly/}\cdot\text{/weakly] efficient} \iff \nexists \hat{x} \in \mathcal{X} \setminus \{x\} : f(\hat{x}) \in f(x) - \mathbb{R}^k_{[\geqq, \geq, >]}.$$

Remark 2.1. A strictly efficient point is also an efficient point. An efficient point is also a weakly efficient point, hence

strictly efficient $\implies$ efficient $\implies$ weakly efficient.

Given the tools already presented, we want to define the uncertain counterpart for multi-objective optimization. Given a set of scenarios $\mathcal{U} \subseteq \mathbb{R}^m$, also referred to as the uncertainty set, an uncertain multi-objective optimization problem is given as the family $(\mathcal{P}(\xi),\ \xi \in \mathcal{U})$ of multi-objective optimization problems

$$\begin{aligned} \textbf{min} \quad & f(x,\xi) \\ \textbf{s.t.} \quad & x \in \mathcal{X}, \end{aligned} \qquad \mathcal{P}(\xi)$$

with objective function $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^k$, a feasible set $\mathcal{X} \subseteq \mathbb{R}^n$ and $\xi \in \mathcal{U}$ to represent one particular scenario of the uncertainty set. From this framework, it is clear that we need to extend our definitions in order to define what an efficient solution is. The reason for this is that uncertain optimization by our definition is a family of problems where the object changes by each scenario $\xi \in \mathcal{U}$.

This gives motivation for defining the set

$$f_{\mathcal{U}}(x) := \{f(x,\xi) : \xi \in \mathcal{U}\} \subseteq \mathbb{R}^k$$

which is the set of all possible objective values for a point $x \in \mathcal{X}$ given each $\xi \in \mathcal{U}$. We now need to find out how to turn the family of uncertain values for each feasible point into a deterministic optimization formulation. One way to do this is to define different set order relations to define what property a feasible set $f_{\mathcal{U}}(x)$ needs to have in order to dominate another feasible set $f_{\mathcal{U}}(\hat{x})$ given all feasible points from $x \in \mathcal{X}$ and $\hat{x} \in \mathcal{X} \setminus \{x\}$.

Chapter 3

Definitions of efficiency based on set order relations

In this section, we want to introduce different set order relations as a way of defining efficient solutions in uncertain multi-objective optimization problems based on various approaches. We will also discuss how this affects the properties affiliated with the solutions that the different relations give us. To guide us through the different set order relations in the chapter, we use Example 3.1.

Example 3.1. Figure 3.1 illustrates an uncertain multi-objective optimization problem for $k = 2$, hence with $\mathbb{R}^2$ as ordering cone.

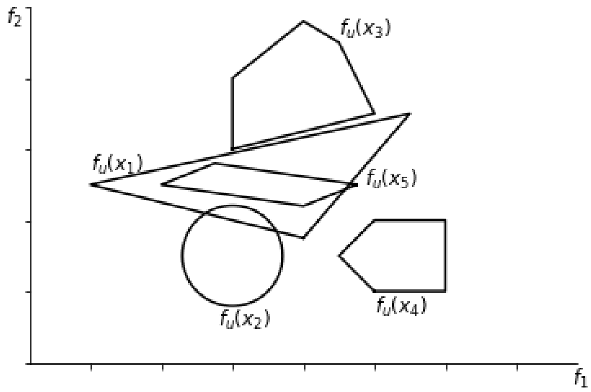

Figure 3.1: Uncertain multi-objective optimization problem with feasible set $\mathcal{X} = \{x_1, x_2, x_3, x_4, x_5\}$.

3.1 Upper set less order relation

3.1.1 Description and problem formulation

The first set order relation we want to define, introduced by Kuroiwa [1], is the upper set less order relation:

Definition 3.1. A set $A \subseteq \mathbb{R}^k$ *dominates* a set $B \subseteq \mathbb{R}^k$ with respect to the upper set less order relation, denoted $A \preceq^u_{[\geqq,\geq,>]} B$, with respect to $\mathbb{R}^k_{[\geqq,\geq,>]}$ if

$$A \preceq^u_{[\geqq,\geq,>]} B \iff A \subseteq B - \mathbb{R}^k_{[\geqq,\geq,>]}.$$

Remark 3.1. The relation can be equivalently written as

$$A \preceq^u_{[\geqq,\geq,>]} B \iff \forall a \in A \exists b \in B : a[\leqq,\leq,<]b.$$

The way to frame $\mathcal{P}(\xi)$ to mirror this property is to take the supremum of the uncertainty set

$$\min_{x \in \mathcal{X}} \sup_{\xi \in \mathcal{U}} \quad f(x,\xi). \qquad \mathcal{P}(\xi)^{\mathrm{u}}$$

3.1.2 Efficiency and interpretation

Definition 3.2. Given an uncertain multi-objective optimization problem $\mathcal{P}(\xi)$, a solution $x \in \mathcal{X}$ to $\mathcal{P}(\xi)$ is *upper set less ordered* [*strictly*/ $\cdot$ / *weakly*] *efficient* if there is no $\bar{x} \in \mathcal{X} \setminus \{x\}$ s.t. $f_{\mathcal{U}}(\bar{x}) \preceq^u_{[\geqq,\geq,>]} f_{\mathcal{U}}(x)$ with respect to $\mathbb{R}^k_{[\geqq,\geq,>]}$, or equivalently written

$$\nexists \bar{x} \in \mathcal{X} \setminus \{x\} : f_{\mathcal{U}}(\bar{x}) \subseteq f_{\mathcal{U}}(x) - \mathbb{R}^k_{[\geqq,\geq,>]}.$$

The way to interpret this set order relation is that a solution set $f_{\mathcal{U}}(x)$ is efficient if there does not exist another solution set $f_{\mathcal{U}}(\bar{x})$ such that the worst case scenario for $\bar{x}$ is better than the worst case scenario for x for the given problem. We can look at this set ordering as pessimistic as the efficient solutions to the problem given the feasible sets $f_{\mathcal{U}}(x), x \in \mathcal{X}$ will be chosen on the grounds of which ones have the greatest worst case scenario. Hence this approach can be used to find risk averse solutions. In our example, we can see that only $f_{\mathcal{U}}(x_2)$ and $f_{\mathcal{U}}(x_4)$ fulfill the criteria. Looking at Figure 3.3 we observe that $f_{\mathcal{U}}(x_1) - \mathbb{R}^2_{[\geqq,\geq,>]}$, $f_{\mathcal{U}}(x_3) - \mathbb{R}^2_{[\geqq,\geq,>]}$ and $f_{\mathcal{U}}(x_5) - \mathbb{R}^2_{[\geqq,\geq,>]}$ contains $f_{\mathcal{U}}(x_2)$. Hence, only x_2 and x_4 are upper set less ordered strictly efficient.

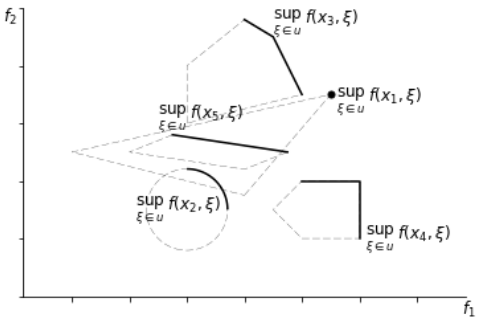

Figure 3.2: Supremum of every feasible set.

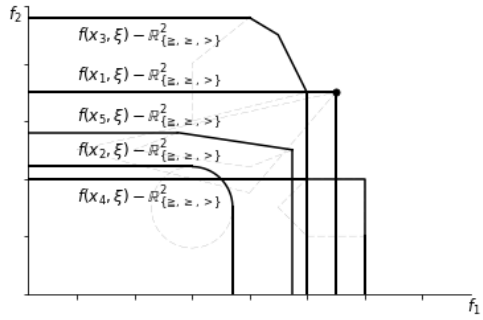

Figure 3.3: All five sets when we subtract $\mathbb{R}^2_{[\geqq,\geq,>]}$.

3.1.3 Computing upper set less ordered efficient solutions

We can use approaches from deterministic multi-objective optimization to compute upper set less ordered efficient solutions by extending their framework from deterministic to uncertain.

Weighted sum scalarization

The idea with this method is to form a single objective optimization problem by multiplying each of the objective functions by some non-negative weight λ_i and sum them together. So with a weight vector $\lambda \in \mathbb{R}^k_{[\geq,>]}$, we consider

$$\min_{x\in\mathcal{X}} \sum_{i=1}^{k} \lambda_i f_i(x) \qquad \mathcal{P}_\lambda$$

The way we extend the framework to use this method to compute upper set less ordered efficient solutions is to insert the problem formulation obtained in 3.1.1, $\mathcal{P}(\xi)^{\mathrm{u}}$:

$$\min_{x\in\mathcal{X}} \sup_{\xi\in\mathcal{U}} \sum_{i=1}^{k} \lambda_i f_i(x,\xi) \qquad \mathcal{P}(\xi)^{\mathrm{u}}_{\lambda}$$

Theorem 3.1. ***(Theorem 4.3. Ehrgott et al. [8])*** *Given an uncertain multi-objective optimization problem $\mathcal{P}(\xi)$, the following statements hold.*

1. *If $\hat{x} \in \mathcal{X}$ is the unique optimal solution to $\mathcal{P}(\xi)^u_\lambda$ for some $\lambda \in \mathbb{R}^k_{\geq}$, then $\hat{x}$ is upper set less ordered strictly efficient solution to $\mathcal{P}(\xi)$.*
2. *If $\hat{x} \in \mathcal{X}$ is an optimal solution to $\mathcal{P}(\xi)^u_\lambda$ for some $\lambda \in \mathbb{R}^k_{\{>,\geq\}}$ and*

$$\max_{\xi \in \mathcal{U}} \sum_{i=1}^k \lambda_i f_i(x,\xi)$$

exists for all $x \in \mathcal{X}$, then $\hat{x}$ is upper set less ordered [·/weakly] efficient solution to $\mathcal{P}(\xi)$.

