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Explorations of Coherent State Path Integral Formulations for Spin Systems Using a Projection Operator Implementation of Occupation Number Constraints

Master's thesis in Physics
Supervisor: John Ove Fjærestad
June 2021

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NTNU
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Abstract

In this thesis, we explore how a projection operator approach to constraint implementation can be used when constructing coherent state path integrals of spin systems represented in terms of Schwinger boson or Abrikosov fermion operators. In particular, we use this construction to calculate the single spin partition function for the zero energy and Zeeman Hamiltonians for a general spin-S system represented by Schwinger boson operators, and the single spin partition function for a two-level system in two different ways for a system represented by Abrikosov fermion operators. Furthermore, we use the projection operator to more rigorously construct the path integral of Bruckmann and Urbina for a spin system, and use this framework to exactly calculate the partition function for a single spin with a Hamiltonian proportional to $\hat{S}_x$, and a ring of Ising spins with a longitudinal field. We also use the framework of Bruckmann and Urbina to do a high temperature expansion for partition functions, and in particular we do the expansion for a system of two spins with a Heisenberg interaction and a ring of Ising spins in a transversal field. Finally, we derive a real time propagator within the framework of Bruckmann and Urbina, and verify that even for a Hamiltonian equal to zero the action picks up a Berry phase, and that for a spin coupled to an external field in the limit of $S \to \infty$, we recover the action of a classical spin in a magnetic field.

Sammendrag

I denne oppgaven utforsker vi hvordan en projeksjonsoperatormetode kan brukes til å implementere begrensninger i konsrtuksjonen av vegintegraler over koherente tilstander for spinnsystemer representert av Schwinger boson- eller Abrikosov fermionoperatorer. Spesifikt bruker vi denne konstruksjonen til å regne ut en-spinnpartisjonsfunksjonen til et system med null energi og Zeeman energi for et generelt spinn-S system representert i Schwinger bosonoperatorer, og en-spinnpartisjonsfunksjonen til et to-nivåsystem for et system representert av Abrikosov fermionoperatorer. Videre bruker vi projeksjonsoperatoren til å mer rigorøst konstruere vegintegralet til Bruckmann og Urbina, og vi bruker dette rammeverket til å regne ut partisjonsfunksjonene til ett spinn med en Hamiltonian proporsjonal til $\hat{S}_x$ og en ring av Ising spinn i et longitudinalt felt eksakt. Vi bruker også rammeverket til Bruckmann og Urbina til å gjøre en høytemperatur utvikling av partisjonsfunksjonen, og vi gjør denne utviklingen for et system av to partikkler med Heisenberg interaksjon, og en ring av Ising spinn i et transversalt felt. Til slutt utleder vi reell-tid-propagatoren til et spinn, og verifiserer at selv for en Hamiltonian lik null plukker virkningen opp en Berryfase, og at for et spinn koblet til et eksternt felt finner vi i grensen $S \to \infty$ virkningen til et klassisk spinn i et magnetfelt.

Preface

This thesis is the product of research conducted over the last two semesters of the Masters programme at the Norwegian University of Science and Technology. First and foremost I would like to thank my supervisor Assoc. Prof. John Ove Fjærestad for giving me the opportunity of doing such an interesting project, and for providing me with excellent guidance throughout the duration of this project. I would also like to thank Julie Marie Bekkevold for proofreading this thesis, and in general for being great support the last years. I thank the friends I have made throughout the last 5 years as a physics student, in particular for making the last two semesters more liveable. Finally, I would like to thank my family for always supporting me.

Contents

Chapter 1

Introduction

1.1 Motivation

The path integral formulation of quantum mechanics, first developed by Richard Feynman in the 1940s [1], has turned out to be one of the most powerful tools in physics. The intuition behind is beautiful, the probability amplitude of a quantum particle going from point a to b is a weighted sum of all the different paths between the two points, where the weight for each path is $e^{iS[x]/\hbar}$, where $S[x]$ is the action of the path x. The formalism has found uses reaching far beyond regular quantum mechanics, something that is well exemplified by the fact that one of the standard books for learning path integral methods is called *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets* [2].

In this thesis we will focus on path integral formulations for spin systems. Including spin degrees of freedom in the path integral is something Feynman himself famously failed to do [3]. One of the ways of describing spin systems in terms of coherent state path integrals is by using the coherent states of Schwinger boson operators. The way one implements the Schwinger boson number constraint, is typically by using a Dirac delta function to make sure that the constraint is always enforced [4]. However, while the delta function takes continuous arguments, the Schwinger boson occupation numbers take discrete values. For this reason, we believe that using a projection operator, that by design let the Schwinger boson occupation numbers be non-negative integers, is a more natural way of implementing the constraint. Towards the very end of this project, we discovered that the projection operator we had been using was first used in the 1990s to show that the Schwinger boson mean field theory for a 2 dimensional Heisenberg antiferromagnet produced a ground state with fluctuating Schwinger boson occupation numbers [5], thus violating the Schwinger boson number constraint. The Schwinger boson mean field theory is by far the standard way of doing calculations whenever one formulate a theory of Schwinger bosons, and often times one resort to relaxing the Schwinger boson number constraint from a local one to a global one, only satisfied on average to simplify calculations. Thus, the results of [5] are hardly surprising. In later years, it has been shown that the lack of Schwinger boson number rigidity can give rise to unphysical excitations [6], or it can completely miss the ground state of the system [7]. Even though it is possible to go beyond mean field theory using different perturbation series, the fact that problems with Schwinger boson number constraints exist still motivates us to explore alternate ways of implementing constraints.

