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Ola Isaac Høgåsen Mæhlen

Travelling and global-in-time solutions of water wave equations with mild and nonlocal dispersion

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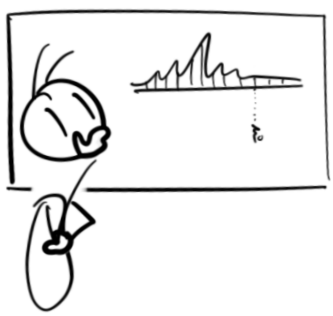

Thesis for the Degree of Philosophiae Doctor

Trondheim, November 2021

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Preface and Acknowledgements

The submission of this thesis is in partial fulfillment of the requirements for the degree Philosophiae Doctor in Mathematics at Norwegian University of Science and Technology.

My two supervisors Prof. Mats Ehrnström and Prof. Espen R. Jakobsen suggested to me, now four years ago, a three-part PhD project that I gladly agreed to. I was to A) work partially with Mats on nonlocal dispersive equations, B) work partially with Espen on nonlocal conservation laws and then C) combine what I had learned to study the nonlocal and dispersive Whitham equation through the lens of conservation laws. Initially, I thought the respective fields of Mats and Espen might merge nicely, with the results unfolding themselves. This was of course naive thinking, but the techniques I learned working with Espen were applicable for weakly dispersive equations, thus forming Paper 3 of this thesis. The remaining three papers are also on dispersive equations, for much remains on my work on nonlocal conservation laws – which has been centered around one tough problem – and so I have chosen to omit it from the thesis. Nevertheless, this last problem has been both fun and educational and is now mostly cracked; it pleases me that a future publication is finally in sight for I remember many weekly meetings spent staring at the blackboard in silence while thinking to myself that "we have no idea what is going on". Luckily, those periods of panic passed.

I give a big thank you to Mats and Espen for their support and expertise and for being so easy going throughout this four year period. An extra thanks goes to Mats for his thorough feedback on my work, suggestions for conferences and workshops and for frequently organizing social events. I am also grateful for meeting Nathaël Alibaud whom I learned a lot from the one month I had in Besançon before the outbreak of COVID-19. On the same note, I am thankful for all the good discussions with Jørgen Endal, Fredrik Hildrum, Kristoffer Varholm and Jun Xue. Finally, a warm thank you to friends and family for their support and company, especially to Maia; it would be tough to spend numerous hours working from the kitchen table without her.

Ola Mæhlen

Ola Mæhlen,
Trondheim, August 2021

Introduction

This thesis is made up of four papers on dispersive equations. Such equations are typically used to model the behaviour of water waves, and in this context, *dispersion* refers to the physical phenomenon that waves of different wavelengths travel at different speeds. One may wonder where the interest in water wave models comes from when the governing equations for fluid dynamics, the Euler equations [26], have been known for two centuries. The short answer is that the Euler equations are complicated and not well suited for providing explanations to seemingly simple observations. For an analogy, suppose one was interested in understanding why planetary orbits are elliptical. As Newton demonstrated at the end of the 17th century, this was a direct corollary of his law of gravity. And while this law has since been superseded by Einstein's far more complicated theory of general relativity, Earth's orbit remains ellipse shaped, for Newton's equation is still an excellent model for how gravity behaves in this particular regime. Similarly for water waves, one may replace the intricate Euler equations with simpler models depending on the regime of interest. For example, in the shallow water regime, the water surface can be modelled by the Whitham equation [24, 37] which features several qualitative properties of shallow water waves including periodic travelling waves [20], solitary waves [17], wave breaking [33] and Stokes waves of maximal amplitude [22].

The Whitham equation is in many ways simpler than the Euler equations. The former features only one spacial dimension (the horizontal) and one unknown (the wave height) while the latter includes the water depth and the unknown velocity field of the water. But one arising difficulty, not present in the Euler equations, is that of nonlocality: The dispersive term of the Whitham equation cannot be calculated locally without knowing the global surface profile. A nonlocal dispersive term is common for water wave models, and can be thought of as a price to pay for dropping a spacial dimension and one unknown. Contrariwise, any equation featuring a fractional Laplacian-type nonlocality can be transformed into a pair of local equations at the cost of introducing one more space variable and one more unknown [9].

The theory on water wave models, both local and nonlocal, covers a broad specter of mathematical tools and techniques; each question asked requires its own meticulous analysis further depending on the equation in focus. This intriguing complexity is partially why water wave problems have gained much attention among mathematicians, especially in recent years.

The thesis will cover various topics and equations, but most of the water wave models considered are included in the general form

$$u_t + (n(u) + Lu)_x = 0. \tag{1}$$

Here, the wave height u is a real valued function in the two variables time and space, the nonlinearity n is a real valued function in one variable and the – typically nonlocal – operator L is a symmetric Fourier multiplier in space. Often L is characterized on the Fourier side by its corresponding real valued and symmetric symbol $\xi \mapsto m(\xi)$. That is, L and m are related through the equation $\widehat{Lf}(\xi) = m(\xi)\hat{f}(\xi)$ for any Schwartz function f and where the hat-notation denotes the Fourier transform. The symbol m is also referred to as the dispersion relation; in the linear setting $n = 0$, a planar wave of wavelength ξ will travel at the velocity $m(\xi)$. In the nonlinear setting, this is still approximately true for small amplitude waves, but in general, the velocity of travelling wave solutions might not even coincide with any value of $m(\xi)$. This last point is discussed more in the introduction to Paper 2.

Multiple one dimensional water wave models are of the form (1), and in the case of a quadratic nonlinearity $n(u) = u^2$, the following equations (here specified by their symbol m) are included:

TABLE 1. Equations of the form (1) with $n(u) = u^2$.

$m(\xi)$	Equation
ξ^2	Korteweg–De Vries
$\lvert\xi\rvert$	Benjamin–Ono
$\sqrt{\frac{(1+T\xi^2)\tanh(\xi)}{\xi}}$	Capillary Whitham
$\sqrt{\frac{\tanh(\xi)}{\xi}}$	Whitham
$\lvert\xi\rvert^{-1}$	Burgers–Hilbert
ξ^{-2}	Ostrovsky–Hunter
$(1+\xi^2)^{-1}$	Burgers–Poisson

where $T > 0$ denotes the surface tension parameter for the capillary Whitham equation. These equations are listed in decreasing order with respect to the 'strength' of the featured dispersion, referring here to the order of growth of $m(\xi)$ as $|\xi| \to \infty$. In particular, the top three feature *positive order* dispersion as $\lim_{|\xi|\to\infty} m(\xi) = \infty$ while the remaining four feature *negative order* dispersion as $\lim_{|\xi|\to\infty} m(\xi) = 0$.

As three of the papers in the thesis examine travelling wave solutions, we give the corresponding definition. Such solutions take the form $(t, x) \mapsto u(x-ct)$ for some velocity $c \in \mathbb{R}$ and a function u in one variable satisfying the stationary equation

$$(-cu + n(u) + Lu)' = 0, \tag{2}$$

that is, the bracket in (2) is a constant. What follows is a short introduction to each paper, where the relevant results and concepts are introduced.

Paper 1: On the bifurcation diagram of the capillary–gravity Whitham equation.

With: Mats Ehrnström, Mathew A. Johnson and Filippo Remonato .

Published in Water Waves [19]

Paper 1 concerns periodic travelling wave solutions of the capillary-gravity Whitham equation, or just capillary Whitham for short, which models shallow water waves when surface tension is included [15,39]. This equation takes the form (1) with a quadratic nonlinearity $n(u) = u^2$ and the symbol

$$m_T(\xi) = \sqrt{\frac{(1 + T\xi^2)\tanh(\xi)}{\xi}},$$

for a positive surface tension parameter $T > 0$. In the original Whitham equation capillary effects are ignored and so the surface tension is set to zero. For any $T > 0$ the symbol m_T grows like $|\xi|^{\frac{1}{2}}$ at infinity and so the corresponding Fourier multiplier $L = M_T$ is a positive order operator (of order $\frac{1}{2}$); this is in stark contrast to the original Whitham equation in which $L = M_0$ is a smoothing operator (of order $-\frac{1}{2}$). An available trick that removes the cumbersome positive order operator, is to apply its inverse M_T^{-1} to this instance of (2), so to obtain

$$M_T^{-1}[-cu + u^2] + u = 0, \tag{3}$$

where we integrated once and set the integrating constant to zero (resulting here in no loss of generality due to a Galilean invariance principle). As M_T^{-1} admits the symbol $1/m_T$ it is a smoothing operator and can be realized as a convolution operator K_T* for an even kernel $K_T \in L^1(\mathbb{R})$. Paper 1 starts off by demonstrating various properties of both M_T^{-1} and

K_T including characterizing for which surface tensions K_T is monotone on $(0, \infty)$. These properties are then used to carry out bifurcation analysis on periodic solutions of (3), here mimicking the approach in for example [16, 21].

Bifurcation analysis in this context relates to the fact that travelling wave solutions of water wave models, such as those provided in Table 1, tend to form connected manifolds in appropriate solutions spaces. Thus, by tracing these manifolds using implicit function theorems one may discover nontrivial solutions *bifurcating* from some trivial starting point. In the quintessential scenario the manifolds are curves, then called bifurcation branches, whose global behaviour can be characterized through the theory developed by Dancer [14] and further improved by Buffoni and Toland [8]. For example, [11] deploys these tools for the Euler equations to conclude that the bifurcation branches of even periodic solutions connecting to trivial constant ones never 'loop'. But there are limitations to analytic techniques, and so numerical results serve as an integral part of bifurcation analysis. By numerical methods, [29] finds isolated bifurcation branches of non-symmetric periodic solutions for the Euler equations; these branches are not connected to any trivial solution and *do* form loops.

Paper 1 provides a local description of even periodic solutions, bifurcating from zero, of the stationary capillary Whitham equation (3). These solutions differ in two qualitative ways from those found for the original Whitham equation in [20, 21]. First, any periodic and bounded solution of (3) is necessarily smooth, while the bifurcation branches for the Whitham equation approach non-smooth solutions of maximal amplitude as shown in [22] (the introduction to Paper 4 gives a description of such *highest waves*). Second, the bifurcation kernel can here be two-dimensional giving rise to bifurcation sheets, a novelty not featured in the Whitham case. When this happens, the (small) solutions found are roughly the sum of two weighted cosines of different wave length. And if one of the wavelengths divide the other, a resonance phenomenon occurs, resulting in 'slit' sheets. The paper also provides some global bifurcation results, both numerical and analytic, using the previously mentioned theory of Dancer, Buffoni and Toland.

Paper 2: Solitary waves for weakly dispersive equations with inhomogeneous nonlinearities.

Published in Discrete and Continuous Dynamical Systems [43]

Paper 2 proves the existence of solitary wave solutions for a sub-family of (1). Solitary waves are travelling wave solutions that vanish at infinity.

As a physical phenomenon, such waves can be observed in shallow water canals. The first research on these waves was conducted by the Scottish engineer John Scott Russel [47] in 1834. At the time, water wave models would not allow for such solutions, leaving the scientific community skeptical of Russel's observations [12, 45]. However, in 1871 and 1872 Joseph Boussinesq derived, in a series of papers [5–7], the first model featuring solitary waves, and today such models are abundant; the key being an intricate balance of nonlinearity and dispersion.

Numerous techniques have been developed to prove the existence of solitary waves for dispersive equations. In [50] bifurcation analysis is used to find solitary waves for the Whitham equation that reach maximal amplitude; again, such highest waves is the topic of Paper 4. Another technique is Lions' very successful concentration-compactness method [41] of which an appropriate variation is applied in Paper 2. Curiously, the question of *uniqueness* of solitary waves is typically far more difficult, and no general method, like that of Lions' for existence, have been discovered. Still, some instances of (1) are well understood. For the Korteweg–De Vries equation, classical ODE-techniques show that the solitary waves are uniquely characterized, up to translation, by their amplitude. This uniqueness result is extended in [28] to a homogeneous sub-family of (1) for corresponding *ground states* (positive and symmetric solitary waves) by exploiting the previously mentioned fact that (1) can be rephrased as a local problem when L is a fractional Laplacian.

In Paper 2, small amplitude solitary waves of (1) are proved to exist under mild assumptions on the *positive* order symbol m and nonlinearity n, where the novelty is in allowing for a nonhomogeneous n. The case of positive order dispersion and a homogeneous nonlinearity has been dealt with by numerous authors, for example [2, 4, 51, 54] which all deploy Lions' concentration-compactness method. While this method does not go through for a badly behaving n (such as one with exponential growth), we overcome this difficulty by first truncating the nonlinearity at a fixed height, and then demonstrate the existence of solitary waves of arbitrarily small amplitude for the truncated equation. Any such sufficiently small solitary wave then necessarily solves the original equation as well.

This work compliments [17] where a similar result is proved, also for a nonhomogeneous n, but for negative order dispersion. Interestingly, the solitary waves found in Paper 2 are *subcritical*, meaning their velocities are less than $m(0)$ (which in our case is the minimum of m), while those from [17] are *supercritical*, meaning their velocities exceed $m(0)$ (which in their case is the maximum of m). That is, these waves move at velocities outside of those provided by the dispersion relation. In [35], *generalized* solitary waves are constructed for the capillary Whitham equation, and the velocities of these waves are indeed of the form $\nu = m(\xi_0)$ for some frequency ξ_0. But the waves do not vanish at infinity, and instead approach a periodic solution of wavelength ξ_0.

Paper 3: One sided Hölder regularity of global weak solutions of negative order dispersive equations.

With: Jun Xue.

Submitted for publication.

Paper 3 examines global weak solutions of negative order dispersive equations taking the form (1) with a quadratic nonlinearity and with $L = G*$, where G can be any function admitting a weak integrable derivative.

As convolution operators are smoothing (making L negative order), one should expect the dispersive effects in these equations to be generally small compared to those of the nonlinearity. A phenomenon then arising is *wave breaking*; this is when a solution attains infinite slope in finite time, while its height remains bounded. In fact, wave breaking occurs in all four of the negative order dispersive equations from Table 1 [30, 33, 42, 53]. As a consequence, neither of these equations are (classically) globally well-posed, although they are locally well-posed [23, 27, 31, 32].

In contrast, the strongly dispersive Korteweg–de Vries equation and Benjamin–Ono equation are both globally well-posed [34, 36] in appropriate Sobolev spaces. While positive order dispersion seems to hinder wave breaking (at least for instances of (1)) it is not sufficient to guarantee global well-posedness. In [44] the authors show that the modified Benjamin–Ono equation – which takes the form (1) with $n(u) = u^3$ and $m(\xi) = |\xi|$ – is *ill-posed* by constructing a solution behaving like an accelerating and growing solitary wave reaching infinite velocity and amplitude in finite time. It is believed [38, 40] that a similar phenomenon occurs for a homogeneous analogue of the capillary Whitham equation; in particular, no global well-posedness result exists for the capillary Whitham equation.

Turning back to the case of negative order dispersive equations, one remedy for dealing with the absence of global classical solutions is to instead consider weak solutions. A powerful approach originally from the theory of hyperbolic conservation laws is then to reformulate (1) in a weaker sense as a family of *entropy conditions*. Remarkably, this concept of entropy solutions gives rise to a global well-posedness theory for the Ostrovsky–Hunter equation [10] and the Burgers–Poisson equation [30]. This latter equation is in fact included in the family we study, but [30] considers this equation in a 'pure' $L^1(\mathbb{R})$ setting, while we have chosen to work in a slightly more natural $L^2(\mathbb{R})$ setting, allowing for different results.

The first half of Paper 3 establishes the uniqueness, existence, and L^2-stability of entropy solutions for the considered family of equations. This analysis, consisting mostly of classical conservation law techniques, is first carried out for $L^2 \cap L^\infty(\mathbb{R})$ initial data, and is by a continuity argument further extended to pure $L^2(\mathbb{R})$ data. In the second half of the paper, an operator splitting argument is used to demonstrate that the acquired weak solutions satisfy explicit one sided Hölder conditions with time decreasing coefficients. This result can be viewed as a generalization of the classical Oleĭnik estimate [13] for Burgers' equation, which states that entropy solutions satisfy the inequality

$$u(t,x) - u(t,y) \leq \frac{x-y}{t},$$

whenever $x \geq y$. But contrary to the Oleĭnik estimate, the one sided Hölder conditions we find do not vanish as $t \to \infty$; such a result is generally unattainable, as the family of equations we consider features solitary waves [17].

The inferred one sided Hölder regularity of the attained weak solutions results in two interesting consequences. First, the solutions satisfy explicit and time decreasing height bounds. And second, the lifespan of a classical solution can be bounded, provided its initial data is sufficiently steep. While it is tempting to think (and very possible) that the finite lifespan of a classical solution is due to wave breaking, this is not proved.

Paper 4: On the precise behaviour of extreme solutions to a family of nonlocal dispersive equations.

With: Mats Ehrnström and Kristoffer Varholm.

In preparation.

Paper 4 establishes previously conjectured limits at the crests of the *highest waves* of the Whitham equation and the bidirectional Whitham equation. This latter water wave model, formally derived in [1, 46] from the Euler equations, is a system of two equations that allow for both left- and rightward wave propagation. In contrast, (small) periodic waves of the original Whitham equation necessarily propagate to the right [20] by the positivity of the dispersion relation.

In 1880, Sir George Stokes conjectured [48] that the Euler equations featured periodic waves reaching a maximal height (relative to the wavelength) and whose crests formed 120° degree interior angles. Just short of a decade later, John F. Toland proved the existence of such highest waves [49] in 1978, and together with Amick and Fraenkel, they verified Stokes' conjecture on the interior angles [3] in 1982.

Similarly, Gerald B. Whitham conjectured [52] that his proposed water wave model, the Whitham equation, featured analogous highest waves, but whose crests formed cusps like that of square roots. And indeed, through bifurcation analysis such periodic waves were found in [22]. More recently, it was proved that the Whitham equation also features highest *solitary* waves [50]. But in both of these works, the crests of the corresponding highest waves were only shown to behave like square roots in '$\eqsim$' sense; more precisely, denoting such a wave with ϕ, whose crest we assume is at zero, the following bounds were demonstrated

$$0 < \liminf_{x\to 0} \frac{\phi(0)-\phi(x)}{|x|^{\frac{1}{2}}} \qquad \limsup_{x\to 0} \frac{\phi(0)-\phi(x)}{|x|^{\frac{1}{2}}} < \infty.$$

This left open the question of whether a full limit exists. Not long ago, such a limit was established in [25] for a 2π-periodic highest wave of the Whitham equation constructed through a computer assisted fixed point argument. For this wave, the authors proved the even stronger result of a convex profile in between the crests.

Paper 4 determines the precise cusp shape (i.e. the full limits) at the crests of the highest waves found in [22, 50]. The paper includes the corresponding limits for the analogous highest waves of the bidirectional Whitham equation; these were found in [18] and conjectured to admit crests forming *logarithmic* cusps, that is, cusps like that of $x \mapsto |x| \log |x|$ at zero.

Curiously, the phenomenon of cusp-shaped surface profiles is not only reserved for the highest waves. Numerical evidence [38] suggests that the solutions of the Whitham equation that undergoes wave breaking, do so by forming cusps in finite time. In fact, it is conjectured in [38] for the Whitham equation that large *positive* initial data results in the formation of square root cusps, while large *negative* initial data results in the formation of cube root cusps. Related is the work [53] which proves the existence of a family of solutions for the Burgers–Hilbert equation undergoing wave breaking by forming cube root cusps in finite time.

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Paper 1

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ON THE BIFURCATION DIAGRAM OF THE CAPILLARY-GRAVITY WHITHAM EQUATION

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Abstract. We study the bifurcation of periodic travelling waves of the capillary-gravity Whitham equation. This is a nonlinear pseudo-differential equation that combines the canonical shallow water nonlinearity with the exact (unidirectional) dispersion for finite-depth capillary-gravity waves. Starting from the line of zero solutions, we give a complete description of all small periodic solutions, unimodal as well bimodal, using simple and double bifurcation via Lyapunov–Schmidt reductions. Included in this study is the resonant case when one wavenumber divides another. Some bifurcation formulas are studied, enabling us, in almost all cases, to continue the unimodal bifurcation curves into global curves. By characterizing the range of the surface tension parameter for which the integral kernel corresponding to the linear dispersion operator is completely monotone (and therefore positive and convex; the threshold value for this to happen turns out to be $T = \frac{4}{\pi^2}$, not the critical Bond number $\frac{1}{3}$), we are able to say something about the nodal properties of solutions, even in the presence of surface tension. Finally, we present a few general results for the equation and discuss, in detail, the complete bifurcation diagram as far as it is known from analytical and numerical evidence. Interestingly, we find, analytically, secondary bifurcation curves connecting different branches of solutions; and, numerically, that all supercritical waves preserve their basic nodal structure, converging asymptotically in $L^2(\mathbb{S})$ (but not in L^∞) towards one of the two constant solution curves.

1. Introduction

We consider periodic travelling wave solutions of the capillary-gravity Whitham equation

$$u_t + M_T u_x + 2uu_x = 0 \tag{1.1}$$

where M_T is a Fourier multiplier operator defined via its symbol m_T as

$$\widehat{M_T f}(\xi) = m_T(\xi)\widehat{f}(\xi) = \left(\frac{(1+T\xi^2)\tanh(\xi)}{\xi}\right)^{\frac{1}{2}} \widehat{f}(\xi), \tag{1.2}$$

and the coefficient $T > 0$ denotes the strength of the surface tension. The symbol m_T arises as the linear dispersion relation for capillary-gravity water waves over a finite depth described by the Euler equations [30]. In the purely gravitational case, that is, when $T = 0$, the use of this symbol was proposed by Whitham as a way to generalise the KdV equation and remedy its strong dispersion [42]. Bifurcation in the gravitational setting has been investigated in [18, 19, 21]. We are here interested in completely characterising the local theory for travelling wave solutions of (1.1), and understanding their global extensions.

The overarching technique follows an approach similar to that used for the gravity Whitham equation in [19] and the Euler equations in [14], where a Lyapunov–Schmidt reduction is used to prove the existence of wave solutions through the application of the implicit function theorem. Here, however, the symbol of the linear dispersion has a different large-frequency behaviour: whereas it is $\sim |\xi|^{-1/2}$ in the gravity case, it changes to $\sim |\xi|^{1/2}$ in the presence of surface tension. Inspired by recent work on large waves for very weakly dispersive equations, we tackle the equation by inverting the linear operator, see (2.3), presenting us with a smoothing operator with good properties but that now acts nonlocally on a nonlinear term. Apart from the results presented in this paper, we see this as a first step toward handling large-amplitude theory for equations with mixed nonlocal and nonlinear terms. A study in that direction, but with a different order and global structure of the solutions, has been carried out in [6].

The organisation of the paper correspond to the development of our theory:

We start, in Section 2, with a study of the *inverse* of the Fourier multiplier operator M in (1.2). This is a smoothing operator of order $-\frac{1}{2}$ on any Fourier-based scale of functions spaces (such as the Sobolev and Zygmund spaces), that is realised as a convolution operator with a surface tension-dependent integral kernel K_T. We characterise the kernel K_T in Theorem 2.3, expressing it as a sum of three terms that are, optimally, in the regularity classes $\mathcal{C}^{-\frac{1}{2}}$, $\mathcal{C}^{\frac{3}{2}}$ and C^ω, respectively, where $\mathcal{C}^s$ is the scale of Zygmund spaces, and C^ω is the class of real-analytic functions. This is different from the regular Whitham symbol which, although of the same order, has only two terms when decomposed in the same manner [21]. Additionally, we estimate the decay rate of K_T and its compactness properties (in suitable spaces) which will play an important role in the global bifurcation analysis in Section 3. Finally, as in [21] we apply complex analysis techniques and the theory of Stieltjes functions to determine further properties of the convolution kernel, in particular the signs of its derivatives to infinite order. When the surface tension is big enough, $T \geqslant \frac{4}{\pi^2}$, we are able in Theorem 2.8 to show that the kernel is *completely monotone*, a delicate structural property shared by the kernel for the linear dispersion in the pure gravity case (not its inverse). Moreover, we can show that neither complete monotonicity nor monotonicity on a half-line is preserved if $0 < T < \frac{4}{\pi^2}$, showing in effect that the critical Bond number $\frac{1}{3}$ separating weak from strong surface tension is *not* the break-off value for the positivity of the kernel (or its stronger properties). How this affects solutions, is further discussed and studied in Section 5.

In Section 3 we perform the one-dimensional bifurcation of periodic waves from simple eigenvalues along the line of zero solutions. After an initial discussion of the eigenvalues of the linearised operator, and a scaling to reduce the problem to a fixed period, we use Lyapunov–Schmidt reduction to prove the existence of small-amplitude solutions in a vicinity of the simple eigenvalues (expressed using the wavespeed) in Theorem 3.1. The constructed waves are all unimodal and bell-shaped in a minimal period. They arise for both strong and weak surface tension; for strong surface tension they are the only type of waves in a $\mathcal{C}^s(\mathbb{S})$-vicinity of the line of zero solutions, $s > 0$. Although one could have carried out the simple bifurcation using the Crandall–Rabinwitz theorem [27], we choose to prove Theorem 3.1 using a Lyapunov–Schmidt reduction as a preparation for the two-dimensional case (which would otherwise be harder to understand). Under a simple condition that relates the wavenumber to the surface tension and period, we prove the continuation of the local solution curves to global ones in Theorem 3.6. This condition may be related to sub- and supercritical bifurcation, and we see in Remark 3.7 that both cases can appear. The modulational stability of these waves in the small-amplitude case has been studied in [24]

A challenge and interesting feature of the capillary-gravity case is that weak surface tension allows for a non-monotone dispersion relation (see Figure 1) and double eigenvalues of the corresponding linearised operator (in spaces of even functions). We handle this case in Section 4. To analytically capture the larger dimension of the space of solutions nearby the trivial ones, one requires an additional free parameter in addition to the wavespeed, used in the one-dimensional bifurcation. In line with [20] we choose to use the period as this extra parameter, while holding the surface tension fixed. The result, presented in Theorem 4.1, depends on the resonances between the two frequencies appearing in the nullspace: if one of the wavenumbers is a multiple of the other, one obtains a slit disk of solutions, excluding bifurcation straight in the direction of the higher wavenumber; if not, one obtains a full open disk of solutions, see Figure 2. These results are in line with similar ones in [14, 34, 38], and include — when projecting the full disk onto a fixed period — a curve of bimodal rippled waves connecting waves of different wavenumbers (secondary bifurcation). This technique has later been used also in [2]. The existence of these interconnecting branches of waves have been corroborated numerically, showing persistence with respect to perturbations in the surface tension parameter [35]. The nonexistence of the pure higher mode in the resonant case of Theorem 4.1 (ii) has also been confirmed numerically in the same paper. More generally, Wilton ripples, as these kinds of waves are sometimes called, have earlier been found to exist for the Euler equations with surface tension [34, 38], and their spectral stability has been numerically investigated in [39]. They also exist in the presence of vorticity [31], even without capillarity [14, 20]. In that case, one may even construct arbitrary large kernels [1, 15], and corresponding multi-dimensional solution sets [29].

Our motivation for this investigation has arisen from two different directions: one is the study of the (very) weakly dispersive equations with nonlocal nonlinearities, and especially their large-amplitude theories; the other is the mathematically qualitative analogues between the full water-wave problem and the family of fully dispersive Whitham-type equations. While numerical bifurcation of steady water waves with surface tension have been earlier carried out [8], and display striking resemblances to our case, it is not known how to control the waves along the bifurcation curves when surface tension is present, and our results show that, at least for weak surface tension, the looping alternative in Theorem 3.6 is possible*. Our initial hope was that, using methods as in [16, 21], one would be able to reach a conclusion for larger waves. In Section 5 we turn to this question, as well as discussing the general picture of bifurcation in the capillary-gravity Whitham equation. While we are indeed able to reach a partial result, preserving the nodal properties to $\mathcal{O}(1)$-height of the solutions in Proposition 5.4, the final evolution of solution curves is very challenging to handle analytically. While both our preliminary calculations and numerical simulations for this paper indicate that one can follow curves of supercritical bell-shaped solutions all the way to $c \to \infty$, and that they converge, asymptotically in $L^2(\mathbb{S})$, towards the curve of constant solutions $u = c - 1$, they do not converge in L^∞, and the analysis is complicated by that the equation lies exactly at the Sobolev-critical balance $s = \frac{1}{2}$, $p = 2$ and $n = 1$. We discuss both our findings

*See also the discussion in Section 5 concerning related results for the Euler equations.

and conjectures in detail in Section 5. For a quick overview, we refer to Figures 3 and 4.

Finally, we give in Appendix A some bifurcation formulas.

2. Properties of the convolution kernel K_T

Traveling-wave solutions of the form $u(x-ct)$ satisfy the (profile) equation

$$-cu + M_T u + u^2 = 0, \tag{2.1}$$

where we have integrated once and used Galilean invariance to set the constant of integration to zero. Since m_T is strictly positive on $\mathbb{R}$, the operator M_T is invertible (for example in any Fourier-based space) with inverse L_T defined via

$$\widehat{L_T f}(\xi) = l_T(\xi)\widehat{f}(\xi), \qquad l_T(\xi) = (m_T(\xi))^{-1}. \tag{2.2}$$

In particular, the capillary-gravity Whitham equation (2.1) can be rewritten in the "smoothing" form

$$u - cL_T(u) + L_T(u^2) = 0, \tag{2.3}$$

where $L_T = K_T *$ and K_T is the convolution kernel corresponding to the symbol l_T. Note that the form (2.3) is resemblant of the Whitham equation itself, but with a nonlocal nonlinearity. *By a solution of* (2.1) *(respectively* (2.3)*), we shall mean a real-valued, continuous and bounded function u that satisfies* (2.1) *(respectively* (2.3)*) everywhere.*

In the rest of this work we shall make heavy use of the properties of the convolution kernel K_T and its symbol. Our choice of Fourier transform is

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}\,\mathrm{d}x.$$

To start, note that $K_T = \mathcal{F}^{-1}l_T$ is smooth away from the origin with

$$\int_{\mathbb{R}} K_T(x)\,\mathrm{d}x = \lim_{\xi\to 0} l_T(\xi) = 1 \tag{2.4}$$

and

$$\lim_{x\to 0} K_T(x) = \frac{1}{2\pi}\int_{\mathbb{R}} l_T(\xi)\,\mathrm{d}\xi = +\infty.$$

Moreover, since l_T is analytic, K_T has rapid decay at $\pm\infty$, whence $K_T \in L^1(\mathbb{R})$ provided that the blow-up at $x=0$ is not too fast. Later in this section, we will show that the singularity at the origin is of order $|x|^{-\frac{1}{2}}$, with a second-leading term of somewhat smoother order, and that the convolution kernel is completely monotone for strong enough surface tension.

2.1. **Analyticity of the symbol.** We start by studying the analytic extension of l_T to the complex plane; the results to come will be important to establish both the decay and the complete monotonicity of K_T. Define the meromorphic function

$$\varrho_T(\zeta) = \frac{\zeta}{(1+T\zeta^2)\tanh(\zeta)}, \tag{2.5}$$

with ζ a complex number. We want to understand the complex extension $\sqrt{\varrho_T}$ of l_T, where $\sqrt{\cdot}$ denotes the principal branch of the square root. Thus we determine the pre-image $\varrho_T^{-1}((-\infty,0])$ and the set of singularities of ϱ_T. As it turns out, the union of these (problematic) sets lie solely on the imaginary axis. To show this, we introduce the sets

$$\begin{aligned} Z_c &= \left\{\pi(k-\tfrac{1}{2})\colon k \in \mathbb{Z}\right\}, \\ Z_s &= \left\{\pi k\colon k \in \mathbb{Z}\setminus\{0\}\right\}, \\ Z_T &= \left\{-\tfrac{1}{\sqrt{T}}, \tfrac{1}{\sqrt{T}}\right\}, \end{aligned}$$

that is, the zeros of $\cos(\zeta)$, $\frac{\sin(\zeta)}{\zeta}$, and $1-T\zeta^2$, respectively. Finally, recall that the *symmetric difference* between two sets A and B is the set $A \triangle B$ of elements either in A and not B, or contrariwise[†]

Lemma 2.1. *Let $\zeta = \xi + i\eta$. Then $\varrho_T(\zeta)$ takes a zero or infinite value exactly if $\xi = 0$ and $\eta \in Z_s \cup (Z_c \triangle Z_T)$. Further, $\varrho_T(\zeta)$ is negative exactly when the following three conditions hold: $\xi = 0$, $\eta \notin Z_s \cup (Z_c \triangle Z_T)$, and the intersection $(0,|\eta|) \cap \big((Z_c \cup Z_s) \triangle Z_T\big)$ contains an odd number of elements.*

Proof. By the infinite product formulas for $\sinh\zeta$ and $\cosh\zeta$ we obtain

$$\varrho_T(\zeta) = \frac{1}{1+T\zeta^2}\prod_{n=1}^{\infty}\frac{1+\frac{\zeta^2}{\pi^2(n-\frac{1}{2})^2}}{1+\frac{\zeta^2}{\pi^2 n^2}}. \tag{2.6}$$

The first part of the lemma now follows immediately, where the symmetric difference accounts for removable singularities should the term $(1+T\zeta^2)$ coincide with a term of the form $1+\frac{\zeta^2}{\pi^2(n-\frac{1}{2})^2}$. For the second part we start by showing that ϱ_T is never negative away from the imaginary axis. As ϱ_T is symmetric about zero, we restrict our attention to $\xi > 0$. We

[†]That is, $(A \triangle B) = (A \cap B^c) \cup (B \cap A^c)$.

have

$$\begin{aligned}\operatorname{Re}\Big[\cosh(\zeta)\overline{\sinh(\zeta)}\Big] &= \frac{1}{2}\sinh(2\xi) > 0,\\ \operatorname{Re}\Big[\zeta\,\overline{(1+T\zeta^2)}\Big] &= \xi + \xi T(\xi^2+\eta^2) > 0,\end{aligned}$$

and consequently $|\arg(\frac{\zeta}{1+T\zeta^2})|, |\arg(\frac{1}{\tanh(\zeta)})| < \frac{\pi}{2}$. This in turn implies that $|\arg(\varrho_T(\zeta))| < \pi$, and so $\varrho_T(\zeta)$ cannot be negative. Restricting our attention to the imaginary axis ($\zeta = i\eta$) and away from zeroes and singularities, it is clear from (2.6) that $\varrho_T(i\eta)$ is real valued and satisfies

$$\operatorname{sgn}(\varrho_T(i\eta)) = \operatorname{sgn}(1-T\eta^2)\prod_{n=1}^{\infty}\operatorname{sgn}\Big(1-\frac{\eta^2}{\pi^2(n-\frac{1}{2})^2}\Big)\operatorname{sgn}\Big(1-\frac{\eta^2}{\pi^2 n^2}\Big).$$

As $\varrho_T(i\eta)$ is positive for $\eta = 0$, it is negative exactly when an odd number of factors in the expression above has swapped sign. This is equivalent to the last part of the lemma. □

In Section 2.2 we will use Paley–Wiener theory to establish the decay rate of K_T; we will need to know the maximal vertical analytic extension of l_T into the complex plane. This is immediate from the previous result, and so we record the following corollary.

Corollary 2.2. *The symbol l_T extends analytically onto the strip $\mathbb{R}\times i(-\delta^*,\delta^*)$, where*

$$\delta^* = \begin{cases}\min\{\frac{1}{\sqrt{T}},\frac{\pi}{2}\}, & T\neq 4/\pi^2,\\ \pi & T = 4/\pi^2.\end{cases}$$

We shall also use decay of symbols on horizontal lines in $\mathbb{R}\times i(-\delta^*,\delta^*)$. While l_T decays too slow ($\sim|\xi|^{-\frac{1}{2}}$) to be in $L^2(\mathbb{R})$, its derivatives decay sufficiently fast (at least as $|\xi|^{-\frac{3}{2}}$). In particular, there is an increasing function $\tau\colon[0,\delta^*)\to\mathbb{R}^+$ such that $|l_T'(\zeta)| \leqslant \tau(|\eta|)(1+|\xi|)^{-\frac{3}{2}}$, which is readily seen by differentiating and exploiting that $\coth'$ decays exponentially along fixed horizontal lines in the complex plane.

2.2. **Regularity properties and decay.** In this subsection we split K_T into three canonical parts, and determine the precise regularity of these. We also record the rapid decay and smoothing properties of K_T. Write

$$l_T = l_{-\frac{1}{2}} + l_{\frac{3}{2}} + l_\omega,$$

with $l_{-\frac{1}{2}}(\xi) = \frac{1}{\sqrt{T|\xi|}}$, $l_{\frac{3}{2}}(\xi) = \sqrt{\frac{|\xi|}{1+T\xi^2}} - \frac{1}{\sqrt{T|\xi|}}$ and $l_\omega(\xi) = l_T(\xi) - \sqrt{\frac{|\xi|}{1+T\xi^2}}$. The subscripts represent the regularity of each corresponding term of K_T, as will be seen. The decay of $l_{-\frac{1}{2}}(\xi) \eqsim |\xi|^{-\frac{1}{2}}$ for $|\xi| \gg 1$ is clear, and for any fixed $T > 0$, it is readily seen that

$$l_{\frac{3}{2}}(\xi) \eqsim -|\xi|^{-\frac{5}{2}},$$

and

$$l_\omega(\xi) = \sqrt{\frac{|\xi|}{1+T\xi^2}} \left(\sqrt{\coth(|\xi|)} - 1 \right) \eqsim |\xi|^{-\frac{1}{2}} e^{-2|\xi|},$$

both for $|\xi| \gg 1$.

To establish the regularity of the corresponding parts of K_T we shall use Zygmund spaces. Let $\{\psi_j^2\}_{j=0}^\infty$ be a partition of unity with $\psi_0(\xi)$ supported in $|\xi| \leqslant 1$, $\psi_1(\xi)$ supported in $\frac{1}{2} \leqslant |\xi| \leqslant 2$, and $\psi_j(\xi) = \psi_1(2^{1-j}\xi)$ for $j \geqslant 2$. Then the support of each ψ_j is concentrated around $\xi \eqsim 2^j$. With $D = -i\partial_x$, the Fourier multiplier operators $\psi_j(D)\colon f \mapsto \mathcal{F}^{-1}(\psi_j \hat{f})$ characterises the Zygmund spaces: we say $u \in \mathcal{C}^s(\mathbb{R})$ if

$$\|u\|_{\mathcal{C}^s(\mathbb{R})} = \sup_j 2^{js} \|\psi_j^2(D)u\|_{L^\infty} \tag{2.7}$$

is finite. For non-integer values of $s \geqslant 0$ the Zygmund spaces coincide with the standard (inhomogeneous) Hölder spaces‡,

$$\mathcal{C}^s(\mathbb{R}) \cong C^s(\mathbb{R}), \qquad s \in \mathbb{R}_+ \setminus \mathbb{N}_0,$$

and one furthermore has the embedding $C^k(\mathbb{R}) \hookrightarrow \mathcal{C}^k(\mathbb{R})$ for integer values of k. We refer the reader to [37, Section 13.8] and [22, Section 1.4] for further details.

Now, the symbols $l_{-\frac{1}{2}}$, $l_{\frac{3}{2}}$ and l_ω all have well-defined Fourier transforms, and we let

$$\begin{aligned} K_{-\frac{1}{2}}(x) &= \mathcal{F}^{-1}(1/\sqrt{T|\cdot|})(x), \\ K_{\frac{3}{2}}(x) &= \mathcal{F}^{-1}(l_{\frac{3}{2}})(x), \\ K_\omega(x) &= \mathcal{F}^{-1}(l_\omega)(x), \end{aligned}$$

so that

$$K_T(x) = \mathcal{F}^{-1}(l_T)(x) = K_{-\frac{1}{2}}(x) + K_{\frac{3}{2}}(x) + K_\omega(x).$$

From Fourier analysis we know that $\mathcal{F}^{-1}(1/\sqrt{|\cdot|})(x) = 1/\sqrt{2\pi|x|}$ and, additionally, that the exponential decay of $l_\omega(\xi)$ for $|\xi| \gg 1$ implies that K_ω is real-analytic by Paley–Wiener's first theorem [33]. The optimal regularity of $K_{\frac{3}{2}}$ follows from the following theorem about the integral kernel K_T.

‡Throughout, we use the notation that $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Theorem 2.3. *The integral kernel* K_T *may be written as*

$$K_T(x) = \frac{1}{\sqrt{2\pi T|x|}} + K_{\frac{3}{2}}(x) + K_\omega(x),$$

where the second term belongs to the optimal Hölder class $C^{\frac{3}{2}}$ *and the third is real-analytic. The singularity of* K_T *thus has the characterization*

$$\lim_{x\to 0} \sqrt{|x|}\, K_T(x) = \frac{1}{\sqrt{2\pi T}}.$$

Moreover,

$$|K_T(x)| \lesssim e^{-\delta|x|} \qquad \text{for } |x| > 1,$$

with $\delta < \delta^*$ *as given in Corollary 2.2. As a consequence,* $K_T \in L^1(\mathbb{R})$.

Proof. Most of the first claim was established in the preceding discussion, and only the regularity of $K_{\frac{3}{2}}$ remains. Notice that $l_{\frac{3}{2}}$ is always of negative sign, and thus so is the product $\psi_j^2(\xi) l_{\frac{3}{2}}(\xi)$. This means

$$\|\psi_j^2(D) K_{\frac{3}{2}}\|_{L^\infty} = \|\psi_j^2(\xi) l_{\frac{3}{2}}\|_{L^1}.$$

Further, we exploit the decay of $l_{\frac{3}{2}}$ and the compact support of ψ_j^2, to obtain

$$\|\psi_j^2(\xi) l_{\frac{3}{2}}\|_{L^1} \eqsim \int_{2^{j-1}}^{2^{j+1}} |\xi|^{-\frac{5}{2}}\, \mathrm{d}\xi \eqsim 2^{-\frac{3}{2}j}.$$

Combining these two equations, we conclude in view of (2.7) and the equivalence between Hölder and Zygmund norms for non-integer indices that $K_{\frac{3}{2}}$ lies in the optimal Hölder class $C^{\frac{3}{2}}(\mathbb{R})$. As for the decay rate of K_T, we instead prove this estimate for the more regular expression $x \mapsto xK_T(x)$, which again proves it for K_T. The exponential decay of $x \mapsto xK_T(x)$ is a direct consequence of Corollary 2.2 and the discussion thereafter combined with Paley–Wiener theory (see, for example, [33, Theorem IV]). One can obtain further asymptotic estimates as in [21, Prop. 2.1 and Cor. 2.26]. $\square$

We conclude this subsection by recording some mapping properties of the convolution operator $L_T = K_T *$ that will be vital to the global bifurcation analysis in Section 3 and additionally employed in the analysis in Section 5. Let $\mathbb{S}$ be the one-dimensional unit sphere of circumference 2π, and note that the Hölder and Zygmund spaces are straightforward to define on the compact manifold $\mathbb{S}$ (these are the 2π-periodic functions in the larger spaces $C^s(\mathbb{R})$ and $\mathcal{C}^s(\mathbb{R})$).

Lemma 2.4. *For each* $T > 0$ *and each* $s \geqslant 0$, L_T *is a continuous linear mapping* $\mathcal{C}^s(\mathbb{R}) \to \mathcal{C}^{s+1/2}(\mathbb{R})$ *and is hence compact on* $\mathcal{C}^s(\mathbb{S})$.

Proof. Consider $T > 0$ fixed. We want to show that the inequality

$$\|\psi_j^2(D)L_T u\|_{L^\infty} \lesssim 2^{-\frac{j}{2}}\|\psi_j^2(D)u\|_{L^\infty}, \tag{2.8}$$

is valid for all $j \in \mathbb{N}_0$. We prove this estimate for $j \geqslant 1$; the case $j = 0$ must be done separately, but the calculation is similar to what follows and so we exclude it. Pick a smooth function φ supported in $\frac{1}{3} \leqslant |\xi| \leqslant 3$ satisfying $\varphi(\xi) = 1$ whenever $\frac{1}{2} \leqslant |\xi| \leqslant 2$. For $j \geqslant 1$, we define $\varphi_j(\xi) = \varphi(2^{1-j}\xi)$, and observe that $\varphi_j\psi_j^2 = \psi_j^2$. Exploiting this relationship, we deduce

$$\begin{aligned}
\|\psi_j^2(D)L_T u\|_{L^\infty} &= \|\mathcal{F}(\psi_j^2(\xi)l_T\hat{u})\,\|_{L^\infty} \\
&= \|\mathcal{F}(\varphi_j\psi_j^2(\xi)l_T\hat{u})\|_{L^\infty} \\
&= \|\mathcal{F}(l_T\varphi_j) * (\psi_j^2(D)u)\|_{L^\infty} \\
&\leqslant \|\mathcal{F}(l_T\varphi_j)\|_{L^1}\|\psi_j^2(D)u\|_{L^\infty},
\end{aligned}$$

where we have used Young's inequality for convolution. The proof will be complete if we can establish $\|\mathcal{F}(l_T\varphi_j)\|_{L^1} \lesssim 2^{-\frac{j}{2}}$; we do this by splitting the integral, $\|\cdot\|_{L^1} = \|\cdot\|_{L^1(|x|\leqslant 2^{-j})} + \|\cdot\|_{L^1(|x|>2^{-j})}$, and then prove the bound for each part separately. From the general fact $\|f\|_{L^p} \leqslant |\mathrm{supp}(f)|^{\frac{1}{p}}\|f\|_{L^\infty}$, we deduce two important inequalities for the calculations to come:

$$\|l_T\varphi_j\|_{L^1} \lesssim 2^{\frac{j}{2}}, \qquad \|(l_T\varphi_j)'\|_{L^2} \lesssim 2^{-j}.$$

These follows from the bounds $|l_T(\xi)| \lesssim |\xi|^{-\frac{1}{2}}$, $|l_T'(\xi)| \lesssim |\xi|^{-\frac{3}{2}}$ and $(\varphi_j)' \eqsim 2^{-j}(\varphi')_j$, and the observation that the support of φ_j (and φ_j') is of size 2^j and located about $|\xi| \eqsim 2^j$. We now conclude the proof with the two calculations promised above; the first is straight forward

$$\|\mathcal{F}(l_T\varphi_j)\|_{L^1(|x|\leqslant 2^{-j})} \lesssim 2^{-j}\|\mathcal{F}(l_T\varphi_j)\|_{L^\infty} \leqslant 2^{-j}\|l_T\varphi_j\|_{L^1} \lesssim 2^{-\frac{j}{2}}.$$

For the second, we use basic Fourier analysis, the Cauchy–Schwarz inequality, and the Plancherel theorem:

$$\begin{aligned}
\|\mathcal{F}(l_T\varphi_j)\|_{L^1(|x|>2^{-j})} &= \|\tfrac{1}{x}\mathcal{F}((l_T\varphi_j)')\|_{L^1(|x|>2^{-j})} \\
&\leqslant \|\tfrac{1}{x}\|_{L^2(|x|>2^{-j})}\|\mathcal{F}((l_T\varphi_j)')\|_{L^2(|x|>2^{-j})} \\
&\lesssim 2^{\frac{j}{2}}\|(l_T\varphi_j)'\|_{L^2} \\
&\lesssim 2^{-\frac{j}{2}},
\end{aligned}$$

and so we have established (2.8). It is immediate that L_T maps $\mathcal{C}^s(\mathbb{R})$ to $\mathcal{C}^{s+\frac{1}{2}}(\mathbb{R})$ continuously, and combining this with the compact embedding $\mathcal{C}^{s+\frac{1}{2}}(\mathbb{S}) \hookrightarrow \mathcal{C}^s(\mathbb{S})$ we get the full result. $\square$

2.3. **Montonicity and complete monotonicity.** We conclude this section by showing that K_T is completely monotone for sufficiently large T. This result will be employed in our analysis in Section 5. A function $g : (0, \infty) \to [0, \infty)$ is called *completely monotone* if g is infinitely differentiable with

$$(-1)^n g^{(n)}(\lambda) \geqslant 0$$

for $n = 0, 1, 2, \ldots$ and all $\lambda > 0$. If it can furthermore be written in the form

$$g(\lambda) = \frac{a}{\lambda} + b + \int_{(0,\infty)} \frac{1}{\lambda + t} \,\mathrm{d}\sigma(t)$$

for some constants $a, b > 0$, with σ a Borel measure on $(0, \infty)$ such that $\int_{(0,\infty)} \frac{1}{1+t} \,\mathrm{d}\sigma(t) < \infty$, then it is called *Stieltjes.* Our interest in such functions is motivated by the following two results, taken from [21] and [36].

