Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Natural Sciences
Department of Physics

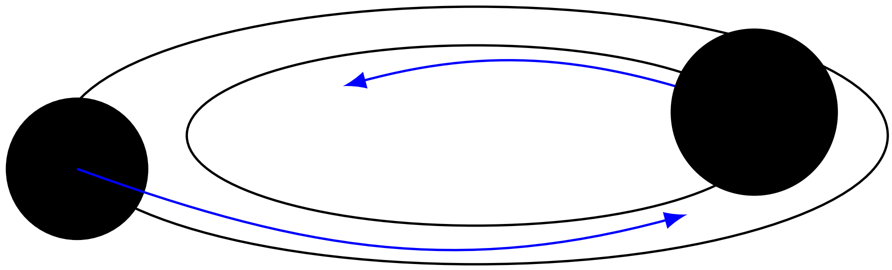
Vegard Undheim

First post-Newtonian correction to gravitational waves produced by compact binaries

How to compute relativistic corrections to gravitational waves using Feynman diagrams

Master's thesis in Physics
Supervisor: Alex Bentley Nielsen
Co-supervisor: Jens Oluf Andersen

June 2021



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Kunnskap for en bedre verden

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Abstract

The purpose of this thesis is to calculate the relativistic correction to the gravitational waves produced by compact binaries in the inspiral phase. The correction is up to the next to leading order, the so-called first post-Newtonian order (1PN), which are correctional terms proportional to $(v/c)^2$ compared to leading order, Newtonian, terms.

These corrections are well known in the literature, even going beyond the first order corrections, so why is it computed again here? In later years, an alternative approach for computing these terms using effective field theory has emerged. This thesis investigates this approach by replicating it, and attempts to make this approach more accessible to those not familiar with effective field theories.

It has been claimed that this approach greatly simplifies the complicated calculations of gravitational waveforms, and even provides the required intuition for 'physical understanding'. By this master student that was found not to be entirely correct. The calculations were made easier for those with a rich background in quantum field theory, but for those who are not well acquainted with quantum field theory this was not the case.

It was, however, found to be a worthwhile method as a means for deepening one's understanding of gravity, and might provide a shorter route for some alternative theories of gravity to testable predictions.

Sammendrag

Hensikten med denne oppgaven er å beregne den relativistiske korreksjonen til gravitasjonsbølger som er produsert av kompakte binærsystemer i spiral-fall fasen. Korreksjonene er av den såkalte første post-Newtonske orden (1PN), som er korreksjonstermer proporsjonal med $(v/c)^2$ sammenlignet med ledende, Newtonske, termerene.

Disse korreksjonene er velkjente i litteraturen, og går til og med utover korreksjonene av første orden, så hvorfor blir de beregnet igjen her? I nyere tid har en alternativ tilnærming for å beregne disse størrelsene ved hjelp av effektiv feltteori dukket opp. Denne oppgaven undersøker tilnærmingen ved å reprodusere dem, og prøver å gjøre metoden mer tilgjengelig for de som ikke er kjent med effektive feltteorier.

Det har blitt hevdet at beregningen av gravitasjonsbølgeformer kan gjøres mye enklere ved å bruke denne tilnærmingen, og til og med gir den nødvendige intuisjonen for 'fysisk forståelse'. Ifølge denne masterstudenten er ikke dette helt riktig. Beregningene ble gjort enklere for de med en spesialisert bakgrunn i kvantefeltteori, og for de som er mindre kjent med kvantefeltteori var dette ikke tilfelle.

Det ble imidlertid funnet å være en verdifull metode som et middel for å utdype forståelsen av tyngdekraften, og kan gi en kortere rute for noen alternative teorier for gravitasjon til testbare forutsigelser.

Acknowledgements

I would like to thank my supervisor Alex Bentley Nielsen for adhering to my wishes of working on gravitational waves, and as a consequence the numerous hours spent guiding me through this project. Of these hours, I am especially thankful for the time he spent discussing gravity, academia, and physics in general with me. I found these talks motivating and educational, and often the highlight of my week.

I would also like to thank the Department of Mathematics and Physics of the University of Stavanger. During the COVID-19 pandemic, the department made the necessary arrangements to let me come visit them, for which I am grateful. The possibility to spend time physically with my supervisor was much appreciated. They welcomed me with open arms, and I thoroughly enjoyed my stay. A special thanks to Germano Nardini for conversations, coffee, and a scoop of ice cream during my visits to Stavanger.

I also extend my thanks to my local supervisor, Jens Oluf Andersen, and NTNU for making the formal facilitations need to make this thesis. Especially for granting travel funds for me to visit Stavanger.

Lastly, I thank Michelle Angell for proofreading the last draft of this thesis. There may still linger some typos in this document, but had it not been for her, it would have been many more.

Contents

Figures

Acronyms

BH black hole. viii, 1, 36, 57

EFT effective field theory. i, 3, 4, 15, 67, 68

EH Einstein–Hilbert. viii, 37

EIH Einstein-Infeld-Hoffmann. 52

EoM equation of motion. 3, 7, 17–22, 24, 27, 29, 34, 35, 37, 50, 53, 75

FP Fierz-Pauli. 16, 18, 19, 27

gf gauge fixing term. 19, 27

GR general relativity. viii, 1, 3, 8, 15, 16, 20, 35, 38, 46

GW gravitational wave. i, iii, vi, viii, 1–4, 6–8, 12–16, 21, 25, 26, 29–31, 33, 34, 36, 38, 39, 55, 57, 58, 66–68

LHS left hand side. 11, 54

LIGO Laser Interferometer Gravitational-Wave Observatory. viii, 1, 12

NS neutron star. 1, 36, 57

PN post-Newtonian. i, 3, 7–12, 14, 15, 27, 33, 36, 40–50, 52–57, 60, 61, 65–67, 76

pp point particle. 16, 26–28, 33, 36, 38, 44, 49, 64

QED quantum electrodynamics. 36

QFT quantum field theory. i, 4, 15, 20, 37, 48, 67, 68

RHS right hand side. 11, 31, 38

SPA stationary phase approximation. 12–14

STF symmetric trace free. 58–65

TT transverse-traceless. 21, 25, 29, 30, 57, 59, 60

Glossary

black hole A region of space-time curved to the point that no matter or radiation can escape. Usually taken to be a gravitationally collapsed star. vii, viii, 1

quantum field theory The theory of fields endowed with quantum properties that can be used to describe forces and matter. i, vii, 4, 15, 68

Einstein's field equations The equations of motion resulting from the Einstein-Hilbert action which dictates the dynamics of space-time. Coupled to a matter source it reads $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$. viii, 2, 3, 15, 20

Einstein-Hilbert action Einstein-Hilbert action is the action which when extremized generates the Einstein's field equations, i.e. the action which governs general relativity $S_{\text{EH}} = \frac{c^4}{16\pi G}\int R\sqrt{-g}\,\mathrm{d}^4x$. viii, 16, 20, 35, 37, 38

field theorist Physicists using fields on a static background space-time to model physical effects like forces and particles. In this thesis especially those who use fields to model gravity. 3, 4, 6, 15, 20, 29, 35, 67

GW150914 The gravitational wave event which occurred 14/09/2015. The first GW event by LIGO [1]. 1, 12

quasi-stable circular orbit Approximating the inspiral as circular orbits with gradually falling radii. The change in radius is negligible unless viewed over several periods. 8, 13, 64, 67

relativist Physicists using a geometrical interpretation of gravity, following in the footsteps of Einstein. 3, 4, 6, 15, 26, 67, 68

Schwarzschild radius The Schwarzschild radius R_S is the radius associated with the event horizon of a non-rotating, static black hole. $R_S = \frac{2GM}{c^2}$. 36

two body problem Name of the physics problem of describing how a system consisting of two bodies (usually taken to be point particles) evolve in time, given they only interact with each other. For r^{-1} potentials the two body problem generally has the solution of conic sections [2]. 1, 74, 75

Chapter 1

Introduction

1.1 Binary inspirals and gravitational waves

One the 14$^{\text{th}}$ of September 2015 the world was shocked, ever so slightly. So slightly in fact that the only reason we know about it is thanks to the effort of the Laser Interferometer Gravitational-Wave Observatory (LIGO), who measured this faint strain in their detectors. After careful testing and retesting, LIGO published their results on the 11$^{\text{th}}$ of February 2016 [1]. They concluded that the event, called GW150914, was a gravitational wave (GW) produced by the merger of two black holes, and was the first directly detected gravitational wave event in human history.

With the announcement of the historic detection of GW150914 came promises of a new era of astronomy, now equipped with a brand new type of data to constrain astronomical theories. Popular science lectures and books were given and written, and at the height of this hype I started my bachelor's degree in physics. Fascinated by these mysterious waves I wanted to learn more about them, and when the time came to pick a topic for my master's thesis I requested to work on gravitational waves.

My supervisor and I decided to work on relativistic corrections to the binary inspiral, using field theoretical methods. To date, all confirmed GW events are thought to be produced by compact binaries. A compact object is a black hole (BH) or neutron star (NS), and a compact binary is a system consisting of two compact objects. When compact objects revolve around each other they produce so called gravitational waves which dissipate orbital energy from the system. As a result the compact objects fall toward each other, and in the end collide and merge together.

The problem with compact binaries is that they are too heavy and fall too close to each other to be adequately described by Newtons law of gravity. Although the two body problem has a general solution in Newtonian mechanics, there is no known equivalent solution for the two body problem in general relativity, only the one body problem. To combat this issue, researchers have followed one of two approaches.[1]

[1] Or tried to find the actual, analytical, solution.

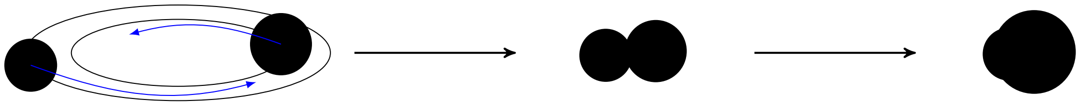


Figure 1.1: The evolution of compact binaries in three phases.

1. Solve the full, non-linear, Einstein's field equations numerically for the binary system in question.
2. Use an approximate, analytical, solution and perturbatively expand it to account for relativistic corrections.

This thesis will focus on analytical approximations. With numerical simulations one obtains a picture of the dynamics at an arguably very high accuracy, but due to the complexity of Einstein's field equations this is computationally costly, i.e. takes a lot of time and computing power. Furthermore, analytical expressions provide information about important quantities and intuition about the most important physical effects at play, that one simply does not gain from computer simulations.

In order to expand an analytical solution relativistically one first needs an approximate solution to expand. For this it is useful to divide the evolution of the compact binary into three phases, see Figure 1.1. The first phase is called the **inspiral phase**. Here the compact objects orbit each other at a distance, gradually falling closer together due to the emission of gravitational waves. Once the bodies are so close that a collision is imminent (typically when they 'touch' or form a common event horizon) the system becomes highly non-linear, and enters the so-called **merger phase**. After the two objects have merged into one, the system enters the **ringdown phase**, in which the system can be described as a one body problem, but with remnant asymmetries from the merger. Typically, the merged object's asymmetries oscillate around the Kerr solution and gradually dampen down, hence the name ringdown.

This is a useful division of the binary evolution as the different phases lend themselves to different approximations. The first phase, the inspiral, can be approximated as Keplerian orbits since the leading order term in the equations of motion is the Newtonian law of gravitation. The last phase can be approximated as a Schwarzschild or Kerr solution with perturbations. The merger phase is sandwiched between these two widely different approximations and is dominated by non-linear effects. Thus the merger phase has no good analytical approximation and must be simulated numerically.

In this thesis I will work with the analytical approximation of the inspiral phase.

1.2 Structure of this thesis

As we will see in Chapter 3 the frequency of the gravitational waves produced by compact binaries are directly dependent on the frequency at which the source oscillates. Therefore, the waveform of GWs measured here on Earth provides information about the dynamics of the binary which produced it, and can be compared with the predicted dynamics according to general relativity. This is why GW observation is a precise tool for constraining theories of gravity.

To motivate these computations, Chapter 2 starts off by computing the waveform, using results from following chapters. Then in Chapter 3 an alternative path to gravity is presented, that of a *gauge field theory* on a static space-time background. It is demonstrated to recover the main results of standard *linearized gravity*, which is the Einstein's field equations expanded to linear order in metric perturbations over flat space-time. In Chapter 4 and 5 the main results needed to compute the waveform in Chapter 2 are derived, using the effective field theory (EFT) based on the material presented in Chapter 3. Then the thesis ends with Chapter 6, which is concluding remarks on the effective field theory approach to gravitational waves.

This is a form of *top down* approach, starting with the final result (the waveform) and working back to the fundamental assumptions behind it. This structure has been chosen because of the large amount of labioursome calculations leading to the gravitational waveform, and it will hopefully provide the overview needed to understand the motivation for each calculation as it appears.

1.3 Why effective field theory?

In 2006, Goldberger and Rothstein [3] wrote a paper showing how the gravitational waveform could systematically be calculated to any post-Newtonian (PN) order using EFT formalism. Post-Newtonian expansion is ordering results like energy, the equation of motion, radiated energy flux, velocity, etc. as the Newtonian result plus relativistic corrections, usually expanded in factors of v/c.

E.g.

$$E = E_{\text{Newt}} \left[1 + \sum_{i=2}^{\infty} E_i \left(\frac{v}{c} \right)^i \right]. \tag{1.1}$$

Here $E_{\text{Newt}} \cdot E_i \left(\frac{v}{c}\right)^i$ would be the $\frac{i}{2}$PN term of the energy. This scaling as half the v/c power is chosen to represent the PN order such that the leading order correction is 1PN.

In this thesis, working with fields on a non-dynamic, flat, space-time will be referred to as field theory, or the approach of field theorists, like Goldberger. This is supposed to be contrary to traditional geometrical theories of gravity, in the spirit of Einstein, which will be referred to as the approach of relativists. By any normal definition however, general relativity and its interpretation by relativists, is a field theory. But they work with dynamical space-times, making it conceptually and mathematically quite differently formulated. Therefore, these constructed labels of field theorists and relativists will be employed in this thesis to emphasize the difference in approach.

Formulating the computations in the language of field theorists, Goldberger and Rothstein unlocked all tools, tricks, and language usually reserved for quantum field theory (QFT). Since then, this approach has been argued by field theorists to be easier and faster than the traditional relativist approach. One of these field theorists, R. Porto, has even claimed [4]

"[...] that adopting an EFT framework, when possible, greatly simplifies the computations and provides the required intuition for 'physical understanding'."

My supervisor, a self-proclaimed relativist, got curious, and wondered just how easy the effective field theory approach would make the computation. Therefore, he asked me if I would try to go through these computations, to test if they made the computation manageable even for master's students. My comments on Porto's claim are given in the discussion of Chapter 6.

With verifying or refuting Porto's claim as the ultimate goal of this thesis, it is mostly written as a relativist's guide to a field theorists' approach to gravitational waves. It should also be useful for those with a field theoretical background who wish to understand how Feynman diagrams can be used in classical gravity, and gravitational wave physics.

1.4 Notation

This thesis uses the mostly positive flat space-time metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{Flat metric}$$

Four-vectors are written with Greek letter indices, and spatial vectors with Latin letter indices. The Einstein summation convention applies.

$$x^\mu = \left(x^0 \ , \ \boldsymbol{x}\right) = \left(ct \ , \ \boldsymbol{x}\right) \qquad \text{Four-vector}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\partial_t \ , \ \nabla\right) \qquad \text{Four-gradienet}$$

$$\mathrm{d}^4 x = \mathrm{d}x^0\,\mathrm{d}^3 x = c\,\mathrm{d}t\,\mathrm{d}^3 x \qquad \text{Integration volume of space-time}$$

Notably, the action is defined as

$$S = \int \mathrm{d}t\, L = \int \frac{\mathrm{d}^4 x}{c}\mathcal{L},$$

with L and $\mathcal{L}$ being the Lagrangian, and Lagrangian density, respectively.

These tensor index notations are also used.

$T_{[\mu\nu]} \equiv \frac{1}{2!}\left(T_{\mu\nu} - T_{\nu\mu}\right) = A_{\mu\nu}$	Antisymmetrizing operation
$T_{\{\mu\nu\}} \equiv \frac{1}{2!}\left(T_{\mu\nu} + T_{\nu\mu}\right) = S_{\mu\nu}$	Symmetrizing operation
$T = T_{\alpha}{}^{\alpha} = \eta_{\alpha\beta}T^{\alpha\beta}$	Trace of tensor
$T_{\mu\nu,\alpha} = \partial_\alpha T_{\mu\nu}$	Partial derivative
$T_{\mu\nu,\alpha}{}^{\alpha} = \partial_\alpha \partial^\alpha T_{\mu\nu} = \Box T_{\mu\nu}$	d'Alembertian operator
$\bar{T}_{\mu\nu} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu} - T^{\sigma}{}_{\sigma}\eta_{\mu\nu}\right)$	Bar operator

Colons will also appear in indices, but these have no mathematical meaning. Colons are simply used to separate pairs of indices that have distinct roles. E.g. could $T^{\mu\nu}x^{\lambda} \equiv S^{\mu\nu:\lambda}$.

Lastly, the Fourier transform, and inverse Fourier transform are defined by[2]

$$
\begin{aligned}
F(x) &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \tilde{F}(k) e^{ik_\sigma x^\sigma}, \\
\tilde{F}(k) &= \int \mathrm{d}^4 x \, F(x) e^{-ik_\sigma x^\sigma}.
\end{aligned}
$$

[2]Note that for most of this thesis, the tilde over the Fourier transformed function will be dropped, as the argument (x or k) gives away whether it is a real-space or Fourier-space function.

Chapter 2

The gravitational waveform

In this chapter the gravitational waveform will be computed, both in the time domain (2.19) and in the frequency domain (2.26).

The computation follows standard methods, like presented in Arun *et al.* [5].

2.1 Setting up the equation for the gravitational waveform

2.1.1 What is a waveform?

As inferred by the name, gravitational waves are waves, which is to say they are solutions of the *wave equation*.

$$\left(-\frac{\partial^2}{\partial(ct)^2} + \nabla^2\right) h_{\mu\nu} = \partial_\alpha \partial^\alpha h_{\mu\nu} \equiv \Box\, h_{\mu\nu} = 0. \tag{2.1}$$

Here the *d'Alembert operator*, also called the *d'Alembertian*, $\Box$ has been defined, which is the operator of the wave equation.

A simple solution to equation (2.1) is $h_{\mu\nu} = \epsilon_{\mu\nu} e^{-ik_\sigma x^\sigma}$, with $k_\mu k^\mu = -k_0^2 + \boldsymbol{k}^2 = 0$, and where $\epsilon_{\mu\nu}$ is some x^μ-independent tensor structure. The exponential is a plane wave solution, according to Euler's formula (C.1).

Gravitational waves are rank two tensors, which means they have two indices and therefore $4 \times 4 = 16$ components. It is also symmetric in these two indices: $h_{\mu\nu} = h_{\nu\mu}$, which means that only 10 of these components are independent. The reason gravitational waves are rank two tensors follows in the relativists' approach because $h_{\mu\nu}$ is a perturbation of the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the flat space-time metric. In the field theorists' approach it is because gravity is the effect of a massless spin two field. $\epsilon_{\mu\nu}$ is the polarization tensor of GWs, and since it is a massless field it only has two independent polarizations. Gravitational waves are transverse, and thus $\epsilon_{ij}k^j = 0$, i.e. the amplitude direction given by the polarization is orthogonal to the direction of propagation $\boldsymbol{k}$.

To solve the wave equation, the wave four-vector had to be null-like. This implies further that the wave itself must travel at the speed of light, $v = \frac{\partial\omega}{\partial|\boldsymbol{k}|} = \frac{\partial c k_0}{\partial|\boldsymbol{k}|} = c$. This is also a consequence of $h_{\mu\nu}$ being a massless field.

Because of the linearity of the wave operator, any sum of such exponential (or trigonometric) terms will also be a solution of the wave equation. The most general solution is thus, a sum over all null-like wave-vectors k_μ, and an expression which also leaves $h_{\mu\nu}$ as a real function.[1]

$$h_{\mu\nu}(x^\alpha) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \cdot 2\omega_k} \left\{ a_{\mu\nu}(\boldsymbol{k}) e^{-ik_\sigma x^\sigma} + a^\dagger_{\mu\nu}(\boldsymbol{k}) e^{ik_\sigma x^\sigma} \right\}. \tag{2.2}$$

The expression above being real follows from the observation $h^\dagger_{\mu\nu}(x^\alpha) = h_{\mu\nu}(x^\alpha)$, which can only hold for real numbers. Here $\omega_k = |\boldsymbol{k}| = k_0$, which is to make the wave null-like, also known as 'on shell'. For a derivation of this solution, see Appendix A.

The coefficients $a_{\mu\nu}(\boldsymbol{k})$ are used to select particular solutions based on some initial condition, and are left to be determined.

The frequency of the wave turns out to be integer multiples of the frequency at which the source binary orbits, which will be demonstrated in Chapter 5. Thus, it can be approximated as

$$h_{ij}(t) \simeq \epsilon_{ij} \sum_{n=1}^{\infty} a_n(t) \cos(n\Phi(t)), \tag{2.3}$$

where $\Phi(t)$ is the phase of the source binary. The *waveform* describes what kind of wave it is. $a_n(t)$ can be found, but the most important factor for detection of gravitational waves is contained in $\Phi(t)$. The reason for this is that gravitational wave detectors receives faint signals with *amplitudes* close to the amplitude of noise. However, the *frequency* of GWs is different from the major noise factors, and can thus be extracted using Fourier analysis. Therefore, in the rest of this chapter, and much of the literature, the word waveform will be used interchangeably about the phase, as it encodes information about the frequency spectrum.

The orbital energy for circular, Newtonian motion is related to the frequency as $E = -\frac{1}{2}\mu v^2 = -\frac{1}{2}\mu(GM\omega)^{2/3}$, using $v = \omega r$ and Kepler's third law,

$$\omega^2 = \frac{GM}{r^3}, \tag{2.4}$$

to eliminate r in favour of ω.[2]

The approximation of circular motion here might seem over idealized, but it turns out that the effect of gravitational wave emission on elliptical orbits is to *circularize* them.By the time the binary's frequency enters the detector range, near the time of coalescence, the orbits have become very circular, making circular orbits a sensible approximation.

Noting that the energy was easier to handle with v rather than ω, as it has integer powers instead of fractional powers, one may use $v = (GM\omega)^{1/3}$ as a proxy variable for the frequency. Note that as a Newtonian approximation this variable coincides with the

[1] The exponential function with an imaginary argument is a great shorthand for trigonometric functions, but all observables must in the end be real valued.

[2] How v, ω, and r are related follows from the EoM, which are presented in their 1PN form in (4.57)-(4.59).

relative velocity parameter, but this is no longer the case after relativistic corrections are accounted for.

Then the phase of the orbit can be expressed as

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \omega = \frac{v^3}{GM} \quad \Rightarrow \quad \mathrm{d}\Phi = \frac{v^3}{GM}\,\mathrm{d}t\,. \tag{2.5}$$

Sadly $v = v(t)$, which at this point is still an unknown function of time. However it is known that v must evolve with time according to how the orbital energy evolves with time.

2.1.2 Time evolution of orbital energy

The differential equation governing the dynamics of the orbital phase is

$$-\frac{\mathrm{d}E}{\mathrm{d}t} = \mathcal{F}, \tag{2.6}$$

with E the energy associated with conserved orbital motion, and $\mathcal{F}$ the total energy flux out of the system by means of GWs. This is nothing but energy conservation for a gravitationally bound system.[3]

Both E and $\mathcal{F}$ can be analytically expanded in a relativistic parameter, like (v/c). This requires a separation in scale, where on the short timescale the motion is conservative and has energy E, while on the long timescale the system loses energy to gravitational radiation at a rate $\mathcal{F}$, leading to an inspiral. This requires the inspiral to happen slowly compared to the orbital motion, so that at any one moment the motion can still adequately be described by Newtonian motion. Thus, it only works for relatively small values of $\mathcal{F}$, such that the objects do not fall down too rapidly.[4]

Luckily, to leading order the flux term is suppressed by a factor of c^{-5} compared to the leading order term of the energy. Thus, the approximation of so called quasi-stable circular orbits and post-Newtonian formalism holds surprisingly well, even when compared to numerical simulations of the full Einstein equations (see Borhanian *et al.* [8]).

As will be demonstrated in Chapter 4 and 5, the orbital energy (4.63) and energy flux (5.36) can be expanded in terms of (v/c) as

---

[3] It is not obvious that energy *should* be conserved however. In full GR there is no trivial argument why there should be a conserved energy quantity [6], but in the post-Newtonian expansion the dynamics are expanded around the Newtonian problem, in which energy is conserved. Thus it it can be taken to be an artifact of the Newtonian background of which the solution is expanded in. Note however that energy conservation is not controversy free [7].

[4] Later in this chapter it will be shown that the requirement of slow infall can be fufilled by having $\dot{\omega}/\omega^2 \ll 1$ (see equation (2.21)), which is equivalent to having the orbital velocity ωr much greater than infall velocity $\dot{r}$.

$$
\begin{aligned}
E &= E_{\text{Newt}} v^2 \left\{ 1 + \sum_{i=2}^{\infty} E_i \left(\frac{v}{c}\right)^i \right\} \\
&= -\frac{\mu}{2} v^2 \left\{ 1 + \left(-\frac{3}{4} - \frac{1}{12}\eta \right) \frac{v^2}{c^2} + \mathcal{O}\left(\frac{v^3}{c^3}\right) \right\},
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
\mathcal{F} &= F_{\text{Newt}} v^{10} \left\{ 1 + \sum_{i=2}^{\infty} F_i \left(\frac{v}{c}\right)^i \right\} \\
&= \frac{32}{5} \frac{\eta^2}{Gc^5} v^{10} \left\{ 1 + \left(-\frac{1247}{336} - \frac{35}{12}\eta \right) \frac{v^2}{c^2} + \mathcal{O}\left(\frac{v^3}{c^3}\right) \right\}.
\end{aligned}
\tag{2.8}
$$

Since v is just a proxy for the frequency the expression (2.7)-(2.8) would be different expressed in terms of the *actual* centre-of-mass frame relative velocity.

Here μ is the reduced mass and η is the symmetric mass ratio, see Appendix B (and specifically equation (B.5)) for their definition and the motivation for introducing such mass terms.

Up to $(v/c)^2$ order corrections define the first post-Newtonian order, or 1PN for short, and is the leading order correction. This has started the convention of calling terms $\sim (v/c)^{2i}$ for iPN order corrections, e.g. the leading order, Newtonian, term is 0PN order. This has a somewhat awkward effect, since not all terms are even powers of v/c, already the next order correction is $\sim (v/c)^3$, and is thus of 1.5PN order. Higher order terms of both the energy and flux, and the final result of this chapter: The waveform, can be found in papers like Arun *et al.* [5].

Using equation (2.6) the time evolution dt can be expressed in terms of v as

$$
\boxed{\mathrm{d}t = -\frac{1}{\mathcal{F}}\,\mathrm{d}E = -\frac{1}{\mathcal{F}}\frac{\mathrm{d}E}{\mathrm{d}v}\,\mathrm{d}v\,.}
\tag{2.9}
$$

Substituting (2.9) for dt in (2.5) results in the final expression for which the waveform can be derived (using (2.7)-(2.8))

$$
\boxed{\mathrm{d}\Phi = -\frac{v^3}{GM}\frac{1}{\mathcal{F}}\frac{\mathrm{d}E}{\mathrm{d}v}\,\mathrm{d}v\,.}
\tag{2.10}
$$

Solving (2.9) will provide v as a function of time. We proceed however by computing Φ as a function of v directly rather than of time, as ultimately to be compared with experiments it is the waveform in the frequency domain (which will be called Ψ) which is needed. As already mentioned, this is because the signal is filtered in the frequency domain, and therefore the highest resolution is in the frequency spectrum.

2.2 Computing the waveform

2.2.1 Computing the waveform as a function of time

In order to compute $\Phi(t)$ it is convenient to first compute $\Phi(v)$ (equation (2.14)), then $v(t)$ (equation (2.17)), and lastly $\Phi(t) = \Phi(v(t))$ (equation (2.19)).

Computing the waveform as a function of frequency

Combining (2.10) with (2.7)-(2.8) yield up to 1PN

$$\begin{aligned}
\mathrm{d}\Phi &= -\frac{v^3}{GM}F_{\mathrm{Newt}}^{-1}v^{-10}\left[1+\left(-\frac{1247}{336}-\frac{35}{12}\eta\right)\frac{v^2}{c^2}\right]^{-1}\frac{\mathrm{d}}{\mathrm{d}v}\left[E_{\mathrm{Newt}}v^2\left(1+\left(-\frac{3}{4}-\frac{1}{12}\eta\right)\frac{v^2}{c^2}\right)\right]\mathrm{d}v \\
&= \frac{-2}{GM}\frac{E_{\mathrm{Newt}}}{F_{\mathrm{Newt}}}\frac{1}{v^6}\frac{1+\left(-\frac{3}{2}-\frac{1}{6}\eta\right)v^2/c^2}{1+\left(-\frac{1247}{336}-\frac{35}{12}\eta\right)v^2/c^2}\mathrm{d}v \equiv \frac{-2}{GM}\frac{E_{\mathrm{Newt}}}{F_{\mathrm{Newt}}}\frac{1}{v^6}\frac{1+\alpha v^2/c^2}{1+\beta v^2/c^2}\mathrm{d}v\,. \qquad (2.11)
\end{aligned}$$

To evaluate this integral it would be advantageous to write the last fraction in an easier form. Utilising that v/c is small the last fraction can be Taylor expanded around $v/c = 0$ up to 1PN.

Preforming the Taylor expansion results in

$$\frac{1+\alpha x}{1+\beta x} \overset{\text{for } x\sim 0}{\simeq} 1+(\alpha-\beta)x+\beta(\beta-\alpha)x^2+\ldots$$

This result inserted in (2.11) yields the easily integratable 1PN expression

$$\mathrm{d}\Phi = -\frac{2}{GM}\frac{E_{\mathrm{Newt}}}{F_{\mathrm{Newt}}}\frac{1}{v^6}\left\{1+\left(\frac{743}{336}+\frac{11}{4}\eta\right)\frac{v^2}{c^2}\right\}\mathrm{d}v\,. \qquad (2.12)$$

Integrating to obtain $\Phi(t) = \Phi(v(t))$ one must choose a reference point in time, usually referred to as t_0. For binary inspirals this reference point is canonically chosen to be the moment of coalescence t_c (see Maggiore [9] chapter 4), which for the duration of the inspiral is in the future. Therefore, the integration variables should go from $v(t)$ to $v_c = v(t_c)$, but a multiplication of -1 to both sides can flip this order. Performing the integral finally provides $\Phi(v)$

$$\begin{aligned}
\Phi(v) &= \Phi_c - \frac{2E_{\mathrm{Newt}}}{GMF_{\mathrm{Newt}}}\int_{v_c}^{v} v'^{-6}\left\{1+\left(\frac{743}{336}+\frac{11}{4}\eta\right)\frac{v'^2}{c^2}\right\}\mathrm{d}v' \\
&= \Phi_c + \frac{2E_{\mathrm{Newt}}}{GMF_{\mathrm{Newt}}}\left[\frac{1}{5}v'^{-5}\left\{1+\frac{5}{3}\left(\frac{743}{336}+\frac{11}{4}\eta\right)\frac{v'^2}{c^2}\right\}\right]_{v'=v_c}^{v'=v}
\end{aligned} \qquad (2.13)$$

Collecting all constant terms into one phase constant Φ_0, writing out E_{Newt} and F_{Newt} from (2.7) and (2.8) respectively, results in the final result for the waveform as a function of v

$$\boxed{\Phi(v) = \Phi_0 - \frac{1}{2^5\eta}\frac{c^5}{v^5}\left\{1+\left(\frac{3715}{1008}+\frac{55}{12}\eta\right)\frac{v^2}{c^2}+\mathcal{O}\left(\frac{v^3}{c^3}\right)\right\}.} \qquad (2.14)$$

The phase is dimensionless, as one should expect.[5] To obtain the waveform as a direct function of time the frequency parameter v must be given as a function of time.