It has long been realized that the mathematical foundations for path integral formulations is on shaky ground, especially when going to the continuum version of the path integral. In 2011 Wilson and Galitski showed that the textbook approach to continuous coherent state path integrals break down in the two simple cases of a single site Bose-Hubbard model and a single spin with a Hamiltonian quadratic in $\hat{S}_z$ [8], and the partition function calculated for the continuous coherent state path integrals is in disagreement with the one calculated in the operator formalism. This spurred some discussion in how coherent state path integral expressions should be constructed, where some of the suggestions caused some controversy [9–11]. However, the suggestion from Bruckmann and Urbina on how to fix the problems raised by Wilson and Galitski has received very little attention from the scientific community [12]. Their approach to solving the problem includes using the so-called P-representation of the Hamiltonian (see eq. (5.1)) and doing a duality transform to variables living on time bonds rather than time steps. Finally, they take the continuous time limit, without the assumption of continuity in the paths of the dual variables. Using this procedure, Bruckmann and Urbina were able to get the correct result for both the one site Hubbard model and the single spin with Hamiltonian proportional to an arbitrary power of $\hat{S}_z$. Thus, this construction fixes the problems of Wilson and Galitski.

1.2 Outline of thesis and connection to other work

The goal of Chapter 2 is to introduce the basics of occupation number theory and coherent states, the basic theory underlying all of the rest of this thesis. In Chapter 3 we go through the text book way of doing coherent state path integrals, thus bringing everyone up to speed on the basic theory.

In Chapter 4 we first develop the projection operator for both Schwinger boson, and Abrikosov fermion systems. We will then use these projection operators to implement constraints in coherent state path integral formulations. The use of the projection operator in coherent state path integrals was first presented in a previous master thesis [13], however, only bosonic systems were considered there. In this thesis, we use a more straightforward method to evaluate the path integrals, by getting rid of the extra set of bosonic variables. This allows us to calculate the single spin partition function for both zero and Zeeman energy for a general S. It also allows us to show how we can go from the Schwinger boson coherent state path integral to the more standard spin coherent state path integral. Furthermore, we are able to generalise to systems represented by Abrikosov fermion operators.

In Chapter 5 we show how the projection operator in a more rigorous way can be used to construct the path integral of Bruckmann and Urbina for spin systems such that the Schwinger boson number constraint on the dual variables follows naturally. We then proceed to use the Bruckmann and Urbina path integral to do calculations. First, we expand upon the work by Bruckmann and Urbina by calculating the partition function of a single spin with a Hamiltonian proportional to $\hat{S}_x$ to infinite order. We then do high temperature expansion of the partition function for a system of two spins with a Heisenberg interaction. We further generalise the Bruckmann-Urbina path integral to a system of many spins, showing that we can compute the partition function of a Ising ring in a longitudinal field exactly, and that we can do a high-temperature expansion for a transversal field. We end the chapter by also developing the real time propagator in a similar fashion to the imaginary time path integral considered until this point. We verify that even though the Hamiltonian of the system is zero, the propagator still picks up a Berry phase factor. Finally, we verify that if we consider a spin-S particle coupled to an external field we get the action of a classical spin in a magnetic field in the limit $S \to \infty$.

1.3 Conventions

Throughout this thesis we will use units such that the reduced Planck constant and the Boltzmann constant are both equal to 1

$$\hbar = k_b = 1.$$

We will often use the shorthand notation

$$z^\dagger z = \sum_i z_i^* z_i,$$

for summation over indices, where * denotes complex conjugation. We will also use the dagger to mean the collection of complex conjugates

$$f(z_1^*, z_2^*, \dots) = f(z^\dagger). \tag{1.1}$$

Chapter 2

Occupation number theory and coherent states

This chapter serves the purpose of introducing the basic machinery underlying the rest of this thesis. The first section introduces the occupation number theory, and thus the concept of creation and annihilation operators. This is the language used in much of modern theoretical physics. For the sake of brevity, only the bare minimum of content needed to understand the rest of the thesis is included.

Next, we turn to the topic of coherent states. We will see how we can construct bosonic, fermionic and spin coherent states, and derive some key properties these states possess.

2.1 Creation and annihilation operators

The first thing we will discuss is the concept of indistinguishable particles. Consider two quantum particles located in the same area. In quantum mechanics, there is no way to distinguish between the two particles. A consequence of this is that the absolute square of the combined wave function of the two particles is unchanged if the two particles are interchanged. This argument is easily extended to multiple particles, and instead of position, we can generalise to any set of quantum numbers $\{\mathbf{r_i}\}$. We then have that

$$\left|\Psi(\mathbf{r_1}, \ldots, \mathbf{r_i}, \ldots, \mathbf{r_j}, \ldots, \mathbf{r_N})\right|^2 = \left|\Psi(\mathbf{r_1}, \ldots, \mathbf{r_j}, \ldots, \mathbf{r_i}, \ldots, \mathbf{r_N})\right|^2. \qquad (2.1)$$

This means that the two states can at most differ by a complex phase factor. If that factor is $+1$, we call the particles bosons, and if it is -1 we call them fermions. In 1 or 2 dimensions we can have more exotic phase factors[14], but we will not get into that here. We can immediately see that a consequence of this, is that at most one fermion can occupy any single particle state[1].