Lemma 2.5. [21] *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : (0, \infty) \to \mathbb{R}$ be two functions satisfying $f(\xi) = g(\xi^2)$ for $\xi \neq 0$. Then f is the Fourier transform of an even, integrable, and completely monotone function if and only if g is Stieltjes with $\lim_{\lambda \searrow 0} g(\lambda) < \infty$ and $\lim_{\lambda\to\infty} g(\lambda) = 0$.*

Lemma 2.6. [36] *Let g be a positive function on $(0, \infty)$. Then g is Stieltjes if and only if $\lim_{\lambda\searrow 0} g(\lambda)$ exists in $[0, \infty]$ and g extends analytically to $\mathbb{C} \setminus (-\infty, 0]$ such that $\mathrm{Im}(z) \cdot \mathrm{Im}(g(z)) \leqslant 0$.*

With $f(\xi) = l_T(\xi)$ and $g(\xi) = l_T(\sqrt{\xi})$ we want to employ the two above results to conclude that $K_T = \mathcal{F}^{-1}(l_T(\xi))$ is completely monotone for T sufficiently large. Since l_T has a unit limit at the origin and a vanshing limit at infinity, it only remains to prove that $l_T(\sqrt{\cdot})$ is Stieltjes. In light of Lemma 2.6 it is useful to note that $l_T(\sqrt{\cdot})$ indeed extends analytically to $\mathbb{C} \setminus (-\infty, 0]$. Its extension is $\zeta \mapsto \sqrt{\varrho_T(\sqrt{\zeta})}$, where ϱ_T is as in (2.5) and $\sqrt{\cdot}$ is the principal branch of the square root. To see that this extension is well defined, note that $\sqrt{\cdot}$ maps $\mathbb{C} \setminus (-\infty, 0]$ into the right half-plane $\mathbb{C}_{\xi>0}$, while Lemma 2.1 guarantees that ϱ_T maps $\mathbb{C}_{\xi>0}$ into $\mathbb{C} \setminus (-\infty, 0]$. Consequently, $\varrho_T(\sqrt{\mathbb{C} \setminus (-\infty, 0]}) \subseteq \mathbb{C} \setminus (-\infty, 0]$, and so it has principal branch square root. We are ready to prove Theorem 2.8, where we determine a critical value $T_* = \frac{4}{\pi^2}$ of the surface tension T, for which K_T is completely monotone whenever $T \geqslant T_*$. Note that T_* *does not* correspond to the, likewise critical, Bond number $T = \frac{1}{3}$ that separates strong from weak surface tension; in fact, $T_* > \frac{1}{3}$. Further, this result is sharp since K_T is not monotone for $T \in (0, T_*)$. As we shall see, the image of K_T in this regime contains negative values which rules out monotonicity as Theorem 2.3 guarantees that K_T is positive near zero and decays to zero at infinity.

In the calculations to come, we will make use of the class of so-called positive definite functions. A function $f\colon \mathbb{R} \to \mathbb{C}$ is said to be *positive definite* if for every $n \in \mathbb{N}$ and $\boldsymbol{\xi} \in \mathbb{R}^n$ the $n \times n$ matrix $[f(\xi_i - \xi_j)]_{i,j=1}^n$ is positive semi-definite. We point out the following standard results [9].

Lemma 2.7. *The following statements are true.*

(i) *[Bochner's Theorem] Any positive definite function is the Fourier transform of a non-negative, finite Borel measure.*
(ii) *[Schur's Theorem] A countable product of positive definite functions is positive definite.*
(iii) *If $f : \mathbb{R} \to \mathbb{C}$ is positive definite, then the global maximum of f occurs at $x = 0$.*
(iv) *The function $f(x) = \frac{1+ax^2}{1+bx^2}$ is positive definite if and only if $b \geqslant a \geqslant 0$.*

With the above preliminaries, we now state the main result for this section.

Theorem 2.8. *For $T \geqslant \frac{4}{\pi^2}$, the kernel K_T is completely monotone on $(0,\infty)$. Further, for $0 < T < \frac{4}{\pi^2}$, the image of K_T includes negative values. Consequently, K_T is not monotone on $(0,\infty)$.*

Proof. We first prove that K_T is completely monotone for $T \geqslant \frac{4}{\pi^2}$. By Lemma 2.5 and Lemma 2.6 and the discussion thereafter, we conclude that K_T is completely monotone exactly if $\mathrm{Im}(\zeta) \cdot \mathrm{Im}\sqrt{\varrho_T(\sqrt{\zeta})} \leqslant 0$ for $\zeta \in \mathbb{C} \setminus (-\infty, 0]$. This property is satisfied for $\sqrt{\varrho_T(\sqrt{\cdot})}$ if and only if it is satisfied for $\varrho_T(\sqrt{\cdot})$, as the latter function maps $\mathbb{C} \setminus (-\infty, 0]$ to itself (Lemma 2.1). Moving the first factor of $\cosh \zeta$ out of the infinite product in (2.6), we obtain

$$\varrho_T(\xi) = \frac{1 + \frac{4}{\pi^2}\xi^2}{1 + T\xi^2} \prod_{n=1}^{\infty} \frac{1 + \frac{\xi^2}{\pi^2(n+\frac{1}{2})^2}}{1 + \frac{\xi^2}{\pi^2 n^2}}. \tag{2.9}$$

Substituting $\xi \mapsto \sqrt{\zeta}$ in (2.9), and taking the complex argument of both sides, we obtain

$$\begin{aligned} \arg\left(\varrho_T(\sqrt{\zeta})\right) =& \left[\arg\left(1 + \frac{4}{\pi^2}\zeta\right) - \arg(1 + T\zeta)\right] \\ &+ \sum_{n=1}^{\infty} \left[\arg\left(1 + \frac{\zeta}{\pi^2(n+\frac{1}{2})^2}\right) - \arg\left(1 + \frac{\zeta}{\pi^2 n^2}\right)\right]. \end{aligned} \tag{2.10}$$

This equation is valid whenever the right hand side takes values in $(-\pi, \pi)$, which in turn is always true when $\zeta \in \mathbb{C} \setminus (-\infty, 0]$ as the RHS is continuous in ζ, zero for $\zeta > 0$ and prevented from taking the values $\pm\pi$ as $\varrho_T(\sqrt{\zeta})$ is non-negative (Lemma 2.1). Moreover, when $\mathrm{Im}(\zeta) > 0$, it is easily seen that $t \mapsto \arg(1 + t\zeta)$ is strictly increasing in $t \in \mathbb{R}$, and so each square bracket in (2.10) is negative (the first non-positive), further implying $\mathrm{Im}(\zeta) \cdot \mathrm{Im}\sqrt{\varrho_T(\sqrt{\zeta})} < 0$. After a similar argument for $\mathrm{Im}(\zeta) < 0$, we obtain the desired conclusion.

For the second part of the theorem, we observe that by Bochner's Theorem in Lemma 2.7(i), K_T is non-negative if and only if its Fourier transform l_T is a positive definite function; we now prove that the latter statement is false when $0 < T < \frac{4}{\pi^2}$. Note first that for $0 < T < \frac{1}{3}$, this follows immediately from Lemma 2.7(iii) as l_T does not have a global maximum at $\xi = 0$ (see Figure 1). Suppose instead that $\frac{1}{3} \leqslant T < \frac{4}{\pi^2}$. If l_T is positive definite, then Lemma 2.7(ii) implies the same would be true for its square $\xi \mapsto \varrho_T(\xi)$. To this end, we write (2.9) as

$$\varrho_T(\xi) = \frac{1 + \frac{4}{\pi^2}\xi^2}{1 + T\xi^2}\ \varphi(\xi),$$

which, after introducing the positive constants $\alpha = 4/(T\pi^2)$ and $\beta = \alpha - 1$, can be further rewritten as

$$\varrho_T(\xi) = \Big(\alpha - \frac{\beta}{1 + T\xi^2}\Big)\varphi(\xi) =: \alpha\varphi(\xi) - \beta\psi(\xi).$$

By Lemma 2.7, both φ and ψ are positive definite as they are (countable) products of positive definite functions, and thus $\hat{\varphi}, \hat{\psi} \geqslant 0$ by Bochner's Theorem. Note that φ has a complex analytic extension to the strip $\mathbb{R} \times i(-\pi, \pi)$, while ψ can not be extended to a larger strip than $\mathbb{R} \times i(\frac{-1}{\sqrt{T}}, \frac{1}{\sqrt{T}})$. Since $\frac{1}{\sqrt{T}} \leqslant \sqrt{3} < \pi$, we can pick some $\gamma \in (\frac{1}{\sqrt{T}}, \pi)$ and use Paley–Wiener theory [33] and Cauchy–Schwarz to conclude that

$$0 < \int_{\mathbb{R}} \widehat{\varphi}(x) e^{\gamma|x|}\,\mathrm{d}x < \infty \quad \text{and} \quad \int_{\mathbb{R}} \widehat{\psi}(x) e^{\gamma|x|}\,\mathrm{d}x = +\infty,$$

which further implies

$$\int_{\mathbb{R}} \widehat{\varrho_T}(x) e^{\gamma|x|}\,\mathrm{d}x = -\infty.$$

By Bochner's Theorem, $\xi \mapsto \varrho_T(\xi)$ is not positive definite, and so neither is l_T, which concludes the proof. □

Before we end this section, we note that there is a range of values of strong surface tension $T \in (\frac{1}{3}, \frac{4}{\pi^2})$ where the kernel K_T is not monotone. As we will see, this has implications when trying to establish monotonicity of solutions along the supercritical global solution branches described in

Section 3.3 below; see Proposition 5.4 and the discussion in Section 5 in general.

3. One-dimensional bifurcation

Since $K \in L^1(\mathbb{R})$, it may be periodised to an arbitrary period. In particular, given a 2π-periodic $f \in L^\infty(\mathbb{R})$ we can define the action of $L_T = K_T *$ on f through a convolution of f with a 2π-periodic kernel K_p over a single period:

$$L_T f(x) = \int_{\mathbb{R}} K_T(x-y) f(y)\,\mathrm{d}y = \int_{-\pi}^{\pi} \left(\sum_{k\in\mathbb{Z}} K_T(x-y+2k\pi) \right) f(y)\,\mathrm{d}y$$
$$=: \int_{-\pi}^{\pi} K_p(x-y) f(y)\,\mathrm{d}y.$$

Clearly K_p is even, strictly positive on $\mathbb{R}$ and satisfies $\|K_p\|_{L^1(-\pi,\pi)} = 1$. Further, by Theorem 2.3 we know that K_p is smooth on $\mathbb{R} \setminus 2\pi\mathbb{Z}$, and that for $T > \frac{4}{\pi^2}$ it follows by Theorem 2.8 and [21, Proposition 3.2] that K_p is completely monotone function on the half period $(0,\pi)$. To find nontrivial solutions of the equation (2.1), or, equivalently, of (2.3), we fix $s > 1/2$ and define a map $F\colon \mathcal{C}^s_{\text{even}}(\mathbb{S}) \times \mathbb{R} \to \mathcal{C}^s_{\text{even}}(\mathbb{S})$ via

$$F(u,c) = u - cL_T(u) + L_T(u^2), \tag{3.1}$$

where $\mathcal{C}^s_{\text{even}}(\mathbb{S})$ is the subspace of even functions in $\mathcal{C}^s(\mathbb{S})$. Note this map is well-defined since $\mathcal{C}^s_{\text{even}}(\mathbb{S})$ is a Banach algebra for any $s > 0$. Then the roots of F correspond to the even and 2π-periodic solutions of (2.1) with wavespeed c. The choice $s > \frac{1}{2}$ is by convenience, as functions of that regularity have absolutely convergent Fourier series [26].

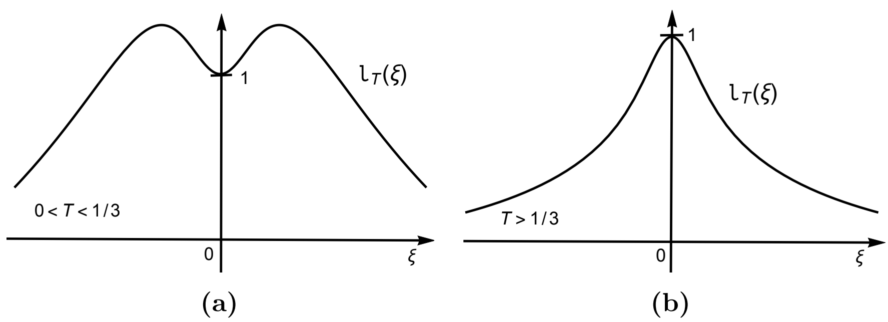

Figure 1. Schematic drawings of the behavior of the symbol $l_T(\xi)$ for (a) weak surface tension $0 < T < 1/3$ and for (b) strong surface tension $T > 1/3$. In both cases, the symbol is strictly positive and decays as $|\xi|^{-1/2}$ as $|\xi| \to \infty$.

Now, we begin with the observation that $F(0, c) = 0$ for all $c \in \mathbb{R}$ and that the linearised operator

$$D_u F[0, c] = \mathrm{Id} - cL_T$$

has a nontrivial kernel in $\mathcal{C}^s_{\text{even}}(\mathbb{S})$ if and only if $c\, l_T(k) = 1$ for some $k \in \mathbb{N}_0$ (we intentionally include the case $k = 0$ as it will play a role in the two-dimensional bifurcation to come). Consequently, for a fixed $c \in \mathbb{R}$ we have

$$\ker D_u F[0, c] = \mathrm{span}\left\{\cos(kx) \colon k \in \mathbb{N}_0 \text{ such that } c l_T(k) = 1\right\}, \quad (3.2)$$

and hence the multiplicity of the kernel depends sensitively on the graph of the function $l_T(\xi)$. In particular, if $T > 1/3$ then $l_T(\xi)$ is monotone decreasing on $\mathbb{R}_+$ and hence the above kernel is simple: see Figure 1. If $0 < T < 1/3$, however, the function l_T has exactly one local extremum (a maximum) in the interior of $\mathbb{R}_+$, whence opening the possibility of two different positive integers for which $l_T(m) = l_T(k)$: again, see Figure 1. A simple calculation shows that for a fixed $k \in \mathbb{N}_0$, the kernel will be simple if and only if $T \notin \{T_*(n; k)\}_{n \in \mathbb{N}_0}$, where[§]

$$T_*(n; k) := \frac{n \tanh(k) - k \tanh(n)}{kn\,(n \tanh(n) - k \tanh(k))},$$

while it will have multiplicity exactly two when $T = T_*(n; k)$ for some $n \in \mathbb{N}_0$.

Note that for each fixed k, the function $T_*(\cdot; k)$ tends to zero as $n \to \infty$, as does $T_*(0; k)$ when $k \to \infty$. It is also not hard to see that $T_*(0; k)$ is a strictly decreasing function of k. Numerical plots indicates that also the function $T_*(\cdot; k)$ is strictly decreasing, but we will not use this monotonicity property in our proofs.

Throughout the remainder of this section, we turn our attention to the branches of solutions $\{(u, c)\}$ bifurcating from the trivial line $u = 0$ at some wavespeed c_* for a fixed value of the surface tension $T > 0$ and where the kernel of $D_u F[0, c_*]$ is one-dimensional; two-dimensional bifurcation in the case $0 < T < \frac{1}{3}$ is dealt with in Section 4. Note that while one-dimensional kernels appear both for sub- and supercritical wave speeds, separated by $c = 1$, two-dimensional kernels only appear for $c \in (0, 1]$: see Section 4 below.

3.1. **The parameters.** To investigate the bifurcations we will make use in the following sections of three positive quantities — the wavespeed c, the surface tension T, and a scaling in the period of the waves, κ. While

[§]Note that the function $T_*(\cdot; \cdot)$ can be extended to the cases $n = 0$ and $k = 0$ through continuity.

the first two appear directly in the steady problem (2.1), the scaling $\xi \mapsto \kappa\xi$ is realised by introducing the corresponding dependence in the convolution operator L, so that

$$\widehat{L_{\kappa,T}}(\xi) = l_{\kappa,T}(\xi) := l_T(\kappa\xi). \tag{3.3}$$

This operator agrees with the original one for $\kappa = 1$. In particular, finding 2π-periodic solutions of (2.1) with symbol $L_{\kappa,T}$ is equivalent to finding $2\pi/\kappa$-periodic solutions of (2.1) with symbol $L_T = L_{1,T}$. This allows us to treat different wavelengths in the same equation by moving the wavelength parameter to $L_{\kappa,T}$. Of course, the family of operators $L_{\kappa,T}$ all enjoy the embedding properties of Lemma 2.4, as the proof is identical for an arbitrary, fixed, $\kappa > 0$. In what follows, we will thus modify (3.1) and seek non-trivial solutions of the map

$$F_\kappa(u,c) = u - cL_{\kappa,T}(u) + L_{\kappa,T}\left(u^2\right) \tag{3.4}$$

in $\mathcal{C}^s_{\mathrm{even}}(\mathbb{S}) \times \mathbb{R}$ for a fixed $\kappa > 0$.

Since surface tension is a property of the medium, while the speed and wavenumber are properties of particular waves, it is physically more relevant to use the two latter as bifurcation parameters, while holding the surface tension fixed. This is what we will do in the following.

3.2. **Local bifurcation via Lyapunov–Schmidt.** The following theorem establishes, for fixed wavelength and surface tension, the local bifurcation of small amplitude steady solutions the capillary-gravity Whitham equation (1.1). Although this is by now a standard Crandall–Rabinowitz type result [27], we prove the result using a direct Lyapunov–Schmidt reduction as to prepare for the two-dimensional bifurcation in Section 4. This is similar to the strategy in [14]. As κ and T will be fixed — assuming that we already have a one-dimensional kernel as described in the beginning of this section — we shall here suppress the dependence upon these parameters.

Theorem 3.1. *Let $k \in \mathbb{N}$ and set $c_0 = l_{\kappa,T}(k)^{-1}$. For any $T, \kappa > 0$ such that* $\dim\ker D_uF_\kappa(0,c_0) = 1$ *there exists a smooth curve*

$$\{(u(t), c(t)) : 0 < |t| \ll 1\}$$

of small-amplitude, 2π-periodic even solutions of the steady capillary-gravity Whitham equation (2.1) *with symbol given by* (3.3)*. These solutions satisfy*

$$\begin{aligned} u(t) &= t\cos(kx) + \mathcal{O}(t^2) \\ c(t) &= c_0 + \mathcal{O}(t). \end{aligned}$$

in $\mathcal{C}^s_{\text{even}}(\mathbb{S}) \times \mathbb{R}$*, and constitute all nontrivial solutions in a neighbourhood of* $(0, c_0)$ *in that space.*

Remark 3.2. *There is an additional but qualitatively different bifurcation taking place at* $c = 1$*, where the straight curve of constant solutions* $(u, c) = (c - 1, c)$ *crosses the trivial solution curve* $(0, c)$*. These solutions must be taken into consideration when constructing non-constant waves at* $c = 1$ *when the kernel is two-dimensional, see Theorem 4.1.*

Remark 3.3. *By considering the role of* κ *in the proof of Theorem 3.1 one can see that by varying* κ *one obtains a one-dimensional family of solution curves, the starting points of which depend smoothly on* κ*. This may be seen also by applying the implicit function theorem directly to 3.1. For each* $k \in \mathbb{N}$ *we thus obtain a two-dimensional sheet of solutions,*

$$\mathcal{S}^k = \{(u(t, \kappa), c(t, \kappa), \kappa) \colon 0 < |t| \ll 1, |\kappa - \kappa_0| \ll 1\} \tag{3.5}$$

parameterised by (t, κ) *in a neighbourhood of a bifurcation point* $(0, \kappa_0)$*.*

Proof. As stated above, we suppress the dependence on the fixed parameters T and κ throughout. According to the assumptions and the discussion after (3.2), on $\mathcal{C}^s_{\text{even}}(\mathbb{S})$ we have

$$\ker D_u F(0, c_0) = \ker(\mathrm{Id} - c_0 L) = \operatorname{span}\{\cos(k\cdot)\}.$$

We first write

$$\begin{aligned} u(t) &= t\cos(kx) + v(t), \\ c(t) &= c_0 + r(t), \end{aligned}$$

with $v(t) \in \mathcal{C}^s_{\text{even}}(\mathbb{S})$ such that $\int_{-\pi}^{\pi} \cos(kx) v \, \mathrm{d}x = 0$ and $r(t) \in \mathbb{R}$, and proceed to show the existence of v and r such that for $|t| \ll 1$ we have

$$F(t\cos(kx) + v(t), c_0 + r(t)) = 0. \tag{3.6}$$

As a subspace of $L^2(\mathbb{S})$, we equip $\mathcal{C}^s_{\text{even}}(\mathbb{S})$ with the L^2 inner product $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} fg \, \mathrm{d}x$ and let $\Pi \colon \mathcal{C}^s_{\text{even}}(\mathbb{S}) \to \ker D_u F(0, c_0)$ be the orthogonal projection onto $\operatorname{span}\{\cos(k\cdot)\}$. Since $D_u F(0, c_0)$ is a symmetric Fredholm operator with index 0 by Corollary 3.5 below, it follows that $\mathcal{C}^s_{\text{even}}(\mathbb{S})$ may be decomposed as a direct sum between its kernel and range. In particular, (3.6) is equivalent to the system of equations

$$\begin{aligned} \Pi F(t\cos(kx) + v, c_0 + r) &= 0, \\ (I - \Pi) F(t\cos(kx) + v, c_0 + r) &= 0, \end{aligned} \tag{3.7}$$

where we have suppressed the t-dependence in v and r. Noting that

$$\begin{aligned}
&F(t\cos(kx)+v, c_0+r)\\
&= t\cos(kx)+v-(c_0+r)L(t\cos(kx)+v)+L(t\cos(kx)+v)^2\\
&= D_uF(0,c_0)(v+t\cos(kx))\\
&\quad - rL(t\cos(kx)+v)+L(t\cos(kx)+v)^2,
\end{aligned}$$

and that $\cos(k\cdot)$ is in the kernel of $D_uF(0,c_0)$, the equation (3.6) may be rewritten as

$$D_uF(0,c_0)v = rL(t\cos(kx)+v) - L(t\cos(kx)+v)^2 =: g(t,r,v) \quad (3.8)$$

and hence, recalling that $v \in (1-\Pi)\mathcal{C}^s_{\text{even}}(\mathbb{S})$, (3.7) is equivalent to the system

$$\begin{aligned}
0 &= \Pi g(t,r,v)\\
D_uF(0,c_0)v &= (\mathrm{Id}-\Pi)g(t,r,v).
\end{aligned} \quad (3.9)$$

Finally, observe that since $D_uF(0,c_0)$ is invertible on $(I-\Pi)\mathcal{C}^s_{\text{even}}(\mathbb{S})$, the second equation in (3.9) can be rewritten as

$$v = [D_uF(0,c_0)]^{-1}(\mathrm{Id}-\Pi)g(t,r,v).$$

Concerning this latter equation, note that at $(t,r)=(0,0)$ we have both that $v=0$ is a solution and that the Fréchet derivative with respect to v is invertible on $(\mathrm{Id}-\Pi)\,\mathcal{C}^s_{\text{even}}(\mathbb{S})$ (because $D_uF(0,c_0)$ is). Therefore, by the implicit function theorem on Banach spaces, the second line of (3.9) has a unique solution $v(t,r) \in (\mathrm{Id}-\Pi)\,\mathcal{C}^s_{\text{even}}(\mathbb{S})$ defined in a neighbourhood of $(t,r)=(0,0)$, and depending analytically on its arguments. By uniqueness, $v(0,r)=0$ for all $|r| \ll 1$. Moreover, differentiation with respect to t at $(t,r)=(0,0)$ in (3.8) shows that $\frac{\partial}{\partial t}v(0,r)=0$, which implies that v has no constant or linear terms in t. As it is smooth in t, it may be expanded in an (at least) quadratic series around $t=0$.

We now need to solve the equation

$$\Pi g(t,r,v(t,r)) = Q(r,t)\cos(kx) = 0$$

for r, with

$$Q(t,r) := \langle g(t,r,v(t,r)), \cos(k\cdot)\rangle.$$

Notice that that $Q(0,r)=0$ since $v(0,r)=0$ for all r, which together with the symmetry of L implies that we can write

$$Q(t,r) = t\left[r\, l(k) + R(t,r)\right],$$

where R is analytic with $R(0,0) = \partial_r R(0,0) = 0$, again due to the properties of v (here, $l = l_{T,\kappa}$). An application of the implicit function theorem to the equation $r\, l(k)\pi + R(t,r) = 0$ at $(t,r)=(0,0)$ then yields the existence of a locally unique smooth function $r\colon t \mapsto r(t)$ with $r(0) = 0$ such that

$$Q(t, r(t))) = t(r(t)\, l(k) + \tilde{R}(t, r(t))) = 0$$

for all $|t| \ll 1$. This concludes the proof. □

3.3. **Global bifurcation (analytic).** We now extend the local bifurcation curves from Section 3.2 to global ones by the means of the analytic bifurcation theory pioneered by Dancer [12, 13] and then developed further by Buffoni and Toland [10]. For fixed $s > 1/2$, we define $N\colon \mathcal{C}^s_{\text{even}}(\mathbb{S}) \times \mathbb{R} \to \mathcal{C}^{s+1/2}_{\text{even}}(\mathbb{S})$ by

$$N(u, c) = L(cu - u^2).$$

Fixed points of N are solutions of the steady capillary-gravity Whitham equation (2.1), and conversely. Let

$$S = \{(u, c) \in \mathcal{C}^s_{\text{even}}(\mathbb{S}) \times \mathbb{R}\colon F(u, c) = 0\}$$

be the set of solutions (fixed points of N). Note that Lemma 2.4 implies that $S \subset \mathcal{C}^\infty_{\text{even}} \times \mathbb{R}$, so that all solutions are smooth: for details, see Proposition 5.1 below. By combining this with a diagonal argument one obtains the following compactness result.

Lemma 3.4. *Bounded and closed sets in S are compact in $\mathcal{C}^s_{\text{even}}(\mathbb{S}) \times \mathbb{R}$.*

Proof. Let $K \subset S \subset \mathcal{C}^s_{\text{even}}(\mathbb{S}) \times \mathbb{R}$ be closed and bounded, and pick a sequence $(u_j, c_j)_j \subset K$. Since $\{c \in \mathbb{R}\colon (u, c) \in K\}$ is a closed and bounded subset of $\mathbb{R}$, it is compact. This means that $(c_j)_j$ has a convergent subsequence, name it $(c_k)_k$. As the map

$$\mathcal{C}^s_{\text{even}}(\mathbb{S}) \times \mathbb{R} \ni (u, c) \mapsto cu - u^2 \in \mathcal{C}^s_{\text{even}}(\mathbb{S})$$

is continuous for $s > 1/2$, and since the map L is compact on $\mathcal{C}^s_{\text{even}}(\mathbb{S})$ thanks to Lemma 2.4, it follows that after passing to a further subsequence $(u_l, c_l)_l \subset K$ that $(N(u_l, c_l))_l$ converges in $\mathcal{C}^s_{\text{even}}(\mathbb{S})$ to some function u. Since $u_l = N(u_l, c_l)$ by definition, passing to limits implies the sequence $(u_l, c_l)_l$ converges in $\mathcal{C}^s_{\text{even}}(\mathbb{S}) \times \mathbb{R}$ with limit $(u, c) \in S$. As K is closed it follows that $(u, c) \in K$, establishing that K is compact. □

Corollary 3.5. *The Fréchet derivative $D_u F(u, c)$ is a Fredholm operator of index 0 at any point $(u, c) \in C^s_{\text{even}}(\mathbb{S}) \times \mathbb{R}$.*

Proof. This follows immediately from Lemma 3.4 as then

$$D_u F(u, c) = \mathrm{Id} - L(c - 2u)$$

is a compact perturbation of the identity. □

Before embarking on to the next theorem, we recall the shorthand $l(\cdot)$ for $l_{\kappa,T}(\cdot) = l_T(\kappa\cdot)$.

Theorem 3.6. *Whenever*

$$\ddot{c}(0) = \frac{3c_0 l(2k) - l(2k) - 2}{(c_0 - 1)(c_0 l(2k) - 1)} \tag{3.10}$$

is finite and non-vanishing the local bifurcation curve $t \mapsto (u(t), c(t))$, $|t| \ll 1$, from Theorem 3.1 extends to a continuous and locally analytically parametrizable curve in $\mathcal{C}^s_{\text{even}}(\mathbb{S}) \times \mathbb{R}$ defined for all $t \in [0, \infty)$. One of the following alternatives holds:

- (i) $\|(u(t), c(t))\|_{\mathcal{C}^s(\mathbb{S})\times\mathbb{R}} \to \infty$ *as* $t \to \infty$.
- (ii) $t \mapsto (u(t), c(t))$ *is P-periodic for some finite P, so that the curve forms a loop.*

Remark 3.7. *We note that*

$$\ddot{c}(0; k) = \begin{cases} \dfrac{10}{(3T-1)(\kappa k)^2} + \mathcal{O}(1) & \text{for } |k| \ll 1 \\ -(\sqrt{2}-1)(T\kappa k)^{-1/2} + \mathcal{O}\left(k^{-1}\right) & \text{for } k \gg 1. \end{cases}$$

For $T > 1/3$ it follows that $(0, c_0)$ undergoes a supercritical pitchform bifurcation for small k, and a subcritical pitchfork bifurcation for large k. Note numerically, we observe there exists a unique $k_* = k_*(T) > 0$ such that $\ddot{c}(0) > 0$ for $0 < k < k_*$ and $\ddot{c}(0) < 0$ for $k > k_*$. For $0 < T < 1/3$, both the numerator and denomenator of (3.10) change signs. Note that one may be able to do global bifurcation when $\ddot{c}(0) = 0$ but inspecting $c^{(4)}(0)$: see, for example, [21, Theorem 6.1]. We do not pursue this here.

Proof. This theorem is a version of the global analytic bifurcation theorem in [10], and — apart from the bifurcation formulas — the proof goes as in the purely gravitation case in [19, 21]. The assumptions are fulfilled from Lemma 3.4 and Corollary 3.5 if one can just show that some derivative $c^{(k)}(0)$ is non-vanishing. We give the calculations for $\dot{c}(0)$ and $\ddot{c}(0)$ in the Appendix; the first is 0, and the second is given by (3.10). Note that a third alternative in the theorem in [10] does not happen here, as the set $\mathcal{C}^s_{\text{even}}(\mathbb{S}) \times \mathbb{R}$ lacks a boundary. □

There are a few more things one can say about the global bifurcation curves, both numerically and analytically, and we discuss the global bifurcation diagram in detail in Section 5. In particular, the cases of strong and weak surface tension are summarised in Figures 3 and 4, respectively.

4. Two-dimensional local bifurcation

We now focus our attention on the case of a two-dimensional bifurcation kernel in $\mathcal{C}^s_{\text{even}}(\mathbb{S})$. To enable the necessary two degrees of freedom we shall make use of the wavelength κ in addition to the wavespeed c, while the surface tension T is assumed to be fixed. We shall therefore study for $\kappa > 0$ the operator

$$F_\kappa(u,c) = u + L_\kappa(u^2 - cu)$$

on $\mathcal{C}^s_{\text{even}}(\mathbb{S}) \times \mathbb{R}$, along with its linearisation

$$\mathcal{L} = D_u F_{\kappa_0}(0, c_0) = \text{Id} - c_0 L_{\kappa_0},$$

assuming that $T, \kappa_0, c_0 > 0$ are constants such that

$$\ker(\mathcal{L}) = \text{span}\{\cos(k_1\cdot), \cos(k_2\cdot)\}, \tag{4.1}$$

which happens when $\kappa_0, c_0 > 0$ and $k_1, k_2 \in \mathbb{N}_0$, $k_1 \neq k_2$, are such that

$$c_0 = l_{\kappa_0}(k_1)^{-1} = l_{\kappa_0}(k_2)^{-1},$$

as described at the start of Section 3 (we suppress the dependence on T, as it will not be used apart from in this assumption). A two-dimensional kernel can arise only for $c_0 \in (0,1]$. Let now $1 \leqslant k_1 \leqslant k_2$. With $\mathcal{S}^k$ being the sheet of $2\pi/k$-periodic solutions defined in (3.5) we shall show that in addition to the solutions in $\mathcal{S}^{k_1}$ and $\mathcal{S}^{k_2}$, we may obtain solutions in a set called $\mathcal{S}^{mixed}$ consisting of perturbations of functions in the span of $\cos(k_1\cdot)$ and $\cos(k_2\cdot)$. Assuming that $k_1 \leqslant k_2$, the resonant case when k_2 is an integer multiple of k_1 (sometimes referred to as Wilton ripples) is more difficult than the generic case, but we follow here the procedure in [14, 20] to construct a slit disk of solutions also in that case. Numerical calculations indicate that this set is optimal [35].

When one of the wavenumbers is zero (meaning $c_0 = 1$), we instead call that one k_2, and we will automatically have the resonant case, as then $k_1 \mid k_2$. That case is included in the below theorem. Hence, at $c = 1$ there is a nontrivial bifurcation, but the arising waves always have a non-zero component in the constant direction.

Theorem 4.1. *Let $T > 0$ be fixed and assume that* (4.1) *holds for some distinct $k_1, k_2 \in \mathbb{N}_0$.*

(i) *When k_1 does not divide k_2 there is a full, smooth, sheet*

$$\mathcal{S}^{mixed} = \{(u(t_1,t_2), c(t_1,t_2), \kappa(t_1,t_2)) : 0 < |(t_1,t_2)| \ll 1\}$$

of solutions in $\mathcal{C}^s_{\text{even}}(\mathbb{S}) \times \mathbb{R} \times \mathbb{R}_+$ *of the form*

$$\begin{aligned} u(t_1,t_2) &= t_1 \cos(k_1 x) + t_2 \cos(k_2 x) + \mathcal{O}(|(t_1,t_2)|^2),\\ c(t_1,t_2) &= c_0 + \mathcal{O}((t_1,t_2)),\\ \kappa(t_1,t_2) &= \kappa_0 + \mathcal{O}((t_1,t_2)), \end{aligned}$$

to the steady capillary-gravity Whitham equation (2.1)*. The set* $\mathcal{S}^{k_1} \cup \mathcal{S}^{k_2} \cup \mathcal{S}^{mixed}$ *contains all nontrivial solutions in* $\mathcal{C}^s_{\text{even}}(\mathbb{S}) \times \mathbb{R} \times \mathbb{R}_+$ *of this equation in a neighbourhood of* $(0, c_0, \kappa_0)$*.*

(ii) *When* k_1 *divides* k_2 *there exists for any* $\delta > 0$ *a small but positive* ε_δ *and a slit, smooth, sheet*

$$\mathcal{S}^{mixed}_\delta = \{(u(\varrho,\vartheta), c(\varrho,\vartheta), \kappa(\varrho,\vartheta)) : 0 < \varrho < \varepsilon_\delta,\ \delta < |\vartheta| < \pi - \delta\}$$

of solutions in $\mathcal{C}^s_{\text{even}}(\mathbb{S}) \times \mathbb{R} \times \mathbb{R}_+$ *of the form*

$$\begin{aligned} u(\varrho,\vartheta) &= \varrho \cos(\vartheta)\cos(k_1 x) + \varrho \sin(\vartheta)\cos(k_2 x) + \mathcal{O}(\varrho^2),\\ c(\varrho,\vartheta) &= c_0 + \mathcal{O}(\varrho),\\ \kappa(\varrho,\vartheta) &= \kappa_0 + \mathcal{O}(\varrho). \end{aligned}$$

to the steady capillary-gravity Whitham equation (2.1)*. In a neighbourhood of* $(0, c_0, \kappa_0)$*, the set* $\mathcal{S} = \mathcal{S}^{k_2} \cup \mathcal{S}^{mixed}_\delta$ *contains all nontrivial solutions in* $\mathcal{C}^s_{\text{even}}(\mathbb{S}) \times \mathbb{R} \times \mathbb{R}_+$ *of* (2.1) *such that* $\delta < |\vartheta| < \pi - \delta$*.*

Remark 4.2. *The order of vanishing of the functions* $c - c_0$ *and* $\kappa - \kappa_0$ *in Theorem 4.1 is analyzed in Section A.2 of Appendix A.*

Remark 4.3. *The bifurcation theorem Theorem 4.1 shows that near a two-dimensional bifurcation point in the case where* $k_2/k_1 \notin \mathbb{N}_0$ *there exists a full disk of solutions (for fixed* κ*), while if* $k_2/k_1 \in \mathbb{N}_0$ *the disk is slit with one axis removed. This situation is summarised in Figure 2. In particular this means that it is possible to find curves connecting solutions with different wavenumbers, consistent with the recent numerical findings in [35].*

Proof. We start by writing

$$\begin{aligned} u(t_1,t_2) &= t_1 \cos(k_1 x) + t_2 \cos(k_2 x) + v,\\ c(t_1,t_2) &= c_0 + r,\\ \kappa(t_1,t_2) &= \kappa_0 + p, \end{aligned}$$

where, generically, we want to find v, r and p parameterised by (t_1, t_2) such that

$$F_{\kappa_0+p}(t_1 \cos(k_1 x) + t_2 \cos(k_2 x) + v, c_0 + r) = 0, \tag{4.2}$$

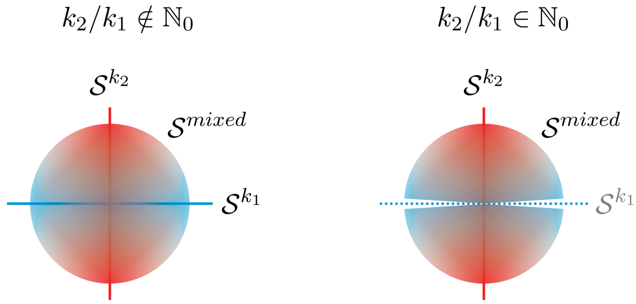

Figure 2. The local solution disks for the steady capillary-gravity Whitham equation (2.1) around a point where the bifurcation kernel is two-dimensional. The left-hand drawing depicts the situation in Theorem 4.1 (i), whereas the right-hand drawing refers to case (ii) of the same theorem. The blue and red colours represent the proximity of the solutions to the pure k_1- and k_2-modes, respectively. In particular, when k_1 divides k_2 we have not found any waves bifurcating in the direction of $\cos(k_1\cdot)$.

for sufficiently small values of (t_1, t_2). As in the proof of Theorem 3.1, we let $\Pi\colon C^s_{\text{even}}(\mathbb{S}) \to \ker(D_u F_{\kappa 0}(0, c_0))$ be the orthogonal projection onto $\ker(D_u F_{\kappa 0}(0, c_0))$ parallel to $\operatorname{ran}(D_u F_{\kappa 0}(0, c_0))$, where we have equipped $C^s_{\text{even}}(\mathbb{S})$ with the L^2 inner product $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} fg \,\mathrm{d}x$. According to Corollary 3.5 equation (4.2) is then equivalent to

$$\begin{cases} \Pi F_{\kappa(t_1,t_2)}\left(u(t_1,t_2), c(t_1,t_2)\right) = 0 \\ (\mathrm{Id} - \Pi) F_{\kappa(t_1,t_2)}\left(u(t_1,t_2), c(t_1,t_2)\right) = 0. \end{cases} \tag{4.3}$$

Note that under the above ansatz, where it is assumed that $\Pi v = 0$,

$$\begin{aligned} F_\kappa\left(u, c\right) = {} & t_1 \cos(k_1 x) + t_2 \cos(k_2 x) + v \\ & + L_{\kappa_0 + p}\left[\left(t_1 \cos(k_1 x) + t_2 \cos(k_2 x) + v\right)^2 \right. \\ & \left. -(c_0 + r)\left(t_1 \cos(k_1 x) + t_2 \cos(k_2 x) + v\right)\right] \\ = {} & \left(v - c_0 L_{\kappa_0+p} v\right) + t_1\left(\cos(k_1 x) - c_0 L_{\kappa_0+p}\cos(k_1 x)\right) \\ & + t_2\left(\cos(k_2 x) - c_0 L_{\kappa_0+p}\cos(k_2 x)\right) \\ & - r L_{\kappa_0+p}\left(t_1 \cos(k_1 x) + t_2 \cos(k_2 x) + v\right) \\ & + L_{\kappa_0+p}\left(t_1 \cos(k_1 x) + t_2 \cos(k_2 x) + v\right)^2, \end{aligned}$$

and writing $L_{\kappa_0+p} = L_{\kappa_0} + (L_{\kappa_0+p} - L_{\kappa_0})$ we have

$$\begin{aligned}F_\kappa(u,c) &= D_uF_{\kappa_0}(0,c_0)v - c_0(L_{\kappa_0+p} - L_{\kappa_0})v \\ &\quad - t_1c_0(L_{\kappa_0+p} - L_{\kappa_0})\cos(k_1x) - t_2c_0(L_{\kappa_0+p} - L_{\kappa_0})\cos(k_2x) \\ &\quad - rL_{\kappa_0+p}(t_1\cos(k_1x) + t_2\cos(k_2x) + v) \\ &\quad + L_{\kappa_0+p}(t_1\cos(k_1x) + t_2\cos(k_2x) + v)^2 \\ &=: D_uF_{\kappa_0}(0,c_0)v - g(t_1,t_2,r,p,v).\end{aligned}$$

Therefore (4.2) is equivalent to

$$D_uF_{\kappa_0}(0,c_0)v = g(t_1,t_2,r,p,v), \tag{4.4}$$

and we can rewrite (4.3) as

$$\begin{cases} 0 = \Pi g(t_1,t_2,r,p,v) \\ D_uF_{\kappa_0}(0,c_0)v = (\mathrm{Id} - \Pi)g(t_1,t_2,r,p,v). \end{cases} \tag{4.5}$$

Note that since v is orthogonal to $\ker(D_uF_{\kappa_0}(0,c_0))$ the second equation in (4.5) reads $v = D_uF_{\kappa_0}(0,c_0)^{-1}(\mathrm{Id} - \Pi)g(t_1,t_2,r,p,v)$. It is clear that

$$D_uF_{\kappa_0}(0,c_0)v - (\mathrm{Id} - \Pi)g(t_1,t_2,r,p,v) = 0$$

has the solution $(t_1,t_2,r,p,v) = (0,0,0,0,0)$ and at that point the Fréchet derivative respect to v is $D_uF_{\kappa_0}(0,c_0)$, which is invertible on $(\mathrm{Id} - \Pi)C^s_{\mathrm{even}}(\mathbb{S})$. The implicit function theorem then ensures the existence of a solution $v = v(t_1,t_2,r,p) \in (\mathrm{Id} - \Pi)C^s_{\mathrm{even}}(\mathbb{S})$. By uniqueness we have that $v(0,0,r,p) = 0$ for all small enough values of r and p. Moreover, note that $\frac{\partial}{\partial t_1}v(0,0,0,0) = 0$ and $\frac{\partial}{\partial t_2}v(0,0,0,0) = 0$. This follows by differentiating (4.4) respect to t_1 or t_2, and evaluating at $(t_1,t_2,r,p) = (0,0,0,0)$ recalling that $D_uF_{\kappa_0}(0,c_0)$ is invertible on its range. As a consequence, v depends at least quadratically on t_1 and t_2.

We are now left with solving the finite-dimensional problem given by the first equation in (4.5). To this end, we decompose the projection Π as $\Pi = \Pi_1 + \Pi_2$, where Π_1 is the projection onto $\cos(k_1\cdot)$, and Π_2 is the projection onto $\cos(k_2\cdot)$. Then

$$\Pi g = \Pi_1 g + \Pi_2 g = Q_1\cos(k_1x) + Q_2\cos(k_2x),$$

with $Q_j = \langle g, \cos(k_j\cdot)\rangle$, and the first line of (4.5) is equivalent to showing that

$$Q_1 = Q_2 = 0. \tag{4.6}$$

To solve (4.6) we consider two cases.

The non-resonant case. Assume that $k_2/k_1 \notin \mathbb{N}_0$. Using the properties of v and Π_1, a direct calculation shows that

$$\begin{aligned} Q_1 = t_1 &\left[c_0\big(l((\kappa_0+p)k_1) - l(\kappa_0 k_1)\big) + r\, l((\kappa_0+p)k_1)\right] \\ &- l((\kappa_0+p)k_1) \left\langle \cos(k_1\cdot), (t_1\cos(k_1\cdot) + t_2\cos(k_2\cdot) + v(t_1,t_2,r,p))^2 \right\rangle. \end{aligned} \tag{4.7}$$

As $v(0,t_2,r,p)$ is $2\pi/k_2$-periodic and $k_2 \neq k_1$, the above inner term product vanishes for $t_1 = 0$. Therefore we may write

$$Q_1(t_1,t_2,r,p) = t_1\, \Psi_1(t_1,t_2,r,p) \tag{4.8}$$

with

$$\Psi_1(t_1,t_2,r,p) = \int_0^1 \frac{\partial Q_1}{\partial t_1}(zt_1,t_2,r,p)\,\mathrm{d}z, \tag{4.9}$$

and note (4.7) implies

$$\Psi_1(0,0,r,p) = c_0\left[l((\kappa_0+p)k_1) - l(\kappa_0 k_1)\right] + r\, l((\kappa_0+p)k_1). \tag{4.10}$$

Similarly, we have

$$\begin{aligned} Q_2 = t_2 &\left[c_0\big(l((\kappa_0+p)k_2) - l(\kappa_0 k_2)\big) + r\, l((\kappa_0+p)k_2)\right] \\ &- l((\kappa_0+p)k_2) \left\langle \cos(k_2\cdot), (t_1\cos(k_1\cdot) + t_2\cos(k_2\cdot) + v(t_1,t_2,r,p))^2 \right\rangle \end{aligned} \tag{4.11}$$

with the inner product term vanishing at $t_2 = 0$ since we assumed $k_2/k_1 \notin \mathbb{N}_0$. We can thus write

$$Q_2(t_1,t_2,r,p) = t_2\, \Psi_2(t_1,t_2,r,p) \tag{4.12}$$

with

$$\Psi_2(t_1,t_2,r,p) = \int_0^1 \frac{\partial Q_2}{\partial t_2}(t_1,zt_2,r,p)\,\mathrm{d}z \tag{4.13}$$

so that

$$\Psi_2(0,0,r,p) = c_0\left[l((\kappa_0+p)k_2) - l(\kappa_0 k_2)\right] + r\, l((\kappa_0+p)k_2). \tag{4.14}$$

Hence, condition (4.6) is equivalent solving the system

$$\begin{cases} t_1\Psi_1(t_1,t_2,r,p) = 0 \\ t_2\Psi_2(t_1,t_2,r,p) = 0 \end{cases}$$

for p and r in a neighborhood of $(t_1,t_2,r,p) = (0,0,0,0)$. There are clearly four cases: $t_1 = t_2 = 0$ represents the trivial solutions. When $\Psi_1 = 0$ and $t_2 = 0$ we can apply Theorem 3.1 concerning one-dimensional bifurcations along with the remark following it to obtain the solutions in $\mathcal{S}^{k_1}$. Similarly, when $t_1 = 0$ and $\Psi_2 = 0$ we instead retrieve the solutions in $\mathcal{S}^{k_2}$. To obtain the mixed-period solutions we apply the implicit function theorem to solve $\Psi_1 = \Psi_2 = 0$ near the origin. Indeed, note that $\Psi_1(0,0,0,0) = \Psi_2(0,0,0,0) = 0$ and that the Jacobian of the map

$$(r,p) \mapsto (\Psi_1(0,0,r,p), \Psi_2(0,0,r,p))$$

at $(r,p) = (0,0)$ is given by

$$\begin{aligned} \det \begin{bmatrix} D_r\Psi_1(0,0,r,p) & D_p\Psi_1(0,0,r,p) \\ D_r\Psi_2(0,0,r,p) & D_p\Psi_2(0,0,r,p) \end{bmatrix}\Bigg|_{(r,p)=(0,0)} \\ = c_0\, l_{\kappa_0}(k_1)\left[l'_{\kappa_0}(k_2)\, k_2 - l'_{\kappa_0}(k_1)\, k_1\right], \end{aligned} \tag{4.15}$$

which is always different from 0 since l_T has only one positive stationary point, $l_{\kappa_0}(k_1) \neq 0$, and that the terms $l'_{\kappa_0}(k_1)$ and $l'_{\kappa_0}(k_2)$ necessarily have opposite signs. Applying the Implicit Function Theorem gives the solutions in $\mathcal{S}^{mixed}$. Note in each of the above four cases, we find $r = r(t_1,t_2)$ and $p = p(t_1,t_2)$ with p and r both vanishing to at least second order at $(t_1,t_2) = (0,0)$, as claimed.