[5] By definition the frequency measure $v = (GM\omega)^{1/3}$ has dimension of velocity, in accordance with the symbol used.

Computing the frequency as a function of time

The frequency parameter v as a function of time can be obtained from the differential equation (2.9), in an equivalent fashion to how (2.14) was derived.

$$\mathrm{d}t = -\frac{1}{\mathcal{F}}\frac{\mathrm{d}E}{\mathrm{d}v}\,\mathrm{d}v = \frac{GM}{v^3}\,\mathrm{d}\Phi \overset{(2.12)}{=} -\frac{2E_{\mathrm{Newt}}}{F_{\mathrm{Newt}}}v^{-9}\left\{1+\left(\frac{743}{336}+\frac{11}{4}\eta\right)\frac{v^2}{c^2}\right\}\mathrm{d}v\,, \tag{2.15}$$

$$\Rightarrow \qquad t_c - t = \frac{5}{2^8}\frac{GMc^5}{\eta}v^{-8}\left\{1+\frac{8}{6}\left(\frac{743}{336}+\frac{11}{4}\eta\right)\frac{v^2}{c^2}\right\}. \tag{2.16}$$

Notice that $t_c - t$ was chosen for the left hand side (LHS) such that the expression on the right hand side (RHS) becomes strictly positive. This is desired because both sides must be raised to the negative one 4^{th} power, in order to produce a quadratic equation of v^2. Taking the square root of the resulting solution for v^2, and Taylor expanding it to the 1PN order yields the expression for $v(t)$.

Following the aforementioned procedure, and using $\tau = t_c - t$, the frequency can be determined to be

$$\boxed{v(\tau) = \frac{c}{2}\left(\frac{5GM}{c^3\eta}\right)^{1/8}\tau^{-1/8}\left\{1+\left(\frac{743}{8064}+\frac{11}{96}\eta\right)\left(\frac{5GM}{c^3\eta}\right)^{1/4}\tau^{-1/4}\right\}.} \tag{2.17}$$

By the definition of v, the actual frequency $\omega(\tau) = v^3/GM$ can be computed as well, for completeness.

$$\omega(\tau) = \frac{5}{8}\left(\frac{5G\mathcal{M}}{c^3}\right)^{-5/8}\tau^{-3/8}\left\{1+\frac{8}{5}\left(\frac{3715}{8064}+\frac{55}{96}\eta\right)\frac{3}{8}\left(\frac{5GM}{c^3\eta}\right)^{1/4}\tau^{-1/4}\right\}. \tag{2.18}$$

This result can be compared with e.g., Maggiore [9] (equation (5.258) on p. 295). Note that he, and most of the rest of the literature, use dimensionless variables[6], but the 1PN part of the expression is equivalent to (2.17) and (2.18).

Computing the waveform as a function of time

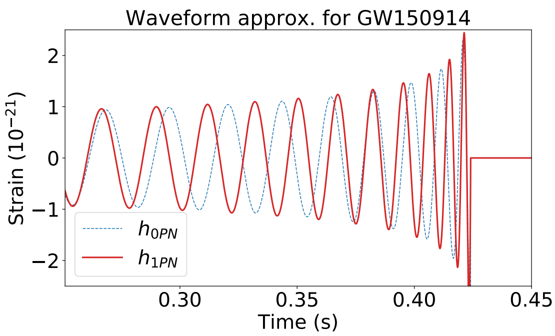
Substituting (2.17) for v in (2.14) yields

$$\boxed{\Phi(\tau) = \Phi_0 - \left(\frac{5G\mathcal{M}}{c^3}\right)^{-5/8}\tau^{5/8}\left\{1+\left(\frac{3715}{8064}+\frac{55}{96}\eta\right)\left(\frac{5GM}{c^3\eta}\right)^{1/4}\tau^{-1/4}\right\}.} \tag{2.19}$$

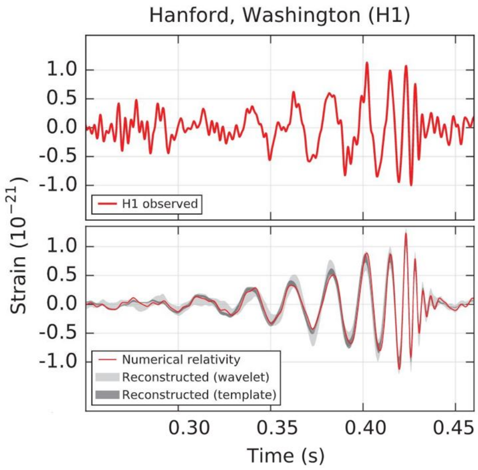
All that remains now to obtain the waveform is to find the amplitude of the different harmonics, and multiply them by $\cos(n\Phi(\tau))$. The result can be seen in Figure 2.1a.

In Figure 2.1a it is clear that 1PN corrections does not affect the amplitude much *directly*, but it has significant effect on the time evolution of the phase, and hence the frequency spectrum. It is however noticeable that the phase of Figure 2.1a does not match up with Figure 2.1b. Either higher order corrections are required, or the model breaks down for such low values of τ.

[6] These are commonly denoted $x = v^2/c^2 = (GM\omega/c^3)^{2/3}$, $\gamma = GM/rc^2$, and $\Theta = (5GM/\eta c^3)^{-1}(t_c - t)$. Preforming the substitutions for (2.17) should be straightforward.



(a) Waveform including (red) and excluding (blue) first order corrections. The waveform was plotted for values matching those of Table 1 of Abbott *et al.* [1]. The plot depicts the waveform as seen in the detector frame, i.e. cosmologically redshifted.



(b) Data and model by LIGO for GW150914, published in Abbott *et al.* [1].

Figure 2.1: Waveform based on computations of this thesis (a) and LIGO's data and model (b) for comparison. The plots share a similar structure, but it is clear by figure (b) that the signal-to-noise ratio is small, and much of the early-time structure is 'washed out' by noise. That is to say, both the signal and the model in (b) has been filtered by frequency, making the plots somewhat jagged. Plot (a) has not been filtered. In figure (a) the amplitude diverges at $t = 0.425$s, while in the in (b) it does not. This is because the inspiral model breaks down here, and the merger phase takes over.

2.2.2 Computing the Fourier transform of the waveform

To obtain the high sensitivities in GW detections the signal is Fourier transformed, in order to show which frequencies dominate the signal. This frequency spectrum can be compared to theoretical predictions to determine factors like the total mass at 0PN, symmetric mass ratio at 1PN, and more parameters at higher orders, e.g. spin at 1.5PN [5] and finite size effects like tidal deformation at 5PN [10].

In order to compare data with theoretical predictions these predictions must also be expressed in the frequency domain. Therefore, the desired waveform is $\Psi(f)$, which is the phase of the Fourier transformed waveform.

The Fourier transform and stationary phase approximation

To compute the Fourier transformed $\tilde{B}(f)$ of some function $B(t)$ the stationary phase approximation (SPA) can be used, and it is commonly utilized to compute the Fourier transform of (2.3). Standardized in GW physics by Cutler and Flanagan [11] it approximates

$$\begin{aligned}
&\text{for} \quad B(t) = A(t)\cos(\Phi(t)), \\
&\Rightarrow \quad \tilde{B}(f) \approx \frac{1}{2}A(t)\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)^{-1/2} \exp\{i\,(2\pi f t - \Phi(f) - \pi/4)\} \\
&\qquad\qquad \equiv \frac{1}{2}A(t)\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)^{-1/2} \exp\{i\Psi(f)\}, \\
&\text{provided} \quad \frac{\mathrm{dln}(A(t))}{\mathrm{d}t} \ll \frac{\mathrm{d}\Phi}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} \ll \left(\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)^2.
\end{aligned} \tag{2.20}$$

This is exactly the type of expression which describes GWs (2.3), and the conditions do indeed apply to the inspiral phase.

The leading order amplitude scales as $v^2 \sim \tau^{-1/4}$ (2.17) (also, see equation (3.67) in the next chapter for why the amplitude scales as v^2), while $\mathrm{d}\Phi/\mathrm{d}t = \omega(\tau) \sim \tau^{-3/8}$ (see (2.18)). Thus, for large τ, which is the time remaining till coalescence, $\mathrm{dln}(a_n(t))\,/\mathrm{d}t \sim \frac{1}{4}\tau^{-1} \ll \omega(\tau) \sim \tau^{-3/8}$.

As for the last prerequisite it can be shown to hold for quasi-stable circular orbits. Taking the time derivative of Kepler's third law (2.4) results in

$$\begin{aligned}
2\omega\dot{\omega} &= -3\dot{r}\frac{GM}{r^4} = -3\frac{\dot{r}}{r}\omega^2, \\
\Rightarrow \quad \frac{-\dot{r}}{\omega r} &= \frac{2}{3}\frac{\dot{\omega}}{\omega^2} \ll 1.
\end{aligned} \tag{2.21}$$

For quasi-stable circular orbits the inspiral *must* be slow compared to the orbital motion, and thus the radial velocity ($\dot{r}$) must be small compared to the tangential velocity (ωr), since for perfectly circular motion their fraction is identically zero. From Kepler's law this implies also that $\dot{\omega}/\omega^2 \ll 1 \rightarrow \ddot{\Phi} \ll \dot{\Phi}^2$, which is exactly the condition required to use the SPA.

This in hand also provides an estimate for the validity of this approximation, as ω is a known function of time (2.18)

$$\frac{2}{3}\frac{\dot{\omega}}{\omega^2} \overset{(2.18)}{\simeq} \frac{2}{5}\left(\frac{5G\mathcal{M}}{c^3}\right)^{5/8}\tau^{-5/8} \ll 1. \tag{2.22}$$

This expression is indeed small for most values of $t < t_c$.

Since the most important part of the waveform for comparisons to experimental data is the frequency spectrum, the last computation of this chapter will be of the Fourier transformed phase $\Psi(f)$.

Computing the SPA waveform

From equation (2.20) the phase of the Fourier transformed waveform can be approximated as

$$\Psi_{\mathrm{SPA}} = 2\pi f t(f) - \Phi(f) = \omega t(\omega) - \Phi(\omega) = \frac{v^3}{GM}t(v) - \Phi(v). \tag{2.23}$$

$\Phi(v)$ being given by equation (2.14), and $t(v)$ by (2.16), $\Psi(v)$ can easily be computed.

$$\frac{v^3}{GM}t(v) = \frac{v^3}{GM}t_c - \frac{5}{2^8}\frac{1}{\eta}\frac{c^5}{v^5}\left\{1 + \frac{8}{6}\left(\frac{743}{336} + \frac{11}{4}\eta\right)\frac{v^2}{c^2} + \dots\right\}, \tag{2.24}$$

$$\boxed{\Psi_{\mathrm{SPA}}(v) = \frac{v^3}{GM}t_c - \Phi_0 + \frac{3}{256}\frac{1}{\eta}\frac{c^5}{v^5}\left\{1 + \left(\frac{3715}{756} + \frac{55}{9}\eta\right)\frac{v^2}{c^2} + \dots\right\}.} \tag{2.25}$$

Lastly the phase can be expressed in terms of the physical frequency by using $v = (GM\omega)^{1/3} = (2\pi GMf)^{1/3}$.

$$\boxed{\begin{aligned}\Psi_{\mathrm{SPA}}(f) = 2\pi f t_c - \Phi_0 + \frac{3}{256}\left(\frac{2\pi G\mathcal{M}f}{c^3}\right)^{-\frac{5}{3}} \\ \cdot\left\{1 + \left(\frac{3715}{756} + \frac{55}{9}\eta\right)\left(\frac{2\pi GMf}{c^3}\right)^{\frac{2}{3}}\right\}.\end{aligned}} \tag{2.26}$$

This is indeed equivalent to the expression found by Arun *et al.* [5] (equation (6.22), page 21) up to 1PN, with some difference in notation.

In order to compute this waveform all that is needed is the PN expansion of the orbital energy, and the GW energy flux, both associated with stable, energy conservative, motion. In [5] these were provided with references to other papers.

In a sense, (2.26) is the final result of this thesis, computation wise, but it now remains to justify the expressions used for the post-Newtonian expansion of the orbital energy (2.7), and energy flux expansion (2.8), which will be derived in Chapter 4 and 5 respectively.

Chapter 3

Gravity as a gauge theory

In this chapter the fundamental theory by which the orbital energy and energy flux will be calculated is derived. How can Einstein's general relativity be described as a classical field theory, and then recast into the language of EFT.

The derivations presented in this chapter largely follows those presented in Feynman [12], with supplements from Maggiore [9] and Porto [4].

3.1 Background

The modern theory of gravity is partially split between two traditions. On the one hand there is the geometrical tradition following Einstein's approach by interpreting gravity as the effect of a curved space-time, which is curved according to the Einstein's field equations. The followers of this tradition may be called relativists. On the other hand there is the tradition of using the formalism of Lorentz invariant fields on a static, Minkowskian, space-time, inspired by its monumental success for electrodynamics and quantum theory. The followers of this tradition may be called field theorists.

Though these traditions are not entirely separated, the two different interpretations lend themselves to different *natural* extensions of general relativity. Thus the two traditions tend to separate relativists and field theorists by which theories they work on.

In this thesis the 1PN phase of GW produced by compact binaries are computed using the formalism of field theory. Familiarity with basic quantum field theory (QFT) is expected, but the derivations are otherwise supposed to be elementary.

Feynman and gravity

One of the more famous field theorists, R. P. Feynman had a "gravity phase" from 1954 to the late 1960s (Di Mauro *et al.* [13]). After having worked on the foundations of *quantum electrodynamics*, Feynman sought to uncover the quantum nature of gravity pursuing a similar method. He reckoned that gravity could, similarly to electromagnetism, be perturbatively expanded with respect to its coupling constant, and then quantized by quantizing the frequencies.

Quantizing gravity turns out to be a little more complicated than that, but Feynman's approach to classical gravity as a massless, spin 2, gauge field has made a lasting impression on gravity physics, especially in the context of GWs. This approach can be studied in the lecture notes from his lecture series of the 60's [12].

3.2 Fierz-Pauli Lagrangian

To linearized order, the Einstein-Hilbert action of general relativity is equivalent to the massless Fierz-Pauli action from field theory [14].

3.2.1 Deriving the graviton Lagrangian

When Feynman set out to study gravity, he took the mindset of a field theorist who until recently was unaware of gravitation, and just now have been presented with data suggesting that all masses attract other masses according to an inverse square law, proportional to the product of their masses,

$$\boldsymbol{F} \sim -\frac{m_1 m_2 \boldsymbol{r}}{r^3}.$$

Feynman envisioned this as the mindset of aliens on Venus who had just now acquired the technology to pierce through the atmosphere and measure the movement of the planets, but were still our equals in particle physics.

Their first impulse would probably be to guess that this is an unknown effect of some known field. After finding no field that could replicate the solar system observation, their next guess would be that there exists a new kind of field which mediates this mysterious force. Calling this hypothetical field *the graviton field*, and its associated quantum particle the graviton, the Venusians would next try to uncover its structure.

To construct the Lagrangian for this new force of nature they would determine that it has to be of even spin, and thus an even tensor rank, for the resulting static force to be attractive for equal charges, where the charge for the graviton field would be mass. For the force to go as an inverse square the field must also be massless.

Lastly, it must couple to all matter equally, but it must do so in a relativistic way. The natural suggestion is to somehow couple the field with the four-velocity of the source, like how the electric charge which the electric field couples to is promoted to the charge density four current $j^\mu = \gamma^{-1}\rho u^\mu$, and couples to the vector potential $A_\mu = (\phi/c\ ,\ A_i)$. See Gourgoulhon [15] or other textbooks on relativistic field theory.

However, to let the graviton field couple to *all* fields a natural candidate is the energy-momentum tensor $T^{\mu\nu}$, induced by field invariance under space-time translations. Incidentally, for a point particle it is constructed by the four-velocity of the source: $T^{\mu\nu} = \gamma^{-1}p^\mu u^\nu = \gamma^{-1}\rho u^\mu u^\nu$. Now for a scalar field it can be contracted to form a scalar, the trace, which is proportional to the mass density. Alternatively, a field of higher tensor rank can couple to the indices, also coupling the field to the mass density in the static frame $T^{\mu\nu} = T^{00}\delta^{\mu 0}\delta^{\nu 0}$.

The spin zero / scalar field is a candidate for the graviton, but fails to couple to the electromagnetic energy-momentum tensor, as the electromagnetic energy-momentum tensor is traceless. It also fails to predict the perihelion procession of Mercury correctly [15].

Thus, the Venusians would probably try a massless spin 2 field next. Since massless fields only have one (spin $s = 0$) or two degrees of freedom (helicity $= \pm s$) the symmetric spin two field should be *easiest* to work with, as it will have $10 - 2 = 8$ redundant degrees of freedom. The antisymmetric field by comparison only have $6 - 2 = 4$ redundant degrees of freedom.

Thus demanding the Lagrangian to be composed of a massless, symmetric, rank two tensor field there are only four unique terms, containing only second / two derivatives, after considering partial integrations:

Two where the index of the tensor and the index of the derivative differ.

1. $h_{\mu\nu,\rho}h^{\mu\nu,\rho}$
2. $h_{\mu\nu,\rho}h^{\mu\rho,\nu}$

And three where two of the indices contract for the individual $h_{\mu\nu}$.

3. $h_{\mu\nu}{}^{,\nu}h^{\mu\rho}{}_{,\rho}$
4. $h_{\mu\nu}{}^{,\nu}h^{,\mu}$ with $h \equiv h_\sigma{}^\sigma$
5. $h_{,\mu}h^{,\mu}$

Note specifically that term 2. and 3. are the same after two successive partial integrations $h_{\mu\nu,\rho}h^{\mu\rho,\nu} = -h_{\mu\nu}h^{\mu\rho,\nu}{}_\rho = h_{\mu\nu}{}^{,\nu}h^{\mu\rho}{}_{,\rho}$. Some texts use term 2. (like Maggiore [9]), but here term 3. will be employed (like in Feynman [12]). Thus, the free part of the Lagrangian[1] must be of the form

$$\mathcal{L} = a_1 h_{\mu\nu,\rho}h^{\mu\nu,\rho} + a_2 h_{\mu\nu}{}^{,\nu}h^{\mu\rho}{}_{,\rho} + a_3 h_{\mu\nu}{}^{,\nu}h^{,\mu} + a_4 h_{,\mu}h^{,\mu}. \tag{3.1}$$

It is possible to determine all the coefficients a_{1-4} by imposing gauge invariance on the equation of motion (EoM). The EoM for fields is determined by the Euler-Lagrange equation (3.2a) (see e.g. Goldstein *et al.* [2], or Kachelrieß [17]), and for (3.1) the equation of motion becomes (3.2c).

$$\partial_\rho \frac{\partial \mathcal{L}}{\partial h_{\mu\nu,\rho}} - \frac{\partial \mathcal{L}}{\partial h_{\mu\nu}} = 0 \tag{3.2a}$$

$$= \partial_\rho \Big(2a_1 h^{\mu\nu,\rho} + a_2 \eta^{\nu\rho} h^{\mu\sigma}{}_{,\sigma} + a_2 \eta^{\mu\rho} h^{\nu\sigma}{}_{,\sigma} + a_3 \eta^{\nu\rho} h^{,\mu} + a_3 \eta^{\mu\nu} h^{\rho\sigma}{}_{,\sigma} + 2a_4 \eta^{\mu\nu} h^{,\rho} \Big) \tag{3.2b}$$

$$= 2a_1 h^{\mu\nu,\rho}{}_\rho + a_2 h^{\mu\rho,\nu}{}_\rho + a_2 h^{\nu\rho,\mu}{}_\rho + a_3 h^{,\mu\nu} + a_3 \eta^{\mu\nu} h^{\rho\sigma}{}_{,\rho\sigma} + 2a_4 \eta^{\mu\nu} h^{,\rho}{}_\rho \equiv \Xi^{\mu\nu}. \tag{3.2c}$$

From the action of $\mathcal{L} + \mathcal{L}_{\text{int}} = \mathcal{L} + \frac{\lambda}{2} h_{\mu\nu} T^{\mu\nu}$ the inferred EoM should be

$$\Xi^{\mu\nu} = -\frac{1}{2}\lambda T^{\mu\nu}, \tag{3.3a}$$

$$T^{\mu\nu}{}_{,\nu} = 0 \quad \Rightarrow \quad \Xi^{\mu\nu}{}_{,\nu} = 0. \tag{3.3b}$$

[1]Terms $\sim h^1$ and h^0 only contribute constants to the equation of motion, and can thus be removed by field shifts. Terms proportional to h^2, but with no derivatives, determine the mass of the field $\sim m^2 hh$, and must therefore be zero for massless fields. Lastly, the Lagrangian must be a scalar in order to be Lorentz invariant. There are no contractions of only 1 derivative and two h's that can produce a scalar. Therefore, to leading order in h^n, the Lagrangian must consist of terms proportional to h^2 with two derivatives. See e.g. Schwichtenberg [16], page 573-575, for a more detailed discussion.

Equation (3.3b) can be used to fix the coefficients of equation (3.2c), and thus also the Lagrangian.

$$\Xi^{\mu\nu}{}_{,\nu} = \Box h^{\mu\nu}{}_{,\nu}(2a_1 + a_2) + \Box h^{,\mu}(a_3 + 2a_4) + h^{\rho\sigma,\mu}{}_{\rho\sigma}(a_2 + a_3) = 0,$$
$$\Rightarrow \quad a_1 = -\frac{1}{2}a_2 = \frac{1}{2}a_3 = -a_4. \tag{3.4}$$

Thus a Lagrangian of a symmetric, massless, rank 2 tensor field which couples to a divergenceless rank 2 tensor field (e.g. $\mathcal{L}_{\text{int}} = -\frac{\lambda}{2}h_{\mu\nu}T^{\mu\nu}$), consisting of only second derivatives, in a flat space-time, must to second power of h take the form of the *massless Fierz-Pauli Lagrangian* [14]

$$\boxed{\mathcal{L}_{\text{FP}} = -\frac{1}{2}h_{\mu\nu,\rho}h^{\mu\nu,\rho} + h_{\mu\nu}{}^{,\nu}h^{\mu\rho}{}_{,\rho} - h_{\mu\nu}{}^{,\nu}h^{,\mu} + \frac{1}{2}h_{,\mu}h^{,\mu}.} \tag{3.5}$$

Here the overall factor has been set to $a_1 = -1/2$.

3.2.2 The equation of motion and gauge condition

The EoM for $\mathcal{L}_{\text{FP}} + \frac{1}{2}\lambda h_{\mu\nu}T^{\mu\nu}$ follows directly from the Euler-Lagrange equation (3.2a) as

$$-\Box h_{\mu\nu} + 2h^{\alpha}{}_{\{\mu,\nu\}\alpha} - h_{,\mu\nu} - \eta_{\mu\nu}\left(h_{\rho\sigma}{}^{,\rho\sigma} - \Box h\right) = \frac{\lambda}{2}T_{\mu\nu}. \tag{3.6}$$

This is equivalent with the equation of motion found in the linear approximation of general relativity, for an appropriate choice of λ (see e.g. equation (1.17) of Maggiore [9], or equation (9.16) of Grøn and Hervik [18])

Varying the Fierz-Pauli action directly should also provide the equations of motion

$$\delta\mathcal{L}_{\text{FP}} = \frac{\delta\mathcal{L}_{\text{FP}}}{\delta h_{\mu\nu,\rho}}\,\delta h_{\mu\nu,\rho} + \frac{\delta\mathcal{L}_{\text{FP}}}{\delta h_{\mu\nu}}\,\delta h_{\mu\nu} = \frac{\delta\mathcal{L}_{\text{FP}}}{\delta h_{\mu\nu,\rho}}\partial_\rho\,\delta h_{\mu\nu} + \frac{\delta\mathcal{L}_{\text{FP}}}{\delta h_{\mu\nu}}\,\delta h_{\mu\nu}$$
$$= \left[\frac{\delta\mathcal{L}_{\text{FP}}}{\delta h_{\mu\nu}} - \partial_\rho\frac{\delta\mathcal{L}_{\text{FP}}}{\delta h_{\mu\nu,\rho}}\right]\delta h_{\mu\nu} = -\Xi^{\mu\nu}\,\delta h_{\mu\nu} = 0. \tag{3.7}$$

This automatically holds because of (3.2c) ($\Xi^{\mu\nu} = 0$). But (3.7) can also be solved using the condition (3.3b), $\Xi^{\mu\nu}{}_{,\nu} = 0$. Letting $\delta h_{\mu\nu} = -\xi_{\mu,\nu} - \xi_{\nu,\mu}$ it is easy to show that the action stays invariant under this type of transformation, using partial integration.

$$\delta\mathcal{L}_{\text{FP}} = \Xi^{\mu\nu}\left(\xi_{\mu,\nu} + \xi_{\nu,\mu}\right) = -\Xi^{\mu\nu}{}_{,\nu}\xi_\mu - \Xi^{\mu\nu}{}_{,\mu}\xi_\nu = -2\Xi^{\mu\nu}{}_{,\nu}\xi_\mu \overset{(3.3b)}{=} 0. \tag{3.8}$$

Thus the following transformation of the field leaves both the EoM and the gauge condition invariant.

$$\boxed{h_{\mu\nu}(x) \to h_{\mu\nu}(x) + \delta h_{\mu\nu}(x) = h_{\mu\nu}(x) - \xi_{\mu,\nu}(x) - \xi_{\nu,\mu}(x).} \tag{3.9}$$

Again, this is equivalent to the gauge condition found in linear theory when linearizing metric invariance under change of coordinates (see equation (9.9) of Grøn and Hervik [18]).

Also introducing the commonly used *bar operator*, which symmetrize tensors and changes the sign of their trace,

$$\bar{S}_{\mu\nu} \equiv \frac{1}{2}\left(S_{\mu\nu} + S_{\nu\mu} - S^{\sigma}{}_{\sigma}\eta_{\mu\nu}\right), \tag{3.10}$$

the gauge condition for the *barred* h-field is obtained by transforming (3.9) as (3.10), resulting with

$$\bar{h}_{\mu\nu}(x) \to \bar{h}_{\mu\nu}(x) - \xi_{\mu,\nu}(x) - \xi_{\nu,\mu}(x) + \eta_{\mu\nu}\xi_{\sigma}{}^{,\sigma}(x) \equiv \bar{h}_{\mu\nu}(x) - \xi_{\mu\nu}(x). \tag{3.11}$$

Thus the divergence of this barred field transforms as

$$\bar{h}_{\mu\nu}{}^{,\nu} \to \bar{h}_{\mu\nu}{}^{,\nu} - \Box\,\xi_{\mu}. \tag{3.12}$$

As ξ_μ can be any vector without changing the EoM, it can be chosen such that $\Box\,\xi_\mu = \bar{h}_{\mu\nu}{}^{,\nu}$, shifting the field such that $\bar{h}'_{\mu\nu}{}^{,\nu} = 0$, which is to impose the *Lorenz gauge*[2].

$$\boxed{\begin{array}{c}\text{Lorenz gauge condition :}\\ \bar{h}_{\mu\nu}{}^{,\nu} = 0.\end{array}} \tag{3.13}$$

The exact expression for ξ_μ can be obtained by method of Green's functions, but this is unnecessary to compute. Simply keeping in mind that $\bar{h}'_{\mu\nu}{}^{,\nu} = 0$ shall suffice to simplify the Lagrangian (3.5). Notice that the following terms must be zero, using again (3.10), but in reverse.

$$\bar{h}_{\mu\nu}{}^{,\nu} = 0 = h_{\mu\nu}{}^{,\nu} - \frac{1}{2}\eta_{\mu\nu}h^{,\nu}, \tag{3.14a}$$

$$\bar{h}_{\mu\nu}{}^{,\nu}\bar{h}^{\mu\rho}{}_{,\rho} = 0^2 = h_{\mu\nu}{}^{,\nu}h^{\mu\rho}{}_{,\rho} - h_{\mu\nu}{}^{,\nu}h^{,\mu} + \frac{1}{4}h_{,\mu}h^{,\mu}. \tag{3.14b}$$

Adding and subtracting 0 from the Lagrangian should change nothing, and thus the following expression can be used as a gauge fixing term (gf)

$$\mathcal{L}_{\text{gf}} = -\bar{h}_{\mu\nu}{}^{,\nu}\bar{h}^{\mu\rho}{}_{,\rho} = \left[-h_{\mu\nu}{}^{,\nu}h^{\mu\rho}{}_{,\rho} + h_{\mu\nu}{}^{,\nu}h^{,\mu} - \frac{1}{4}h_{,\mu}h^{,\mu}\right], \tag{3.15}$$

$$\Rightarrow \quad \mathcal{L}_{(2)} + \mathcal{L}_{\text{int}} \equiv \mathcal{L}_{\text{FP}} + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{int}} = -\frac{1}{2}h_{\mu\nu,\rho}h^{\mu\nu,\rho} + \frac{1}{4}h_{,\mu}h^{,\mu} + \frac{\lambda}{2}h_{\mu\nu}T^{\mu\nu}. \tag{3.16}$$

with the subscript (2) to signify that this is the action to quadratic order in h.

The EoM of $\mathcal{L}_{(2)} + \mathcal{L}_{\text{int}}$ is the familiar

$$\boxed{\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) = \Box\,\bar{h}_{\mu\nu} = -\frac{\lambda}{2}T_{\mu\nu}\,,} \tag{3.17}$$

[2]The divergenceless gauge can be refered to by many names, but the most common is to use the same name as in electro dynamics: Lorenz. Other names include Hilbert, De Donder and Harmonic gauge, though the latter two are more assosciated with curved backgrounds.

from linearized theory (see equation (1.24) of Maggiore [9] or equation (9.22) of Grøn and Hervik [18]).

Comparing with the EoM of linearized GR it is tempting to conclude that $\lambda \equiv 4\kappa = 32\pi G/c^4$, but then it is also common to make the Einstein-Hilbert action dimensionless by scaling it with a factor of $(16\pi G/c^4)^{-1}$. Comparing (3.16) with the Einstein-Hilbert action expanded to second order[3] the Lagrangian (3.16) carries an additional factor of 2, and is dimensionful. Field theorists usually fix the dimensionality of the action by rescaling their fields to become dimensionful. Doing this

$$h_{\mu\nu}^{\text{dim.ful}} = \left(\frac{32\pi G}{c^4}\right)^{-1/2} h_{\mu\nu}^{\text{dim.less}}, \tag{3.18}$$

h adsorbs the dimensionful prefactor. To compare (3.17) with the linearized Einstein's field equations, it must first be rescaled back to a dimensionless field according to (3.18), and then the coupling constant is revealed to be

$$\boxed{\lambda \equiv \left(\frac{32\pi G}{c^4}\right)^{1/2} = \sqrt{4\kappa},} \tag{3.19}$$

where $\kappa = \frac{8\pi G}{c^4}$ is the constant which appears in Einstein's field equations.