Proof. 1. Assume $\hat{x}$ is not upper set less ordered [strictly/·/weakly] efficient for $\mathcal{P}(\xi)$. Then there exists an $x' \in \mathcal{X}$ such that

$$f_{\mathcal{U}}(x') \subseteq f_{\mathcal{U}}(\hat{x}) - \mathbb{R}^k_{[\geqq,\geq,>]} \implies \forall \xi \in \mathcal{U} \exists \eta \in \mathcal{U} : f(x',\xi)[\leqq,\leq,<] f(\hat{x},\eta)$$

Now choose $\lambda \in \mathbb{R}^k_{\{\geq,>,\geq\}}$ arbitrary but fixed.

$$\begin{aligned}
&\implies \forall \xi \in \mathcal{U} \exists \eta \in \mathcal{U} : \sum_{i=1}^k \lambda_i f(x',\xi)[\leqq,\leq,<] \sum_{i=1}^k \lambda_i f(\hat{x},\eta) \\
&\iff \forall \xi \in \mathcal{U} : \sum_{i=1}^k \lambda_i f(x',\xi)[\leqq,\leq,<] \sup_{\eta' \in \mathcal{U}} \sum_{i=1}^k \lambda_i f(\hat{x},\eta') \\
&\iff \sup_{\xi' \in \mathcal{U}} \sum_{i=1}^k \lambda_i f(x',\xi')[\leqq,\leq,<] \sup_{\eta' \in \mathcal{U}} \sum_{i=1}^k \lambda_i f(\hat{x},\eta')
\end{aligned}$$

The last equivalence holds because for 2. because

$$\max_{\xi' \in \mathcal{U}} \sum_{i=1}^k \lambda_i f(x',\xi')$$

exists. However, this means that $\hat{x}$ is not [the unique/an/an] optimal solution to $\mathcal{P}(\xi)^{\mathrm{u}}_\lambda$ for $\lambda \in \mathbb{R}^k_{\{\geq,>,\geq\}}$. □

Given a set of scalarization vectors Λ, we can now compute upper set less ordered efficient solutions by solving $\mathcal{P}(\xi)^{\mathrm{u}}_\lambda$ for every $\lambda \in \Lambda$. One challenge with the method is the choice of Λ. On the other hand, the technique does not add any additional constraints to the problem formulation and thus preserves the problem structure.

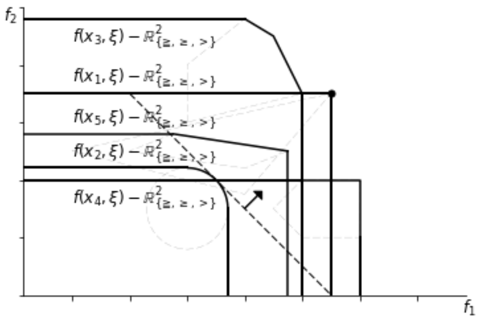

Figure 3.4: Examples of weights to find x_2 as upper set less strictly efficient. Here $\lambda = [1/2, 1/2]$.

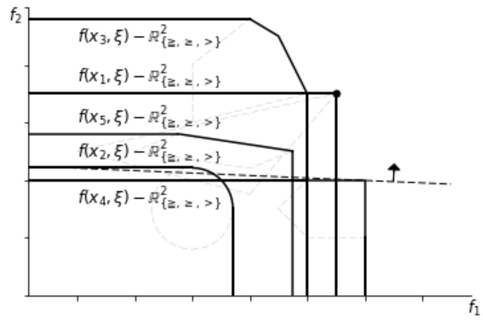

Figure 3.5: Examples of weights to find x_4 as upper set less strictly efficient. Here $\lambda = [3/73, 70/73]$.

ϵ-constraint scalarization

This approach uses the idea of minimizing one of the objective functions, the i^{th} objective function, while the others are less than a value $\epsilon_j, j \neq i$. By doing this for every value from $i \in \{1,...,k\}$ and $\epsilon \in \mathbb{R}^k_{\geqq}$, we consider the problem formulation

$$\begin{aligned} \min_{x \in \mathcal{X}} \quad & f_i(x) \\ \textbf{s.t.} \quad & f_i(x) \leq \epsilon_j \forall j \neq i. \end{aligned} \qquad \mathcal{P}_{(\epsilon,i)}$$

The way we extend the framework to use this method to compute upper less ordered efficient solutions is to insert the problem formulation obtained in 3.1.1, $\mathcal{P}(\xi)^{\mathrm{u}}$:

$$\begin{aligned} \min_{x \in \mathcal{X}} \sup_{\xi \in \mathcal{U}} \quad & f_i(x,\xi) \\ \textbf{s.t.} \sup_{\xi \in \mathcal{U}} \quad & f_i(x,\xi) \leq \epsilon_j \forall j \neq i. \end{aligned} \qquad \mathcal{P}(\xi)^{\mathrm{u}}_{(\epsilon,i)}$$

Theorem 3.2. ***(Theorem 4.7. Ehrgott et al. [8])*** *Given an uncertain multi-objective optimization problem $\mathcal{P}(\xi)$, the following statements hold.*

1. *If $\hat{x} \in \mathcal{X}$ is the unique optimal solution to $\mathcal{P}(\xi)^u_{(\epsilon,i)}$ for some $\epsilon \in \mathbb{R}^k$ and some $i \in \{1,..,k\}$, then $\hat{x}$ is upper set less strictly efficient solution to $\mathcal{P}(\xi)$.*

2. *If $\hat{x} \in \mathcal{X}$ is an optimal solution to $\mathcal{P}(\xi)^u_{(\epsilon,i)}$ for some $\epsilon \in \mathbb{R}^k$ and some $i \in \{1,..,k\}$ and*

$$\max_{\xi \in \mathcal{U}} f_i(x,\xi)$$

exists for all $x \in \mathcal{X}$, then $\hat{x}$ is upper set less weakly efficient solution to $\mathcal{P}(\xi)$.

Proof. 1. Assume $\hat{x}$ is not upper set less ordered strictly efficient for $\mathcal{P}(\xi)$. Then there exists an $x' \in \mathcal{X}$ such that

$$\begin{aligned} f_{\mathcal{U}}(x') \subseteq f_{\mathcal{U}}(\hat{x}) - \mathbb{R}^k_{\geqq} &\Longrightarrow \forall \xi \in \mathcal{U} \exists \eta \in \mathcal{U} : f(x',\xi) \leqq f(\hat{x},\eta) \\ &\Longrightarrow \sup_{\xi' \in \mathcal{U}} f(x',\xi') \leqq \sup_{\eta' \in \mathcal{U}} f(\hat{x},\eta') \text{ and} \\ &\quad\ \forall \xi \in \mathcal{U} \exists \eta \in \mathcal{U} : f_j(x',\xi) \leqq f_j(\hat{x},\eta) \leqq \epsilon_j, j \neq i. \end{aligned}$$

In this scenario, x' is feasible for $\mathcal{P}(\xi)^{\mathrm{u}}_{(\epsilon,i)}$ and has an equal of better objective value than $\hat{x}$. This is a contradiction to the assumption that $\hat{x}$ is the unique optimal solution to $\mathcal{P}(\xi)^{\mathrm{u}}_{(\epsilon,i)}$.

2. Assume $\hat{x}$ is not upper set less ordered weakly efficient for $\mathcal{P}(\xi)$. Then there exists an $x' \in \mathcal{X}$ such that

$$\begin{aligned} f_{\mathcal{U}}(x') \subseteq f_{\mathcal{U}}(\hat{x}) - \mathbb{R}^k_{>} &\Longrightarrow \forall \xi \in \mathcal{U} \exists \eta \in \mathcal{U} : f(x',\xi) < f(\hat{x},\eta) \\ &\Longrightarrow \max_{\xi' \in \mathcal{U}} f(x',\xi') < \max_{\eta' \in \mathcal{U}} f(\hat{x},\eta') \text{ and} \\ &\quad\ \forall \xi \in \mathcal{U} \exists \eta \in \mathcal{U} : f_j(x',\xi) < f_j(\hat{x},\eta) \leqq \epsilon_j, j \neq i. \end{aligned}$$

In this scenario, x' is feasible for $\mathcal{P}(\xi)^{\mathrm{u}}_{(\epsilon,i)}$ and has an equal of better objective value than $\hat{x}$. This is a contradiction to the assumption that $\hat{x}$ is an optimal solution to $\mathcal{P}(\xi)^{\mathrm{u}}_{(\epsilon,i)}$. □

Given a set $\mathcal{E}$ of vectors $\epsilon \in \mathbb{R}^k_{\geqq}$, we can now compute upper set less ordered efficient solutions by solving $\mathcal{P}(\xi)^{\mathrm{u}}_{(\epsilon,i)}$ for each $i \in \{1,\ldots,k\}$ and every $\epsilon \in \mathcal{E}$. One challenge with the method is to choose the set $\mathcal{E}$ correctly. If the elements in $\mathcal{E}$ are chosen too small, then the set of feasible solutions may be empty, but if the elements in $\mathcal{E}$ are chosen too large, then the optimality of the functions representing the constraints decreases.