The resonant case. Assume now that $k_2/k_1 \in \mathbb{N}_0$. In this case, we are not guaranteed that $Q_2(t_1,0,r,p) = 0$ for all $|t_1| \ll 1$ due to a possible resonance in the inner product term in (4.11). Nevertheless, we do know that $Q_2(0,0,r,p) = 0$. Using polar coordinates to introduce the function

$$\widetilde{Q}_2(\varrho,\vartheta,r,p) = Q_2(\varrho\cos(\vartheta), \varrho\sin(\vartheta), r, p),$$

defined for $0 \leqslant \varrho \ll 1$ and $|(\vartheta,r,p)| \ll 1$, we find from (4.11) that

$$\begin{aligned} \widetilde{Q}_2(\varrho,\vartheta,r,p) = {} & \varrho\sin(\vartheta)c_0\big(l((\kappa_0+p)k_2) - l(\kappa_0 k_2)\big) \\ & + \varrho\sin(\vartheta) r\, l((\kappa_0+p)k_2) \\ & - l((\kappa_0+p)k_2)\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(k_2 x)\Big[\varrho\cos(\vartheta)\cos(k_1 x) \\ & + \varrho\sin(\vartheta)\cos(k_2 x) + v(\varrho\cos(\vartheta), \varrho\sin(\vartheta), r, p)\Big]^2\, \mathrm{d}x. \end{aligned} \tag{4.16}$$

Since $\widetilde{Q}_2(0,\vartheta,r,p) = 0$, we may as before write

$$\widetilde{Q}_2(\varrho,\vartheta,r,p) = \varrho\, \widetilde{\Psi}_2(\varrho,\vartheta,r,p) \tag{4.17}$$

with

$$\widetilde{\Psi}_2(\varrho,\vartheta,r,p) = \int_0^1 \frac{\partial \widetilde{Q}_2}{\partial\varrho}(z\varrho,\vartheta,r,p)\,\mathrm{d}z \tag{4.18}$$

so that

$$\begin{aligned} \widetilde{\Psi}_2(0,\vartheta,r,p) = {} & \sin(\vartheta)\, c_0\left[l((\kappa_0+p)k_2) - l(\kappa_0 k_2)\right] \\ & + r\sin(\vartheta)\, l((\kappa_0+p)k_2). \end{aligned} \tag{4.19}$$

For Q_1, instead, all the previous calculations remain true and hence, similarly defining the function

$$\widetilde{\Psi}_1(\varrho, \vartheta, r, p) := \Psi_1(\varrho \cos(\vartheta), \varrho \sin(\vartheta)), \tag{4.20}$$

it follows in this resonant case that (4.6) is equivalent to solving the system

$$\begin{cases} \varrho \cos(\vartheta) \widetilde{\Psi}_1(\varrho, \vartheta, r, p) = 0 \\ \varrho \, \widetilde{\Psi}_2(\varrho, \vartheta, r, p) = 0. \end{cases}$$

for r and p in a neighborhood of $(\varrho, \vartheta, r, p) = (0, 0, 0, 0)$. The case $\varrho = 0$ clearly corresponds to trivial solutions, while the case $\cos(\vartheta) = 0, \widetilde{\Psi}_2 = 0$ corresponds to solutions in $\mathcal{S}^{k_2}$ via the application of Theorem 3.1. For the case that $\widetilde{\Psi}_1 = 0, \widetilde{\Psi}_2 = 0$ we again apply the implicit function theorem near the origin. Indeed, note that both $\widetilde{\Psi}_1$ and $\widetilde{\Psi}_2$ both vanish at the origin and that the Jacobian of the map

$$(r, p) \mapsto (\widetilde{\Psi}_1(0, 0, r, p), \widetilde{\Psi}_2(0, 0, r, p))$$

at $(r, p) = (0, 0)$ is given by

$$\begin{aligned} &\det \begin{bmatrix} D_r \widetilde{\Psi}_1(0, \vartheta, r, p) & D_p \widetilde{\Psi}_1(0, \vartheta, r, p) \\ D_r \widetilde{\Psi}_2(0, \vartheta, r, p) & D_p \widetilde{\Psi}_2(0, \vartheta, r, p) \end{bmatrix} \Bigg|_{(r,p)=(0,0)} \\ &\quad = \sin(\vartheta) \, c_0 \, l(\kappa_0 k_1) \left[l'(\kappa_0 k_2) \, k_2 - l'(\kappa_0 k_1) \, k_1 \right], \end{aligned} \tag{4.21}$$

which, by the same considerations we applied to (4.15), is non-zero so long as $\sin(\vartheta) \neq 0$ Therefore, for any fixed $\delta > 0$, restricting to $\delta < |\vartheta| < \pi - \delta$ gives the solutions in $\mathcal{S}_\delta^{mixed}$, as desired □

5. Global bifurcation diagram

In this section we give some additional properties of solutions of (2.1), that is, of continuous and finitely periodic solutions. Our goal is to communicate the global bifurcation picture, as gathered from both analytic and numerical evidence, as well as to relate this to some comparable studies. We first present and prove the additional analytic results, after which we discuss the bifurcation diagram of the periodic capillary-gravity Whitham with the help of Figures 3 and 4.

Proposition 5.1. *Any $L^\infty(\mathbb{R})$-solution of the steady capillary-gravity Whitham equation* (2.1) *is smooth.*

Proof. This is immediate from writing the equation in the form (2.3). For any $T > 0$, the operator L_T is a smoothing Fourier multiplier operator of order $-\frac{1}{2}$. This applies in particular to the scale of Zygmund spaces $\mathcal{C}^s(\mathbb{R})$, $s \geqslant 0$, see Lemma 2.4. As $L^\infty(\mathbb{R})$ is an algebra embedded in

$\mathcal{C}^0(\mathbb{R})$ [37, Section 13.8], and the spaces $\mathcal{C}^s(\mathbb{R})$ are Banach algebras for $s > 0$, the result follows by bootstrapping. □

Proposition 5.2.

(i) *There are no periodic solutions of* (2.1) *in the region*

$$\max u < \min\{0, c-1\}.$$

(ii) *Except for the bifurcation points when $c = \frac{1}{l_T(k)} > 0$ there are no small periodic solutions in a vicinity of any point along the curve of trivial solutions $(u, c) = (0, c)$, $c \in \mathbb{R}$. Similarly, there are no periodic solutions that are small perturbations of the constant solutions $(u, c) = (c-1, c)$, $c \in \mathbb{R}$, except for the bifurcation points that appear along this line for $c < 2$.*

(iii) *The solution $u = 0$ is the only periodic solution for $c = 1$.*

(iv) *For $T \geqslant \frac{4}{\pi^2}$, all periodic solutions satisfy*

$$\max u \leqslant \tfrac{c^2}{4},$$

with equality if and only if u is a constant solution and either $c = 0$ or $c = 2$.

Remark 5.3. *The qualifier 'periodic' is here used only to guarantee that solutions, which we have defined to be continuous, are integrable over their period.*

Proof. As all steady solutions are smooth, and the symbol of L_T satisfies $l_T(0) = 1$, one may as in [21] integrate over any finite period to obtain

$$(c-1)\int_{-\pi}^{\pi} u \,\mathrm{d}x = \int_{-\pi}^{\pi} u^2 \,\mathrm{d}x. \tag{5.1}$$

(The same argument works for other periods as well.) This is an immediate contradiction for $u < \min\{0, c-1\}$.

For the second statement, consider first $c < 1$. As the symbol l_T is positive, and the operator L_T is a linear isomorphism $\mathcal{C}^s(\mathbb{S}) \to \mathcal{C}^{s+\frac{1}{2}}(\mathbb{S})$ unless $cl_T(k) = 1$ (cf. (3.2)), the implicit function theorem implies that there are no small solutions in a vicinity except for the bifurcation points found in Theorems 3.1 and 4.1 when $c < 1$. In particular, there are no such solutions for $c < 1$ in the case of strong surface tension $T \geqslant \frac{1}{3}$, and none for $c < 0$ in the case of weak surface tension $0 < T < \frac{1}{3}$. By Galilean invariance, the corresponding result applies to the line $u = c-1$ for $c \geqslant 1$.

The proposition (iii) is immediate from (5.1).

For (iv), note that

$$u(x) = L(cu - u^2) = \frac{c^2}{4} - L\left(\frac{c}{2} - u\right)^2 \leqslant \frac{c^2}{4},$$

when $T \geqslant \frac{4}{\pi^2}$, as the integral kernel of L is then everywhere positive. This proves that $\max u \leqslant \frac{c^2}{4}$, with equality if and only if $(u, c) = (1, 2)$ or $(u, c) = (0, 0)$, as these are the only constant solutions along the line $\max u = \frac{c}{2}$. □

Proposition 5.4. *If the surface tension satisfies $T \geqslant \frac{4}{\pi^2}$, then the bifurcation curve found in Theorem 3.6 for $k = 1$ can be constructed such that it contains a subsequence of solutions that are all single-crested (bell-shaped) in each minimal period and that either:*

(i) *is bounded in wavespeed but with* $\min u$ *unbounded; or*

(ii) *eventually leaves every set* $\{\max u \leqslant \lambda c\}$ *for* $\lambda < \frac{1}{2}$.

Proof. For even and periodic solutions u one may as in [16, 21] use (2.1) to write

$$u'(x) = 2\int_0^\pi (K_p(x-y) - K_p(x+y))\left(\frac{c}{2} - u(y)\right) u'(y)\,\mathrm{d}y. \tag{5.2}$$

When K_p is completely monotone, and u is decreasing on $(0, \pi)$ with $u \leqslant \frac{c}{2}$, this implies that u is strictly decreasing on the same interval (unless u is a constant), and a standard argument [16, Lemma 5.5] yields that looping as in alternative (ii) is ruled out.

Let us therefore, for a contradiction, assume that the bifurcation curve remains within the set $\{\max u < \frac{c}{2}\}$. Recalling that Theorem 2.8 and [21, Proposition 3.2] together imply that K_p is completely monotone on $(0, \pi)$ when $T \geqslant \frac{4}{\pi^2}$, it follows that alternative (i) in Theorem 3.6 has to hold. As solutions are smooth, this is equivalent to a sequence of solutions $(u_n, c_n) = (u(t_n), c(t_n))$ satisfying $|u_n|_\infty + |c_n| \to \infty$ as $n \to \infty$.

Assume first that $\{c_n\}_n$ is bounded. Then $\{u_n\}_n$ is unbounded in $L^\infty(\mathbb{R})$, and therefore $\min u_n \to -\infty$ as $n \to \infty$ is the only possibility, by Proposition 5.2 (iv).

If, on the other hand, $\{c_n\}_n$ is unbounded, pick a subsequence such that $\lim_{n\to\infty} |c_n| = \infty$. Note that c_n cannot pass $c = 1$, as Proposition 5.2 (iii) shows that it would have to pass via $(u, c) = (0, 1)$, but near that point there are only small constant solutions (see Remark 3.2 and Theorem 4.1). Hence, the solution curve would first have to connect to either the curve $u = c - 1$ or $u = 0$. But, as described in Proposition 5.2 (ii), the first of these has no bifurcation points for strong surface tension and $c > 1$, and connection back to the bifurcation points of the second is excluded by the argument used in [16, Lemma 5.5] (no looping). Hence, $\lim_{n\to\infty} c_n = \infty$.

We now show that this is impossible when $\max u_n \leqslant \lambda c_n$, $\lambda < \frac{1}{2}$. Recall that we are following a branch of the curve for which u is even, and strictly increasing on the half-period $(-\pi, 0)$, in view of the positivity of the integrand in (5.2). According to our assumptions, there exists $\delta > 0$ such that $\frac{c_n}{2} - \max u_n \geqslant \delta c_n$, pick $x_n \in (0, \pi)$ such that

$$-u_n'(x_n) = \min_{y \in [\delta, \pi-\delta]} (-u_n'(y)).$$

$$\begin{aligned}
-u_n'(x_n) &= 2\int_0^\pi (K_p(x_n - y) - K_p(x_n + y)) \left(\frac{c_n}{2} - u_n(y)\right) (-u'(y))\,\mathrm{d}y \\
&\geqslant 2\delta c_n \int_\delta^{\pi-\delta} (K_p(x_n - y) - K_p(x_n + y)) (-u_n'(y))\,\mathrm{d}y \\
&\geqslant -2\delta c_n u_n'(x_n) \int_\delta^{\pi-\delta} (K_p(x_n - y) - K_p(x_n + y))\,\mathrm{d}y.
\end{aligned}$$

On the interval of consideration, $K_p(x_n - y) - K_p(x_n + y)$ is bounded from below by a positive constant (it is zero only for $y = k\pi$, $k \in \mathbb{Z}$). Although it has a singularity at $x_n = y$, it tends to ∞ there, so we may estimate it from below, uniformly in x_n, by

$$\min\{(K_p(x_n - y) - K_p(x_n + y)) : (x, y) \in [\delta, \pi-\delta] \times [\delta, \pi-\delta]\} \gtrsim 1.$$

Consequently,

$$-u_n'(x_n) \gtrsim -c_n u_n'(x_n),$$

which is not possible, as $c_n \to \infty$ and $-u_n'(x_n) > 0$ for all n. □

5.1. **Discussion and summary of results.** Analytically, we have determined almost completely[¶] the solution set near the lines of constant solutions $u = 0$ and $u = c - 1$. The result depends crucially on the strength of surface tension T, and, apart from the easily seen change in the dispersion relation at $T = \frac{1}{3}$, we have seen in Section 2 that there is a more subtle change at $T = \frac{4}{\pi^2}$, at which the integral kernel of the dispersive operator L loses its positivity and monotonicity; that has made it possible to prove some additional, but not complete, results for the case of (very) strong surface tension $T \geqslant \frac{4}{\pi^2}$. To complete the picture where our analytical methods have so far proved insufficient, we have additionally run a spectral bifurcation code similar to the one used in [35]: a Fourier-collocation scheme is employed to discretise and solve the equation, while a pseudo-arclength strategy allows us to follow the branch of solutions in the presence of turning points and other complex behaviours. In these computations the wavelength 2π has been used, that

[¶]We lack a proof of non-existence of the k_1-modal waves in the resonant case of Theorem 4.1, but these waves do not seem to exist numerically.

is, $\kappa = 1$. We will present the main result of these calculations as well, but only in overview form.

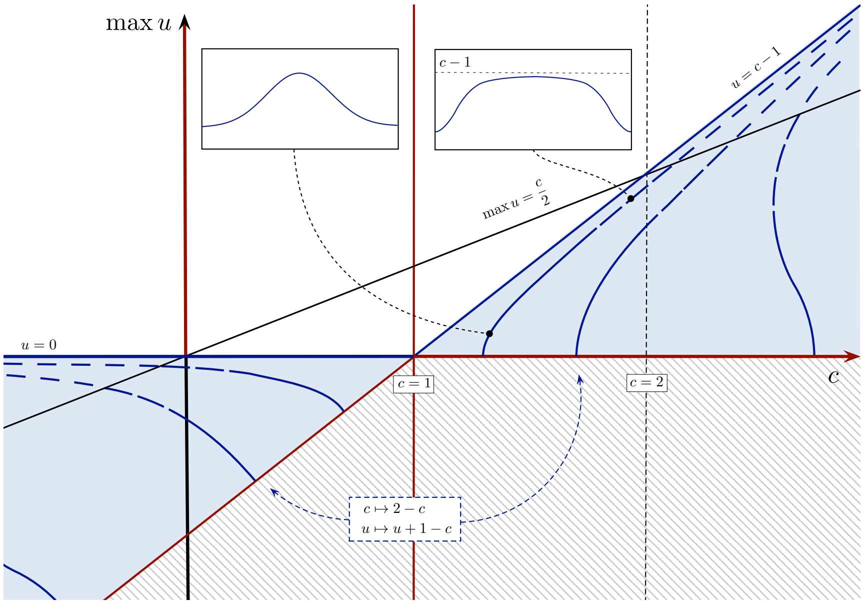

Figure 3. A schematic drawing of the global bifurcation diagram in the case of strong surface tension $T > \frac{1}{3}$ (partly $T = \frac{4}{\pi^2}$). The diagram is discussed in detail in Section 5.2.

To start our discussion, focus first on one of the Figures 3 or 4. Just as the regular Whitham equation, the capillary-gravity Whitham equation (2.1) admits two lines of constant solutions, namely $u = 0$ and $u = c - 1$. These cross at $c = 1$, the point of a transcritical bifurcation (see Remark 3.2), and also a bifurcation point for solitary [7] and generalised solitary [25] waves ; additionally, $c = 1$ is the symmetry line for the Galilean invariance

$$c \mapsto 2 - c, \qquad u \mapsto u + 1 - c,$$

that leaves (2.1) invariant, and is shared by the regular Whitham equation [21]. The two constants 0 and $c - 1$ correspond to the two natural depths that appear for steady flows in the water wave problem, see for example [28]. In addition to these two lines, there is a third, mathematical, constant arising from the structure of (2.1) when completing the square, namely $\frac{c}{2}$. While this constant is of physical and absolute importance in the regular Whitham equation — being the height above surface of a highest wave — and while it appears as a technical difficulty when trying to expand the result of Proposition 5.4, numerical evidence indicate that this construct is probably only artificial in the presence of capillarity. Still, we have indicated it in Figure 3 using the line $\max u = \frac{c}{2}$ (but not in Figure 4, as it did not prove any help in communicating our results). Additionally, in both Figures 3 and 4 the greyed-out area illustrates Proposition 5.2, that there are no solutions in the region where

$$\max u < \min\{0, c-1\}.$$

A final common feature of the strong and weak surface tension case is that solutions cannot pass $c = 1$, except via the transcritical bifurcation point $(u, c) = (0, 1)$, where, locally, the only solutions are given by the constant functions $u = 0$ and $u = c - 1$. This fact may be induced from Proposition 5.2 (iii) and Remark 3.2, and is indicated in the figures with a solid red line (no solutions pass). Note that both figures are for a fixed and finite period.

5.2. **The case of strong surface tension.** Now, let us focus on the strong surface tension case and especially the case $T \geqslant \frac{4}{\pi^2}$, which is depicted in Figure 3. As described in Theorem 3.1, we have small waves of the approximate linear form $\cos(k\cdot)$ bifurcating at

$$c_k = \frac{1}{l_T(k)} > 1.$$

The bifurcation curves of these waves are indicated by solid blue lines, with a zoom-in on a small wave along the main bifurcation branch $k = 1$. The red line $\{u = 0, 1 < c \neq c_k\}$ shows the result of Proposition 5.2 (ii), that there are no other supercritical solutions in a $\mathcal{C}^s$-vicinity of the line of vanishing solutions. By Galilean invariance, each of these curves (and non-existence results) has an exact counterpart for $c < 1$ along the line $u = c - 1$, and we do not comment more on that in the case of strong surface tension.

The initial direction of the curves is calculated in Remark 3.7: analytically, sub-critical bifurcation is established for small enough values of k, and super-critical bifurcation as $k \to \infty$; numerically, this shift happens at exactly one value, and we have illustrated this with the last visible (third) curve bending leftwards from the bifurcation point, while the two first bend right-wards (the direction after the Galilean shift is opposite).

The result of the global bifurcation theory as carried out in Theorem 3.6 is that each curve, when considered in a space of $2\pi/k$-periodic functions, is either unbounded in $\mathcal{C}^s \times \mathbb{R}$, or returns (loops) back to $(u, c) = (0, c_k)$ in a finite period of the bifurcation parameter. The standard tool for ruling out looping is by preserving the unimodal nodal pattern along the main bifurcation branch, an argument for which one relies on maximum principles/positivity of the underlying operators. As we prove in Theorem 2.8 that this property is present when the surface tension coefficient satisfies $T \geqslant \frac{4}{\pi^2}$ (and only then)[‖], the complete monotonicity of the kernel K established in Theorem 2.8 for that case provides hope for stronger results. Note that, regardless of the exact value of $T > 0$, it follows from Lemma 2.4 that all solutions of (2.1) are smooth, so that alternative (i) in Theorem 3.6 is equivalent, by bootstrapping in (2.3), to a sequence of solutions satisying $|u|_\infty + |c| \to \infty$ along the bifurcation curve.

While we cannot rule out alternative (ii) in Theorem 3.1 completely, see Proposition 5.4, we can at least show that looping would require leaving every set of the form $\max u < \lambda c$ for $\lambda < \frac{1}{2}$ (that is the consequence of Proposition 5.4, as an unbounded continuous bifurcation curve cannot be finitely periodic). Although alternative (i) in Proposition 5.4 is very unlikely, and never appears in our numerical calculations, we have been unable to rule it out (the reason for this might be that the balance between Mu and u^2 is exactly at the critical threshold for Gagliardo–Nirenberg, so that control of a higher Sobolev norm of u in terms of a lower seems to require using precise properties of the integral kernel.) We have illustrated this with long-dashed lines in Figure 3, showing the curves (probably) leaving the cone $\max u \leqslant \frac{c}{2}$.

After that point, our calculations are purely numerical, showing the solution curves asymptotically approaching the second curve of constant solutions $u = c - 1$. Indeed, if the quotient

$$\frac{u}{c-1}$$

should converge to any constant along the bifurcation curve, it is immediate from (5.1) that the limit is either 0 or 1. The numerics indicate that the quotient $\frac{\max(u)}{c-1}$ increases along the bifurcation curve to cover all of the interval $(0, 1)$, with wave profiles that are monotone on a minimal half-period even though, by far, we have passed $u = \frac{c}{2}$. Such a result, we believe, would be new in the setting of capillary-gravity water waves, but it is so far out of reach for us when u crosses $\frac{c}{2}$. Interestingly enough, the same pattern seems to persist even when the kernel is not everywhere positive and monotone, that is, for $T < \frac{4}{\pi^2}$.

As a comparison, for the Euler equations — in the presence of interfacial waves or waves with surface tension — analytically all alternatives along a global bifurcation curve are open: waves could be steepening, looping, speeding, lengthening or develop surface or vorticity singularities [4]; for interfacial waves without surface tension, unboundedness in speed, slope

[‖]It is possible that the periodised kernel is positive even when the original kernel is not, depending on the period, but we have not investigated that here.

or in the form of a surge is necessary [5]. There is an indirect proof, however, of connection between the trivial state and waves with infinite slopes/overhanging profiles [3] and even self-touching surface (so-called splash singularities) [11], in that the former are perturbations of Crapper waves, the Crapper family being a continuum from undisturbed water [32]. Numerical investigation have further shown that waves with infinite slopes can re-appear higher up along bifurcation branches [40]. As the model we are dealing with cannot capture multi-valued profiles, the increased steepening visible in the numerical calculations is probably the closest one can come. Interestingly, in [4], an alternative is that two different flat states connect in a way very much resemblant to our curves approaching the line $u = c - 1$.

Finally, for surface tension $T \geqslant \frac{4}{\pi^2}$, Proposition 5.2 shows that no solutions pass the line $c = 0$ with $\max u \geqslant 0$, indicated by red in Figure 3.

5.3. **The case of weak surface tension.** When the surface tension is weak, $T < \frac{1}{3}$, several things are very different. First of all, the first single bifurcation points c_k might, depending on the period, appear in the interval $0 < c < 1$, although for large enough values of the wavenumber k the waves will all be supercritical. Just as as in the case of strong surface tension, Proposition 5.2 guarantees that solutions do not cross the lines marked with red in Figure 4 (although these now do not include the positive vertical axis $\max u > 0$), and there are no solutions in the grey area. Similarly, there are no small, non-constant, solutions in a neighbourhood of any point along the constant solution axes $u = 0$ and $u = c - 1$, except at the countable bifurcation points.

A peculiarity in the case of weak surface tension is the appearance of multimodal waves connecting different curves of k-modal waves. Analytically, we find a full disk of solutions by two-dimensional bifurcation in Theorem 4.1 (i), by varying the wavelength. Fixing the fundamental period, however, this yields a one-dimensional subset of this disk, where we continuously transform via only a curve between two main modes of waves. Numerically, this effect persists even for values slightly off the exact points of two-dimensional bifurcation: as the numerical investigation [35] shows, the looping alternative (i) in the global one-dimensional Theorem 3.6 happens in the form of one bifurcation curve of k-modal waves transforming into one of n-modal waves and thereby connecting back to the line of zero states. The same kind of connections have been found for the Euler equations, analytically for small waves [38], and numerically for small and large waves [8, Figures 4 and 5] (see also [23,41] for perturbation theory and numerical calculations showing the rippling and non-uniqueness of small waves). These branch-to-branch connections are illustrated in Figure 4 by a curve of small bimodal waves connecting two curves of unimodal waves bifurcating off the 0-axis for $c \in (0, 1)$. (In numerical calculations for this manuscript, there have even been instances of curves of waves bridging, consecutively, three different unimodal bifurcation curves, that is, a nontrivial path that connects three separate bifurcation points, but that is not indicated in the graphics.)

The curves of subcritical waves can be followed, again numerically, past zero wave speed, going left-ward without any indication to stop. In $L^2(\mathbb{S})$, they seem to flatten out to 0, but not in L^∞. This feature reappears again and again in both numerics and our calculations: while L^∞-bounds easily yield bounds on higher norms, and one has control of solutions in L^2 with respect to the wave speed, it is extremely difficult to relate the L^∞-norm of solutions to their $L^2(\mathbb{S})$-norm, even when the wave speed is bounded. Generally, all curves of solutions appear to asymptotically approach one of the curve of constant solutions ($u = 0$ or $u = c - 1$) in $L^2(\mathbb{S})$, while an actual connection in a space of higher regularity is impossible for almost all wavespeeds because of the invertibility of the linear operator $D_u F$ (note that it is not obvious how to make sense of the nonlinear mapping F in $L^2(\mathbb{S})$).

Finally, in the case of supercritical bifurcation, we find only single-crested (bell-shaped) waves even though the surface tension is weak. When these waves are small it is a result of Theorem 3.1. These curves may be continued globally (Theorem 3.6), but the information about them is purely numerical. Just as in the case of strong surface tension, these supercritical waves show no ripples, and they asymptotically approach $u = c - 1$ in $L^2(\mathbb{S})$, but not in L^∞. Any proof of preservation of the nodal properties in the case of supercritical bifurcation when the surface tension is weak is for the moment entirely out of our reach, even though it would be very interesting to obtain.

Appendix A. Bifurcation formulas

This appendix contains higher order expansions of the quantities in Theroem 3.1 and Theorem 4.1. We start with the first and second order terms in the expansion for the speed $c(t)$ in the one-dimensional bifurcation case, which is required by the proof of the global extension in Theorem 3.10. We then proceed to study the first order terms for the expansions of the functions r and p in the two-dimensional bifurcation case.

A.1. **One-dimensional bifurcation case.**

We begin by determining the derivatives $\dot{c}(0)$ and $\ddot{c}(0)$ associated to the bifurcation curve constructed in Theorem 3.1. This can be done either directly using the

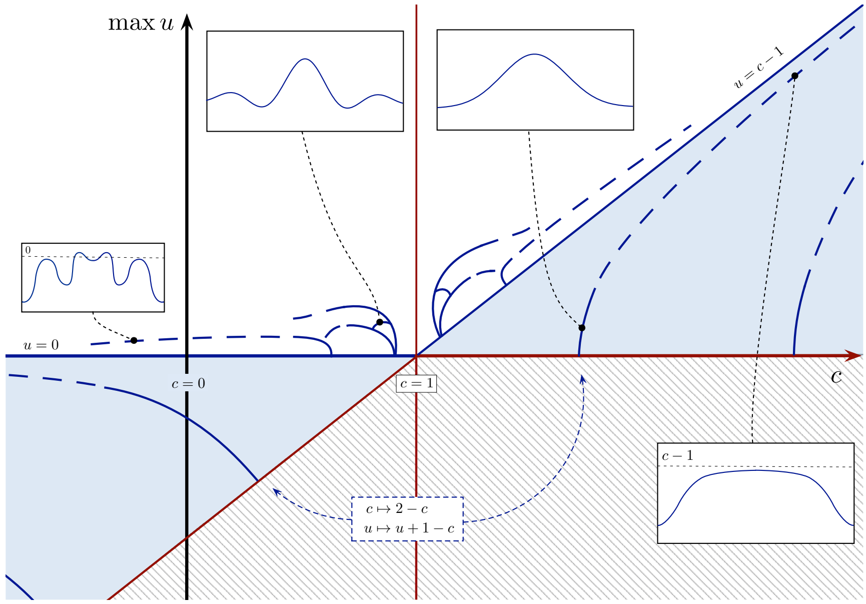

Figure 4. A schematic drawing of the global bifurcation diagram in the case of weak surface tension $T < \frac{1}{3}$. The diagram is discussed in detail in Section 5.3.

Lyapunov–Schmidt reduction carried out in the proof of Theorem 3.1 or by the means of bifurcation formulas given for example in [27]. The latter requires an identification between the bifurcation function $\phi(u, c) = \Pi F(u + \psi(u, c), c)$ used in [27] and the functions v and r used in the proof of Theorem 3.1. This relation is given by $v(t) = \psi(t\cos(kx), c(t))$.

Here, start from the Lyapunov–Schmidt representation

$$\begin{aligned} 0 &= F(t\cos(kx) + v(t), c_0 + r(t)) \\ &= t\cos(kx) + v(t) \\ &\quad + L\left[(t\cos(kx) + v(t))^2 - (c_0 + r(t))(t\cos(kx) + v(t))\right], \end{aligned} \tag{A.1}$$

where here it is understood that for each t small the function $v(t)$ is a $2\pi/k$-periodic function of x. Differentiating (A.1) once with respect to t, evaluating at $t = 0$ and using that $v(0) = \dot{v}(0) = r(0) = 0$ yields the equation

$$(1 - c_0 L)\cos(kx) = 0,$$

which holds by our choice of c_0. Similarly, differentiating (A.1) twice with respect to t and evaluating at $t=0$ yields

$$\begin{aligned}(1-c_0L)\ddot{v}(0) &= 2\dot{r}(0)L\cos(kx) - 2L\cos^2(kx)\\ &= 2\dot{r}(0)l(k)\cos(kx) - (1+l(2k)\cos(2kx)).\end{aligned} \tag{A.2}$$

Since $\int_{-\pi}^{\pi} v(t)\cos(kx)dx = 0$ for all $|t| \ll 1$, the above implies that $\dot{r}(0)=0$. Returning to (A.2), it now follows that

$$\ddot{v}(0) = \frac{1}{c_0-1} + \frac{l(2k)\cos(2kx)}{c_0 l(2k)-1}. \tag{A.3}$$

Continuing, we observe that taking the third derivative of (A.1) with respect to t and evaluating at $t=0$ yields

$$(1-c_0L)\dddot{v}(0) = 3\ddot{r}(0)L\cos(kx) - 6L(\ddot{v}(0)\cos(kx)).$$

Using (A.3), we compute that

$$L(\ddot{v}(0)\cos(kx)) = \frac{l(k)\cos(kx)}{c_0-1} + \frac{l(2k)\,(l(k)\cos(kx)+l(3k)\cos(3kx))}{2(c_0 l(2k)-1)}.$$

Using again that $\int_{-\pi}^{\pi} v(t)\cos(kx)\,\mathrm{d}x = 0$ for all $|t| \ll 1$, it follows that

$$\ddot{r}(0) = \frac{3}{c_0-1} + \frac{l(2k)}{c_0 l(2k)-1} = \frac{3c_0 l(2k) - l(2k) - 2}{(c_0-1)(c_0 l(2k)-1)},$$

which is the expression (3.10) for $\ddot{c}(0)$ given in Theorem 3.6. Note that the above procedure could be continued to obtain asymptotic expansions of $r(t)$ and $v(t)$ to arbitrarily high order in t. We also note that the above result is consistent with the asymptotic formulas in [24].

A.2. **Two-dimensional bifurcation case.** We now consider the case of a two-dimensional bifurcation as considered in Section 4 above. Recall that the solutions constructed in Theorem 4.1 can be written as

$$\begin{aligned}u(t_1,t_2) &= t_1\cos(k_1x) + t_2\cos(k_2x) + v(t_1,t_2),\\ c(t_1,t_2) &= c_0 + r(t_1,t_2),\\ \kappa(t_1,t_2) &= \kappa_0 + p(t_1,t_2),\end{aligned}$$

with v of order $\mathcal{O}(|(t_1,t_2)|^2)$ and r, p of order $\mathcal{O}(|(t_1,t_2)|)$. We now characterize the order of vanishing of the functions r and p at the origin.

Proposition A.1. *Let the functions r and p be as in Theorem 4.1. If $k_2/k_1 \notin \mathbb{N}_0$, then*

$$\nabla r(0,0) = 0, \qquad \nabla p(0,0) = 0$$

so that, in particular, r and p are of order $\mathcal{O}(|(t_1,t_2)|^2)$ near the origin. If instead $k_2/k_1 \in \mathbb{N}_0$, then for any $\delta > 0$ small we have that, in polar coordinates,

$$r_\varrho(0,\vartheta) = 0, \qquad p_\varrho(0,\vartheta) = 0$$

if and only if either $k_2 \notin \{0, 2k_1\}$ or $(k_2,\vartheta) = \left(2k_1, \frac{\pi}{2}\right)$.

Proof. We begin the non-resonant case, $k_2/k_1 \notin \mathbb{N}_0$. From the proof of Theorem 4.1, we know for all $0 < |(t_1,t_2)| \ll 1$ the functions r and p satisfy

$$\Psi_i(t_1,t_2,r(t_1,t_2),p(t_1,t_2)) = 0 \quad \text{for } i = 1,2,$$

where the Ψ_i are defined in (4.9) and (4.13). Fixing $j \in \{1,2\}$ we find that differentiating the above with respect to t_j and evaluating at $(t_1,t_2) = (0,0)$ gives the system of equations

$$\begin{pmatrix} \Psi_{1,r}(\mathbf{0}) & \Psi_{1,p}(\mathbf{0}) \\ \Psi_{2,r}(\mathbf{0}) & \Psi_{2,p}(\mathbf{0}) \end{pmatrix} \begin{pmatrix} r_{t_j}(0,0) \\ p_{t_j}(0,0) \end{pmatrix} = - \begin{pmatrix} \Psi_{1,t_j}(\mathbf{0}) \\ \Psi_{2,t_j}(\mathbf{0}) \end{pmatrix}, \tag{A.4}$$

where here $\mathbf{0}$ denotes the origin in $\mathbb{R}^4$. Since the above system matrix is invertible by (4.15), it remains to determine the values of $\Psi_{i,t_j}(\mathbf{0})$ for $i = 1,2$. This can be accomplished by recalling (4.9) and (4.13) and noting that (4.7) implies that

$$\frac{\partial^2 Q_i}{\partial t_j^2}(\mathbf{0}) = -\frac{2}{\pi}\, l(\kappa_0 k_i) \int_{-\pi}^{\pi} \cos^3(k_i x)\, \mathrm{d}x$$

and

$$\frac{\partial^2 Q_i}{\partial t_1 \partial t_2}(\mathbf{0}) = \begin{cases} -\dfrac{2}{\pi} l(\kappa_0 k_2) \displaystyle\int_{-\pi}^{\pi} \cos^2(k_1 x)\cos(k_2 x)\, \mathrm{d}x, & i = 1, \\ -\dfrac{2}{\pi} l(\kappa_0 k_1) \displaystyle\int_{-\pi}^{\pi} \cos^2(k_2 x)\cos(k_1 x)\, \mathrm{d}x, & i = 2 \end{cases}$$

Consequently, since $k_2/k_1 \notin \mathbb{N}_0$ it follows that $\Psi_{i,t_j}(\vec{0}) = 0$ for $i = 1,2$ and hence (A.4) implies that $r_{t_j}(0,0) = p_{t_j}(0,0) = 0$ as claimed. Since $j \in \{1,2\}$ was arbitrary, this proves the proposition in the non-resonant case.

Now, consider the resonant case when $k_2/k_1 \in \mathbb{N}_0$ and fix $\delta > 0$ small. In this case, for each $\delta < |\vartheta| < \pi - \delta$ and $0 < \varrho \ll 1$ the functions $r(\varrho,\vartheta)$ and $p(\varrho,\vartheta)$ satisfy the system

$$\widetilde{\Psi}_i\left(\varrho,\vartheta,r(\varrho,\vartheta),p(\varrho,\vartheta)\right) = 0 \quad \text{for } i = 1,2,$$

where here the $\widetilde{\Psi}_i$ are as in (4.20) and (4.18). Differentiating this system with respect to ϱ at $\varrho = 0$ gives the system of equations

$$\begin{aligned}&\begin{pmatrix} \widetilde{\Psi}_{1,r}(0,\vartheta,0,0) & \widetilde{\Psi}_{1,p}(0,\vartheta,0,0) \\ \widetilde{\Psi}_{2,r}(0,\vartheta,0,0) & \widetilde{\Psi}_{2,p}(0,\vartheta,0,0) \end{pmatrix} \begin{pmatrix} r_\varrho(0,\vartheta) \\ p_\varrho(0,\vartheta) \end{pmatrix} \\ &= - \begin{pmatrix} \widetilde{\Psi}_{1,\varrho}(0,\vartheta,0,0) \\ \widetilde{\Psi}_{2,\varrho}(0,\vartheta,0,0) \end{pmatrix}.\end{aligned} \tag{A.5}$$

As in the non-resonant case, the above system matrix is invertible, this time thanks to (4.21), and hence it remains to determine the values of $\widetilde{\Psi}_{i,\varrho}(0,\vartheta,0,0)$ for $i = 1, 2$. Let us begin by determining the value in the case $i = 1$. From (4.20) and the preceding discussion, we know we can write

$$\widetilde{\Psi}_1(\varrho,\vartheta,0,0) = \int_0^1 \frac{\partial \widetilde{Q}_1}{\partial \varrho}(z\varrho,\vartheta,0,0)\,\mathrm{d}z$$

where, using (4.7), we have explicitly

$$\begin{aligned}\widetilde{Q}_1(\varrho,\vartheta,0,0) &= Q_1(\varrho\cos(\vartheta),\varrho\sin(\vartheta),0,0) \\ &= -\frac{2\varrho^2 l(k_0k_1)\cos(\vartheta)\sin(\vartheta)}{\pi}\int_{-\pi}^{\pi}\cos^2(k_1x)\cos(k_2x)\,\mathrm{d}x.\end{aligned}$$

Clearly then, $\widetilde{Q}_{2,\varrho\varrho}(0,\vartheta,0,0)$ is equal to zero if and only if either $\vartheta = \frac{\pi}{2}$ or $k_2 \notin \{0, 2k_1\}$. Since

$$\widetilde{\Psi}_{1,\varrho}(0,\vartheta,0,0) = \frac{1}{2}\frac{\partial^2 \widetilde{Q}_1}{\partial \varrho^2}(0,\vartheta,0,0)$$

by above, we have shown that $\widetilde{\Psi}_{1,\varrho}(0,\vartheta,0,0) = 0$ if and only if either of the conditions $\vartheta = \frac{\pi}{2}$ or $k_2 \notin \{0, 2k_1\}$ hold.

Similarly, we have

$$\widetilde{\Psi}_{2,\varrho}(0,\vartheta,0,0) = \frac{1}{2}\frac{\partial^2 \widetilde{Q}_2}{\partial \varrho^2}(0,\vartheta,0,0)$$

where, using (4.16), we have

$$\begin{aligned}\widetilde{Q}_2(\varrho,\vartheta,0,0) = &-\frac{\varrho^2 l(k_0k_2)}{\pi}\times \\ &\int_{-\pi}^{\pi}\cos(k_2z)\left[\cos^2(\vartheta)\cos^2(k_1x) + \sin^2(\vartheta)\cos^2(k_2x)\right]\,\mathrm{d}x.\end{aligned}$$

Clearly, $\widetilde{Q}_{2,\varrho\varrho}(0,\vartheta,0,0)$ vanishes whenever $k_2 \notin \{0, 2k_1\}$. When $k_2 = 0$, $\widetilde{Q}_{2,\varrho\varrho}(0,\vartheta,0,0)$ does not vanish for any ϑ, and when $k_2 = 2k_1$ it only vanishes when $\vartheta = \frac{\pi}{2}$. Consequently, $\widetilde{\Psi}_{2,\varrho}(0,\vartheta,0,0)$ vanishes only when either $k_2 \notin \{0, 2k_1\}$ or $(k_2, \vartheta) = (2k_1, \frac{\pi}{2})$. Together with the results concerning $\widetilde{\Psi}_{1,\varrho}$, this completes the proof. □

Remark A.2. *The special case $k_2 = 2k_1$ has been found also in the Euler equations (with gravity and vorticity) by the authors of [1]. The special case $k_2 = 0$ is instead due to the transcritical double bifurcation allowed by the capillary-gravity Whitham equation.*

Remark A.3. *An explicit example where $r_\varrho(0, \vartheta) \neq 0$ can be seen in [35, Figure 6], where the branch of nontrivial solutions has a non-vertical tangent at the bifurcation point in the speed-height plane.*

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Paper 2

Paper 2

SOLITARY WAVES FOR WEAKLY DISPERSIVE EQUATIONS WITH INHOMOGENEOUS NONLINEARITIES

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Abstract. We show existence of solitary-wave solutions to the equation

$$u_t + (Lu - n(u))_x = 0\,,$$

for weak assumptions on the dispersion L and the nonlinearity n. The symbol m of the Fourier multiplier L is allowed to be of low positive order ($s > 0$), while n need only be locally Lipschitz and asymptotically homogeneous at zero. We shall discover such solutions in Sobolev spaces contained in H^{1+s}.

1. Introduction

A great deal of model equations for the evolution of water waves in one spacial dimension can be compactly written as

$$u_t + (Lu - n(u))_x = 0\,, \tag{1.1}$$

where the dispersion L is a Fourier multiplier in space with real-valued symmetric symbol m, that is,

$$\widehat{Lu}(\xi) = m(\xi)\hat{u}(\xi),$$

and n is a local nonlinear term. Solutions of (1.1) tend to enjoy a variety of qualitative properties of water, see [12], but our focus will be on the existence of *solitary waves*. Traveling at constant velocity ν, these solutions take the form $(x,t) \mapsto u(x - \nu t)$, where $u(y) \to 0$ as $|y| \to \infty$. For such solutions (1.1) means

$$-\nu u + Lu - n(u) = 0\,, \tag{1.2}$$

in light of the assumption that u vanish at infinity.

A common approach to prove solitary waves in equations of the form (1.2) is Lion's concentration-compactness method introduced in [15]. Weinstein used this in 1987 to prove existence and orbital stability in the case of a monomial nonlinearity and a symbol of order $s \geq 1$ [19]. The limit $s = 1$ is not only superficial: In [2] the authors study an equation corresponding to $s = 1$, and that method was later put in a more general framework in [1], again for $s \geq 1$. Zeng [20] later used a different energy functional (and different conserved quantity) to relax some of the conditions, but still for $s \geq 1$.

These works led a number of different authors to consider the case when $s < 1$: in [14] and [3] the authors treat equations with positive-order Fourier operators ($s > 0$) — the case of homogeneous and inhomogeneous symbols respectively – and in both cases with homogeneous nonlinearities; whereas in [7] smoothing operators ($s < 0$) with mildly inhomogeneous nonlinearities are allowed. The method for positive-order operator is indeed based upon Weinstein's paper [19], whereas the method for negative-order operators is different, and more closely related to works on the Euler equations and other systems with dispersion of very weak type [10]. A main difference between the works [3, 14] and [7] is the requirement that the waves in the latter should be small. This is related to scalings/homogeneity of the nonlinearity, and an essential part of the method of proof in [7]. A later work, related to the investigations for positive s, is [6], in which the authors look at (1.1) when the nonlinearity is polynomial, cubic or higher, and the symbol m grows at least as $|\xi|^{\frac{1}{2}}$ at infinity. This growth may be slightly lowered: in the case of a quadratic pure-power nonlinearity and a homogeneous symbol m (the fractional KdV equation), the optimal assumption in terms of growth is $m(\xi) = |\xi|^s$, $s > \frac{1}{3}$ [9]; below this value one does not have solitary waves for the (homogeneous) fKdV equation [13]. This coincides with our assumption on s below; for the assumption on s', see our remarks in Section 1.3.2.

Our goal has been twofold. First, to combine ideas from [3] and [7] to allow for more inhomogeneous nonlinearities in the theory for lower-order ($s > 0$) symbols; and, second, to improve upon the required assumptions on both the linear and nonlinear terms by a slightly different method of proof. The last point is made visible mostly in that the theory for low-order s is carried out in corresponding low-order Sobolev spaces (below the L^∞ embedding), for which we use a cut-off of the nonlinearity n which is different from the 'small ball' used in [7]. (Our solutions will eventually be somewhat more regular, but the near-minimizers we work with might not exhibit the same regularity). In effect, we are able to reduce the assumptions on (1.2) to the following.

1.1. **The assumptions and the main theorem.** Throughout the paper, we will assume the following:

(A) The nonlinearity $n\colon \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, and decomposes into $n = n_p + n_r$, where n_p is homogeneous of one of the two forms:

(A1) $x \mapsto c|x|^{1+p}$ and $c \neq 0$,

(A2) $x \mapsto cx|x|^{p}$ and $c > 0$,

for a real number $p > 0$, while the remainder term satisfies $n_r(x) = \mathcal{O}(|x|^{1+r})$, as $x \to 0$, for some $r > p$.

(B) The symbol $m \colon \mathbb{R} \to \mathbb{R}$ is even and satisfies the growth bounds

$$\begin{cases} m(\xi) - m(0) \simeq |\xi|^{s'}, & \text{for } |\xi| < 1, \\ m(\xi) - m(0) \simeq |\xi|^{s}, & \text{for } |\xi| > 1, \end{cases}$$

with $s' > p/2$ and $s > p/(2+p)$. We also require $\xi \mapsto m(\xi)/\langle\xi\rangle^s$ to be uniformly continuous on $\mathbb{R}$.

We will discuss these assumptions in detail below. Given them, we will prove the following existence result.

Theorem 1.1. *There exists $\mu_* > 0$ so that for every $\mu \in (0, \mu_*)$, there is a solution $u \in H^{1+s}$ of* (1.2)*, with wave speed $\nu \in \mathbb{R}$, satisfying*

(i) $\|u\|_{H^{1+s}}^2 \lesssim \|u\|_2^2 = 2\mu$,

(ii) $m(0) - \nu \simeq \mu^\beta$, *with* $\beta = \frac{s'p}{2s'-p}$,

where the implicit constants in (*i*) *and* (*ii*) *are independent of* $\mu \in (0, \mu_*)$.

An interesting special case of Theorem 1.1 is the case of the capillary-gravity Whitham equation with strong surface tension, for which $p = 1$ and the symbol is

$$m(\xi) = \left((1 + T\xi^2)\frac{\tanh(\xi)}{\xi}\right)^{\frac{1}{2}}, \qquad T \geq \tfrac{1}{3},$$

which corresponds to $s = \frac{1}{2}$ and $s' = 2$. Modelled on the water wave problem with surface tension, the capillary-gravity Whitham equation is known to admit generalized solitary waves in the case $T < \frac{1}{3}$ (weak surface tension) [11], and decaying solitary waves for $T > 0$ (both weak and strong surface tension) [3], as well as periodic steady waves, including rippled solutions in the case of weak surface tension [8]. In the case $T < \frac{1}{3}$ the solitary waves have wave speeds ν smaller than $m(0)$ (called subcritical), whereas the generalized waves exhibit supercritical wave speeds $\nu > m(0)$; for strong surface tension we are only aware of sub-critical solutions. As we also prove the existence of sub-critical solutions, in the case of strong surface tension $T \geq \frac{1}{3}$, there currently seems to lack super-critical truly solitary waves in the capillary-gravity Whitham equation. The same waves have also not been found for the capillary-gravity Euler equations (although we have not found a source actually stating this), but a proof of general non-existence is lacking. What has been shown is that there are no small-amplitude, exponentially decaying, even, supercritical solitary-wave solutions of the Euler equations in the slightly weak case when T is close to, but less than, $\frac{1}{3}$ [18].

On a related note, it might be worth noticing that Theorem 1.1 is also an existence result for solitary waves tending to a general value c, not necessarily zero, at infinity. For if $\tilde{n}(x) = n(c+x) - n'(c)x - n(c)$ satisfies the assumptions, then there is a solitary-wave solution u, with velocity ν, of the equation $u_t + (Lu - \tilde{n}(u))_x = 0$, and thus, $u + c$ is a traveling wave solution of (1.2) with velocity $\nu - n'(c)$.

1.2. **The method.** In this subsection, the framework used to prove Theorem 1.1 will be introduced. In particular, we develop a constrained minimization problem whose solutions satisfy (1.2), and in fact, it is exactly solutions of this minimization problem that we shall prove the existence of. For this purpose, we will be working with two 'extra' assumptions on (1.2), namely

(C_1) n is globally Lipschitz continuous,
(C_2) $m(0) = 0$.

While these auxiliary assumptions (especially the first) excludes many instances of (1.1) where we would like to prove the existence of solitary wave solutions, it turns out that proving our main theorem for this smaller class implies the result in the more general setting, as we now demonstrate.