Some call this coupling constant M_{Pl}^{-1} rather than λ ([4], [3], [19]). However it does not have dimesion of mass, nor is the Planck constant anywhere in the expression, so why do they do this? These articles use natural units $\hbar = c = 1$, and $M_{\text{Pl}}^{-1} = \sqrt{G/\hbar c} = \lambda \cdot \sqrt{\hbar c^3/32\pi}$, which is just a numerical factor off from λ (in natural units).

Furthermore, in natural units, legths (L) are dimesionally equal to inverse mass $(M)^{-1}$ (using $[x]$ as dimesion of x: $L = [x] = [ct] = 1 \cdot T$, and $E = [\hbar\omega] = 1 \cdot T^{-1} \overset{\text{also}}{=} [mc^2] = M \cdot 1^2, \quad \Rightarrow \quad M = T^{-1} = L^{-1} = E$), and thus the action has dimension of $[S] = [\int \mathrm{d}^4x\, \mathcal{L}] = L^4[\mathcal{L}] = M^{-4}[\mathcal{L}] = 1$. The action must be dimesionless in QFT, since in the path integral approach it is exponated. Every field Lagrangian has a kinetic term $\sim \partial\phi\partial\phi$, which scale as $[\mathcal{L}] = M^4 = [\partial\phi]^2 = L^{-2}[\phi]^2 = M^2[\phi]^2 \quad \Rightarrow \quad [\phi] = M$. Thus for couplings $\mathcal{L}_{\text{int}} \sim \lambda h(\partial h)^2$ to have the same dimension as the kinetic term; $[\lambda] = M^{-1}$.

Calling the coupling constant M_{Pl}^{-1} might have the unfortunate consequense of making it look like a quantum theory, but make no mistake, this is all classical field theory. Therefore, it is simply labelled λ in this thesis.

[3] Which is the necessary order needed to derive the linearized Einstein's field equations.

3.3 Solutions of the graviton field

3.3.1 Gravitational waves in vacuum, and their polarization

According to the equation of motion (3.17) the field will in a vacuum ($T_{\mu\nu} = 0$) behave as a relativistic wave

$$\Box\, \bar{h}_{\mu\nu} = 0, \tag{3.20}$$

which admits solutions of the form (2.2). See Appendix A for a derivation of this solution.

Up until now the gauge has only been used to make sure the entire EoM (3.17) remains divergence free, just like the source term $T_{\mu\nu}{}^{,\nu} = 0$. In doing so it was determined that the field might only be shifted according to (3.9). Furthermore, the divergence of the barred h-field could be eliminated only imposing further that $\Box\, \xi_\mu = 0$.

Keeping $\Box\, \xi_\mu = 0$ still leaves

$$\xi_{\mu\nu}(x) \equiv \xi_{\mu,\nu}(x) + \xi_{\nu,\mu}(x) - \eta_{\mu\nu}\xi_\sigma{}^{,\sigma}(x) \tag{3.21}$$

with four degrees of freedom, as it is a function of the four independent parameters ξ_μ, which satisfy $\Box\, \xi_\mu = 0$. To make the graviton field divergence free imposes four additional conditions on $h_{\mu\nu}$, by the four equations $h_{\mu\sigma}{}^{,\sigma} = 0$. This leaves $h_{\mu\nu}$ with $10-4 = 6$ degrees of freedom. Subtracting further the four gauge freedoms reduces $h_{\mu\nu}$ to only two effective degrees of freedom, as any massless spin two field should have. These four gauge freedoms ξ_μ can be used to impose four additional conditions on $\bar{h}_{\mu\nu}$. ξ_0 can be used to set the trace $\bar{h} = 0$, and since $\bar{h}_{\mu\nu}$ is just $h_{\mu\nu}$ with the reversed sign trace, in this gauge $h_{\mu\nu} = \bar{h}_{\mu\nu}$.

The three remaining freedoms, ξ_i, can be used to set $h_{0i} = 0$ as well. Since $h_{\mu\nu}{}^{,\nu} = 0$ this implies $h_{0\nu}{}^{,\nu} = \partial^0 h_{00} + \partial^i h_{0i} = 0 = \partial^0 h_{00}$, making h_{00} a constant of time. A constant contribution to a GW are uninteresting and for all intents and purposes it can be considered to be zero, making all $h_{0\mu} = 0$.

This specific gauge is referred to as the *transverse-traceless (TT) gauge*, and is defined by

$$\boxed{\begin{array}{l}\text{TT gauge condition:}\\ h_{0\mu} = h_i{}^i = h_{ij}{}^{,j} = 0.\end{array}} \tag{3.22}$$

Note that this gauge can only be imposed in a vacuum, since the vacuum condition $\Box\, \bar{h}_{\mu\nu} = 0 \Rightarrow \Box\, \xi_{\mu\nu} = 0$ was used.

Assuming $h_{\mu\nu}(x^\alpha) = \epsilon_{\mu\nu} h(x^\alpha)$ with $h(x^\alpha)$ as the scalar solution to the wave equation

$$h(x^\alpha) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 \cdot 2\omega_k} \left\{ a(\boldsymbol{k})e^{-ik_\sigma x^\sigma} + a^\dagger(\boldsymbol{k})e^{ik_\sigma x^\sigma} \right\}, \tag{3.23}$$

which is (A.6) from Appendix A. The TT gauge condition then implies

$$\begin{aligned} h_{mn}{}^{,n}(x^\alpha) &\overset{(3.22)}{=} 0 = \epsilon_{mn}\partial^n h(x^\alpha) \\ = \int &\frac{\mathrm{d}^3k}{(2\pi)^3 \cdot 2\omega_k} \epsilon_{mn} i k^n \left\{ -a(\boldsymbol{k})e^{-ik_\sigma x^\sigma} + a^\dagger(\boldsymbol{k})e^{ik_\sigma x^\sigma} \right\}, \end{aligned}$$

hence

$$k^j \epsilon_{ij} = 0. \tag{3.24}$$

For a wave travelling in the z-direction $k_\mu = \big(k, \quad 0, \quad 0, \quad k\big)$, making $\epsilon_{\mu 3} = 0$. With (3.22) this is enough to determine the polarization down to the two essential degrees of freedom

$$\boxed{\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_+ & \epsilon_\times & 0 \\ 0 & \epsilon_\times & -\epsilon_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \epsilon_+ & \epsilon_\times \\ \epsilon_\times & -\epsilon_+ \end{pmatrix}_{\text{in plane} \perp \text{to } \boldsymbol{k}} .} \tag{3.25}$$

3.3.2 Source of gravitational waves

The general solution of the equation of motion (3.17) with sources can be obtained by method of Green's functions.

Green's functions in general

The Green's function of a linear differential operator $\mathcal{LO}_x$ is defined as the function that satisfies

$$\mathcal{LO}_x \Delta(x, x') = \delta^n\big(x - x'\big) , \tag{3.26}$$

where $\mathcal{LO}_x$ only acts on x.

The differential equation in question

$$\mathcal{LO}_x \psi(x) = f(x)$$

admits solutions of the form

$$\psi(x) = \int \mathrm{d}^n x' \, \Delta(x, x') f(x').$$

This solution can easily be demonstrated to recover the original differential equation by using the definition of the Green's function,

$$\begin{aligned} \mathcal{LO}_x \psi(x) = \mathcal{LO}_x \int \mathrm{d}^n x' \, \Delta(x, x') f(x') &= \int \mathrm{d}^n x' \, f(x') \mathcal{LO}_x \Delta(x, x') \\ \overset{(3.26)}{=} \int \mathrm{d}^n x' \, f(x') \delta^n\big(x - x'\big) &= f(x), \end{aligned}$$

which was the original differential equation.

Green's function of the d'Alembert operator

In our case $\mathcal{LO}_x = \Box_x$, which is invariant under translation. Thus $\Delta(x, x') = \Delta(x - x')$. In order to find the corresponding Green's function $\Delta(x - x')$ the easiest way is to go through Fourier space.

$$\underset{x}{\Box}\,\Delta(x - x') = \underset{x}{\Box} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \tilde{\Delta}(k) e^{ik_\sigma(x^\sigma - x'^\sigma)} = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \tilde{\Delta}(k) \left(i^2 k_\mu k^\mu\right) e^{ik_\sigma(x^\sigma - x'^\sigma)}$$
$$= \delta^4(x - x') = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{ik_\sigma(x^\sigma - x'^\sigma)}.$$

Matching the last equality of both lines indicates that the Fourier transform of $\Delta(x - x')$ must be[4]

$$\boxed{\Delta(k) = \frac{-1}{k_\mu k^\mu}.} \tag{3.27}$$

Obtaining the Green's function in real space is now just a matter of transforming (3.27).

One last remark about Green's functions is in the context of four dimensional space-time the solution in terms of Green's functions can be understood as counting up contributions from source terms, all over space, and across all time

$$h(x^\alpha) = \int \mathrm{d}^4 x'\, \Delta(x^\alpha - x'^\alpha) T(x'^\alpha) = \int_{-\infty}^{\infty} \mathrm{d}ct' \int \mathrm{d}^3 x'\, \Delta(x^\alpha - x'^\alpha) T(x'^\alpha).$$

Thus $\Delta(x - x')$ *weighs the importance* of source contributions at different points in space and time. For physical solutions only contributions of source configurations from the *past* contribute to $h(t)$. This is imposed by demanding $t \geq t'$.

Back to deriving the real space Green's function. Performing the transform, and defining $r^\alpha = x^\alpha - x'^\alpha$

$$\Delta(r^\alpha) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-1}{k_\mu k^\mu} e^{ik_\sigma r^\sigma} = \int \frac{\mathrm{d}k_0}{2\pi} e^{-ik_0 r^0} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{k_0^2 - \boldsymbol{k}^2} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$
$$= \int \frac{\mathrm{d}k_0}{2\pi} e^{-ik_0 r^0} \int_0^\infty \int_{-1}^{1} \int_0^{2\pi} \frac{|\boldsymbol{k}|^2\, \mathrm{d}|\boldsymbol{k}|\, \mathrm{d}\cos\theta\, \mathrm{d}\phi}{(2\pi)^3} \frac{1}{k_0^2 - \boldsymbol{k}^2} e^{i|\boldsymbol{k}||\boldsymbol{r}|\cos\theta}$$
$$= \int \frac{\mathrm{d}k_0}{2\pi} e^{-ik_0 r^0} \int_0^\infty \frac{\mathrm{d}|\boldsymbol{k}|}{(2\pi)^2} \frac{|\boldsymbol{k}|^2}{i|\boldsymbol{k}||\boldsymbol{r}|} \frac{e^{i|\boldsymbol{k}||\boldsymbol{r}|} - e^{-i|\boldsymbol{k}||\boldsymbol{r}|}}{k_0^2 - \boldsymbol{k}^2},$$

where the spatial integral was preformed in spherical coordinates. Relabelling $|\boldsymbol{k}| = k$ and $|\boldsymbol{r}| = r$, the expression can be worked further

$$\Delta(r^\alpha) = \int \frac{\mathrm{d}k_0}{2\pi} e^{-ik_0 r^0} \int_{-\infty}^{\infty} \mathrm{d}k\, \frac{k}{i(2\pi)^2 r} \frac{e^{ikr}}{k_0^2 - k^2}.$$

[4]From here on out the tilde over $\tilde{\Delta}(k)$ will be dropped, and whether it is the Green's function in real or Fourier space will be expected to be understood by its argument.

Extending to the complex plane, this integral can be evaluated using *Cauchy's residue theorem*, from complex analysis. Shifting $k_0 \to k_0 + i\varepsilon$ is equivalent to imposing the *retardation condition:* $t \geq t' \Leftrightarrow r^0 \geq 0$. Why this is the case should become apparent soon. According to the residue theorem

$$\oint \mathrm{d}z\, f(z) = i2\pi \sum_k \mathrm{Res}(f, a_k),$$
$$\text{where } a_k \text{ is a pole of } f(z) \text{ enclosed by the integral, and} \tag{3.28}$$
$$\mathrm{Res}(f, a_k) = \lim_{z \to a_k} (z - a_k) \cdot f(z).$$

Utilizing this theorem and integrating over the upper complex plane, the Green's function becomes

$$\begin{aligned}
\Delta(r^\alpha) &= \lim_{\varepsilon \to 0} \int \frac{\mathrm{d}k_0}{2\pi} e^{-ik_0 r^0} \frac{-1}{i(2\pi)^2 r} \oint \mathrm{d}k\, k \frac{e^{ikr}}{(k - k_0 - i\varepsilon)(k + k_0 + i\varepsilon)} \\
&= \lim_{\varepsilon \to 0} \int \frac{\mathrm{d}k_0}{2\pi} e^{-ik_0 r^0} \frac{-1}{i(2\pi)^2 r} \frac{i2\pi}{2} e^{ir(k_0 + i\varepsilon)} \\
&= \frac{-1}{4\pi r} \int \frac{\mathrm{d}k_0}{2\pi} e^{-ik_0(r^0 - r)} = \frac{-\delta(ct - r)}{4\pi r}
\end{aligned}$$

It is now apparent that this is the *retarded solution.* Had k_0 rather been shifted by $-i\varepsilon$, the Dirac delta function would have been $\delta(ct + r)$. The delta function picks out contributions on the light cone of x^α, the retarded Green's function picks out on the past light cone, while the *advanced* Green's function picks out on the future light cone.

$$\Delta_{\mathrm{ret}}(r^\alpha) = \frac{-\delta(ct - |\boldsymbol{r}|)}{4\pi|\boldsymbol{r}|}. \tag{3.29a}$$

$$\Delta_{\mathrm{adv}}(r^\alpha) = \frac{-\delta(ct + |\boldsymbol{r}|)}{4\pi|\boldsymbol{r}|}. \tag{3.29b}$$

Beyond singling out contributions from the light cone, it is apparent that the importance of each contribution to the field is weighted by how far away it is from the point in question, according to the inverse power of the spatial distance.

Solving the inhomogeneous equation of motion

Back to the EoM (3.17), using the retarded Green's functions it admits solution of the form

$$\begin{aligned}
\Box\, \bar{h}_{\mu\nu}(x) &= -\frac{\lambda}{2} T_{\mu\nu}(x) \\
\Rightarrow \quad \bar{h}_{\mu\nu}(x) &= -\frac{\lambda}{2} \int \mathrm{d}^4x'\, \Delta_{\mathrm{ret}}(x - x') T_{\mu\nu}(x') \\
&= \frac{\lambda}{8\pi} \int \mathrm{d}^3x' \frac{T_{\mu\nu}(t_{\mathrm{ret}}, \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|}, \quad \text{where } t_{\mathrm{ret}} \equiv t - \frac{|\boldsymbol{x} - \boldsymbol{x'}|}{c}.
\end{aligned} \tag{3.30}$$

Assuming the GW is measured far away from the source compared to the size of the source system, then the approximation $|\boldsymbol{x} - \boldsymbol{x'}| \approx |\boldsymbol{x}| \equiv R$ holds. This simplifies the integral in equation (3.30) to only be dependent on the energy distribution of the source, and not where it is measured.

Furthermore, waves measured in vacuum can be set into the TT gauge, which implies that all physical information of the source can be captured by its spatial indices $\bar{h}_{ij}$.

$$\bar{h}_{ij}(t, r) = \frac{\lambda}{8\pi R} \int_{\mathcal{V}} T_{ij}\left(t_{\text{ret}}, \boldsymbol{x'}\right) \mathrm{d}^3 x' \,. \tag{3.31}$$

To simplify the expression of equation (3.31) further it is useful to note some properties of the energy-momentum tensor.

First: it is divergenceless.

$$T^{\mu\nu}{}_{,\nu} = T^{\mu 0}{}_{,0} + T^{\mu i}{}_{,i} = 0. \tag{3.32}$$

Second: use of the following integral will be made.

$$\int_{\mathcal{V}} \left(T^{ik} x^j\right)_{,k} \mathrm{d}^3 x = \int_{\mathcal{V}} T^{ik}{}_{,k}\, x^j \,\mathrm{d}^3 x + \int_{\mathcal{V}} T^{ij} \,\mathrm{d}^3 x = \oint_{\partial\mathcal{V}} T^{ik} x^j \,\mathrm{d}A_k \,. \tag{3.33}$$

In the last line the divergence theorem has been used.

If the integration boundary is taken to encapsulate the entire source, $T^{ik}\big|_{\partial V} = 0$, equation (3.33) becomes equal to 0. It then follows

$$\begin{aligned}\int_{\mathcal{V}} T^{ij} \,\mathrm{d}^3 x &= -\int_{\mathcal{V}} T^{k\{i}{}_{,k}\, x^{j\}} \,\mathrm{d}^3 x = \int_{\mathcal{V}} T^{0\{i}{}_{,0}\, x^{j\}} \,\mathrm{d}^3 x = \frac{\mathrm{d}}{\mathrm{d}ct} \int_{\mathcal{V}} T^{0\{i} x^{j\}} \,\mathrm{d}^3 x \\ &= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}ct} \int_{\mathcal{V}} \left(T^{i0} x^j + T^{j0} x^i\right) \mathrm{d}^3 x \,.\end{aligned} \tag{3.34}$$

In the last line the symmetry $T^{\{ij\}} = \frac{1}{2}(T^{ij} + T^{ji})$ was written out explicitly.

So far not much has been accomplished, but notice how this procedure can be repeated to eliminate *all* dependence of the spatial components of $T^{\mu\nu}$.

$$\begin{aligned}\int_{\mathcal{V}} \left(T^{k0} x^i x^j\right)_{,k} \mathrm{d}^3 x &= \int_{\mathcal{V}} T^{k0}{}_{,k}\, x^i x^j \,\mathrm{d}^3 x + \int_{\mathcal{V}} \left(T^{i0} x^j + T^{j0} x^i\right) \mathrm{d}^3 x \\ &= \oint_{\partial\mathcal{V}} T^{k0} x^i x^j \,\mathrm{d}A_k = 0\end{aligned} \tag{3.35}$$

$$\Rightarrow \quad \int_{\mathcal{V}} \left(T^{i0} x^j + T^{j0} x^i\right) \mathrm{d}^3 x = -\int_{\mathcal{V}} T^{0k}{}_{,k}\, x^i x^j \,\mathrm{d}^3 x = \int_{\mathcal{V}} T^{00}{}_{,0}\, x^i x^j \,\mathrm{d}^3 x \tag{3.36}$$

Using that $T^{00} = \rho c^2$ is the mass-energy density, the integral in equation (3.31) takes a simple form of the second time derivative of the so-called *quadrupole moment*

$$Q_{ij}(t) \equiv \int_{\mathcal{V}} \rho(t, \boldsymbol{x}) x_i x_j \,\mathrm{d}^3 x \,, \tag{3.37a}$$

$$\ddot{Q}_{ij}(t) = 2 \int_{\mathcal{V}} T_{ij}\left(t, \boldsymbol{x}\right) \mathrm{d}^3 x \,. \tag{3.37b}$$

Finally, the source of linearized GWs is provided as

$$\boxed{\bar{h}_{ij}(t,\boldsymbol{R}) = \frac{\lambda}{16\pi R}\ddot{Q}_{ij}(t_{\text{ret}}).} \tag{3.38}$$

Utilizing this result the GWs generated from any source, with a non-vanishing energy-momentum tensor, can be calculated at distances sufficiently far away from the source.

3.4 Gravity from gravitons

Equipped with a field Lagrangian it is now desirable to check that it in fact reproduces the Newtonian law of universal gravity for non-relativistic sources.

In special relativity the action of point particles are the *geodesics*[5] of Minkowski space-time. The geodesic of some non-trivial space-time, with metric $g_{\mu\nu} = \eta_{\mu\nu} + \lambda h_{\mu\nu}$ is

$$\begin{aligned}
S_{pp} &= -mc\int \mathrm{d}s = -mc\int \sqrt{-g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu} && (3.39a)\\
&= -mc\int \sqrt{-(\eta_{\mu\nu}+\lambda h_{\mu\nu})\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu} && (3.39b)\\
&= -mc\int \sqrt{-\,\mathrm{d}\tau^2\,(\eta_{\mu\nu}+\lambda h_{\mu\nu})\dot{x}^\mu\dot{x}^\nu} && (3.39c)\\
&= -mc\int \mathrm{d}\tau\,\sqrt{c^2-\lambda h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} && (3.39d)\\
&= -mc^2\int \mathrm{d}\tau\,\sqrt{1-\lambda h_{\mu\nu}\frac{\dot{x}^\mu}{c}\frac{\dot{x}^\nu}{c}} && (3.39e)\\
&\approx -mc^2\int \mathrm{d}t\,\gamma^{-1} + \frac{\lambda}{2}\int \mathrm{d}t\,h_{\mu\nu}m\gamma^{-1}\dot{x}^\mu\dot{x}^\nu + \dots && (3.39f)\\
&\approx \int\left[\left(-mc^2+\frac{1}{2}mv^2+\dots\right)+\frac{\lambda}{2}h_{\mu\nu}T^{\mu\nu}_{pp}\right]\mathrm{d}t\,. && (3.39g)
\end{aligned}$$

Why compute the geodesic for a perturbed Minkowski space? In the relativists' approach GWs is such a perturbation of the metric (at least for vacuum solutions), thus this is the action of point particles to linearized order in $h_{\mu\nu}$, according to relativists. It is included here to motivate the interaction term $\mathcal{L}_{\text{int}} = \frac{\lambda}{2}h_{\mu\nu}T^{\mu\nu}$, with the point particle energy momentum-tensor[6]

$$T^{\mu\nu}_{pp}(x) = \sum_a \gamma_a^{-1}m_a\dot{x}^\mu\dot{x}^\nu\delta^3(\boldsymbol{x}-\boldsymbol{x}_a(t))\,, \tag{3.40a}$$

$$\text{with}\quad \dot{x}^\mu_a = \gamma_a\,(c\,,\ \boldsymbol{v}) = \left(1-\frac{|\boldsymbol{v}|^2}{c^2}\right)^{-\frac{1}{2}}(c\,,\ \boldsymbol{v})\,. \tag{3.40b}$$

[5]Extrema, e.g. shortest, path between two points in space-time. In Euclidian space it is the straigh line connecting the two points.

[6]The factor of γ_a^{-1} is a result of rewriting the proper time integral into a generic time integral $\frac{\mathrm{d}t}{\mathrm{d}\tau_a} = \gamma_a$, common for all particles. Writing out the four-velocities should result in an overall factor of γ_a^1. For more info on the pp energy-momentum tensor see e.g. Gourgoulhon [15].

Otherwise, it just shows that the free point particle Lagrangian (the kinetic part) is just $L_{\text{free }pp} \simeq \frac{1}{2}m_a v_a^2 = \int \frac{1}{2}m_a \dot{x}^2 \delta^3(\boldsymbol{x}-\boldsymbol{x}_a)\,\mathrm{d}^3x$. The constant term $-mc^2$ does not contribute to the EoM, and can therefore be neglected.

Thus, the total Lagrangian of two point particles interacting only via the graviton field, up to the 0PN order, is

$$\mathcal{L}_{pp} = \sum_{a=1}^{2}\left[\frac{1}{2}m_a\dot{\boldsymbol{x}}^2\delta^3(\boldsymbol{x}-\boldsymbol{x}_a) + \frac{\lambda}{2}h_{\mu\nu}m_a\dot{x}^\mu\dot{x}^\nu\delta^3(\boldsymbol{x}-\boldsymbol{x}_a)\right] + \mathcal{L}_{\mathrm{FP}} + \mathcal{L}_{\mathrm{gf}} \tag{3.41}$$

The EoM for the h field is known from (3.17). Since the bar operator is its own inverse operator when used on symmetric tensors, $\bar{\bar{S}}_{\mu\nu} = S_{\mu\nu}$, equation (3.17) may also be written as

$$\Box\, h = -\frac{\lambda}{2}\bar{T}_{\mu\nu}\,, \tag{3.42a}$$

$$\begin{aligned}\Rightarrow \quad h_{\mu\nu}(x) &= -\frac{\lambda}{2}\int \mathrm{d}^4y\,\Delta_{\mathrm{ret}}(x-y)\bar{T}_{\mu\nu}(y)\\ &= -\frac{\lambda}{2}\int \mathrm{d}^4y\,\Delta_{\mathrm{ret}}(x-y)P_{\mu\nu:\alpha\beta}T^{\alpha\beta}(y),\end{aligned} \tag{3.42b}$$

$$\text{with} \quad P_{\mu\nu:\alpha\beta} = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right). \tag{3.42c}$$

Substituting (3.42b) for $h_{\mu\nu}$ in the interaction term results in the following action for the point particles.

$$\begin{aligned} S_{pp} =& \sum_{a=1}^{2}\int\frac{\mathrm{d}^4x}{c}\Bigg\{\frac{1}{2}m_a\dot{x}^2\delta^3(\boldsymbol{x}-\boldsymbol{x}_a)\\ &-\frac{\lambda}{2}\left[\int \mathrm{d}^4y\,\frac{\lambda}{2}\Delta_{\mathrm{ret}}(x-y)P_{\mu\nu:\alpha\beta}T_{\alpha\beta}(y)\right]m_a\dot{x}^\mu\dot{x}^\nu\delta^3(\boldsymbol{x}-\boldsymbol{x}_a)\Bigg\}\\ =& \sum_{a=1}^{2}\int\frac{1}{2}m_a\dot{x}^2\delta^3(\boldsymbol{x}-\boldsymbol{x}_a)\,\frac{\mathrm{d}^4x}{c} + \sum_{b>a}\frac{\lambda^2}{4}\iint\left[m_a\dot{x}^\mu\dot{x}^\nu\delta^3(\boldsymbol{x}-\boldsymbol{x}_a)\right]\\ &\cdot\left\{P_{\mu\nu:\alpha\beta}\,\frac{\delta^3(x^\rho-y^\rho)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\right\}\left[m_b\dot{y}^\alpha\dot{y}^\beta\delta^3(\boldsymbol{y}-\boldsymbol{x}_b)\right]\frac{\mathrm{d}^4x}{c}\,\mathrm{d}^4y \end{aligned} \tag{3.43}$$

The next trick is to first approximate the space-time distance to mostly be in time, for slow moving, not too far separated point particles. Then the middle Dirac delta is approximately $\delta(x^0-y^0)$. After using that approximation to eliminate the y^0 integral, the spatial integrals are next, which due to the remaining Dirac deltas just makes all $\boldsymbol{x}\to\boldsymbol{x}_a(t)$, $\boldsymbol{y}\to\boldsymbol{x}_b(t)$, and $r = |\boldsymbol{x}_1-\boldsymbol{x}_2|$. Lastly, to leading order in powers of c the only contributing factor of $\dot{x}^\mu$ is the time component $\dot{x}_a^0 = \gamma_a c^2 \approx c^2$. The result is

$$\boxed{\begin{aligned} S_{\text{Newt }pp} &= \int\left(\frac{1}{2}\left(m_1v_1^2+m_2v_2^2\right) + \frac{\lambda^2}{8}\frac{m_1m_2c^4}{4\pi r}\right)\mathrm{d}t\\ &= \int\left(\frac{1}{2}\left(m_1v_1^2+m_2v_2^2\right) + \frac{Gm_1m_2}{r}\right)\mathrm{d}t\,, \end{aligned}} \tag{3.44}$$

which is exactly the action of Newtonian theory for two point particles with mass.

Notice in equation (3.43), the potential can be understood as the energy-momentum tensor $T_a^{\mu\nu}(x)$ connected to the energy momentum tensor $T_b^{\alpha\beta}(y)$ by some sort of *propagator*, defined by the contents of the curly brackets. This can be expressed graphically, as in Figure 4.1, and will be further developed in Chapter 4.

Once the action is known, the energy can simply be derived by finding the corresponding Hamiltonian

$$\boxed{H \equiv \sum_i \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L = \dot{q}^i p_i - L,} \tag{3.45}$$

where p_i is the canonical momentum, and q^i are generalized coordinates. This is nothing but a Legendre transformation of the Lagrangian for $\dot{q}^i \to p_i$.

Unsurprisingly, for the Newtonian action it reads

$$H_{\text{Newt}} = \sum_{i=1}^{2} \boldsymbol{v}_i \cdot m_i \boldsymbol{v}_i - L_{\text{Newt}} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - \frac{G m_1 m_2}{r} = E_{\text{Newt}}. \tag{3.46}$$

In relative coordinates it reads (see Appendix B for derivations and tricks)

$$\boxed{E_{\text{Newt}} = \frac{1}{2}\mu v^2 - \frac{GM\mu}{r} \overset{(2.4)}{=} \frac{1}{2}\mu v^2 - r^2\omega^2\mu = -\frac{1}{2}\mu v^2,} \tag{3.47}$$

which is exactly the first order term of the energy expansion (2.7).

3.5 The energy-momentum tensor of gravitational waves

Equipped with the Lagrangian for linearized theory (3.16) it is straight forward to utilize Noether's theorem to obtain the energy-momentum tensor of the graviton field.

$$\text{For coordinate transformations} \quad x^\nu \to x^\nu + \epsilon^a A_a^\nu(x) \tag{3.48a}$$

$$\text{And field transformations} \quad \phi_i(x) \to \phi_i(x) + \epsilon^a F_{i,a}(\phi, \partial\phi) \tag{3.48b}$$

$$\Rightarrow \quad \partial_\nu j_a^\nu = 0, \quad \text{where}$$

$$j_a^\nu = \left[\frac{\partial \mathcal{L}}{\partial \phi_{i,\nu}} \phi_{i,\rho} - \delta_\rho^\nu \mathcal{L}\right] A_a^\rho(x) - \frac{\partial \mathcal{L}}{\partial \phi_{i,\nu}} F_{i,a}(\phi, \partial\phi). \tag{3.48c}$$

Noether's theorem is a central result of modern physics, and for a detailed derivation consult any field theory book, e.g. Kachelrieß [17], Maggiore [9] or Goldstein *et al.* [2].

For pure translations $A_\alpha^\beta = \delta_\alpha^\beta$, $F_{i,\alpha} = 0$, and thus

$$j_\alpha{}^\beta \equiv -t_\alpha{}^\beta = \mathcal{L}_{(2)} \delta_\alpha{}^\beta - \frac{\partial \mathcal{L}_{(2)}}{\partial h_{\mu\nu,\beta}} h_{\mu\nu,\alpha}. \tag{3.49}$$

The terms in this equation is known, the first from (3.2b),[7] and the second from (3.16).

$$t^{\mu\nu} = h_{\sigma\rho}{}^{,\mu} h^{\sigma\rho,\nu} - \frac{1}{2} h^{,\mu} h^{,\nu} + \mathcal{L}_{(2)} \eta^{\mu\nu}. \tag{3.50}$$

[7] To match this expression to (3.16) set $a_1 = -2a_4 = -1/2$ and $a_2 = a_3 = 0$, which is to impose the Lorenz gauge. This gauge can only be imposed *after* evaluating the $\partial\mathcal{L}/\partial h_{\mu\nu,\beta}$ term.