Remark 3.2. ϵ-constraint method is able to find all the efficient solutions to a deterministic multi-objective optimization problem, but this is not necessarily the case for uncertain multi-objective optimization problems - even when the sets are convex. To illustrate this, we can look at Example 3.2.

Another problem with this approach lies in the altered problem structure. Since the problem structure of the original problem is not preserved as constraints are added to the problem, it may further complicate the decision process.

Example 3.2. Figure 3.6 shows an uncertain multi-objective optimization problem for $k = 2$ with both x_1 and x_2 as upper set less strictly efficient solutions.

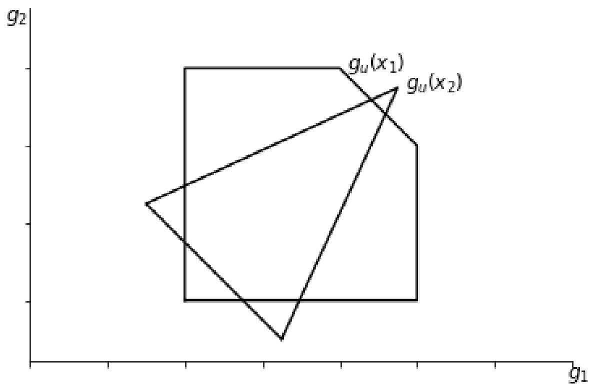

Figure 3.6: Uncertain multi-objective optimization problem with feasible set $\mathcal{X} = \{x_1, x_2\}$.

From what we have learned, we observe in Figure 3.6 that both x_1 and x_2 are upper set less strictly efficient. However, since x_2 has a lower supremum in both objective functions individually and is therefore feasible for whenever x_1 is. As shown in Figure 3.7, we can therefore not obtain x_1 as an upper set less strictly efficient solution with this method even though it is.

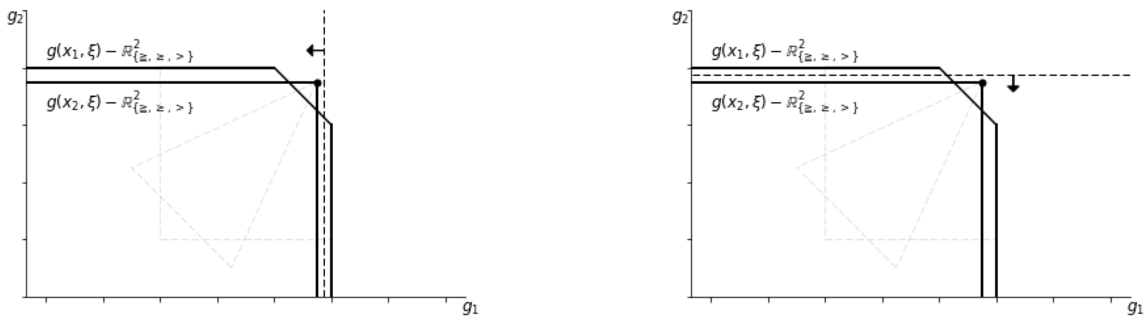

Figure 3.7: Illustration of how x_2 is always feasible for every value $\epsilon \in \mathcal{E}$ where x_1 is feasible for both objective functions. Hence it is not possible to obtain x_1 as an upper set less strictly efficient solution by using the ϵ-constraint scalarization in this example.

3.2 Lower set less order relation

3.2.1 Description and problem formulation

The second set order relation we are looking at is the lower set less order relation, first introduced by Kuroiwa [1].

Definition 3.3. A set $A \subseteq \mathbb{R}^k$ *dominates* a set $B \subseteq \mathbb{R}^k$ with respect to the lower set less order relation, denoted $A \preceq^l_{[\geqq,\geq,>]} B$, with respect to $\mathbb{R}^k_{[\geqq,\geq,>]}$ if

$$A \preceq^l_{[\geqq,\geq,>]} B \iff A + \mathbb{R}^k_{[\geqq,\geq,>]} \supseteq B.$$

Remark 3.3. The relation can be equivalently be written as

$$A \preceq^l_{[\geqq,\geq,>]} B \iff \forall b \in B \exists a \in A : a[\leqq,\leq,<]b.$$

This time, the way to frame the problem to obtain this property is to take the infimum on the uncertainty set

$$\min_{x \in \mathcal{X}} \inf_{\xi \in \mathcal{U}} \quad f(x,\xi). \qquad \mathcal{P}(\xi)^1$$

3.2.2 Efficiency and interpretation

Definition 3.4. Given an uncertain multi-objective optimization problem $\mathcal{P}(\xi)$, a solution $x \in \mathcal{X}$ to $\mathcal{P}$ is *lower set less ordered* [*strictly*/ · /*weakly*] *efficient* if there is no $\bar{x} \in \mathcal{X} \setminus \{x\}$ s.t. $f_{\mathcal{U}}(\bar{x}) \preceq^l_{[\geqq,\geq,>]} f_{\mathcal{U}}(x)$ with respect to $\mathbb{R}^k_{[\geqq,\geq,>]}$, or equivalently written

$$\nexists \bar{x} \in \mathcal{X} \setminus \{x\} : f_{\mathcal{U}}(\bar{x}) + \mathbb{R}^k_{[\geqq,\geq,>]} \supseteq f_{\mathcal{U}}(x).$$

When we examine the solution sets $f_{\mathcal{U}}(x)$ for this set order relation we observe that in order for a solution to be efficient, there can not exist another solution set $f_{\mathcal{U}}(\bar{x})$ such that the best case scenario for $\bar{x}$ is better than the best case scenario for x. Hence, we can interpret this ordering as optimistic as the efficient solutions to the problem given the feasible sets $f_{\mathcal{U}}(x), x \in \mathcal{X}$ will be evaluated based on greatest best case scenario. Because of this, it is possible to obtain solutions for situations where we are looking to be risk seeking. For our example, the risk seeking solutions are therefore $f_{\mathcal{U}}(x_1)$ and $f_{\mathcal{U}}(x_2)$. Conversely, as seen in Figure 3.9, $f_{\mathcal{U}}(x_3)$ and $f_{\mathcal{U}}(x_5)$ is contained in $f_{\mathcal{U}}(x_1) + \mathbb{R}^2_{[\geqq,\geq,>]}$ while $f_{\mathcal{U}}(x_3)$ and $f_{\mathcal{U}}(x_4)$ is contained in $f_{\mathcal{U}}(x_2) + \mathbb{R}^2_{[\geqq,\geq,>]}$. Hence, only x_1 and x_2 are lower set less ordered strictly efficient.

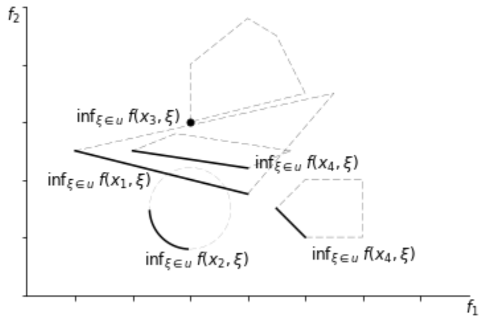

Figure 3.8: Infimum of every feasible set.

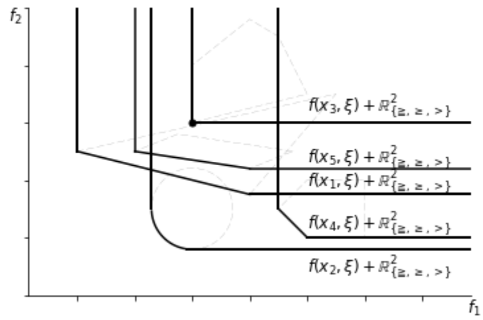

Figure 3.9: All five sets when we add $\mathbb{R}^2_{[\geqq,\geq,>]}$.

3.2.3 Computing lower set less ordered efficient solutions

To compute lower set less ordered efficient solutions we can use the same extension of framework as for upper set less ordered efficient solutions in 3.1.3.

Weighted sum scalarization

The way we extend the framework from $\mathcal{P}_\lambda$ to use this method for computing lower set less ordered efficient solutions is to insert the problem formulation obtained in 3.2.1, $\mathcal{P}(\xi)^{\mathrm{l}}$:

$$\min_{x\in\mathcal{X}} \inf_{\xi\in\mathcal{U}} \quad \sum_{i=1}^{k} \lambda_i f_i(x,\xi) \qquad \mathcal{P}(\xi)^{\mathrm{l}}_\lambda$$

As for the upper set less relation, if we are given a set of scalarization vectors Λ, we can now compute lower set less ordered efficient solutions by solving $\mathcal{P}(\xi)^{\mathrm{l}}_\lambda$ for every $\lambda \in \Lambda$.

Theorem 3.3. ***(Theorem 11 Ide et al.*** **[9]*)*** *Given an uncertain multi-objective optimization problem $\mathcal{P}(\xi)$, the following statements hold.*

1. *If $\hat{x} \in \mathcal{X}$ is the unique optimal solution to $\mathcal{P}(\xi)^{l}_\lambda$ for some $\lambda \in \mathbb{R}^k_{\geq}$, then $\hat{x}$ is lower set less ordered strictly efficient solution to $\mathcal{P}(\xi)$.*

2. *If $\hat{x} \in \mathcal{X}$ is an optimal solution to $\mathcal{P}(\xi)^l_\lambda$ for some $\lambda \in \mathbb{R}^k_{\{>,\geq\}}$ and*

$$\min_{\xi \in \mathcal{U}} \sum_{i=1}^{k} \lambda_i f_i(x,\xi)$$

exists for all $x \in \mathcal{X}$, then $\hat{x}$ is lower set less ordered [·/weakly] efficient solution to $\mathcal{P}(\xi)$.