Lemma 1.2. *If Theorem 1.1 holds true under the assumptions (A), (B), (C_1) and (C_2), then it also holds true when only (A) and (B) are satisfied.*

Proof. Assume n and m satisfy (A) and (B). Define

$$\tilde{n}(x) = \begin{cases} n(x), & |x| \leq 1, \\ n(\pm 1), & \pm x > 1, \end{cases} \qquad \tilde{m}(\xi) = m(\xi) - m(0),$$

and notice that $\tilde{n}$ and $\tilde{m}$ satisfy (A), (B), (C_1) and (C_2). By assumption, Theorem 1.1 now holds for the modified equation

$$-\tilde{\nu} u + \tilde{L} u - \tilde{n}(u) = 0,$$

where $\tilde{L}$ is the Fourier multiplier whose symbol is $\tilde{m}$. Thus there is a $\tilde{\mu}_* > 0$ so that for each $\mu \in (0, \tilde{\mu}_*)$ we have a solution u with velocity $\tilde{\nu}$ satisfying

$$\|u\|_{H^{1+s}}^2 \lesssim \mu,$$
$$-\tilde{\nu} \simeq \mu^\beta,$$

where we omitted $\tilde{m}(0) = 0$ from the second expression. As $H^{1+s} \hookrightarrow L^\infty$, we can pick $\mu_* \in (0, \tilde{\mu}_*)$ so that $\|u\|_\infty \leq 1$ for all $\mu \in (0, \mu_*)$. For such solutions u, we have $\tilde{n}(u) = n(u)$, and setting $\nu = \tilde{\nu} - m(0)$ we see that

$$\begin{aligned} 0 &= -\tilde{\nu} u + \tilde{L} u - \tilde{n}(u), \\ &= -\nu u + (\tilde{L} + m(0))u - n(u), \\ &= -\nu u + Lu - n(u). \end{aligned}$$

Thus, for $\mu < \mu_*$ the solutions provided by Theorem 1.1 for the modified equation are solutions of the original equation, but with a shifted velocity ν satisfying

$$m(0) - \nu \simeq \mu^\beta.$$

□

We now construct the minimization problem mentioned above, whose well-posedness is assured when the assumption (C_1) is added to (A) and (B). We will work in the Sobolev space $H^{\frac{s}{2}}$ of measurable functions $f \colon \mathbb{R} \to \mathbb{R}$ with finite Sobolev norm

$$\|f\|_{H^{\frac{s}{2}}} = \|\langle \cdot \rangle^{\frac{s}{2}} \hat{f}\|_2,$$

where we use the Japanese bracket $\langle \xi \rangle = \left(1 + \xi^2\right)^{1/2}$. Our main tools shall be the functionals $\mathcal{Q}, \mathcal{L}, \mathcal{N} \colon H^{\frac{s}{2}} \to \mathbb{R}$, defined by

$$\begin{aligned} \mathcal{Q}(u) &= \frac{1}{2} \int_{\mathbb{R}} u^2 \, dx, \\ \mathcal{L}(u) &= \frac{1}{2} \int_{\mathbb{R}} m(\xi) |\hat{u}|^2 \, d\xi, \\ \mathcal{N}(u) = \mathcal{N}_p(u) + \mathcal{N}_r(u) &= \int_{\mathbb{R}} N_p(u) \, dx + \int_{\mathbb{R}} N_r(u) \, dx, \end{aligned}$$

where $N_p(x) = \int_0^x n_p \, dt$, and $N_r(x) = \int_0^x n_r \, dt$. We will prove the above functionals to be Fréchet differentiable with $H^{\frac{s}{2}}$-derivatives

$$\mathcal{Q}'(u) = u, \qquad \mathcal{L}'(u) = Lu, \qquad \text{and} \qquad \mathcal{N}'(u) = n(u).$$

Consider now the constraint minimization problem

$$I_\mu = \inf_{u \in U_\mu} \mathcal{E}(u) \,, \tag{1.3}$$

where $\mathcal{E} = \mathcal{L} - \mathcal{N}$ and

$$U_\mu = \{u \in H^{\frac{s}{2}} \colon \mathcal{Q}(u) = \mu\}, \tag{1.4}$$

and where we restrict $\mu \in (0, \mu_*)$, for some fixed upper bound μ_* that we shall require to be sufficiently small. Our strategy shall be to find minimizers of (1.3); a minimizer u must for some Lagrange multiplier $\nu \in \mathbb{R}$ satisfy

$$0 = -\nu \mathcal{Q}'(u) + \mathcal{E}'(u) = -\nu u + Lu - n(u),$$

thus solving (1.2). Note that, although our solutions are 'discovered' in $H^{\frac{s}{2}}$, we additionally prove they lie in the more regular space H^{1+s} (or, in an even more regular space, see Prop. 8.2). Had we been working on a compact domain, then any "uniformly regular" minimizing sequence of (1.3), would admit a converging subsequence, implying the existence of a minimizer. As $\mathbb{R}$ is not compact, we instead use Lion's concentration–compactness theorem (see Section 2). Informally, any bounded sequence $(\rho_k) \subset L^1$ admits a subsequence (again indexed with k) that will, as $k \to \infty$, either

- *vanish* (the mass spreads out),
- *dichotomize* (the mass splits in two parts that separate), or
- *concentrate* (the mass remains uniformly concentrated in space).

We will show that for a 'concentrated' minimizing sequence, we can pick a converging subsequence. Thus, the existence of a minimizer of (1.3) follows if we can for minimizing sequences rule out the possibility of vanishing and dichotomy. To achieve this, we use a "long-wave ansatz" to find a low enough upper bound for I_μ that will allows us to compare the size of μ, $\mathcal{L}$ and $\mathcal{N}$ on 'near minimizers'. This size comparison will directly exclude vanishing and also imply that $\mu \mapsto I_\mu$ is subadditive for small $\mu > 0$, which excludes dichotomy. The paper concludes with some regularity estimates for our solutions (see Prop. 8.2).

We end this section with some discussion regarding the main assumptions (A) and (B).

1.3. **A technical look at the assumptions (A) and (B).**

In this subsection, we discuss our main assumptions on the the pair n and m; we mention what role the different parts play and whether some could be weakened. This discussion is easier to follow after a read through.

1.3.1. *The nonlinearity n.* The continuity of n is needed for $\mathcal{N}$ to be Fréchet differentiable. The stronger local Lipschitz continuity is used to obtain the estimate $\|u\|_{H^{1+s}}^2 \lesssim \mu$ for our solutions in Prop. 8.1; this important estimate gives us Lemma 1.2 which is what we use to guarantee the well-posedness of (1.3) in the case $s \leq 1$. Still, there are two alternative ways of proving solitary waves when we assume n to be merely continuous:

(i) If $s > 1$, we have $H^{\frac{s}{2}} \hookrightarrow BC$, and so one could use Prop. 4.1 (specifically equation (4.3)) in place of Prop. 8.1 to attain Lemma

1.2.

(ii) Alternatively, if $|n_r(x)| \lesssim |x|^{1+p}$ for $|x| > 1$, all steps in this paper (apart from Prop. 8.1) go through, granted we include the restriction $\|u\|_{H^{\frac{s}{2}}} < R$ to our minimization problem for some arbitrary constant $R > 0$, which only plays a role in proving Prop. 4.1.

We choose to assume local Lipschitz continuity of n to avoid these other conditions, and to provide a somewhat different technique in comparison to earlier proofs.

Finally, the reason for excluding the case $n_p(x) = cx|x|^p$, $c < 0$, is the same as in [3] and [7]. Our method breaks down at the first step in that regime, as we cannot hope to obtain the low upper bound for I_μ in Prop. 3, because $-\mathcal{N}_p(u) > 0$ for all $u \neq 0$.

1.3.2. *The symbol m.* The upper bound of the growth at zero and the corresponding inequality $s' > p/2$ are needed to find a satisfactorily low upper bound for I_μ by a long-wave ansatz (see Prop. 3), while the lower bound is necessary for Prop. 4.1, which is crucial for the remainder term n_r to be negligible for sufficiently small μ.

As for the growth bounds when $|\xi| > 1$, the lower bound is chosen to control the $H^{\frac{s}{2}}$-norm by $\mathcal{Q}$ and $\mathcal{L}$, which together with $s > p/(2+p)$ gives control of the L^{2+p}-norm by Sobolev embedding. This is used in the proof of Prop. 4.1 and in (5.4) to exclude vanishing.

The upper growth bound is instead needed when excluding dichotomy: Indeed, if $m(\cdot) - m(0)$ was bounded by $\langle \cdot \rangle^{\tilde{s}}$, $\tilde{s} > s$, we would need to work in $H^{\tilde{s}/2}$ (for $\mathcal{E}(u)$ to be well defined). Then equation (4.3), which bounds the $H^{\frac{s}{2}}$-norm, would still be the best regularity estimate on a minimizing sequence, but Lemma 6.2 (now, for operators $B_r\colon H^{\tilde{s}/2} \to H^{-\tilde{s}/2}$), would require a bound on the stronger $H^{\tilde{s}/2}$-norm to be of any use when proving Prop. 6.3.

Finally, the uniform continuity of $\xi \mapsto m(\xi)/\langle \xi \rangle^s$ is necessary for excluding dichotomy. It assures that L is not 'too' non-local, as described in Lemma 6.2. Note that a sufficient estimate for our regularity constraint is $|m'(\xi)| \lesssim \langle \xi \rangle^s$, as it implies that $\xi \mapsto m(\xi)/\langle \xi \rangle^s$ is globally Lipschitz.

2. Preliminaries

In this section, we presents bounds and regularity estimates for the functionals $\mathcal{Q}, \mathcal{L}, \mathcal{N}, \mathcal{E}$ introduced in subsection 1.2. Throughout section 2-7, we assume (only) that n and m satisfies the assumptions (A), (B), (C_1) and (C_2), introduced in subsection 1.1 and 1.2. In light of Lemma 1.2, proving Theorem 1.1 in this case, implies the validity of the theorem when either (C_1) or (C_2) fails.

Proposition 2.1. *For $u \neq 0$, we have*

$$(i)\ 0 < \mathcal{L}(u) \lesssim \|u\|^2_{H^{\frac{s}{2}}}, \qquad (iii)\ |\mathcal{N}_p(u)| \lesssim \|u\|^{2+p}_{2+p},$$
$$(ii)\ |\mathcal{N}(u)| \lesssim \mathcal{Q}(u), \qquad (iv)\ |\mathcal{N}_r(u+v)| \lesssim \|u\|^{2+r}_{2+r} + \|v\|^{2+p}_{2+p}.$$

Proof. Combining the growth bounds on m from (B) with (C_2), we see that $0 < m(\xi) \lesssim \langle \xi \rangle^s$ for $\xi \neq 0$, and so bound (i) follows. By (A) and (C_1), we have $|n(x)| \lesssim |x|$, and so we obtain (ii). From $|n_p(x)| \lesssim |x|^{1+p}$ we immediately get (iii). For (iv), we note that

$$|N_r(x)| \lesssim |x|^{2+r}, \quad |x| \leq 1, \quad \text{and} \quad |N_r(x)| \lesssim |x|^{2+p}, \quad |x| \geq 1,$$

where the the first bound follows from $n_r(x) = O(|x|^{1+r})$, while the latter follows from $|n_r(x)| = |n(x) - n_p(x)| \lesssim |x| + |x|^{1+p}$. With this, and the fact that $r > p$, we obtain

$$|N_r(x)| \lesssim \min\{|x|^{2+r}, |x|^{2+p}\},$$

or equivalently

$$\frac{|N_r(x+y)|}{|x|^{2+r} + |y|^{2+p}} \lesssim \min\left\{\frac{|x+y|^{2+r}}{|x|^{2+r} + |y|^{2+p}}, \frac{|x+y|^{2+p}}{|x|^{2+r} + |y|^{2+p}}\right\}$$
$$=: \min\Big\{a(x,y), b(x,y)\Big\}.$$

Note that $a(x,y)$ and $b(x,y)$ are bounded for $|y| \leq 1$ and $|y| \geq 1$ respectively, and so $|N_r(x+y)| \lesssim |x|^{2+r} + |y|^{2+p}$. □

From here on, we will refrain from explicitly referring to the assumptions as done in the previous proof, so to attain a more straight forward presentation.

Proposition 2.2. *The Fréchet derivative of $\mathcal{Q}, \mathcal{L}, \mathcal{N}$ and $\mathcal{E}$ at $u \in H^{\frac{s}{2}}$ are the elements in the (dual) space $H^{\frac{-s}{2}}$ given by*

(i) $\mathcal{Q}'(u) = u$,

(ii) $\mathcal{L}'(u) = Lu$,

(iii) $\mathcal{N}'(u) = n(u)$,

(iv) $\mathcal{E}'(u) = Lu - n(u)$.

Proof. The Fréchet derivative of $\mathcal{Q}$ and $\mathcal{E}$ follows from an elementary calculation and linearity of the Fréchet derivative respectively. Turning to $\mathcal{L}$, we note that L is self-adjoint, $\langle Lu, v\rangle = \langle u, Lv\rangle$, due to the symmetry of m. Consequently $\mathcal{L}(u+v) = \mathcal{L}(u) + \langle Lu, v\rangle + \mathcal{L}(v)$. We then obtain

$$\frac{|\mathcal{L}(u+v) - \mathcal{L}(u) - \langle Lu, v\rangle|}{\|v\|_{H^{\frac{s}{2}}}} = \frac{\mathcal{L}(v)}{\|v\|_{H^{\frac{s}{2}}}} \lesssim \|v\|_{H^{\frac{s}{2}}} \to 0,$$

as $v \to 0$, in $H^{\frac{s}{2}}$, where we used (i) from Prop. 2.1. For $\mathcal{N}$, we exploit the global Lipschitz-continuity of n and calculate

$$\begin{aligned}\frac{|\mathcal{N}(u+v)-\mathcal{N}(u)-\langle n(u),v\rangle|}{\|v\|_{H^{\frac{s}{2}}}} &\leq \frac{1}{\|v\|_{H^{\frac{s}{2}}}}\int_{\mathbb{R}}|v|\int_0^1|n(u+tv)-n(u)|\,dt\,dx\\ &\lesssim \frac{\|v\|_2^2}{\|v\|_{H^{\frac{s}{2}}}} \to 0,\end{aligned}$$

as $v \to 0$, in $H^{\frac{s}{2}}$. □

One important implication of the previous proposition is the following description of the continuity of $\mathcal{E}$ on $H^{\frac{s}{2}}$, that we shall utilize when excluding dichotomy.

Corollary 2.3. *For $u, v \in H^{\frac{s}{2}}$ we have*

$$|\mathcal{E}(u)-\mathcal{E}(v)|\lesssim (\|u\|_{H^{\frac{s}{2}}}+\|v\|_{H^{\frac{s}{2}}})\|u-v\|_{H^{\frac{s}{2}}}.$$

Proof. Using $|n(u)|\lesssim |u|$ and $m(\xi)\lesssim\langle\xi\rangle^s$, we have for arbitrary $u, v \in H^{\frac{s}{2}}$

$$\begin{aligned}|\langle\mathcal{E}'(u),v\rangle| &\leq |\langle Lu,v\rangle|+|\langle n(u),v\rangle|\\ &\lesssim \|u\|_{H^{\frac{s}{2}}}\|v\|_{H^{\frac{s}{2}}}+\|u\|_2\|v\|_2\lesssim \|u\|_{H^{\frac{s}{2}}}\|v\|_{H^{\frac{s}{2}}}.\end{aligned}$$

We then conclude

$$\begin{aligned}|\mathcal{E}(u)-\mathcal{E}(v)| &\leq \max_{0\leq t\leq 1}|\langle\mathcal{E}'(v+(u-v)t),u-v\rangle|\\ &\lesssim (\|u\|_{H^{\frac{s}{2}}}+\|v\|_{H^{\frac{s}{2}}})\|u-v\|_{H^{\frac{s}{2}}}.\end{aligned}$$

□

The uniform continuity of $\xi\mapsto m(\xi)/\langle\xi\rangle^s$ is a simple assumption to state, but not directly convenient to work with. Instead we shall use an implied regularity constraint on m, described by the next lemma.

Lemma 2.4. *There is a function $\omega\colon\mathbb{R}\to[0,\infty)$, bounded above by a polynomial, with $\lim_{t\to 0}\omega(t)=0$, such that*

$$|m(\xi)-m(\eta)|\leq \omega(\xi-\eta)\langle\xi\rangle^{\frac{s}{2}}\langle\eta\rangle^{\frac{s}{2}}. \tag{2.1}$$

Proof. Firstly, the bound $|\langle\xi\rangle^s-\langle\eta\rangle^s|\lesssim (\langle\xi\rangle^s+\langle\eta\rangle^s)|\xi-\eta|$, is easily obtained by the mean value theorem together with crude upper bounds. By assumption, there is a modulus of continuity $\tilde{\omega}$ so that

$$\Big|\frac{m(\xi)}{\langle\xi\rangle^s}-\frac{m(\eta)}{\langle\eta\rangle^s}\Big|\leq\tilde{\omega}(\xi-\eta), \tag{2.2}$$

and $\lim_{\lambda\to 0}\tilde{\omega}(\lambda)=0$. As $m(\cdot)/\langle\cdot\rangle^s$ is a bounded function, we can assume $\tilde{\omega}$ to also be bounded. We arrive at

$$\begin{aligned}|m(\xi)-m(\eta)| &\leq \Big|\frac{m(\xi)}{\langle\xi\rangle^s}-\frac{m(\eta)}{\langle\eta\rangle^s}\Big|\langle\xi\rangle^s+\frac{m(\eta)}{\langle\eta\rangle^s}|\langle\xi\rangle^s-\langle\eta\rangle^s| \\ &\lesssim \tilde{\omega}(\xi-\eta)\langle\xi\rangle^s+|\xi-\eta|(\langle\xi\rangle^s+\langle\eta\rangle^s) \\ &\lesssim (\tilde{\omega}(\xi-\eta)+|\xi-\eta|)\langle\xi-t\rangle^{\frac{s}{2}}\langle\xi\rangle^{\frac{s}{2}}\langle\eta\rangle^{\frac{s}{2}}, \\ &=: \omega(\xi-\eta)\langle\xi\rangle^{\frac{s}{2}}\langle\eta\rangle^{\frac{s}{2}},\end{aligned}$$

where we used the estimate $\langle x\rangle\lesssim\langle x-y\rangle\langle y\rangle$, when going from second to third line. □

By a more careful argument, it is possible to show that the two regularity constraints (2.1) and (2.2) are equivalent without any a priori knowledge of m, although we shall not prove this.

We conclude this section with the concentration-compactness theorem; the foundation of our proof of Theorem 1.1.

Theorem 2.5 (Lions [15], concentration-compactness)**.** *Any sequence $(\rho_k)\subset L^1$ of non-negative functions with the property*

$$\int_{\mathbb{R}}\rho_k dx=\mu>0,$$

admits a subsequence, denoted again by (ρ_k), for which one of the following phenomena occurs.
Vanishing: For each $r>0$, $k\to\infty$ implies that

$$\sup_{x_0\in\mathbb{R}}\int_{-r}^{r}\rho_k(x-x_0)dx\to 0.$$

Dichotomy: There exist a real number $\lambda\in(0,\mu)$ and three sequences $(x_k)\subset\mathbb{R}$ and $(r_k),(\tilde{r}_k)\subset\mathbb{R}^+$, so that when $k\to\infty$

$$\begin{aligned}&\int_{-r_k}^{r_k}\rho_k(x-x_k)dx\to\lambda, &\qquad r_k\to\infty,\\ &\int_{-\tilde{r}_k}^{\tilde{r}_k}\rho_k(x-x_k)dx\to\lambda, &\qquad \tilde{r}_k/r_k\to\infty,\end{aligned}$$

Concentration: There is a sequence $(x_k)\subset\mathbb{R}$, so that for each $\varepsilon>0$ there exists $r<\infty$ satisfying for all $k\in\mathbb{N}$

$$\int_{-r}^{r}\rho_k(x-x_k)\,dx\geq\mu-\varepsilon.$$

3. Upper and lower bounds for I_μ

In this section, we prove that the infimum I_μ of the minimization problem (1.3) satisfies $-\infty < I_\mu < -\kappa\mu^{1+\beta}$, for two positive constants κ and β. The upper bound will give us Prop. 4.1, which declares some fruitful bounds on near minimizers. The importance of also having a lower bound is the trivial consequence $I_\mu \neq -\infty$, allowing Prop. 6.1 to be meaningful. For clarity, we note that μ_*, as of now, is an arbitrary fixed positive upper bound for μ. The proof of the following proposition is inspired by [7].

Proposition 3.1. *There exists $\kappa > 0$, so that for $\mu \in (0, \mu_*)$, we have $-\infty < I_\mu < -\kappa\mu^{1+\beta}$, where the exponent $\beta = s'p/(2s'-p)$.*

Proof. Note that (i) and (ii) in Prop. 2.1, immediately gives us that $I_\mu > -C\mu$ for some $C < \infty$. For the upper bound, we pick a function φ, satisfying $\operatorname{supp}(\hat{\varphi}) \subset (-1,1)$, $\mathcal{Q}(\varphi) = 1$ and $c\varphi(x) \geq 0$. This last inequality implies that $\mathcal{N}_p(\varphi) = \frac{|c|}{2+p}\|\varphi\|_{2+p}^{2+p}$. An example of such a function would be an appropriately scaled version of $x \mapsto \operatorname{sinc}(x)^2$. We define the ansatz function $\varphi_{\mu,t}(x) = \sqrt{\frac{\mu}{t}}\varphi(x/t)$, for $t \geq 1$. By a substitution of variables we obtain

$$\|\varphi_{\mu,t}\|_k^k = \mu\left[\frac{\mu}{t}\right]^{\frac{k}{2}-1}\|\varphi\|_k^k\,. \tag{3.1}$$

When $k = 2$, we get $\mathcal{Q}(\varphi_{\mu,t}) = \mu$, and moreover

$$\mathcal{N}_p(\varphi_{\mu,t}) = \frac{|c|}{2+p}\|\varphi_{\mu,t}\|_{2+p}^{2+p} =: C_1\mu\left[\frac{\mu}{t}\right]^{\frac{p}{2}},$$

$$\mathcal{N}_r(\varphi_{\mu,t}) \lesssim \|\varphi_{\mu,t}\|_{2+r}^{2+r} = \mathcal{O}(\mu)\left[\frac{\mu}{t}\right]^{\frac{r}{2}}.$$

Exploiting the local growth of m, a simple computation gives the inequality $\mathcal{L}(\varphi_{\mu,t}) \leq C_2\mu/t^{s'}$, for some $C_2 < \infty$. We evaluate the ansatz to obtain

$$I_\mu \leq \mathcal{E}(\varphi_{\mu,t}) \leq -\left[C_1\left[\frac{\mu}{t}\right]^{\frac{p}{2}} - \frac{C_2}{t^{s'}}\right]\mu + \mathcal{O}(\mu)\left[\frac{\mu}{t}\right]^{\frac{r}{2}}.$$

We set $t^{-s'} = B\mu^\beta$ with $\beta = s'p/(2s'-p)$, where $B > 0$ is small enough to guarantee $t \geq 1$ for $\mu \in (0,\mu_*)$. The inequality above becomes

$$I_\mu \leq -\underbrace{\left[C_1 B^{\frac{p}{2s'}} - C_2 B\right]}_{2\kappa}\mu^{1+\beta} + B^{\frac{r}{2s'}}\mathcal{O}\left(\mu^{1+\beta+\frac{r-p}{2}}\right).$$

Without loss of generality, we can choose B small enough so that $\kappa > 0$ and $\kappa\mu^{1+\beta}$ is greater than the $\mathcal{O}$-term for all values of $\mu \in (0, \mu_*)$; this is possible as $p < \min\{2s', r\}$ and $\mu_* < \infty$ is fixed. We get the desired result:

$$I_\mu < -\kappa\mu^{1+\beta}. \tag{3.2}$$

□

Remark 3.2. From here on, we assume to have picked a constant $\kappa > 0$ as described in the last proposition. It is important to note that if we replace μ_* by a lower upper bound $\mu_*' < \mu_*$, then (3.2) would still hold for the same κ, as $(0, \mu_*') \subset (0, \mu_*)$. This allows us to later assume μ_* to be 'sufficiently' small, without having to worry about the effect on κ. Similarly, the implicit constants in Prop. 4.1 will also remain fixed when lowering μ_*.

4. Near minimizers

A consequence of the preceding proposition is that the feasible region $U_\mu = \{u \in H^{\frac{s}{2}} : Q(u) = \mu\}$ of the the minimization problem (1.3) contains elements u satisfying

$$\mathcal{E}(u) < -\kappa\mu^{1+\beta}, \qquad \text{with } \beta = \frac{s'p}{2s'-p},$$

where κ is some fixed positive constant independent of $\mu \in (0, \mu_*)$. We will refer such functions as *near minimizers*. Only these functions are of interest to us; any minimizing sequence $(u_k) \subset U_\mu$ must consist solely of near minimizers, except for a finite number of exceptions. Proposition 4.1 will give important bounds of such functions, that will serve as the main building blocks for excluding vanishing and dichotomy. We stress that throughout this paper, the implicit constants associated with our usage of $\lesssim, \gtrsim$ and $\simeq$ are independent of $\mu \in (0, \mu_*)$.

Proposition 4.1. *A near minimizer $u \in U_\mu$ satisfies*

$$\mathcal{L}(u) \simeq \mathcal{N}(u) \simeq \|u\|_{2+p}^{2+p} \simeq \mu^{1+\beta}, \tag{4.1}$$

$$\mathcal{N}_r(u) = o(\mu^{1+\beta}), \tag{4.2}$$

$$\|u\|_{H^{\frac{s}{2}}}^2 \simeq \mu. \tag{4.3}$$

Proof. Obtaining the bounds (4.1). As $\mathcal{L} > 0$, we immediately get from the definition of a near minimizer that

$$\max\{\mathcal{L}(u), \mu^{1+\beta}\} \lesssim \mathcal{N}(u) \lesssim \|u\|_{2+p}^{2+p}, \tag{4.4}$$

where the last inequality follows from Prop. 2.1. It remains to show $\|u\|_{2+p}^{2+p} \lesssim \min\{\mathcal{L}(u), \mu^{1+\beta}\}$. Let the indicator function on $[-1,1]$ be denoted χ and partition $u = u_1 + u_2$ with $\widehat{u_1} = \chi\hat{u}$ and $\widehat{u_2} = (1-\chi)\hat{u}$. By the Gagliardo–Nirenberg interpolation inequality,

$$\|u_1\|_{2+p}^{2+p} \lesssim \|u_1\|_{\dot{H}^{\frac{s'}{2}}}^{\frac{p}{s'}} \|u_1\|_2^{2+p-\frac{p}{s'}} \lesssim \mathcal{L}(u)^{\frac{p}{2s'}} \mu^{1+\frac{p}{2}-\frac{p}{2s'}}. \tag{4.5}$$

For u_2, we use Sobolev embedding to obtain

$$\|u_2\|_{2+p}^{2+p} \lesssim \|u_2\|_{H^{\frac{s}{2}}}^{2+p} \lesssim \mathcal{L}(u)^{1+\frac{p}{2}}. \tag{4.6}$$

As $\mathcal{L}(u) \lesssim \mathcal{N}(u)$, and $\mathcal{N}(u) \lesssim \mu$ by (ii) in Prop. 2.1, the expression (4.6) can be reduced further to

$$\|u_2\|_{2+p}^{2+p} \lesssim \mathcal{L}(u)^{\frac{p}{2s'}} \mu^{1+\frac{p}{2}-\frac{p}{2s'}}. \tag{4.7}$$

Exploiting the connection $1 + \frac{p}{2} - \frac{p}{2s'} = (1 - \frac{p}{2s'})(1+\beta)$, we combine inequality (4.5) and (4.7) to obtain

$$\|u\|_{2+p}^{2+p} \lesssim \|u_1\|_{2+p}^{2+p} + \|u_2\|_{2+p}^{2+p} \lesssim \mathcal{L}(u)^{\frac{p}{2s'}} \left[\mu^{1+\beta}\right]^{1-\frac{p}{2s'}}. \tag{4.8}$$

Combining (4.4) with (4.8), we conclude that $\|u\|_{2+p}^{2+p} \lesssim \min\{\mathcal{L}(u), \mu^{1+\beta}\}$.

Obtaining the bound (4.2). Now that (4.1) is established, we get $\|u_1\|_{2+p}^{2+p} \lesssim \mu^{1+\beta}$ by (4.5). Moreover, $\|u_1\|_\infty^2 \leq \|\widehat{u_1}\|_1^2 \leq 4\mu$, and so

$$\|u_1\|_{2+r}^{2+r} \leq \|u_1\|_{2+p}^{2+p} \|u_1\|_\infty^{r-p} \lesssim \mu^{1+\beta+(r-p)/2}.$$

Looking back at (4.6), we also obtain $\|u_2\|_{2+p}^{2+p} \lesssim \mu^{(1+\frac{p}{2})(1+\beta)}$. Finally, by (iv) in Prop. 2.1,

$$|\mathcal{N}_r(u)| \lesssim \|u_1\|_{2+r}^{2+r} + \|u_2\|_{2+p}^{2+p} = o(\mu^{1+\beta}).$$

Obtaining the bound (4.3). This is also a consequence of (4.1) together with $\|\cdot\|_{H^{\frac{s}{2}}}^2 \simeq \mathcal{Q}(\cdot) + \mathcal{L}(\cdot)$ and the fact that the upper bound μ_* is fixed. □

5. A congestion result for near minimizers

In this section, we show that a minimizing sequence (u_k) of (1.3) will never *vanish* in accordance with the Concentration-Compactness Theorem 2.5. We start by demonstrating some 'uniform' congestion of mass in L^{2+p}-norm of each element in (u_k). To formalize, we pick a smooth function φ, satisfying $\text{supp}(\varphi) \subset [-1,1]$ and $\sum_{j\in\mathbb{Z}} \varphi(x-j) = 1$. An example would be the convolution of the characteristic function on $[-\frac{1}{2}, \frac{1}{2}]$ with a mollifier supported in $[-\frac{1}{4}, \frac{1}{4}]$. For brevity, we set $\varphi_j(x) = \varphi(x-j)$.

Proposition 5.1. *For any near minimizer $u \in U_\mu$ we have*

$$\max_{j\in\mathbb{Z}} \|\varphi_j u\|_{2+p} \gtrsim \mu^{\frac{\beta}{p}}.$$

Proof. Consider the operator $T\colon f \mapsto (\varphi_j f)_j$, mapping functions to sequences of functions. It is a fact that $\|T\|_{H^\alpha \to \ell^2(H^\alpha)} < \infty$ for all $\alpha \geq 0$; this is a trivial calculation when $\alpha \in \mathbb{N}_0$ if one replaces $\|\cdot\|_{H^\alpha}$ with the equivalent norm $f \mapsto \|f\|_2 + \|f^{(\alpha)}\|_2$. For non-integer values of $\alpha > 0$, the result follows immediately from the (so called) 'complex interpolation method'; in particular, the two results [4, Theorem 5.1.2. on p. 107] and [4, Theorem 6.4.5.(7) on p. 152] combined with the boundness of T for $\alpha \in \mathbb{N}_0$, implies the general bound. Setting $\alpha = s/2$, we conclude

$$\sum_{j\in\mathbb{Z}} \|\varphi_j u\|^2_{H^{\frac{s}{2}}} \lesssim \|u\|^2_{H^{\frac{s}{2}}}. \tag{5.1}$$

By (4.3) and (4.1) we also obtain

$$\mu^\beta \|u\|^2_{H^{\frac{s}{2}}} \simeq \|u\|^{2+p}_{2+p} \simeq \sum_{j\in\mathbb{Z}} \|\varphi_j u\|^{2+p}_{2+p}, \tag{5.2}$$

where the last equivalence uses $\sum_{j\in\mathbb{Z}} |\varphi_j(x)|^{2+p} \simeq 1$. Combining (5.1) and (5.2), we get

$$\mu^\beta \sum_{j\in\mathbb{Z}} \|\varphi_j u\|^2_{H^{\frac{s}{2}}} \leq C \sum_{j\in\mathbb{Z}} \|\varphi_j u\|^{2+p}_{2+p},$$

for some $C < \infty$ independent of our choice of near minimizer u. At least one $j_0 \in \mathbb{Z}$ must then satisfy

$$\mu^\beta \|\varphi_{j_0} u\|^2_{H^{\frac{s}{2}}} \leq C \|\varphi_{j_0} u\|^{2+p}_{2+p}. \tag{5.3}$$

Combining (5.3) with the Sobolev embedding, $\|\varphi_{j_0} u\|^2_{2+p} \lesssim \|\varphi_{j_0} u\|^2_{H^{\frac{s}{2}}}$, we are done. ☐

To exclude vanishing we would need congestion of mass in L^2-norm; this is achievable from the previous result through the Gagliardo–Nirenberg inequality inequality. Indeed, setting $j_0 = \arg\max_{j\in\mathbb{Z}} \|\varphi_j u\|_{2+p}$ we obtain

$$\|\varphi_{j_0} u\|^{2+p}_{2+p} \lesssim \|\varphi_{j_0} u\|^{\frac{p}{s}}_{\dot{H}^{\frac{s}{2}}} \|\varphi_{j_0} u\|_2^{2+p-\frac{p}{s}}. \tag{5.4}$$

By the boundness of T in the previous proof, and (4.3), we have the estimate $\|\varphi_{j_0} u\|^2_{\dot{H}^{\frac{s}{2}}} \lesssim \mu$; together with the previous proposition, equation (5.4) now implies

$$\mu^{\frac{\beta}{p}(2+p)} \lesssim \mu^{\frac{p}{2s}} \|\varphi_{j_0} u\|_2^{2+p-\frac{p}{s}}.$$

As $2+p-p/s>0$, we conclude that $\mu^\delta \lesssim \|\varphi_{j_0}u\|_2$, for some appropriate exponent $\delta>0$, and so we get the following corollary.

Corollary 5.2. *No minimizing sequence of* (1.3) *has a subsequence for which vanishing occurs in accordance with Theorem 2.5.*

6. Strict subadditivity of the mapping $\mu \mapsto I_\mu$

Excluding *dichotomy* from a minimizing sequence is a more difficult task than that of vanishing, reflected by the laborious calculations in this subsection. The main idea however, is a simple one: Suppose dichotomy (as described in Theorem 2.5) occurs on a minimizing sequence $(u_k) \subset U_\mu$ of (1.3), then we shall see it can be 'split' in two $(u_k^1) \subset U_\lambda$, $(u_k^2) \subset U_{\mu-\lambda}$ so that $\lim_{k\to\infty} \mathcal{E}(u_k^1)+\mathcal{E}(u_k^2) = I_\mu$. This will contradict that the mapping $\mu \mapsto I_\mu$ is strictly subadditive for small μ, a fact we now prove.

Proposition 6.1. *For $\mu_*>0$ sufficiently small, the mapping $\mu \mapsto I_\mu$ is strictly subadditive on $(0,\mu_*)$, that is,*

$$I_{\mu_1+\mu_2} < I_{\mu_1} + I_{\mu_2},$$

for $\mu_1,\mu_2>0$ satisfying $\mu_1+\mu_2<\mu_*$.

Proof. We begin by finding a $\mu_*>0$ so that $\mu \mapsto I_\mu$ is strictly subhomogenous on $(0,\mu_*)$. Pick a near minimizer $u \in U_\mu$ and $t \in [1,2]$. Notice that $\mathcal{L}(\sqrt{t}u) = t\mathcal{L}(u)$ and $\mathcal{N}_p(\sqrt{t}u) = t^{1+\frac{p}{2}}\mathcal{N}_p(u)$. As $\mathcal{Q}(\sqrt{t}u) = t\mu$, we calculate

$$\begin{aligned} I_{t\mu} &\leq \mathcal{L}(\sqrt{t}u) - \mathcal{N}(\sqrt{t}u) \\ &= t\mathcal{L}(u) - t^{1+\frac{p}{2}}\mathcal{N}(u) + t^{1+\frac{p}{2}}\mathcal{N}_r(u) - \mathcal{N}_r(\sqrt{t}u) \\ &= t\mathcal{E}(u) - \underbrace{[t^{1+\frac{p}{2}} - t]\mathcal{N}(u)}_{\varphi(t,u)} + \underbrace{t^{1+\frac{p}{2}}\mathcal{N}_r(u) - \mathcal{N}_r(\sqrt{t}u)}_{\phi(t,u)} \end{aligned} \tag{6.1}$$

By (4.1) we get $\varphi(t,u) \gtrsim (t-1)\mu^{1+\beta}$, where we exploited that $t^{1+\frac{p}{2}} - t \gtrsim t-1$, when $t \in [1,2]$. As for ϕ, we see that $\phi(1,u)=0$ and so we use the mean value theorem for some $t_* \in [1,t]$ (and Leibniz integral rule) to get

$$\begin{aligned} \phi(t,u) &= (t-1)\frac{d\phi}{dt}(t_*,u) \\ &= (t-1)\int_{\mathbb{R}} (1+\tfrac{p}{2})t_*^{\frac{p}{2}} N_r(u) - \frac{u}{2\sqrt{t_*}} n_r(\sqrt{t_*}u)\,dx. \end{aligned}$$

It should be clear that $u \mapsto \int_{\mathbb{R}} u n_r(\sqrt{t}u)\,dx$ also satisfies an inequality of the form (iv) in Prop. 2.1, uniformly in $t \in [1,2]$. This in turn means it satisfies an inequality of the form (4.2) uniformly in $t \in [1,2]$. Thus the above calculation implies that $|\varphi(t,u)|= (t-1)o(\mu^{1+\beta})$. These two bounds on φ and ϕ implies we can pick $\mu_*>0$ small enough so that

$$-\varphi(t,u)+\phi(t,u)\le -\delta(t-1)\mu^{1+\beta},$$

is satisfied for some $\delta>0$, all $t\in[1,2]$ and all near minimizers $u\in U_\mu$ with $\mu\in(0,\mu_*)$. Assuming we have chosen such a $\mu_*>0$, then (6.1) becomes

$$I_{t\mu}\le t\mathcal{E}(u)-\delta(t-1)\mu^{1+\beta}.$$

Picking a minimizing sequence $(u_k)\subset U_\mu$ and assuming $1<t\le 2$, this last inequality implies

$$I_{t\mu}<tI_\mu, \tag{6.2}$$

on $(0,\mu_*)$. Finally, for a general $t>1$ and μ satisfying $t\mu\in(0,\mu_*)$, we can pick an integer $k>0$, so that $\sqrt[k]{t}\le 2$, which combined with (6.2) implies

$$I_{t\mu}<t^{\frac{1}{k}}I_{t^{1-\frac{1}{k}}\mu}<t^{\frac{2}{k}}I_{t^{1-\frac{2}{k}}\mu}<\cdots<tI_\mu,$$

that is, $\mu\mapsto I_\mu$ is strictly subhomogenous on $(0,\mu_*)$. To show that strict subhomogeneity implies strict subadditivity, we assume without loss of generality that $0<\mu_1\le\mu_2$ and $\mu_1+\mu_2<\mu_*$, and calculate

$$I_{\mu_1+\mu_2}<\Big(\frac{\mu_1}{\mu_2}+1\Big)I_{\mu_2}=\frac{\mu_1}{\mu_2}I_{\frac{\mu_2}{\mu_1}\mu_1}+I_{\mu_2}\le I_{\mu_1}+I_{\mu_2}\,.$$

□

Now that strict subadditivity of $\mu\mapsto I_\mu$ has been established, we shall create the contradiction as described at the beginning of this section. It will be essential that the non-local component of $\mathcal{E}$, namely $\mathcal{L}$, behaves almost like a local operator on sums of functions whose mass is 'sufficiently' separated. It is exactly the regularity of m that allows $\mathcal{L}$ to enjoy such a property. This result is encapsulated in the next lemma, which roughly states that the commutator operator $[L,\varphi(\cdot/r)]$ tends to zero as $r\to\infty$, for any Schwartz function φ. Here, the multiplication operator $f\mapsto\varphi f$ is defined for any distribution f in the canonical sense.

Lemma 6.2. *For a Schwartz function φ, let $B_r\colon H^{\frac{s}{2}}\to H^{\frac{-s}{2}}$ be the commutator of the operators L and $f\mapsto\varphi(\cdot/r)f$. Then*

$$\|B_r\|_{op}\to 0,\quad r\to\infty.$$

Proof. Set $\varphi_r = \varphi(\cdot/r)$. Using the bound (2.1), we have for any $u, v \in H^{\frac{s}{2}}$,

$$\begin{aligned}
|\langle [L, \varphi_r]u, v\rangle| &= \Big| \int_{\mathbb{R}} \int_{\mathbb{R}} \check{v}(\xi)\widehat{\varphi_r}(t)\hat{u}(\xi - t)(m(\xi) - m(\xi - t))dtd\xi \Big| \\
&\lesssim \int_{\mathbb{R}} |\widehat{\varphi_r}(t)|\omega(t) \int_{\mathbb{R}} \langle \xi\rangle^{\frac{s}{2}} |\check{v}(\xi)| \langle \xi - t\rangle^{\frac{s}{2}} |\hat{u}(\xi - t)| d\xi dt \\
&\lesssim \underbrace{\int_{\mathbb{R}} |\hat{\varphi}(t)|\omega(t/r)dt}_{\gtrsim \|B_r\|_{op}} \|u\|_{H^{\frac{s}{2}}} \|v\|_{H^{\frac{s}{2}}}.
\end{aligned}$$

As ω is bounded above by a polynomial and $\lim_{t\to 0} \omega(t) = 0$, the statement of the lemma follows. ☐

We are now ready to prove that a dichotomized minimizing sequence can be 'split' in two as described at the beginning of the section.

Proposition 6.3. *Suppose a minimizing sequence $(u_k) \subset U_\mu$ undergoes dichotomy, then there exist a real number $0 < \lambda < \mu$ and two sequences $(u_k^1) \subset U_\lambda$ and $(u_k^2) \subset U_{\mu-\lambda}$, so that*

$$\mathcal{E}(u_k^1) + \mathcal{E}(u_k^2) \to I_\mu, \quad k \to \infty.$$

Proof. By the Concentration-Compactness principle, we can pick $(r_k) \subset \mathbb{R}^+$ with $r_k \to \infty$, and $(x_k) \subset \mathbb{R}$ so that

$$\int_X |u_k(x - x_k)|^2 dx \to \begin{cases} \lambda, & X = \{x : |x| \leq r_k\}, \\ 0, & X = \{x : r_k \leq |x| \leq 2r_k\}, \\ \mu - \lambda, & X = \{x : 2r_k \leq |x|\}, \end{cases} \tag{6.3}$$

as $k \to \infty$; without loss of generality, we assume $x_k = 0$ for all k. Next, we pick two smooth symmetrical functions $\varphi, \psi: \mathbb{R} \to [0, 1]$, satisfying $\varphi(x) = 1$ when $|x| \leq 1$, $\varphi = 0$ when $|x| \geq 2$ and $\varphi^2 + \psi^2 = 1$. We denote φ_k and ψ_k for $\varphi(\cdot/r_k)$ and $\psi(\cdot/r_k)$, and set $v_k^1 = \varphi_k u_k$ and $v_k^2 = \psi_k u_k$. By (6.3), these function automatically satisfies

$$\mathcal{Q}(v_k^1) \to \lambda, \quad \mathcal{Q}(v_k^2) \to \mu - \lambda, \qquad k \to \infty.$$

It is easily verified that if ϕ is Schwartz and symmetric, then $\langle v, \phi u\rangle = \langle \phi v, u\rangle$ for any $v \in H^{\frac{-s}{2}}$ and $u \in H^{\frac{s}{2}}$, and so we may write

$$\begin{aligned}
\mathcal{L}(v_k^1) - \langle Lu_k, \varphi_k^2 u_k\rangle &= \langle [L, \varphi_k]u_k, \varphi_k u_k\rangle, \\
\mathcal{L}(v_k^2) - \langle Lu_k, \psi_k^2 u_k\rangle &= \langle [L, (1 - \psi_k)]u_k, (1 - \psi_k)u_k\rangle.
\end{aligned}$$

By Lemma 6.2, the RHS of these equations tend to zero, provided we can uniformly bound the $H^{\frac{s}{2}}$-norm of u_k, $\varphi_k u_k$ and $(1 - \psi_k)u_k$ in k. By (4.3), this again is guaranteed if multiplication by φ_k and $(1 - \varphi_k)$ are uniformly bounded (in k) as operators on $H^{\frac{s}{2}}$. This is indeed true and follows by similar reasoning as in the proof of Prop. 5.1; it is trivially proven when $s/2 \in \mathbb{N}_0$, and the result for general $s > 0$ follows from interpolation. Thus $\mathcal{L}(v_k^1) + \mathcal{L}(v_k^2) - \mathcal{L}(u_k) \to 0$, as $k \to \infty$. Turning to $\mathcal{N}$, we have

$$\mathcal{N}(v_k^1) + \mathcal{N}(v_k^2) - \mathcal{N}(u) = \int_{r_k<|x|<2r_k} N(v_k^1) + N(v_k^2) - N(u_k)dx.$$

By Prop. 2.1, we have $|N(x)| \lesssim x^2$, and so (6.3) guarantees the RHS of this equation to tend to zero as $k \to \infty$. As (u_k) is a minimizing sequence, we conclude that

$$\mathcal{E}(v_k^1) + \mathcal{E}(v_k^2) \to I_\mu,$$

for $k \to \infty$. By the same reasoning as before, the $H^{\frac{s}{2}}$-norm of v_k^1 and v_k^2 is uniformly bounded in k, and so by Corollary 2.3 the proposition is proved for the two sequences $u_k^1 = v_k^1\sqrt{\lambda/\mathcal{Q}(v_k^1)}$ and $u_k^2 = v_k^2\sqrt{(\mu-\lambda)/\mathcal{Q}(v_k^2)}$. □

With these two results at hand, we can exclude dichotomy; picking $\mu_* > 0$ so that $\mu \mapsto I_\mu$ is strictly subadditive and assuming $(u_k), (u_k^1)$ and (u_k^2) to be as in the previous proposition, we arrive at the contradiction

$$I_\mu = \lim_{k\to\infty} \mathcal{E}(u_k^1) + \mathcal{E}(u_k^2) \geq \liminf_{k\to\infty} \mathcal{E}(u_k^1) + \liminf_{k\to\infty} \mathcal{E}(u_k^2) \geq I_\lambda + I_{\mu-\lambda}.$$

Corollary 6.4. *Provided $\mu_* > 0$ is sufficiently small, no minimizing sequence of* (1.3) *has a subsequence for which dichotomy occurs in accordance with Theorem 2.5.*

7. Solutions from concentrated minimizing sequences

Theorem 2.5 provided us with the three possible phenomena that could occur for a minimizing sequence of (1.3); the previous two sections excluded vanishing and dichotomy, and so it remains to see that we can construct a minimizer from a *concentrating* minimizing sequence. This is straight forward:

Proposition 7.1. *Provided $\mu_* > 0$ is sufficiently small, any minimizing sequence $(u_k) \subset U_\mu$ of* (1.3) *admits a subsequence converging in L^2-norm to a minimizer $u \in U_\mu$.*

Proof. For μ_* sufficiently small, the two preceding sections guarantees that (u_k) admits a subsequence, again denoted (u_k), that concentrates in accordance with Theorem 2.5. Without loss of generality, we assume (u_k) to consist solely of near minimizers and shifted appropriately to concentrate about zero ($x_k = 0$ for all k). By the Kolmogorov-Riesz-Fréchet compactness theorem, (u_k) is relatively compact in L^2, as it is bounded, concentrated about zero and uniformly continuous with respect to translation:

$$\begin{aligned}\|u_k(\cdot+y)-u_k(\cdot)\|_2 &= \|(e^{-i(\cdot)y}-1)\hat{u}_k\|_2\\ &\leq \|(e^{-i(\cdot)y}-1)\langle\cdot\rangle^{\frac{-s}{2}}\|_\infty\|u_k\|_{H^{\frac{s}{2}}}\\ &\to 0,\end{aligned}$$

uniformly in k as $y \to 0$, as guaranteed by (4.3). We conclude that (u_k) admits a subsequence, yet again denoted (u_k), so that $u_k \to u$, for some $u \in L^2$ with $\mathcal{Q}(u) = \mu$. We now demonstrate that u is a minimizer of (1.3). As the positive functions $m(\cdot)|\hat{u}_k|^2$ converges locally in measure to $m(\cdot)|\hat{u}|^2$, Fatou's lemma implies

$$\mathcal{L}(u) \leq \liminf_{k\to\infty} \mathcal{L}(u_k).$$

Using the Fréchet derivative (Prop. 2.2) of $\mathcal{N}$, and that $|n(x)| \lesssim |x|$, we also obtain

$$\begin{aligned}|\mathcal{N}(u)-\mathcal{N}(u_k)| &= \Big|\int_0^1\int_{\mathbb{R}} n(tu+(1-t)u_k)(u-u_k)dxdt\Big|\\ &\lesssim \int_0^1 \|tu+(1-t)u_k\|_2\|u-u_k\|_2 dt\\ &\to 0,\end{aligned}$$

as $k\to\infty$. We now have $I_\mu \leq \mathcal{E}(u) \leq \liminf_{k\to\infty}\mathcal{E}(u_k) = I_\mu$. □

Not only is a minimizer of (1.3) a solutions of (1.2), we are also provided some additional control over the respective velocity ν, as described in the next proposition.