It turns out that working with the many-body wave function of eq. (2.1) is very cumbersome in practise. Instead, we will assume we have a set of single particle states labelled by some ordered set of quantum numbers $\{\mathbf{r_i}\}$. We will then label the states by how many particles are occupying each of the states, the occupation number $n_{\mathbf{r_i}}$. We write a N-particle state as

$$\left|n_{\mathbf{r_1}}, n_{\mathbf{r_2}}, n_{\mathbf{r_3}}, \ldots\right\rangle = \left|n_1, n_2, n_3, \ldots\right\rangle, \qquad (2.2)$$

where we need $\sum_i n_i = N$. If the state describes fermions, it is clear that n_i can only take the values 0 or 1, while for bosons it can take any non-negative integer value. The state in eq. (2.2) live in a N-particle Hilbert space denoted by $\mathcal{H}^N$. In general we can have states, $|\Psi\rangle$, that are in a superposition of states with a varying number of particles. To accommodate such states, we need to generalise our idea of a Hilbert space, to allow for a varying number of particles. We say that $|\Psi\rangle$ lives in a Fock space $\mathcal{F}$, which is a “collection” of n-particle Hilbert spaces for all values of n. Mathematically we write this as a direct sum

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^n.$$

[1] This is known from quantum mechanics as the Pauli exclusion principle, and remarkably was proposed by Wolfgang Pauli in 1925 in the very early stages of the development of quantum mechanics[15]

Included in the definition of a Fock space is the zero-particle Hilbert space containing only the state where all occupation numbers are zero. We call this the vacuum state, and denote it by $|0\rangle$. We will use the vacuum state to build all other states. Another thing to note, is that due to the fact that the single particle states form a complete, orthonormal basis of their Hilbert spaces, the states of eq. (2.2) form a complete and orthonormal basis of $\mathcal{F}$. We now define a creation operator $a_\nu^\dagger$ by the way it acts on an arbitrary bosonic or fermionic state

$$\begin{aligned} a_\nu^\dagger \left|n_1, \ldots, n_\nu, \ldots\right\rangle &= \sqrt{n_\nu + 1} \left|n_1, \ldots, n_\nu + 1, \ldots\right\rangle, && \text{bosons} \\ a_\nu^\dagger \left|n_1, \ldots, n_\nu, \ldots\right\rangle &= (1 - n_\nu)(-1)^{\sum_{\mu<\nu} n_\mu} \left|n_1, \ldots, n_\nu + 1, \ldots\right\rangle, && \text{fermions,} \end{aligned}$$

where we note the phase factor in the fermionic case due to the anti-symmetric nature of this state. Starting from the zero-particle state, we can get any state by repeated use of the creation operator

$$\left|n_1, n_2, \ldots\right\rangle = \prod_\nu \frac{1}{\sqrt{n_\nu!}} \left(a_\nu^\dagger\right)^{n_\nu} |0\rangle . \tag{2.3}$$

There is one last piece needed to complete the picture, namely the annihilation operator. We define it as the adjoint of the creation operator, and its action on a state is

$$\begin{aligned} a_\nu \left|n_1, \ldots, n_\nu, \ldots\right\rangle &= \sqrt{n_\nu} \left|n_1, \ldots, n_\nu - 1, \ldots\right\rangle, && \text{bosons} \\ a_\nu \left|n_1, \ldots, n_\nu, \ldots\right\rangle &= n_\nu(-1)^{\sum_{\mu<\nu} n_\mu} \left|n_1, \ldots, n_\nu + 1, \ldots\right\rangle, && \text{fermions.} \end{aligned} \tag{2.4}$$

It is worth noting that the operator $\hat{n}_\nu = a_\nu^\dagger a_\nu$ returns n_ν when acting on any state, thus counting the number of particles in a given state. By considering eq. (2.3), it is clear that the creation operators for fermions can not commute. In fact, all creation and annihilation operators obey a set of commutation relations. If we define $[A, B]_\zeta = AB - \zeta BA$, we have the important relations

$$\begin{aligned}\left[a_\nu, a_\mu^\dagger\right]_\zeta &= \delta_{\mu\nu} \\ \left[a_\nu, a_\mu\right]_\zeta &= 0 \\ \left[a_\nu^\dagger, a_\mu^\dagger\right]_\zeta &= 0,\end{aligned} \tag{2.5}$$

where $\zeta = 1$ for bosons and -1 for fermions.