For a wave-packet centred around a reduced wavelength $\lambda \equiv \lambda/2\pi$, the total 4-momentum flux is obtained by integrating over a volume $\mathcal{V} \sim L^3$ where $L \gg \lambda$, such that $t^{\mu\nu}$ is zero on the boundary $\partial\mathcal{V}$. Then the effective energy-momentum tensor is[8]

$$t^{\mu\nu} = \left\langle \mathcal{L}_{(2)} \right\rangle \eta^{\mu\nu} - \left\langle \frac{\partial \mathcal{L}_{(2)}}{\partial h_{\sigma\rho,\nu}} h_{\sigma\rho,\mu} \right\rangle = \left\langle h_{\sigma\rho}{}^{,\mu} h^{\sigma\rho,\nu} \right\rangle - \frac{1}{2} \left\langle h^{,\mu} h^{,\nu} \right\rangle \tag{3.51}$$

Here the $\langle x \rangle$ is to be understood as spatial averaging. Also, for radiation detectable far away from the source the EoM can be considered to be for a vacuum, and thus $\Box \bar{h}_{\mu\nu} = 0$. Setting the vacuum as the zero solution of the h-field the Lagrangian averages to zero $\left\langle \mathcal{L}_{(2)} \right\rangle = 0$, and is thus dropped from (3.51).

Additionally, in vacuum the Lorenz gauge can be promoted to the TT gauge, further imposing that $h = h_{0\mu} = h_{ij}{}^{,j} = 0$ (3.22), results in the final expression for the energy-momentum (pseudo-)[9] tensor

$$t^{\mu\nu} = \left\langle \partial^\mu h_{ij}^{\mathrm{TT}} \partial^\nu h^{ij}_{\mathrm{TT}} \right\rangle, \tag{3.52a}$$

$$P^\mu = \frac{1}{c} \int_{\mathcal{V}} \mathrm{d}^3x \, t^{\mu 0} = \frac{1}{c} \int_{\mathcal{V}} \mathrm{d}^3x \left\langle \partial^\mu h_{ij}^{\mathrm{TT}} \partial^0 h^{ij}_{\mathrm{TT}} \right\rangle . \tag{3.52b}$$

3.5.1 Total radiated energy flux

The total radiated energy flux $\mathcal{F}$ can be obtained from $P^0 = E/c$ and the divergencelessness of the energy-momentum tensor, which follows from Noether's theorem

$$t^{\mu 0}{}_{,\mu} = 0,$$

$$\Rightarrow \quad \frac{\partial P^0}{\partial ct} = -P^i{}_{,i} = -\frac{1}{c} \int_{\mathcal{V}} \mathrm{d}^3x \, \partial_i t^{i0} = -\frac{1}{c} \int_{\partial\mathcal{V}} t^{r0} \, \mathrm{d}A \, , \tag{3.53}$$

where the last equality used the divergence theorem, expressed in spherical coordinates. For the next step notice that $t^{r0} = \left\langle \partial^r h_{mn}^{\mathrm{TT}} \partial^0 h^{mn}_{\mathrm{TT}} \right\rangle$, and recall that h's sufficiently far from the source can be written as $h_{mn}^{\mathrm{TT}} = \frac{1}{r} f_{mn}(t_{\mathrm{ret}})$ with $t_{\mathrm{ret}} = t - r/c$ and f_{mn} some function. Then observe that $\frac{\partial}{\partial r} f_{mn}(t_{\mathrm{ret}}) = -\frac{\partial}{\partial ct} f_{mn}(t_{\mathrm{ret}})$, which makes

$$\frac{\partial}{\partial r} h_{mn}^{\mathrm{TT}}(t, r) = -\partial_0 h_{mn}^{\mathrm{TT}} + \mathcal{O}(r^{-2}) = \partial^0 h_{mn}^{\mathrm{TT}} + \mathcal{O}(r^{-2}). \tag{3.54}$$

[8] The Noehter current need not be physical *in itself*, it is the volume integral of j^ν_a, where j^ν_a goes sufficently fast to zero on the boundary, which is conserved, and thus physical. In field theory one defines the *effective* energy-momentum tensor, which is the average value over the volum integral, i.e. the spatial average, as the physical energy-momentum tensor. All terms lost under spatial averaging would not contribute to physical effects anyway. This is why (3.51) is averaged over space.

An interesting lesson here is that the energy of GWs can not be isolated to one space-time point, which can be understood by them being non-localisable.

[9] Note that this is the energy-momentum tensor in the TT gauge, which is to say it is gauge dependent! This might sound strange, but as Maggiore [9] points out this is also the case for electromagnetism.

In the geometric approach the gauge dependence of $t_{\mu\nu}$ is a coordinate dependence, since the gauge transformation is a coordinate transform in the geometrical picture. Thus $t_{\mu\nu}$ is not a real tensor, but rather a *pseudotensor*. Also in the geometrical picture, $t_{\mu\nu}$ is averaged over several wavelengths to extract the GW contribution to the background metric, making it equivalent to the energy-momentum tensor obtained by field theorists.

Thus in the TT gauge, and sufficiently far from the source, $t^{r0} = t^{00} + \mathcal{O}(r^{-3})$, and

$$-\mathcal{F} = \frac{\mathrm{d}E}{\mathrm{d}t} = c^2 \frac{\partial P^0}{\partial ct} = -c \int \left(t^{00} + \mathcal{O}(r^{-3})\right) r^2 \,\mathrm{d}\Omega = \frac{-r^2}{c} \int \left\langle \dot{h}^{\mathrm{TT}}_{ij} \dot{h}^{ij}_{\mathrm{TT}} \right\rangle \mathrm{d}\Omega \,,$$

$$\boxed{\mathcal{F} = \frac{r^2}{c^2} \int \left\langle \dot{h}^{\mathrm{TT}}_{ij} \dot{h}^{ij}_{\mathrm{TT}} \right\rangle \mathrm{d}\Omega \overset{(3.38)}{=} \frac{\lambda^2}{2^8 \pi^2 c} \int \Lambda^{ij,kl}(\boldsymbol{n}) \left\langle \dddot{Q}_{ij} \dddot{Q}_{kl} \right\rangle \mathrm{d}\Omega \,.} \tag{3.55}$$

Note that $\dot{h} = \frac{\mathrm{d}h}{\mathrm{d}t}$ does not include any additional factors of c. It is apparent that GWs carry energy *out* of a volume, hence the negative sign convention for $\mathcal{F}$. The term of order r^{-3} can be neglected for large values of r, while in the remaining term the r dependence cancels, making it non-vanishing.

The Lambda tensor

Before computing the angular integral it is important to note that the radiation is not isotropic, but rather a function of direction $\boldsymbol{n}$. Following the outline of Maggiore [9] this can be accounted for by introducing the so-called *Lambda tensor*

$$\begin{aligned} \Lambda_{ij:kl}(\boldsymbol{n}) = \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} \\ + \frac{1}{2} n_k n_l \delta_{ij} + \frac{1}{2} n_i n_j \delta_{kl} + \frac{1}{2} n_i n_j n_k n_l. \end{aligned} \tag{3.56}$$

The Lambda tensor is a projection operator which is defined in such a way that it projects a wave already in the Lorenz gauge (3.13) into the TT gauge (3.22).

$$h^{\mathrm{TT}}_{ij}(\boldsymbol{n}) = \Lambda_{ij}{}^{kl}(\boldsymbol{n}) h_{kl}, \tag{3.57}$$

which makes sure the wave is transverse with respect to the direction of propagation $\boldsymbol{n}$.

The Lambda tensor has the property

$$\Lambda_{ij}{}^{kl} \Lambda_{kl}{}^{mn} = \Lambda_{ij}{}^{mn}, \tag{3.58}$$

making it a projection operator. Thus, the contraction of two waves in the TT gauge becomes

$$h^{\mathrm{TT}}_{ij} h^{ij}_{\mathrm{TT}} = \Lambda_{ij}{}^{kl} h_{kl} \Lambda^{ij}{}_{mn} h^{mn} = \Lambda_{mn}{}^{ij} \Lambda_{ij}{}^{kl} h^{mn} h_{kl} = \Lambda_{mn}{}^{kl} h^{mn} h_{kl}, \tag{3.59}$$

i.e. the Lambda tensors can be used to contract tensors not in the TT to get the contraction of the TT gauged tensors, which is exactly what is needed for equation (3.55). The last equality utilized the property $\Lambda_{ij:kl} = \Lambda_{kl:ij}$ of the Lambda tensor.

When integrating over the Lambda tensor the following integral appears

$$\int \frac{\mathrm{d}\Omega}{4\pi} n_{i_1} n_{i_2} \cdots n_{i_{2\ell-1}} n_{i_{2\ell}} = \frac{1}{(2\ell+1)!!} \left(\delta_{i_1 i_2} \cdots \delta_{i_{2\ell-1} i_{2\ell}} + \dots\right). \tag{3.60}$$

Here $\boldsymbol{n}$ is taken to be a unit vector $|\boldsymbol{n}| = 1$. To illustrate the solution take $\ell = 1 \rightarrow \int \mathrm{d}\Omega\,/(4\pi) n_i n_j = S_{ij}$. The trace of $S_i{}^i = 1$, because $n_i n^i = |\boldsymbol{n}| = 1$, reducing the integral to the scalar integral $\int \mathrm{d}\Omega = 4\pi$. The tensor structure of S_{ij} should be that of the Kronecker delta, as it should be symmetric in its indices, and have a non-vanishing trace. Thus, $S_{ij} = \delta_{ij}/3$.

Generalizing this argument it follows that the integral (3.60) should be the sum of all possible Kronecker delta combinations, to maintain index symmetry. This is indicated by the dots on the RHS of (3.60). The symbol !! is here meant to imply the product of every second number: $(2\ell+1)!! = 1\cdot 3\cdots(2\ell-1)\cdot(2\ell+1)$, and $2\ell!! = 2\cdot 4\cdots(2\ell-2)\cdot 2\ell$. This fixes the correct trace value, just like the factor of $1/3$ for the $\ell = 1$ case. Note that all integrals over an odd number of unit vector components are 0 because the integral is also odd then.

Thus, the integral of the Lambda tensor becomes

$$\begin{aligned}
\int \frac{\mathrm{d}\Omega}{4\pi}\Lambda_{ij:kl} &= \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - \delta_{ik}\int\frac{\mathrm{d}\Omega}{4\pi}n_j n_l - \delta_{jl}\int\frac{\mathrm{d}\Omega}{4\pi}n_i n_k \\
&\quad + \frac{1}{2}\delta_{ij}\int\frac{\mathrm{d}\Omega}{4\pi}n_k n_l + \frac{1}{2}\delta_{kl}\int\frac{\mathrm{d}\Omega}{4\pi}n_i n_j + \frac{1}{2}\int\frac{\mathrm{d}\Omega}{4\pi}n_i n_j n_k n_l \\
&= \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - \delta_{ik}\frac{1}{3}\delta_{jl} - \delta_{jl}\frac{1}{3}\delta_{ik} + \frac{1}{2}\delta_{ij}\frac{1}{3}\delta_{kl} + \frac{1}{2}\delta_{kl}\frac{1}{3}\delta_{ij} \\
&\quad + \frac{1}{2}\frac{1}{3\cdot 5}\left(\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}\right) \\
&= \frac{11}{30}\delta_{ik}\delta_{jl} - \frac{2}{15}\delta_{ij}\delta_{kl} + \frac{1}{30}\delta_{il}\delta_{kj}.
\end{aligned} \tag{3.61}$$

Utilizing this result for (3.55), and the fact that $Q_{ij} = Q_{ji}$, yields the flux in terms of the mass quadrupole

$$\boxed{\mathcal{F} = \frac{\lambda^2}{2^5\cdot 5\pi c}\left\langle \dddot{Q}_{ij}\dddot{Q}^{ij} - \frac{1}{3}\dddot{Q}^2\right\rangle = \frac{G}{5c^5}\left\langle \dddot{Q}_{ij}\dddot{Q}^{ij} - \frac{1}{3}\dddot{Q}^2\right\rangle,} \tag{3.62}$$

with $Q \equiv Q_i^{\ i}$.

This is an important dynamical property of GWs, that they can carry energy, momentum, and angular momentum out of a system. For a compact binary this influences the orbital energy, extracting energy from it over time, make them spiral in towards each other. The fact that there is no analytical exact solution to the relativistic binary problem is often attributed to this effect. The orbit is affected by GW radiation, which is determined by the orbit, making the problem complicated.

3.6 Illustrative example: Binary system with circular orbits

As an illustrative example this section will calculate the exact form of GW produced by binary systems in circular orbits, according to linearized theory (3.16).

Those already familiar with GWs may skip this section, as this system will be covered in greater detail in the following chapters, and is here mearly used as an example to better illustrate the result (3.38) and (3.62).

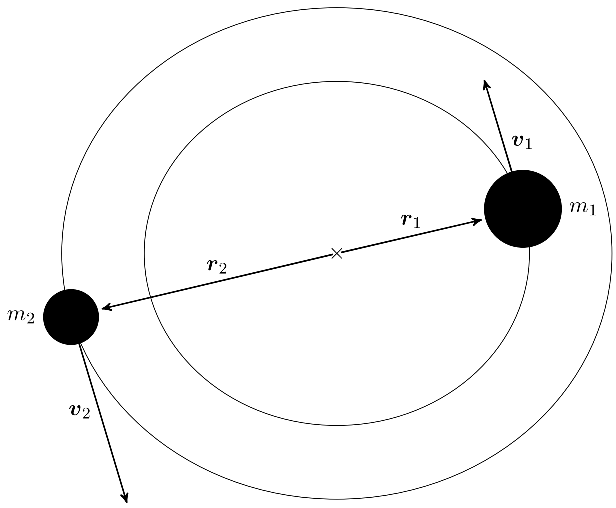


Figure 3.1: Diagram of a binary system.

Taking the centre of mass as the origin, the positions of the stars are

$$\boldsymbol{r}_1 = \big(r_1 \cos(\omega t), \quad r_1 \sin(\omega t), \quad 0\big)\,, \tag{3.63a}$$

$$\boldsymbol{r}_2 = -\,\big(r_2 \cos(\omega t), \quad r_2 \sin(\omega t), \quad 0\big)\,. \tag{3.63b}$$

This determines the mass density as

$$\rho(t, \boldsymbol{r}) = \sum_a m_a \delta^3(\boldsymbol{r} - \boldsymbol{r}_a(t)) \tag{3.64}$$

The quadrupole moment of the binary system is thus

$$\begin{aligned} Q_{ij}(t) &= \int \rho x_i x_j \, \mathrm{d}^3 x = \sum_a m_a (x_a)_i (x_a)_j \\ &= (m_1 r_1^2 + m_2 r_2^2) \begin{pmatrix} \cos^2(\omega t) & \cos(\omega t)\sin(\omega t) & 0 \\ \cos(\omega t)\sin(\omega t) & \sin^2(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \frac{\mu r^2}{2} \begin{pmatrix} \cos(2\omega t) & \sin(2\omega t) & 0 \\ \sin(2\omega t) & -\cos(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{3.65}$$

$$\Rightarrow \quad \ddot{Q}_{ij}(t) = 2\mu(\omega r)^2 \begin{pmatrix} -\cos(2\omega t) & -\sin(2\omega t) & 0 \\ -\sin(2\omega t) & \cos(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.66}$$

Details of mass and trigonometric term manipulations can be found in Appendix B and C respectively.

Thus, the GWs produced by a circular binary system is determined by (3.38) and (3.66) to be

$$h_{ij}(t,\boldsymbol{R}) = -\frac{\lambda}{8\pi}\frac{\mu v^2}{R}\begin{pmatrix}\cos(2\omega t_{\rm ret}) & \sin(2\omega t_{\rm ret}) & 0\\ \sin(2\omega t_{\rm ret}) & -\cos(2\omega t_{\rm ret}) & 0\\ 0 & 0 & 0\end{pmatrix} \tag{3.67}$$

For convenience $r\omega = (r_1 + r_2)\omega$ has been replaced by v, which is the sum of the velocities of the stars. This is equivalent to the frequency parameter $v^3 = GM\omega$ introduced in Chapter 2.

It should be noted that the wave frequency $\omega_{\rm GW} = 2\omega = 2\omega_{\rm s}$, with $\omega_{\rm s}$ as the source frequency, is the dominant frequency for circular orbits, and that the amplitude is proportional to the frequency $v^2 = (GM\omega)^{2/3}$. This makes the GW highly dependent on the frequency of the source binary, both for the amplitude, and frequency spectrum.

The energy flux produced by such a system is now easily computable by using equation (3.62) together with (3.66).

$$\begin{aligned}\mathcal{F} &= \frac{\lambda^2}{2^5\cdot 5\pi c}\left\langle \dddot{Q}_{ij}\dddot{Q}^{ij} - \frac{1}{3}\dddot{Q}^2\right\rangle\\ &= \frac{\lambda^2}{2^5\cdot 5\pi c}2^4\mu^2 v^4\omega^2\left\langle 2\sin^2(2\omega t_{\rm ret}) + 2\cos^2(2\omega t_{\rm ret}) - 0\right\rangle\\ &= \frac{\lambda^2\mu^2 v^{10}}{5\pi G^2M^2c} = \frac{32}{5}\frac{\eta^2}{G}\frac{v^{10}}{c^5}.\end{aligned} \tag{3.68}$$

This is exactly the first order term of the flux expansion (2.8).

Furthermore, circular orbits *is* a solution of the Newtonian, point particle, action (3.44). If the source interact with the graviton field, and only the graviton field, this implies that the orbital energy of the system is $E_{\rm Newt} = -\frac{1}{2}\mu v^2$ (3.47).

With (3.68) and (3.47), the Newtonian order (0PN) version of the phase (2.26) can be derived, assuming compact binaries can effectively be treated as point masses.

The flux is obviously not zero, which means that energy is dissipated out of the system. However, according to the Lagrangian (3.16) the energy of the source $T^{\mu\nu}$ *is* conserved. Therefore, this energy radiation is an inconsistency of the theory.

This hits deeply into the problem of gravity. As a field theory it couples to energy, while as a field it itself stores energy, and should therefore couple to itself. Amending this feature is the topic of the next section.

3.7 Graviton action beyond quadratic order

In Section 3.5 it was found that gravitons / GWs can carry energy, and can even be given an effective energy-momentum (pseudo-)tensor of its own. But since the graviton field couples to energy-momentum tensors, should it not couple to itself?

In Section 3.6 it was even found that a binary system of point particles radiated energy in the form of GWs, but where does this energy come from? One natural assumption would be that the energy is extracted from the orbital energy associated with the system, leading to an inspiral. But this would mean that the energy-momentum tensor of the binary $T_{\mu\nu}{}^{,\nu} \neq 0$, as it changes over time. Furthermore, $T^{\mu\nu}{}_{,\nu} = 0$ was used to determine the graviton action in the first place, making the conservation of energy of the source, and the energy carrying capability of the graviton field a theoretical inconsistency!

The solution is to *change* the energy conservation criteria (3.3b) to

$$(T^{\mu\nu} + t^{\mu\nu})_{,\nu} = 0, \tag{3.69}$$

which is to make the energy of the source *plus* the energy stored in gravitons (i.e. in GWs) conserved, together.

To implement this in a new and improved Lagrangian for the graviton field the first naive idea would be to couple the graviton directly to the graviton energy-momentum tensor $\mathcal{L}_{\text{int}} \to \frac{\lambda}{2} h_{\mu\nu} (T^{\mu\nu} + t^{\mu\nu})$. This will however lead to problems for the EoM, as

$$\begin{aligned} &\frac{\delta}{\delta h_{\mu\nu}} \frac{\lambda}{2} h_{\mu\nu} t^{\mu\nu} = \frac{\lambda}{2} t^{\mu\nu} + \frac{\lambda}{2} h_{\rho\sigma} \frac{\delta}{\delta h_{\mu\nu}} t^{\rho\sigma} \neq \frac{\lambda}{2} t^{\mu\nu} \\ &\text{since} \quad t^{\mu\nu} \overset{(3.50)}{=} h_{\sigma\tau}{}^{,\mu} h^{\sigma\tau,\nu} - \frac{1}{2} h^{,\mu} h^{,\nu} + \mathcal{L}_{(2)} \eta^{\mu\nu}. \end{aligned} \tag{3.70}$$

Since it is the EoM which dictates the physics of any given action, one would rather demand that

$$\frac{\delta}{\delta h_{\mu\nu}} \mathcal{L}^{*}_{\text{int}} = \frac{\lambda}{2} (T^{\mu\nu} + t^{\mu\nu}). \tag{3.71}$$

This requires the addition of a more general cubic term to the Lagrangian $\frac{\lambda}{2}\mathcal{L}_{(3)}$ such that

$$\frac{\delta}{\delta h_{\mu\nu}} \frac{\lambda}{2} \mathcal{L}_{(3)} = \frac{\lambda}{2} t^{\mu\nu}. \tag{3.72}$$

Writing out all possible terms cubic in h, containing only two derivatives (necessary to obtain $t^{\mu\nu}$), which also produces $t^{\mu\nu}$ when varied (3.72), together with new energy conservation condition (3.69) fixes the 18 coefficients uniquely (up to the same overall factor of $\mathcal{L}_{(2)}$). This is obviously a lot of work, but the result can be quoted from Feynman [12] (equation (6.1.13)).[10]

$$\begin{aligned} \frac{\lambda}{2}\mathcal{L}_{(3)} = \frac{\lambda}{2}\Big[& h^{\alpha\beta} h^{\gamma\delta} h_{\alpha\beta,\gamma\delta} - \frac{1}{2} h h^{\alpha\beta} h_{,\alpha\beta} - \frac{1}{2} h h^{\alpha\beta} \Box h_{\alpha\beta} - \frac{3}{8} h^2 \Box h \\ & + h_{\gamma}{}^{\beta} h^{\gamma\alpha} \Box h_{\alpha\beta} - \frac{3}{4} h_{\alpha\beta} h^{\alpha\beta} \Box h - 2 h^{\alpha\beta} h_{\beta\delta} h_{\alpha\gamma}{}^{,\gamma\delta} + h^{\alpha\beta} h_{\beta\delta} h_{,\alpha}{}^{\delta} \\ & + 2 h_{\alpha\beta} h^{\sigma\alpha}{}_{,\sigma} h^{\tau\alpha}{}_{,\tau} - 2 h_{\alpha\beta} h^{\sigma\alpha}{}_{,\sigma} h^{,\beta} + \frac{1}{2} h_{\alpha\beta} h^{,\alpha} h^{,\beta} - h h^{\sigma\alpha}{}_{,\sigma} h^{\tau}{}_{\alpha,\tau} \\ & + h h^{\sigma\alpha}{}_{,\sigma} h_{,\alpha} - \frac{1}{4} h h_{,\alpha} h^{,\alpha} + \frac{1}{2} h_{\alpha\beta} h^{\alpha\beta} h^{\sigma\tau}{}_{,\sigma\tau} + \frac{1}{4} h^2 h^{\sigma\tau}{}_{,\sigma\tau} \Big]. \end{aligned} \tag{3.73}$$

[10] Note that Feynman uses a different overall scaling factor for his graviton action ($a_1 = 1$, rather than $a_1 = -1/2$). In (3.73) all barred factors of h have been written out, also in contrast to (6.1.13) of [12].

Using again Noether's theorem on this new Lagrangian $\mathcal{L}_{(2)} + \mathcal{L}_{(3)}$, the new effective energy-momentum tensor picks up a term cubic in h:

$$t^{\mu\nu} = t^{\mu\nu}_{(2)} + \frac{\lambda}{2} t^{\mu\nu}_{(3)}. \tag{3.74}$$

Again it can be argued that the cubic term here is not conserved, and the process of this section can be started all over again. In fact, it turns out that

$$\boxed{\mathcal{L}_{\text{grav}} = \mathcal{L}_{\text{int}} + \mathcal{L}_{(2)} + \frac{\lambda}{2}\mathcal{L}_{(3)} + \frac{\lambda^2}{2^2}\mathcal{L}_{(4)} + \dots,} \tag{3.75}$$

with the EoM

$$\boxed{\Box\,\bar{h}_{\mu\nu} = -\frac{\lambda}{2}\left(T_{\mu\nu} + t^{(2)}_{\mu\nu} + \frac{\lambda}{2} t^{(3)}_{\mu\nu} + \dots\right).} \tag{3.76}$$

This makes the equation of motion nonlinear, and in general both the action and the EoM contain infinitely high powers of h.

By now the Venusians would probably be disappointed, but for terrestrial physicists this should be expected, as GR is also a nonlinear theory. When expanded in powers of metric perturbations, also the Einstein-Hilbert action contain infinitely high powers of h, which coincide with the action found thus far in this thesis.[11]

An important observation is that all these corrections scale with increasing powers of λ, which is small. Thereby the more powers of h it contains, the less it contributes to the final result. Thus, the theory can be perturbatively expanded in powers of λ.

[11] There are many who argue that the procedure outlined here can be carried out ad infinitum, and then reproduces the Einstein-Hilbert action, but this is disputed. The Einstein-Hilbert action can definitely be expanded in this manner, but it is disputed wether or not the field theorists' method produces *uniquely* the Einstein-Hilbert action. [20]

Chapter 4

Calculating the orbital energy

In this chapter the 1PN energy of compact binaries in circular motion is computed.

The derivation follows closely those presented in Porto [4], and Goldberger [19], [3].

4.1 Effective field theory

In the last chapter examples focused on point particles. Compact objects like BHs and NSs are however not point particles. But for sufficiently far separated binaries, the separation distance r will be much greater than the 'size' of the compact object, which can be approximated as the Schwarzschild radius $\sim R_S \ll r$. Then the system can be described by an *effective action*, treating the compact object as a point mass at the scale $\sim r$.

On the other end of the scale the system is producing GWs, carrying energy out of the system. But when calculating the energy flux in Section 3.5, the source was assumed to effectively be a point endowed with quadrupole structure (which will be expanded to a general multipole structure in Chapter 5), since the GW was measured far away. That is, the flux is measured at a scale $L \gg \lambda \gg r$.[1] Thus, in this chapter the effective action at the scale r, with $R_S \ll r \ll L$, will be derived, which approximates the immediate[2] orbital dynamics.

To compute the 1PN orbital energy the procedure of Section 3.4 will be expanded. In Section 3.4 it was found that the graviton potential could be expressed graphically like Figure 4.1. In Figure 4.1 the two sources are depicted as solid lines, like in a space-time diagram, while the Green's function is depicted as a squiggly line, connecting the two point particles. Graphical representations of this kind are called *Feynman diagrams*, as they were introduced by Feynman [21] to illustrate expansion terms of QED. Note that the Green's function has been given the subscript 'inst', to remind that the Newtonian action was recovered by approximating $\delta^4(x^\mu - y^\mu) \approx \delta(t - t')$, making the interaction *instantaneous*.

[1] The wavelength can be shown to be greater than the size of the binary by noticing two things. First: $\omega_{\rm GW} \sim 2\omega_{\rm s}$, as approximated in (3.67). The reduced wavelength $\lambda = \lambda/2\pi$ is related to the angular frequency as $\lambda = c/\omega$. Second: Kepler's third law (2.4) relate ω and r: $\lambda_{\rm GW} = \frac{c}{2\sqrt{GM}} r^{3/2} = \sqrt{\frac{r}{2R_S}}\, r \gg r$.

[2] Immediate because over time the energy loss through GW emssion can not be ignored. However over short periodes of time, the orbital energy can be approximated to be conserved.

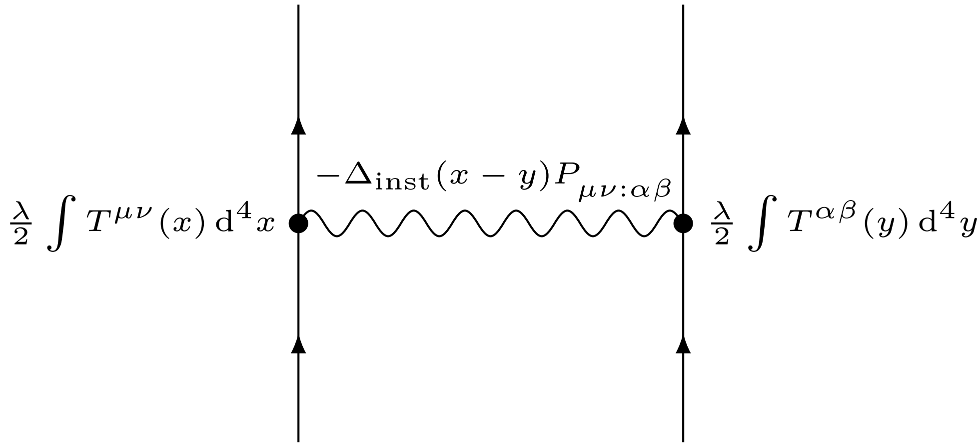


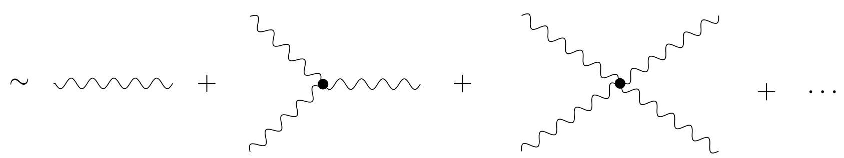
Figure 4.1: Newtonian Feynman diagram / 'H-diagram'.

For those familiar with Feynman diagrams, note that the solid lines are *not propagating*, even though they are depicted the same way as propagating fermions in QFT. This can be confusing, but it is the convention introduced by Goldberger and Rothstein [3], and has by now become the standard.

4.1.1 Expand the action in powers of h

Based on the findings of Section 3.7, the Einstein-Hilbert action S_{EH} can be used to find the expression of the *propagator* for the $h_{\mu\nu}$ field, by solving equation (3.76). Expanding the Einstein-Hilbert action in powers of h yields the contribution for the different levels of self interaction, which all scale with additional powers of the coupling constant λ.

$$S_{\text{EH}} \sim \frac{1}{2}\int \mathrm{d}^4x \left[(\partial h)^2 + \lambda h(\partial h)^2 + \lambda^2 h^2(\partial h)^2 + \dots\right] \tag{4.1}$$



Note that the terms in (4.1) are symbolic representations for the powers of $h_{\mu\nu}$, and that all terms contain only two derivatives.

Why does the first order correction lead to a three graviton vertex? Looking at the leading order correction in the EoM (3.76)

$$\begin{aligned} \Box\, h_{\mu\nu} &= -\frac{\lambda}{2}P_{\mu\nu:\alpha\beta}\left(t^{\alpha\beta}_{(2)}\right) \sim \frac{\lambda}{2}P_{\mu\nu:\alpha\beta}\left(\partial h \partial h\right), \\ \Rightarrow \quad h_{\mu\nu}(x) &\sim -\int \mathrm{d}^4y_1\,\mathrm{d}^4y_2\,\partial T^{\alpha\beta}(y_1)\partial T^{\alpha\beta}(y_2)\Delta_{\text{inst}}(x-y_1)\Delta_{\text{inst}}(x-y_2), \end{aligned} \tag{4.2}$$

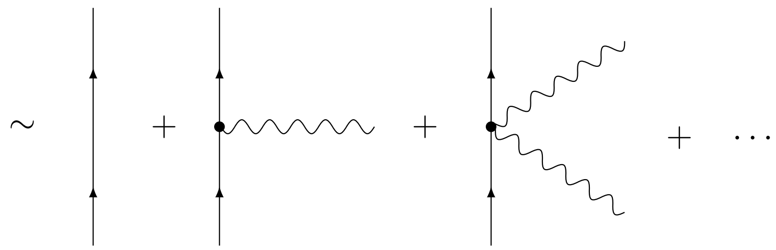
where again $\sim$ implies symbolic relation of powers of $h_{\mu\nu}$, rather than the exact relation. The leading order term of this three-point propagator will be derived in Section 4.2.7.