Proposition 7.2. *Any minimizer $u \in U_\mu$ of the minimization problem* (1.3)*, solves* (1.2) *in distribution sense, with velocity $\nu = \langle\mathcal{E}'(u), u\rangle/2\mu$. Provided $\mu_* > 0$ is small enough, we additionally have $-\nu \simeq \mu^\beta$.*

Proof. As the feasible set U_μ is a Hilbert submanifold of $H^{\frac{s}{2}}$, it follows that there must be a Lagrange multiplier $\nu \in \mathbb{R}$ (depending on the minimizer u), so that

$$\mathcal{E}'(u) - \nu\mathcal{Q}'(u) = 0, \tag{7.1}$$

in $H^{-\frac{s}{2}}$. In particular, if we pair (7.1) with u and insert for $\mathcal{Q}'$ we obtain

$$\nu = \frac{\langle\mathcal{E}'(u), u\rangle}{2\mu},$$

and so we attain the first part of the proposition. For the latter, note that

$$n(u)u = (2+p)N(u) + n_r(u)u - (2+p)N_r(u),$$

and as argued in the proof of Prop. 6.1, we have

$$\int_{\mathbb{R}} n_r(u)u - (2+p)N_r(u)dx = o(\mu^{1+\beta}).$$

Then

$$\begin{aligned}\langle \mathcal{E}'(u), u\rangle &= \langle Lu, u\rangle - \langle n(u), u\rangle \\ &= 2\mathcal{L}(u) - (2+p)\mathcal{N}(u) + o(\mu^{1+\beta}) \\ &= 2I_\mu - p\mathcal{N}(u) + o(\mu^{1+\beta}) \\ &< -C\mu^{1+\beta} + o(\mu^{1+\beta}),\end{aligned}$$

for some fixed $C > 0$, by Prop. 3 and (4.1). Thus, for a sufficiently small $\mu_* > 0$ we obtain $-\nu \gtrsim \mu^\beta$ when $\mu \in (0, \mu_*)$. The upper bound on $-\nu$ follows trivially from

$$-\nu \lesssim \frac{1}{\mu}\Big(\mathcal{L}(u) + \|u\|_{2+p}^{2+p}\Big) \lesssim \mu^\beta,$$

where we used $|n(x)x| \lesssim |x|^{2+p}$ and (4.1). □

8. Regularity of solutions

Before moving on, we summarize what has been proved so far. For the class of equations (1.2) that satisfies the assumptions (A) and (B) (see subsection 1.1) and the 'auxiliary' assumptions (C_1) and (C_2) (see subsection 1.2), we have proved all parts of Theorem 1.1, except the estimate $\|u\|_{H^{1+s}}^2 \lesssim \mu$. By Lemma 1.2, when this estimate is proven, the theorem automatically holds in the case when only (A) and (B) are satisfied. Hence, we now introduce the final piece, concluding the proof of Theorem 1.1.

Proposition 8.1. *Provided $\mu_* > 0$ is sufficiently small, minimizers $u \in U_\mu$ of (1.3) satisfies*

$$\|u\|_{H^{1+s}}^2 \lesssim \mu.$$

Proof. By Prop. 7.2, minimizers are solutions of (1.2), and so by a little rewriting, we have

$$\underbrace{(L - \nu + 1)u}_{\Lambda_\nu u} = \underbrace{n(u) + u}_{\eta(u)}. \tag{8.1}$$

Proposition 7.2 also guarantees that $-\nu+1>\delta$ for a positive constant δ independent of $\mu\in(0,\mu_*)$, provided $\mu_*>0$ is small enough. The inverse of Λ_ν then defines a bounded linear Fourier multiplier, $\Lambda_\nu^{-1}\colon H^\alpha\to H^{\alpha+s}$ for any $\alpha\in\mathbb{R}$, whose norm has the upper bound

$$\|\Lambda_\nu^{-1}\|_{H^\alpha\to H^{\alpha+s}}=\sup_{\xi\in\mathbb{R}}\frac{\langle\xi\rangle^s}{m(\xi)-\nu+1}\le\sup_{\xi\in\mathbb{R}}\frac{\langle\xi\rangle^s}{m(\xi)+\delta}=:C.$$

Clearly C is independent of $\mu\in(0,\mu_*)$. We also note that $T_\eta\colon u\mapsto\eta(u)$, is a bounded operator on H^α, whenever $0\le\alpha\le1$, as η is globally Lipschitz continuous with $\eta(0)=0$. Looking back at (8.1), a minimizer $u\in U_\mu$ satisfies

$$\|u\|_{H^{\alpha+s}}=\|\Lambda_\nu^{-1}\circ T_\eta(u)\|_{H^{\alpha+s}}\lesssim\|u\|_{H^\alpha},\tag{8.2}$$

whenever $0\le\alpha\le1$ (where the implicit constant in (8.2) can depend on α). We now obtain the desired conclusion by the following 'bootstrap' argument. Pick $k\in\mathbb{N}$ and $0\le r<s$ so that $1+s=ks+r$. By a (finite) repeated use of (8.2), we obtain

$$\|u\|_{H^{1+s}}=\|u\|_{H^{ks+r}}\lesssim\|u\|_{H^{(k-1)s+r}}\lesssim\cdots\lesssim\|u\|_{H^r}\le\|u\|_{H^s}\lesssim\|u\|_{L^2},$$

and so we are done. □

8.1. **Further regularity.** We conclude this paper with a regularity result on the solutions we have constructed. Clearly, if equation (8.2) was satisfied for large α, we could (as done in the previous proof) bootstrap to corresponding regularity. It is ultimately the regularity of n that determines how large α can be in (8.2). In [5], the authors prove that for any $\gamma>3/2$, the composition operator $T_f:u\mapsto f(u)$ maps H^γ to itself if, and only if, $f(0)=0$ and $f\in H^\gamma_{loc}$; in particular, if we restrict $\|u\|_\infty<R<\infty$, then we have

$$\|f(u)\|_{H^\alpha}\le C\|u\|_{H^\alpha},\tag{8.3}$$

for some constant C depending only on f,R and $\alpha\in(\frac{3}{2},\gamma]$. Moreover, using the result of [17], we can extend the inequality (8.3) to the case $\alpha\in[1,\gamma]$ (still with $\gamma>3/2$). It is now an easy task to improve the regularity of our solutions when $n\in H^{\alpha_*}_{loc}$ for some $\alpha_*>3/2$; note that functions in these spaces are necessarily locally Lipschitz continuous. We present the final proposition of this paper.

Proposition 8.2. *If $n\in H^{\alpha_*}_{loc}$ with $\alpha_*>3/2$, then the solutions u of* (1.2) *provided by Theorem 1.1, satisfies*

$$\|u\|_{H^{\alpha_*+s}}\lesssim\|u\|_2.$$

Proof. Looking back at (8.2), this equation is now valid for $0 \leq \alpha \leq \alpha_*$. This follows from the previous discussion as: 1) $\eta \in H^{\alpha_*}_{loc}$ with $\eta(0) = 0$, and 2) by Theorem 1.1 we have a uniform upper bound on the L^∞-norm of our solutions u (μ_* is fixed). The result is then attained by a similar bootstrap argument as the one used in the proof of Prop. 8.1. ☐

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Paper 3

<u>**Paper 3**</u>

ONE SIDED HÖLDER REGULARITY OF GLOBAL WEAK SOLUTIONS OF NEGATIVE ORDER DISPERSIVE EQUATIONS

Submitted for publication

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ABSTRACT. We prove global existence, uniqueness and stability of entropy solutions with $L^2 \cap L^\infty$ initial data for a general family of negative order dispersive equations. It is further demonstrated that this solution concept extends in a unique continuous manner to all L^2 initial data. These weak solutions are found to satisfy one sided Hölder conditions whose coefficients decay in time. The latter result controls the height of solutions and further provides a way to bound the maximal lifespan of classical solutions from their initial data.

1. INTRODUCTION

We consider the initial value problem

$$\begin{cases} u_t + \frac{1}{2}(u^2)_x = (G * u)_x, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

for initial data $u_0 \in L^2(\mathbb{R})$ and an even convolution kernel $G \in L^1(\mathbb{R})$ admitting an integrable weak derivative $G' =: K \in L^1(\mathbb{R})$. A classical family of examples for (1.1) is attained when we for G or $-G$ insert a Bessel kernel G_α with $\alpha > 1$, as defined by its Fourier transform

$$\widehat{G_\alpha}(\xi) = \frac{1}{(1 + 4\pi^2\xi^2)^{\frac{\alpha}{2}}}, \tag{1.2}$$

using the normalization $\mathcal{F}(f) = \hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}\mathrm{d}x$. In particular, setting $G = -G_2$ yields the Burgers–Poisson equation, which in [24] is derived as a model for shallow water waves. Central questions in the study of water wave model equations include well-posedness, persistence and non-persistence of solutions, the latter two in particular exemplified by solitary and breaking waves. The answers depend intricately on the type of nonlinearity and dispersive term featured in the equation.

In the case of a quadratic nonlinearity, the fractional Korteweg–de Vries equation (fKdV)

$$u_t + \tfrac{1}{2}(u^2)_x = (|D|^\beta u)_x \tag{1.3}$$

where $\mathcal{F}(|D|^\beta u) = |\xi|^\beta \hat{u}$ and $\beta \in \mathbb{R}$, has been suggested [19] as a scale for studying how the strength of the dispersion affects the questions of well-posedness and water-wave features. To connect (1.1) to the fKdV setting, observe that our assumption on G implies that $\widehat{G}(\xi) = o(|\xi|^{-1})$ as $|\xi| \to \infty$ and so in this sense one may place (1.1) in the region $\beta < -1$ for fKdV. However, $\widehat{G}$ will in our case be bounded, while $|\xi|^\beta \to \infty$ as $\xi \to 0$ for $\beta < 0$, and thus (1.1) can not match the low-frequency effect of negative order fKdV which assigns (very) high velocities to (very) low frequencies. This qualitative difference disappears in a periodic setting; the dispersion of fKdV on the torus is for $\beta < -1$ precisely of the form assumed in (1.1). It should be noted that the methods in this paper can, after a few modifications, be carried out on the torus, and thus our results can be extended to *periodic* solutions of fKdV for $\beta < -1$. With the relation between (1.1) and (1.3) accounted for, we now summarize a few results for the latter to sketch what one may expect of well-posedness and water-wave features in our case.

The fractional KdV equation of order $\beta \in (\frac{6}{7}, 2]$ is globally well-posed in appropriate function spaces; the regions $\beta \in (\frac{6}{7}, 1)$ and $\beta \in (1, 2)$ are treated in [23] and [12] respectively, and there are numerous works on the well posedness for $\beta = 1$ (Benjamin–Ono equation) and $\beta = 2$ (KdV equation), see for example [15] and [16] and the references therein. For values $\beta \leq \frac{6}{7}$ only local well-posedness results have been established [10, 23]. Still, numerical investigation [17] suggests that fKdV is globally well-posed for dispersion as weak as $\beta > \frac{1}{2}$, but not for $\beta \leq \frac{1}{2}$; this is also conjectured in [19]. One might expect the culprit of this loss of global well-posedness for weak dispersion, to be the appearance of breaking waves (shock formation), i.e. bounded solutions that develop infinite slope in finite time. In the negative order regime $\beta < 0$ this might be true: the occurrence of breaking waves have been proved for the case $\beta = -2$ (Ostrovsky–Hunter equation) by [20], for the case $\beta = -1$ (Burgers–Hilbert) by [25] and for the region $\beta \in (-1, -\frac{1}{3})$ by [14]. However, no such results exists in the positive order regime $\beta > 0$, and it is believed that instead other blowup phenomena occur in the range $\beta \in (0, \frac{1}{2}]$ inhibiting global well-posedness; see the discussion in [17, 19] or [22] where an example of L^∞ blowup in finite time is constructed for the modified Benjamin–Ono equation. In the absence of classical global solutions, several authors have for the $\beta < 0$ regime turned to the concept of *entropy* solutions. Adapted from the study of hyperbolic conservation laws, entropy solutions are weak solutions that satisfy extra conditions – the entropy inequalities – automatically satisfied by classical solutions when the latter exist. This solution concept allows for continuation past wave breaking and so global well-posedness may again be achieved. In [6] existence and uniqueness of global entropy solutions for the Ostrovsky–Hunter equation ($\beta = -2$) is established for appropriate initial data. Similarly, [4] provides global entropy solutions for the Burgers–Hilbert equation ($\beta = -1$) and a partial uniqueness result. Finally, the Burgers–Poisson equation mentioned above is in [11] shown to admit unique global entropy solutions for integrable initial data. The authors also provide sufficient conditions on the initial data leading to wave breaking. This equation is not an isolated instance of (1.1) featuring wave breaking; [7] shows that the phenomena is present whenever $G \in C \cap L^1(\mathbb{R})$ is symmetric and monotone on $\mathbb{R}^+$. More generally Corollary 2.7, which provides maximal lifespans for classical solutions, hints that every instance of (1.1) features wave breaking as is explained in more detail below.

We now give a brief discussion of our results presented in Section 2. Theorem 2.1 provides existence, uniqueness and L^2 stability of entropy solutions of (1.1) – as defined by Def. 1.1 – for initial data in $L^2 \cap L^\infty(\mathbb{R})$. The result is proved in Section 3. Here, existence follows from an operator splitting argument as done in [11], while uniqueness follows from a variation of the Kružkov's doubling of variables device [18] yielding a weighted L^1-contraction. The L^2 stability follows from a variation of the L^1-contraction combined with an L^2 tightness estimate of these solutions. The stability result is strong enough to allow the solution concept to be extended – in a unique continuous manner – to all L^2 initial data; this is Corollary 2.2.

Theorem 2.3 infers one sided Hölder regularity for weak solutions of (1.1) with L^2 initial data, and it is a generalization of the Oleĭnik estimate (4.1) for Burgers' equation. The result is proved in Section 4. Here, the idea is to introduce for a solution u an object $\omega(t,h) \geq \sup_x[u(t,x+h) - u(t,x)]$ bounding the one sided growth rate of u, and through an operator splitting argument the evolution of ω can be controlled. As Lemma 4.3 shows, the nonlinearity has a smoothing effect on ω. The dispersion on the other hand, is treated as perturbative source term (Lemma 4.4) that we are able to limit – and this is the key – through Lemma 4.2 using ω itself and the non-increasing L^2 norm of u. Letting then the iterative steps of the operator splitting go to zero, one attains an autonomous equation (4.14) for ω, which can be replaced by a coarser but simpler equation (4.16) resulting in Theorem 2.3. This result has two interesting consequences. Corollary 2.6 bounds the height of the entropy solutions by an expression dependent only on $K = G'$, the L^2 norm of the initial data and the time t. Said expression is decreasing in t, but does not tend to zero; this would generally be impossible due to the existence of solitary waves [9] for several instances of (1.1). Corollary 2.7 bounds the lifespan of classical solutions of (1.1) provided the initial data satisfies a skewness condition (2.8). The idea is to exploit the time-reversibility for classical solutions of (1.1): as Theorem 2.3 is valid also for reversed solutions this poses one sided Hölder conditions on the original solution's initial data. The implication is that a classical solution will break down before any contradiction is reached. One may ask 'how' these classical solutions break down, and wave breaking rise as the natural candidate, but proving this rigorously is beyond the scope of this paper. That said, one can expect a classical solution of (1.1) to break down at $t = T$ only if $\inf_x u(t,x) \to -\infty$ as $t \nearrow T$, which is the case for Burgers' equation with a C^1 source. We also point out that our skewness condition (2.8) differ from that of both [11] and [7]; neither imply the other.

1.1. **The entropy formulation.** We shall restrict the concept of entropy solutions to the function class $L^\infty_{\mathrm{loc}}([0,\infty), L^\infty(\mathbb{R}))$, which we here define as the subspace of $L^\infty_{\mathrm{loc}}([0,\infty)\times\mathbb{R})$ of functions $u = u(t,x)$ that are essentially bounded on $[0,T]\times\mathbb{R}$ for each $T > 0$. Necessary is the notion of an entropy pair (η, q) of (1.1), which is to say that

$$\eta\colon \mathbb{R}\to\mathbb{R} \text{ is smooth and convex, while } q'(u) = \eta'(u)u.$$

Definition 1.1. For bounded initial data $u_0 \in L^\infty(\mathbb{R})$, we say that a function $u \in L^\infty_{\mathrm{loc}}([0,\infty), L^\infty(\mathbb{R}))$ is an entropy solution of (1.1) if:

(1) it satisfies for all non-negative $\varphi \in C_c^\infty(\mathbb{R}^+\times\mathbb{R})$ and all entropy pairs (η, q) of (1.1) the entropy inequality

$$\int_0^\infty \int_{\mathbb{R}} \eta(u)\varphi_t + q(u)\varphi_x + \eta'(u)(K * u)\varphi \,\mathrm{d}x\mathrm{d}t \geq 0, \tag{1.4}$$

(2) it assumes the initial data in L^1_{loc} sense, that is

$$\operatorname*{ess\,lim}_{t\searrow 0} \int_{-r}^{r} |u(t,x) - u_0(x)|\mathrm{d}x = 0,$$

for all $r > 0$.

The concept of entropy solutions lies between that of strong and weak solutions. If $u \in L^\infty_{\mathrm{loc}}([0,\infty), L^\infty(\mathbb{R})) \cap C^1(\mathbb{R}^+\times\mathbb{R})$ is a classical solution of (1.1) then it is necessarily an entropy solution as multiplying (1.1) with $\eta'(u)\varphi$ and integrating by parts yields (1.4) as an equality. And if

u is an entropy solution of (1.1) then it is necessarily a weak solution as follows from considering the two entropy pairs $(\eta(u), q(u)) = (u, \frac{1}{2}u^2)$ and $(\eta(u), q(u)) = (-u, -\frac{1}{2}u^2)$ respectively.

1.2. **A fractional variation.** The exponents of the one sided Hölder conditions provided by Theorem 2.3 depend on the regularity of $K = G'$; the smoother K is, the higher the exponent. More precisely, we attain the Hölder exponent $\frac{1+s}{2}$ if $|K|_{TV^s} < \infty$ where the latter seminorm is for $s \in [0, 1]$ defined by

$$|K|_{TV^s} = \sup_{h>0} \frac{\|K(\cdot + h) - K\|_{L^1(\mathbb{R})}}{h^s}. \tag{1.5}$$

When $s = 1$ this seminorm coincides with the classical total variation of K, while $s = 0$ gives twice the L^1 norm of K, and thus we necessarily have $|K|_{TV^0} < \infty$ as we assume $K \in L^1(\mathbb{R})$. For $s \in (0, 1)$ the seminorm is a measure of intermediate regularity between $L^1(\mathbb{R})$ and $BV(\mathbb{R})$; in particular Lemma A.3 bounds this seminorm by the one associated with $W^{s,1}(\mathbb{R})$. The seminorm also satisfies the scaling property $|K(\lambda\cdot)|_{TV^s} = |\lambda|^{s-1}|K|_{TV^s}$ and so does not coincide with the scaling invariant fractional variation from [21] used in [3] to attain maximal smoothing effects for one-dimensional scalar conservation laws.

2. Main results

We here present the two main results, Theorem 2.1 and Theorem 2.3 and corresponding corollaries. For a general discussion of the content given here, see the end of the above introduction. We start with Theorem 2.1, which provides a global well-posedness theory for entropy solutions of (1.1) with initial data in $L^2 \cap L^\infty(\mathbb{R})$. The theorem is established in Section 3.

Theorem 2.1. *For every initial data $u_0 \in L^2 \cap L^\infty(\mathbb{R})$ there exists a unique entropy solution u of (1.1). The mapping $t \mapsto u(t)$ is continuous from $[0, \infty)$ to $L^2(\mathbb{R})$ and $u(t)$ satisfies for all $t \geq 0$ the bounds*

$$\|u(t)\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}, \qquad \|u(t)\|_{L^\infty(\mathbb{R})} \leq e^{t\kappa}\|u_0\|_{L^\infty(\mathbb{R})}, \tag{2.1}$$

where $\kappa := \|K\|_{L^1(\mathbb{R})}$. Moreover, we have the following stability result: if two sequences $(t_k)_{k\in\mathbb{N}} \subset [0, \infty)$ and $(u_{0,k})_{k\in\mathbb{N}} \subset L^2 \cap L^\infty(\mathbb{R})$ admit the limits

$$\lim_{k\to\infty} t_k = t, \quad \textit{and} \quad \lim_{k\to\infty} u_{0,k} = u_0 \quad \textit{in } L^2(\mathbb{R}),$$

where $u_0 \in L^2 \cap L^\infty(\mathbb{R})$, then the corresponding entropy solutions satisfy

$$\lim_{k\to\infty} u_k(t_k) = u(t) \quad \textit{in } L^2(\mathbb{R}).$$

It is worth mentioning that this theorem is also valid on a time-bounded domain $(0,T)\times\mathbb{R}$; see the discussion following the proof of Proposition 3.1. The tools used to prove the stability result of the theorem do not depend on the height of the initial data, thus allowing for the following corollary which is proved at the end of Subsection 3.3.

Corollary 2.2 (Global L^2 well-posedness)**.** *Equation* (1.1) *is globally well-posed for $L^2(\mathbb{R})$ initial data in the following sense: The solution map $S\colon (t,u_0)\mapsto u(t)$ mapping $L^2\cap L^\infty(\mathbb{R})$ initial data to the corresponding entropy solution at time $t\geq 0$, extends uniquely to a jointly continuous mapping $S\colon [0,\infty)\times L^2(\mathbb{R})\to L^2(\mathbb{R})$. In particular, the L^2-bound, -continuity and -stability of Theorem 2.1 carries over to all weak solutions provided by S. Moreover, for any $u_0\in L^2(\mathbb{R})$, the corresponding weak solution $u(t,x):=S(t,u_0)(x)$ is locally bounded in $(0,\infty)\times\mathbb{R}$ and satisfies the entropy inequalities* (1.4)*.*

The second theorem infers one sided Hölder regularity for the weak solutions provided by Corollary 2.2. The result depends on the regularity of $K=G'$ which is measured using the fractional variation $|K|_{TV^s}$ defined in (1.5). The theorem is proved in Section 4.

Theorem 2.3. *For initial data $u_0\in L^2(\mathbb{R})$, let u be the corresponding weak solution of* (1.1) *provided by Corollary 2.2, and let $s\in[0,1]$ be such that $|K|_{TV^s}<\infty$. Then for all $t>0$, $x\mapsto u(t,x)$ coincides a.e. with a left-continuous function satisfying for all $x\geq y$ the one sided Hölder condition*

$$u(t,x)-u(t,y)\leq a(t)(x-y)^{\frac{1+s}{2}}, \tag{2.2}$$

for a Hölder coefficient $a(t)$ decreasing in t.

As we assume $K\in L^1(\mathbb{R})$, the case $s=0$ of Theorem 2.3 is valid for all instances of (1.1). Also, when either G or $-G$ coincides with a Bessel kernel G_α, as introduced in (1.2), and $\alpha\in(1,2]$ the induced one sided Hölder regularity from Theorem 2.3 takes the form

$$u(t,x)-u(t,y)\leq a(t)(x-y)^{\frac{\alpha}{2}}, \qquad x\geq y,$$

as follows from Lemma A.4 when setting $s=\alpha-1$. In particular, for the Burgers–Poisson equation (where $G=-G_2$) L^2 data results in weak solutions that are one sided Lipschitz continuous with a Lipschitz constant that can be read off from the second part of Corollary 2.5 when using $|(G_2)'|_{TV}=2$. We conclude this section with a few corollaries of Theorem 2.3 including a decaying height bound for weak solutions of (1.1) and a maximal lifespan estimate for classical solutions.

Remark 2.4. As Corollary 4.10 states, the Hölder coefficient in (2.2) can be set to

$$a(t) = C_1(s)|K|_{TV^s}^{\frac{2+s}{3+2s}} \|u_0\|_{L^2(\mathbb{R})}^{\frac{1+s}{3+2s}} + C_2(s)\frac{\|u_0\|_{L^2(\mathbb{R})}^{\frac{1-s}{3}}}{t^{\frac{2+s}{3}}}, \tag{2.3}$$

where the two coefficients $C_1(s)$ and $C_2(s)$ are given by

$$C_1(s) = \frac{2^{\frac{3+s}{6+4s}}[(2+s)(3+s)]^{\frac{1+s}{6+4s}}}{1+s}, \quad C_2(s) = \frac{2^{\frac{4+2s}{3+3s}}(2+s)^{\frac{5+s}{6}}(3+s)^{\frac{1-s}{6}}}{2^{\frac{1-s}{6}}3^{\frac{2+s}{3}}(1+s)}. \tag{2.4}$$

Alternatively, one may use the sharper expression for $a(t)$ provided by Lemma 4.9 where the shorthand notation $\mu = \|u_0\|_{L^2(\mathbb{R})}$ and $\kappa_s = |K|_{TV^s}$ is used.

For clarity, we now use the explicit expressions from this remark to write out the content of Theorem 2.3 for the special case $s = 0$ where we may use the identity $|K|_{TV^0} = 2\|K\|_{L^1(\mathbb{R})}$, and the case $s = 1$ where we may use $|K|_{TV^1} = |K|_{TV}$.

Corollary 2.5 (Explicit regularity when $s = 0$ and $s = 1$)**.**

- *The $s = 0$ case: For initial data $u_0 \in L^2(\mathbb{R})$, let u be the corresponding weak solution of* (1.1) *provided by Corollary 2.2. Then for all $t > 0$, $x \mapsto u(t, x)$ coincides a.e. with a left-continuous function satisfying for all $x \geq y$ the one sided Hölder condition*

$$\frac{u(t,x) - u(t,y)}{(x-y)^{\frac{1}{2}}} \leq 2^{\frac{4}{3}} 3^{\frac{1}{6}} \|K\|_{L^1(\mathbb{R})}^{\frac{2}{3}} \|u_0\|_{L^2(\mathbb{R})}^{\frac{1}{3}} + \frac{4\|u_0\|_{L^2(\mathbb{R})}^{\frac{1}{3}}}{3^{\frac{1}{2}} t^{\frac{2}{3}}}.$$

- *The $s = 1$ case: If $|K|_{TV} < \infty$, then the above u further satisfies for all $t > 0$ and $x \geq y$ the one sided Lipschitz condition*

$$\frac{u(t,x) - u(t,y)}{x-y} \leq \frac{3^{\frac{1}{5}}}{2^{\frac{1}{5}}} |K|_{TV}^{\frac{3}{5}} \|u_0\|_{L^2(\mathbb{R})}^{\frac{2}{5}} + \frac{1}{t}.$$

Note that the second part of the corollary generalizes the classical Oleĭnik estimate $u(t,x) - u(t,y) \leq \frac{x-y}{t}$ for Burgers' equation (where $K = 0$). Next, we introduce an L^∞ bound for the weak solutions provided by Corollary 2.2 which, in contrast to the one from (2.1), is decreasing in time.

Corollary 2.6 (Height bound)**.** *For initial data $u_0 \in L^2(\mathbb{R})$, let u be the corresponding weak solution of* (1.1) *provided by Corollary 2.2. Then for* *all $t > 0$ we have the height bound*

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq \left[2^{\frac{11}{12}} 3^{\frac{1}{3}} \|K\|_{L^1(\mathbb{R})}^{\frac{1}{3}} + \frac{2^{\frac{5}{4}}}{t^{\frac{1}{3}}}\right] \|u_0\|_{L^2(\mathbb{R})}^{\frac{2}{3}}. \tag{2.5}$$

More generally, for any $s \in [0, 1]$ such that $|K|_{TV^s} < \infty$, we have for the above u and all $t > 0$ the height bound

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq \tilde{C}_1(s) |K|_{TV^s}^{\frac{1}{3+2s}} \|u_0\|_{L^2(\mathbb{R})}^{\frac{2+2s}{3+2s}} + \tilde{C}_2(s) \frac{\|u_0\|_{L^2(\mathbb{R})}^{\frac{2}{3}}}{t^{\frac{1}{3}}}, \tag{2.6}$$

where the coefficients $\tilde{C}_1(s)$ and $\tilde{C}_2(s)$ are expressions similar to $C_1(s)$ and $C_2(s)$ from Remark 2.4, and they are both written out in (A.2) *in the appendix.*

Proof. See Appendix B.

Observe that together, the two height bounds (2.1) and (2.5) imply that when $u_0 \in L^2 \cap L^\infty(\mathbb{R})$ the corresponding weak solution of (1.1) is globally bounded. The next and final result of this section establishes a maximal lifespan for classical solutions of (1.1). For brevity we introduce the following seminorm

$$[u_0]_s := \operatorname*{ess\,sup}_{\substack{x \in \mathbb{R} \\ h > 0}} \left[\frac{u_0(x-h) - u_0(x)}{h^{\frac{1+s}{2}}}\right], \tag{2.7}$$

which is a (left) one sided Hölder seminorm of exponent $\frac{1+s}{2}$.

Corollary 2.7 (Maximal lifespan)**.** *There are universal constants $C, c > 0$ such that: if initial data $u_0 \in L^2 \cap L^\infty(\mathbb{R})$ satisfies the skewness condition*

$$[u_0]_s^{3+2s} > c |K|_{TV^s}^{2+s} \|u_0\|_{L^2(\mathbb{R})}^{1+s}, \tag{2.8}$$

for some $s \in [0, 1]$ such that $|K|_{TV^s} < \infty$, then the lifespan T of a classical solution $u \in L^\infty \cap C^1((0, T) \times \mathbb{R})$ of (1.1) *admitting u_0 as initial data must satisfy*

$$T < C \left[\frac{\|u_0\|_{L^2(\mathbb{R})}^{1-s}}{[u_0]_s^3}\right]^{\frac{1}{2+s}}. \tag{2.9}$$

Proof. See Appendix B.

3. Well posedness of entropy solutions

In this section, we provide for (1.1) a global well-posedness theory of entropy solutions as defined by Def. 1.1. In particular, the content of Theorem 2.1 follows from Proposition 3.1, Corollary 3.6 and Proposition 3.9; see the summary at the beginning of Subsection 3.3. Corollary 2.2 is also proved here at the end of Subsection 3.3. For entropy solutions of (1.1), the proofs of existence and uniqueness is the same for $L^2 \cap L^\infty$ data as for L^∞ data; only the L^1 setting allows for 'shortcuts'. Thus for generality, many results in the two coming subsections will be presented for initial data $u_0 \in L^\infty(\mathbb{R})$. We also note that in these two subsections only Lemma 3.3 exploits the dispersive nature of (1.1), that is, that $K = G'$ is odd.

3.1. **Uniqueness of entropy solutions.**

It is natural to start with the proof of uniqueness, as this equips us with a weighted L^1-contraction that can further be used in the existence proof. The involved weight $w_M^r(t,x)$ can be interpreted as a bound on the propagation of information for solutions of (1.1). Its technical role in the coming proof is to serve as a subsolution of a dual equation, namely the one obtained from setting the square bracket in (3.17) to zero. A similar method can be found in [1] where nonlocal conservation laws are treated. The weight is constructed as follows. Writing $|K|$ to denote the function $x \mapsto |K(x)|$, we introduce for a parameter $t \geq 0$ the operator $e^{t|K|*}$ mapping $L^p(\mathbb{R})$ to itself for any $p \in [1,\infty]$, defined by

$$\left(e^{t|K|*}f\right)(x) = f(x) + \sum_{n=1}^{\infty}\left((|K|*)^n f\right)(x)\frac{t^n}{n!}, \tag{3.1}$$

where $(|K|*)^n$ represents the operation of convolving with $|K|$ repeatedly n times. Observe that by repeated use of Young's convolution inequality we have for any $p \in [1,\infty]$ and $f \in L^p(\mathbb{R})$

$$\|e^{t|K|*}f\|_{L^p(\mathbb{R})} \leq e^{t\kappa}\|f\|_{L^p(\mathbb{R})}, \tag{3.2}$$

where $\kappa := \|K\|_{L^1(\mathbb{R})}$. For parameters $r, M \geq 0$, we further introduce

$$\chi_M^r(t,x) = \begin{cases} 1, & |x| < r + Mt, \\ 0, & \text{else}, \end{cases} \tag{3.3}$$

and set

$$w_M^r(t,x) = \left(e^{t|K|*}\chi_M^r(t,\cdot)\right)(x). \tag{3.4}$$

By (3.2), this weight satisfies for $p \in [1,\infty]$ the bound

$$\|w^r_M(t,\cdot)\|_{L^p(\mathbb{R})} \leq e^{t\kappa}(2r+2Mt)^{\frac{1}{p}}, \tag{3.5}$$

where the case $p=\infty$ is evaluated in a limit sense. Thus, $w^r_M(t,\cdot) \in L^1 \cap L^\infty(\mathbb{R})$ for all $t, r, M \geq 0$. With w^r_M defined, we are ready to state Proposition 3.1 establishing the uniqueness of entropy solutions. It should be noted that although the following result is stated to hold for a.e. $t \geq 0$, it can be extended to all $t \geq 0$, as we shall later prove that entropy solutions of (1.1) are continuous when viewed as $L^1_{\mathrm{loc}}(\mathbb{R})$-valued time-dependent functions.

Proposition 3.1. *Let $u, v \in L^\infty_{\mathrm{loc}}([0,\infty), L^\infty(\mathbb{R}))$ be entropy solutions of (1.1) with $u_0, v_0 \in L^\infty(\mathbb{R})$ as initial data. Then, for any $r>0$ and a.e. $t \geq 0$ we have the weighted L^1-contraction*

$$\int_{-r}^{r} |u(t,x)-v(t,x)|dx \leq \int_{-\infty}^{\infty} |u_0(x)-v_0(x)|w^r_M(t,x)dx, \tag{3.6}$$

where w^r_M is given by (3.4), and M is any parameter satisfying

$$M \geq \frac{\|u\|_{L^\infty([0,t]\times\mathbb{R})} + \|v\|_{L^\infty([0,t]\times\mathbb{R})}}{2}. \tag{3.7}$$

Thus, there is at most one entropy solution of (1.1) for each $u_0 \in L^\infty(\mathbb{R})$.

Proof. We begin by reformulating (1.4) in terms of the Kružkov entropies; parameterized over $k \in \mathbb{R}$, they are given by $(\eta_k(u), q_k(u)) = (|u-k|, F(u,k))$ where

$$F(u,k) := \tfrac{1}{2}\mathrm{sgn}(u-k)(u^2-k^2).$$

These entropy pairs lack the required smoothness, but are still applicable in (1.4) as they can be smoothly approximated. Indeed, consider for $\delta > 0$ and $k \in \mathbb{R}$ the entropy pairs $\eta^\delta_k(u) = \sqrt{(u-k)^2+\delta^2}$ and $q^\delta_k(u) = \int_k^u (\eta^\delta_k)'(y)y\mathrm{d}y$. As we have the pointwise limits

$$\lim_{\delta\to 0} \eta^\delta_k(u) = |u-k|, \quad \lim_{\delta\to 0} q^\delta_k(u) = F(u,k), \quad \lim_{\delta\to 0} (\eta^\delta_k)'(u) = \mathrm{sgn}(u-k),$$

we can substitute $(\eta, q) \mapsto (\eta^\delta_k, q^\delta_k)$ in (1.4) and let $\delta \to 0$ to conclude through dominated convergence that u satisfies

$$0 \leq \int_0^\infty \int_{\mathbb{R}} |u-k|\varphi_t + F(u,k)\varphi_x + \mathrm{sgn}(u-k)(K*u)\varphi \mathrm{d}x\mathrm{d}t, \tag{3.8}$$

for all $k \in \mathbb{R}$ and all non-negative $\varphi \in C^\infty_c(\mathbb{R}^+ \times \mathbb{R})$. For brevity, we set $U = \mathbb{R}^+ \times \mathbb{R}$ for use throughout the proof. Let $\psi \in C^\infty_c(U \times U)$ be non-negative, and consider u and v as functions in (t,x) and (s,y)

respectively. For fixed $(s,y) \in U$, we can in (3.8) insert the test-function $\varphi\colon (t,x) \mapsto \psi(t,x,s,y)$ and the constant $k = v(s,y)$ so to obtain

$$0 \leq \int_U |u-v|\psi_t + F(u,v)\psi_x + \operatorname{sgn}(u-v)(K *_x u)\psi \mathrm{d}x\mathrm{d}t, \tag{3.9}$$

where we write $K *_x u$ to stress that the operator $K*$ is applied with respect to the x-variable. As (3.9) holds for all $(s,y) \in U$ we can integrate (3.9) over $(s,y) \in U$ to further attain

$$0 \leq \int_U \int_U |u-v|\psi_t + F(u,v)\psi_x + \operatorname{sgn}(u-v)(K *_x u)\psi \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s. \tag{3.10}$$

Next, we swap the role of $u(t,x)$ and $v(s,y)$: rewriting (3.8) in terms of the variables (s,y) and replacing u by v, we can fix $(t,x) \in U$ and insert the test-function $\varphi\colon (s,y) \mapsto \psi(t,x,s,y)$ and the constant $k = u(t,x)$ so to obtain

$$0 \leq \int_U \int_U |u-v|\psi_s + F(v,u)\psi_y + \operatorname{sgn}(v-u)(K *_y v)\psi \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s, \tag{3.11}$$

where we also integrated over $(t,x) \in U$. As $F(u,v) = F(v,u)$ and $\operatorname{sgn}(v-u) = -\operatorname{sgn}(u-v)$ we can add (3.10) to (3.11) so to further obtain

$$\begin{aligned} 0 \leq &\int_U \int_U |u-v|(\psi_t + \psi_s) + F(u,v)(\psi_x + \psi_y) \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\ &+ \int_U \int_U \operatorname{sgn}(u-v)(K *_x u - K *_y v)\psi \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{3.12}$$

Let $\rho \in C_c^\infty(\mathbb{R}^2)$ be non-negative and satisfy $\|\rho\|_{L^1(\mathbb{R}^2)} = 1$, and let ρ_ε denote the expression

$$\rho_\varepsilon = \rho_\varepsilon(t-s, x-y) = \frac{1}{\varepsilon^2}\rho\left(\frac{t-s}{\varepsilon}, \frac{x-y}{\varepsilon}\right),$$

for $\varepsilon > 0$. For a fixed $T \in (0,\infty)$, we further let φ denote a non-negative element of $C_c^\infty((0,T) \times \mathbb{R})$ and set

$$\psi(t,x,s,y) = \varphi(t,x)\rho_\varepsilon(t-s, x-y),$$

or simply $\psi = \varphi\rho_\varepsilon$ for short. Note that, for $\varepsilon > 0$ sufficiently small, this ψ is non-negative, smooth and of compact support in $U \times U$; in particular, it satisfies the prior assumptions posed on it. From the observation that $(\partial_t + \partial_s)\rho_\varepsilon = 0 = (\partial_x + \partial_y)\rho_\varepsilon$, we conclude that

$$(\psi_t + \psi_s) = \varphi_t\rho_\varepsilon, \qquad (\psi_x + \psi_y) = \varphi_x\rho_\varepsilon,$$

and so inserting for ψ in (3.12) we attain

$$\begin{aligned} 0 \leq & \int_U \int_U \Big[|u-v|\varphi_t + F(u,v)\varphi_x\Big]\rho_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\ & + \int_U \int_U \operatorname{sgn}(u-v)(K *_x u - K *_y v)\varphi\rho_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{3.13}$$

We now wish to 'go to the diagonal' by taking $\limsup_{\varepsilon\to 0}$ of (3.13); for simplicity we study each line separately. For the first one we pick $M \in (0,\infty)$ satisfying the inequality (3.7) with T replacing t, and use $(u^2 - v^2) = (u+v)(u-v)$ to calculate

$$\begin{aligned} & \int_U \int_U \Big[|u-v|\varphi_t + F(u,v)\varphi_x\Big]\rho_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\ \leq & \int_U \int_U |u-v|\Big[\varphi_t + M|\varphi_x|\Big]\rho_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\ \leq & \int_U |u(t,x)-v(t,x)|\Big[\varphi_t + M|\varphi_x|\Big]\mathrm{d}x\mathrm{d}t \\ & + \int_U \int_U |v(t,x)-v(s,y)|\Big[\varphi_t + M|\varphi_x|\Big]\rho_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s, \end{aligned} \tag{3.14}$$

where we in the last step added and subtracted $v(t,x)$ followed by the triangle inequality. As $\rho_\varepsilon(t-s, x-y)$ is supported in the region $|(t-s, x-y)| \leq \varepsilon$ and satisfies $\|\rho_\varepsilon\|_{L^1(\mathbb{R}^2)} = 1$, the very last integral in (3.14) is bounded by

$$\sup_{|(\epsilon,\delta)|\leq\varepsilon} \int_U |v(t,x) - v(t+\epsilon, x+\delta)|\Big[\varphi_t + M|\varphi_x|\Big]\mathrm{d}x\mathrm{d}t \to 0, \quad \varepsilon \to 0,$$

where the limit holds as translation is a continuous operation on $L^1_{\mathrm{loc}}(\mathbb{R})$ and $\varphi \in C_c^\infty((0,T)\times\mathbb{R})$. Thus we have established

$$\begin{aligned} & \limsup_{\varepsilon\to 0} \int_U \int_U \Big[|u-v|\varphi_t + F(u,v)\varphi_x\Big]\rho_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\ \leq & \int_U |u-v|\Big[\varphi_t + M|\varphi_x|\Big]\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{3.15}$$

where the v on the right-hand side of (3.15) is a function in (t,x). Turning our attention to the second line of (3.13), we start by observing

$$\begin{aligned}
&\int_U\int_U \operatorname{sgn}(u-v)(K *_x u - K *_y v)\varphi\rho_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\
&\le \int_U\int_U\int_{\mathbb{R}} |K(z)||u(t,x-z)-v(s,y-z)| \\
&\qquad\qquad\qquad \times \varphi(t,x)\rho_\varepsilon(t-s,x-y)\mathrm{d}z\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\
&= \int_U\int_U\int_{\mathbb{R}} |K(z)||u(t,x)-v(s,y)| \\
&\qquad\qquad\qquad \times \varphi(t,x+z)\rho_\varepsilon(t-s,x-y)\mathrm{d}z\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\
&= \int_U\int_U |u-v|\Big[|K| *_x \varphi\Big]\rho_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s,
\end{aligned}$$

where the third line holds by the substitution $(x,y)\mapsto(x+z,y+z)$ and the last by the symmetry of $z\mapsto|K(z)|$. By similar reasoning used to attain (3.14), we conclude

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_U\int_U \operatorname{sgn}(u-v)(K *_x u - K *_y v)\rho_\varepsilon\varphi \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\
&\le \int_U |u-v|(|K| * \varphi)\mathrm{d}x\mathrm{d}t,
\end{aligned}\tag{3.16}$$

where the v on the right-hand side of (3.16) is a function in (t,x). Combining (3.13) with (3.15) and (3.16), yields the inequality

$$0 \le \int_U |u-v|\Big[\varphi_t + M|\varphi_x| + |K| * \varphi\Big]\mathrm{d}x\mathrm{d}t, \tag{3.17}$$

where again, both u and v are now functions in (t,x). By density, we may extend (3.17) to hold for all non-negative $\varphi \in W_0^{1,1}((0,T)\times\mathbb{R})$. Thus, we can set $\varphi(t,x) = \theta(t)\phi(t,x)$ for two non-negative functions $\theta\in W_0^{1,1}((0,T))$ and $\phi\in W^{1,1}((0,T)\times\mathbb{R})$ where we note that ϕ need not vanish at $t=0$ and $t=T$. In doing so, (3.17) yields

$$0 \le \int_U |u-v|\theta'\phi \mathrm{d}x\mathrm{d}t + \int_U |u-v|\theta\Big[\phi_t + M|\phi_x| + |K| * \phi\Big]\mathrm{d}x\mathrm{d}t, \tag{3.18}$$

To rid ourselves of the second integral, we now construct a particular ϕ such that the square bracket in (3.18) is non-positive in $(0,T)\times\mathbb{R}$. Let $f\colon \mathbb{R}\to[0,1]$ be smooth, non-increasing and satisfy $f(x)=1$ for $x\le 0$ and $f(x)=0$ for sufficiently large x, and define

$$g(t,x) = f(|x| + M(t-T)). \tag{3.19}$$

By the properties of f, it is readily checked that $g \in C_c^\infty([0,T]\times\mathbb{R})$. We now define the function ϕ to be

$$\phi(t,x) = \left(e^{(T-t)|K|*}g(t,\cdot)\right)(x), \tag{3.20}$$

where we used the operator defined in (3.1). Observe that ϕ is non-negative and smooth on $[0,T]\times\mathbb{R}$ with integrable derivatives; this last part follows when using (3.2). That the square bracket in (3.18) is non-positive, can be seen as follows: note first from (3.19) that

$$\begin{aligned} g_t(t,x) =& Mf'(|x| + M(t-T)),\\ g_x(t,x) =& \operatorname{sgn}(x)f'(|x| + M(t-T)). \end{aligned}$$

As f' is non-positive, we find $g_t = -M|g_x|$. Thus, using (3.20) we calculate for $t \in (0,T)$

$$\begin{aligned} \phi_t + |K| * \phi =& e^{(T-t)|K|*}g_t,\\ =& -M\left(e^{(T-t)|K|*}|g_x|\right),\\ \leq& -M\left|e^{(T-t)|K|*}g_x\right|\\ =& -M|\phi_x|, \end{aligned}$$

where the last equality holds as differentiation commutes with convolution. In conclusion, the second integral in (3.18) is non-positive. Next, for a small parameter $\epsilon > 0$ we set $\theta = \theta_\epsilon$ where θ_ϵ is given by

$$\theta_\epsilon(t) = \begin{cases} t/\epsilon, & t \in (0,\epsilon),\\ 1, & t \in (\epsilon, T-\epsilon),\\ (T-t)/\epsilon, & t \in (T-\epsilon, T). \end{cases} \tag{3.21}$$

Inserting this in (3.18), removing the non-positive integral and letting $\epsilon \to 0$, we conclude

$$\begin{aligned} &\liminf_{\epsilon\to 0} \int_{T-\epsilon}^{T} \left(\int_{\mathbb{R}} |u(t,x) - v(t,x)|\phi(t,x)\mathrm{d}x\right)\frac{\mathrm{d}t}{\epsilon}\\ \leq& \limsup_{\epsilon\to 0} \int_0^{\epsilon} \left(\int_{\mathbb{R}} |u(t,x) - v(t,x)|\phi(t,x)\mathrm{d}x\right)\frac{\mathrm{d}t}{\epsilon} \end{aligned} \tag{3.22}$$

where we moved the negative term over to the left-hand side. As u and v are bounded on $(0,T)\times\mathbb{R}$ and continuous at $t=0$ in L^1_{loc} sense, it is easy to see that $|u(t,\cdot) - v(t,\cdot)|\phi(t,\cdot) \to |u_0(\cdot) - v_0(\cdot)|\phi(0,\cdot)$ in $L^1(\mathbb{R})$ when $t \to 0$ since the same is true for $\phi(t,x)$ and $\phi(0,x)$. Thus the right-hand side of (3.22) is given by

$$\limsup_{\epsilon\to 0} \int_0^{\epsilon} \left(\int_{\mathbb{R}} |u(t,x) - v(t,x)|\phi(t,x)\mathrm{d}x\right)\frac{\mathrm{d}t}{\epsilon} = \int_{\mathbb{R}} |u_0 - v_0|\phi(0,x)\mathrm{d}x.$$

As for the left-hand side, we wish to apply the Lebesgue differentiation theorem so to get convergence for a.e. $T > 0$, but this can not be directly done due to the implicit T-dependence of ϕ. Instead, we observe from (3.19) and (3.20) that $\phi(T,x) = g(T,x) = f(|x|)$ where the latter function is independent of T. Since $\varphi(t,\cdot) \to f(|\cdot|)$ in $L^1(\mathbb{R})$ as $t \to T$, the boundness of u and v means that $|u(t,\cdot) - v(t,\cdot)|(\varphi(t,\cdot) - f(|\cdot|)) \to 0$ in $L^1(\mathbb{R})$ as $t \to T$ and so we may estimate

$$\begin{aligned}
&\limsup_{\epsilon\to 0} \int_{T-\epsilon}^{T} \left(\int_{\mathbb{R}} |u(t,x) - v(t,x)|\phi(t,x)\mathrm{d}x \right) \frac{\mathrm{d}t}{\epsilon} \\
&= \limsup_{\epsilon\to 0} \int_{T-\epsilon}^{T} \left(\int_{\mathbb{R}} |u(t,x) - v(t,x)|f(|x|)\mathrm{d}x \right) \frac{\mathrm{d}t}{\epsilon} \\
&= \int_{\mathbb{R}} |u(T,x) - v(T,x)|f(|x|)\mathrm{d}x, \quad \text{a.e. } T \geq 0,
\end{aligned}$$

where the last equality used the Lebesgue differentiation theorem. Thus we conclude from (3.22) that we for a.e. $T \geq 0$ have

$$\begin{aligned}
&\int_{\mathbb{R}} |u(T,x) - v(T,x)|f(|x|)\mathrm{d}x \\
&\leq \int_{\mathbb{R}} |u_0(x) - v_0(x)| \Big(e^{T|K|*} f(|\cdot| - MT) \Big)(x)\mathrm{d}x,
\end{aligned} \tag{3.23}$$

where we inserted for $\phi(0,x)$ using (3.19) and (3.20). As f was any smooth, non-negative, non-increasing function satisfying $f(x) = 1$ for $x \leq 0$ and $f(x) = 0$ for sufficiently large x, we may in (3.23) set $f = \mathbb{1}_{(-\infty,r)}$ through a standard approximation argument. Doing this, we observe that $f(|x| - MT) = \chi_M^r(T,x)$ where the latter is defined in (3.3), and so we obtain from (3.23) exactly (3.6), with T substituting for t. This concludes the proof. $\square$

While we in this paper are concerned with global entropy solutions, one may wish to study entropy solutions on a time-bounded domain $(0,T) \times \mathbb{R}$. Such solutions would be defined as in Def. 1.1, but with the test-functions in (1.4) restricted to $C_c^\infty((0,T) \times \mathbb{R})$. Still, no new solutions are attained this way: the uniqueness of entropy solutions on a time-bounded domain follows from the same argument as above, and thus an entropy solution on $(0,T) \times \mathbb{R}$ is the restriction of a global one which the following section establishes the existence of.