Remark 3.4. To prove Theorem 3.3, we can use a proof with similar reasoning as for Theorem 3.1, only with the assumption that $\hat{x}$ is lower set less ordered [strictly/·/weakly] efficient.

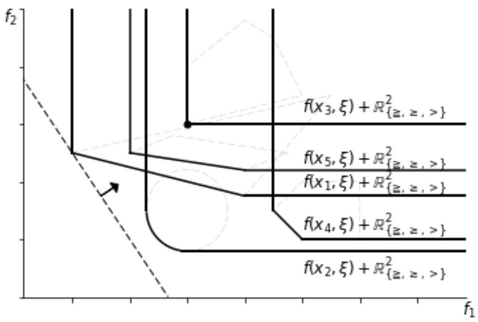

Figure 3.10: Examples of weights to find x_1 as lower set less strictly efficient. Here $\lambda = [2/5, 3/5]$.

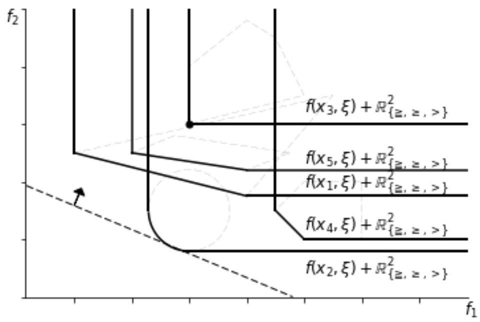

Figure 3.11: Examples of weights to find x_2 as lower set less strictly efficient. Here $\lambda = [5/8, 3/8]$.

ϵ-constraint scalarization

The way we extend the framework from $\mathcal{P}_{(\epsilon,i)}$ to use this method in order for computing lower set less ordered efficient solutions is to insert the problem formulation obtained in 3.2.1, $\mathcal{P}(\xi)^{\mathrm{l}}$:

$$\begin{aligned} \min_{x \in \mathcal{X}} \inf_{\xi \in \mathcal{U}} \quad & f_i(x,\xi) \\ \textbf{s.t.} \inf_{\xi \in \mathcal{U}} \quad & f_i(x,\xi) \leq \epsilon_j \forall j \neq i. \end{aligned} \qquad \mathcal{P}(\xi)^{\mathrm{l}}_{(\epsilon,i)}$$

Given a set of vectors $\mathcal{E}$, we can now compute lower set less ordered efficient solutions by solving $\mathcal{P}(\xi)^{\mathrm{l}}_{(\epsilon,i)}$ for each $i \in \{1,\ldots,k\}$ and every $\epsilon \in \mathcal{E}$.

Theorem 3.4. *Theorem 26 Köbis [10] Given an uncertain multi-objective optimization problem $\mathcal{P}(\xi)$, the following statements hold.*

1. *If $\hat{x} \in \mathcal{X}$ is the unique optimal solution to $\mathcal{P}(\xi)^l_{(\epsilon,i)}$ for some $\epsilon \in \mathbb{R}^k$ and some $i \in \{1,..,k\}$, then $\hat{x}$ is lower set less strictly efficient solution to $\mathcal{P}(\xi)$.*

2. *If* $\hat{x} \in \mathcal{X}$ *is an optimal solution to* $\mathcal{P}(\xi)^{l}_{(\epsilon,i)}$ *for some* $\epsilon \in \mathbb{R}^k$ *and some* $i \in \{1,..,k\}$ *and*

$$\min_{\xi \in \mathcal{U}} f_i(x,\xi)$$

exists for all $x \in \mathcal{X}$*, then* $\hat{x}$ *is lower set less weakly efficient solution to* $\mathcal{P}(\xi)$.

Remark 3.5. To prove Theorem 3.4, we can use a proof that will look similar to the one used for Theorem 3.2, only with the assumption that $\hat{x}$ is lower set less ordered [strictly/$\cdot$/weakly] efficient.

3.3 Set less order relation

3.3.1 Description and problem formulation

As our first two set order relations were pessimistic and optimistic respectively, it is then natural to define set order relations where we combine the two in order to obtain a sort of compromise between the two opposites. The first one is the less restrictive, namely the set less order relation which Young [2] introduced

Definition 3.5. A set $A \subseteq \mathbb{R}^k$ *dominates* a set $B \subseteq \mathbb{R}^k$ with respect to the set less order relation, denoted $A \preceq^s_{[\geqq,\geq,>]} B$, with respect to $\mathbb{R}^k_{[\geqq,\geq,>]}$ if

$$A \preceq^s_{[\geqq,\geq,>]} B \iff A \preceq^l_{[\geqq,\geq,>]} B \wedge A \preceq^u_{[\geqq,\geq,>]} B.$$

Remark 3.6. The relation can be equivalently be written as

$$A \preceq^s_{[\geqq,\geq,>]} B \iff (\forall b \in B \exists a \in A : a[\leqq,\leq,<]b) \wedge (\forall a \in A \exists b \in B : a[\leqq,\leq,<]b).$$

This time, the way to frame the problem to obtain this property is to take both the infimum and the supremum respectively of the uncertainty set

$$\min_{x\in\mathcal{X}} \begin{pmatrix} \inf_{\xi\in\mathcal{U}} & f(x,\xi) \\ \sup_{\xi\in\mathcal{U}} & f(x,\xi) \end{pmatrix}. \qquad \mathcal{P}(\xi)^{\mathrm{s}}$$

3.3.2 Efficiency and interpretation

Definition 3.6. Given an uncertain multi-objective optimization problem $\mathcal{P}(\xi)$, a solution $x \in \mathcal{X}$ to $\mathcal{P}(\xi)$ is *set less ordered* [*strictly*/ · / *weakly*] *efficient* if there is no $\bar{x} \in \mathcal{X} \setminus \{x\}$ s.t. $f_{\mathcal{U}}(\bar{x}) \preceq^s_{[\geqq,\geq,>]} f_{\mathcal{U}}(x)$ with respect to $\mathbb{R}^k_{[\geqq,\geq,>]}$, or equivalently written

$$\nexists \bar{x} \in \mathcal{X} \setminus \{x\} : f_{\mathcal{U}}(\bar{x}) + \mathbb{R}^k_{[\geqq,\geq,>]} \supseteq f_{\mathcal{U}}(x) \wedge f_{\mathcal{U}}(\bar{x}) \subseteq f_{\mathcal{U}}(x) - \mathbb{R}^k_{[\geqq,\geq,>]}.$$

We can look at this ordering as a middle ground compared to the relations it is a mixture of, as the efficient solutions for the previous two is a subset of the efficient solutions in this relation. So if the solution is either upper or lower set less ordered efficient it is efficient for this relation as well. It is however important to notice that in this relation a solution is efficient if not any other solution dominates it in *both* the worst and best case. Therefore it might be possible that a solution is efficient for this relation without being efficient for the two other relations. This is actually shown in Example 3.1. As a result of the relationship between this relation and the other two mentioned, we know that since x_2 and x_4 is upper set less ordered strictly efficient and x_1 and x_2 is lower set less ordered strictly efficient these three are also automatically set less ordered strictly efficient. However, we also have another feasible solution that is neither but still set less ordered strictly efficient. If we look more closely at Figures 3.3 and 3.9, we observe that only x_2 dominate x_5's worst case scenario while only x_1 dominate x_5's best case scenario respectively. That said, none of the solutions dominate in both relations. Hence, x_5 is set less ordered strictly efficient even though it is neither upper set less ordered strictly efficient nor lower set less ordered strictly efficient.

3.3.3 Computing set less ordered efficient solutions

To compute set less ordered efficient solutions we can use the same extension of framework as for the former two ordered efficient solutions in 3.1.3 and 3.2.3.

Weighted sum scalarization

The way we extend the framework from $\mathcal{P}_\lambda$ to use this method to compute set less ordered strictly efficient solutions is to insert the problem formulation obtained in 3.3.1, $\mathcal{P}(\xi)^{\mathrm{s}}$:

$$\min_{x\in\mathcal{X}} \begin{pmatrix} \inf_{\xi\in\mathcal{U}} & \sum_{i=1}^{k}\lambda_i f_i(x,\xi) \\ \sup_{\xi\in\mathcal{U}} & \sum_{i=1}^{k}\lambda_i f_i(x,\xi) \end{pmatrix} \qquad \mathcal{P}(\xi)^{\mathrm{s}}_{\lambda}$$

Given a set of scalarization vectors Λ, we can now compute set less ordered efficient solutions by solving $\mathcal{P}(\xi)^{\mathrm{s}}_{\lambda}$ for every $\lambda \in \Lambda$. As discussed in 3.3.2, x_5 is a set less ordered strictly efficient solution even though it is neither upper nor lower set ordered strictly efficient.

Theorem 3.5. ***(Theorem 23 Ide et al. [9])*** *Given an uncertain multi-objective optimization problem $\mathcal{P}(\xi)$, the following statements hold.*

1. *If $\hat{x} \in \mathcal{X}$ is strictly efficient to $\mathcal{P}(\xi)^s_\lambda$ for some $\lambda \in \mathbb{R}^k_{\geq}$, then $\hat{x}$ is set less ordered strictly efficient solution to $\mathcal{P}(\xi)$.*

2. *If $\hat{x} \in \mathcal{X}$ is weakly efficient to $\mathcal{P}(\xi)^s_\lambda$ for some $\lambda \in \mathbb{R}^k_{\{>,\geq\}}$ and*

$$\min_{\xi \in \mathcal{U}} \sum_{i=1}^{k} \lambda_i f_i(x,\xi) \text{ and } \max_{\xi \in \mathcal{U}} \sum_{i=1}^{k} \lambda_i f_i(x,\xi)$$

exists for all $x \in \mathcal{X}$, then $\hat{x}$ is set less ordered [·/weakly] efficient solution to $\mathcal{P}(\xi)$.