It can however be seen from this simple relation, where the h's of the RHS was substituted for the linear order solution, that at this order h is produced by two sources, which in principle can be at different space-time points. Therefore, an interaction $h_{\mu\nu}T^{\mu\nu}(x)$ in the action will to next order connect three energy-momentum tensors, using two Green's functions. This can neatly be visualized by a three graviton vertex.

In Section 3.4 the action of point particles (3.39) was found to also be expandable in powers of h, similarly to the Einstein-Hilbert action.

$$S_{pp} = -\, mc^2 \int \mathrm{d}\tau \sqrt{1 - \lambda h_{\mu\nu} \frac{\dot{x}^\mu}{c} \frac{\dot{x}^\nu}{c}} \tag{4.3a}$$

$$= -\, mc^2 \int \gamma^{-1}\, \mathrm{d}t + \frac{\lambda}{2} \int \mathrm{d}t\, h_{\mu\nu} \gamma^{-1} m \dot{x}^\mu \dot{x}^\nu + \frac{m\lambda^2}{8c^2} \int \mathrm{d}t\, \gamma^{-1} \left(h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu\right)^2 + \dots \tag{4.3b}$$



Here again the choice of having two gravitons coupling to the world line for the $\sim \left(h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right)^2$ term is natural as the integral can be split in two integrals multiplied together
$\sim \int \mathrm{d}^3x \left(h_{\mu\nu}(t,\boldsymbol{x})\dot{x}^\mu\dot{x}^\nu\right)\delta^3(\boldsymbol{x}-\boldsymbol{x}_a(t)) \int \mathrm{d}y \left(h_{\mu\nu}(t,\boldsymbol{y})\dot{y}^\mu\dot{y}^\nu\delta^3(\boldsymbol{y}-\boldsymbol{x}_a(t))\right)$, which couples this source effectively to two different gravitons.

In Section 3.4 this was argued to be a natural choice for the point particle action, as it is the expansion of the action of GR. It is also a natural choice for a classical field theory with a coupling to a symmetric rank 2 tensor field, and is equivalent to equation (11.40) of Gourgoulhon [15].

4.1.2 Separation of scale

Because of the separation of scale, $\sim r \gg R_S$ and $\sim L \gg r$, it will be useful to split the graviton field into a short-range, potential field ($H_{\mu\nu}$) and a long range, radiation field ($\mathcal{H}_{\mu\nu}$)

$$\boxed{h_{\mu\nu} = H_{\mu\nu} + \mathcal{H}_{\mu\nu}.} \tag{4.4}$$

From here on out the potential field will be drawn using dashed lines, while the radiation field will continue to be drawn using squiggly lines.

The frequency of GWs are proportional to the frequency of the source binary $\omega_{\mathrm{GW}} \simeq 2\omega = 2\omega_{\mathrm{s}}$, and the relative velocity of a binary in circular motion is $v = \omega r$. Using the relation between null wave frequency and wavelength $\omega = c|\boldsymbol{k}| = c\lambda^{-1}$ the wavelength of GWs scale as

$$\lambda_{\rm GW} \simeq \frac{c}{2\omega} = \frac{rc}{2v} \gg r, \tag{4.5}$$

for binaries moving at non-relativistic speeds.

Because $\mathcal{H}$ is null-like, it follows that $k_\sigma k^\sigma = 0$, and

$$\boxed{\partial_\alpha \mathcal{H}_{\mu\nu} = k_\alpha \mathcal{H}_{\mu\nu} \sim \lambda_{\rm GW}^{-1} \mathcal{H}_{\mu\nu} \simeq \frac{v}{r}\mathcal{H}_{\mu\nu},} \tag{4.6}$$

i.e. $\mathcal{H}$ must be *on shell.*

H on the other hand is a potential field, and *can not* be on shell, since it shall reproduce the gravitational potential in the static limit. In Section 3.4 this was achieved by approximating $\delta^4(x-y) \approx \delta(t-t')$. This 'instantaneous' propagator is *not* the Green's function of the d'Alembertian operator

$$\Delta_{\rm inst}(x-y) = \frac{-\delta(t-t')}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{ik_\sigma(x^\sigma - y^\sigma)}}{-\boldsymbol{k}^2}, \tag{4.7}$$

but it *is* the Green's function of the *Laplace operator* $\nabla^2 \equiv \partial_i \partial^i$. This integral scales proportional to $|\boldsymbol{k}|$

$$\int \mathrm{d}^3 k \, \frac{1}{\boldsymbol{k}^2} \sim \frac{|\boldsymbol{k}|^3}{\boldsymbol{k}^2} \sim k_i \sim \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}. \tag{4.8}$$

Assuming k_0 is small compared to k_i, the instantaneous propagator may be obtained as a leading order term of an expansion in $k_0/|\boldsymbol{k}|$

$$\begin{aligned} \Delta_{\rm inst}(k) \equiv \frac{1}{-k_\mu k^\mu} &= \frac{1}{k_0^2 - \boldsymbol{k}^2} = \frac{1}{-\boldsymbol{k}^2}\frac{1}{1 - k_0^2/\boldsymbol{k}^2} \\ &= \frac{1}{-\boldsymbol{k}^2}\left(1 + \left(\frac{k_0}{\boldsymbol{k}}\right)^2 + \left(\frac{k_0}{\boldsymbol{k}}\right)^4 + \dots\right). \end{aligned} \tag{4.9}$$

$\sim$ ---------- $+$ ----$\otimes$---- $+$ --$\otimes$--$\otimes$-- $+$ $\cdots$

Not only is this a highly constructed expansion to obtain a desired leading order term, it will be demonstrated that this expansion *actually* scales as an expansion in $(v/c)^2$, and will be drawn graphically by $\otimes$ on the propagator to show which order in the $(k_0/|\boldsymbol{k}|)^2$ expansion it represents. This also implies that $k_0/k_i \sim v \Rightarrow k_0 \sim vk_i \sim v/r$, and in conclusion

$$\boxed{\partial_0 H_{\mu\nu} = k_0 H_{\mu\nu} \sim \frac{v}{r} H_{\mu\nu}, \quad \partial_i H_{\mu\nu} = k_i H_{\mu\nu} \sim \frac{1}{r} H_{\mu\nu}.} \tag{4.10}$$

Now Feynman diagrams may be constructed by putting together terms as presented in (4.1), (4.3), and (4.9), but the following three rules must be upheld to make sense as an expansion term in the point particle action.

1. Diagrams must remain connected if the particle lines are stripped off.
2. Diagrams may only contain internal $H_{\mu\nu}$ lines.
3. Diagrams may only contain external $\mathcal{H}_{\mu\nu}$ lines.

Rule 2-3 follow by definition of $H_{\mu\nu}$ and $\mathcal{H}_{\mu\nu}$. Rule 1 however seems more mysterious, but it is a consequence of the solid lines *not propagating*, and thus is just a requirement that the Feynman diagram is connected, as all Feynman diagrams must. Unconnected diagrams are simply separate diagrams, and represents multiple terms in the expansion at once.

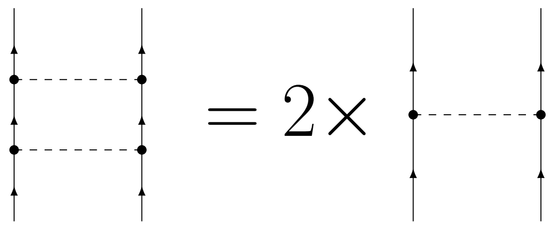


Figure 4.2: 'Ladder' Feynman diagram. With non-propagating sources, unconnected graviton lines simply represent different diagrams.

4.2 The 1PN Lagrangian

In order to acquire relativistic, post-Newtonian, corrections to the Newtonian Lagrangian (3.44) it is a straightforward matter to just add additional terms of the form found in expansion (4.1), (4.3), and (4.9) to the action. But (4.1) and (4.3) are expansions in λ and H rather than in $(v/c)^2$, so how can it be determined which Feynman diagrams contribute at which PN order?

4.2.1 Assigning PN order to Feynman diagrams

Using the scaling of k_μ for the potential field (4.10) and Kepler's law (2.4), it turns out that it is possible to assign a power of v, r and m to each Feynman diagram, and thus select the appropriate diagrams and terms relevant to each PN order, all without doing the full calculation! This is why the diagrams are introduced in the first place, as they are tools to make it easier to order expansion terms.

The scaling of coordinate (integration variables) follows from the relations of k_μ.

$$\begin{aligned}\int \mathrm{d}x^0 \int \frac{\mathrm{d}k_0}{2\pi} e^{ik_0 x^0} &= \int \mathrm{d}x_0\, \delta(x_0) = 1 \sim 1, \\ &\sim x^0 \cdot k_0 \sim x^0 \cdot \frac{v}{r}\end{aligned} \tag{4.11a}$$

$$\begin{aligned}\int \mathrm{d}^3x \int \frac{\mathrm{d}^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} &= \int \mathrm{d}^3x\, \delta^3(\boldsymbol{x}) = 1 \sim 1 \\ &\sim \left(x^i k_i\right)^3 \sim \left(x^i \cdot \frac{1}{r}\right)^3\end{aligned} \tag{4.11b}$$

$$\Rightarrow \boxed{x^0 \sim \frac{r}{v}, \quad x^i \sim r.} \tag{4.11c}$$

Notice that Dirac delta functions carries inverse dimension and scaling of its argument (and the exponential function is of course dimensionless)[3].

For each graviton $H_{\mu\nu}$ in a diagram, it scales as

$$\left\langle T H_{\mu\nu}(x) H_{\alpha\beta}(y)\right\rangle = \Delta_{\text{inst}}(x-y) P_{\mu\nu:\alpha\beta} = \frac{-\delta(t-t')}{4\pi r} P_{\mu\nu:\alpha\beta} \sim \frac{v}{r} \cdot \frac{1}{r} \sim \frac{v}{r^2}, \tag{4.12a}$$

$$\Rightarrow \boxed{H_{\mu\nu}(x) \sim \frac{\sqrt{v}}{r}.} \tag{4.12b}$$

Note that to leading order in interaction terms $H_{\mu\nu}$ couples only to temporal components $\mu = \nu = 0$, since $\dot{x}^0 = \gamma c \sim c$, and $\dot{x}^i = \gamma v^i \sim v$, relegating spatial indices to higher PN orders compared to the temporal ones.

Since (4.1) and (4.3) expands in powers of λ, it would be useful to associate a scaling with the coupling constant. This can be achieved using Kepler's third law (2.4), and $v = \omega r$.

$$v^2 = \omega^2 r^2 = \frac{GM}{r^3} r^2 = \frac{GM}{r} = \frac{(\lambda^2 c^4/32\pi)M}{r}, \tag{4.13a}$$

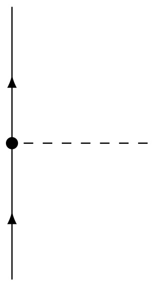
$$\lambda^2 \sim \frac{v^2 r}{m} \sim \frac{(rmv)\cdot v}{m^2} \tag{4.13b}$$

$$\Rightarrow \boxed{\lambda \sim \frac{\sqrt{Lv}}{m}.} \tag{4.13c}$$

The orbital angular momentum scale $L \sim rmv$ has been introduced as a convenient scaling, as will be demonstrated shortly.

It is now a straightforward exercise to assign PN orders to different diagrams:

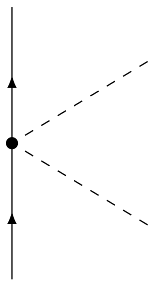
[3] Since $e^x = 1 + x + \frac{x^2}{2!} + \ldots$, and all the terms in a sum must have the same dimesion.



$$\sim \frac{m\lambda}{2}\int \mathrm{d}x^0\, H_{00}\dot{x}^0_a\dot{x}^0_a \sim \frac{m\sqrt{Lv}}{m}\cdot\frac{r}{v}\cdot\frac{\sqrt{v}}{r}\sim\sqrt{L}. \tag{4.14}$$

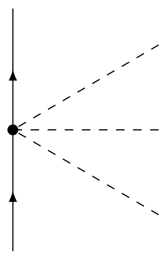
Thus the Newtonian diagram, Figure 4.1, scales as (4.14) squared, making L the scaling of the 0PN order. This is why introducing $L = rmv$ as a scaling is convenient, as it easily makes the leading order scaling apparent. Now, all 1PN diagrams should scale as Lv^2

The next interaction graph is similarly found to scale



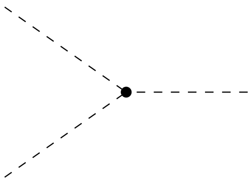
$$\sim \frac{m\lambda^2}{8}\int \mathrm{d}x^0 \left(H_{00}(x)\dot{x}^0_a\dot{x}^0_a\right)^2 \sim \frac{(rmv)v}{m}\cdot\frac{r}{v}\cdot\frac{v}{r^2}\sim v^2. \tag{4.15}$$

And so it goes on...



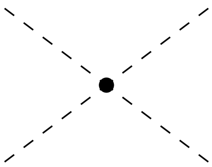
$$\begin{aligned}&\sim \frac{m\lambda^3}{16}\int \mathrm{d}x^0 \left(H_{00}(x)\dot{x}^0_a\dot{x}^0_a\right)^3 \sim m\frac{(rmv^2)^{\frac{3}{2}}}{m^3}\cdot\frac{r}{v}\cdot\frac{v^{\frac{3}{2}}}{r^3}\\ &\sim v^4/\sqrt{L}.\end{aligned} \tag{4.16}$$

The last type of diagram that needs to be assigned a scaling is the multi graviton vertex propagators. Note the inclusion of a three-dimensional Dirac delta, to make sure momentum ($\boldsymbol{k}_i$ for graviton i) is conserved. This is because the gravitons only transfers momentum from one source to another (Newton's third law), and therefore the sum of momentum in and out of this vertex should be zero. Otherwise, the graviton field would spontaneously generate momentum and energy. The two spatial derivatives comes from the definition of $t^{\mu\nu}_{(2)}$, which consist of two derivatives $\partial_j \sim x_j^{-1} \sim k_j$, and two factors of H, which is the reason the propagator exists in the first place. Notice that $\partial_j \sim r^{-1}$, while $\partial_0 \sim \frac{v}{r}$, relegating temporal derivatives to higher PN orders.



$$
\begin{aligned}
&\sim \lambda \int \mathrm{d}x^0\, \delta^3\!\left(\sum_{i=1}^{3} \boldsymbol{k}_i\right) \partial_j^2 \left(H_{00}\right)^3 \\
&\sim \frac{\sqrt{Lv}}{m} \cdot \frac{r}{v} \cdot r^3 \cdot r^{-2} \cdot \left(\frac{\sqrt{v}}{r}\right)^3 = v^2/\sqrt{L}
\end{aligned}
\tag{4.17}
$$

At the next order, the only change is an additional factor of λ, and H. This follows from the expansion of the graviton energy-momentum tensor (3.76).



$$
\begin{aligned}
&\sim \lambda^2 \int \mathrm{d}x^0\, \delta^3\!\left(\sum_{i=1}^{3} \boldsymbol{k}_i\right) \partial_j^2 \left(H_{00}\right)^4 \\
&\sim \frac{(rmv)v}{m^2} \cdot \frac{r}{v} \cdot r^3 \cdot r^{-2} \cdot \frac{v^2}{r^4} = v^4/L
\end{aligned}
\tag{4.18}
$$

As already mentioned, spatial indices in the interaction term between the source and graviton leads to additional powers of velocity ($H_{0i}\dot{x}^0\dot{x}^i = H_{0i}\gamma c\gamma v^i$), and thus belong to higher PN orders. These will graphically be represented by a v^n next to the interaction vertex, with n describing the power of v correction the diagram represents.

All thinkable diagrams belonging to the 1PN correction scale as Lv^2, and are depicted in Figure 4.3. Summing them all up will result with the 1PN Lagrangian (4.48).

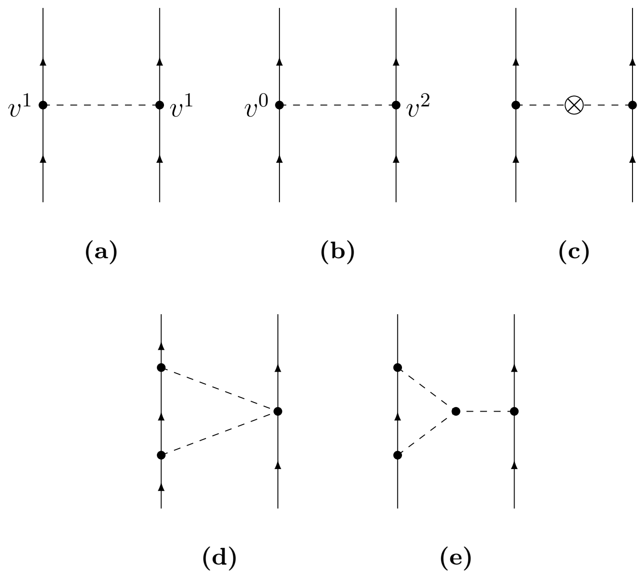
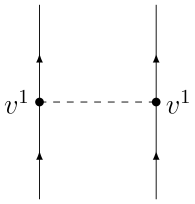


Figure 4.3: The Feynman diagrams contributing to 1PN order orbital energy. The three first diagrams (a)-(c) are Newtonian, 'H'-type, diagrams with relativistic corrections to the interaction terms and propagator. Diagram (d) and (e) represent new types of diagrams, and are of the 'V'-type, and 'Y'-type respectively.

4.2.2 Computing Feynman diagram (a)

In this first diagram nothing has changed from the Newtonian diagram (see Figure 4.1), except a coupling between the velocity of the two sources[4]



$$\begin{aligned} cS^{\text{eff}}_{(\text{a})} = \frac{m_1\lambda}{2} \int \mathrm{d}^4x\, \dot{x}_1^i \dot{x}_1^0 \delta^3\big(\boldsymbol{x} - x_1(x^0)\big) \\ \cdot \frac{m_2\lambda}{2} \int \mathrm{d}^4y\, \frac{\delta\big(x^0 - y^0\big)\, P_{0i:0j}}{4\pi|\boldsymbol{x} - \boldsymbol{y}|} \dot{x}_2^j \dot{x}_2^0 \delta^3\big(\boldsymbol{y} - x_2(y^0)\big) \end{aligned} \tag{4.19}$$

Integrals over Dirac deltas should be straightforward, leaving $x_1^i P_{0i:0j}\, x_2^j$ as the only new and interesting part. Note that $P_{\mu\nu:\alpha\beta}$ is symmetric in $\mu \leftrightarrow \nu$, $\alpha \leftrightarrow \beta$, and $\mu\nu \leftrightarrow \alpha\beta$, and therefore all permutations of the indices belong to this same diagram. The last symmetry, $\mu\nu \leftrightarrow \alpha\beta$, is just a relabelling of $m_1 m_2 \leftrightarrow m_2 m_1$, and is superfluous. The other symmetries also result in the same potential, but should be summed over, producing a factor of $2{\cdot}2 = 4$.

Recalling the definition of the projector $P_{\mu\nu:\alpha\beta}$, (3.42c)[5], this is

$$v_1^i P_{0i:0j}\, v_2^j = v_1^i \left(\frac{1}{2} \cdot (-1) \cdot \eta_{ij} \right) v_2^j = \frac{-1}{2} \boldsymbol{v}_1 {\cdot} \boldsymbol{v}_2 \tag{4.20}$$

Thus the potential of diagram 4.3a is

$$\begin{aligned} V_{(a)} &= \frac{m_1 m_2 \lambda^2}{4} \left(\frac{1}{4\pi r} \cdot \frac{4c\boldsymbol{v}_1 \cdot c\boldsymbol{v}_2}{2} \right) = \frac{m_1 m_2 c^2 \lambda^2}{32\pi r} \left(4\boldsymbol{v}_1 {\cdot} \boldsymbol{v}_2\right) \\ &= 4 \frac{G m_1 m_2}{r} \frac{\boldsymbol{v}_1 {\cdot} \boldsymbol{v}_2}{c^2} = -4 V_{\text{Newt}} \frac{\boldsymbol{v}_1 {\cdot} \boldsymbol{v}_2}{c^2}. \end{aligned} \tag{4.21}$$

According to the sign, this is seemingly a *repulsive* force, but it is dependent on the dot product of the two velocity vectors, which for any Keplerian orbit will be negative (see Figure 3.1). Of course, it also scales by an additional factor of v^2/c^2 compared to the Newtonian term, as a 1PN term should.

But for particles moving in the same direction it *is* a repulsive force, making it comparable to a magnetic type force. For oppositely charged particles, which should attract in the static approximation, will also repel each other magnetically when moving in the same direction, quite analogously. The analogy goes even further, as magnetic forces can also be interpreted as relativistic corrections to the electric force [22].

4.2.3 Computing Feynman diagram (b)

This diagram mostly follows suit of Section 4.2.2, but with a few additional details. These are

- The relativistic expansion of the free point particle kinetic energy is added here, as it also scales $\sim \boldsymbol{v}_a^2$ compared to 0PN.

[4]The additional factor of c with the effective action is to allow $S = \int \mathcal{L}\, \mathrm{d}^3x\, \mathrm{d}t \;\to\; cS = \int \mathcal{L}\, \mathrm{d}^4x$.

[5]For the readers convinience: $P_{\mu\nu:\alpha\beta} = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right)$.

- The relativistic expansion of the Lorentz factor of the energy-momentum tensor is expanded, as it also scales $\sim \boldsymbol{v}_a^2$ compared to 0PN.
- And of course, the $P_{00:ij}$ and $P_{ij:00}$ couplings belong to this diagram.

The first term is just the velocity expansion of the free point particle action

$$-m_a c^2 \int \mathrm{d}\tau_a = -m_a c^2 \int \mathrm{d}x^0\, \gamma_a^{-1} = -m_a c^2 \int \mathrm{d}x^0 \left(1 - \frac{1}{2}\frac{\boldsymbol{v}_a^2}{c^2} - \frac{1}{8}\frac{\boldsymbol{v}_a^4}{c^4} + \dots\right), \tag{4.22}$$

which for the 1PN expansion is $\frac{1}{8} m_a \frac{\boldsymbol{v}_a^4}{c^2}$, as it has an extra factor of $\frac{\boldsymbol{v}_a^2}{c^2}$ compared to the 0PN kinetic term $\frac{1}{2} m_a v_a^2$.

Next notice that there is also a Lorentz factor in the energy-momentum tensor, that until now has been approximated to 1.

$$\begin{aligned}\frac{m_a\lambda}{2c}\int \mathrm{d}^4x\, \gamma_a^{-1} H_{00}\, \dot{x}_a^0 \dot{x}_a^0 \delta^3(\boldsymbol{x} - \boldsymbol{x}_a(t)) &= \frac{m_a\lambda}{2c}\int \mathrm{d}^4x\, \gamma_a H_{00}\, c^2 \delta^3(\boldsymbol{x} - \boldsymbol{x}_a(t)) \\ &= \frac{m_a\lambda}{2c}\int \mathrm{d}^4x \left(1 + \frac{1}{2}\frac{\boldsymbol{v}_a^2}{c^2} + \frac{3}{8}\frac{\boldsymbol{v}_a^4}{c^4} + \dots\right) H_{00}\, c^2 \delta^3(\boldsymbol{x} - \boldsymbol{x}_a(t))\,.\end{aligned} \tag{4.23}$$

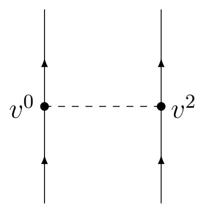
Other than the factor of $\frac{1}{2}\frac{\boldsymbol{v}_a^2}{c^2}$ this is exactly the same as the 0PN potential.

Lastly, the velocity dependent coupling. Using again (3.42c)

$$P_{ij:00}\, \dot{x}_a^i \dot{x}_a^j = \frac{1}{2}\boldsymbol{v}_a \cdot \boldsymbol{v}_a = \frac{1}{2}\boldsymbol{v}_a^2. \tag{4.24}$$

To the first PN order, only one particle at the time may be expanded this way, therefore the v^0-v^2 vertices in the diagram. But really both the v^0-v^2 diagram and the v^2-v^0 diagram belong to the 1PN potential, therefore these will both be summed up here, under the same diagram.

Summing up all these contributions to the 4.3b diagram yields



$$cS^{\text{eff}}_{(b)} = \sum_a \Bigg[\int \mathrm{d}x^0\, \frac{1}{8} m_a \frac{\boldsymbol{v}_a^4}{c^2} \tag{4.25a}$$

$$+ \frac{m_a\lambda}{2} \int \mathrm{d}x^0 \Bigg\{ H_{ij}(\boldsymbol{x}_a)\dot{x}_a^i \dot{x}_a^j \tag{4.25b}$$

$$+ \frac{1}{2}\frac{\boldsymbol{v}_a^2}{c^2} H_{00}(\boldsymbol{x}_a)\dot{x}_a^0 \dot{x}_a^0 \Bigg\} \Bigg]. \tag{4.25c}$$

Here (4.25a)[6] is the result of (4.22), (4.25c) of (4.23), and (4.25b) of (4.24).

[6]It could be argued that this contributon of the action should be an expansion of the free particle diagram (with no propagators), but for streamlining it is included here.

Substituting $H_{00}(x^0, \boldsymbol{x}_a)$ for the usual term (3.42b) yields the potential

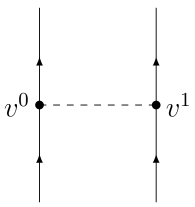
$$\begin{aligned} V_{(b)} &= -\frac{m_1 m_2 \lambda^2}{2 \cdot 2 \cdot 4\pi r} \sum_a \frac{1}{2} \left[c^2 \cdot \boldsymbol{v}_a^2 + \frac{1}{2} c^4 \cdot \frac{\boldsymbol{v}_a^2}{c^2} \right] \\ &= -\frac{m_1 m_2 \lambda^2 c^2}{32\pi} \frac{3}{2} \left(\boldsymbol{v}_1^2 + \boldsymbol{v}_2^2 \right) \\ &= -\frac{G m_1 m_2}{r} \frac{3}{2} \frac{\boldsymbol{v}_1^2 + \boldsymbol{v}_2^2}{c^2} = V_{\text{Newt}} \frac{3}{2} \frac{\boldsymbol{v}_1^2 + \boldsymbol{v}_2^2}{c^2}. \end{aligned} \tag{4.26}$$

This potential is attractive, and proportional to the Newtonian kinetic energy. Thus, this diagram can be thought of as the gravitational attraction from the kinetic energy of one of the particles on the other, showing that in GR gravity is an attraction of *energies*, and not just masses. Using Einstein's mass-energy equivalence, the 'Newtonian kinetic mass' is $E = \frac{1}{2} m v^2 = m_{\text{kin}} c^2$, and inserted into Newton's law of gravity produces $V_{(b)}/3$. This analogy can not explain the missing factor of 3, because the analogy only account for the (4.23) part of the potential.

Notice that terms of the type (4.23) can in principle be expanded to infinite orders, thus there is no reason to stop expanding vertices past the v^2 order.

4.2.4 But wait, what about 0.5PN diagrams?

After having computed diagram a and b, a natural idea of a v^0-v^1 coupling emerges as a 0.5PN contribution.



$$\begin{aligned} c S^{\text{eff}}_{0.5\text{PN}} = \sum_{a \neq b} \frac{m_a \lambda}{2} \int & \mathrm{d}^4 x \, \dot{x}_a^0 \dot{x}_a^0 \delta^3\big(\boldsymbol{x} - x_a(x^0)\big) \\ \cdot \frac{m_b \lambda}{2} \int & \mathrm{d}^4 y \, \frac{\delta\big(x^0 - y^0\big) \, P_{00:0i}}{4\pi |\boldsymbol{x} - \boldsymbol{y}|} \dot{x}_b^i \dot{x}_b^0 \delta^3\big(\boldsymbol{y} - x_b(y^0)\big) \end{aligned} \tag{4.27}$$

The situation is analogous to Section 4.2.2, only here the connection is not symmetric. Thus, the potential is also here the Newtonian potential multiplied by some velocity factor.

$$V_{0.5\text{PN}} = \frac{m_1 m_2 \lambda^2}{16\pi r} \sum_{a \neq b} \left[\dot{x}_a^0 \dot{x}_a^0 \left(P_{00:i0} + P_{00:0i} \right) \dot{x}_b^i \dot{x}_b^0 \right] = 0, \tag{4.28}$$

which follows from $P_{00:i0} = \frac{1}{2} \left(2\eta_{00}\eta_{0i} - \eta_{00}\eta_{i0} \right) = P_{00:0i} = 0$.

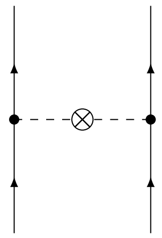
A lesson to be taken from this is that $P_{\mu\nu:\alpha\beta}$ only couples sources which both have an even or both an odd number of temporal indices. E.g. 00:00, 00:ij, and 0i:0j. Terms like 0i:00, or 0i:ij will turn out to be zero. Therefore, this kind of interaction expansion can only produce even powers of v/c.