3.2. **Existence of entropy solutions.** In this subsection, we prove the existence of an entropy solution of (1.1) for arbitrary initial data $u_0 \in L^\infty(\mathbb{R})$. The strategy goes as follows: we first introduce for a parameter $\varepsilon > 0$ an approximate solution map $S_{\varepsilon,t}\colon L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$

whose key properties are collected in Proposition 3.2. Next, we show in Lemma 3.4 that when $S_{\varepsilon,t}$ is applied to sufficiently regular initial data u_0, we attain approximate entropy solutions. Further, in Proposition 3.5 we establish the convergence (as $\varepsilon \to 0$) of these approximations to an entropy solution, and the result is extended to general L^∞ data in Corollary 3.6. Throughout the section, we occasionally refer to the space $C([0,\infty), L^1_{\mathrm{loc}}(\mathbb{R}))$ of functions $u \in L^1_{\mathrm{loc}}([0,\infty) \times \mathbb{R})$ such that $t \mapsto u(t,\cdot)$ is a continuous mapping from $[0,\infty)$ to $L^1_{\mathrm{loc}}(\mathbb{R})$. By an operator splitting argument, we aim to build entropy solutions of (1.1) from those of Burgers' equation, $u_t + \frac{1}{2}(u^2)_x = 0$, and the linear convolution equation, $u_t = K * u$. On that note, we introduce two families of operators $(S^B_t)_{t\geq 0}$ and $(S^K_t)_{t\geq 0}$ parameterized over $t \geq 0$. The operator $S^B_t \colon L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is the solution map for Burgers' equation restricted to L^∞ data at time t; that is,

$$S^B_t \colon f \mapsto u^f(t,\cdot), \tag{3.24}$$

where $(t,x) \mapsto u^f(t,x)$ is the unique bounded entropy solution of the problem

$$\begin{cases} u_t + \frac{1}{2}(u^2)_x = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = f(x), & x \in \mathbb{R}, \end{cases}$$

which necessarily lies in $C([0,\infty), L^1_{\mathrm{loc}}(\mathbb{R}))$ (see [8]). Note that S^B_t is a flow map in the sense that $S^B_{t_1} \circ S^B_{t_2} = S^B_{t_1+t_2}$ for all $t_1, t_2 \geq 0$. The second map $S^K_t \colon L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is for $t \geq 0$ defined by

$$S^K_t \colon f \mapsto f + tK * f. \tag{3.25}$$

The actual solution map for the equation $u_t = K * u$ is the operator e^{tK*} defined as (3.1) with K replacing $|K|$; the reason we have instead chosen S^K_t as (3.25) (which can be seen as a first order approximation of e^{tK*}) is for our calculations to be slightly tidier. Note however, S^K_t is not a flow mapping. With these two families of operators, we build a third family of operators $S_{\varepsilon,t}$: for fixed parameters $\varepsilon > 0$ and $t \geq 0$, pick $n \in \mathbb{N}_0$ and $s \in [0,\varepsilon)$ such that $t = s + n\varepsilon$, and define

$$S_{\varepsilon,t} = S^B_s \circ \left[S^K_\varepsilon \circ S^B_\varepsilon \right]^{\circ n}, \tag{3.26}$$

where the notation $\circ n$ implies that the square bracket is composed with itself $(n-1)$ times; if $n = 0$, then the square bracket should be replaced by the identity. We shall demonstrate that as $\varepsilon \to 0$ the map $S_{\varepsilon,t}$ converges in an appropriate sense to the solution map for entropy solutions of (1.1). We begin by collecting a few properties of $S_{\varepsilon,t}$ when applied to the space $BV(\mathbb{R})$; this subspace of $L^1(\mathbb{R})$ is equipped with the norm

$\|\cdot\|_{BV(\mathbb{R})} = \|\cdot\|_{L^1(\mathbb{R})} + |\cdot|_{TV}$, where the total variation seminorm $|\cdot|_{TV}$ coincides with $|\cdot|_{TV^1}$ as defined in (1.5). A short and effective discussion of $BV(\mathbb{R})$ can be found in either [8] or [13]; we note that functions in $BV(\mathbb{R})$ have essential right and left limits at each point, and their height is bounded by their total variation, thus $BV(\mathbb{R}) \hookrightarrow L^1 \cap L^\infty(\mathbb{R})$.

Proposition 3.2. *With $S_{\varepsilon,t}$ as defined in (3.26), we have for all $\varepsilon > 0$, $t \geq \tilde{t} \geq 0$, $f \in BV(\mathbb{R})$ and $p \in [1,\infty]$*

$$\begin{aligned} \|S_{\varepsilon,t}(f)\|_{L^p(\mathbb{R})} &\leq e^{t\kappa}\|f\|_{L^p(\mathbb{R})}, && (L^p \textit{ bound}),\\ \|S_{\varepsilon,t}(f)\|_{TV} &\leq e^{t\kappa}\|f\|_{TV}, && (TV \textit{ bound}),\\ \|S_{\varepsilon,t}(f) - S_{\varepsilon,\tilde{t}}(f)\|_{L^1(\mathbb{R})} &\leq (t - \tilde{t} + \varepsilon)C_f(t), && (\textit{Approx. time continuity}), \end{aligned}$$

where $\kappa := \|K\|_{L^1(\mathbb{R})}$ and where the factor $C_f(t)$ only depends on f and t.

Proof. Consider $\varepsilon > 0$ fixed. We will be using the following properties of the mappings S_t^B and S_t^K

$$\|S_t^B(f)\|_{L^p(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}, \qquad \|S_t^K(f)\|_{L^p(\mathbb{R})} \leq e^{t\kappa}\|f\|_{L^p(\mathbb{R})}, \tag{3.27}$$

$$|S_t^B(f)|_{TV} \leq |f|_{TV}, \qquad |S_t^K(f)|_{TV} \leq e^{t\kappa}|f|_{TV}, \tag{3.28}$$

$$\|S_t^B(f) - f\|_{L^1(\mathbb{R})} \leq t|f|_{TV}^2, \qquad \|S_t^K(f) - f\|_{L^1(\mathbb{R})} \leq t\kappa\|f\|_{L^1(\mathbb{R})}, \tag{3.29}$$

valid for all $t \geq 0$, $p \in [1,\infty]$ and $f \in BV(\mathbb{R})$. The inequalities involving S_t^B are well known and can be found for example in [13]. As for the inequalities involving S_t^K, these estimates follow directly from the definition of S_t^K (3.25) together with Young's convolution inequality and $1 + t\kappa \leq e^{t\kappa}$. We start by proving the L^p and TV bound of the proposition. For this we fix $t \geq 0$ and pick $n \in \mathbb{N}_0$ and $s \in [0,\varepsilon)$ such that $t = s + n\varepsilon$, and pick an arbitrary $f \in BV(\mathbb{R})$. By iteration of the two inequalities in (3.27) we attain

$$\|S_{\varepsilon,t}(f)\|_{L^p(\mathbb{R})} = \|S_s^B \circ [S_\varepsilon^K \circ S_\varepsilon^B]^{\circ n}(f)\|_{L^p(\mathbb{R})} \leq e^{n\varepsilon\kappa}\|f\|_{L^p(\mathbb{R})}, \tag{3.30}$$

for all $p \in [1,\infty]$, and by iteration of the inequalities in (3.28) we similarly get

$$|S_{\varepsilon,t}(f)|_{TV} = |S_s^B \circ [S_\varepsilon^K \circ S_\varepsilon^B]^{\circ n}(f)|_{TV} \leq e^{n\varepsilon\kappa}|f|_{TV}. \tag{3.31}$$

This gives the first two bounds of the proposition. For the time continuity, we pick $\tilde{t} \in [0,t]$ and $\tilde{n} \in \mathbb{N}$ and $\tilde{s} \in [0,\varepsilon)$ such that $\tilde{t} = \tilde{s} + \tilde{n}\varepsilon$. Suppose first that $t - \tilde{t} \leq \varepsilon$, and set $\tilde{f} = S_{\varepsilon,\tilde{n}\varepsilon}(f)$. Then either $S_{\varepsilon,t}(f) = S_{s-\tilde{s}}^B(\tilde{f})$ or $S_{\varepsilon,t}(f) = S_s^B \circ S_\varepsilon^K \circ S_{\varepsilon-\tilde{s}}^B(\tilde{f})$ corresponding to the two situations $n = \tilde{n}$ and $n = \tilde{n} + 1$; we will only deal with the latter as the other case is dealt with similarly. By the triangle inequality we then have

$$\begin{aligned}\|S_{\varepsilon,t}(f)-S_{\varepsilon,\tilde{t}}(f)\|_{L^1(\mathbb{R})}\leq{}&\|S_s^B\circ S_\varepsilon^K\circ S_{\varepsilon-\tilde{s}}^B(\tilde{f})-S_\varepsilon^K\circ S_{\varepsilon-\tilde{s}}^B(\tilde{f})\|_{L^1(\mathbb{R})}\\&+\|S_\varepsilon^K\circ S_{\varepsilon-\tilde{s}}^B(\tilde{f})-S_{\varepsilon-\tilde{s}}^B(\tilde{f})\|_{L^1(\mathbb{R})}\\&+\|S_{\varepsilon-\tilde{s}}^B(\tilde{f})-\tilde{f}\|_{L^1(\mathbb{R})}.\end{aligned}$$

The three terms on the right-hand side can be directly dealt with using the two inequalities (3.29) followed by the estimates (3.30) and (3.31). Doing so in a straight forward manner results in the bound

$$\begin{aligned}&se^{2n\varepsilon\kappa}|f|_{TV}^2+\varepsilon\kappa e^{\tilde{n}\varepsilon\kappa}\|f\|_{L^1(\mathbb{R})}+(\varepsilon-\tilde{s})e^{2\tilde{n}\varepsilon\kappa}|f|_{TV}^2\\&\leq\varepsilon e^{2t\kappa}(2|f|_{TV}^2+\kappa\|f\|_{L^1(\mathbb{R})}).\end{aligned}$$

Thus, setting for example $C_f(t)=e^{2t\kappa}(2|f|_{TV}^2+\kappa\|f\|_{L^1(\mathbb{R})})$ the time continuity estimate holds whenever $t-\tilde{t}\leq\varepsilon$. By breaking any large time step into steps of size no larger than ε, the general case follows by the triangle inequality. □

The L^p bound provided by the previous proposition was attained by applying Young's convolution inequality on the operator $K*$; in doing so, we miss possible cancellations that might take place as K, after all, is an odd function. While efficient L^p bounds might not be feasible for general $p\geq 1$, these cancellations are easily exploited for the L^2 norm as seen from the following lemma. This L^2 control is crucial for the analysis of Section 4.

Lemma 3.3. *With $S_{\varepsilon,t}$ as defined in* (3.26), *we have for all $\varepsilon>0$, $t\geq 0$ and $f\in L^2\cap L^\infty(\mathbb{R})$*

$$\|S_{\varepsilon,t}(f)\|_{L^2(\mathbb{R})}\leq e^{\frac{1}{2}\varepsilon t\kappa^2}\|f\|_{L^2(\mathbb{R})},$$

where $\kappa:=\|K\|_{L^1(\mathbb{R})}$.

Proof. Consider $\varepsilon>0$ and $t\geq 0$ fixed. As K is odd, real valued and in $L^1(\mathbb{R})$, it is readily checked that $K*$ is a skew-symmetric operator on $L^2(\mathbb{R})$, that is

$$\langle f,K*g\rangle=-\langle K*f,g\rangle,$$

for all $f,g\in L^2(\mathbb{R})$, and consequently $\langle f,K*f\rangle=0$ for all $f\in L^2(\mathbb{R})$. In particular,

$$\begin{aligned}\|S_\varepsilon^K(f)\|_{L^2(\mathbb{R})}^2&=\langle f+\varepsilon K*f,f+\varepsilon K*f\rangle\\&=\langle f,f\rangle+\varepsilon^2\langle K*f,K*f\rangle\\&\leq(1+\varepsilon^2\kappa^2)\|f\|_{L^2(\mathbb{R})}^2.\end{aligned}$$

Combined with $1+\varepsilon^2\kappa^2 \leq e^{\varepsilon^2\kappa^2}$ and the fact that S_ε^B is non-expansive on $L^2(\mathbb{R})$ (left-most inequality in (3.27)), the result follows by iteration. □

When $u_0 \in BV(\mathbb{R})$, we can use $S_{\varepsilon,t}$ to construct a family of approximate entropy solutions of (1.1) as follows. For an arbitrary, but fixed, $u_0 \in BV(\mathbb{R})$, let the family $(u^\varepsilon)_{\varepsilon>0} \subset L^\infty_{\text{loc}}([0,\infty), L^\infty(\mathbb{R}))$ be defined by

$$u^\varepsilon(t) = S_{\varepsilon,t}(u_0), \tag{3.32}$$

where $u^\varepsilon(t)$ is compact notation for $x \mapsto u^\varepsilon(t,x)$ and $S_{\varepsilon,t}$ is as defined in (3.26). Although $(u^\varepsilon)_{\varepsilon>0}$ is considered a family in $L^\infty_{\text{loc}}([0,\infty), L^\infty(\mathbb{R}))$, we stress that each member is for all $t \geq 0$ well defined in $L^\infty(\mathbb{R})$. For small $\varepsilon > 0$ these functions are not far off from satisfying the entropy inequality (1.4), as we now show.

Lemma 3.4. *With $(u^\varepsilon)_{\varepsilon>0}$ as defined in* (3.32) *for some $u_0 \in BV(\mathbb{R})$, we have for every entropy pair (η, q) of* (1.1) *and non-negative $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ the approximate entropy inequality*

$$\int_0^\infty \int_{\mathbb{R}} \eta(u^\varepsilon)\varphi_t + q(u^\varepsilon)\varphi_x + \eta'(u^\varepsilon)(K * u^\varepsilon)\varphi \mathrm{d}x\mathrm{d}t \geq O(\varepsilon).$$

Proof. Fixing $\varepsilon > 0$, we observe from the definition of $S_{\varepsilon,t}$ (3.26) that u^ε is an entropy solution of Burgers' equation on the open sets $(t_{n-1}^\varepsilon, t_n^\varepsilon) \times \mathbb{R}$ for $n \in \mathbb{N}$, where $t_n^\varepsilon = n\varepsilon$; thus

$$\int_{t_{n-1}^\varepsilon}^{t_n^\varepsilon} \int_{\mathbb{R}} \eta(u^\varepsilon)\varphi_t + q(u^\varepsilon)\varphi_x \mathrm{d}x\mathrm{d}t \geq 0, \tag{3.33}$$

for every non-negative $\varphi \in C_c^\infty((t_{n-1}^\varepsilon, t_n^\varepsilon) \times \mathbb{R})$ and every entropy pair (η, q) of Burgers' equation, which coincides with the entropy pairs of (1.1) as the convection term of the two equations agree. Moreover, by the time continuity of S_t^B (3.28) and the TV bound from Proposition 3.2, we see that $u^\varepsilon \in C([t_{n-1}^\varepsilon, t_n^\varepsilon), L^1_{\text{loc}}(\mathbb{R}))$; at $t = t_n^\varepsilon$ it is discontinuous from the left, as the left limit is given by $u^\varepsilon(t_n^\varepsilon-) = S_\varepsilon^B(u^\varepsilon(t_{n-1}^\varepsilon))$, while we have defined

$$u^\varepsilon(t_n^\varepsilon) = u^\varepsilon(t_n^\varepsilon-) + \varepsilon K * u^\varepsilon(t_n^\varepsilon-). \tag{3.34}$$

The continuity in time allows us, by a similar trick used to attain (3.22), to extend (3.33) to

$$\begin{aligned}\int_{t_{n-1}^\varepsilon}^{t_n^\varepsilon} \int_{\mathbb{R}} \eta(u^\varepsilon)\varphi_t + q(u^\varepsilon)\varphi_x \mathrm{d}x\mathrm{d}t \geq \int_{\mathbb{R}} &\eta(u^\varepsilon(t_n^\varepsilon-))\varphi(t_n^\varepsilon, x)\mathrm{d}x \\ &- \int_{\mathbb{R}} \eta(u^\varepsilon(t_{n-1}^\varepsilon))\varphi(t_{n-1}^\varepsilon, x)\mathrm{d}x,\end{aligned} \tag{3.35}$$

for all non-negative $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$. For the remainder of the proof, consider the entropy pair (η, q) and $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ fixed. Summing (3.35) over $n \in \mathbb{N}$ and using $\varphi(0,x) = 0$, we get

$$\begin{aligned} &\int_{\mathbb{R}^+\times\mathbb{R}} \eta(u^\varepsilon)\varphi_t + q(u^\varepsilon)\varphi_x \mathrm{d}x\mathrm{d}t \\ &\geq \sum_{n=1}^\infty \int_{\mathbb{R}} \Big[\eta(u^\varepsilon(t_n^\varepsilon-)) - \eta(u^\varepsilon(t_n^\varepsilon))\Big]\varphi(t_n^\varepsilon, x)\mathrm{d}x. \end{aligned} \tag{3.36}$$

By Proposition 3.2, the family $(u^\varepsilon)_{\varepsilon>0}$ is uniformly bounded on the support of φ, and so we can assume without loss of generality that $|\eta'|, |\eta''| < C_1$ for some large C_1. Using the relation (3.34), the square bracket from (3.36) can thus be estimated

$$\begin{aligned} &\eta(u^\varepsilon(t_n^\varepsilon-)) - \eta(u^\varepsilon(t_n^\varepsilon)) \\ &\geq -\varepsilon\eta'(u^\varepsilon(t_n^\varepsilon-))\Big[K * u^\varepsilon(t_n^\varepsilon-)\Big] - \frac{C_1\varepsilon^2}{2}|K * u^\varepsilon(t_n^\varepsilon-)|^2, \end{aligned}$$

which, again by the uniform bound of u^ε on the compact support of φ, further implies

$$\begin{aligned} &\int_{\mathbb{R}} \Big[\eta(u^\varepsilon(t_n^\varepsilon-)) - \eta(u^\varepsilon(t_n^\varepsilon))\Big]\varphi(t_n^\varepsilon, x)\mathrm{d}x \\ &\geq -\varepsilon\int_{\mathbb{R}} \eta'(u^\varepsilon(t_n^\varepsilon-))\Big[K * u^\varepsilon(t_n^\varepsilon-)\Big]\varphi(t_n, x)\mathrm{d}x - C_2\varepsilon^2, \end{aligned} \tag{3.37}$$

for some $C_2 > 0$ independent of n and ε. Combining the uniform time regularity of Proposition 3.2 and the compact support of φ, we see that the function

$$g_\varepsilon(t) := \int_{\mathbb{R}} \eta'(u^\varepsilon(t))\Big[K * u^\varepsilon(t)\Big]\varphi(t,x)\mathrm{d}x, \tag{3.38}$$

satisfies for all $t \geq \tilde{t} \geq 0$ an inequality $|g_\varepsilon(t) - g_\varepsilon(\tilde{t})| \leq C_3(t - \tilde{t} + \varepsilon)$ for some sufficiently large C_3 independent of ε. Thus, the integral on the right-hand side of (3.37) can be bounded from below as such

$$\begin{aligned} &-\varepsilon\int_{\mathbb{R}} \eta'(u^\varepsilon(t_n^\varepsilon-))\Big[K * u^\varepsilon(t_n^\varepsilon-)\Big]\varphi(t_n, x)\mathrm{d}x \\ &= -\int_{t_{n-1}^\varepsilon}^{t_n^\varepsilon}\int_{\mathbb{R}} \eta'(u^\varepsilon(t_n^\varepsilon-))\Big[K * u^\varepsilon(t_n^\varepsilon-)\Big]\varphi(t_n, x)\mathrm{d}x\mathrm{d}t \\ &\geq -\int_{t_{n-1}^\varepsilon}^{t_n^\varepsilon}\int_{\mathbb{R}} \eta'(u^\varepsilon(t))\Big[K * u^\varepsilon(t)\Big]\varphi(t, x)\mathrm{d}x\mathrm{d}t - 2C_3\varepsilon^2. \end{aligned} \tag{3.39}$$

Picking the smallest $N(\varepsilon) \in \mathbb{N}$ such that $\operatorname{supp}\varphi \cap (\varepsilon N(\varepsilon), \infty) \times \mathbb{R} = \emptyset$, we combine (3.36), (3.37) and (3.39) to deduce

$$\int_{\mathbb{R}^+ \times \mathbb{R}} \eta(u^\varepsilon)\varphi_t + q(u^\varepsilon)\varphi_x + \eta'(u^\varepsilon)(K * u^\varepsilon)\varphi \mathrm{d}x\mathrm{d}t \geq CN(\varepsilon)\varepsilon^2,$$

for some sufficiently large $C > 0$. And as $N(\varepsilon)\varepsilon^2 \sim \varepsilon$ the proof is complete. □

With the previous result at hand, it is natural to look for a limit function of $(u^\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0$; this would be a suitable candidate for an entropy solution of (1.1) with initial data $u_0 \in BV(\mathbb{R})$. In the next proposition, we do exactly this and collect a few properties about the resulting solution.

Proposition 3.5. *For any initial data $u_0 \in BV(\mathbb{R})$, let $(u^\varepsilon)_{\varepsilon>0}$ be as defined in* (3.32)*. Then, for all $t \geq 0$ the following limit holds in $L^1_{\mathrm{loc}}(\mathbb{R})$*

$$u^\varepsilon(t) \to u(t), \quad \varepsilon \to 0, \tag{3.40}$$

where u is an entropy solution of (1.1) *with initial data u_0. Moreover, u is an element of $C([0,\infty), L^1(\mathbb{R})) \cap L^\infty_{\mathrm{loc}}([0,\infty), L^\infty(\mathbb{R}))$ and satisfies for all $t \geq 0$*

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq e^{t\kappa}\|u_0\|_{L^\infty(\mathbb{R})}, \tag{3.41}$$

$$\|u(t)\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}, \tag{3.42}$$

where $\kappa := \|K\|_{L^1(\mathbb{R})}$.

Proof. We first prove the limit (3.40) for a special subsequence of $(u^\varepsilon)_{\varepsilon>0}$ and then generalize afterwards. Fixing $t \geq 0$, we see from Proposition 3.2 that the functions $(u^\varepsilon(t))_{\varepsilon>0}$ satisfy for any $p \in [1, \infty]$

$$\|u^\varepsilon(t)\|_{L^p(\mathbb{R})} \leq e^{t\kappa}\|u_0\|_{L^p(\mathbb{R})}, \tag{3.43}$$

and in particular, they are uniformly bounded in $L^1(\mathbb{R})$. Moreover, they are equicontinuous with respect to translation

$$\|u^\varepsilon(t, \cdot + h) - u^\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq he^{t\kappa}|u_0|_{TV},$$

for all $h > 0$, and so by the Kolmogorov–Riesz compactness Theorem, any infinite subset of $(u^\varepsilon(t))_{\varepsilon>0}$ is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R})$; as we have skipped developing a tightness estimate for $(u^\varepsilon(t))_{\varepsilon>0}$, we can not claim the family to be relatively compact in $L^1(\mathbb{R})$. The family $(u^\varepsilon)_{\varepsilon>0}$ is not equicontinuous in time and so we can not directly apply the Arzelà-Ascoli theorem, however, the family is for small ε arbitrary close to be equicontinuous and so the proof of the theorem is still applicable; for clarity we perform the steps. By a standard diagonalization argument, we can select a sub-sequence $(u^{\varepsilon_j})_{j\in\mathbb{N}} \subset (u^\varepsilon)_{\varepsilon>0}$ such that $\lim_{j\to\infty} \varepsilon_j = 0$

and $u^{\varepsilon_j}(t)$ converges in $L^1_{\mathrm{loc}}(\mathbb{R})$ for every $t \in E$ with E being a countable dense subset of $\mathbb{R}^+$. Next, we claim that $u^{\varepsilon_j}(t)$ converges in $L^1_{\mathrm{loc}}(\mathbb{R})$ for every $t \geq 0$. Indeed, fix $r > 0$ for locality and pick $s \in E$ such that $|s - t| < \epsilon$ for some arbitrary $\epsilon > 0$. By the time regularity estimate of Proposition 3.2, we have

$$
\begin{aligned}
&\limsup_{j,i\to\infty} \int_{-r}^{r} |u^{\varepsilon_j}(t) - u^{\varepsilon_i}(t)| \mathrm{d}x \\
\leq &\limsup_{j,i\to\infty} \int_{-r}^{r} |u^{\varepsilon_j}(t) - u^{\varepsilon_j}(s)| + |u^{\varepsilon_j}(s) - u^{\varepsilon_i}(s)| + |u^{\varepsilon_i}(s) - u^{\varepsilon_i}(t)| \mathrm{d}x \\
\leq &\limsup_{j,i\to\infty} (2\epsilon + \varepsilon_j + \varepsilon_i) C_{u_0}(t+\epsilon) + \limsup_{j,i\to\infty} \int_{-r}^{r} |u^{\varepsilon_j}(s) - u^{\varepsilon_i}(s)| \mathrm{d}x \\
= &2\epsilon C_{u_0}(t+\epsilon),
\end{aligned}
$$

and since r and ϵ were arbitrary, we conclude that $u^{\varepsilon_j}(t)$ converges in $L^1_{\mathrm{loc}}(\mathbb{R})$ to some $u(t)$. Moreover, as $u^{\varepsilon_j}(t)$ converges locally to $u(t)$, the bound (3.43) necessarily carries over to $u(t)$, and so in particular

$$
\|u(t)\|_{L^p(\mathbb{R})} \leq e^{t\kappa} \|u_0\|_{L^p(\mathbb{R})},
$$

and further by Fatou's lemma we infer for all $t \geq \tilde{t} \geq 0$

$$
\begin{aligned}
\|u(t) - u(\tilde{t})\|_{L^1(\mathbb{R})} &\leq \liminf_{j\to\infty} \|u^{\varepsilon_j}(t) - u^{\varepsilon_j}(\tilde{t})\|_{L^1(\mathbb{R})} \\
&\leq \liminf_{j\to\infty} (t - \tilde{t} + \varepsilon_j) C_{u_0}(t) \\
&= (t - \tilde{t}) C_{u_0}(t).
\end{aligned}
\tag{3.44}
$$

Thus $u \in C([0,\infty), L^1(\mathbb{R})) \cap L^\infty_{\mathrm{loc}}([0,\infty), L^\infty(\mathbb{R}))$. Next, we prove that u is, in accordance with Def. 1.1, an entropy solution of (1.1) with initial data u_0; the latter part follows from $u(0) = u_0$ and (3.44). To see that u satisfies the entropy inequalities (1.4), we pick an arbitrary entropy pair (η, q) of (1.1) and a non-negative $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ and recall Lemma 3.4 to calculate

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} \eta(u)\varphi_t + q(u)\varphi_x + \eta'(u)(K * u)\varphi \mathrm{d}x\mathrm{d}t \\
= &\lim_{j\to 0} \int_0^\infty \int_{\mathbb{R}} \eta(u^{\varepsilon_j})\varphi_t + q(u^{\varepsilon_j})\varphi_x + \eta'(u^{\varepsilon_j})(K * u^{\varepsilon_j})\varphi \mathrm{d}x\mathrm{d}t \\
\geq &\lim_{j\to 0} O(\varepsilon_j) = 0,
\end{aligned}
\tag{3.45}
$$

where the second line holds as the integrand converges in $L^1(\mathbb{R})$; after all, $(u^{\varepsilon_j})_{j\in\mathbb{N}}$ is uniformly bounded on the compact support of φ. By Proposition 3.1 we conclude that u is the unique entropy solution of (1.1) with u_0 as initial data. What remains to show, is the general limit

(3.40) and the L^2 bound of u (3.42); the latter follow by Lemma 3.3 and Fatou's lemma. We prove (3.40) by contradiction; if this limit does not exist, then there is a subsequence $(u^{\varepsilon_j})_{j\in\mathbb{N}} \subset (u^\varepsilon)_{\varepsilon>0}$, a $t>0$ and an $r>0$ such that

$$\liminf_{j\to\infty} \int_{-r}^{r} |u(t) - u^{\varepsilon_j}(t)|\mathrm{d}x > 0.$$

But as argued above, the infinite set $(u^{\varepsilon_j})_{j\in\mathbb{N}}$ must be precompact in $L^1_{\mathrm{loc}}(\mathbb{R})$ for every $t \geq 0$, and thus we can pick a subsequence converging for every $t \geq 0$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ to the unique (Proposition 3.1) entropy solution u which contradicts the above limit inferior. $\square$

The existence of entropy solutions for general L^∞ data now follows from the previous proposition together with the weighted L^1-contraction provided by Proposition 3.1. It is useful to observe that the weight w^r_M (3.4) is increasing in t. In particular, with u, v, t, r and M as in (3.6), we have the contraction

$$\int_{-r}^{r} |u(t,x) - v(t,x)|\mathrm{d}x \leq \int_{\mathbb{R}} |u_0(x) - v_0(x)| w^r_M(T,x)\mathrm{d}x, \tag{3.46}$$

where $T \in (0,\infty)$ is any parameter satisfying $T > t$. We note that while Proposition 3.1 only implies the validity of (3.46) for a.e. $t \in [0,T]$, we will in the following corollary apply it on entropy solutions with BV data; as the previous proposition guaranteed that these functions are continuous from $[0,\infty)$ to $L^1_{\mathrm{loc}}(\mathbb{R})$, the above contraction holds for all $t \in [0,T]$.

Corollary 3.6. *For any initial data $u_0 \in L^\infty(\mathbb{R})$, there exists a corresponding entropy solution $u \in C([0,\infty), L^1_{\mathrm{loc}}(\mathbb{R}))$ of (1.1) satisfying for all $t \geq 0$*

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq e^{t\kappa}\|u_0\|_{L^\infty(\mathbb{R})}, \tag{3.47}$$

where $\kappa := \|K\|_{L^1(\mathbb{R})}$. If $u_0 \in L^2 \cap L^\infty(\mathbb{R})$, it also satisfies for all $t \geq 0$

$$\|u(t)\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}. \tag{3.48}$$

Proof. For $u_0 \in L^\infty(\mathbb{R})$, let $(u^j)_{j\in\mathbb{N}}$ be a sequence of entropy solutions of (1.1) whose corresponding initial data $(u^j_0)_{j\in\mathbb{N}} \subset BV(\mathbb{R})$ satisfies $\sup_j \|u^j_0\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}$ and $u^j_0 \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ as $j \to \infty$. For a fixed $T > 0$, set

$$M = e^{T\kappa}\|u_0\|_{L^\infty(\mathbb{R})},$$

and observe from (3.41) that $\sup_j \|u^j(t)\|_{L^\infty(\mathbb{R})} \leq M$ for all $t \in [0,T]$. In particular, (3.46) is valid for all substitutions $(u,v) \mapsto (u^j, u^i)$ and all parameters $r > 0$, $t \in [0,T]$ and $x_0 \in \mathbb{R}$. Using this contraction, we may estimate for any $r > 0$

$$\limsup_{j,i\to\infty} \sup_{0\le t\le T} \int_{-r}^{r} |u^j(t,x) - u^i(t,x)|\mathrm{d}x$$
$$\le \limsup_{j,i\to\infty} \int_{\mathbb{R}} |u_0^j(x) - u_0^i(x)| w_M^r(T,x)\mathrm{d}x = 0,$$

where the last limit holds by the dominated convergence theorem. This shows that $(u^j)_{j\in\mathbb{N}}$ is Cauchy in the Fréchet space $C([0,\infty), L^1_{\mathrm{loc}}(\mathbb{R}))$ and so the sequence converges to some $u \in C([0,\infty), L^1_{\mathrm{loc}}(\mathbb{R}))$. Moreover,

$$\|u(t)\|_{L^\infty(\mathbb{R})} \le \liminf_{j\to\infty} \|u^j(t)\|_{L^\infty(\mathbb{R})} \le e^{t\kappa}\|u_0\|_{L^\infty(\mathbb{R})},$$

by (3.41), and so $u \in L^\infty_{\mathrm{loc}}([0,\infty), L^\infty(\mathbb{R}))$ too. That u takes u_0 as initial data in L^1_{loc}-sense follows from the time-continuity of u and $u(0) = \lim_{j\to\infty} u_0^j = u_0$ where the limit is taken in $L^1_{\mathrm{loc}}(\mathbb{R})$. Moreover, as each member $(u^j)_{j\in\mathbb{N}}$ satisfies the entropy inequalities (1.4), the same can be said for u by a similar calculation as (3.45). Thus the corollary is proved, save for the L^2 estimate; this is attained through Fatou's lemma and (3.42) as we may assume $\sup_j \|u_0^j\|_{L^2(\mathbb{R})} \le \|u_0\|_{L^2(\mathbb{R})}$. □

3.3. **L^2 continuity and stability of entropy solutions.** For clarity, we summarize what of Theorem 2.1 has been proved so far and what remains to be proved. Combining Proposition 3.1 and Corollary 3.6, we conclude that there exists a unique entropy solution of (1.1) in accordance with Def. 1.1 for every initial data $u_0 \in L^\infty(\mathbb{R})$ and thus also for $u_0 \in L^2 \cap L^\infty(\mathbb{R})$. Furthermore, Corollary 3.6 guarantees that these solutions are continuous from $[0,\infty)$ to $L^1_{\mathrm{loc}}(\mathbb{R})$ so that the restriction $u(t) := u(t,\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R})$ makes sense for all $t \ge 0$. The same corollary also provides the bounds (2.1) of Theorem 2.1. It remains to prove that entropy solutions with $L^2 \cap L^\infty$ data are continuous from $[0,\infty)$ to $L^2(\mathbb{R})$ and that they satisfy the stability result of Theorem 2.1. To do so, we shall exploit the height bound of Corollary 2.6. As explained at the beginning of Section 4, Corollary 2.6 can be proved for the case $u_0 \in L^2 \cap L^\infty(\mathbb{R})$ independently of this subsection; thus we may here use the height bound (2.5) for entropy solutions of (1.1) without risking a circular argument. From here til the end of the section, we take the above properties of entropy solutions for granted. We begin with a variant of Proposition 3.1 which makes use of the above discussed height bound.

Lemma 3.7. *There is a function $\Psi\colon [0,\infty)^3 \to [0,\infty)$, increasing in all arguments, such that for any pair of entropy solutions u, v of* (1.1) *with* *respective initial data* $u_0, v_0 \in L^2 \cap L^\infty(\mathbb{R})$ *one has for any* $t, r \geq 0$ *and* $N \geq \max\{\|u_0\|_{L^2(\mathbb{R})}, \|v_0\|_{L^2(\mathbb{R})}\}$ *the inequality*

$$\|u(t) - v(t)\|_{L^1([-r,r])} \leq \Psi(t, N, r)\|u_0 - v_0\|_{L^2(\mathbb{R})}. \tag{3.49}$$

Proof. Let u, v, u_0, v_0 and N be as described in the lemma. By (2.5) from Corollary 2.6, and the property of N, we have for all $t > 0$

$$\frac{\|u(t)\|_{L^\infty(\mathbb{R})} + \|v(t)\|_{L^\infty(\mathbb{R})}}{2} \leq CN^{\frac{2}{3}}\left(1 + \frac{1}{t^{\frac{1}{3}}}\right) =: m(t), \tag{3.50}$$

where $C \coloneqq \max\{2^{\frac{11}{12}}3^{\frac{1}{3}}\|K\|_{L^1(\mathbb{R})}^{\frac{1}{3}}, 2^{\frac{5}{4}}\}$. With $F(u,v) \coloneqq \frac{1}{2}\mathrm{sgn}(u-v)(u^2 - v^2)$, we have for any non-negative $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$ the inequality

$$0 \leq \int_0^\infty \int_{\mathbb{R}} |u-v|\varphi_t + F(u,v)\varphi_x + |u-v|(|K| * \varphi)\mathrm{d}x\mathrm{d}t. \tag{3.51}$$

This is attained by following the first half of the proof of Proposition 3.1 without using the bound $|F(u,v)| \leq M|u-v|$ as done in the first inequality of (3.14); one may instead, when 'going to the diagonal', subtract $F(u(t,x), v(t,x))$ from $F(u(t,x), v(s,y))$ and use

$$|F(u(t,x), v(s,y)) - F(u(t,x), v(x,y))| \lesssim |v(s,y) - v(t,x)|,$$

which follows from local Lipschitz continuity of F and the fact that u and v are globally bounded (as pointed out after Corollary 2.6). With (3.51) established, we may *now* filter out $(u+v)/2$ from F using the more precise bound (3.50), that is

$$|F(u(t,x), v(t,x))| \leq m(t)|u(t,x) - v(t,x)|.$$

Doing so, and additionally setting $\varphi(t,x) = \theta(t)\phi(t,x)$ for two arbitrary non-negative functions $\theta \in C_c^\infty((0,T))$ and $\phi \in C_c^\infty((0,T)\times\mathbb{R})$, with $T > 0$ also arbitrary, we conclude from (3.51) that

$$\begin{aligned} 0 \leq &\int_0^T \int_{\mathbb{R}} |u-v|\theta'\phi\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{R}} |u-v|\theta\Big[\phi_t + m(t)|\phi_x| + |K| * \phi\Big]\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.52}$$

Observe that (3.52) resembles (3.18); for brevity, we skip minor details in the following steps due to their similarity of those following (3.18). Let $f\colon \mathbb{R} \to [0,1]$ be a smooth and non-increasing function satisfying $f(x) = 1$ for $x \leq 0$ and $f(x) = 0$ for sufficiently large x, and set

$$g(t,x) \coloneqq f(|x| + M(t) - M(T)),$$

where we have here defined $M(t)$ by

$$M(t) := \int_0^t m(s)\mathrm{d}s = CN^{\frac{2}{3}}\Big(t + \tfrac{3}{2}t^{\frac{2}{3}}\Big),$$

not to be confused with the constant M from the proof of Proposition 3.1. Analogous to (3.20), we then set

$$\phi(t,x) = \Big(e^{(T-t)|K|*}g(t,\cdot)\Big)(x), \tag{3.53}$$

and while this ϕ is not of compact support, both it, and its derivatives, are integrable on $(0,T)\times\mathbb{R}$ and so by an approximation argument it can be used in (3.52) (the compact support of θ means the singularity of $m(t)$ at $t=0$ is not seen). By similar arguments as those following (3.20) we find also here that the second integral in (3.52) is non-positive, and so we may remove it. Letting then θ approximate $\mathbb{1}_{(0,T)}$ in a similar (smooth) manner as done by the sequence (3.21), we may from (3.52) conclude

$$\int_{\mathbb{R}} |u(T,x)-v(T,x)|\phi(T,x)\mathrm{d}x \leq \int_{\mathbb{R}} |u_0(x)-v_0(x)|\phi(0,x)\mathrm{d}x, \tag{3.54}$$

where we used that $t\mapsto |u(t,\cdot)-v(t,\cdot)|\phi(t,\cdot)$ is continuous from $[0,T]$ to $L^1(\mathbb{R})$ which holds as the same is true for $t\mapsto\phi(t,\cdot)$ while u and v are both globally bounded and continuous from $[0,T]$ to $L^1_{\mathrm{loc}}(\mathbb{R})$. Note that $\phi(0,x)=f(|x|)$, and so letting $f\to\mathbb{1}_{(-\infty,r)}$ in L^1 sense, the left-hand side of (3.54) becomes the left-hand side of (3.49). When $f\to\mathbb{1}_{(-\infty,r)}$ we also get from (3.53) that

$$\phi(0,x) \to \Big(e^{T|K|*}\mathbb{1}_{(-\infty,r)}(|\cdot|-M(T))\Big)(x), \tag{3.55}$$

in L^1 sense. Denoting the right-hand side of (3.55) also by $\phi(0,x)$, it follows by Young's convolution inequality that

$$\|\phi(0,x)\|_{L^2(\mathbb{R})} \leq e^{T\kappa}[2r+2M(T)]^{\frac{1}{2}} = e^{T\kappa}\Big[2r+2CN^{\frac{2}{3}}\Big(T+\tfrac{3}{2}T^{\frac{2}{3}}\Big)\Big]^{\frac{1}{2}}, \tag{3.56}$$

where $\kappa := \|K\|_{L^1(\mathbb{R})}$. Applying then the Cauchy–Schwarz inequality to the right-hand side of (3.54), and using the above L^2 bound for $\phi(0,x)$, we attain (3.49) (with T substituting for t) for $\Psi(T,N,r)$ given by the right-hand side of (3.56). □

We follow up with a tightness bound for entropy solutions with $L^2\cap L^\infty$ data.

Lemma 3.8. *There is a function $\Phi\colon [0,\infty)^2\times\mathbb{R}\to[0,\infty)$, increasing in all arguments, such that if u is an entropy solution of* (1.1) *with initial data $u_0\in L^2\cap L^\infty(\mathbb{R})$, then for any $t,r\geq 0$ and $N\geq \|u_0\|_{L^2(\mathbb{R})}$*

$$\int_{|x|>r} u^2(t,x)\mathrm{d}x \leq \int_{\mathbb{R}} u_0^2(x)\Phi(t,N,|x|-r)\mathrm{d}x. \tag{3.57}$$

Moreover,

$$\lim_{\xi\to-\infty}\Phi(t,N,\xi)=0, \qquad \Phi(t,N,\xi)=e^{2t\kappa}, \quad \xi>0,$$

where $\kappa:=\|K\|_{L^1(\mathbb{R})}$, and in particular, $\xi\mapsto\Phi(t,M,\xi)$ is a bounded function.

Proof. Pick arbitrary initial data $u_0\in L^2\cap L^\infty(\mathbb{R})$ and let u denote the corresponding entropy solution of (1.1). Writing out the entropy inequality (1.4) for u using the entropy pair $(\eta(u),q(u))=(u^2,\frac{2}{3}u^3)$ and a non-negative test function $\varphi\in C_c^\infty((0,T)\times\mathbb{R})$, with $T\in(0,\infty)$ fixed, we get

$$0\leq\int_0^T\int_{\mathbb{R}} u^2\varphi_t+\tfrac{2}{3}u^3\varphi_x+2u(K*u)\varphi\,\mathrm{d}x\mathrm{d}t. \tag{3.58}$$

By the height bound (2.5) of Corollary 2.6, we have $\|u(t)\|_{L^\infty(\mathbb{R})}\leq m(t)$ where $m(t)$ is as defined in (3.50), and so the second term of the above integrand satisfies

$$\tfrac{2}{3}u^3\varphi_x\leq u^2\Big[\tfrac{2}{3}m(t)|\varphi_x|\Big].$$

Additionally, the third part of the integrand satisfies

$$\begin{aligned}\int_{\mathbb{R}}2u(K*u)\varphi\,\mathrm{d}x&=\int_{\mathbb{R}}\int_{\mathbb{R}}2u(t,x)u(t,y)K(x-y)\varphi(t,x)\,\mathrm{d}y\mathrm{d}x\\&\leq\int_{\mathbb{R}}\int_{\mathbb{R}}\Big[|u(t,x)|^2+|u(t,y)|^2\Big]|K(x-y)|\varphi(t,x)\,\mathrm{d}y\mathrm{d}x\\&=\int_{\mathbb{R}}u^2\Big[\kappa\varphi+|K|*\varphi\Big]\mathrm{d}x.\end{aligned}$$

Inserting these two bounds in (3.58) we get for any non-negative $\varphi\in C_c^\infty((0,T)\times\mathbb{R})$

$$0\leq\int_0^T\int_{\mathbb{R}}u^2\Big[\varphi_t+\tfrac{2}{3}m(t)|\varphi_x|+\mathcal{K}*\varphi\Big]\,\mathrm{d}x\mathrm{d}t, \tag{3.59}$$

where we introduced the measure $\mathcal{K}:=\kappa\delta+|K|$, where δ is the Dirac measure. We also here, like the previous proof, proceed in a manner similar to the second half of the proof of Proposition 3.1, though some necessary changes are made. We set $\varphi(t,x)=\theta(t)\rho(x)\phi(t,x)$ for three smooth non-negative functions on $[0,T]\times\mathbb{R}$ with θ and ρ having compact support in $(0,T)$ and $\mathbb{R}$ respectively. Additionally, while ϕ need not be compactly supported, we require ϕ and its derivatives to be bounded. Inserting this in (3.59) we get

$$\begin{aligned} 0 \leq &\int_0^T \int_{\mathbb{R}} u^2\theta'\rho\phi \mathrm{d}x\mathrm{d}t \\ &+ \int_0^\infty \int_{\mathbb{R}} u^2\theta\Big[\rho\phi_t + \tfrac{2}{3}m(t)|(\rho\phi)_x| + \mathcal{K}*(\rho\phi)\Big]\,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.60}$$

By approximation, (3.60) is still valid for a non-negative $\theta \in W_0^{1,1}((0,T))$ and so we may set $\theta = \theta_\epsilon$ where the latter given by (3.21), followed by letting $\epsilon \to 0$ to conclude from (3.60) that

$$\begin{aligned} 0 \leq &\int_{\mathbb{R}} \Big[u_0^2(x)\phi(0,x) - u^2(T,x)\phi(T,x)\Big]\rho(x)\mathrm{d}x \\ &+ \int_0^\infty \int_{\mathbb{R}} u^2\Big[\rho\phi_t + \tfrac{2}{3}m(t)|(\rho\phi)_x| + \mathcal{K}*(\rho\phi)\Big]\,\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{3.61}$$

where we used that $t\mapsto u^2(t,\cdot)\rho(\cdot)\phi(t,\cdot)$ is continuous in L^1 sense as follows from the L^1_{loc} continuity and boundness of u, the smoothness of ϕ and the compact support of ρ. Next, we set $\rho(x) = \tilde{\rho}(x/N)$ where $\tilde{\rho} \in C_c^\infty(\mathbb{R})$ is non-negative and satisfies $\tilde{\rho}(0) = 1$. Letting $N \to \infty$, (3.61) yields through the dominated convergence theorem

$$\begin{aligned} \int_{\mathbb{R}} u^2(T,x)\phi(T,x)\mathrm{d}x \leq &\int_{\mathbb{R}} u_0^2(x)\phi(0,x)\mathrm{d}x \\ &+ \int_0^\infty \int_{\mathbb{R}} u^2\Big[\phi_t + \tfrac{2}{3}m(t)|\phi_x| + \mathcal{K}*\phi\Big]\,\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{3.62}$$

where the convergence of the integrals follows from the boundness of ϕ (and its derivatives) combined with $\|u(t)\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}$ for all $t \in [0,T]$. To rid ourselves of the last integral in (3.62), we perform a similar trick as done for (3.18) and (3.52), but with a different f; we here let $f\colon \mathbb{R}\to[0,1]$ be a *non-decreasing* function with bounded derivatives. Define further g by

$$g(t,x) \coloneqq f(|x| + M(T) - M(t)),$$

where $M(t)$ denotes

$$M(t) \coloneqq \int_0^t \tfrac{2}{3}m(s)\mathrm{d}s = CN^{\frac{2}{3}}\Big(\tfrac{2}{3}t + t^{\frac{2}{3}}\Big), \tag{3.63}$$

and analogues to (3.20), we set ϕ to be

$$\phi(t,x) = \Big(e^{(T-t)\mathcal{K}*}g(t,\cdot)\Big)(x).$$

As $t \mapsto g(t,x)$ is still a non-increasing function, we conclude by similar arguments as those following (3.20) that the square bracket in (3.62) is non-positive. Thus, removing the non-positive integral in (3.62) we conclude

$$\int_{\mathbb{R}} u^2(T,x) f(|x|)\mathrm{d}x \leq \int_{\mathbb{R}} u_0^2(x)\Big(e^{T\mathcal{K}*} f(|\cdot| + M(T))\Big)(x)\mathrm{d}x, \tag{3.64}$$

where we used the explicit expressions for $\phi(T,x)$ and $\phi(0,x)$. Letting $f \to \mathbb{1}_{(r,\infty)}$ pointwise a.e. it is clear that the left-hand side of (3.64) converges to $\int_{|x|>r} u^2(T)\mathrm{d}x$. As for the right-hand side, we get the cumbersome term $e^{T\mathcal{K}*}\mathbb{1}_{(r,\infty)}(|\cdot|+MT)$ which we now simplify. Let the Borel measure ν_T be defined by the relation $\nu_T* = e^{T\mathcal{K}*}$ and observe that we for $x\in\mathbb{R}$ have

$$\begin{aligned}\Big(\nu_T * \mathbb{1}_{(r,\infty)}(|\cdot| + M(T))\Big)(x) &= \int_{|x-y|+M(T)>r} \mathrm{d}\nu_T(y)\\ &\leq \int_{|x|-r+M(T)>-|y|} \mathrm{d}\nu_T(y).\end{aligned} \tag{3.65}$$

We thus define $\Phi(T,N,|x|-r)$ to be the latter expression after substituting for $M(T)$ using (3.63). Inserting this in (3.64) we get exactly (3.57) with T substituting for t. The properties of Φ stated in the lemma can be read directly from (3.65) when setting $\xi = |x|-r$ together with the fact that $T\mapsto \nu_T$ is increasing (in the canonical sense) and $\int_{\mathbb{R}} d\nu_T = e^{T\mathcal{K}*}1 = e^{2T\kappa}$. □

We may now prove the remaining part of Theorem 2.1.