Remark 3.7. To prove Theorem 3.5, we use a similar proof as for Theorem 3.1, but with the assumption that $\hat{x}$ is both lower set less ordered [strictly/·/weakly] efficient *and* upper set less ordered [strictly/·/weakly] efficient and then use both these two facts in the proof.

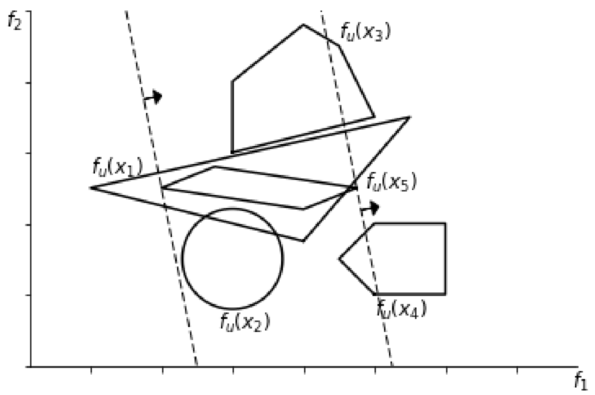

Figure 3.12: Examples of weights to find x_5 as set less strictly efficient. Here $\lambda = [1/6, 5/6]$.

As we can observe in Figure 3.12 that there exist weights where only x_1 is better than x_5 for the best case, and only x_2 that is better than x_5 in the worst case. Hence, since none of the solutions are better than x_5 in both cases given the weights, x_5 is here found to be set less ordered strictly efficient using the weighted sum scalarization method.

ϵ-constraint scalarization

The way we extend the framework from $\mathcal{P}_{(\epsilon,i)}$ to use this method to compute set less ordered efficient solutions is to insert the problem formulation obtained in 3.3.1, $\mathcal{P}(\xi)^s$:

$$
\begin{aligned}
\min_{x \in \mathcal{X}} \quad & \begin{pmatrix} \inf_{\xi \in \mathcal{U}} & \lambda_i f_i(x,\xi) \\ \sup_{\xi \in \mathcal{U}} & \lambda_i f_i(x,\xi) \end{pmatrix} \\
\textbf{s.t.} \quad & \inf_{\xi \in \mathcal{U}} \quad f_j(x,\xi) \leq \epsilon_j \forall j \neq i \qquad\qquad\qquad\qquad \mathcal{P}(\xi)^{\mathrm{s}}_{(\epsilon,i)} \\
& \sup_{\xi \in \mathcal{U}} \quad f_j(x,\xi) \leq \epsilon_j \forall j \neq i.
\end{aligned}
$$

Given a set of vectors $\mathcal{E}$, we can now compute lower set less ordered efficient solutions by solving $\mathcal{P}(\xi)^{\mathrm{s}}_{(\epsilon,i)}$ for each $i \in \{1,\dots,k\}$ and every $\epsilon \in \mathcal{E}$.

Theorem 3.6. *Given an uncertain multi-objective optimization problem $\mathcal{P}(\xi)$, the following statements hold.*

1. *(a) If $\hat{x} \in \mathcal{X}$ is strictly efficient to $\mathcal{P}(\xi)^{s}_{(\epsilon,i)}$ for some $\epsilon \in \mathbb{R}^k$ and some $i \in \{1,..,k\}$, then $\hat{x}$ is set less strictly efficient solution to $\mathcal{P}(\xi)$.*
2. *(b) If $\hat{x} \in \mathcal{X}$ is weakly efficient to $\mathcal{P}(\xi)^{s}_{(\epsilon,i)}$ for some $\epsilon \in \mathbb{R}^k$ and some $i \in \{1,..,k\}$ and $\max_{\xi \in \mathcal{U}} f_i(x,\xi)$ exists for all $x \in \mathcal{X}$, then $\hat{x}$ is set less weakly efficient solution to $\mathcal{P}(\xi)$.*

Remark 3.8. The proof we use to prove Theorem 3.6 is similar as the one we use to prove Theorem 3.2, but instead with the assumption that $\hat{x}$ is both lower set less ordered [strictly/·/weakly] efficient *and* upper set less ordered [strictly/·/weakly] efficient and then combine those two.

3.4 Strict set less order relation

3.4.1 Description

The other combination of the first two relations is the more restrictive, namely the strict set less order relation which was first introduced in Ide et al. [3] as alternative set less order relation.

Definition 3.7. A set $A \subseteq \mathbb{R}^k$ *dominates* a set $B \subseteq \mathbb{R}^k$ with respect to the strict set less order relation, denoted $A \preceq^{ss}_{[\geqq,\geq,>]} B$, with respect to $\mathbb{R}^k_{[\geqq,\geq,>]}$ if

$$A \preceq^{ss}_{[\geqq,\geq,>]} B \iff A \preceq^{l}_{[\geqq,\geq,>]} B \vee A \preceq^{u}_{[\geqq,\geq,>]} B.$$

Remark 3.9. The relation can be equivalently be written as

$$A \preceq^{ss}_{[\geqq,\geq,>]} B \iff (\forall b \in B \exists a \in A : a[\leqq,\leq,<]b) \vee (\forall a \in A \exists b \in B : a[\leqq,\leq,<]b)$$

3.4.2 Efficiency and interpretation

Definition 3.8. Given an uncertain multi-objective optimization problem $\mathcal{P}(\xi)$, a solution $x \in \mathcal{X}$ to $\mathcal{P}(\xi)$ is *strict set less ordered* [*strictly*/ $\cdot$ /*weakly*] *efficient* if there is no $\bar{x} \in \mathcal{X} \setminus \{x\}$ s.t. $f_{\mathcal{U}}(\bar{x}) \preceq^{ss}_{[\geqq,\geq,>]} f_{\mathcal{U}}(x)$ with respect to $\mathbb{R}^k_{[\geqq,\geq,>]}$, or equivalently written

$$\nexists \bar{x} \in \mathcal{X} \setminus \{x\} : f_{\mathcal{U}}(\bar{x}) + \mathbb{R}^k_{[\geqq,\geq,>]} \supseteq f_{\mathcal{U}}(x) \vee f_{\mathcal{U}}(\bar{x}) \subseteq f_{\mathcal{U}}(x) - \mathbb{R}^k_{[\geqq,\geq,>]}.$$

For a solution set $f_{\mathcal{U}}(x)$ to be efficient in this relation there can not exist another solution set $f_{\mathcal{U}}(\bar{x})$ such that neither the best or worst case scenario for $\bar{x}$ is lower than the best or worst case scenario for x. This is a really strict property and therefore this solution set might be sparse or even empty for some problems. To see this, we notice that in order to be efficient in this relation, the solution needs to be *both* upper and lower set less ordered efficient. Hence, this solution set can be seen as extremely good as it is in the top in both the worst case and the best case compared to the other feasible solutions. If we look at Example 3.1, we actually have such a solution, x_2. It is both upper and lower set less ordered strictly efficient and hence also strict set less ordered strictly efficient.

3.4.3 Computing strict set less ordered efficient solutions

To obtain strict set less ordered efficient solutions we use the efficient solutions computed in 3.1.3 and 3.2.3 and find the intersection of the two solution sets. The resulting set is the set of strict set less ordered efficient solutions.

3.5 Relationship between the solutions in the different relations

In this chapter we have introduced different set order relations. During the different sections it has been discussed relationship between some of the relations. In this section, we want to illustrate with a picture how the different relations are related to each other.

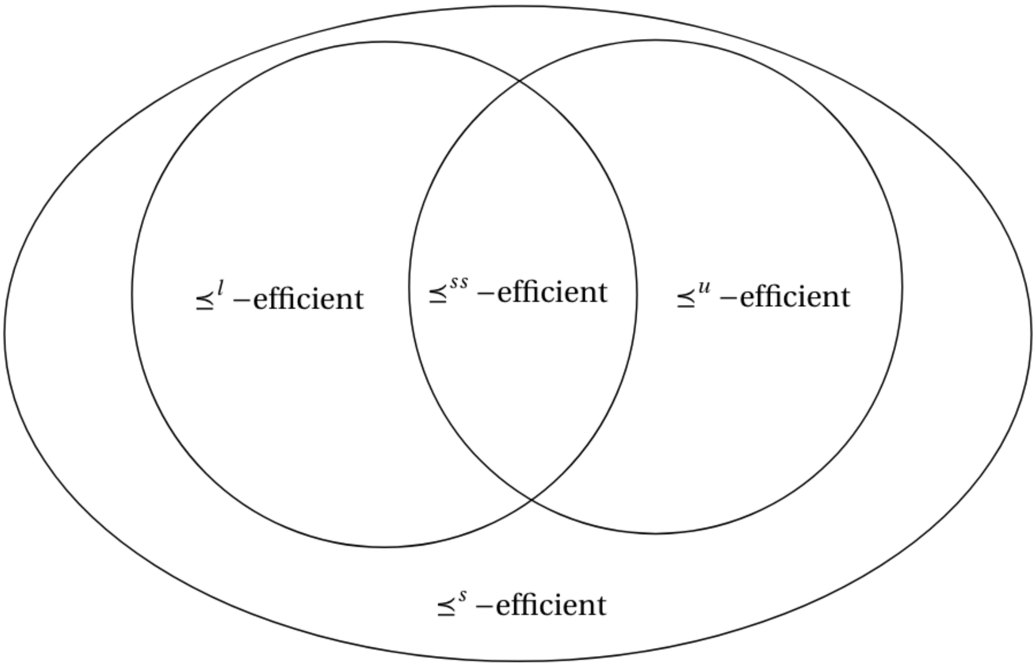

Figure 3.13: Venn diagram of the different set order relations introduced in the chapter.