4.2.5 Computing Feynman diagram (c)

The last 'H'-shaped diagram is not expanded in its vortexes, but rather the propagator of H is expanded to second order in v, according to expansion (4.9). In that expansion, the propagator was only expanded in Fourier space, so it remains to determine the value of the first order correction of the propagator in real space.

$$\Delta^{(2)}_{\text{inst}}(x-x') \equiv \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-k_0^2}{\boldsymbol{k}^4}e^{ik_\nu(x^\nu - x'^\nu)}. \tag{4.29}$$

From the gradient relations (4.10) it is evident that $(k^0/|\boldsymbol{k}|)^2 \sim \left(\frac{v}{r}/\frac{1}{r}\right)^2 = v^2$ and thus belong to the 1PN correction, but the scaling of $k_0 \sim \frac{v}{r}$ was not truly argued for. So this section will demonstrate this scaling by computing the propagator in real space.



$$\begin{aligned} cS^{\text{eff}}_{(c)} &= \frac{m_1 m_2 \lambda^2 c^4}{8}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big\{\delta^3(\boldsymbol{x}-\boldsymbol{x}_1(t)) \\ &\qquad \cdot\,\delta^3\big(\boldsymbol{y}-\boldsymbol{x}_2(t')\big)\,\Delta^{(2)}_{\text{inst}}(x-y)\Big\} &\text{(4.30a)} \\ &= \frac{m_1 m_2 \lambda^2 c^4}{8}\int \mathrm{d}ct\,\mathrm{d}ct'\,\Delta^{(2)}_{\text{inst}}\big(t-t',\boldsymbol{x}_1(t)-\boldsymbol{x}_2(t')\big) &\text{(4.30b)} \end{aligned}$$

To evaluate the integral in (4.29) some tricks are in order. Writing out the exponent as $e^{ik_\mu(x^\mu - x'^\mu)} = e^{-ik_0(ct-ct')}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}$, a useful way to proceed is to rewrite the integral as $k_0^2 e^{-ik_0(ct-ct')} = -\frac{\partial^2}{\partial ct\partial ct'}e^{-ik_0(ct-ct')}$. In the following note that the time derivative is *only* operating on the k_0 integral.

$$\begin{aligned} \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k_0)^2}{\boldsymbol{k}^4}e^{ik_\mu(x^\mu - x'^\mu)} &= -\frac{\partial^2}{\partial ct\partial ct'}\left[\int\frac{\mathrm{d}k_0}{2\pi}e^{-ik_0(ct-ct')}\right]\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\boldsymbol{k}^4}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} &\text{(4.31a)} \\ &= -\frac{\partial^2}{\partial ct\partial ct'}\big[\delta(ct-ct')\big]\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\boldsymbol{k}^4}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} &\text{(4.31b)} \end{aligned}$$

To proceed the derivative of Dirac's delta function must be determined. To that end, partial integration is the key.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\frac{\mathrm{d}\delta(x-x_0)}{\mathrm{d}x}\,\mathrm{d}x &= \int_{-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)\delta(x-x_0)\right]\mathrm{d}x - \int_{-\infty}^{\infty}\delta(x-x_0)\frac{\mathrm{d}f(x)}{\mathrm{d}x}\,\mathrm{d}x &\text{(4.32a)} \\ &= f(x)\delta(x-x_0)\Big|_{-\infty}^{\infty} - \int\delta(x-x_0)f'(x)\,\mathrm{d}x = -\int\delta(x-x_0)f'(x)\,\mathrm{d}x\,, &\text{(4.32b)} \end{aligned}$$

the last equality only holding for $x_0 \notin \pm\infty$. The result generalizes for any number of derivatives as

$$\int_{-\infty}^{\infty} f(x)\frac{\mathrm{d}^n\delta(x-x_0)}{\mathrm{d}x^n}\,\mathrm{d}x = \int_{-\infty}^{\infty}(-1)^n\delta(x-x_0)\frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x)\,\mathrm{d}x\,. \tag{4.33}$$

Back to equation (4.31b), the situation is analogous, thus partial integrations can also be utilized here to rewrite the derivative of the Dirac delta function in the same manner.

$$cS^{\text{eff}}_{(c)} = -\frac{m_1 m_2 \lambda^2 c^4}{8} \int \mathrm{d}ct\, \mathrm{d}ct'\, \delta\big(ct - ct'\big) \frac{\partial^2}{\partial ct \partial ct'} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\boldsymbol{k}^4} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t'))} \tag{4.34a}$$

$$= 4\pi G m_1 m_2 \int \mathrm{d}ct\, \mathrm{d}ct'\, \delta\big(ct - ct'\big) \frac{v_1^i}{c} \frac{v_2^j}{c} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{k_i k_j}{\boldsymbol{k}^4} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t'))} \tag{4.34b}$$

$$= 4\pi G m_1 m_2 \int \mathrm{d}ct\, \frac{v_1^i}{c} \frac{v_2^j}{c} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{k_i k_j}{\boldsymbol{k}^4} e^{i\boldsymbol{k}\cdot\boldsymbol{r}(t)} \tag{4.34c}$$

$$= -4\pi G m_1 m_2 \int \mathrm{d}ct\, \frac{v_1^i}{c} \frac{v_2^j}{c} \frac{\partial^2}{\partial r^i \partial r^j} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\boldsymbol{k}^4} e^{i\boldsymbol{k}\cdot\boldsymbol{r}(t)}. \tag{4.34d}$$

This final integral might not look like that much of an improvement, but these are the kinds of integrals often encountered in QFT, and there exists a known solution [4].

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\boldsymbol{k}^{2\alpha}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(\frac{d}{2} - \alpha\right)}{\Gamma(\alpha)} \left(\frac{\boldsymbol{r}^2}{4}\right)^{\alpha - d/2}. \tag{4.35}$$

Here $\Gamma(z)$ is the gamma-function, and d is the spatial dimension. In the integral of equation (4.34d), $d = 3$ and $\alpha = 2$. Recalling that $\Gamma(1) = 1$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, and $z\Gamma(z) = \Gamma(z+1)$, it follows that $\Gamma(2) = 1 \cdot \Gamma(1) = 1 = \Gamma(\alpha)$. In the same fashion $-\frac{1}{2}\Gamma\left(-\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \rightarrow \Gamma\left(\frac{d}{2} - \alpha\right) = \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$.

Then, the final expression of the potential follows as

$$V_{(c)} = 4\pi G m_1 m_2 \frac{v_1^i}{c} \frac{v_2^j}{c} \frac{\partial^2}{\partial r^i \partial r^j} \left[\frac{1}{8\pi^{3/2}} \frac{-2\pi^{1/2}}{1} \frac{|\boldsymbol{r}|}{2}\right] \tag{4.36a}$$

$$= -\frac{G m_1 m_2}{2} \frac{v_1^i}{c} \frac{v_2^j}{c} \frac{1}{r^3} \left(r^2 \delta_{ij} - r_i r_j\right) \tag{4.36b}$$

$$= -\frac{G m_1 m_2}{2r} \left(\frac{\boldsymbol{v}_1 \cdot \boldsymbol{v}_2}{c^2} - \frac{(\boldsymbol{v}_1 \cdot \boldsymbol{r})(\boldsymbol{v}_2 \cdot \boldsymbol{r})}{c^2 r^2}\right) \tag{4.36c}$$

$$= V_{\text{Newt}} \frac{1}{2c^2} \left(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 - \frac{(\boldsymbol{v}_1 \cdot \boldsymbol{r})(\boldsymbol{v}_2 \cdot \boldsymbol{r})}{r^2}\right). \tag{4.36d}$$

Again, this is obviously of first post-Newtonian order, as it has an additional factor of $\frac{v^2}{c^2}$ compared to the Newtonian potential. From the procedure it is hopefully clear how additional powers of $(k_0/\boldsymbol{k})^2$ leads to additional time-derivatives, additional powers of c^{-2}, and additional powers of v^2. Thereby the scaling $k_0 \sim \frac{v}{r}$ should be justified.

Like diagram 4.3a, this potential is also dependent on $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2$, but has the opposite sign. Thus, it is attractive for particles moving in the same direction, and repulsive for particles moving in opposite directions.

Some insight might be gained from the *projector* in line (4.36b)

$$P_{ij}(\boldsymbol{n}) \equiv \delta_{ij} - n_i n_j, \tag{4.37}$$

where $\boldsymbol{n} = \boldsymbol{r}/r$. Contracting $P_{ij}(\boldsymbol{n})$ with a vector $\boldsymbol{x}$ has the effect of projecting $\boldsymbol{x}$ onto the orthogonal plane of $\boldsymbol{n}$

$$\begin{aligned} P_{ij}(\boldsymbol{n})x^j &= x_i - (\boldsymbol{n}\cdot\boldsymbol{x})\,n_i, \\ \Rightarrow \quad n^i P_{ij}(\boldsymbol{n})x^j &= (\boldsymbol{n}\cdot\boldsymbol{x}) - (\boldsymbol{n}\cdot\boldsymbol{x}) = 0. \end{aligned}$$

Ergo, $P_{ij}(\boldsymbol{n})x^j$ is orthogonal to $\boldsymbol{n}$.

With this in mind, the potential only couples velocities orthogonal to $\boldsymbol{r}$, and is equivalent to

$$\begin{aligned} V_{(c)} &= V_{\text{Newt}}\frac{(\boldsymbol{r}\times\boldsymbol{v}_1)\cdot(\boldsymbol{r}\times\boldsymbol{v}_2)}{2r^2c^2} = \frac{G}{2r^3c^2}(\boldsymbol{r}\times\boldsymbol{p})\cdot(\boldsymbol{r}\times\boldsymbol{p}) \\ &= \frac{G\mu}{rc^2}\left(\frac{1}{2}\mu r^2\omega^2\right), \end{aligned} \tag{4.38}$$

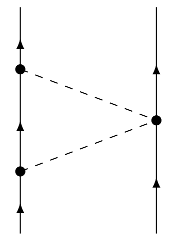
where $\boldsymbol{v}_i$ was exchanged for $\boldsymbol{v}$ according to (B.4), and $\boldsymbol{p} \equiv \mu\boldsymbol{v}$. In the last line, $\boldsymbol{\omega} = \boldsymbol{r}\times\boldsymbol{v}/r^2$ was used, and the term inside the parenthesis is the kinetic energy associated with rotational motion of a particle with effective mass μ.

A last observation is that the Lambda tensor, introduced in Section 3.5 (3.56), can be defined using this projection operator (Maggiore [9])

$$\Lambda_{ij:kl}(\boldsymbol{n}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\,. \tag{4.39}$$

4.2.6 Computing Feynman diagram (d)

The last two diagrams have a higher order of H's and are thus *non-linear* in λ. Therefore, these diagrams should be proportional to G^2. Diagram 4.3d represent a second order in H coupling between the graviton field and the point particle, while the last diagram represents the higher order propagator, where H couples with itself. It was argued in equation (4.15) and (4.17) that these diagrams belong to the 1PN correction, but then only coupling to the 00 component of $T^{\mu\nu}$. Thus, the same expansions preformed for diagram (a)-(c) will need to be implemented to these non-linear diagrams, when computing higher order PN corrections.



$$\begin{aligned} cS^{\text{eff}}_{(d)} = &\sum_{a\neq b}\frac{m_a^2\lambda^2}{4}\frac{m_b\lambda^2}{8}\int \mathrm{d}x^0\,\mathrm{d}\tilde{x}^0\,\mathrm{d}y^0\,c^6P^2_{00:00} \\ &\cdot\Delta_{\text{inst}}\left(\boldsymbol{x}_a(x^0)-\boldsymbol{x}_b(y^0)\right)\cdot\Delta_{\text{inst}}\left(\boldsymbol{x}_a(\tilde{x}^0)-\boldsymbol{x}_b(y^0)\right) \end{aligned} \tag{4.40}$$

Notice that particle a interacts with the graviton field *twice*, and possibly at different times, while particle b only interacts once, but to a higher order, thus connecting it to the other particle through two graviton propagators, as depicted in the diagram. The sum adds the mirrored diagram as well.

There is nothing surprising in this diagram, and the integrals can be carried out without any fuzz, eliminating two of the three time integrals, making both graviton exchanges instantaneous, and simultaneous.

$$V_{(d)} = -\sum_{a\neq b} \frac{m_a^2 m_b \lambda^4 c^6}{4\cdot 8\cdot 4} \frac{1}{4\pi r}\frac{1}{4\pi r} = -\sum_{a\neq b}\frac{G^2 m_a^2 m_b}{2r^2c^2} \tag{4.41a}$$

$$= -\frac{G^2 m_1 m_2 (m_1+m_2)}{2r^2c^2} = V_{\text{Newt}}\frac{GM}{2rc^2} \tag{4.41b}$$

One way to interpret this result is as a gravitational coupling between the total mass of the system, and half the Newtonian potential energy itself

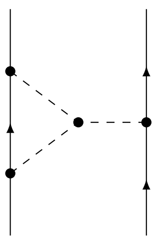
$$V_{(d)} = -\frac{GM(|V_{\text{Newt}}|/2c^2)}{r}. \tag{4.42}$$

This does not however make much sense as being separated the distance r, but it is an interesting analogy.

4.2.7 Computing Feynman diagram (e)

Note that in this section spacial indices are suppressed, as subscripts are used to enumerate vector variables.

The last diagram makes use of the three graviton propagator, and is the first one not to use the linear propagator.



$$cS_{(e)} = \frac{m_1^2 m_2 \lambda^3}{8}\int \mathrm{d}^4x\,\mathrm{d}^4\tilde{x}\,\mathrm{d}^4y\,\left(\delta^3(\boldsymbol{x}-\boldsymbol{x}_1(t_1)) + \dots\right)c^6$$
$$\int \langle T\left\{H_{00}(t_1,\boldsymbol{k}_1)H_{00}(t_2,\boldsymbol{k}_2)H_{00}(t_3,\boldsymbol{k}_3)\right\}\rangle \prod_{j=1}^{3}\frac{\mathrm{d}^3k_j}{(2\pi)^3} \tag{4.43}$$
$$+ \text{ the mirrored diagram.}$$

Here the propagator in the second line needs a lot of work. It can be read off from Goldberger and Rothstein [3], equation (37)-(39). But it will also be derived in the following.

Much of the structure is already given in (4.2), which was found inserting leading order results[7] for h in the expanded EoM.

$$H_{\mu\nu}(y) = -\frac{\lambda}{2}P_{\mu\nu:\alpha\beta}\int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2\,\mathrm{d}^4k_3}{(2\pi)^{12}}\frac{-e^{ik_1y}}{\boldsymbol{k}_1^2}P_{\rho\sigma:\tau\delta}\,k_1^{\alpha}H_{\boldsymbol{k}_2\,\sigma\rho}k_3^{\beta}H_{\boldsymbol{k}_1\,\tau\delta}e^{ik_2x}e^{ik_3\tilde{x}} \tag{4.44}$$

To make sure momentum and energy is not spontaneously generated by the propagator, one should demand the momentum vectors $\boldsymbol{k}_i$ sum to zero. This is achieved by multiplying the integral by $(2\pi)^4\delta^4(\sum_i \boldsymbol{k}_i)$.

[7] Leading order here referes to the PN expansion, not the λ expansion of the EoM (3.76).

$$\begin{aligned} H_{00}(y) &= -\frac{\lambda}{4} P_{00:\alpha\beta} \int \left[\prod_{j=1}^{3} \frac{\mathrm{d}^4 k_j}{(2\pi)^4}\right] \frac{-e^{ik_1 y}}{\boldsymbol{k}_1^2} \frac{-e^{ik_2 x}}{\boldsymbol{k}_2^2} \frac{-e^{ik_3 \tilde{x}}}{\boldsymbol{k}_3^2} k_1^\alpha k_1^\beta (2\pi)^4 \delta^4\left(\sum_i k_i\right) \\ &= -\frac{\lambda}{4} \int \left[\prod_{j=1}^{3} \frac{\mathrm{d}^4 k_j}{(2\pi)^4}\right] \frac{-e^{ik_1 y}}{\boldsymbol{k}_1^2} \frac{-e^{ik_2 x}}{\boldsymbol{k}_2^2} \frac{-e^{ik_3 \tilde{x}}}{\boldsymbol{k}_3^2} \frac{(k_1^0)^2 + \boldsymbol{k}_1^2}{2} (2\pi)^4 \delta^4\left(\sum_i k_i\right). \end{aligned} \tag{4.45}$$

The factor of $(k_1^0)^2 + \boldsymbol{k}_1^2 = \boldsymbol{k}_1^2\left(1 + (k_1^0)^2/\boldsymbol{k}_1^2\right)$, which was found to induce factors of v^2 past leading order in Section 4.2.5. Therefore, this factor will be approximated as $\boldsymbol{k}_1^2$.

Furthermore, when one of the k_i integrals are preformed; the Dirac delta function will eliminate that vector by $k_1 + k_2 + k_3 = 0$. For example preforming the k_1 integral yields

$$H_{00}(y) = \frac{\lambda}{8} \int \left[\prod_{j=2}^{3} \frac{\mathrm{d}^4 k_j}{(2\pi)^4}\right] e^{-i(k_2+k_3)y} \frac{e^{ik_2 x}}{\boldsymbol{k}_2^2} \frac{e^{ik_3 \tilde{x}}}{\boldsymbol{k}_3^2} \tag{4.46a}$$

$$= \frac{\lambda}{8} \int \frac{\mathrm{d}k_2^0\,\mathrm{d}k_3^0}{(2\pi)^2} e^{-ik_2^0(x^0 - y^0)} e^{-ik_3^0(\tilde{x}^0 - y^0)} \int \frac{\mathrm{d}^3 k_2}{(2\pi)^3} \frac{e^{i\boldsymbol{k}_2 \cdot (\boldsymbol{x}_2 - \boldsymbol{y})}}{\boldsymbol{k}_2^2} \int \frac{\mathrm{d}^3 k_3}{(2\pi)^3} \frac{e^{i\boldsymbol{k}_3 \cdot (\tilde{\boldsymbol{x}}_2 - \boldsymbol{y})}}{\boldsymbol{k}_3^2} \tag{4.46b}$$

$$= \frac{\lambda}{8} \frac{\delta\left(x^0 - y^0\right)}{4\pi|\boldsymbol{x}_2 - \boldsymbol{y}|} \frac{\delta\left(\tilde{x}^0 - y^0\right)}{4\pi|\boldsymbol{x}_3 - \boldsymbol{y}|} \tag{4.46c}$$

$$= \frac{\lambda}{2} P_{00:00}\, \Delta_{\text{inst}}(x_2 - y) \cdot P_{00:00}\, \Delta_{\text{inst}}(x_3 - y) \tag{4.46d}$$

It is the product of two propagators, connecting two different points to the same third point, just like in the diagram. This result could also have been argued to result from *Wick's theorem*, like Porto [4] does, but then it would be all pairwise combinations of points, including $\Delta_{\text{inst}}(x_2 - x_3)$. It was discarder here because the k_1 propagator was eliminated by the derivative, which again followed from k_1 being the main transform, i.e. the lone graviton going in the final interaction term of the action. In [4] this 'missing' contribution was handled using *dimensional regularization*, and turns out to be zero, as demanded by our result.

The potential should now follow straightforwardly as

$$\begin{aligned} V_{(e)} &= \sum_{a \neq b} \frac{m_a^2 m_b \lambda^4 c^6}{2^{10} \pi^2 r^2} \\ &= \frac{G^2 m_1 m_2 (m_1 + m_2)}{r^2 c^2} = -V_{\text{Newt}} \frac{GM}{rc^2} \\ &= -2V_{(d)}. \end{aligned} \tag{4.47}$$

Surprisingly, this diagram has the same potential as diagram 4.3d, times negative 2. Thus, the joined effect of these two last diagrams is a positive, and thus repulsive, potential proportional to the total mass and the Newtonian potential.

It is surprising that the effect of non-linear terms is to weaken the static force, but this is the case.

4.2.8 The total 1PN Lagrangian

Summing up all the potentials $V_{(a)}$-$V_{(e)}$ ((4.21), (4.26), (4.36d), (4.41), and (4.47)), and remembering to add the kinetic energy expansion (4.25a), the final 1PN Lagrangian will be the result

$$\boxed{\begin{aligned} L_{1\mathrm{PN}} = L_{\mathrm{EIH}} =& \frac{1}{8}\sum_a m_a \frac{v_a^4}{c^2} + \frac{Gm_1m_2}{2rc^2}\left[3(\boldsymbol{v}_1^2+\boldsymbol{v}_2^2) - 7\boldsymbol{v}_1\cdot\boldsymbol{v}_2 - \frac{(\boldsymbol{v}_1\cdot\boldsymbol{r})(\boldsymbol{v}_2\cdot\boldsymbol{r})}{r^2c^2}\right] \\ &- \frac{G^2m_1m_2(m_1+m_2)}{2r^2c^2}, \end{aligned}} \tag{4.48}$$

$$\text{and with total effective action} \quad S^{\mathrm{eff}} = \int \mathrm{d}t\,\{L_{0\mathrm{PN}} + L_{1\mathrm{PN}} + \dots\} \tag{4.49}$$

This is the *Einstein-Infeld-Hoffmann* Lagrangian from 1938 [23], derived in an entirely different manner, warranting some confidence in the result.

In order to determine the orbital energy of the binary system it is useful to reduce this Lagrangian to its equivalent one body problem.

As it is already *just* spatially dependent on the relative displacement of the two bodies r, the last thing needed is just to express their velocities by the relative velocity.

Using the centre of mass frame the position of each body can be expressed through the relative displacement $\boldsymbol{r}$ as[8]

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2, \quad \boldsymbol{r}_1 = \frac{m_2}{M}\boldsymbol{r}, \quad \boldsymbol{r}_2 = -\frac{m_1}{M}\boldsymbol{r}. \tag{4.50}$$

The velocity of particle number i is defined as the time derivative of its position $\boldsymbol{v}_i \equiv \dot{\boldsymbol{r}}_i$. Defining the time derivative of the relative displacement as the relative velocity one finds

$$\boldsymbol{v} \equiv \dot{\boldsymbol{r}}, \quad \boldsymbol{v}_1 = \frac{m_2}{M}\boldsymbol{v}, \quad \boldsymbol{v}_2 = -\frac{m_1}{M}\boldsymbol{v}. \tag{4.51}$$

Substituting (4.51) into the 1PN (4.48), and the 0PN (3.44) Lagrangian, the equivalent, total, one-body version is obtained

$$\begin{aligned} L^{\mathrm{eff}} =& \frac{1}{2}\left(m_1\frac{m_2^2}{M^2} + m_2\frac{m_1^2}{M^2}\right)\boldsymbol{v}^2 + \frac{Gm_1m_2}{r} + \frac{1}{8c^2}\left(m_1\frac{m_2^4}{M^4} + m_2\frac{m_1^4}{M^4}\right)\boldsymbol{v}^4 \\ &+ \frac{Gm_1m_2}{2rc^2}\left[3\left(\frac{m_2^2+m_1^2}{M^2}\right)\boldsymbol{v}^2 + 7\frac{m_2m_1}{M^2}\boldsymbol{v}^2 + \frac{m_2m_1}{M^2}\boldsymbol{v}^2\left(\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{r}}\right)^2\right] \\ &- \frac{G^2m_1m_2(m_1+m_2)}{2r^2c^2} \end{aligned} \tag{4.52a}$$

$$\begin{aligned} =& \frac{\mu}{2}\boldsymbol{v}^2 + \frac{GM\mu}{r} + \frac{\mu}{8c^2}\left(1-3\eta\right)\boldsymbol{v}^4 \\ &+ \frac{GM\mu}{2r}\left[3+\eta\left(1+\left(\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{r}}\right)^2\right)\right]\frac{\boldsymbol{v}^2}{c^2} - \frac{G^2M^2\mu}{2r^2c^2}. \end{aligned} \tag{4.52b}$$

[8] For details on how to derive these relations, and how to do the soon to come mass-term manipulation, see Appendix B.

4.3 Computing the 1PN equations of motion and energy

Equipped with the effective Lagrangian up to first post-Newtonian order (4.52b), all that remains is to determine the equation of motion and associated energy.

4.3.1 Finding the associated equations of motion

The corresponding equation of motion can be obtained by finding the extremum of the action. Using polar coordinates it is obvious that θ is a cyclic coordinate, as it does not appear in the Lagrangian.

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \omega} = \frac{\partial L}{\partial \theta} = 0, \tag{4.53a}$$

$$\begin{aligned} \ell \equiv \frac{\partial L}{\partial \omega} &= \mu r^2 \omega + \frac{\mu}{2c^2}(1-3\eta)(r^4\omega^3 + \dot{r}^2 r^2 \omega) + \frac{GM\mu}{rc^2}(3+\eta) r^2 \omega \\ &= \mu r^2 \omega \left[1 + \frac{1}{c^2}\left\{ \frac{1-3\eta}{2}(r^2\omega^2 + \dot{r}^2) + \frac{GM}{r}(3+\eta) \right\} \right]. \end{aligned} \tag{4.53b}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \omega} &= \mu(r^2\dot{\omega} + 2r\omega\dot{r}) \left[1 + \frac{1}{c^2}\left\{ \frac{1-3\eta}{2}(r^2\omega^2 + \dot{r}^2) + \frac{GM}{r}(3+\eta) \right\} \right] \\ &\quad + \frac{1}{c^2}\mu r^2 \omega \left[(1-3\eta)(r\omega^2\dot{r} + r^2\omega\dot{\omega} + \dot{r}\ddot{r}) - \frac{GM}{r^2}(3+\eta)\dot{r} \right]. \end{aligned} \tag{4.53c}$$

Approximating $\ell \approx \mu r^2 \omega + \mathcal{O}\left(\frac{1}{c^2}\right)$ it is clear that ℓ is the angular momentum of Newtonian theory, with a 1PN correction.

The radial equation of motion is similarly obtained by

$$\frac{\partial L}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} = 0. \tag{4.54a}$$

$$\begin{aligned} \frac{\partial L}{\partial r} &= \mu r \omega^2 - \frac{GM\mu}{r^2} + \frac{\mu}{2c^2}(1-3\eta)(r^3\omega^4 + r\omega^2\dot{r}^2) \\ &\quad + \frac{GM\mu}{2c^2}\left[(3+\eta)\omega^2 - (3+2\eta)\left(\frac{\dot{r}}{r}\right)^2 + \frac{2GM}{r^3} \right], \end{aligned} \tag{4.54b}$$

$$\begin{aligned} \frac{\partial L}{\partial \dot{r}} &= \mu\dot{r} + \frac{\mu}{2c^2}(1-3\eta)(\dot{r}^3 + r^2\omega^2\dot{r}) + \frac{GM\mu}{c^2}(3+2\eta)\frac{\dot{r}}{r} \\ &= \mu\dot{r}\left[1 + \frac{1}{c^2}\left\{ \frac{1-3\eta}{2}(\dot{r}^2 + r^2\omega^2) + \frac{GM}{r}(3+2\eta) \right\} \right], \end{aligned} \tag{4.54c}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} &= \mu\ddot{r}\left[1 + \frac{1}{c^2}\left\{ \frac{1-3\eta}{2}(\dot{r}^2 + r^2\omega^2) + \frac{GM}{r}(3+2\eta) \right\} \right] \\ &\quad + \frac{1}{c^2}\mu\dot{r}\left[(1-3\eta)\left(\dot{r}\ddot{r} + r\omega^2\dot{r} + r^2\omega\dot{\omega}\right) - \frac{GM}{r^2}\dot{r}\,(3+2\eta) \right]. \end{aligned} \tag{4.54d}$$

Imposing circular motion entails $\dot{r} = \ddot{r} = 0$, thus the EoM simplifies to

$$0 = \left\{ \frac{G^2M^2}{r^3c^2} - \frac{GM}{r^2} \right\} + \left\{ r + \frac{GM(3+\eta)}{2c^2} \right\}\omega^2 + \left\{ \frac{r^3(1-3\eta)}{2c^2} \right\}(\omega^2)^2 \tag{4.55}$$

The solution to this equation follows as

$$\omega^2 = \frac{GM}{r^3}\left\{1-(3-\eta)\frac{GM}{rc^2}+\mathcal{O}\left(\frac{1}{c^4}\right)\right\}, \tag{4.56a}$$

$$\Rightarrow \quad v^2 = r^2\omega^2 = \frac{GM}{r}\left\{1-(3-\eta)\frac{GM}{rc^2}+\mathcal{O}\left(\frac{1}{c^4}\right)\right\}. \tag{4.56b}$$

Notice that to the 0PN order equation (4.56a) reduces to Kepler's third law for circular orbits (2.4).

Using (4.56), r, ω, and v can be related with 1PN corrections. To compute the other relations, organize the equation into a quadratic equation, and solve for the quadratic parameter (e.g. solve for GM/r in (4.56b) to obtain GM/r as a function of v^2).

The 1PN correct relations of ω, r, and v turns out to be

$$\omega^2 = \frac{GM}{r^3}\left\{1-(3-\eta)\frac{GM}{rc^2}+\mathcal{O}\left(\frac{1}{c^4}\right)\right\}, \tag{4.57a}$$

$$GM\omega = v^3\left\{1+(3-\eta)\frac{v^2}{c^2}+\mathcal{O}\left(\frac{1}{c^4}\right)\right\}. \tag{4.57b}$$

$$\frac{GM}{r} = (GM\omega)^{2/3}\left\{1+\left(1-\frac{\eta}{3}\right)\frac{(GM\omega)^{2/3}}{c^2}+\mathcal{O}\left(\frac{1}{c^4}\right)\right\}, \tag{4.58a}$$

$$\frac{GM}{r} = v^2\left\{1+(3-\eta)\frac{v^2}{c^2}+\mathcal{O}\left(\frac{1}{c^4}\right)\right\}. \tag{4.58b}$$

$$v^2 = (GM\omega)^{2/3}\left\{1-\left(2-\frac{2}{3}\eta\right)\frac{(GM\omega)^{2/3}}{c^2}+\mathcal{O}\left(\frac{1}{c^4}\right)\right\}, \tag{4.59a}$$

$$v^2 = \frac{GM}{r}\left\{1-(3-\eta)\frac{GM}{rc^2}+\mathcal{O}\left(\frac{1}{c^4}\right)\right\}. \tag{4.59b}$$

To get the relations of other order of the LHS, is simply a matter of raising the equation to the desired power, and then Taylor expanding away terms that are not of first post-Newtonian order. But the relations as written here are those that usually come up, many having already been used in this thesis at the 0PN approximation.

One illustrative example to get a sense of the scale of the 1PN correction is to use (4.58a) to compute the correction of the lunar orbital distance. The Moon does not follow a circular orbit, but rather has an eccentricity of $0.02 < e_☾ < 0.08$, so it will not be an exact approximation. It can however give an idea of the scale of the effect.

The Moon has a (sidereal) period of $T_☾ = 27.32$ days, and mass of $m_☾ = 1.23 \cdot 10^{-2} M_\oplus$, where the Earth mass is $M_\oplus = 5.97 \cdot 10^{24}$ kg [24].

$$r_☾ = \frac{GM}{(GM\omega_☾)^{2/3}}\left\{1-\left(1-\frac{\eta}{3}\right)\frac{(GM\omega_☾)^{2/3}}{c^2}\right\} \tag{4.60a}$$

$$\begin{aligned} &\simeq 3.85\cdot 10^8\left\{1-1.17\cdot 10^{-11}\right\}\,\mathrm{m} \\ &\simeq 3.85\cdot 10^8\ \mathrm{m}\ -4.47\cdot 10^{-3}\ \mathrm{m} \end{aligned} \tag{4.60b}$$

Thus, even though the approximation is crude, this shows that the correction is in order of *millimetres* for the Earth-Moon system. It is perhaps not surprising considering that the Moon is not exactly moving at relativistic speeds.

However, due to reflective mirrors left by the Apollo missions, the Earth-Moon distance *is* measured at a millimetre precision [25].

The best models for the Earth-Moon system operates at this precision, and thus needs to account for 1PN corrections like this one [26].

4.3.2 Computing the Hamiltonian

To obtain the orbital energy it will suffice to derive the corresponding Hamiltonian of the 1PN Lagrangian (4.52b) by Legendre transformation (3.45). Utilizing the results of equations (4.53b) and (4.54c) the Hamiltonian is found to be

$$H(r,v) = \dot{r}\frac{\partial L}{\partial \dot{r}} + \omega\frac{\partial L}{\partial \omega} - L \tag{4.61a}$$

$$\begin{aligned} &= \mu\dot{r}^2\left[1+\frac{1}{c^2}\left\{\frac{1-3\eta}{2}(\dot{r}^2+r^2\omega^2)+\frac{GM}{r}(3+2\eta)\right\}\right] \\ &\quad + \mu r^2\omega^2\left[1+\frac{1}{c^2}\left\{\frac{1-3\eta}{2}(r^2\omega^2+\dot{r}^2)+\frac{GM}{r}(3+\eta)\right\}\right] - L \end{aligned} \tag{4.61b}$$

$$= \mu v^2 + \frac{4\mu}{8}(1-3\eta)\frac{v^4}{c^2} + 2\frac{GM}{2r}\left[3+\eta\left(1+\frac{\dot{r}^2}{v^2}\right)\right]\frac{v^2}{c^2} - L \tag{4.61c}$$

$$\begin{aligned} &= \frac{\mu}{2}v^2 - \frac{GM\mu}{r} + \frac{3\mu}{8}(1-3\eta)\frac{v^4}{c^2} \\ &\quad + \frac{GM\mu}{2r}\left[3+\eta\left(1+\frac{\dot{r}^2}{v^2}\right)\right]\frac{v^2}{c^2} + \frac{G^2M^2\mu}{2r^2c^2}. \end{aligned} \tag{4.61d}$$

The Hamiltonian is expressed in terms of the relative velocity $v^2 = \dot{r}^2 + r^2\omega^2$ instead of the canonical momentum because the end goal is simply to obtain the 1PN energy in terms of the frequency. Note that this expression is valid for all type of motion, not just circular.