2.1.1 The Schwinger boson formalism

In this thesis we will be interested in studying spin systems, and a particular way of representing spin operators is the Schwinger boson formalism [16] which we will use heavily throughout the thesis. Suppose we have a spin-S particle. The key idea, is to introduce two kinds of bosons, one denoted by $\uparrow$ and one by $\downarrow$. We associate creation and annihilation operators $a_{\uparrow/\downarrow}, a_{\uparrow/\downarrow}^\dagger$ with these bosons. From these operators we can define spin operators

$$\hat{\mathbf{S}} = \begin{pmatrix}\hat{S}_x & \hat{S}_y & \hat{S}_z\end{pmatrix}^T = \begin{pmatrix}a_\uparrow^\dagger & a_\downarrow^\dagger\end{pmatrix}\boldsymbol{\sigma}\begin{pmatrix}a_\uparrow \\ a_\downarrow\end{pmatrix}, \tag{2.6}$$

where $\boldsymbol{\sigma}$ is a vector of the three Pauli matrices. The creation and annihilation operators of eq. (2.6) obey the normal commutation relations for bosons, and the components of the spin operator obey the angular momentum commutation relation[2]

$$\left[\hat{S}_\alpha, \hat{S}_\beta\right] = i\epsilon_{\alpha\beta\gamma}\hat{S}_\gamma, \tag{2.7}$$

where $\epsilon_{\alpha\beta\gamma}$ is the totally anti-symmetric tensor. This can be checked by simply inserting the definition from eq. (2.6). Using the definition of the spin operator we find

$$\hat{S}^2 = \frac{\hat{n}}{2}\left(\frac{\hat{n}}{2} + 1\right),$$

[2]This is also called the $SU(2)$ algebra, as the components of the spin operator are the generators of the $SU(2)$ group. It is also possible to generalise the Schwinger boson construction to other groups like $SU(N)$ or $Sp(N)$, see for instance [4, 17].

where $\hat{n} = \sum_\sigma a^\dagger_\sigma a_\sigma$. We know that the total spin operator must obey $\hat{S}^2 = S(S+1)$, so at this point we get the very important result

$$\sum_\sigma a^\dagger_\sigma a_\sigma = \sum_\sigma \hat{n}_\sigma = 2S, \tag{2.8}$$

the Schwinger boson number constraint. The construction we have defined here will be very useful throughout the rest of this thesis.

2.1.2 The Abrikosov fermion formalism

In a similar fashion to the Schwinger boson formalism, we can construct spin operators from fermionic creation and annihilation operators. We will refer to this as the Abrikosov fermion formalism [18]. The idea is completely analogous to the Schwinger bosons, we have two kinds of fermions associated with fermionic creation and annihilation operators $c_{\uparrow/\downarrow}, c^\dagger_{\uparrow/\downarrow}$, and we define spin operators

$$\hat{\mathbf{S}} = \begin{pmatrix} c^\dagger_\uparrow & c^\dagger_\downarrow \end{pmatrix} \boldsymbol{\sigma} \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}, \tag{2.9}$$

and these operators do obey the spin algebra of eq. (2.7). However, due to the fermionic statistics, the constraint on the occupation numbers is in this case

$$\sum_\sigma c^\dagger_\sigma c_\sigma = 1, \tag{2.10}$$

and we see that we can only treat spin-1/2 particles using this formalism.

2.2 Bosonic coherent states

Now that we know the basic theory of creation and annihilation operators, we are ready to move on to coherent states. Suppose we have some system where we can build any state using bosonic creation operators $\{a^\dagger_i\}$, where i is some arbitrary collection of quantum numbers. We then define the coherent states to be states that are eigenstates of the annihilation operators.

We write the state

$$|z_1, z_2, \ldots, z_N\rangle = |z\rangle\,,$$

and this state must then obey

$$a_i\,|z\rangle = z_i\,|z\rangle \tag{2.11}$$

where z_i is a complex number. This holds for any i. We can of course construct this state from the vacuum state by the use of creation operators, we will now show that the bosonic coherent states takes the form

$$|z\rangle = e^{\sum_i z_i a_i^\dagger}\,|0\rangle\,. \tag{2.12}$$

Here, the exponential of the creation operator is defined as the Taylor series of the exponential (see Appendix A.2). Note that the coherent state is a linear combination of states with different number of particles. That eq. (2.12) is indeed the correct way to write the coherent state can be verified by acting on the expansion of the exponential with a_i. Using the definition from eq. (2.4), we see after some algebra that what we end up with is z_i times the original state. Thus, this is indeed the form of the coherent state. We will now investigate some of the properties of this state. First, it is clear that from this ket we can construct the corresponding bra by taking the Hermitian conjugate and the bra can be written

$$\langle z| = \langle 0|\,e^{\sum_i z_i^* a_i}.$$

We can then show that the overlap between coherent states $|z\rangle$ and $|w\rangle$ is given by

$$\langle w|z\rangle = e^{\sum_i w_i^* z_i}, \tag{2.13}$$

by remembering that for two operators $\hat{A}$ and $\hat{B}$, where the commutator $\left[\hat{A}, \hat{B}\right]$ commutes with both operators, we have