First of all we have the least strict relation, set less order relation. The solution set of this relation is the union of the lower and upper set less ordered efficient solutions. In addition to this, as discussed in 3.3.1, it it possible for a solution to be neither lower nor upper set ordered strictly efficient but still be set less ordered strictly efficient solution. Next, we have the solutions that are lower and upper set less ordered efficient. These are both a subset, but not necessarily equal to the set less ordered efficient solutions. Lastly, we have the strict set less ordered efficient solutions. These solutions are the intersection of the lower and upper set less ordered efficient solutions. To be precise, a strict set less ordered efficient solution needs to be both a lower and upper set less ordered efficient solution as well.

Chapter 4

Non-convex feasible sets

We want to have methods that are able to find all [upper/lower/·/strict] set less ordered solutions given any uncertain multi-objective optimization problem. As we saw with Example 3.2, ϵ-constraint method is not able to fulfill this. It is also well known that for problems where the feasible set is non-convex, the weighted sum scalarization method is not always able to find all efficient solutions. This will be illustrated by several examples which we are going to look at throughout this chapter. Therefore, we want to introduce methods we can use to solve problems when the feasible set is not convex. These frameworks can also solve problems with disconnected feasible sets as well. In this chapter, we are going to focus on introducing two methods. These approaches have been primarily been used in multi-objective optimization problems, not uncertain. Hence we are going to look at the deterministic version while we get to know the methods. In further work we want to see if they are able to be used in order to obtain all efficient solutions for our different set order relations for uncertain multi-objective optimization. So given our already introduced problem formulation $(\mathcal{P})$, we introduce the first example of the chapter

Example 4.1. Let $\mathcal{X} = \{x \in \mathbb{R}^2_{\geqq} : x_1^2 + x_2^2 - 1 \leq 0, 1 - x_1^2 - x_2^2 \leq 0\}$ and $f_1(x) = x_1, f_2(x) = x_2$. Then the efficient solutions is given by $\mathcal{X}_E = \{x \in \mathcal{X} : 1 - x_1^2 - x_2^2 = 0\}$.

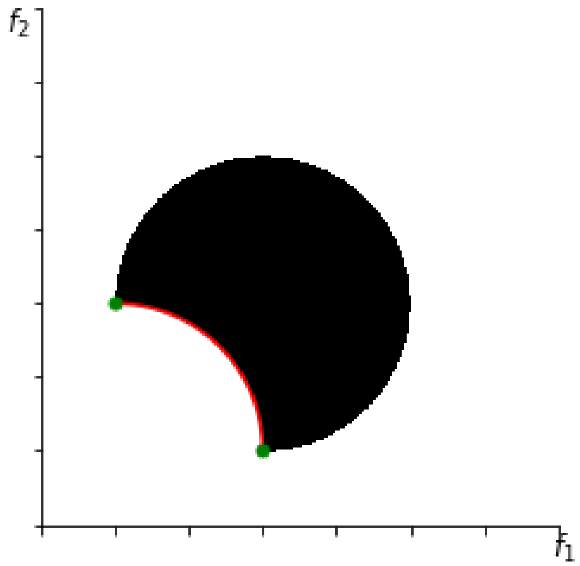

Figure 4.1: Feasible set and Pareto set for Example 4.1 visualized. Feasible set is black while Pareto set is red. The points that are obtainable using weighted sum approach, $\{(1,0),(0,1)\}$, is highlighted in green.

For Example 4.1, the only points which are possible to obtain as efficient solutions with the weighted sum scalarization approach, $\mathcal{P}_\lambda$, is $\{(1,0),(0,1)\}$. The rest of $\mathcal{X}_E$ is unobtainable using this method.

We also want to discuss an example where the feasible set is disconnected. We already know that the weighted sum methods are not necessarily able to find the whole Pareto set, the efficient solutions, for all problems but it is a nice illustration of how powerful the methods we are going to introduce in this chapter are.

Example 4.2. Let $\mathcal{X} = \{x \in \mathbb{R}^2_{\geqq} : (x_1 - 0.5)^2 + (x_2 - 0.5)^2 - 1 \leq 0, x_1^2 + x_2^2 - 2 \leq 0, x_1^2 + x_2^2 - 2x_1x_2 - 0.05 \geq 0\}$ and $f_1(x) = x_1, f_2(x) = x_2$. Then the efficient solutions is given by $\mathcal{X}_E = \{x \in \mathcal{X} : x_1^2 + x_2^2 - 2 = 0\}$.

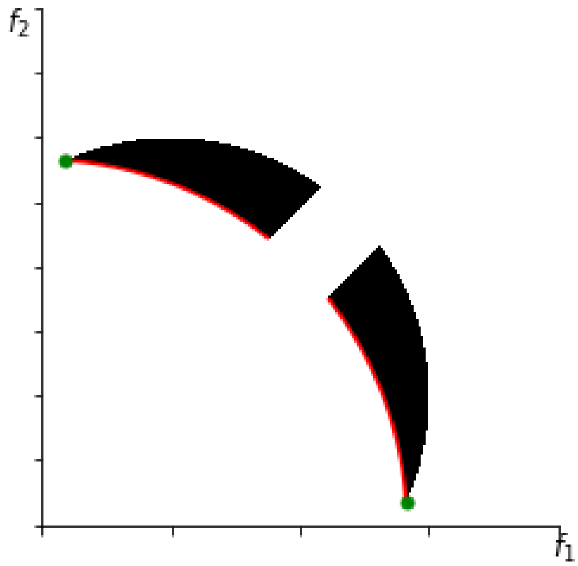

Figure 4.2: Illustration of both the feasible set and Pareto set for Example 4.2. Feasible set is shown as black while Pareto set is in red. The two points that are obtainable using weighted sum approach, $\{(\frac{3+\sqrt{7}}{4}, \frac{3-\sqrt{7}}{4}), (\frac{3-\sqrt{7}}{4}, \frac{3+\sqrt{7}}{4})\}$, is set in green.

For Example 4.2, the only points which are possible to obtain as efficient solutions with the weighted sum method $\mathcal{P}_\lambda$ is $\{(\frac{3+\sqrt{7}}{4}, \frac{3-\sqrt{7}}{4}), (\frac{3-\sqrt{7}}{4}, \frac{3+\sqrt{7}}{4})\}$. The rest of $\mathcal{X}_E$ is unobtainable using this method.

To deal with both these examples and find the whole Pareto set for both of them, we finally introduce the two methods.

4.1 Weighted-Constraint Method

This method was first introduced by Burachik et al. [4].

4.1.1 Description

Given the different objective functions f_i, $i = 1, \ldots, k$ and some positive weights $w \in W^{++} := w \in \mathbb{R}^k | w_i > 0, \sum_{i=1}^{k} w_i = 1$, we consider the problem

$$\begin{array}{rl} \textbf{min} & w_d f_d(x) \\ \textbf{s.t.} & w_i f_i(x) \le w_d f_d(x), i = 1, \ldots, k, i \ne d \\ & x \in \mathcal{X}, \end{array} \qquad \mathcal{P}_{(w,d)}$$

which we refer to as the d^{th}-objective weighted constraint problem. Then for a fixed d and w we define the feasible set as

$$\mathcal{X}_w^d := \{x \in \mathcal{X} | w_i f_i(x) \le w_d f_d(x), \forall i \ne d\}$$

and the solution set of $(\mathcal{P}_{(w,d)})$ as

$$\mathcal{S}_w^d := \{x \in \mathcal{X} | x \text{ solves } (\mathcal{P}_{(w,d)})\}.$$

For each of the fixed $w \in W^{++}$, we have that

$$\mathcal{X} := \bigcup_{d=1}^{k} \mathcal{X}_w^d, \tag{4.1}$$

Which means that the feasible set is the union of the feasible set for each of the different d's given a w. We also define

$$W(x) := \{w \in W^{++} | x \in \mathcal{S}_w^d, \forall d = 1, \ldots, k\}.$$

It is possible that $W(x) = \emptyset$ for some $x \in \mathcal{X}$. To now generate an approximation of the Pareto front, we just solve $\mathcal{P}_{(w,d)}$ for all $d \in \{1, \ldots, k\}$ over a grid of values w. Then for all $w' \in W^{++}$, we have

$$\bigcap_{d=1}^{k} \mathcal{S}_{w'}^{d} \subseteq \mathrm{WE}(\mathcal{P}).$$

Where we have denoted WE($\mathcal{P}$) as the weak efficient solutions of $\mathcal{P}$. This relationship thus shows a way to compute weak efficient solutions by solving $\mathcal{P}_{(w,d)}$ for all $d = 1,...,k$ for some $w' \in W^{++}$. If $\bigcap_{d=1}^{k} \mathcal{S}_{w'}^{d} \neq \emptyset$ then we have obtained at least one weak efficient solution.

Theorem 4.1. ***(Theorem 3.1. Burachik et al.*** *[4])* $\hat{x} \in \mathcal{X}$ *is a weak efficient solution of* $\mathcal{P} \iff$ *there exist some* $w \in W^{++}$ *such that* $\hat{x}$ *solves* $\mathcal{P}_{(w,d)}$ *for all* $d \in \{1,...,k\}$.