Proposition 3.9. *Let two sequences $(t_k)_{k\in\mathbb{N}} \subset [0,\infty)$ and $(u_{0,k})_{k\in\mathbb{N}} \subset L^2\cap L^\infty(\mathbb{R})$ admit limits*

$$\lim_{k\to\infty} |t_k - t| = 0, \qquad \lim_{k\to\infty} \|u_{0,k} - u_0\|_{L^2(\mathbb{R})} = 0,$$

with $t\in[0,\infty)$ and $u_0 \in L^2\cap L^\infty(\mathbb{R})$. Letting $(u_k)_{k\in\mathbb{N}}$ and u denote the entropy solutions of (1.1) corresponding to the initial data $(u_{0,k})_{k\in\mathbb{N}}$ and u_0 respectively, we have

$$\lim_{k\to\infty} \|u_k(t_k) - u(t)\|_{L^2(\mathbb{R})} = 0.$$

In particular, entropy solutions of (1.1) with $L^2\cap L^\infty$ data are continuous from $[0,\infty)$ to $L^2(\mathbb{R})$.

Proof. Suppose first that $t > 0$. As $t_k \to t$ there is a $T \in (0,\infty)$ such that $(t_k)_{k\in\mathbb{N}} \subset [0,T]$. Similarly, there is an N such that $N \geq \|v_0\|_{L^2(\mathbb{R})}$ for every $v_0 \in \{u_{0,1}, u_{0,2}, \ldots, u_0\}$; observe that such an N satisfies $N \geq \|v(t)\|_{L^2}$ for all $t \in [0,T]$ and v ranging over the corresponding entropy solutions. As the function Φ from Lemma 3.8 was increasing in its arguments, we infer for all $k \in \mathbb{N}$ and $r > 0$ that

$$\int_{|x|>r} u_k^2(t_k,x)\mathrm{d}x \leq \int_{\mathbb{R}} u_{0,k}^2(x)\Phi(T,N,|x|-r).$$

Furthermore, as $\xi \mapsto \Phi(T,M,\xi)$ is bounded while $u_{0,k}^2 \to u_0^2$ in $L^1(\mathbb{R})$ as $k \to \infty$, it follows that

$$\limsup_{k\to\infty} \int_{|x|>r} u_k^2(t_k,x)\mathrm{d}x \leq \int_{\mathbb{R}} u_0^2(x)\Phi(T,M,|x|-r), \tag{3.66}$$

for any $r > 0$. Since u_0^2 is integrable and $\lim_{\xi\to-\infty}\Phi(T,M,\xi) = 0$, we may for any $\varepsilon > 0$ pick a sufficiently large $r > 0$ such that the right-hand side of (3.66) is smaller than ε^2. For such a couple of constants $\varepsilon, r > 0$ we find

$$\limsup_{k\to\infty} \|u_k(t_k) - u(t)\|_{L^2(\mathbb{R})} \leq 2\varepsilon + \limsup_{k\to\infty} \|u_k(t_k) - u(t)\|_{L^2([-r,r])}. \tag{3.67}$$

To deal with the rightmost term in (3.67), we yet again let m be the function defined in (3.50) using the above N. As $t > 0$, there are only a finite number of elements in $(t_k)_{k\in\mathbb{N}}$ smaller than $t/2$; without loss of generality, we shall assume there are none. By the height bound (2.5) from Corollary 2.6 and m being decreasing in t, it then follows that $\|v\|_{L^\infty(\mathbb{R})} \leq m(t/2)$ for every $v \in \{u_1(t_1), u_2(t_2), \ldots, u(t)\}$. Thus,

$$\|u_k(t_k) - u(t)\|_{L^2([-r,r])}^2 \leq 2m(t/2)\|u_k(t_k) - u(t)\|_{L^1([-r,r])},$$

and by the triangle inequality, we further have

$$\begin{aligned}\|u_k(t_k) - u(t)\|_{L^1([-r,r])} \leq{}& \|u_k(t_k) - u(t_k)\|_{L^1([-r,r])} \\ &+ \|u(t_k) - u(t)\|_{L^1([-r,r])}.\end{aligned}$$

As $t \mapsto u(t)$ is continuous in L^1_{loc} sense, we have $\|u(t_k) - u(t)\|_{L^1([-r,r])} \to 0$ as $k \to \infty$, while Lemma 3.7 gives us

$$\|u_k(t_k) - u(t_k)\|_{L^1([-r,r])} \leq \Psi(T,N,r)\|u_{0,k} - u_0\|_{L^2(\mathbb{R})} \to 0,$$

as $k \to \infty$. The last term of (3.67) is thus zero, and as $\varepsilon > 0$ was arbitrary, we conclude $\limsup_{k\to\infty} \|u_k(t_k) - u(t)\|_{L^2(\mathbb{R})} = 0$. Suppose next $t = 0$. We have the two immediate properties

$$\lim_{k\to\infty} \|u_k(t_k) - u_0\|_{L^1([-r,r])} = 0, \quad \forall r > 0,$$

$$\limsup_{k\to\infty}\Big[\|u_k(t_k)\|_{L^2(\mathbb{R})}-\|u_0\|_{L^2(\mathbb{R})}\Big]\le 0,$$

the first following by the triangle inequality, Lemma 3.7 and the fact that $t\mapsto u(t)$ is continuous in L^1_{loc} sense, while the second follows from using $\|u_k(t_k)\|_{L^2(\mathbb{R})}\le\|u_{0,k}\|_{L^2(\mathbb{R})}$. Moreover, for any $w\in C_c^\infty(\mathbb{R})$ we have

$$\begin{aligned}\langle u_k(t_k)-u_0,u_0\rangle &= \langle u_k(t_k)-u_0,w\rangle+\langle u_k(t_k)-u_0,u_0-w\rangle\\ &\le \|u_0-u_k(t_k)\|_{L^1(\mathrm{supp}(w))}\|w\|_{L^\infty(\mathbb{R})}\\ &\quad+\Big[\|u_k(t_k)\|_{L^2(\mathbb{R})}+\|u_0\|_{L^2(\mathbb{R})}\Big]\|u_0-w\|_{L^2(\mathbb{R})},\end{aligned}$$

and so, together with the above properties, we see from approximating u_0 (in L^2 sense) by elements in $C_c^\infty(\mathbb{R})$ that

$$\lim_{k\to\infty}\langle u_k(t_k)-u_0,u_0\rangle=0.$$

This last limit, and the above limit superior, then give us

$$\begin{aligned}&\lim_{k\to\infty}\|u_k(t_k)-u_0\|^2_{L^2(\mathbb{R})}\\ &=\lim_{k\to\infty}\Big[\langle u_k(t_k),u_k(t_k)\rangle-\langle u_0,u_0\rangle\Big]-2\lim_{k\to\infty}\langle u_k(t_k)-u_0,u_0\rangle\le 0.\end{aligned}$$

Thus, the stability result of the proposition has been demonstrated. That this implies the continuity of $t\mapsto u(t)$ in L^2 sense follows by considering the sequence of initial data where $u_{0,k}=u_0$ for all $k\in\mathbb{N}$. □

We end the section by proving Corollary 2.2.

Proof of Corollary 2.2. The solution mapping S is by Proposition 3.9 jointly continuous from $[0,\infty)\times(L^2\cap L^\infty(\mathbb{R}))^*$ to $L^2(\mathbb{R})$, where $(L^2\cap L^\infty(\mathbb{R}))^*$ denotes the set $L^2\cap L^\infty(\mathbb{R})$ equipped with its L^2 subspace-topology. Seeking to extend S to all of $[0,\infty)\times L^2(\mathbb{R})$ in a continuous manner, we note that we have only one choice: whenever a sequence $(u_{0,k})_{k\in\mathbb{N}}\in L^2\cap L^\infty(\mathbb{R})$ converges in $L^2(\mathbb{R})$, it follows from Lemma 3.7 that the corresponding entropy solutions $(u_k)_{k\in\mathbb{N}}$ form a Cauchy sequence in the Fréchet space $C([0,\infty),L^1_{\mathrm{loc}}(\mathbb{R}))$, and thus they converge to a unique element $u\in C([0,\infty),L^1_{\mathrm{loc}}(\mathbb{R}))$ in the appropriate topology. We now argue that u inherits all the nice properties of entropy solutions of (1.1) established so far, apart from being bounded at $t=0$. Denoting $u_0\in L^2(\mathbb{R})$ for the L^2 limit of $(u_{0,k})_{k\in\mathbb{N}}$, we have by Fatou's lemma

$$\|u(t)\|_{L^2(\mathbb{R})}\le\liminf_{k\to\infty}\|u_k(t)\|_{L^2(\mathbb{R})}\le\liminf_{k\to\infty}\|u_{0,k}\|_{L^2(\mathbb{R})}=\|u_0\|_{L^2(\mathbb{R})}.$$

Moreover, as each u_k satisfy the height bound (2.5) this bound also carries over to u, and thus u is locally bounded in $(0,\infty)\times\mathbb{R}$. Similarly, as each u_k satisfy the entropy inequalities (1.4), the same is true for

u by a limit argument exploiting the uniform bound of $(u_k)_{k\in\mathbb{N}}$ on the support of φ and the fact that η and q are smooth; in particular, u is a weak solution of (1.1). Even Lemma 3.7 and Lemma 3.8 carries over to u by approximation. In conclusion, u – and all other weak solutions obtained this way – satisfy every property used for entropy solutions in the proof of Proposition 3.9, and so the proposition extends to these weak solutions. Consequently, S is continuous on the larger set $[0,\infty)\times L^2(\mathbb{R})$, and the proof is complete. □

4. One sided Hölder regularity for entropy solutions

In this section we show that entropy solutions of (1.1) with $L^2\cap L^\infty$ data satisfy one sided Hölder conditions with time-decreasing coefficients. As Subsection 3.3 exploits Corollary 2.6, which is proved using the results established here, we stress that the coming analysis will only depend on the results of Subsection 3.1 and 3.2, thus avoiding a circular argument. In Subsection 4.1 we introduce the necessary building blocks and provide an informal discussion of the idea behind the analysis of Subsection 4.2 where the Hölder conditions are constructed; Theorem 2.3 is proved in the summary following Corollary 4.8. Central in this section is the following object, which in classical terms can be described as a modulus of right upper semi-continuity.

Definition 4.1. We say that a smooth and strictly increasing function $\omega\colon(0,\infty)\to(0,\infty)$ is a *modulus of growth* for $v\colon\mathbb{R}\to\mathbb{R}$ if for all $h>0$

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}}\Big[v(x+h)-v(x)\Big]\le\omega(h).$$

The requirement that ω be smooth and strictly increasing is for technical convenience. Note also that we did not require $\omega(0+)=0$; this is to include the expression (4.10) when $s=0$.

4.1. Preliminary results.

The classical Oleĭnik estimate [8] for entropy solutions of Burgers' equation is for $(t,x)\in\mathbb{R}^+\times\mathbb{R}$ and $h\ge0$ given by

$$u(t,x+h)-u(t,x)\le\frac{h}{t}.\tag{4.1}$$

For a fixed $t>0$, this one sided Lipschitz condition (or modulus of growth) restricts how fast $x\mapsto u(t,x)$ can grow, but not how fast it can decrease, thus allowing for jump discontinuities (shocks) whose left limit is above the right. Interestingly, when the initial data of Burgers' equation satisfies $u_0\in L^p(\mathbb{R})$ for some $p\in[1,\infty)$, one can for the corresponding entropy solution u use (4.1) to attain

$$\|u(t)\|^{p+1}_{L^\infty(\mathbb{R})}\le\tfrac{p+1}{t}\|u(t)\|^p_{L^p(\mathbb{R})}\le\tfrac{p+1}{t}\|u_0\|^p_{L^p(\mathbb{R})},\tag{4.2}$$

where the rightmost inequality is just the classical L^p bound for Burgers' equation, and thus, the height of $u(t) = u(t,\cdot)$ tends to zero as $t \to \infty$. We omit the proof of (4.2), which is similar to that of the next lemma where we provide a general method for bounding the height of a function $u \in L^2(\mathbb{R})$ admitting a modulus of growth ω. Our focus on $L^2(\mathbb{R})$ is because the other L^p norms might fail to be non-increasing for entropy solutions of (1.1); the generalization of (4.1) will (surprisingly) require a generalization of (4.2), so $p = 2$ is the natural choice as $\|u(t)\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}$ for entropy solutions of (1.1). In the coming lemma we also provide for later convenience a bound on the following seminorm defined for $v \in L^\infty(\mathbb{R})$ by

$$|v|_\infty \coloneqq \operatorname*{ess\,sup}_{x,y\in\mathbb{R}} \frac{v(x) - v(y)}{2}. \tag{4.3}$$

As $|v|_\infty \leq \|v\|_{L^\infty(\mathbb{R})}$, any bound on $\|v\|_{L^\infty(\mathbb{R})}$ obviously carries over to $|v|_\infty$. Note however, that the next lemma bounds $|v|_\infty$ sharper than it does $\|v\|_{L^\infty(\mathbb{R})}$. Finally, we mention that the extra assumptions posed on ω in the lemma are only for technical simplicity, as the lemma holds more generally.

Lemma 4.2. *Let $v \in L^2(\mathbb{R})$ admit a modulus of growth ω that satisfies $\omega(0+) = 0$ and $\omega(\infty) = \infty$. Then $v \in L^2 \cap L^\infty(\mathbb{R})$ and moreover*

$$\|v\|^2_{L^2(\mathbb{R})} \geq F\Big(\|v\|_{L^\infty(\mathbb{R})}\Big), \tag{4.4}$$

$$\tfrac{1}{2}\|v\|^2_{L^2(\mathbb{R})} \geq F\Big(|v|_\infty\Big), \tag{4.5}$$

where F is the strictly increasing and convex function

$$F(y) \coloneqq 2\int_0^y \int_0^{y_1} \omega^{-1}(y_2)\mathrm{d}y_2\mathrm{d}y_1. \tag{4.6}$$

Proof. By Lemma A.1 from the appendix we may assume v to be left-continuous, and in particular, well defined at every point. Now, if $x \in \mathbb{R}$ is such that $v(x) \geq 0$ we have

$$v(x-h) \geq v(x) - \omega(h) \geq 0,$$

for all $h \in (0, \omega^{-1}(v(x))]$, while if x is such that $v(x) < 0$ we have

$$v(x+h) \leq v(x) + \omega(h) \leq 0,$$

for all $h \in (0, \omega^{-1}(-v(x))]$. Squaring each of these inequalities (the bottom one would flip direction) and integrating over $h \in (0, \omega^{-1}(|v(x)|)]$, yields in both cases

$$\|v\|^2_{L^2(\mathbb{R})} \geq \int_0^{\omega^{-1}(|v(x)|)} (|v(x)| - \omega(h))^2\mathrm{d}h, \tag{4.7}$$

where the left-hand side has been replaced by the upper bound $\|v\|^2_{L^2(\mathbb{R})}$. Performing the change of variables $h = \omega^{-1}(y)$ the right-hand side of (4.7) can further be written

$$\begin{aligned}\int_0^{|v(x)|} (|v(x)|-y)^2 \mathrm{d}\omega^{-1}(y) &= 2\int_0^{|v(x)|} (|v(x)|-y)\omega^{-1}(y)\mathrm{d}y \\ &= 2\int_0^{|v(x)|}\int_0^y \omega^{-1}(z)\mathrm{d}z\mathrm{d}y,\end{aligned}$$

where we integrated by parts twice. This last expression is exactly $F(|v(x)|)$, and so letting this replace the right-hand side of (4.7) followed by taking the supremum with respect to $x \in \mathbb{R}$ yields (4.4). For (4.5), we write v_+ and v_- for the positive and negative part of v respectively, and observe that $v \in L^2 \cap L^\infty(\mathbb{R})$ implies $|v|_\infty = \frac{1}{2}(\|v_+\|_{L^\infty(\mathbb{R})} + \|v_-\|_{L^\infty(\mathbb{R})})$ and $\|v\|^2_{L^2(\mathbb{R})} = \|v_+\|^2_{L^2(\mathbb{R})} + \|v_-\|^2_{L^2(\mathbb{R})}$. Furthermore, as both v_+ and $-v_-$ admit ω as a modulus of growth, we can use (4.4) followed by Jensen's inequality to calculate

$$\begin{aligned}\tfrac{1}{2}\|v\|^2_{L^2(\mathbb{R})} &= \tfrac{1}{2}\Big[\|v_+\|^2_{L^2(\mathbb{R})} + \|v_-\|^2_{L^2(\mathbb{R})}\Big] \\ &\geq \tfrac{1}{2}\Big[F\Big(\|v_+\|_{L^\infty(\mathbb{R})}\Big) + F\Big(\|v_-\|_{L^\infty(\mathbb{R})}\Big)\Big] \\ &\geq F\Big(\tfrac{1}{2}\Big[\|v_+\|_{L^\infty(\mathbb{R})} + \|v_-\|_{L^\infty(\mathbb{R})}\Big]\Big) \\ &= F\Big(|v|_\infty\Big).\end{aligned}$$

□

The calculations of the next subsection, where Theorem 2.3 is proved, can be boiled down to the three lemmas of this subsection (Lemma 4.2 being the first). The remaining Lemma 4.3 and Lemma 4.4, induce a natural evolution of a modulus of growth from the mappings S_t^B and S_t^K, introduced in (3.24) and (3.25). The relevance of these results should come as no surprise; the previous section showed that entropy solutions could be approximated by repeated compositions of said mappings.

Lemma 4.3. *Suppose $v \in BV(\mathbb{R})$ admits a concave modulus of growth ω. Then for any $\varepsilon > 0$, the function $w = S_\varepsilon^B(v)$, admits the modulus of growth*

$$h \mapsto \frac{\omega(h)}{1+\varepsilon\omega'(h)}. \tag{4.8}$$

Proof. As $v \in BV(\mathbb{R})$ it admits for each $x \in \mathbb{R}$ an essential left limit $v(x-)$ and right limit $v(x+)$, and since S_t^B is non-expansive on $BV(\mathbb{R})$, the same can be said for w. Thus we assume without loss of generality that v and w are left continuous. For $x \in \mathbb{R}$, $h > 0$ and $t \in [0, \varepsilon]$, introduce the two (minimal) backward characteristics of $S_t^B(v)$ emanating from (ε, x) and $(\varepsilon, x+h)$ respectively

$$\begin{aligned} \xi_1(t) &= x + (t-\varepsilon)w(x), \\ \xi_2(t) &= x + h + (t-\varepsilon)w(x+h). \end{aligned}$$

As v and w are left continuous, it follows from Theorem 11.1.3. in [8] that

$$v(\xi_1(0)) \leq w(x), \qquad\qquad w(x+h) \leq v(\xi_2(0)+).$$

Moreover, by the Oleĭnik estimate of w (4.1), we find

$$\xi_2(0) - \xi_1(0) = h - \varepsilon[w(x+h) - w(x)] \geq 0,$$

and so exploiting ω we can calculate

$$\begin{aligned} w(x+h) - w(x) &\leq v(\xi_2(0)+) - v(\xi_1(0)) \\ &\leq \omega(h - \varepsilon[w(x+h) - w(x)]) \\ &\leq \omega(h) - \varepsilon\omega'(h)(w(x+h) - w(x)), \end{aligned} \tag{4.9}$$

where the last inequality holds as ω is concave. We conclude that

$$w(x+h) - w(x) \leq \frac{\omega(h)}{1 + \varepsilon\omega'(h)},$$

for all $x \in \mathbb{R}$ and $h > 0$. That (4.8) is positive, smooth and strictly increasing follows from ω being positive, smooth, strictly increasing and concave. $\square$

We follow immediately with a similar result for the operator S_t^K, which will depend on the fractional variation $|K|_{TV^s}$ as defined in (1.5) and the seminorm $|\cdot|_\infty$ defined in (4.3).

Lemma 4.4. *Let $s \in [0,1]$ and assume $|K|_{TV^s} < \infty$. Suppose $v \in L^\infty(\mathbb{R})$ admits a modulus of growth ω. Then for any $\varepsilon > 0$, the function $w = S_\varepsilon^K(v)$ admits the modulus of growth*

$$h \mapsto \omega(h) + \varepsilon|K|_{TV^s}|v|_\infty h^s. \tag{4.10}$$

Proof. For simple notation, we introduce the shift operator $T_h \colon f \mapsto f(\cdot + h)$. As shifts commute with convolution, and since $\int_{\mathbb{R}} T_h K - K \mathrm{d}x = 0$, we start by noting that for any $k \in \mathbb{R}$

$$(T_h - 1)(K * v) = [(T_h - 1)K] * (v - k).$$

Next, we introduce $\overline{v} = \operatorname{ess\,sup}_x v(x)$ and $\underline{v} = \operatorname{ess\,inf}_x v(x)$, and observe that

$$\|v-k\|_{L^\infty(\mathbb{R})} = \max\{\overline{v}-k, k-\underline{v}\},$$

and so setting $k = \frac{1}{2}(\overline{v}+\underline{v})$, we get $\|v-k\|_{L^\infty(\mathbb{R})} = \frac{1}{2}(\overline{v}-\underline{v}) = |v|_\infty$. By Young's convolution inequality and the above calculations we infer

$$\begin{aligned}\|(T_h-1)(K*v)\|_{L^\infty(\mathbb{R})} &\leq \|K(\cdot+h)-K\|_{L^1(\mathbb{R})}\|v-k\|_{L^\infty(\mathbb{R})}\\ &\leq |K|_{TV^s}|v|_\infty h^s.\end{aligned}$$

Thus, for any $h>0$ we have

$$(T_h-1)w = (T_h-1)v + \varepsilon(T_h-1)(K*v) \leq \omega(h) + \varepsilon|K|_{TV^s}|v|_\infty h^s,$$

where the last inequality holds pointwise for a.e. $x \in \mathbb{R}$. □

We conclude this subsection with an informal discussion to motivate the technicalities of Subsection 4.2. For initial data $u_0 \in L^2 \cap L^\infty(\mathbb{R})$ let u denote the corresponding entropy solution of (1.1). For a fixed $t>0$, suppose the function $\omega\colon [0,t]\times(0,\infty)\to(0,\infty)$ is such that for every $\tau\in[0,t]$ the function $\omega(\tau,\cdot)$ serves as a concave modulus of growth for $u(\tau)=u(\tau,\cdot)$. Seeking to extend the time domain of ω, we let $\Delta t>0$ denote an infinitesimal time step, and write $u(t+\Delta t) = S^K_{\Delta t}\circ S^B_{\Delta t}(u(t))$ which is informally justified by the previous section. Lemma 4.3 and 4.4 now suggests how to extend ω to $\omega(t+\Delta t,h)$, and combining the two lemmas we conclude for some fixed $s\in[0,1]$ that

$$\omega(t+\Delta t,h) = \frac{\omega(t,h)}{1+\Delta t\omega_h(t,h)} + \Delta t|K|_{TV^s}|u(t)|_\infty h^s, \tag{4.11}$$

serves as a modulus of growth for $u(t+\Delta t)$. Using (4.11) to extend ω has the disadvantage of requiring one to calculate $|u(t)|_\infty$. To overcome this difficulty, we replace $|u(t)|_\infty$ with the upper bound $m[\omega]$, dependent only on $\omega(t,\cdot)$ and $\|u_0\|_{L^2(\mathbb{R})}$, defined by

$$\tfrac{1}{2}\|u_0\|^2_{L^2(\mathbb{R})} = \int_0^{m[\omega]}\int_0^y \omega^{-1}(t,z)\mathrm{d}z\mathrm{d}y, \tag{4.12}$$

where $\omega^{-1}(t,\cdot)$ is the inverse of $\omega(t,\cdot)$. By (4.5) and $\|u(t)\|_{L^2(\mathbb{R})}\leq\|u_0\|_{L^2(\mathbb{R})}$ we indeed get $|u(t)|_\infty\leq m[\omega]$. Additionally, by Taylor expansion we have

$$\begin{aligned}\frac{\omega}{1+\Delta t\omega_h} &= \omega - \Delta t\omega_h\omega + O((\Delta t)^2)\\ &= \omega - \frac{\Delta t}{2}(\omega^2)_h + O((\Delta t)^2),\end{aligned}$$

and so subtracting $\omega(t,h)$ on each side of (4.11), dividing by Δt and replacing $|u(t)|_\infty$ with $m[\omega]$, we get

$$\frac{\Delta\omega}{\Delta t} = -\tfrac{1}{2}(\omega^2)_h + \kappa_s m[\omega]h^s + O(\Delta t), \tag{4.13}$$

where $\Delta\omega := \omega(t+\Delta t,h) - \omega(t,h)$ and $\kappa_s := |K|_{TV^s}$. In summary, equation (4.13) describes for an infinitesimal time step Δt a sufficiently large change $\Delta\omega$ such that $\omega + \Delta\omega$ serves as a modulus of growth for $u(t+\Delta t)$ whenever the same can be said for ω and $u(t)$. As t was general, one can expect from this discussion that if ω satisfies the partial differential equation

$$\begin{cases} \omega_t + \frac{1}{2}(\omega^2)_h = \kappa_s m[\omega]h^s, \\ \qquad \omega(0,h) = \omega_0(h), \end{cases} \tag{4.14}$$

where ω_0 is a modulus of growth for u_0, then $h \mapsto \omega(t,h)$ serves as a modulus of growth for $u(t)$ for all $t > 0$. Unfortunately, working directly with (4.14) is cumbersome due to the term $m[\omega]$, which can be viewed as a nonlinear and nonlocal operator in space applied to ω. Nevertheless, we can make the following observation: assume that a solution ω of (4.14) admits a limit $\lim_{t\to\infty}\omega(t,h) = \overline{\omega}(h)$, which in turn yields a limit $m[\omega] \to m[\overline{\omega}] =: \overline{m}$. Then (4.14) reduces to

$$\tfrac{1}{2}(\overline{\omega}^2)_h = \kappa_s\overline{m}h^s, \qquad \Longrightarrow \qquad \overline{\omega}(h) = \sqrt{\frac{2\kappa_s\overline{m}}{1+s}}\,h^{\frac{1+s}{2}},$$

where we assume $\overline{\omega}(0) = 0$. If one wanted, this expression for $\overline{\omega}$ could be used in (4.12) to calculate $\overline{m}$, thus also calculating the coefficient of $\overline{\omega}$ explicitly; the resulting expression would coincide with the limit of (2.3). As the 'limit modulus of growth' is of the form $h \mapsto ah^{\frac{1+s}{2}}$, one may hope that a solution of (4.14) is of the similar form $\omega(t,h) = a(t)h^{\frac{1+s}{2}}$. In Lemma 4.5 we show that $m[\omega] = c_0 a(t)^{\frac{1}{2+s}}$, for an appropriate $c_0 > 0$, whenever $\omega(t,h) = a(t)h^{\frac{1+s}{2}}$, and so seeking a solution of (4.14) of this special form, we insert for ω and $m[\omega]$ in (4.14) and get

$$\dot{a}h^{\frac{1+s}{2}} + \tfrac{(1+s)}{2}a^2h^s = \kappa_s c_0 a^{\frac{1}{2+s}}h^s.$$

Setting $c_1 := \kappa_s c_0$ and $c_2 := \frac{1+s}{2}$, we further divide each side by $h^{\frac{1+s}{2}}$ and rearrange to get

$$\dot{a} = \left[c_1 a^{\frac{1}{2+s}} - c_2 a^2\right] h^{\frac{s-1}{2}}. \tag{4.15}$$

Apart from the special case $s = 1$, the h-dependence on the right-hand side of (4.15) means that the only non-trivial solution $a(t)$ of this form is the constant one where the square bracket is zero. As we do not wish to impose regularity constraints on the initial data, this constant solution is not of value to us. Instead we make a second observation: for $a \gg 1$ the right-hand side of (4.15) is negative and thus increasing in h. This roughly suggests that if we relax our search to instead find a function $h \mapsto a(t)h^{\frac{1+s}{2}}$ serving as a modulus of growth for $u(t)$ when $h < H$, for some finite H, then H can replace h in (4.15) and we can solve thereafter. Conveniently, Lemma 4.5 demonstrates that there is an $H > 0$ such that if $h \mapsto a(t)h^{\frac{1+s}{2}}$ serves as a modulus of growth for $u(t)$ when $h \in (0, H)$ then the same holds for all $h > 0$. This H depends on $a(t)$ and is given by $H(a) = c_3 a^{-\frac{2}{2+s}}$ for an appropriate $c_3 > 0$. Replacing h with $H(a)$ in (4.15) gives

$$\dot{a} = \left[c_1 a^{\frac{1}{2+s}} - c_2 a^2\right] c_3^{\frac{s-1}{2}} a^{\frac{1-s}{2+s}}, \tag{4.16}$$

which indeed is the equation (4.37) that the Hölder coefficients constructed in the next subsection solve. In conclusion, this informal argument suggests that if u_0 admits the modulus of growth $h \mapsto a(0)h^{\frac{1+s}{2}}$, then the same can be said for $u(t)$ and $h \mapsto a(t)h^{\frac{1+s}{2}}$ where $a(t)$ solves (4.16) for $t > 0$. We stress that this discussion is only meant to coarsely summarize the idea behind the steps in the following subsection.

4.2. **Deriving a modulus of growth for entropy solutions.** In this subsection we consider $s \in [0, 1]$ fixed and assume that $|K|_{TV^s}$ is finite. Further, we let $\mu, \kappa_s \in (0, \infty)$ denote arbitrary fixed values, though we impose the requirement $\kappa_s \geq |K|_{TV^s}$ on the latter. The role of μ and κ_s will essentially be that of placeholders for the L^2 norm of the initial data and of $|K|_{TV^s}$ respectively, but note that μ and κ_s are strictly positive (even if the quantities they represent might be zero). This positivity is for technical convenience as some of the coming expressions would otherwise need a limit sense interpretation. Motivated by the previous discussion, we shall for an arbitrary entropy solution u of (1.1) with $L^2 \cap L^\infty$ data, seek an expression $a(t)$ such that $h \mapsto a(t)h^{\frac{1+s}{2}}$ serves as a modulus of growth (Def. 4.1) for $x \mapsto u(t, x)$. We begin with an important result, which among other things rephrases Lemma 4.2 for the more explicit case $\omega(h) = ah^{\frac{1+s}{2}}$. For this purpose, we introduce the constant

$$c_s = \left[\frac{(2+s)(3+s)}{2(1+s)^2}\right]^{\frac{1+s}{4+2s}}, \tag{4.17}$$

and the function

$$H(a) = \frac{(2c_s)^{\frac{2}{1+s}}\mu^{\frac{2}{2+s}}}{a^{\frac{2}{2+s}}}, \tag{4.18}$$

defined for all $a > 0$. We recall for the following lemma definition (4.3) of the seminorm $|\cdot|_\infty$.

Lemma 4.5. *With fixed $a > 0$, let $\omega(h) = ah^{\frac{1+s}{2}}$ for $h \in (0,\infty)$. Suppose $v \in L^2(\mathbb{R})$ satisfies $\|v\|_{L^2(\mathbb{R})} \leq \mu$ and admits ω as a modulus of growth for the restricted values $h \in (0, H(a))$. Then v admits ω as a modulus of growth for all $h \in (0,\infty)$ and moreover*

$$\|v\|_{L^\infty(\mathbb{R})} \leq 2^{\frac{1+s}{4+2s}} c_s \mu^{\frac{1+s}{2+s}} a^{\frac{1}{2+s}}, \tag{4.19}$$

$$|v|_\infty \leq c_s \mu^{\frac{1+s}{2+s}} a^{\frac{1}{2+s}}. \tag{4.20}$$

Proof. We begin by proving the two inequalities, so let us assume for now that v admits ω as a modulus of growth for all $h \in (0,\infty)$. Since $\omega^{-1}(y) = a^{-\frac{2}{1+s}} y^{\frac{2}{1+s}}$ the function F from (4.6) can here be written

$$F(y) = \left[\frac{2(1+s)^2}{(3+s)(4+2s)}\right] \frac{y^{\frac{4+2s}{1+s}}}{a^{\frac{2}{1+s}}} = \frac{1}{2}\left(\frac{y}{c_s a^{\frac{1}{2+s}}}\right)^{\frac{4+2s}{1+s}},$$

with inverse

$$F^{-1}(y) = 2^{\frac{1+s}{4+2s}} c_s a^{\frac{1}{2+s}} y^{\frac{1+s}{4+2s}}.$$

Combined with $\|v\|_{L^2(\mathbb{R})} \leq \mu$, (4.4) and (4.5) give $\|v\|_{L^\infty(\mathbb{R})} \leq F^{-1}(\mu^2)$ and $|v|_\infty \leq F^{-1}(\frac{1}{2}\mu^2)$, which coincides with (4.19) and (4.20) respectively. Next, assume we only know that v admits ω as a modulus of growth for $h \in (0, H(a))$. The steps in the proof of Lemma 4.2 can still be carried out if one lets the role of $\omega^{-1}(y) = a^{-\frac{2}{1+s}} y^{\frac{2}{1+s}}$ be taken by the truncated version

$$y \mapsto \min\left\{a^{-\frac{2}{1+s}} y^{\frac{2}{1+s}}, H(a)\right\},$$

to yield the inequalities $\|v\|_{L^\infty(\mathbb{R})} \leq \tilde{F}^{-1}(\mu^2)$ and $|v|_\infty \leq \tilde{F}^{-1}(\frac{1}{2}\mu^2)$ with

$$\tilde{F}(y) := 2\int_0^y \int_0^{y_1} \min\left\{a^{-\frac{2}{1+s}} y_2^{\frac{2}{1+s}}, H(a)\right\} \mathrm{d}y_2 \mathrm{d}y_1.$$

As $\tilde{F}$ is strictly increasing and agrees with F on $(0, aH(a)^{\frac{1+s}{2}})$, we necessarily have both $\tilde{F}^{-1}(\mu^2) = F^{-1}(\mu^2)$ and $\tilde{F}^{-1}(\frac{1}{2}\mu^2) = F^{-1}(\frac{1}{2}\mu^2)$ provided $F^{-1}(\mu^2) < aH(a)^{\frac{1+s}{2}}$. As $F^{-1}(\mu^2)$ is exactly the right-hand side of (4.19), we see that the latter inequality holds since

$$F^{-1}(\mu^2) = 2^{\frac{1+s}{4+2s}} c_s \mu^{\frac{1+s}{2+s}} a^{\frac{1}{2+s}} < 2c_s \mu^{\frac{1+s}{2+s}} a^{\frac{1}{2+s}} = aH(a)^{\frac{1+s}{2}}.$$

Thus, the bounds for $\|v\|_{L^\infty(\mathbb{R})}$ and $|v|_\infty$ attained now coincides again with (4.19) and (4.20). It then follows that v admits ω as a modulus of growth for all $h \in (0, \infty)$. Indeed, for any $h \in [H(a), \infty)$ we have the two trivial inequalities

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}} \Big[v(x+h) - v(x)\Big] \le 2|v|_\infty, \qquad aH(a)^{\frac{1+s}{2}} \le ah^{\frac{1+s}{2}},$$

and so we would be done if $2|v|_\infty \le aH(a)^{\frac{1+s}{2}}$, which is precisely the already established inequality (4.20) multiplied by two. □

The most essential part of the previous lemma, is in allowing us to extend the domain for which a homogeneous modulus of growth is valid. Its utility will become apparent in the proof of the next proposition which in short combines Lemma 4.3 and 4.4 to attain a corresponding result for the operator $S_\varepsilon^B \circ S_\varepsilon^K$. While it in Section 3 was natural to work with iterations of $S_\varepsilon^K \circ S_\varepsilon^B$, it will here be easier to work with its counterpart $S_\varepsilon^B \circ S_\varepsilon^K$. We now introduce the useful limit value $\underline{a}$ defined by

$$\underline{a} = \left(\frac{2c_s\kappa_s}{1+s}\right)^{\frac{2+s}{3+2s}} \mu^{\frac{1+s}{3+2s}}. \tag{4.21}$$

This quantity will naturally occur in our calculations to come and relate to the expression $a(t)$ we seek by $\lim_{t\to\infty} a(t) = \underline{a}$. In particular, it coincides with the first term on the right-hand side of (2.3) from Remark 2.4, though it is here expressed in different notation.

Proposition 4.6. *For every $A \in (\underline{a}, \infty)$, there are constants $C_A, \varepsilon_A > 0$ such that: if $v \in BV(\mathbb{R})$ satisfies $\|v\|_{L^2(\mathbb{R})} \le \mu$ and admits the modulus of growth $h \mapsto ah^{\frac{1+s}{2}}$ for some $a \in [\underline{a}, A]$, then for every $\varepsilon \in (0, \varepsilon_A]$ the function $w = S_\varepsilon^B \circ S_\varepsilon^K(v)$ admits the modulus of growth*

$$h \mapsto \Big(a - \varepsilon f(a) + \varepsilon^2 C_A\Big) h^{\frac{1+s}{2}}, \tag{4.22}$$

where $f(a) \ge 0$ is given by

$$f(a) = \left[\frac{(1+s)a^{\frac{2-s}{2+s}}}{2^{\frac{2}{1+s}} c_s^{\frac{1-s}{1+s}} \mu^{\frac{1-s}{2+s}}}\right]\left[a^{\frac{3+2s}{2+s}} - \underline{a}^{\frac{3+2s}{2+s}}\right]. \tag{4.23}$$

Proof. For fixed $A > \underline{a}$, let $v \in BV(\mathbb{R})$ and $a \in [\underline{a}, A]$ be as described in the lemma. We fix the pair v and a for convenience, but it should be clear from the proof that the construction of C_A and ε_A do not in fact depend on said pair. Introduce for $\varepsilon > 0$ the auxiliary function

$\tilde{v} = S_\varepsilon^K(v)$. Combining Lemma 4.4 and (4.20), $\tilde{v}$ admits the concave modulus of growth

$$\tilde{\omega}(h) = ah^{\frac{1+s}{2}} + \varepsilon c_s \kappa_s a^{\frac{1}{2+s}} \mu^{\frac{1+s}{2+s}} h^s,$$

where $|K|_{TV^s}$ was replaced by the larger κ_s introduced at the beginning of this subsection. And since $\tilde{v} \in BV(\mathbb{R})$, as follows from (3.27) and (3.28), we can further apply Lemma 4.3 to $w = S_\varepsilon^B(\tilde{v})$, which combined with $\tilde{\omega}'(h) > (\frac{1+s}{2})ah^{\frac{s-1}{2}}$, allows us to conclude that w admits the modulus of growth

$$\begin{aligned}
\omega(h) &= \frac{ah^{\frac{1+s}{2}} + \varepsilon c_s \kappa_s a^{\frac{1}{2+s}} \mu^{\frac{1+s}{2+s}} h^s}{1 + \varepsilon(\frac{1+s}{2})ah^{\frac{s-1}{2}}} \\
&= ah^{\frac{1+s}{2}} + \frac{-\varepsilon(\frac{1+s}{2})a^2 h^s + \varepsilon c_s \kappa_s a^{\frac{1}{2+s}} \mu^{\frac{1+s}{2+s}} h^s}{1 + \varepsilon(\frac{1+s}{2})ah^{\frac{s-1}{2}}} \\
&= ah^{\frac{1+s}{2}} - \varepsilon \underbrace{\left[\frac{(1+s)a^2 - 2c_s \kappa_s a^{\frac{1}{2+s}} \mu^{\frac{1+s}{2+s}}}{2h^{\frac{1-s}{2}} + \varepsilon(1+s)a}\right]}_{B(a,h,\varepsilon)} h^{\frac{1+s}{2}},
\end{aligned} \tag{4.24}$$

where $B(a,h,\varepsilon)$ denotes the square bracket. With $\underline{a}$ as given by (4.21), this square bracket can further be factored

$$B(a,h,\varepsilon) = \left[\frac{(1+s)a^{\frac{1}{2+s}}}{2h^{\frac{1-s}{2}} + \varepsilon(1+s)a}\right]\left[a^{\frac{3+2s}{2+s}} - \underline{a}^{\frac{3+2s}{2+s}}\right]. \tag{4.25}$$

Since $a \geq \underline{a}$ it follows that $B(a,h,\varepsilon)$ is non-negative and thus non-increasing in $h > 0$. Consequently, we read from (4.24) the inequality

$$\omega(h) \leq \Big(a - \varepsilon B(a,\overline{h},\varepsilon)\Big)h^{\frac{1+s}{2}}, \qquad 0 < h < \overline{h}. \tag{4.26}$$

Since (4.26) can be viewed as implying that w admits a homogeneous modulus of growth on bounded intervals, we would like to make use of Lemma 4.5; however, we do not necessarily have $\|w\|_{L^2(\mathbb{R})} \leq \mu$ (as is assumed by said lemma). We deal with this small inconvenience as follows: define $\tilde{w}$ by

$$\tilde{w} := \rho^{-1} w, \qquad \rho := \max\Big\{1, \mu^{-1}\|w\|_{L^2(\mathbb{R})}\Big\}, \tag{4.27}$$

that is, $\tilde{w}$ is the renormalized version of w if the L^2 norm of w exceeds μ. We proceed by proving the proposition for $\tilde{w}$ and then extend the result to w. Observe that ω must serve as a modulus of growth also for $\tilde{w}$ since

$\rho \geq 1$, and consequently by (4.26), $\tilde{w}$ further admits for any fixed $\overline{h} > 0$ the modulus of growth

$$h \mapsto \Big(a - \varepsilon B(a, \overline{h}, \varepsilon)\Big) h^{\frac{1+s}{2}}, \tag{4.28}$$

for the restricted values $h \in (0, \overline{h})$. Lemma 4.5 then tells us that $\tilde{w}$ must additionally admit (4.28) as a modulus of growth for all $h > 0$ provided

$$H\Big(a - \varepsilon B(a, \overline{h}, \varepsilon)\Big) \leq \overline{h}, \tag{4.29}$$

where the function H is as defined by (4.18). As H is a decreasing function while B is non-negative, all $\overline{h}$ satisfying (4.29) necessarily also satisfy $\overline{h} \geq H(a)$; we now show that we for small ε can pick such a $\overline{h}$ close to $H(a)$. To do so, we start by introducing the closed set of points (a, h, ε) defined by

$$S_A = [\underline{a}, A] \times [H(A), \infty) \times [0, \infty),$$

where we abuse notation slightly by reusing a as a dummy variable for referring to elements in $[\underline{a}, A]$ (although the original $a \in [\underline{a}, A]$ is fixed). From (4.25) we see that both $(a, h, \varepsilon) \mapsto B(a, h, \varepsilon)$ and its partial derivatives are bounded on the set S_A. We exploit the additional smoothness of B later; for now we need only $\|B\|_{L^\infty(S_A)} < \infty$. Pick $\varepsilon_A > 0$ such that

$$\varepsilon_A \|B\|_{L^\infty(S_A)} \leq \tfrac{1}{2}\underline{a},$$

and observe that the argument of H in (4.29) must then lie in $[\frac{1}{2}\underline{a}, A]$ for all $(a, \overline{h}, \varepsilon) \in [\underline{a}, A] \times [H(a), \infty) \times [0, \varepsilon_A] \subset S_A$. Moreover, as H is smooth on $[\frac{1}{2}\underline{a}, A]$ we conclude for any such triplet $(a, \overline{h}, \varepsilon)$ that

$$\begin{aligned} H\Big(a - \varepsilon B(a, \overline{h}, \varepsilon)\Big) &\leq H(a) + \varepsilon \|H'\|_{L^\infty([\frac{1}{2}\underline{a}, A])} \|B\|_{L^\infty(S_A)} \\ &=: H(a) + \varepsilon D_A. \end{aligned}$$

Thus, this calculation guarantees that the choice $\overline{h} = H(a) + \varepsilon D_A$ satisfies (4.29) for every $a \in [\underline{a}, A]$ and $\varepsilon \in (0, \varepsilon_A]$, and so substituting for $\overline{h}$ in (4.28), we conclude that $\tilde{w}$ admits the modulus of growth

$$h \mapsto \Big(a - \varepsilon B(a, H(a) + \varepsilon D_A, \varepsilon)\Big) h^{\frac{1+s}{2}}, \tag{4.30}$$

for all $h > 0$, provided $\varepsilon \in (0, \varepsilon_A]$ (we already assume $a \in [\underline{a}, A]$). Recalling that the partial derivatives of B are bounded on S_A, we can write

$$\begin{aligned} B(a, H(a) + \varepsilon D_A, \varepsilon) \geq{}& B(a, H(a), 0) \\ &- \varepsilon \Big[D_A \|\tfrac{\partial B}{\partial h}\|_{L^\infty(S_A)} + \|\tfrac{\partial B}{\partial \varepsilon}\|_{L^\infty(S_A)}\Big], \end{aligned} \tag{4.31}$$

and so letting C_A denote a constant no smaller than the square bracket in (4.31), we combine this inequality with (4.30) to further conclude that

$$h \mapsto \Big(a - \varepsilon B(a, H(a), 0) + \varepsilon^2 C_A\Big) h^{\frac{1+s}{2}}, \tag{4.32}$$

also serves as a modulus of growth for $\tilde{w}$, provided $\varepsilon \in (0, \varepsilon_A]$. Using the explicit expressions (4.25) and (4.18) one attains the identity $B(a, H(a), 0) = f(a)$, where f is as defined in (4.23), and so the proposition has been proved for the renormalized function $\tilde{w}$. It remains to extend the result to w; assume from here on out that $\varepsilon \in (0, \varepsilon_A]$. Introducing $\tilde{a} = (a - \varepsilon f(a) + \varepsilon^2 C_A)$ for brevity, it is clear from the relation $w = \rho\tilde{w}$, where ρ is as defined in (4.27), that w admits $h \mapsto \rho\tilde{a}h^{\frac{1+s}{2}}$ as a modulus of growth, as the same can be said for $\tilde{w}$ and $h \mapsto \tilde{a}h^{\frac{1+s}{2}}$. Moreover, by a similar and coarser calculation as in the proof of Lemma 3.3, we have $\|w\|_{L^2(\mathbb{R})} \leq (1 + \varepsilon^2\kappa^2)\|u\|_{L^2(\mathbb{R})}$ where $\kappa = \|K\|_{L^1(\mathbb{R})}$, and so $\rho \leq 1 + \varepsilon^2\kappa^2$. Thus

$$\rho\tilde{a} \leq (1 + \varepsilon^2\kappa^2)\tilde{a} = a - \varepsilon f(a) + \varepsilon^2[C_A + \kappa^2\tilde{a}] \leq a - \varepsilon f(a) + \varepsilon^2\tilde{C}_A,$$

where $\tilde{C}_A := [C_A + \kappa^2(A + \varepsilon_A^2 C_A)]$, and so this calculation shows that the proposition also holds for w after choosing a larger constant C_A. $\square$

Together with a few results from Section 3, the previous proposition equips us with all we need to construct moduli of growth for entropy solutions of (1.1). Roughly speaking, we can for small $\varepsilon > 0$ iterate Proposition 4.6 repeatedly to construct a modulus of growth for an approximate entropy solution (3.32), and further letting $\varepsilon \to 0$ this construction carries over to the entropy solution itself. To formalize, we shall introduce some notation and assume from here on that a pair of constants ε_A, C_A, as described by Proposition 4.6, has been chosen for each $A > \underline{a}$. Define the function

$$g_A^\varepsilon(a) := a - \varepsilon f(a) + \varepsilon^2 C_A, \tag{4.33}$$

which is parameterized over $A > \underline{a}$ and $\varepsilon \in (0, \varepsilon_A]$ and where

$$f(a) = \gamma a^{\frac{2-s}{2+s}}\Big(a^{\frac{3+2s}{2+s}} - \underline{a}^{\frac{3+2s}{2+s}}\Big), \qquad \gamma = \frac{1+s}{2^{\frac{2}{1+s}} c_s^{\frac{1-s}{1+s}} \mu^{\frac{1-s}{2+s}}}. \tag{4.34}$$

The function f in (4.34) is indeed the same as in (4.23), and so $g_A^\varepsilon(a)$ is the new Hölder coefficient provided by Proposition 4.6. In the coming proposition, we carry out the above sketched argument consisting in part of repeated iterations of Proposition 4.6, and consequently, we will encounter repeated compositions of g_A^ε. We point out two relevant facts about g_A^ε. First off, for any $A > \underline{a}$ and sufficiently small $\varepsilon > 0$, the function g_A^ε maps $[\underline{a}, A]$ to itself. To see this, note from (4.33) that $(g_A^\varepsilon)'$ is strictly positive on $[\underline{a}, A]$ for small $\varepsilon > 0$. Moreover, we have

$$g_A^\varepsilon(\underline{a}) = \underline{a}, \qquad g_A^\varepsilon(A) = A - \varepsilon f(A) + \varepsilon^2 C_A,$$

and since $f(A) > 0$, it is clear that $\varepsilon > 0$ can be made sufficiently small such that

$$\underline{a} = g_A^\varepsilon(\underline{a}) \leq g_A^\varepsilon(a) \leq g_A^\varepsilon(A) \leq A, \tag{4.35}$$

for all $a \in [\underline{a}, A]$. Our second fact, rigorously justified in the coming proposition, is that repeated compositions of g_A^ε applied to the starting value $a = A$ will, as $\varepsilon \to 0$, result in a smooth function $a_A \colon [0, \infty) \to (\underline{a}, A]$, implicitly defined by

$$t = \int_{a_A(t)}^{A} \frac{\mathrm{d}a}{f(a)}. \tag{4.36}$$

That (4.36) yields a unique value $a_A(t) \in (\underline{a}, A]$ for each $t \in [0, \infty)$ follows as the positive integrand has a non-integrable singularity at $a = \underline{a}$. Alternatively, the function a_A can be viewed as the solution of the differential equation

$$\begin{cases} a'(t) = -f(a(t)), & t > 0, \\ a(0) = A, \end{cases} \tag{4.37}$$

which coincides with the equation (4.16) from the discussion of the previous subsection. For the next proposition, we shall exploit the two constants

$$M_A = \max_{a \in [\underline{a}, A]} |f'(a)|, \qquad \tilde{M}_A = \max_{a \in [\underline{a}, A]} |f(a) f'(a)|, \tag{4.38}$$

both well defined as f is smooth on $\mathbb{R}^+$. Note that the latter serves as a bound on $(a_A)'' = f(a_A) f'(a_A)$, and so by Taylor expansion, we infer

$$|a_A(t + \varepsilon) - a_A(t) + \varepsilon f(a_A(t))| \leq \frac{\varepsilon^2}{2} \tilde{M}_A, \tag{4.39}$$

for all $t \geq 0$ and $\varepsilon \geq 0$.