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]}.$$

This follows immediately from the Baker-Campbell-Hausdorff formula (see Appendix A.2). We can immediately notice that an important consequence of this is that the states are in general not normalized. Another very important feature of the coherent states, is that they form an over complete basis set of the Fock space. The identity operator can be written as

$$I = \int \prod_i \left[\frac{\mathrm{d}^2 z_i}{\pi} e^{-|z_i|^2} \right] |z\rangle\langle z| \,, \tag{2.14}$$

where $\mathrm{d}^2 z_i$ is shorthand notation for $\mathrm{d}\,\mathrm{Re}\, z_i\, \mathrm{d}\,\mathrm{Im}\, z_i$. Furthermore, we can use the identity operator to write the trace of any operator $\hat{O}$ as an integral over bosonic coherent states. In general, the trace is given by

$$\mathrm{tr}\, \hat{O} = \sum_n \langle n|\hat{O}|n\rangle \,,$$

where $\{|n\rangle\}$ is some complete set of the Fock space. By inserting identity after $\langle n|$ we get

$$\begin{aligned} \mathrm{tr}\, \hat{O} &= \sum_n \int \prod_i \left[\frac{\mathrm{d}^2 z_i}{\pi} e^{-|z_i|^2} \right] \langle n|z\rangle \; \langle z|\hat{O}|n\rangle \\ &= \int \prod_i \left[\frac{\mathrm{d}^2 z_i}{\pi} e^{-|z_i|^2} \right] \langle z|\hat{O}|z\rangle \,, \end{aligned} \tag{2.15}$$

where the last equality is obtained by moving $\langle n|z\rangle$ past the operator. Because we have assumed that $\{|n\rangle\}$ is complete, we have that $\sum_n |n\rangle\langle n| = 1$ and thus we have been able to write the trace of an arbitrary operator in terms of an integral over coherent states.

2.3 Fermionic coherent states and the Grassmann algebra

We will now consider a system where the states can be constructed by fermionic creation operators $\{c_i\}$. We would like to define a state

$$|\psi\rangle = |\psi_1, \psi_2, \ldots, \psi_N\rangle$$

as a coherent state, and similarly to the bosonic case we want this state to obey

$$c_i \left|\psi\right\rangle = \psi_i \left|\psi\right\rangle . \tag{2.16}$$

One problem we notice at once is that since

$$c_i c_j \left|\psi\right\rangle = -c_j c_i \left|\psi\right\rangle ,$$

due to the anti-commutation relations for fermionic operators the eigenvalues ψ_i can not be regular complex numbers. Instead they will be Grassmann numbers. The study of the algebra of Grassmann numbers, or the exterior algebra as mathematicians call it, is a big field in itself [19]. However, we will only go through the bare minimum needed for this thesis. The first fact of Grassmann numbers is that they anti-commute with other Grassmann numbers and fermionic operators

$$\begin{aligned} \psi_i \psi_j &= -\psi_j \psi_i \\ \psi_i c_j &= -c_j \psi_i. \end{aligned} \tag{2.17}$$

Furthermore, we see that a product of an even number of Grassmann numbers commute with other Grassmann numbers and fermionic operators. An important consequence of eq. (2.17), is that any Grassmann number squared is equal to 0. All higher powers are then obviously also zero. Consider then a function of a Grassmann number $f(\psi)$. Since any power of ψ higher than 1 yields zero, we can Taylor expand f to first order, and the result is exact

$$f(\psi) = a + b\psi. \tag{2.18}$$

An example we will encounter many times is the exponential function. We can clearly see that

$$e^{a\psi} = 1 + a\psi$$

holds true for a Grassmann number ψ. Due to eq. (2.18), if we want to integrate a function of Grassmann variables, there are only two integrals we need to consider. The first is the integral of 1, and the second is the integral of ψ. We define these to be

$$\begin{aligned} \int \mathrm{d}\psi &= 0 \\ \int \mathrm{d}\psi \, \psi &= 1 \end{aligned} \tag{2.19}$$

motivated by the fact that theses definitions produce results that are similar to those of complex variables. It is, however, important to remember that as Grassmann variables carry no size, there is no geometric interpretations of these integrals. It is also important to note that the differentials themselves are Grassmann numbers, and switching the differential with the variable yields

$$\int \psi \, \mathrm{d}\psi = -1.$$

We now have everything we need to write down the coherent state for fermionic operators. We will write down the state, and then quickly show that it is indeed the correct state. The coherent state can be written

$$|\psi\rangle = e^{-\sum_i \psi_i c_i^\dagger} |0\rangle = \prod_i (1 - \psi_i c_i^\dagger) |0\rangle \,, \tag{2.20}$$

where ψ_i are Grassmann numbers. That this state indeed obeys eq. (2.16) can be verified by acting upon it with c_j, and use that it commutes with the product of a Grassmann number and a creation operator, and the fact that $\psi^2 = 0$. Note the similarity of eq. (2.20) to its bosonic counterpart in eq. (2.12). In addition to the coherent state ket, we can define the bra $\langle\psi|$, such that

$$\langle\psi| \, c_i^\dagger = \langle\psi| \, \psi_i^*, \tag{2.21}$$

where the notation * is employed, even though ψ_i^* is not the complex conjugate of ψ_i. The coherent state bra can be constructed in a similar way to the ket, and takes the form

$$\langle\psi| = \langle 0| \, e^{-\sum_i c_i \psi_i^*} = \langle 0| \prod_i (1 + \psi_i^* c_i).$$