Imposing circular motion again, the relations between v, r, ω from (4.57)-(4.59) may be used to express the energy in terms of one of these variables. The most commonly used variable is the frequency ω, as it is most directly related to the observable: the GW frequency. However here the velocity will be used for more convenient calculations.

$$E = \frac{\mu}{2}v^2 - \left(v^2 + (3-\eta)\frac{v^4}{c^2}\right)\mu + \frac{3\mu}{8}(1-3\eta)\frac{v^4}{c^2} + \frac{\mu}{2}\frac{v^4}{c^2}(3+\eta) + \frac{\mu}{2}\frac{v^4}{c^2} \tag{4.62a}$$

$$= -\frac{\mu}{2}v^2 + \frac{\mu}{2}\left[\left(-6+\frac{3}{4}+3+1\right)+\left(2-\frac{9}{4}+1\right)\eta\right]\frac{v^4}{c^2} \tag{4.62b}$$

$$= -\frac{\mu}{2}v^2\left[1+\left\{\frac{5}{4}-\frac{3}{4}\eta\right\}\frac{v^2}{c^2}\right]. \tag{4.62c}$$

This is *not* the energy expansion (2.7) presented in Chapter 2, so what is going on?

Recalling that in Chapter 2 v^* was only used as a proxy variable for the orbital frequency, and was defined as $v^* \equiv (GM\omega)^{1/3}$. In the Newtonian theory $v = v^*$, but at 1PN the relative velocity and orbital frequency are related according to (4.59a), hence $v \neq v^*$. Therefore, (4.62c) is the orbital energy in terms of the *actual* relative velocity.

Using equation (4.59a) to transform $v \to \omega$ the energy in terms of frequency is obtained to be

$$\boxed{E = -\frac{\mu}{2}(GM\omega)^{2/3}\left\{1 + \left\{-\frac{3}{4} - \frac{1}{12}\eta\right\}\frac{(GM\omega)^{2/3}}{c^2} + \mathcal{O}\left(\frac{1}{c^3}\right)\right\}.} \tag{4.63}$$

This is the energy presented in (2.7), where $(GM\omega)^{1/3}$ was named v, somewhat confusingly from the point of view of this chapter.

Beware that in the literature, energy and flux can, and are, presented in terms of $\frac{GM}{r}$, $(GM\omega)^{1/3}$, or v. But they are all the frequency energy/flux, relabelled using the 0PN approximation of the relations (4.57) - (4.59). This is of course since they are ultimately used to compute waveforms, which are computed from differential equations of the frequency. And in the end, the frequency is the directly observable parameter.

To get a sense of the scale of this energy correction, lets use this on the Earth-Moon system.

$$\begin{aligned} E_{☾} &\simeq -3.81 \cdot 10^{28}\left(1 - 8.76 \cdot 10^{-12}\right) \text{J} \\ &\simeq -3.81 \cdot 10^{28}\text{J} + 3.34 \cdot 10^{17}\ \text{J}. \end{aligned} \tag{4.64}$$

Which is of course comparably tiny. Using the mass-energy equivalence, the correction is comparable to $\sim$ 3kg, of an $\sim 4 \cdot 10^8$ metric tonnes 0PN energy.

Chapter 5

Calculating the energy flux

In order to fully describe the 1PN dynamics of the compact binary the energy dissipation by generated GWs need to be accounted for. In this section this total radiated power is to be calculated.

Derivations presented here closely follows those presented in Maggiore [9], Porto [4], and Ross [27].

5.1 The graviton field evaluated at large scales

5.1.1 Seperation of scales

In Section 4.1 it was argued that the binary system could be separated into three different length scales $\sim L$, $\sim r$, and $\sim R_S$, related by $L \gg r \gg R_S$. In Chapter 4 the $\sim R_S$ scale was 'integrated out', leaving BHs and NSs only with a point mass structure at the scale of the orbit $\sim r$. Similarly, in Section 3.5 the total energy flux of a system was found evaluating the graviton field at a scale $L \gg \lambda_{\mathrm{GW}}$, leaving the source effectively as a point source, endowed with a quadrupole structure.

By the requirement of evaluating at a scale $L \gg \lambda$, it is also automatically realized to be evaluated at a scale much larger than the binary system that created it, $r \gg \lambda$.[1]

5.1.2 Modifying the source of gravitational waves

In Section 3.5 the solution of the graviton field was found to be (3.30), which reads

$$\bar{\mathcal{H}}_{ij}^{\mathrm{TT}}(t, \boldsymbol{R}) = \frac{\lambda}{8\pi R} \Lambda_{ij}{}^{kl} \int_{\mathcal{V}} T_{kl}\left(t_{\mathrm{ret}}, \boldsymbol{x}\right) \mathrm{d}^3 x \,, \quad \text{where} \quad t_{\mathrm{ret}} = t - \frac{|\boldsymbol{R} - \boldsymbol{x}|}{c}. \tag{5.1}$$

With $\mathcal{V} \sim L^3$, such that T_{ij} evaluated at $\partial \mathcal{V}$ is zero.

On the other hand, in the far region $\square \mathcal{H} = 0$, which admits solution of the general

[1] To see why this relation holds for the inspiral, see footnote 1 from Chapter 4.

form

$$\begin{aligned}\bar{\mathcal{H}}_{ij}(t,\boldsymbol{R}) &= \frac{F_{kl}(t-R/c)}{R} - \partial_{i_1}\left[\frac{F_{kl}{}^{i_1}(t-R/c)}{R}\right] + \frac{1}{2}\partial_{i_1}\partial_{i_2}\left[\frac{F_{kl}{}^{i_1 i_2}(t-R/c)}{R}\right] + \ldots \\ &\equiv \sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}}{\ell!}\partial_L\left[\frac{F^L_{kl}(t-R/c)}{R}\right], \\ &\text{with} \quad \Box\left[\frac{F^L_{ij}(t-R/c)}{R}\right] = 0.\end{aligned} \tag{5.2}$$

Here the *multi index notation* has been introduced, which is to say a capital letter index L represent a number of ℓ indices.

By comparing these two expressions, which should be equivalent for $R \gg r$, F^L can be demonstrated to be [28]

$$F^L_{ij}(t_{\text{ret}}) = \int \mathrm{d}^3x\, x^L_{\text{STF}} \sum_{p=0}^{\infty} \frac{(2\ell+1)!!}{2^p p!(2\ell+2p+1)!!}\left(\frac{|\boldsymbol{x}|}{c}\frac{\partial}{\partial t}\right)^{2p} T_{ij}\left(t_{\text{ret}}, \boldsymbol{x}\right). \tag{5.3}$$

Here $t_{\text{ret}} = t - R/c$. The subscript STF stands for symmetric trace free. This is the only part that is not eliminated by the Lambda tensor (3.56). Recall that in the flux, the Lambda tensor eliminated the trace of the quadrupole moment.

As a remainder $\ell!! = \ell \cdot (\ell-2) \cdot (\ell-4) \cdots 2$ or 1, depending on whether ℓ is even or odd respectively.

Before proceeding, it will be useful to investigate STF tensors.

5.1.3 STF tensor decomposition

The STF part of a tensor is the irreducible representation of the tensor under rotations. Therefore, in GW physics it represents the physical degrees of freedom, where the other terms can be gauged away.

As an example, a rank two tensor can be decomposed into three parts

$$T_{ij} = T_{[ij]} + T_{\{ij\}} = A_{[ij]} + S_{\{ij\}} = \frac{1}{3}S_k{}^k\delta_{ij} + \varepsilon_{ijk}A^k + \left(S_{\{ij\}} - \frac{1}{3}S_k{}^k\delta_{ij}\right). \tag{5.4}$$

The first term in the last equality is the trace part, the second the anti-symmetric part, and finally the third term is the STF part.

Now, noticing $\Lambda^{ij}{}_{kl}\delta^{kl} = 0$, and since the Lambda tensor is symmetric in $k \leftrightarrow l$ $\Lambda^{ij}{}_{kl}\varepsilon^{kl}{}_m = 0$. This is why only the STF part of F^L_{ij} contribute to the final flux.

The STF part of a rank n tensor can be obtained by [29]

$$\begin{aligned} T^{i_1\cdots i_n}_{\text{STF}} &= \sum_{p=0}^{\lfloor n/2 \rfloor} c_p^{(n)} \delta^{\{i_1 i_2} \ldots \delta^{i_{2p-1} i_{2p}} T^{i_{2p+1}\cdots i_n\} a_1 a_1 \cdots a_p a_p}, \\ c_p^{(n)} &\equiv (-1)^p \frac{n!(2n-4p+1)!!}{(n-2p)!(2n-2p+1)!!(2p)!!}. \end{aligned} \tag{5.5}$$

The construction of this expression is not self-evident, but calculating it for the quadrupole and octupole moments will be instructive. The operator $\lfloor x \rfloor$ rounds x down to the closest integer, and is called the floor function. E.g. $\lfloor 3/2 \rfloor = 1$.

First notice that the $p = 0$ term always correspond to the symmetric version of the tensor in question

$$\text{for } p = 0: \qquad \frac{n!(2n+1)!!}{n!(2n+1)!!} T^{\{i_1 \cdots i_n\}} = T^{i_1 \cdots i_n}_{\text{sym}}. \tag{5.6}$$

Not surprisingly, the terms of $p > 0$ in (5.5) is used to subtract all possible traces, thus making the expression symmetric and trace free. E.g. the quadrupole moment becomes

$$Q^{ij}_{\text{STF}} = Q^{\{ij\}} - \frac{2!(1)!!}{0!(3)!!(2)!!} \delta^{ij} Q^a{}_a = Q^{ij}_{\text{sym}} - \frac{1}{3} \delta^{ij} \operatorname{tr}(Q), \tag{5.7}$$

which is equivalent to the expression used in the quadrupole radiation (3.62).

The octupole moment follows similarly as

$$\begin{aligned} O^{ijk}_{\text{STF}} &= O^{\{ijk\}} - \frac{3!(3)!!}{(1)!(5)!!(2)!!} \frac{1}{3} \left(\delta^{ij} O^{ka}{}_a + \delta^{ik} O^{ja}{}_a + \delta^{jk} O^{ia}{}_a \right) \\ &= O^{ijk}_{\text{sym}} - \frac{1}{5} \left(\delta^{ij} O^{ka}{}_a + \delta^{ik} O^{ja}{}_a + \delta^{jk} O^{ia}{}_a \right). \end{aligned} \tag{5.8}$$

The factor of 1/3 comes from the symmetrizing of $\delta^{\{ij} O^{k\}a}{}_a = \frac{1}{3!} \left(\left(\delta^{ij} + \delta^{ji} \right) O^{ka}{}_a + \dots \right) = \frac{1}{3} \left(\delta^{ij} O^{ka}{}_a + \dots \right)$. Hopefully these examples provide some familiarity with formula (5.5).

5.1.4 The multipole structure of GWs

Working the expression further, following the somewhat complicated steps of Ross [27] the result is

$$\begin{aligned} \bar{\mathcal{H}}^{\text{TT}}_{ij} = -\frac{4G}{Rc^2} \Lambda_{ij:k_{\ell-1}k_\ell} \sum_{\ell=2}^{\infty} \frac{1}{\ell!} \Bigg[& n_{L-2} \partial_0^\ell I_{ij}{}^{L-2}(t_{\text{ret}}) \\ & - \frac{2\ell}{\ell+1} \varepsilon^a{}_{b\{k_{\ell-1}} n_{aL-2} \partial_0^\ell J_{k_\ell\}}{}^{L-2}(t_{\text{ret}}) \Bigg]. \end{aligned} \tag{5.9}$$

Here I is the mass multipole, while J is current multipole, defined as

$$\begin{aligned} I^L(t) = \sum_{p=0}^{\infty} & \\ & \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!} \left(1 + \frac{8p(\ell+p+1)}{(\ell+1)(\ell+2)} \right) \left[\int \mathrm{d}^3x \, \partial_0^{2p} \mathcal{T}^{00}(t, \boldsymbol{x}) r^{2p} x^L \right]_{\text{STF}} \\ & + \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!} \left(1 + \frac{4p}{(\ell+1)(\ell+2)} \right) \left[\int \mathrm{d}^3x \, \partial_0^{2p} \mathcal{T}^k_{\ k}(t, \boldsymbol{x}) r^{2p} x^L \right]_{\text{STF}} \\ & - \frac{(2\ell+1)!!4}{(2p)!!(2\ell+2p+1)!!(\ell+1)} \left(1 + \frac{2p}{(\ell+2)} \right) \left[\int \mathrm{d}^3x \, \partial_0^{2p+1} \mathcal{T}^0_{\ i}(t, \boldsymbol{x}) r^{2p} x^L x^i \right]_{\text{STF}} \\ & + \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!} \left(\frac{2}{(\ell+1)(\ell+2)} \right) \left[\int \mathrm{d}^3x \, \partial_0^{2p+2} \mathcal{T}_{ij}(t, \boldsymbol{x}) r^{2p} x^L x^i x^j \right]_{\text{STF}}. \end{aligned} \tag{5.10}$$

$$J^L(t) = \sum_{p=0}^{\infty}$$
$$\frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!}\left(1+\frac{2p}{(\ell+2)}\right)\left[\int \mathrm{d}^3x\, \epsilon^{k_\ell}{}_{mn}\partial_0^{2p}\mathcal{T}^{0m}(t,\boldsymbol{x})r^{2p}x^{L-1}x^n\right]_{\mathrm{STF}}$$
$$-\frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!(\ell+2)}\left[\int \mathrm{d}^3x\, \epsilon^{k_\ell}{}_{ms}\partial_0^{2p+1}\mathcal{T}^{mn}(t,\boldsymbol{x})r^{2p}x^{L-1}x_n x^s\right]_{\mathrm{STF}}. \tag{5.11}$$

Here $\mathcal{T}^{\mu\nu}$ is the energy-momentum tensor of the source.

Now, using (3.55) the total energy flux is determined as [29]

$$\mathcal{F} = \frac{R^2}{c^2}\int \left\langle \dot{\mathcal{H}}_{ij}^{\mathrm{TT}}\dot{\mathcal{H}}^{ij}_{\mathrm{TT}}\right\rangle \mathrm{d}\Omega \tag{5.12a}$$

$$= \frac{G}{c^3}\sum_{\ell=2}^{\infty}\frac{(\ell+1)(\ell+2)}{\ell(\ell-1)\ell!(2\ell+1)!!}\left\langle\left(\frac{\mathrm{d}^{\ell+1}I^L(t)}{\mathrm{d}(ct)^{\ell+1}}\right)^2\right\rangle + \frac{4\ell(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!}\left\langle\left(\frac{\mathrm{d}^{\ell+1}J^L(t)}{\mathrm{d}(ct)^{\ell+1}}\right)^2\right\rangle \tag{5.12b}$$

5.2 The 1PN flux terms

To assign PN orders to the different terms, notice that every time derivative contributes with a factor of c^{-1}, $\partial_0^{2p} \sim 1/c^{2p}$. Also since $T^{\mu\nu} \sim m\dot{x}^\mu\dot{x}^\nu$, and $\dot{x}^0 \sim c$, we can expect every spatial index of the energy-momentum tensor to contribute with a factor of c^{-1} compared to the 00 term. Utilizing these observations it should be clear at which PN order the various terms of (5.10) and (5.11) enter.

5.2.1 Leading order term, the quadrupole moment

For the leading order term, only moments with the lowest power of $(c^{-1})^n$ can contribute. From the general flux expression (5.12b) it is clear that every derivative of the multipole moments contributes with additional factors of c^{-1}, thus the leading order term must be of only two indices, a quadrupole. Because the leading order term in the energy-momentum tensor is the point particle contributions, and since the point particle energy-momentum tensor is proportional to the tensor product of the particle's four velocity (3.40) any spatial index of $\mathcal{T}^{\mu\nu}$ contributes with an additional factor of c^{-1}. This excludes the current multipole (5.11) entirely, and all but the first line of the mass multipole (5.10).

Since the ct derivatives contribute with superfluous factors of c^{-1}, only the $p = 0$ term of the first line of (5.10) for $L = 2$ contributes to the leading order energy flux.

Using this leading order term of (5.10) and (3.40) for the $\mathcal{T}^{00}$ term the resulting leading order[2] expression for the mass quadrupole moment is

$$
\begin{aligned}
I^{ij}_{(0)}(t) &= \frac{(5)!!}{(5)!!}\int \mathrm{d}^3x\,\mathcal{T}^{00}(t,\boldsymbol{x})\,\left[x^i x^j\right]_{\mathrm{STF}} \\
&= \int \mathrm{d}^3x \sum_a \gamma_a m_a c^2 \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_a(t))\left[x^i x^j - \frac{1}{3}r^2\delta^{ij}\right] \\
&= \mu c^2 r^2 \left[n^i n^j - \frac{1}{3}\delta^{ij}\right]
\end{aligned} \tag{5.13}
$$

In the last line the masses was rewritten to the reduced mass (see Appendix B for more details).

To compute the resulting flux, equation (5.12b) requires the third time derivative of this term. Applying circular motion implies $n^x = \cos(\omega t)$, $n^y = \sin(\omega t)$, and $n^z = 0$. Thus, after consulting (C.2) for how to rewrite squared trigonometric functions, the result is

$$
\begin{aligned}
\frac{\mathrm{d}^3 I^{ij}_{(0)}}{\mathrm{d}(ct)^3} &= \frac{\mu r^2}{2c}\frac{\mathrm{d}^3}{\mathrm{d}t^3}\begin{pmatrix} \frac{1}{3}+\cos(2\omega t) & \sin(2\omega t) & 0 \\ \sin(2\omega t) & \frac{1}{3}-\cos(2\omega t) & 0 \\ 0 & 0 & -\frac{2}{3}\end{pmatrix} \\
&= \frac{2^2\mu r^2\omega^3}{c}\begin{pmatrix} \sin(2\omega t) & -\cos(2\omega t) & 0 \\ -\cos(2\omega t) & -\sin(2\omega t) & 0 \\ 0 & 0 & 0\end{pmatrix}.
\end{aligned} \tag{5.14}
$$

Notice that I^{ij} is indeed symmetric and trace free. To calculate the energy flux the sum over squares of each component is needed.

$$
\begin{aligned}
\left\langle \frac{\mathrm{d}^3 I^{ij}_{(0)}}{\mathrm{d}(ct)^3}\frac{\mathrm{d}^3 I_{(0)ij}}{\mathrm{d}(ct)^3}\right\rangle &= \frac{2^4\mu^2 r^4\omega^6}{c^2}\left(2\sin^2(2\omega t)+2\cos^2(2\omega t)\right) \\
&= 2^5\mu^2\frac{v^4\omega^2}{c^2} = \frac{2^5\eta^2}{G^2}\frac{v^{10}}{c^2}.
\end{aligned} \tag{5.15}
$$

In the last line $v = \omega r$ was used, and finally Kepler's third law (2.4) to exchange ω for v.

Then the leading order term of the energy flux is

$$
\mathcal{F}_{\mathrm{Newt}} = \frac{G}{c^3}\frac{3\cdot 4}{2\cdot 2!\cdot(5)!!}\left\langle \dddot{I}^{ij}\dddot{I}_{ij}\right\rangle = \frac{2^5}{5}\frac{\eta^2}{G}\frac{v^{10}}{c^5} = \frac{32}{5}\frac{\eta^2}{G}\frac{v^{10}}{c^5} \equiv F_{\mathrm{Newt}}v^{10} \tag{5.16}
$$

which is the well established result (3.68). It is worth noting that $[c^5/G]$ does indeed have the dimension of energy per time, as expected from the energy flux term.

For the next to leading order correction the octupole moment $I^{ijk}(t)$ and the current quadrupole moment $J^{ij}(t)$ must be added, and also there are relativistic corrections to the quadrupole formula used here for the 0PN flux term, like the other terms in (5.10) and relativistic corrections to $\mathcal{T}^{\mu\nu}$.

[2] To leading order $\gamma_a = 1$. Recall that $\gamma_a = (1-v^2/c^2)^{-1/2} \approx 1 + \frac{1}{2}v^2/c^2 + \frac{3}{8}v^4/c^4 + \dots$.

5.2.2 Next to leading order term, the octupole moment

Using (5.10) and (3.40) the mass octupole moment reads

$$\begin{aligned} I^{ijk}_{(2)}(t) &= \frac{(7)!!}{(7)!!}\cdot(1)\cdot\int \mathrm{d}^3x\,\mathcal{T}^{00}(t,\boldsymbol{x})\left[x^ix^jx^k\right]_{\text{STF}} \\ &= \int \mathrm{d}^3x \sum_a \gamma_a m_a c^2 \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_a(t))\left[x^ix^jx^k - \frac{r^2}{5}\left(\delta^{ij}x^k+\delta^{ik}x^j+\delta^{jk}x^i\right)\right] \\ &= \mu c^2 r^3\sqrt{1-4\eta}\left[n^in^jn^k - \frac{1}{5}\left(\delta^{ij}n^k+\delta^{ik}n^j+\delta^{jk}n^i\right)\right]. \end{aligned} \tag{5.17}$$

Inserting circular motion ($n^x = \cos(\omega t)$, $n^y = \sin(\omega t)$ and $n^z = 0$) and then taking the 4^{th} time derivative, as necessitated by equation (5.12b) produces

$$\frac{\mathrm{d}^4 I^{xxx}_{(2)}}{\mathrm{d}(ct)^4} = \frac{\mu r^3}{c^2}\sqrt{1-4\eta}\left[\frac{(3\omega)^4}{4}\cos(3\omega t) + \frac{3\omega^4}{20}\cos(\omega t)\right], \tag{5.18a}$$

$$\frac{\mathrm{d}^4 I^{xyy}_{(2)}}{\mathrm{d}(ct)^4} = \frac{\mu r^3}{c^2}\sqrt{1-4\eta}\left[-\frac{(3\omega)^4}{4}\cos(3\omega t) + \frac{\omega^4}{20}\cos(\omega t)\right], \tag{5.18b}$$

$$\frac{\mathrm{d}^4 I^{xzz}_{(2)}}{\mathrm{d}(ct)^4} = \frac{\mu r^3}{c^2}\sqrt{1-4\eta}\left[-\frac{\omega^4}{5}\cos(\omega t)\right]. \tag{5.18c}$$

$$\frac{\mathrm{d}^4 I^{yyy}_{(2)}}{\mathrm{d}(ct)^4} = \frac{\mu r^3}{c^2}\sqrt{1-4\eta}\left[\frac{(3\omega)^4}{4}\sin(3\omega t) + \frac{3\omega^4}{20}\sin(\omega t)\right], \tag{5.18d}$$

$$\frac{\mathrm{d}^4 I^{yxx}_{(2)}}{\mathrm{d}(ct)^4} = \frac{\mu r^3}{c^2}\sqrt{1-4\eta}\left[-\frac{(3\omega)^4}{4}\sin(3\omega t) + \frac{\omega^4}{20}\sin(\omega t)\right], \tag{5.18e}$$

$$\frac{\mathrm{d}^4 I^{yzz}_{(2)}}{\mathrm{d}(ct)^4} = \frac{\mu r^3}{c^2}\sqrt{1-4\eta}\left[-\frac{\omega^4}{5}\sin(\omega t)\right]. \tag{5.18f}$$

Because of the symmetric property of I^L, terms like I^{xyx} are equivalent to I^{yxx}. Thus, any term not appearing in equation (5.18) are either equivalent to one of the listed terms by symmetry, or zero (like odd numbers of z-indices). Notice also that the sum of (5.18a)-(5.18c) is zero. Similarly the sum of (5.18d)-(5.18f) is also zero, as they should be since I^L is trace free.

Note that for the square sum $\langle\cos(n\omega t)\cos(m\omega t)\rangle = \delta_{nm}/2$, thus all contributing terms will be of the form $\sin^2(n\omega t)$, and $\cos^2(n\omega t)$. For example $\left(\frac{\mathrm{d}^4 I^{xxx}_{(2)}}{\mathrm{d}(ct)^4}\right)^2 = \frac{\mu^2 r^6\omega^8}{c^4}(1-4)\left(\frac{3^8}{2^4}\cos^2(3\omega t) + \frac{3^2}{2^4\cdot 5^2}\cos^2(\omega t)\right)$, i.e. cross terms can be dropped.

$$\begin{aligned}\frac{\mathrm{d}^4 I^{ijk}_{(2)}}{\mathrm{d}(ct)^4}\frac{\mathrm{d}^4 I_{(2)ijk}}{\mathrm{d}(ct)^4} &= \frac{\mu^2}{c^4}(1-4\eta)r^6\omega^8\frac{(3^8+3^9)\cdot 5^2+(3^2+3)+3\cdot 2^4}{2^4\cdot 5^2}\\ &= \frac{2\cdot 3\cdot 1367}{5}\frac{\eta^2(1-4\eta)}{G^2}\frac{v^{12}}{c^4},\end{aligned} \tag{5.19}$$

$$\begin{aligned}\Rightarrow\quad \mathcal{F}^{\text{oct.}}_{(2)} &= \frac{G}{c^3}\frac{1}{3^3\cdot 7}\frac{2\cdot 3\cdot 1367}{5}\frac{\eta^2(1-4\eta)}{G^2}\frac{v^{12}}{c^4} = \frac{2\cdot 1367}{3^2\cdot 5\cdot 7}\frac{\eta^2(1-4\eta)}{G}\frac{v^{12}}{c^7}\\ &= \mathcal{F}_{\text{Newt}}\frac{1367}{2^4\cdot 3^2\cdot 7}(1-4\eta)\frac{v^2}{c^2}.\end{aligned} \tag{5.20}$$

5.2.3 Next to leading order term, the current quadrupole moment

Using (5.11) and (3.40) the current quadrupole moment reads

$$\begin{aligned}J^{ij}_{(2)}(t) &= \frac{(5)!!}{(5)!!}\left[\int \mathrm{d}^3x\, \epsilon^j{}_{mn}\mathcal{T}^{0m}(t,\boldsymbol{x})x^i x^n\right]_{\text{STF}} = \sum_a m_a c\left[x^i_a \epsilon^j{}_{mn} v^m_a x^n_a\right]_{\text{STF}}\\ &= \sum_a m_a c\left[x^i_a \epsilon^j{}_{mn}\left(\epsilon^m{}_{kl}\omega^k x^l_a\right)x^n_a\right]_{\text{STF}} = \sum_a m_a c\left[x^i_a\left(\delta_{nk}\delta^j{}_l - \delta_{nl}\delta^j{}_k\right)\omega^k x^l_a x^n_a\right]_{\text{STF}}\\ &= -\sum_a m_a c r_a^2\omega\frac{1}{2}\left(x^i_a\delta^j{}_z + x^j_a\delta^i{}_z\right) = -\frac{\mu c r^3\omega}{2}\sqrt{1-4\eta}\left[n^i\delta^j{}_z + n^j\delta^i{}_z\right].\end{aligned} \tag{5.21}$$

Notice that circular motion is here already assumed as $\boldsymbol{v}_a = \boldsymbol{\omega}\times\boldsymbol{x}_a$ is used to simplify the expression of the first equality of the second line. Again, using $n^x = \cos(\omega t)$, $n^y = \sin(\omega t)$, and $n^z = 0$, and taking the third time derivative as instructed by formula (5.12b) results in

$$\begin{aligned}\frac{\mathrm{d}^3 J^{ij}_{(2)}(t)}{\mathrm{d}(ct)^3} &= -\frac{\mu r^3\omega}{2c^2}\sqrt{1-4\eta}\frac{\mathrm{d}^3}{\mathrm{d}t^3}\begin{pmatrix}0 & 0 & \cos(\omega t)\\ 0 & 0 & \sin(\omega t)\\ \cos(\omega t) & \sin(\omega t) & 0\end{pmatrix}\\ &= \frac{\mu r^3\omega^4}{2c^2}\sqrt{1-4\eta}\begin{pmatrix}0 & 0 & -\sin(\omega t)\\ 0 & 0 & \cos(\omega t)\\ -\sin(\omega t) & \cos(\omega t) & 0\end{pmatrix}.\end{aligned} \tag{5.22}$$

Notice that $J^{ij}_{(2)}$ is trace free and symmetric, as it should be. The flux is determined by the sum of squares of all the tensor components, which is

$$\begin{aligned}\frac{\mathrm{d}^3 J^{ij}_{(2)}(t)}{\mathrm{d}(ct)^3}\frac{\mathrm{d}^3 J_{(2)ij}(t)}{\mathrm{d}(ct)^3} &= \frac{\mu^2 r^6\omega^8}{2^2c^4}(1-4\eta)\left(2\sin^2(\omega t)+2\cos^2(\omega t)\right)\\ &= \frac{\mu^2(1-4\eta)}{2c^4}v^6\omega^2 = \frac{\eta^2(1-4\eta)}{2G^2}\frac{v^{12}}{c^4}\end{aligned} \tag{5.23}$$

$$\begin{aligned}\Rightarrow\quad \mathcal{F}^{\text{curr.quad}}_{(2)} &= \frac{2^4 G}{3^2\cdot 5c^3}\frac{\eta^2(1-4\eta)}{2G^2}\frac{v^{12}}{c^4} = \frac{8}{45}\frac{\eta^2(1-4\eta)}{G}\frac{v^{12}}{c^7}\\ &= \mathcal{F}_{\text{Newt}}\frac{1}{2^2\cdot 3^2}(1-4\eta)\frac{v^2}{c^2}.\end{aligned} \tag{5.24}$$

5.2.4 Next to leading order term, the quadrupole moment corrections

Circling back to the mass quadrupole moment, all first order assumption that went into 5.2.1 must now be expanded to next to leading order. This primarily means 3 things:

1. Relativistic corrections to $\mathcal{T}^{\mu\nu}$, like kinetic energy, and gravitational energy.
2. Including other terms from (5.10), like $p = 1$, $\mathcal{T}^{0i}$ and $\mathcal{T}^{k}_{\ k}$.
3. Relativistic corrections to the inserted motion of the source. For quasi-stable circular orbits the motion does not change, but the relation between v, ω, and r pick up some relativistic corrections (4.57)-(4.59).