Proof. $\implies$ Assume $\hat{x} \in \mathcal{X}$ is a weak efficient solution to $\mathcal{P}$. Without loss of generality we say that $f_i(x) > 0, i = 1,...,k \forall x \in \mathcal{X}$. Then we define

$$w_i : \frac{1/f_i(\hat{x})}{\sum_{j=1}^{k} 1/f_j(\hat{x})}.$$

For this choice, $w \in W^{++}$ and $\hat{x}$ satisfies all constraints as equalities, or in other words

$$w_i f_i(\hat{x}) = w_d f_d(\hat{x}), i = 1,..,k, i \neq d. \tag{1}$$

If $\hat{x}$ is not a solution of $\mathcal{P}_{(w,d)}$ for some d, then there exist $\bar{x} \in \mathcal{X}$ such that

$$w_d f_d(\bar{x}) < w_d f_d(\hat{x}) \tag{2}$$

and

$$w_i f_i(\bar{x}) \leq w_d f_d(\bar{x}), i = 1,...,k, i \neq d.$$

Hence, we can write

$$w_i f_i(\bar{x}) \leq w_d f_d(\bar{x}) < w_d f_d(\hat{x}), i = 1,...,k, i \neq d.$$

Then, by (1), we can write

$$w_i f_i(\bar{x}) \le w_d f_d(\bar{x}) < w_i f_i(\hat{x}), i = 1, ..., k, i \neq d, \tag{3}$$

and since $w_i > 0$, if we combine (2) with (3) we get

$$f_i(\bar{x}) < f_i(\hat{x}), i = 1, ..., k.$$

This contradicts the weak efficiency of $\hat{x}$.

$\Longleftarrow$ Assume that $w \in W^{++}$ is such that $\bar{x}$ solves $\mathcal{P}_{(w,d)}$ for all d. Suppose that $\bar{x} \in \mathcal{X}$ is not a weak efficient point of $\mathcal{P}$. Then there must exist $\hat{x} \in \mathcal{X} \setminus \bar{x}$ such that

$$f_i(\hat{x}) < f_i(\bar{x}), i = 1, ..., k. \tag{4}$$

Then from 4.1 there exists d such that $\hat{x} \in \mathcal{X}_w^d$. Therefore, from 4 we can write $w_d f_d(\hat{x}) < w_d f_d(\bar{x})$ where $w_d > 0$. This contradicts that $\bar{x}$ solves $\mathcal{P}_{(w,d)}$. □

Remark 4.1. **(Remark 3.1. Burachik et al. [4])** The $\Longrightarrow$ part of Theorem 4.1 holds for efficient solutions since every efficient solution is a weak efficient solution. However, if a point solves $\mathcal{P}_{(w,d)}$ for all $d \in 1, ..., k$, then this does not necessarily imply that the solution is efficient unless all objective functions are strictly convex.

4.1.2 Computation of efficient points and interpretation

Now we show an illustration of how efficient solutions are found in Example 3.1 and Example 3.2 with the method.

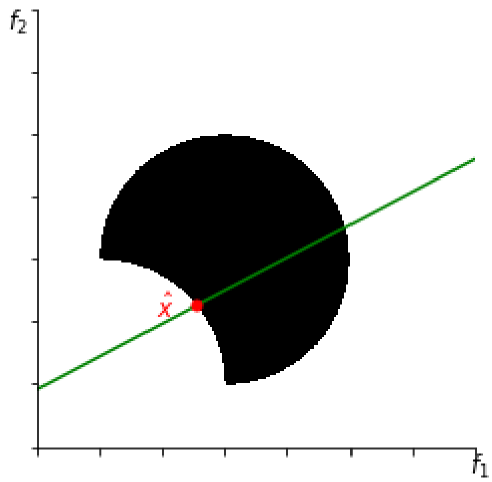

Figure 4.3: Example of obtaining an efficient solution $\hat{x} = (\frac{11}{\sqrt{202}}, \frac{9}{\sqrt{202}})$ for a given $w = [\frac{9}{20}, \frac{11}{20}]$. The green line shows the line $w_1 x_1 = w_2 x_2$.

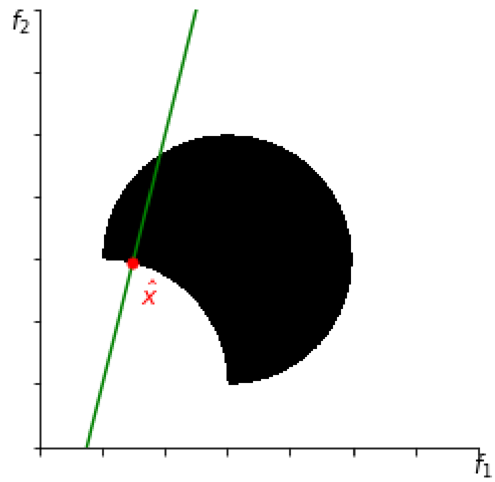

Figure 4.4: Example of obtaining an efficient solution $\hat{x} = (\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}})$ for a given $w = [\frac{4}{5}, \frac{1}{5}]$. The green line shows the line $w_1 x_1 = w_2 x_2$.

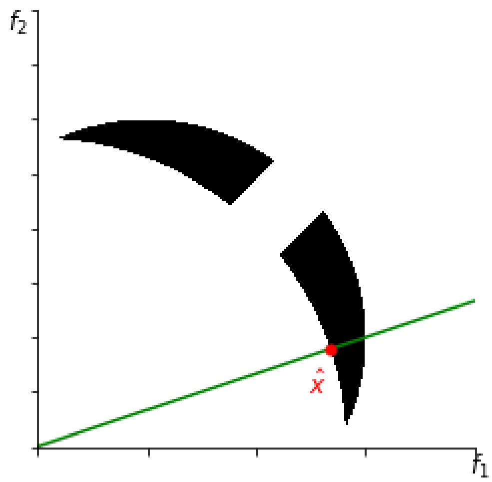

Figure 4.5: Example of obtaining an efficient solution $\hat{x} = (\frac{3}{\sqrt{5}}, \frac{1}{\sqrt{5}})$ for a given $w = [\frac{1}{4}, \frac{3}{4}]$. The green line shows the line $w_1 x_1 = w_2 x_2$.

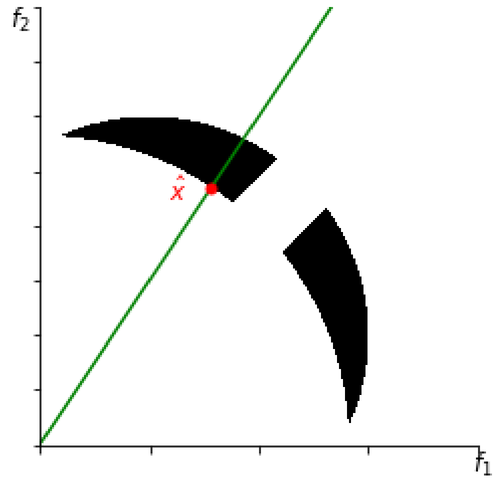

Figure 4.6: Example of obtaining an efficient solution $\hat{x} = (\sqrt{\frac{8}{13}}, \frac{3}{2}\sqrt{\frac{8}{13}})$ for a given $w = [\frac{3}{5}, \frac{2}{5}]$. The green line shows the line $w_1 x_1 = w_2 x_2$.

If we look at $\mathcal{P}_{(w,d)}$ and combine that formulation with the definition of an efficient solution for the method, $\bigcap_{d=1}^{k} \mathcal{S}_{w'}^{d}$, we see that the only set the efficient solution can lie on in both examples for a given w' is the line defined by $w_1 x_1 = w_2 x_2$. This can be explained by the fact that for $\mathcal{P}_{(w',1)}$, the constraint is given by $w_2 x_2 - w_1 x_1 \leq 0$, while for $\mathcal{P}_{(w',2)}$, the constraint is given by $w_1 x_1 - w_2 x_2 \leq 0$. The points $x \in \mathcal{X}$ eligible for being an efficient solution for that given w' are therefore the set of points which fulfill the equation $w_1 x_1 = w_2 x_2$. Hence, by choosing a range of $w \in W^{++}$, it is possible to estimate the Pareto front. In Figure 4.3 and Figure 4.4, two different w's have been chosen to find two of the solutions $\hat{x}$ that make up the efficient solutions in Example 3.1, while in Figure 4.5 and Figure 4.6, two different w's have been chosen to find two of the solutions $\hat{x}$ that make up the efficient solutions in Example 3.2.

One way one can compute an estimation of the Pareto front in the two examples using this method is to first find the $\hat{x}$ which optimize the different objective functions isolated. For Example 3.1 and 3.2 these points are illustrated in 4.1 and 4.2 in green. By finding these points, we know that the rest of the Pareto front is in-between the weights which is used to obtain these two endpoints of the set. By then choosing how many points you want to use to approximate the set, choose the increment in weights you want. If we set the weight w_{f_1} at the weight used in finding the point optimizing the first objective function and w_{f_1} at the weight used in finding the point optimizing the second objective function we can now approximate the Pareto front by setting the weights as $w_n = w_{f_1} + \frac{n(w_{f_2} - w_{f_1})}{N}, n = 1, \ldots, N-1$. By doing this we get an approximation of the Pareto front with N points given that all optimization problems have a solution. For Example 3.1 this is true, while in Example 3.2 we can observe that some of the weight might be set such that $\bigcap_{d=1}^{k} \mathcal{S}_{w'}^{d} = \emptyset$. Then the approximation will contain less points if one does not adapt the algorithm to not divide the interval between w_{f_1} and w_{f_1} equally to avoid this. The problem with trying to fix this issue on a general basis is that one is not sure which weights does not produce an efficient solution.