We can now find the overlap between two states

$$\langle\psi_1|\psi_2\rangle = \langle 0| \prod_{i,j} (1+\psi_j^* c_j)(1-\psi_i c_i^\dagger)\,|0\rangle = \prod_i (1+\psi_i^*\psi_i) = e^{\sum_i \psi_i^*\psi_i}. \tag{2.22}$$

Then, the identity operator in terms of fermionic coherent states is

$$I = \int \prod_i \mathrm{d}\psi_i\,\mathrm{d}\psi_i^*\, e^{-\sum_i \psi_i^*\psi_i}\,|\psi_i\rangle\langle\psi_i|\,. \tag{2.23}$$

Finally, we will find an expression for the trace of a general operator in terms of an integral over coherent fermionic states. The derivation is very similar to the bosonic case of eq. (2.15), as we will insert identity in the same way. We get

$$\begin{aligned}\operatorname{tr}\hat{O} &= \sum_n \int \prod_i \left[\mathrm{d}^2\psi_i\, e^{-\psi_i^*\psi_i}\right] \langle n|\psi\rangle\ \langle\psi|\hat{O}|\psi\rangle \\ &= \int \prod_i \left[\mathrm{d}^2\psi_i\, e^{-\psi_i^*\psi_i}\right] \langle -\psi|\hat{O}|\psi\rangle\,.\end{aligned} \tag{2.24}$$

Note that we pick up a minus sign in the leftmost state when we move $\langle n|\psi\rangle$ through the expression. This concludes our rather short introduction to fermionic coherent states and the Grassmann algebra, as we now have all the properties that we will need later.

2.4 Spin coherent states

Finally we will give a short introduction to the topic of coherent spin states. We will adopt the notation used by Shankar [20], and denote the normalized and fully polarized state by $|SS\rangle$. By fully polarized, we mean that the expectation value of the spin operator in this state is given by

$$\langle SS|\hat{\mathbf{S}}|SS\rangle = S\hat{\mathbf{z}},$$

where $\hat{\mathbf{z}}$ is the unit vector in the z-direction. Thus, the expectation value is a classical spin pointing upwards. We can then get all other coherent states by acting on this by a unitary operator $\hat{R}(\theta, \phi)$. This is the well known $SU(2)$ rotation operator[3]

$$R(\theta, \phi) = e^{-\phi \hat{S}_z} e^{-\theta \hat{S}_y}.$$

Thus, any coherent state can be written $|\Omega\rangle_S = \hat{R}(\theta, \phi) |SS\rangle$, and it is clear that the expectation value of the spin operator is a vector of length S pointing in the direction parametrised by θ, ϕ on the unit sphere. We can therefore say that the spin coherent states are the closest we get to a classical spin vector. To make this point even more clear, we consider the case of spin 1/2, and write the spin coherent state as

$$|\Omega\rangle_{1/2} = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\phi} \end{pmatrix}. \tag{2.25}$$

Note that the state is invariant under multiplication by a phase factor, so other authors might have a different ϕ-dependence, see for instance [21] for a different convention. As the only difference is a phase factor, the states are of course equivalent. The next natural question to ask is how to create higher spin states. The answer is to take the direct product of $2S$ spin-1/2 states

$$|\Omega\rangle_S = \underbrace{|\Omega\rangle_{1/2} \otimes |\Omega\rangle_{1/2} \otimes \cdots \otimes |\Omega\rangle_{1/2}}_{2S \text{ times}}, \tag{2.26}$$

to create a spin-S state. It should then be clear that the overlap between two spin coherent states is

$$_S\langle\Omega_1|\Omega_2\rangle_S = \left(\cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i(\phi_1 - \phi_2)} \right)^{2S}. \tag{2.27}$$

Finally, we write down the identity operator for the spin coherent states. This is given by

$$I = \frac{2S+1}{4\pi} \int_0^\pi \mathrm{d}\theta \sin \theta \int_0^{2\pi} \mathrm{d}\phi \, |\Omega\rangle_S \, {}_S\langle\Omega| \, . \tag{2.28}$$

[3] Actually, there is a third parameter ψ needed to represent the group $SU(2)$, but it only enters as a gauge factor and we can choose $\psi = 0$.

In a similar fashion to the bosonic and fermionic coherent states, we write the trace of any operator as

$$\operatorname{tr}\hat{O} = \frac{2S+1}{4\pi}\int_0^{\pi} \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi \, {}_S\langle\Omega|\hat{O}|\Omega\rangle_S \,. \tag{2.29}$$

Chapter 3

Coherent state path integrals

The coherent state path integral is of paramount importance to this thesis, and in this chapter we will go through the textbook way of constructing the coherent state path integral. First, we go through the construction of the real time propagator and imaginary time partition function for a general bosonic and fermioninc system, and we will then see how this changes when we deal with spin systems. In particular we will go through the standard way of constructing the Schwinger boson, Abrikosov fermion and spin coherent state path integrals. We will also briefly look at the continuous form of these path integral representations.