Proposition 4.7. *Let u be an entropy solution of* (1.1)*, whose initial data $u_0 \in BV(\mathbb{R})$ satisfies $\|u_0\|_{L^2(\mathbb{R})} \leq \mu$ and admits a modulus of growth $h \mapsto Ah^{\frac{1+s}{2}}$ for some $A > \underline{a}$. Then for all $t > 0$, the function $x \mapsto u(t, x)$ admits the modulus of growth*

$$h \mapsto a_A(t) h^{\frac{1+s}{2}},$$

with a_A given by (4.36)*.*

Proof. Consider $t > 0$ fixed, and assume without loss of generality that $\|u_0\|_{L^2(\mathbb{R})} < \mu$; if the proposition holds in this case, it necessarily also holds in the case $\|u_0\|_{L^2(\mathbb{R})} \leq \mu$ as the implicit μ-dependence of $a_A(t)$ is a continuous one. Pick a large $n \in \mathbb{N}$, set $\varepsilon = \frac{t}{n}$ and consider the family of functions $u_n^k \in BV(\mathbb{R})$ defined inductively by

$$\begin{cases} u_n^0 = S_\varepsilon^B(u_0), \\ u_n^k = S_\varepsilon^B \circ S_\varepsilon^K(u_n^{k-1}), \quad k = 1, 2, \dots, n, \end{cases}$$

As u_0 admits $h \mapsto Ah^{\frac{1+s}{2}}$ as a modulus of growth, so does u_n^0 by Lemma 4.3. Observe also that each $u_n^k \in BV(\mathbb{R})$ as follows by induction and the properties of S_ε^B and S_ε^K listed at the very beginning in the proof of Proposition 3.2. Moreover, by similar reasoning as in the proof of Lemma 3.3, we have

$$\|u_n^k\|_{L^2(\mathbb{R})} \leq e^{\frac{k}{2}\varepsilon^2\kappa^2}\|u_0\|_{L^2(\mathbb{R})} \leq e^{\frac{t}{2n}\kappa^2}\|u_0\|_{L^2(\mathbb{R})}, \quad k = 0, 1, \dots, n,$$

where $\kappa = \|K\|_{L^1(\mathbb{R})}$. Since we have a strict inequality $\|u_0\|_{L^2(\mathbb{R})} < \mu$, we can assume n large enough such that $\|u_n^k\|_{L^2(\mathbb{R})} \leq \mu$ for every k. We define further the coefficients a_n^k inductively by

$$\begin{cases} a_n^0 = A, \\ a_n^k = g_A^\varepsilon(a_n^{k-1}), \quad k = 1, 2, \dots, n, \end{cases}$$

where g_A^ε is given by (4.33). We assume n large enough such that $\varepsilon = \frac{t}{n}$ is both less than $\varepsilon_A > 0$ and small enough such that g_A^ε maps $[\underline{a}, A]$ to itself (see the discussion leading up to (4.35)). In particular, each a_n^k is in $[\underline{a}, A]$. We may now apply Proposition 4.6 inductively to each pair (u_n^k, a_n^k), starting with (u_n^0, a_n^0). As u_n^0 admits $h \mapsto a_n^0 h^{\frac{1+s}{2}}$ as a modulus of growth, Proposition 4.6 infers the same relationship for the pair (u_n^1, a_n^1), and by repeating the argument, the same can be said for all pairs (u_n^k, a_n^k). Most importantly, u_n^n admits $h \mapsto a_n^n h^{\frac{1+s}{2}}$ as a modulus of growth. The proposition will now follow if we can, as $n \to \infty$, establish the limits

$$a_n^n \to a_A(t), \tag{4.40}$$

$$u_n^n \to u(t), \tag{4.41}$$

where $u(t) = u(t, \cdot)$ and the latter limit is taken in $L^1_{\mathrm{loc}}(\mathbb{R})$. Indeed, in this scenario we can let φ denote any non-negative smooth function of compact support that satisfies $\int_{\mathbb{R}} \varphi \mathrm{d}x = 1$ so to calculate for $h > 0$

$$\begin{aligned}\operatorname*{ess\,sup}_{x\in R}\Big[u(t,x+h)-u(t,x)\Big] &= \sup_{\varphi}\langle u(t,\cdot+h)-u(t,\cdot),\varphi\rangle \\ &= \sup_{\varphi}\lim_{n\to\infty}\langle u_n^n(\cdot+h)-u_n^n,\varphi\rangle \\ &\le \sup_{\varphi}\lim_{n\to\infty} a_n^n h^{\frac{1+s}{2}} \\ &= a_A(t)h^{\frac{1+s}{2}}.\end{aligned}\tag{4.42}$$

We first prove (4.40). Using the explicit form (4.33) of g_A^ε with $\varepsilon = \frac{t}{n}$, the constants (4.38) and the inequality (4.39) we can calculate for $k \geq 1$,

$$\begin{aligned}&\Big|a_n^k - a_A\Big(\tfrac{kt}{n}\Big)\Big| \\ &= \Big|g_A^\varepsilon\Big(a_n^{k-1}\Big) - a_A\Big(\tfrac{(k-1)t}{n}+\tfrac{t}{n}\Big)\Big| \\ &\le \Big|a_n^{k-1} - a_A\Big(\tfrac{(k-1)t}{n}\Big)\Big| + (\tfrac{t}{n})\Big|f\Big(a_n^{k-1}\Big) - f\Big(a_A\Big(\tfrac{(k-1)t}{n}\Big)\Big)\Big| \\ &\quad + (\tfrac{t}{n})^2\Big(C_A + \tfrac{1}{2}\tilde{M}_A\Big) \\ &\le \Big[1+(\tfrac{t}{n})M_A\Big]\Big|a_n^{k-1} - a_A\Big(\tfrac{(k-1)t}{n}\Big)\Big| + (\tfrac{t}{n})^2 D_A,\end{aligned}\tag{4.43}$$

with $D_A \coloneqq C_A + \frac{1}{2}\tilde{M}_A$. By repeated use of (4.43), and the fact that $a_n^0 = a_A(0) = A$, we conclude

$$|a_n^n - a_A(t)| \le (\tfrac{t}{n})^2 D_A \sum_{k=0}^{n-1}\Big[1+(\tfrac{t}{n})M_A\Big]^k \le \tfrac{1}{n}\Big[t^2 D_A e^{tM_A}\Big],$$

and thus (4.40) is established. To prove (4.41), we recall definition (3.26) of the approximate solution map $S_{\varepsilon,t}$ and observe the relation

$$u_n^n = S_\varepsilon^B \circ S_{\varepsilon,t}(u_0) =: S_\varepsilon^B(u^\varepsilon(t)),\tag{4.44}$$

where the definition of u^ε coincides with (3.32), although we now work with a particular u_0 and $\varepsilon = \frac{t}{n}$. As Proposition 3.5 ensures that $u^\varepsilon(t) \to u(t)$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ as $\varepsilon \to 0$, the same limit then carries over to u_n^n (as $n \to \infty$) by (4.44) and the time continuity of the map S_ε^B (3.29) together with the TV bound of u^ε provided by Proposition 3.2. With the two limits (4.40) and (4.41) established, the proof is complete. □

For a fixed $t > 0$, it is not hard to see from (4.36) that $A \mapsto a_A(t)$ is strictly increasing. In particular, each $a_A(t)$ is bounded above by the pointwise limit $b(t) \coloneqq \lim_{A\to\infty} a_A(t)$. This function $b\colon (0,\infty) \to (\underline{a},\infty)$

is implicitly given by

$$t = \frac{1}{\gamma \underline{a}^{\frac{3}{2+s}}} \int_{\frac{b(t)}{\underline{a}}}^{\infty} \frac{\mathrm{d}\xi}{\xi^{\frac{2-s}{2+s}}\left(\xi^{\frac{3+2s}{2+s}} - 1\right)}, \tag{4.45}$$

which can be read from (4.36) by letting $A \to \infty$ and performing the change of variables $a = \underline{a}\xi$. That $b(t) \in (\underline{a}, \infty)$ is well defined for $t > 0$ follows as the integrand in (4.45) is positive with an integrable tail at $\xi = \infty$ and a non-integrable singularity at $\xi = 1$. The utility of b is that it allows us to generalize the previous proposition to initial data $u_0 \in L^2 \cap L^\infty(\mathbb{R})$.

Corollary 4.8. *Let u be an entropy solution of* (1.1)*, whose initial data $u_0 \in L^2 \cap L^\infty(\mathbb{R})$ satisfies $\|u_0\|_{L^2(\mathbb{R})} \le \mu$. Then for all $t > 0$, the function $x \mapsto u(t,x)$ admits the modulus of growth*

$$h \mapsto b(t) h^{\frac{1+s}{2}},$$

where $b(t)$ is defined by (4.45)*.*

Proof. Consider $t > 0$ fixed. Let $(u^n)_{n\in\mathbb{N}}$ be a sequence of entropy solutions of (1.1) whose corresponding initial data $(u_0^n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ satisfies

$$\|u_0^n\|_{L^2(\mathbb{R})} \le \|u_0\|_{L^2(\mathbb{R})}, \qquad \|u_0^n\|_{L^\infty(\mathbb{R})} \le \|u_0\|_{L^\infty(\mathbb{R})},$$

and yields, in L^1_{loc} sense, the limit

$$u_0^n \to u_0, \quad n \to 0.$$

For a fixed $n \in \mathbb{N}$, we can apply Proposition 4.7 to conclude for a sufficiently large $A > 0$ that $u^n(t) = u^n(t,\cdot)$ admits the modulus of growth $h \mapsto a_A(t) h^{\frac{1+s}{2}}$, which in turn can be replaced by the upper bound $h \mapsto b(t) h^{\frac{1+s}{2}}$. This modulus of growth carries over to $u(t)$ by a calculation similar to (4.42), if we can show that $u^n(t) \to u(t)$ in $L^1_{\mathrm{loc}}(\mathbb{R})$. By the weighted L^1-contraction of Proposition 3.1 and the uniform L^∞ bound of $(u_0^n)_{n\in\mathbb{N}}$, this latter limit follows from the corresponding limit of the initial data. $\square$

We pause here to note that Theorem 2.3 follows.

Proof of Theorem 2.3. We start by proving the theorem for $u_0 \in L^2 \cap L^\infty(\mathbb{R})$ in which case the corresponding weak solution u provided by Corollary 2.2 is the entropy solution provided by Theorem 2.1. For any two positive constants $\mu \ge \|u_0\|_{L^2(\mathbb{R})}$ and $\kappa_s \ge |K|_{TV^s}$ where s is such that $|K|_{TV^s} < \infty$, all calculations of this subsection go through. In particular, Corollary 4.8 then implies for all $t > 0$ that $x \mapsto u(t,x)$ admits $h \mapsto b(t) h^{\frac{1+s}{2}}$ as a modulus of growth. By Lemma A.1, $x \mapsto u(t,x)$ then coincides a.e. with both a left-continuous function and a right-continuous function whenever $t > 0$; associating $u(t, \cdot)$ with either, the inequality $u(t, x+h) - u(t, x) \leq b(t)h^{\frac{1+s}{2}}$ holds for all $t, h > 0$ and $x \in \mathbb{R}$. Finally, the fact that $t \mapsto b(t)$ is decreasing can be read directly from (4.45), and so Theorem 2.3 has been proved for the case $u_0 \in L^2 \cap L^\infty(\mathbb{R})$. The height bound from Corollary 2.6 may now be proved for entropy solutions of (1.1) with $L^2 \cap L^\infty$ data, and so the calculations of Section 3.3 can be carried out resulting in the jointly continuous solution map $S\colon [0, \infty) \times L^2(\mathbb{R}) \to L^2(\mathbb{R})$. Exploiting the continuity of S, Theorem 2.3 holds for every $u_0 \in L^2(\mathbb{R})$ by a density argument. □

Next, we shall establish the content of Remark 2.4. The implicit description (4.45) of $b(t)$ makes its dependence on t somewhat convoluted; whether or not there exists a simple and explicit representation of $b(t)$, for general $s \in [0, 1]$, will not be pursued here. Instead, we provide an explicit bound $b(t) \leq a(t)$, which in the case $s = 0$ turns out to be an equality. The trick is to exploit the identity

$$-\frac{\mathrm{d}}{\mathrm{d}\xi}\left[c_1 \log\left(1 + \frac{c_2}{\xi^{c_3} - 1}\right)\right] = \frac{c_1 c_2 c_3 \xi^{c_3 - 1}}{(\xi^{c_3} - 1)^2 + c_2(\xi^{c_3} - 1)}, \tag{4.46}$$

and that the right-hand side can approximate the integrand in (4.45) from above, for the particular choice of parameters

$$c_1 = \frac{2+s}{3+2s}, \qquad c_2 = \frac{3+2s}{3}, \qquad c_3 = \frac{3}{2+s}. \tag{4.47}$$

It is worth mentioning that these parameters are chosen such that (4.46) preserves the tail and singularity of the integrand in (4.45). This is to make $a(t)$ and $b(t)$ behave qualitatively the same; see Subsection 4.3 for a more precise discussion.

Lemma 4.9. *We have for all $t > 0$ the pointwise bound $b(t) \leq a(t)$, where $b(t)$ is as in (4.45), while $a(t)$ is the quantity defined by*

$$a(t) = \underline{a}\left[1 + \frac{1 + \frac{2}{3}s}{e^{\tau t} - 1}\right]^{\frac{2+s}{3}}, \tag{4.48}$$

where the limit value $\underline{a}$ and the exponent τ are given by

$$\underline{a} = C_1(s)\kappa_s^{\frac{2+s}{3+2s}}\mu^{\frac{1+s}{3+2s}}, \qquad \tau = C_0(s)\kappa_s^{\frac{3}{3+2s}}\mu^{\frac{2s}{3+2s}}, \tag{4.49}$$

and where $C_1(s)$ is as it appears in (2.4) while $C_0(s)$ is given by (A.1) in the appendix. Moreover, for all $t > 0$, if $s = 0$ then the two expressions coincide $b(t) = a(t)$, while if $s \in (0, 1]$ we have a strict inequality $b(t) < a(t)$.

Proof. Note first that the above $\underline{a}$ is the same as the one used throughout this section; Lemma A.2 shows the equivalence between how it is originally defined (4.21) and (4.49). Next, the integral in (4.45) can be bounded by first observing that the integrand satisfies for all $\xi > 1$ the pointwise inequality

$$\frac{1}{\xi^{\frac{2-s}{2+s}}\left(\xi^{\frac{3+2s}{2+s}}-1\right)} \leq \frac{\xi^{\frac{1-s}{2+s}}}{\left(\xi^{\frac{3}{2+s}}-1\right)^2+(1+\frac{2}{3}s)\left(\xi^{\frac{3}{2+s}}-1\right)}. \tag{4.50}$$

To see this, one can multiply each side of (4.50) with the two denominators followed by some cleaning up to find that (4.50) is for $\xi > 1$ equivalent to

$$\xi^{\frac{3-2s}{2+s}} \leq \tfrac{2s}{3} + (\tfrac{3-2s}{3})\xi^{\frac{3}{2+s}}. \tag{4.51}$$

Setting $x = 1$, $y = \xi^{\frac{3-2s}{2+s}}$, $p = \frac{3}{2s}$ and $q = \frac{3}{3-2s}$, Young's inequality $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$ guarantees the validity of (4.51). Moreover, we observe for all $\xi > 1$ that (4.51) is an equality if $s = 0$ and a strict inequality otherwise; this, together with the calculations to come, justifies the last assertion of the lemma. Combining (4.45) with (4.50) we find

$$\begin{aligned}\gamma\underline{a}^{\frac{3}{2+s}}t &\leq \int_{\frac{b(t)}{\underline{a}}}^{\infty} \frac{\xi^{\frac{1-s}{2+s}}}{\left(\xi^{\frac{3}{2+s}}-1\right)^2+(1+\frac{2}{3}s)\left(\xi^{\frac{3}{2+s}}-1\right)}\mathrm{d}\xi \\ &= \left(\frac{2+s}{3+2s}\right)\log\left(1+\frac{1+\frac{2}{3}s}{(b(t)/\underline{a})^{\frac{3}{2+s}}-1}\right),\end{aligned}$$

where the integral was solved by the formula (4.46) with the specific parameters (4.47). Rearranging this inequality, we get (4.48), but with $\tau = \frac{3+2s}{2+s}\gamma\underline{a}^{\frac{3}{2+s}}$. Lemma A.2 shows that this expression for τ is equivalent with (4.49). $\square$

The expression (4.48) is the sharpest explicit Hölder coefficient – appropriate for use in Theorem 2.3 – that we give here, and it is a close approximation of $b(t)$ as pointed out in Subsection 4.3. The much simpler expression (2.3) follows directly from (4.48) if one use the following inequalities

$$\left(1+\frac{c_1}{e^{c_2}-1}\right)^{\frac{2+s}{3}} < \left(1+\frac{c_1}{c_2}\right)^{\frac{2+s}{3}} \leq 1+\left(\frac{c_2}{c_3}\right)^{\frac{2+s}{3}}, \tag{4.52}$$

valid for all $c_1, c_2 > 0$.

Corollary 4.10. *With u and $s \in [0,1]$ as in Theorem 2.3, one may set the Hölder coefficient in* (2.2) *to*

$$C_1(s)|K|_{TV^s}^{\frac{2+s}{3+2s}}\|u_0\|_{L^2(\mathbb{R})}^{\frac{1+s}{3+2s}} + C_2(s)\frac{\|u_0\|_{L^2(\mathbb{R})}^{\frac{1-s}{3}}}{t^{\frac{2+s}{3}}}, \tag{4.53}$$

where $C_1(s)$ and $C_2(s)$ are as the appear in (2.4).

Proof. As explained in the above given proof of Theorem 2.3, one can for any two positive constants $\mu \geq \|u_0\|_{L^2(\mathbb{R})}$ and $\kappa_s \geq |K|_{TV^s}$ use $b(t)$, defined in (4.45), as a valid Hölder coefficient in (2.2). By Lemma 4.9 one may then also use the larger coefficient $a(t)$, here defined by (4.48). Further exploiting (4.52) we find

$$a(t) < \underline{a} + \underline{a}\left(\frac{1+\frac{2}{3}s}{\tau t}\right)^{\frac{2+s}{3}} = C_1(s)\kappa_s^{\frac{2+s}{3+2s}}\mu^{\frac{1+s}{3+2s}} + C_2(s)\frac{\mu^{\frac{1-s}{3}}}{t^{\frac{2+s}{3}}}, \tag{4.54}$$

where the first term on the right hand side is the expression for $\underline{a}$ as given by (4.49) while the second term follows from the identity (A.5) of Lemma A.2. In conclusion, the right hand side of (4.54) serves as a valid Hölder coefficient for u in (2.2). That we may generally set $\mu = \|u_0\|_{L^2(\mathbb{R})}$ and $\kappa_s = |K|_{TV^s}$, even when either of the two are zero, now follows from a continuity argument. $\square$

4.3. **The error of the approximation.** We conclude the section with a short discussion regarding the approximation $a(t)$ of $b(t)$ from Lemma 4.9. One can think of $a(t)$ as a delayed version of $b(t)$; defining the delay $\varepsilon(t)$ through the relation $a(t+\varepsilon(t)) = b(t)$ it follows that $\varepsilon(t)$ satisfies

$$\varepsilon(t) = \frac{1}{\gamma\underline{a}^{\frac{3}{2+s}}}\int_{\frac{b(t)}{\underline{a}}}^{\infty} \frac{\frac{2s}{3} + (\frac{3-2s}{3})\xi^{\frac{3}{2+s}} - \xi^{\frac{3-2s}{2+s}}}{\xi^{\frac{2-s}{2+s}}\left(\xi^{\frac{3+2s}{2+s}}-1\right)\left(\xi^{\frac{3}{2+s}}-1\right)\left(\xi^{\frac{3}{2+s}}+\frac{2}{3}\right)}\mathrm{d}\xi. \tag{4.55}$$

This identity is attained by subtracting the implicit representation of $b(t)$ from that of $a(t+\varepsilon(t))$, that is, the integrand is exactly the difference between the right- and left-hand side of (3.37). From (4.55) we observe that $\varepsilon(t)$ is strictly increasing and bounded above by

$$\overline{\varepsilon} := \lim_{t\to\infty}\varepsilon(t) = \frac{1}{\gamma\underline{a}^{\frac{3}{2+s}}}\int_{1}^{\infty} \frac{\frac{2s}{3} + (\frac{3-2s}{3})\xi^{\frac{3}{2+s}} - \xi^{\frac{3-2s}{2+s}}}{\xi^{\frac{2-s}{2+s}}\left(\xi^{\frac{3+2s}{2+s}}-1\right)\left(\xi^{\frac{3}{2+s}}-1\right)\left(\xi^{\frac{3}{2+s}}+\frac{2}{3}\right)}\mathrm{d}\xi,$$

where the integral is finite as the integrand is bounded (the numerator has a second order zero at $\xi = 1$) and decays sufficiently fast. Moreover, as $b(t) \simeq t^{-\frac{2+s}{3}}$ for small t while the integrand satisfies $\simeq \xi^{-\frac{8+s}{2+s}}$ for large

ξ, we infer from (4.55) that $\varepsilon(t) \lesssim t^2$, and so combining this with the boundness of $\varepsilon(t)$ we get

$$\varepsilon(t) \lesssim \min\{t^2, 1\}.$$

In conclusion $a(t)$ and $b(t)$ behaves very similar for small t, and approach the same limit – at the same exponential rate – as $t \to \infty$ with $b(t)$ being at most a time-step $\overline{\varepsilon}$ ahead of $a(t)$, that is, $b(t) \in [a(t+\overline{\varepsilon}), a(t)]$.

Appendix A. Auxiliary results

In the coming lemma we work with the concept of a modulus of growth as defined by Def. 4.1.

Lemma A.1. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ admit a modulus of growth ω that satisfies $\omega(0+) = 0$. Then f admits essential left and right limits at each point $x \in \mathbb{R}$. In particular, there are functions f^- and f^+, respectively left- and right-continuous, that coincides a.e. with f.*

Proof. For any $x \in \mathbb{R}$ the existence of an essential left limit $f(x-)$ of f at x, follows from the calculation

$$\begin{aligned}
&\operatorname*{ess\,lim\,sup}_{\substack{y<0\\ y\to 0}} f(x+y) - \operatorname*{ess\,lim\,inf}_{\substack{y<0\\ y\to 0}} f(x+y)\\
&= \operatorname*{ess\,lim\,sup}_{\substack{y_2<y_1<0\\ y_2,y_1\to 0}} \Big[f(x+y_1) - f(x+y_2)\Big]\\
&\le \limsup_{\substack{y_2<y_1<0\\ y_2,y_1\to 0}} \omega(y_1-y_2) = 0.
\end{aligned}$$

By the Lebesgue differentiation theorem, the function $f^-(x) := f(x-)$ can only differ from f on a null set, and moreover, must be left continuous as the above calculation could be repeated for f^- with essential limits replaced by limits. A similar argument yields the existence of an essential right limit $f(x+)$ of f at each $x \in \mathbb{R}$ and further that $f^+(x) := f(x+)$ is a right-continuous function agreeing a.e. with f. ☐

The next lemma deals with quantities appearing throughout the paper and the relations between them. For convenience, we here list the definition of each relevant quantity; some of them given for the first time. The quantities c_s and γ were in (4.17) and (4.34) defined to be

$$c_s = \left[\frac{(2+s)(3+s)}{2(1+s)^2}\right]^{\frac{1+s}{2(2+s)}}, \qquad \gamma = \frac{1+s}{2^{\frac{2}{1+s}} c_s^{\frac{1-s}{1+s}} \mu^{\frac{1-s}{2+s}}}.$$

The coefficient $C_0(s)$ from Lemma 4.9 is defined here by

$$C_0(s) := \frac{2^{\frac{3-s}{3+2s}}(3+s)^{\frac{s}{3+2s}}(3+2s)}{2^{\frac{2}{1+s}}(2+s)^{\frac{3+s}{3+2s}}}. \tag{A.1}$$

The two coefficients $C_1(s)$ and $C_2(s)$ were in Remark (2.4) defined to be

$$C_1(s) = \frac{2^{\frac{3+s}{6+4s}}[(2+s)(3+s)]^{\frac{1+s}{6+4s}}}{1+s}, \quad C_2(s) = \frac{2^{\frac{4+2s}{3+3s}}(2+s)^{\frac{5+s}{6}}(3+s)^{\frac{1-s}{6}}}{2^{\frac{1-s}{6}}3^{\frac{2+s}{3}}(1+s)}.$$

Finally, the two coefficients $\tilde{C}_1(s)$ and $\tilde{C}_2(s)$ from Corollary 2.6 are defined here by

$$\begin{aligned} \tilde{C}_1(s) &:= \frac{2^{\frac{3+s}{(3+2s)(4+2s)}}[(2+s)(3+s)]^{\frac{1+s}{3+2s}}}{1+s}, \\ \tilde{C}_2(s) &:= \frac{2^{\frac{2}{3+3s}}(2+s)^{\frac{2}{3}}(3+s)^{\frac{1}{3}}}{2^{\frac{1-s}{12+6s}}3^{\frac{1}{3}}(1+s)}. \end{aligned} \tag{A.2}$$

In the coming lemma, we will also see the quantities μ and κ_s; these are simply placeholders for the expressions $\|u_0\|_{L^2(\mathbb{R})}$ and $|K|_{TV^s}$ respectively and will not affect the algebra in any non-trivial way.

Lemma A.2. *With $c_s, \gamma, C_0(s), C_1(s), C_2(s), \tilde{C}_1(s), \tilde{C}_2(s), \mu$ and κ_s as they appear above, we have the relations*

$$\left(\frac{2c_s\kappa_s}{1+s}\right)^{\frac{2+s}{3+2s}}\mu^{\frac{1+s}{3+2s}} =: \underline{a} = C_1(s)\kappa_s^{\frac{2+s}{3+2s}}\mu^{\frac{1+s}{3+2s}}, \tag{A.3}$$

$$\left(\frac{3+2s}{2+s}\right)\gamma\underline{a}^{\frac{3}{2+s}} =: \tau = C_0(s)\kappa_s^{\frac{3}{3+2s}}\mu^{\frac{2s}{3+2s}}, \tag{A.4}$$

$$\underline{a}\left(\frac{3+2s}{3\tau}\right)^{\frac{2+s}{3}} = C_2(s)\mu^{\frac{1-s}{3}}, \tag{A.5}$$

$$2^{\frac{1+s}{4+2s}}c_sC_1(s)^{\frac{1}{2+s}} = \tilde{C}_1(s), \tag{A.6}$$

$$2^{\frac{1+s}{4+2s}}c_sC_2(s)^{\frac{1}{2+s}} = \tilde{C}_2(s). \tag{A.7}$$

Proof. We start with (A.3): inserting for c_s on the left-hand side of (A.3) we get

$$\begin{aligned} &\left(\frac{2}{1+s}\right)^{\frac{2+s}{3+2s}}\left(\frac{(2+s)(3+s)}{2(1+s)^2}\right)^{\frac{1+s}{2(3+2s)}}\kappa_s^{\frac{2+s}{3+2s}}\mu^{\frac{1+s}{3+2s}} \\ &= \underbrace{\left[\frac{2^{\frac{3+s}{6+4s}}[(2+s)(3+s)]^{\frac{1+s}{6+4s}}}{1+s}\right]}_{C_1(s)}\kappa_s^{\frac{2+s}{3+2s}}\mu^{\frac{1+s}{3+2s}}, \end{aligned}$$

and so (A.3) is established. Next, we prove (A.4): on the left-hand side of (A.4) we substitute for $\underline{a}$ the left-hand side of (A.3) and insert for γ to attain

$$\begin{aligned}&\left(\frac{3+2s}{2+s}\right)\left(\frac{1+s}{2^{\frac{2}{1+s}}c_s^{\frac{1-s}{1+s}}\mu^{\frac{1-s}{2+s}}}\right)\left(\frac{2c_s\kappa_s}{1+s}\right)^{\frac{3}{3+2s}}\mu^{\frac{3+3s}{(2+s)(3+2s)}}\\ &=\left[\left(\frac{2^{\frac{3}{3+2s}}(1+s)^{\frac{2s}{3+2s}}(3+2s)}{2^{\frac{2}{1+s}}(2+s)}\right)c_s^{\frac{2s(2+s)}{(1+s)(3+2s)}}\right]\kappa_s^{\frac{3}{3+2s}}\mu^{\frac{2s}{3+2s}}.\end{aligned}$$

Inserting for c_s, this last square bracket can further be written

$$\begin{aligned}&\left(\frac{2^{\frac{3}{3+2s}}(1+s)^{\frac{2s}{3+2s}}(3+2s)}{2^{\frac{2}{1+s}}(2+s)}\right)\left(\frac{(2+s)(3+s)}{2(1+s)^2}\right)^{\frac{s}{3+2s}}\\ &=\underbrace{\left[\frac{2^{\frac{3-s}{3+2s}}(3+s)^{\frac{s}{3+2s}}(3+2s)}{2^{\frac{2}{1+s}}(2+s)^{\frac{3+s}{3+2s}}}\right]}_{C_0(s)},\end{aligned}$$

and so (A.4) is established. Next, we prove (A.5): if we on the left-hand side of (A.5) replace τ with the left-hand side of (A.4) we attain

$$\begin{aligned}\underline{a}\left(\frac{3+2s}{3}\right)^{\frac{2+s}{3}}\left[\frac{2+s}{(3+2s)\gamma\underline{a}^{\frac{3}{2+s}}}\right]^{\frac{2+s}{3}}&=\left(\frac{2+s}{3\gamma}\right)^{\frac{2+s}{3}}\\ &=\left(\frac{2^{\frac{2}{1+s}}(2+s)}{3(1+s)}\right)^{\frac{2+s}{3}}c_s^{\frac{(1-s)(2+s)}{3(1+s)}}\mu^{\frac{1-s}{3}},\end{aligned}$$

where the second equality follows from inserting for γ. Inserting for c_s in this last expression, we get

$$\begin{aligned}&\left(\frac{2^{\frac{2}{1+s}}(2+s)}{3(1+s)}\right)^{\frac{2+s}{3}}\left[\frac{(2+s)(3+s)}{2(1+s)^2}\right]^{\frac{1-s}{6}}\mu^{\frac{1-s}{3}}\\ &=\underbrace{\left[\frac{2^{\frac{4+2s}{3+3s}}(2+s)^{\frac{5+s}{6}}(3+s)^{\frac{1-s}{6}}}{2^{\frac{1-s}{6}}3^{\frac{2+s}{3}}(1+s)}\right]}_{C_2(s)}\mu^{\frac{1-s}{3}},\end{aligned}$$

and so (A.5) is established. Next, we prove (A.6): inserting for c_s and $C_1(s)$ on the left-hand side of (A.6) we get

$$2^{\frac{1+s}{2(2+s)}}\left(\frac{(2+s)(3+s)}{2(1+s)^2}\right)^{\frac{1+s}{2(2+s)}}\left(\frac{2^{\frac{3+s}{2(3+2s)}}[(2+s)(3+s)]^{\frac{1+s}{2(3+2s)}}}{1+s}\right)^{\frac{1}{2+s}}$$

$$= \underbrace{\left[\frac{2^{\frac{3+s}{(3+2s)(4+2s)}}\left[(2+s)(3+s)\right]^{\frac{1+s}{3+2s}}}{1+s}\right]}_{\tilde{C}_1(s)},$$

and so (A.6) is established. Finally, we prove (A.7): inserting for c_s and $C_2(s)$ on the left-hand side of (A.7) we get

$$2^{\frac{1+s}{2(2+s)}}\left(\frac{(2+s)(3+s)}{2(1+s)^2}\right)^{\frac{1+s}{2(2+s)}}\left(\frac{2^{\frac{2(2+s)}{3+3s}}(2+s)^{\frac{5+s}{6}}(3+s)^{\frac{1-s}{6}}}{2^{\frac{1-s}{6}}3^{\frac{2+s}{3}}(1+s)}\right)^{\frac{1}{2+s}}$$
$$= \underbrace{\left[\frac{2^{\frac{2}{3+3s}}(2+s)^{\frac{2}{3}}(3+s)^{\frac{1}{3}}}{2^{\frac{1-s}{12+6s}}3^{\frac{1}{3}}(1+s)}\right]}_{\tilde{C}_2(s)},$$

demonstrating the last equation (A.7). □

In the next lemma, we show that $W^{s,1}$ regularity is sufficient to bound the fractional variation (1.5). For this, we recall the Slobodeckij seminorm

$$[f]_{s,1} := \int_{\mathbb{R}^2} \frac{|f(x+y)-f(x)|}{|y|^{1+s}}\mathrm{d}x\mathrm{d}y,$$

associated with $W^{s,1}(\mathbb{R})$ for $s \in (0,1)$. Note however, the two seminorms are not equivalent as setting $f(x) = |x|^{s-1}$ with $s \in (0,1)$ yields $|f|_{TV^s} < \infty = [f]_{s,1}$.

Lemma A.3. *For $f \in W^{s,1}(\mathbb{R})$ with $s \in [0,1]$, we have the relations*

$$\begin{aligned} |f|_{TV^0} &= 2\|f\|_{L^1(\mathbb{R})}, \\ |f|_{TV^s} &\le C_s[f]_{s,1}, \quad s \in (0,1), \\ |f|_{TV^1} &= |f|_{TV(\mathbb{R})}, \end{aligned}$$

where the constant C_s only depends on $s \in (0,1)$.

Proof. The cases $s = 0$ and $s = 1$ are trivial and so we focus on $s \in (0,1)$. Set $\tau(y) := \|f(\cdot + y) - f\|_{L^1(\mathbb{R})}$ and observe that τ is sub-additive by the triangle inequality. For any $h > 0$ we find

$$\begin{aligned} 2[f]_{s,1} &= \int_{\mathbb{R}} \frac{\tau(y)}{|y|^{1+s}}\mathrm{d}y + \int_{\mathbb{R}} \frac{\tau(h-y)}{|h-y|^{1+s}}\mathrm{d}y \\ &\ge \int_{\mathbb{R}} \frac{\tau(y)+\tau(h-y)}{\max\{|y|,|h-y|\}^{1+s}}\mathrm{d}t \\ &\ge \tau(h)\int_{\mathbb{R}} \frac{1}{\max\{|y|,|h-y|\}^{1+s}}\mathrm{d}y \end{aligned}$$

$$
=\frac{\tau(h)}{h^s}\int_{\mathbb{R}}\frac{1}{\max\{|y|,|1-y|\}^{1+s}}\mathrm{d}y,
$$

and so taking the supremum over $h>0$ gives the result. $\square$

The Bessel kernels $G_\alpha \in L^1(\mathbb{R})$ are for general $\alpha > 0$ defined by the formula (1.2) if one evaluates the integral in a principle value sense. We list a few properties regarding these kernels that can be found in [2, Chap. 2.4.]; for all $\alpha, \beta > 0$ we have the two identities

$$
\|G_\alpha\|_{L^1(\mathbb{R})} = 1, \qquad G_{\alpha+\beta} = G_\alpha * G_\beta,
$$

and for $\alpha > 1$, the distributional derivatives $K_\alpha \coloneqq (G_\alpha)'$ satisfy

$$
\|K_\alpha\|_{L^1(\mathbb{R})} < \infty, \qquad |K_\alpha(x)| \lesssim_\alpha |x|^{\alpha-2}.
$$

Additionally, G_α is symmetric and completely monotone on $(0,\infty)$ when $\alpha \in (0,2]$ (see [5]). We use these properties in the coming lemma to bound the seminorm (1.5) when evaluated on K_α.

Lemma A.4. *For $\alpha > 1$ and $0 \le s \le \min\{\alpha-1, 1\}$ we have*

$$
|K_\alpha|_{TV^s} < \infty.
$$

Proof. First let $\alpha \in (1,2)$. As G_α is symmetric and completely monotone on $(0,\infty)$, K_α is positive on $(-\infty,0)$, negative on $(0,\infty)$ and strictly increasing on both. Thus we may calculate for $h>0$

$$
\|K_\alpha(\cdot+h)-K_\alpha\|_{L^1(\mathbb{R})} = 4\int_{-h}^{0} K_\alpha(x)\mathrm{d}x \le C_\alpha h^{\alpha-1},
$$

for some constant $C_\alpha < \infty$. For $s \in [0,\alpha-1]$ we then have

$$
\begin{aligned}
|K_\alpha|_{TV^s} =& \sup_{h>0}\frac{\|K_\alpha(\cdot+h)-K_\alpha\|_{L^1(\mathbb{R})}}{h^s}\\
\le&(2\|K_\alpha\|_{L^1(\mathbb{R})})^{\frac{1-s}{\alpha-1}}\sup_{h>0}\left(\frac{\|K_\alpha(\cdot+h)-K_\alpha\|_{L^1(\mathbb{R})}}{h^{\alpha-1}}\right)^{\frac{s}{\alpha-1}},
\end{aligned}
$$

where the last quantity is bounded by the above calculation. For $\alpha = 2$, we have $G_\alpha(x) = \frac{1}{2}e^{-|x|}$ and thus $K_2(x) = -\frac{1}{2}\mathrm{sgn}(x)e^{-|x|}$. This gives $|K_2|_{TV^1} = |K_2|_{TV} = 2$, which together with a similar interpolation argument as above implies the lemma for the $\alpha = 2$ case. Finally, for $\alpha > 2$ we can use the identity

$$
K_\alpha(\cdot+h)-K_\alpha = G_{\alpha-2} * (K_2(\cdot+h)-K_2),
$$

to conclude by Young's convolution inequality that $|K_\alpha|_{TV^s} \le |K_2|_{TV^s}$ for all $s \in [0,1]$. $\square$

Appendix B. Proof of Corollary 2.6 and Corollary 2.7

We prove Corollary 2.6 which provides a decaying L^∞ bound for the weak solutions of (1.1) provided by Corollary 2.2.

Proof of Corollary 2.6. We prove only (2.6) as (2.5) follows directly from the former when setting $s = 0$ and using $|K|_{TV^0} = 2\|K\|_{L^1(\mathbb{R})}$. With $s \in [0,1]$ such that $|K|_{TV^s} < \infty$, we have by Theorem 2.3 that $u(t)$ admits the modulus of growth (Def. 4.1) $h \mapsto a(t)h^{\frac{1+s}{2}}$, where we set $a(t)$ to be the the explicit expression (2.3) provided by Remark 2.4. The parameter $\mu > 0$ from Lemma 4.5 is arbitrary (see the beginning of Subsection 4.2) and so we may set it to $\mu = \|u_0\|_{L^2(\mathbb{R})}$. Using $\|u(t)\|_{L^2(\mathbb{R})} \le \|u_0\|_{L^2(\mathbb{R})}$, we infer from said lemma – more specifically (4.19) – that

$$\|u(t)\|_{L^\infty(\mathbb{R})} \le 2^{\frac{1+s}{2(2+s)}} c_s \|u_0\|_{L^2(\mathbb{R})}^{\frac{1+s}{2+s}} a(t)^{\frac{1}{2+s}},$$

for all $t > 0$. Using the sub-additivity of $y \mapsto |y|^{\frac{1}{2+s}}$ we infer that

$$a(t)^{\frac{1}{2+s}} \le C_1(s)^{\frac{1}{2+s}} |K|_{TV^s}^{\frac{1}{3+2s}} \|u_0\|_{L^2(\mathbb{R})}^{\frac{1+s}{(2+s)(3+2s)}} + C_2(s)^{\frac{1}{2+s}} \frac{\|u_0\|_{L^2(\mathbb{R})}^{\frac{1-s}{3(2+s)}}}{t^{\frac{1}{3}}},$$

and so inserting this in the above inequality we get

$$\begin{aligned}\|u(t)\|_{L^\infty(\mathbb{R})} \le{}& \left[2^{\frac{1+s}{2(2+s)}} c_s C_1(s)^{\frac{1}{2+s}}\right] |K|_{TV^s}^{\frac{1}{3+2s}} \|u_0\|_{L^2(\mathbb{R})}^{\frac{2+2s}{3+2s}} \\ &+ \left[2^{\frac{1+s}{2(2+s)}} c_s C_2(s)^{\frac{1}{2+s}}\right] \frac{\|u_0\|_{L^2(\mathbb{R})}^{\frac{2}{3}}}{t^{\frac{1}{3}}},\end{aligned}$$

for all $t > 0$. That these square brackets coincide with $\tilde{C}_1(s)$ and $\tilde{C}_2(s)$ is precisely the two identities (A.6) and (A.7) of Lemma A.2. □

Next, we prove Corollary 2.7 which established a maximal lifespan for classical solutions of (1.1) with $L^2 \cap L^\infty$ data.

Proof of Corollary 2.7. Consider $s \in [0,1]$ fixed for now, and assume $|K|_{TV^s} < \infty$. As (bounded) classical solutions are entropy solutions, we may associate $u \in L^\infty \cap C^1((0,T) \times \mathbb{R})$ with the global entropy solution admitting u_0 as initial data, provided by Theorem 2.1; the discussion following the proof of Proposition 3.1 justifies this viewpoint. Referring to this solution also as u, we have by (2.1) that $x \mapsto u(T,x)$ is a well defined element of $L^2 \cap L^\infty(\mathbb{R})$ approximated in L^2 sense by $u(t)$ as $t \nearrow T$. Setting $v(t,x) \coloneqq u(T-t,-x)$, we see through pointwise evaluation that v also is a classical solution of (1.1) (and thus an entropy solution) on

$(0,T)\times\mathbb{R}$ with initial data $v_0(x) := u(T,-x)$. From (2.1) we then infer $\|v_0\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}$ since

$$\|v_0\|_{L^2(\mathbb{R})} = \|u(T)\|_{L^2(\mathbb{R})} \le \|u_0\|_{L^2(\mathbb{R})} = \|v(T)\|_{L^2(\mathbb{R})} \le \|v_0\|_{L^2(\mathbb{R})}.$$

Using the identity $u_0(x) = v(T,-x)$ for a.e. $x\in\mathbb{R}$ and applying Theorem 2.3 to v we further find for all $h>0$ and a.e. $x\in\mathbb{R}$ that

$$u_0(x-h) - u_0(x) = v(T,-x+h) - v(T,-x) \le a(T)h^{\frac{1+s}{2}}, \tag{B.1}$$

where we for $a(T)$ use the following explicit expression from Remark 2.4

$$a(T) = C_1(s)|K|_{TV^s}^{\frac{2+s}{3+2s}}\|u_0\|_{L^2(\mathbb{R})}^{\frac{1+s}{3+2s}} + C_2(s)\frac{\|u_0\|_{L^2(\mathbb{R})}^{\frac{1-s}{3}}}{T^{\frac{2+s}{3}}} =: \underline{a} + \frac{q}{T^{\frac{2+s}{3}}}, \tag{B.2}$$

and where we have substituted $\|u_0\|_{L^2(\mathbb{R})}$ for $\|v_0\|_{L^2(\mathbb{R})}$ as the two quantities agree. Dividing each side of (B.1) by $h^{\frac{1+s}{2}}$ and taking the essential supremum with respect to $x\in\mathbb{R}$ we get

$$[u_0]_s := \operatorname*{ess\,sup}_{\substack{x\in\mathbb{R}\\ h>0}}\left[\frac{u_0(x-h)-u_0(x)}{h^{\frac{1+s}{2}}}\right] \le \underline{a} + \frac{q}{T^{\frac{2+s}{3}}}, \tag{B.3}$$

and if $[u_0]_s > \underline{a}$ then (B.3) can be rewritten as

$$T \le \left[\frac{q}{[u_0]_s - \underline{a}}\right]^{\frac{3}{2+s}} = \left(\frac{C_2(s)}{1-\frac{\underline{a}}{[u_0]_s}}\right)^{\frac{3}{2+s}}\frac{\|u_0\|_{L^2(\mathbb{R})}^{\frac{1-s}{2+s}}}{[u_0]_s^{\frac{3}{2+s}}} =: F\Big(\tfrac{\underline{a}}{[u_0]_s}\Big)\frac{\|u_0\|_{L^2(\mathbb{R})}^{\frac{1-s}{2+s}}}{[u_0]_s^{\frac{3}{2+s}}}, \tag{B.4}$$

where the first equality replaced q by its explicit expression as given by (B.2). We now show that this gives for any $\rho\in(0,1)$ the following implication

$$[u_0]_s^{3+2s} > \left(\frac{C_1(s)}{\rho}\right)^{3+2s}|K|_{TV^s}^{2+s}\|u_0\|_{L^2(\mathbb{R})}^{1+s}, \quad\implies\quad T\le F(\rho)\frac{\|u_0\|_{L^2(\mathbb{R})}^{\frac{1-s}{2+s}}}{[u_0]_s^{\frac{3}{2+s}}}. \tag{B.5}$$

Indeed, using the explicit expression (B.2) for $\underline{a}$ we see that the left-hand side of (B.5) is equivalent to $[u_0]_s > \underline{a}/\rho$ which, as $\rho\in(0,1)$, implies that $[u_0]_s > \underline{a}$ and so (B.4) holds. By observing that $\rho\mapsto F(\rho)$ is strictly increasing on $(0,1)$ and that $\rho > \underline{a}/[u_0]_s$ we see that the right-hand side of (B.5) then follows from (B.4). With (B.5) established, the corollary follows: for any $\rho\in(0,1)$ we get such universal constants c and C by setting

$$c = \sup_{s\in[0,1]} \left(\frac{C_1(s)}{\rho}\right)^{3+2s}, \quad C = \sup_{s\in[0,1]} F(\rho) = \sup_{s\in[0,1]} \left(\frac{C_2(s)}{1-\rho}\right)^{\frac{3}{2+s}}. \quad \text{(B.6)}$$

The free parameter ρ allows us to shrink one of the two constants at the cost of enlarging the other; in particular, c is at its smallest for $\rho \to 1$ while C is at its smallest for $\rho \to 0$. □

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