4.2 Pascoletti-Serafini Method

4.2.1 Description

The next method we want to analyze is a method derived first by Pascoletti and Serafini [5], which is a more generalized framework and therefore very applicable to many optimization problems. Given $\mathcal{P}$, the method can be stated as

$$\begin{aligned} &\min_{x \in \mathcal{X}} \xi \\ &\textbf{s.t. } p + \xi q - f(x) \in \mathbb{R}^k_{\geqq} \\ &\quad \xi \in \mathbb{R} \end{aligned} \qquad \mathcal{P}_{(p,q)}$$

This problem is solved by moving along the line $p+\xi q$ for a given pair $\{p, q\} \in \mathbb{R}^k \times \mathbb{R}^k_{\geqq}$ and $\xi \in \mathbb{R}$. We first start with $\xi = 0$, hence at p, and from there we move ξ until the set $(p+\xi q-\mathbb{R}^k_{\geqq}) \cap f(\mathcal{X}) = \emptyset$. $\hat{\xi}$ is then defined as the smallest value of ξ such that the relation $(p+\hat{\xi} q-\mathbb{R}^k_{\geqq}) \cap f(\mathcal{X}) \neq \emptyset$ which then will be the optimal value for the problem with a corresponding $\hat{x}$ which is then an efficient solution to $\mathcal{P}$.

Theorem 4.2. *$\hat{x} \in \mathcal{X}$ is an efficient solution of $\mathcal{P}$ $\iff$ there is some $\hat{\xi} \in \mathbb{R}$ such that $(\hat{\xi}, \hat{x})$ is a minimal solution of $\mathcal{P}_{(p,q)}$ for some pair of parameters $(p, q) \in \mathbb{R}^k \times \mathbb{R}^k_{\geqq}$.*

Proof. $\Longrightarrow$ Assume $\hat{x} \in \mathcal{X}$ is a weak efficient solution to $\mathcal{P}$. If we then set $p' = f(\hat{x})$ and $q' \in \mathbb{R}^k_{\geqq} \setminus \{0\}$, then $(0, \hat{x})$ solves $\mathcal{P}_{(p,q)}$ for $\{p', q'\}$. Suppose that $\hat{x}$ is not the minimal solution given the pair $\{p', q'\}$, then there exist a $\bar{x} \in \mathcal{X}$ such that $(\bar{\xi}, \bar{x})$ is a minimal solution with $\bar{\xi} < 0$. However, if $\bar{x} \in \mathcal{X}$, then by design of $\{p', q'\}$ it contradicts that $\hat{x}$ is an efficient solution to $\mathcal{P}$.

$\Longleftarrow$ Assume that $\hat{\xi}$ is such that $(\hat{\xi}, \hat{x})$ solves $\mathcal{P}_{(p,q)}$ for a pair of $\{p', q'\}$. Suppose that $\hat{x} \in \mathcal{X}$ is not a weak efficient point of $\mathcal{P}$. Then there must exist $\bar{x} \in \mathcal{X}$ such that $(\bar{\xi}, \bar{x})$ solves $\mathcal{P}_{(p,q)}$ for $\{p', q'\}$ with $\bar{\xi} < \hat{\xi}$. Then we can write $p' + \bar{\xi} q' - \mathbb{R}^k_{\geqq} \cap \mathcal{X} = \bar{x} \neq \emptyset$. Since $\bar{\xi} < \hat{\xi}$, this contradicts that $(\hat{\xi}, \hat{x})$ solves $\mathcal{P}_{(p,q)}$. $\square$

4.2.2 Computation of efficient points and interpretation

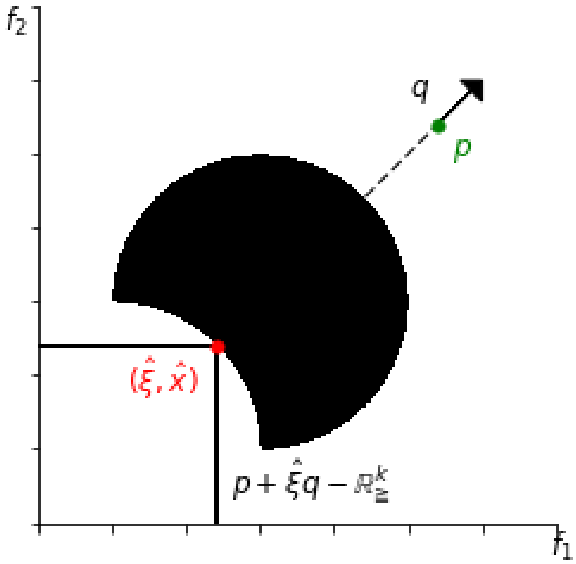

Figure 4.7: Example of obtaining an efficient solution $(\hat{\xi},\hat{x}) = (-\frac{17}{10}\sqrt{2},(\frac{1}{2},\frac{1}{2}))$ for a given pair of $(p,q) = ((\frac{11}{5},\frac{11}{5}),[\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}])$

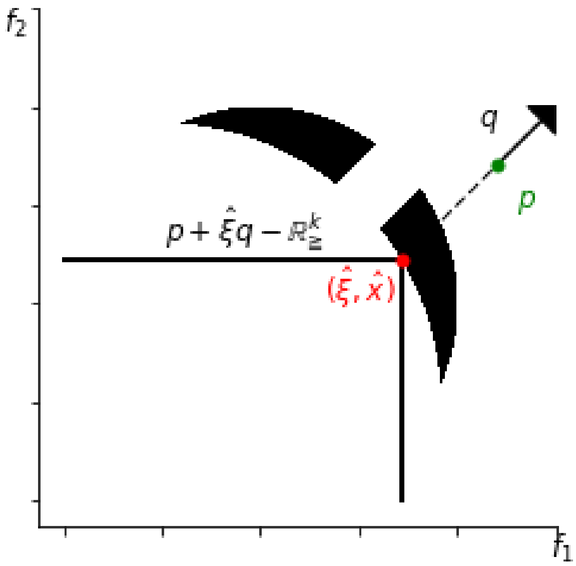

Figure 4.8: Example of obtaining an efficient solution $(\hat{\xi},\hat{x}) = (-\frac{29-5\sqrt{15}}{10\sqrt{2}},(\frac{1+\sqrt{15}}{4},\frac{\sqrt{15}-1}{4}))$ for a given pair of $(p,q) = ((\frac{17}{10},\frac{6}{5}),[\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}])$

As shown in Figure 4.7 and 4.8, the method is able to find a point $\hat{x}$ that is a part of the Pareto set while also being in the non-convex part of the feasible set, something $\mathcal{P}_\lambda$ are unable to do. By varying the two parameters $\{p,q\}$, it would be possible to obtain all efficient points $\hat{x}$ of $\mathcal{P}$ where each $\hat{x}'$ would be a solution corresponding to a specific $\mathcal{P}_{(p,q)}$ given a pair $\{p, q\}$.

Now the question becomes, how many pairs do we need to obtain the whole set of efficient points or at least a good approximation. We can do this by either by setting q static and instead vary p and solve all the different problems given that pair of p and q to obtain the desired number of efficient points we want, or we could do it the other way around and instead set p to a point and vary q. One way to do this is to work your way through different q1s on the form $[\frac{N-n}{N}, \frac{n}{N}]$ for $n = 0, 1, ..., N$ for an arbitrary N.

Chapter 5

Conclusion and further work

In this paper we firstly introduced different set order relations for finding different efficient solutions in uncertain multi-objective optimization problems. The upper set less order relation can be connected with risk aversion as the efficient solutions in this relation are not dominated in the worst case scenario. Hence the solutions you get from this are a way of hedging against the scenarios where the worst happens. Next, we introduced the lower set less order relation, which can be connected with being risk affine as the efficient solutions in this relation are not dominated in the best case scenario. Hence the solutions you get from this have the biggest upside, but on the other hand you are not safe from these solutions having the worst downsides as well. If you are risk neutral instead then the set less order relation might be a good relation for you to use. This is the union of the risk averse and risk affine solutions. As we saw in examples, it is also possible to have solutions in this relation that are neither upper nor lower set less ordered efficient. These solutions are often the ones that are neither the very best but neither the very worst. The last relation we have introduced is the strict set less order relation. This is the intersection of the risk averse and the risk affine solutions. As one can tell, these are the very best solutions as they have the best upside and at the same guaranteed not to have the worst outcomes. Depending on the problem at hand, these solutions can be rare but if they are achievable they are considered the best no matter the risk assessment.

Secondly we dived into two scalarization techniques for approximating the Pareto front in multi-objective optimization problems, Weighted-Constraint and Pascoletti-Serafini. The techniques are non-linear and therefore very useful in the cases where the feasible set is non-convex and even work when the feasible set is disconnected.

This is important as the previously discussed linear approach, the weighted sum scalarization, is not necessarily able to approximate the Pareto front when the problem has a nonconvex feasible set as discussed in Example 4.1 and 4.2. The other method discussed, the ϵ-constraint method, is able to approximate the Pareto front even when the feasible set is non-convex for deterministic multi-objective optimization problems. However, as seen in Example 3.2, it is not always able to obtain all [upper/lower/$\cdot$/strict] set less ordered solutions in uncertain multi-objective optimization problems which is what we are looking for.

In further work, we want to look at how we can combine these two topics in an attempt to obtain all [upper/lower/$\cdot$/strict] set less ordered efficient solutions for all uncertain multi-objective optimization problems - even when the feasible sets for the different solutions are non-convex - by using Weighted-Constraint or Pascoletti-Serafini. This is applicable as there are many problems where the feasible sets are non-convex. Real world problems are often more chaotic than the examples presented in this paper and even then it is easy to construct a non-convex feasible set as seen in Chapter 4.

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