3.1 The coherent state path integral

To build some intuituion for the path integral, we will first go through the derivation in the position basis in 1 dimension, and then generalise our expression to coherent states. From basic quantum mechanics, we know that the probability amplitude of going from one initial state $|x_i\rangle$ to some final state $|x_f\rangle$ in some time t is given by the matrix element of the time-evolution operator $\hat{U}$ [20]

$$U_{i\to f}(t) = \langle x_i|\hat{U}|x_f\rangle = \langle x_i|e^{-i\hat{H}t}|x_f\rangle \,. \tag{3.1}$$

We will call this object the propagator. We divide the time t into N parts each of size $\varepsilon = \frac{t}{N}$. We can write[1]

$$e^{-i\hat{H}t} = \left[e^{-i\hat{H}\varepsilon}\right]^N = \underbrace{e^{-i\hat{H}\varepsilon}e^{-i\hat{H}\varepsilon}\dots e^{-i\hat{H}\varepsilon}}_{N \text{ times}}.$$

We can interpret this expression as the fact that the time evolution operator from 0 to t is the same as the product of time evolution operators from 0 to ε, ε to 2ε etc. The idea is then to insert the identity operator

$$I = \int \mathrm{d}x\,|x\rangle\langle x| \tag{3.2}$$

between each pair of operators. Mathematically we can write this

$$U(t) = \int \mathrm{d}x_1\,\mathrm{d}x_2\dots\mathrm{d}x_{N-1}\,\langle x_0|e^{-i\hat{H}\varepsilon}|x_1\rangle\,\langle x_1|\dots\,\langle x_{N-1}|e^{-i\hat{H}\varepsilon}|x_N\rangle\,, \tag{3.3}$$

where we have renamed $x_i \to x_0$ and $x_f \to x_N$ and we see that the integral is taken over all possible paths form x_0 to x_N.

3.1.1 The quantum partition function

We will now see how we can write the quantum partition function in the style of eq. (3.3). First, remember the definition of the partition function as the trace of the operator $e^{-\beta\hat{H}}$, where β is the inverse temperature. We write

$$\mathcal{Z} = \int \langle x|e^{-\beta\hat{H}}|x\rangle\,\mathrm{d}x\,, \tag{3.4}$$

and notice the similarity with eq. (3.1). It is clear that we can interpret the quantum partition function as the probability amplitude of a state returning

[1] Often times the Hamiltonian will be a sum of a kinetic and potential term. One then has to be a little bit careful, see the Suzuki-Trotter decomposition in Appendix A.2.

to its original state, where we integrate over all possible initial states. The probability amplitude is evaluated at the imaginary time $t = -i\beta$. The transformation between t and β is often referred to as a Wick rotation. We may now split our imaginary time interval into pieces of size $\varepsilon = \frac{\beta}{N}$, and in similar fashion to eq. (3.3), we get an expression for the partition function

$$\mathcal{Z} = \int \prod_{n=1}^{N} \mathrm{d}x_n \, \langle x_n | e^{-\varepsilon \hat{H}} | x_{n-1} \rangle \,, \tag{3.5}$$

with $x_0 = x_N$.

We now generalise the discussion to coherent states, rather than the position states considered this far. We already know how to take the trace of any operator, which means we can immediately write down the expression for the partition function in terms of integrals over coherent states

$$\mathcal{Z} = \int \mathrm{d}[\psi] \, \langle \zeta\psi | e^{-\beta \hat{H}} | \psi \rangle \, e^{-\psi^\dagger \psi}. \tag{3.6}$$

Here, we have introduced ζ equal to $+1$ for bosons and -1 for fermions, and $\mathrm{d}[\psi]$ which is $\frac{\mathrm{d}^2\psi}{\pi}$ for bosons and $\mathrm{d}\psi\,\mathrm{d}\psi^*$ for fermions. Now, we split the imaginary time interval into pieces of size $\varepsilon = \frac{\beta}{N}$ and insert identity after each of the exponential operators. By using the defining property of coherent states as eigenstates to the annihilation operator, and assuming the Hamiltonian is normal ordered (all creation operators are to the left), we get

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}[\psi] \exp\left[\sum_{n=1}^{N} \Big(-\psi_n^\dagger \psi_n + \psi_n^\dagger \psi_{n-1} - \varepsilon H(\psi_n^\dagger, \psi_{n-1}) \Big) \right] \\ &= \int \mathcal{D}[\psi] e^{-S[\psi^\dagger, \psi]}, \end{aligned} \tag{3.7}$$

with $\psi_0 = \zeta\psi_N$ and $\psi_0^* = \zeta\psi_N^*$. The function $H(\psi_n^\dagger, \psi_{n-1})$ is what you get if you replace all creation operators by the corresponding ψ^* and annihilation operators by ψ in the normal ordered Hamiltonian. $S[\psi^\dagger, \psi]$ is the action associated with the "paths" of the ψ and conjugate variables. We have also defined

$$\mathcal{D}[\psi] = \prod_{n=1}^{N} \mathrm{d}[\psi_n] \tag{3.8}$$

3.1.2 Continuum version of the path integral

The textbook way forward from eq. (3.7) is to go to the limit of $N \to \infty$. The key technical steps of the continuum approximation, is to replace the sum over imaginary time steps by an integral, and also introduce the derivative the following way

$$\begin{aligned} \psi_n &\to \psi(\tau) \\ \varepsilon \sum_{n=1}^{N} &\to \int_0^\beta \mathrm{d}\tau \\ \psi_n^\dagger \frac{\psi_n - \psi_{n-1}}{\varepsilon} &\to \psi^\dagger(\tau)\dot{\psi}(\tau), \end{aligned}$$

where $\tau = n\varepsilon$, and the dot represents differentiation with respect to τ. It is worth noting that if ψ are Grassmann variables, differentiation with respect to imaginary time has no meaning other than the definition above. To make progress, one usually do one more approximation, namely that

$$H(\psi_n^\dagger, \psi_{n-1}) \approx H(\psi^\dagger(\tau), \psi(\tau)). \tag{3.9}$$