Starting with the relativistic corrections to $\mathcal{T}^{00}$, and $\mathcal{T}^{k}_{\ k}$ (it will be clear in a moment why these are lumped together) recall that (3.40)

$$\begin{aligned} T^{00}_{pp}(t,\boldsymbol{x}) &= \sum_a \gamma_a m_a c^2 \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_a(t)) \\ &= \sum_a \left(1+\frac{1}{2}\frac{v_a^2}{c^2}+\frac{3}{8}\frac{v_a^4}{c^4}+\dots\right) m_a c^2 \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_a(t)) \\ &= \sum_a \left(m_a c^2+\frac{1}{2}m_a v_a^2+\frac{3}{8}m_a\frac{v_a^4}{c^2}+\dots\right)\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_a(t)), \end{aligned} \tag{5.25}$$

simply Taylor expanding $\gamma_a = \left(1-v_a^2/c^2\right)^{-1/2}$ around $v/c = 0$. The next to leading order terms in $\mathcal{T}^{00}$ is thus proportional to $(c^{-1})^0$. From the Virial theorem, or equivalently from (4.58b), the leading order (Newtonian) term of the gravitational potential scales also as v^2, and should therefore also be included. This concludes point 1., the relativistic correction of $\mathcal{T}^{\mu\nu}$. Finally, to leading order $\mathcal{T}^{k}_{\ k}$ is the point particle tensor (3.40)

$$T_{pp}{}^{k}{}_{k} = \sum_a \gamma_a m_a v_a^k v_{ak}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_a(t)) \simeq \sum_a m_a v_a^2 \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_a(t)), \tag{5.26}$$

which is just twice the kinetic energy. Thus, the trace of $\mathcal{T}^{\mu}_{\ \mu}$ part of the quadrupole reads

$$\begin{aligned} \text{Tr. part of } I^{ij}_{(2)} &= \int \mathrm{d}^3x \left(\mathcal{T}^{00}+\mathcal{T}^{k}_{\ k}\right)\left[x^i x^j\right]_{\text{STF}} \\ &= \left[n^i n^j\right]_{\text{STF}} \sum_a m_a c^2 r_a^2 \left(1+\frac{3}{2}\frac{v_a^2}{c^2}-\sum_{b>a}\frac{Gm_b}{c^2|\boldsymbol{x}_a-\boldsymbol{x}_b|}\right) \\ &= \mu c^2 r^2 \left(1+\frac{3}{2}(1-3\eta)\frac{v^2}{c^2}-(1-2\eta)\frac{GM}{rc^2}\right)\left[n^i n^j\right]_{\text{STF}} \\ &= \mu c^2 r^2 \left(1+\frac{1}{2}(1-5\eta)\frac{v^2}{c^2}\right)\left(n^i n^j-\frac{1}{3}\delta^{ij}\right). \end{aligned} \tag{5.27}$$

Since the time dependent part (n^i) is equivalent to the first order term of the quadrupole formula, the third derivative of the expression correct to first order in $(v^2/c^2)^1$ can be inferred directly from (5.14)

$$\frac{\mathrm{d}^3 I^{ij}_{(2)}}{\mathrm{d}(ct)^3} = \frac{2^2\mu r^2\omega^3}{c}\left(1+\frac{1-5\eta}{2}\frac{v^2}{c^2}\right)\begin{pmatrix}\sin(2\omega t) & -\cos(2\omega t) & 0\\ -\cos(2\omega t) & -\sin(2\omega t) & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{5.28}$$

For the $p = 0$ terms this only leaves the $\partial_0 \mathcal{T}^{0k}$ term in line three of (5.10). The last line containing $\partial_0^2 \mathcal{T}^{ij}$ will not contribute as it scales as $(\partial_0^2 \mathcal{T}^{ij})/\mathcal{T}^{00} \sim c^{-4}$. To leading order also this energy-momentum tensor component is the free point particle tensor, and thus

$$0k\text{-part of } I^{ij}_{(2)}(t) = -\frac{4}{3} \int \mathrm{d}^3x\, \partial_0 \sum_a m_a c v_a^k x_k \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_a(t)) \left[x^i x^j\right]_{\mathrm{STF}} . \tag{5.29}$$

For circular motion $\boldsymbol{v}_a = \pm\omega r_a \big(-\sin(\omega t),\ \ \cos(\omega t),\ \ 0\big)$, and is thus orthogonal to $\boldsymbol{x}_a$: $\boldsymbol{v}_a \cdot \boldsymbol{x}_a = 0$. Therefore the leading order contribution of the $\mathcal{T}^{0k}$-part of I^L is 0.

To finish point 2. only accounting for the $p = 1$ term remains. This term follows as

$$\begin{aligned} p = 1 \text{ part of } I^{ij}_{(2)} &= \frac{5!!}{(2)!!(7)!!}\left(1 + \frac{8}{3}\right) \int \mathrm{d}^3x\, \partial_0^2 \mathcal{T}^{00}(t, \boldsymbol{x}) r^2 \left[x^i x^j\right]_{\mathrm{STF}} \\ &= \frac{11}{42} \sum_a m_a c^2 r_a^4 \partial_0^2 \left[n^i n^j\right]_{\mathrm{STF}} \\ &= \frac{11}{2^2 \cdot 3 \cdot 7} \mu (1 - 3\eta) r^4 \frac{\mathrm{d}^2}{\mathrm{d}t^2} \begin{pmatrix} \frac{1}{3} + \cos(2\omega t) & \sin(2\omega t) & 0 \\ \sin(2\omega t) & \frac{1}{3} - \cos(2\omega t) & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \\ &= -\frac{11}{3 \cdot 7} \mu (1 - 3\eta) r^4 \omega^2 \begin{pmatrix} \cos(2\omega t) & \sin(2\omega t) & 0 \\ \sin(2\omega t) & -\cos(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} , \end{aligned} \tag{5.30}$$

$$\frac{\mathrm{d}^3 I^{ij}_{(2)}}{\mathrm{d}(ct)^3} = \frac{2^2 \mu r^2 \omega^3}{c} \left(-\frac{2 \cdot 11}{3 \cdot 7}(1 - 3\eta)\frac{v^2}{c^2}\right) \begin{pmatrix} \sin(2\omega t) & -\cos(2\omega t) & 0 \\ -\cos(2\omega t) & -\sin(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{5.31}$$

This leaves the totally 1PN correct third derivative of the quadrupole moment

$$\frac{\mathrm{d}^3 I^{ij}_{(2)}}{\mathrm{d}(ct)^3} = \frac{2^2 \mu r^2 \omega^3}{c} \left(1 - \frac{23 - 27\eta}{42}\frac{v^2}{c^2}\right) \begin{pmatrix} \sin(2\omega t) & -\cos(2\omega t) & 0 \\ -\cos(2\omega t) & -\sin(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{5.32}$$

Here the subtlety of point 3., corrections to the equations of motion, enters. At this stage equations (4.57)-(4.59) must be used to convert between r, ω, and v. One might expect at this point to use $\omega r = v$ and (4.59a) to convert the last factor of $\omega = v^3/GM$, but this is not the case. Recalling that v is just a proxy variable for the frequency, one should expand the flux in terms of ω, and perhaps change ω to $v = (GM\omega)^{2/3}$.

Doing that

$$\begin{aligned} \frac{2^2 \mu r^2 \omega^3}{c} = \frac{2^2 \mu}{c} \omega v^2 &= \frac{2^2 \mu}{c} \omega (GM\omega)^{2/3} \left\{1 - \left(2 - \frac{2}{3}\eta\right) \frac{(GM\omega)^{2/3}}{c^2}\right\} \\ &= \frac{4\mu}{GMc} v^5 \left\{1 - \left(2 - \frac{2}{3}\eta\right) \frac{v^2}{c^2}\right\} . \end{aligned}$$

Inserting this into (5.32), discarding terms $\mathcal{O}\left(\frac{v^4}{c^4}\right)$, provides the final result

$$\frac{\mathrm{d}^3 I^{ij}_{(2)}}{\mathrm{d}(ct)^3} = \frac{2^2\eta}{G}\frac{v^5}{c}\left(1 - \frac{107 - 5\cdot 11\eta}{2\cdot 3\cdot 7}\frac{v^2}{c^2}\right)\begin{pmatrix}\sin(2\omega t) & -\cos(2\omega t) & 0\\ -\cos(2\omega t) & -\sin(2\omega t) & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{5.33}$$

$$\Rightarrow \frac{\mathrm{d}^3 I^{ij}_{(2)}}{\mathrm{d}(ct)^3}\frac{\mathrm{d}^3 I_{(2)ij}}{\mathrm{d}(ct)^3} = \frac{2^5\eta^2}{G^2}\frac{v^{10}}{c^2}\left(1 - \frac{107 - 5\cdot 11\eta}{3\cdot 7}\frac{v^2}{c^2}\right). \tag{5.34}$$

And thus the energy flux from the quadrupole at next to leading order is

$$\mathcal{F}^{\text{quad}}_{(2)} = \frac{2^5\eta^2}{5G}\frac{v^{10}}{c^5}\left(1 - \frac{107 - 5\cdot 11\eta}{3\cdot 7}\frac{v^2}{c^2}\right) = \mathcal{F}_{\text{Newt}}\left(1 - \frac{107 - 5\cdot 11\eta}{3\cdot 7}\frac{v^2}{c^2}\right). \tag{5.35}$$

5.2.5 The total 1PN energy flux

The total energy flux correct to 1PN is then the sum of (5.15), (5.20), (5.24), and (5.35),

$$\boxed{\mathcal{F} = \frac{32}{5}\frac{\eta^2}{G}\frac{v^{10}}{c^5}\left\{1 - \left(\frac{1247}{336} + \frac{35}{12}\eta\right)\frac{v^2}{c^2} + \mathcal{O}\left(\frac{1}{c^3}\right)\right\}.} \tag{5.36}$$

Which is exactly the flux (2.8) presented in Chapter 2.

To get some perspective, lets consider the Earth-Moon system again. According to (5.36), the energy flux due to GWs is

$$\mathcal{F}_{☾} = 6.03\cdot 10^{-4}\left(1 - 4.37\cdot 10^{-11}\right)\ \mathrm{J/s} \tag{5.37}$$

Using equation (2.18) for the change in orbital frequency, and relation (4.58a) of r and ω, the time evolution of the relative separation due to GW emission is found to be

$$r_{☾}(\tau) = \frac{GM}{(GM\omega)^{2/3}}\left\{1 - \left(1 - \frac{\eta}{3}\right)\frac{(GM\omega)^{2/3}}{c^2} + \mathcal{O}\left((GM\omega)^{3/2}\right)\right\} \tag{5.38a}$$

$$\approx \frac{4(GM)^{1/3}}{5^{2/3}}\left(\frac{5G\mathcal{M}}{c^3}\right)^{\frac{5}{12}}\tau^{1/4}, \tag{5.38b}$$

$$\Rightarrow \quad \dot{r}_{☾}(\tau) \approx -\frac{(GM)^{1/3}}{5^{2/3}}\left(\frac{5G\mathcal{M}}{c^3}\right)^{\frac{5}{12}}\tau^{-3/4} = \frac{G^3M^3\eta}{5c^5}r^{-3} \tag{5.38c}$$

$$\approx 1.14\cdot 10^{-27}\frac{\mathrm{m}}{s} \approx 3.60\cdot 10^{-17}\frac{\mathrm{mm}}{\mathrm{year}}. \tag{5.38d}$$

This is, not surprisingly, very slowly. At this rate, it would take $\sim 2.8\cdot 10^{16}$ years until the effect would be in the range of millimetres.

Chapter 6

Discussion and conclusion

We have now seen how the 1PN gravitational waveform (2.19) can be obtained from the 1PN energy (4.63) and flux (5.36), assuming quasi-stable circular orbits and separation of scales. We have also demonstrated how the 1PN energy can be determined using Feynman diagrams (Chapter 4), and how the 1PN flux can be computed using multipoles (Chapter 5), all based on an effective field theory of gravity as a gauge field (Chapter 3).

In his text [4], Porto claimed

"[...] that adopting an EFT framework, when possible, greatly simplifies the computations and provides the required intuition for 'physical understanding'."

In my own experience, the computations do not seem all that more simplified compared to the more traditional geometrical approach (see Maggiore [9] for an outline, or Blanchet [30] for more details). For someone without a deep background in EFT, like a master's student, any simplification of the calculation is outweighed by the work of familiarizing oneself with standard results and conventions from QFT.

Of course, if one *does* have a deep familiarity with EFTs, the field theorist approach presented in this thesis is a great way to transfer those skills to gravitational wave physics. These are after all powerful tools for handling perturbative phenomena. The use of Feynman diagrams makes the terms in the perturbation series more manageable, and can give intuition for what kind of physical effects the different terms account for. In this sense, the computations can be considered to have been 'simplified', and provided the intuition for 'physical understanding'.

It is also possible that the field theorists' approach becomes significantly simpler than the relativists' approach at higher PN orders. In order to verify this, I would need to compute higher order corrections.

Even so, for relativists, some familiarity with the field theory way of thinking of gravitational dynamics is helpful for deepening their understanding of gravity. As Feynman once said at a Cornell lecture during his gravity phase:

"Every theoretical physicist who is any good knows six or seven different theoretical representations for exactly the same physics. He knows that they are all equivalent, and that nobody is ever going to be able to decide which one is right at that level, but he keeps them in his head, hoping that they will give him different ideas for guessing." - Feynman [31]

For such reasons, it is valuable to have alternative ways of thinking about gravity and gravitational waves. Some extensions of, and alternative theories for, Einstein's theory of gravity might present themselves more naturally in the language of field theory, rather than differential geometry. E.g. quantum 'loop' corrections of gravity [32]. With gravitational wave data imposing some of the strongest constraints on gravity theories, having an alternative route for translating theories of gravity to gravitational waveforms is a useful tool.

In conclusion, this effective field theory approach to computing gravitational waveforms will probably not replace the more traditional relativist approach as the standard or introductory way of deriving these results any time soon. As a method it is however worth developing, as it provides an alternative perspective on the physics of gravitational waves. It might also provide a shorter path for some alternative theories of gravity to testable predictions, and can be used by physicists with a heavier quantum field theory background to simplify the computations of such theories.

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Appendix A

Solution of the wave equation

This derivation can be found in most textbooks on field theory, e.g. Kachelrieß [17], or Schwichtenberg [16].

Indices will be ignored in this appendix, as the spatio-temporal dependence of the solution is *assumed* to be independent of indices. Thus, $h_{\mu\nu}(x^\alpha) = \epsilon_{\mu\nu} h(x^\alpha)$ and $\Box\, h_{\mu\nu}(x^\alpha) = \epsilon_{\mu\nu} \Box\, h(x^\alpha)$.

Assuming the solution to be a superposition of plane waves $e^{-ik_\sigma x^\sigma}$, the most general form the wave can take is

$$h(x^\alpha) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left\{ a(k_\mu) e^{-ik_\sigma x^\sigma} + b(k_\mu) e^{ik_\sigma x^\sigma} \right\}. \tag{A.1}$$

For (A.1) to be a solution of the wave equation (3.20) the following must hold

$$\begin{aligned} \Box\, h(x^\alpha) = 0 &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \Box \left\{ a(k_\mu) e^{-ik_\sigma x^\sigma} + b(k_\mu) e^{ik_\sigma x^\sigma} \right\} \\ &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \left\{ a(k_\mu) e^{-ik_\sigma x^\sigma} + b(k_\mu) e^{ik_\sigma x^\sigma} \right\} \\ &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \eta^{\mu\nu} \left(i^2 k_\mu k_\nu \right) \left\{ a(k_\mu) e^{-ik_\sigma x^\sigma} + b(k_\mu) e^{ik_\sigma x^\sigma} \right\} \\ &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left(k_0^2 - |\boldsymbol{k}|^2 \right) \left\{ a(k_\mu) e^{-ik_\sigma x^\sigma} + b(k_\sigma) e^{ik_\sigma x^\sigma} \right\}, \end{aligned} \tag{A.2}$$

which is a solution as long as $k_0^2 = |\boldsymbol{k}|^2$. Since k_0 is identified as the temporal frequency it is required to be positive $k_0 = \omega/c \geq 0$ in order to be physical. Both these conditions can be imposed by $\delta\big(k_0^2 - \omega_k^2\big) \cdot \Theta(k_0)$, where $\delta(x)$ is the *Dirac delta function*, $\Theta(x)$ is the *Heaviside step function*, and ω_k is determined by the dispersion relation and is $\omega_k = |\boldsymbol{k}|$ for massless fields.[1]

[1] Massive fields must satisfie the *Klein–Gordon equation* $(\Box - m^2)\phi = 0$. The solution is the same as that of massless fields shown here, but with $\omega_k^2 = |\boldsymbol{k}|^2 + m^2$. That is why $|\boldsymbol{k}|$ is renamed ω_k here, to make the result more easily transferable.

Implementing these restrictions (A.1) becomes

$$h(x^\alpha) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \delta\big(k_0^2 - \omega_k^2\big)\, \Theta(k_0) \left\{ a(k_\mu) e^{-ik_\sigma x^\sigma} + b(k_\mu) e^{ik_\sigma x^\sigma} \right\}. \tag{A.3}$$

Preforming the k_0 integral results with

$$\int \frac{\mathrm{d}k_0}{2\pi} \delta\big(k_0^2 - \omega_k^2\big)\, \Theta(k_0)\, f(k_0) \tag{A.4a}$$

$$= \int \frac{\mathrm{d}k_0}{2\pi} \delta((k_0 - \omega_k)(k_0 + \omega_k))\, \Theta(k_0)\, f(k_0) \tag{A.4b}$$

$$= \int \frac{\mathrm{d}k_0}{2\pi} \frac{1}{2k_0} \left[\delta(k_0 - \omega_k) + \delta(k_0 + \omega_k)\right] \Theta(k_0)\, f(k_0) \tag{A.4c}$$

$$= \frac{1}{2\omega_k} f(\omega_k). \tag{A.4d}$$

Step-by-step the above calculation first, (A.4a), collapse all dependence on k_0 into a function $f(k_0)$, other than the Dirac delta function and Heaviside step function. In line (A.4b) the argument of the Dirac delta was expanded, and in line (A.4c) the Dirac delta was *itself* expanded according to the relation

$$\delta(f(x)) = \sum_i \frac{\delta(x - a_i)}{\frac{\mathrm{d}f}{\mathrm{d}x}}, \quad \forall a_i : f(a_i) = 0. \tag{A.5}$$

Lastly, in line (A.4d), the $k_0 \geq 0$ term was singled out by $\Theta(k_0)$.

All the steps of (A.4) can be preformed for (A.3). Also requiring $h(x^\alpha)$ to be a real function can easily be done by demanding $h^\dagger(x^\alpha) = h(x^\alpha)$, which is obtained most generally by having $b(k_\mu) = a^\dagger(k_\mu)$.

Thus, the most general solution of the wave equation for a real scalar field is

$$\Box\, h(x^\alpha) = 0,$$

$$\Rightarrow \quad h(x^\alpha) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \cdot 2\omega_k} \left\{ a(\boldsymbol{k}) e^{-ik_\sigma x^\sigma} + a^\dagger(\boldsymbol{k}) e^{ik_\sigma x^\sigma} \right\}, \tag{A.6}$$

with $k_0 = |\boldsymbol{k}| = \omega_k$. This is the solution presented in equation (2.2).

This is still a large class of solutions, but it is restricted to travel in the $\boldsymbol{k}$-direction through space, with a velocity of

$$v_g = \frac{\partial \omega}{\partial |\boldsymbol{k}|} = \frac{\partial c k_0}{\partial |\boldsymbol{k}|} = c \frac{\partial |\boldsymbol{k}|}{\partial |\boldsymbol{k}|} = c. \tag{A.7}$$

Notice that both the group velocity $v_g \equiv \frac{\partial \omega}{\partial |\boldsymbol{k}|}$ and the phase velocity $v_p \equiv \frac{\omega}{|\boldsymbol{k}|}$ are both equal to c.

Appendix B

Equivalent one body problem and mass term manipulation

B.1 Rewriting to the equivalent one body problem

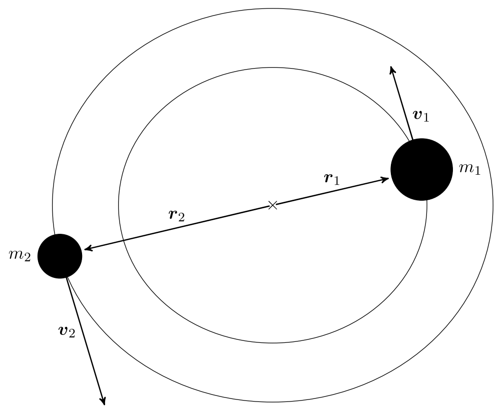

Figure B.1: Diagram of a binary system.

To solve the equation of motion of the two body problem it is useful to rewrite the equations in terms of relative quantities, like the spatial separation $\boldsymbol{r}$, and relative velocity $\boldsymbol{v} \equiv \dot{\boldsymbol{r}}$. This will be done for the centre of mass frame in this appendix.

Letting $\boldsymbol{r}_i$, $i \in \{1, 2\}$ be the position of object i relative the centre of mass, which is placed in the origin, as in Figure B.1 the following identity holds

$$\boldsymbol{r} \equiv \boldsymbol{r}_2 - \boldsymbol{r}_1, \tag{B.1}$$

$$\begin{aligned} \mathbf{0} &\equiv (m_1 + m_2)\boldsymbol{R}_{\text{CM}} = m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2, \\ \Rightarrow \quad m_1\boldsymbol{r}_1 &= -m_2\boldsymbol{r}_2 = -m_2\left(\boldsymbol{r} + \boldsymbol{r}_1\right). \end{aligned} \tag{B.2}$$

In the last line of (B.2) equation (B.1) was used to eliminate $\boldsymbol{r}_2$. Similarly $\boldsymbol{r}_1$ can be eliminated in favour of $\boldsymbol{r}_2$. Thus $\boldsymbol{r}_i$ can be expressed as

$$\boldsymbol{r}_1 = -\frac{m_2}{m_1+m_2}\boldsymbol{r} = -\frac{m_2}{M}\boldsymbol{r}, \tag{B.3a}$$

$$\boldsymbol{r}_2 = \frac{m_1}{m_1+m_2}\boldsymbol{r} = \frac{m_1}{M}\boldsymbol{r}. \tag{B.3b}$$

Here M is the total mass of the binary. Since the velocity of each object in the centre of mass frame is $\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i$ it directly follows

$$\boldsymbol{v}_1 = -\frac{m_2}{M}\boldsymbol{v}, \tag{B.4a}$$

$$\boldsymbol{v}_2 = \frac{m_1}{M}\boldsymbol{v}, \tag{B.4b}$$

where again $\boldsymbol{v} = \dot{\boldsymbol{r}}$.

Substituting $\boldsymbol{r}_i$ and $\boldsymbol{v}_i$ for the expressions of equations (B.3)-(B.4) the Lagrangian, and thus the EoM, becomes a function of *just* $\boldsymbol{r}$ and $\boldsymbol{v}$. Thus the two body problem is reduced to solving for just the relative motion of one object, an equivalent one body problem.

Explicitly the Newtonian Lagrangian becomes

$$\begin{aligned} L_{\text{Newt}} &= \frac{1}{2}\left(m_1 v_1^2 + m_2 v_2^2\right) + \frac{G m_1 m_2}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|} = \frac{1}{2}\left(m_1 \frac{m_2^2}{M^2} + m_2 \frac{m_1^2}{M^2}\right) v^2 + \frac{G m_1 m_2}{r} \\ &= \frac{1}{2}\frac{m_1 m_2}{M}\left(\frac{m_2 + m_1}{M}\right) v^2 + \frac{GM\frac{m_1 m_2}{M}}{r} = \frac{1}{2}\frac{m_1 m_2}{M} v^2 + \frac{GM\frac{m_1 m_2}{M}}{r} \\ &\equiv \frac{1}{2}\mu v^2 + \frac{GM\mu}{r}. \end{aligned}$$

Bottom line is that preforming the substitution to r and v reduces the problem to describing the motion of *one* particle with an effective mass of $\mu = \frac{m_1 m_2}{M}$ in a gravitational potential produced by an effective mass of $M = m_1 + m_2$, which is static and located at the position of the other particle.

B.2 Mass term manipulation

Following the previous section it is hopefully clear what motivates the introduction of the total and reduced mass M and μ. The name reduced mass follows from the observation that in the extreme mass ratio, $m_1 \gg m_2$, $M = m_1 + m_2 \simeq m_1$ and $\mu = \frac{m_1 m_2}{M} \simeq \frac{m_1 m_2}{m_1} = m_2$. I.e. in the test mass regime M is the gravitational source and μ is the test mass exactly. Of course $M > \mu$ in all cases, with the largest value of $\mu_{\text{max}} = \frac{1}{4}M$ when $m_1 = m_2$.

Moving beyond the Newtonian approximation there will appear other mass terms that are common in the literature. These are listen for convenience in equation (B.5).

In this thesis there will appear terms of the form $m_1 v_1^{n+1} + m_2 v_2^{n+1}$ and $m_1(-r_1)^{n+1} + m_2 r_2^{n+1}$. In this section there will be tips for strategies to convert these expressions into (B.5) type mass terms. The result can be read of equations (B.6)-(B.7).

$M \equiv m_1 + m_2$	Total mass	(B.5a)
$\mu \equiv \dfrac{m_1 m_2}{m_1 + m_2} = \dfrac{m_1 m_2}{M}$	Reduced mass	(B.5b)
$\eta \equiv \dfrac{m_1 m_2}{(m_1 + m_2)^2} = \dfrac{\mu}{M}$	Symmetric mass ratio	(B.5c)
$\mathcal{M} \equiv \dfrac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} = \left(\mu^3 M^2\right)^{1/5} = M\eta^{3/5}$	Chirp mass	(B.5d)

Here is a stepwise approach to deal with $m_1 v_1^{n+1} + m_2 v_2^{n+1}$ and $m_1(-r_1)^{n+1} + m_2 r_2^{n+1}$ type expressions.

1. The product $m_1 m_2 = M\mu$. Identify and extract all common factors of μ from the expression.
2. This will leave something like $(m_2^n \pm m_1^n)/M^n$.
   a. If it is a sum, calculate $M^n = m_1^n + m_2^n + \ldots$. Thus $m_2^n + m_1^n = M^n - \ldots$ Then again look out for reduced masses and remember that $\mu/M = \eta$.
   b. If it is a difference this will usually imply that n is even. Let $n = 2k$ and then $m_2^{2k} - m_1^{2k} = (m_2^k - m_1^k)(m_2^k + m_1^k)$. The $(m_2^k + m_1^k)$ term can be expanded as in step a, and hopefully this will be enough. The difference can be further expanded using $m_1^k - m_2^k = \sqrt{(m_1^k - m_2^k)^2}$ to get mixed terms which can be factored as μ.

To get some concrete examples, lets consider $(m_1 m_2^4 + m_2 m_1^4)/M^4$.

$$\begin{aligned} \frac{m_1 m_2^4 + m_2 m_1^4}{M^4} &= \frac{m_1 m_2}{M}\frac{m_2^3 + m_1^3}{M^3} = \mu\frac{M^3 - 3m_1 m_2(m_1 + m_2)}{M^3} \\ &= \mu\left(1 - 3\mu/M\right) = \mu\left(1 - 3\eta\right). \end{aligned}$$

$m_2^3 + m_1^3$ was rewritten using equation (B.6c).

Most of the expressions encountered at 1PN will follow the same approach as the example above, with the exception of a term $(m_2 m_1^3 - m_1 m_2^3)/M^3$ which appear in the octupole moment of the flux (see Chapter 5). It goes like this

$$\begin{aligned} \frac{m_2 m_1^3 - m_1 m_2^3}{M^3} &= \mu\frac{m_1^2 - m_2^2}{M^2} = \mu\frac{(m_1 - m_2)(m_1 + m_2)}{M^2} = \mu\frac{m_1 - m_2}{M} \\ &= \mu\sqrt{\frac{(m_1 - m_2)^2}{M^2}} = \mu\sqrt{\frac{m_1^2 + m_2^2 - 2m_1 m_2}{M^2}} = \mu\sqrt{\frac{M^2 - 4M\mu}{M^2}} \\ &= \mu\sqrt{1 - 4\eta}. \end{aligned}$$

In equation (B.6) the different sum of powers are listed, and in (B.7) the mixed products are listed. These expressions are useful for mass term manipulations like those that appear in this thesis.

$$\boxed{\begin{aligned} m_1 + m_2 &= M, && \text{(B.6a)}\\ m_1^2 + m_2^2 &= M^2 - 2m_1m_2 = M^2 - 2M\mu, && \text{(B.6b)}\\ m_1^3 + m_2^3 &= M^3 - 3m_1^2m_2 - 3m_1m_2^2 = M^3 - 3M^2\mu, && \text{(B.6c)}\\ m_1^4 + m_2^4 &= M^4 - 4m_1^3m_2 - 6m_1^2m_2^2 - 4m_1m_2^3 \\ &= M^4 + 8M\mu^2 - 4M^2\mu - 6M^2\mu^2. && \text{(B.6d)} \end{aligned}}$$

$$\boxed{\begin{aligned} m_1m_2 &= M\mu, && \text{(B.7a)}\\ m_1m_2^2 + m_1^2m_2 &= M^2\mu, && \text{(B.7b)}\\ m_1m_2^3 - m_1^3m_2 &= M^3\mu\sqrt{1-4\eta}, && \text{(B.7c)}\\ m_1m_2^4 + m_1^4m_2 &= M^4\mu\left(1-3\eta\right). && \text{(B.7d)} \end{aligned}}$$

A handy trick is to combine equation (B.7a) with expressions from (B.6) to obtain (B.7)-type expressions.

Appendix C

Trigonometric identities

The trigonometric identities that are used in this thesis can all be derived using Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta). \tag{C.1}$$

For the squared trigonometric functions one only needs to square this formula

$$\begin{aligned}\left(e^{i\theta}\right)^2 &= e^{i2\theta} = \cos(2\theta) + i\sin(2\theta)\\ &= (\cos(\theta) + i\sin(\theta))^2 = \cos^2(\theta) - \sin^2(\theta) + i2\sin(\theta)\cos(\theta)\\ &= 2\cos^2(\theta) - 1 + i2\sin(\theta)\cos(\theta) = 1 - 2\sin^2(\theta) + i2\sin(\theta)\cos(\theta),\end{aligned}$$

where in the last line the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ was used to write the expression only by second powers in cosine or sine respectively. Comparing the real parts and imaginary parts of the first and third line produces the useful identities

$$\boxed{\begin{aligned}\cos^2(\theta) &= \frac{1}{2}\left(1 + \cos(2\theta)\right), && \text{(C.2a)}\\ \sin^2(\theta) &= \frac{1}{2}\left(1 - \cos(2\theta)\right), && \text{(C.2b)}\\ \sin(\theta)\cos(\theta) &= \frac{1}{2}\sin(2\theta). && \text{(C.2c)}\end{aligned}}$$

Likewise the identities for the third power trigonometric functions can be obtained

$$\begin{aligned}\left(e^{i\theta}\right)^3 &= e^{i3\theta} = \cos(3\theta) + i\sin(3\theta) = (\cos(3\theta) + i\sin(3\theta))^3\\ &= \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) + i3\cos^2(\theta)\sin(\theta) - i\sin^3(\theta)\\ &= \left[4\cos^3(\theta) - 3\cos(\theta)\right] + i\left[3\sin(\theta) - 4\sin^3(\theta)\right],\end{aligned}$$

and then once again comparing the imaginary part of the first and third line the following identities are obtained.

$$\boxed{\begin{aligned}\cos^3(\theta) &= \frac{1}{4}\left(3\cos(\theta) + \cos(3\theta)\right), && \text{(C.3a)}\\ \sin^3(\theta) &= \frac{1}{4}\left(3\sin(\theta) + \sin(3\theta)\right). && \text{(C.3b)}\end{aligned}}$$

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