What this approximation is telling us, is that the "paths" $\psi(\tau)$ and $\psi^\dagger(\tau)$ should be approximately continuous. This approximation will be discussed more later in this chapter. What we end up with is an expression for the action given by

$$S[\psi^\dagger, \psi] = \int_0^\beta \mathrm{d}\tau \left[\psi^\dagger(\tau)\dot{\psi}(\tau) - H\Big(\psi^\dagger(\tau), \psi(\tau)\Big)\right]. \tag{3.10}$$

However, the assumption of continuous paths is not always good, and it has been shown by Wilson and Galitski that the continuum version of the path integral can in many cases lead to wrong results [8]. This problem, is however not present in the time-discretized version of the path integral.

3.2 Path integrals for spin systems

3.2.1 Schwinger boson coherent states

We will now investigate how the path integral representation of the partition function change for spin systems. To be general, we consider a lattice of spins. Let the state $|z\rangle$ be a coherent state, such that

$$c_{i\sigma}\,|z\rangle = z_{i\sigma}\,|z\rangle$$

for all sites i and $\sigma \in \{\uparrow, \downarrow\}$. We can then write down a coherent state path integral representation similar to before, but we have to be careful that the Schwinger boson number constraint is fulfilled. The way this constraint is usually enforced in the path integral, is by multiplying with delta functions $\delta(2S - \sum_\sigma z^*_{i\sigma} z_{i\sigma})$, such that the number of Schwinger bosons at each site is consistent with the constraint [4]. If we then write the delta functions in their Fourier form

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, e^{i\lambda(x-a)},$$

we can write down the partition function as

$$\mathcal{Z} = \int \mathcal{D}[z,\lambda] e^{-S[z^\dagger, z, \lambda]},$$

where now the action includes a term to enforce the constraint, and is given by

$$S[z^\dagger, z, \lambda] = \int_0^\beta \mathrm{d}\tau \left[z^\dagger \dot{z} + H\Big(z^\dagger, z\Big) - i \sum_i \lambda_i \left(2S - \sum_\sigma z^*_{i\sigma} z_{i\sigma} \right) \right]. \quad (3.11)$$

The interpretation of this is that λ_i acts as a chemical potential at each lattice site, fixing the number of bosons.

3.2.2 Abrikosov fermion coherent states

In a similar way to the Schwinger boson case, delta functions are used to implement constraints in fermionic theories [22]. Let $|\psi\rangle$ be a coherent state of the Abrikosov fermion operators such that

$$c_{i\sigma}\,|\psi\rangle = \psi_{i\sigma}\,|\psi\rangle\,. \tag{3.12}$$

In a completely analogous way to the Schwinger boson case, we write the path integral as

$$\mathcal{Z} = \int \mathcal{D}[\psi,\lambda]e^{-S[\psi^\dagger,\psi,\lambda]}, \tag{3.13}$$

where the action includes the term to enforce the constraint, and is given by

$$S[\psi^\dagger,\psi,\lambda] = \int_0^\beta \mathrm{d}\tau\left[\psi^\dagger\dot{\psi} + H\Big(\psi^\dagger,\psi\Big) - i\sum_i \lambda_i\left(2S - \sum_\sigma \psi^*_{i\sigma}\psi_{i\sigma}\right)\right],$$

where we have to be careful to remember that ψ are Grassmann numbers.

3.2.3 Spin coherent states

We could also write the path integral partition function in terms of the spin coherent states of sec. 2.4. We will here limit our attention to Hamiltonians linear in the spin operator $\hat{\mathbf{S}}$, because the defining property of spin coherent states is that it has "nice" expectation values of the spin operator. Note that there is no reason to believe that the spin coherent states have nice expectation values of higher order spin operators[2]. We will write the Hamiltonians we are interested in as

$$\hat{H} = \mathbf{B}\cdot\hat{\mathbf{S}}.$$

[2]To go beyond the linear Hamiltonian is in general only possible in the limit of $S\to\infty$, because it is only then that the expectation value of higher order of spin operators behave nicely. For a further discussion on this point, see for instance chapter 21.3 of [20].

Similar to what we have done before, we can write the path integral representation of the partition function in the following way

$$\begin{aligned}\mathcal{Z} &= \frac{2S+1}{4\pi}\int\prod_{n=1}^{N}\mathrm{d}\phi_n\sin\theta_n\,\mathrm{d}\theta_n\,{}_S\langle\Omega_n|e^{-\varepsilon\hat{H}}|\Omega_{n-1}\rangle_S \\ &= \frac{2S+1}{4\pi}\int\prod_{n=1}^{N}\mathrm{d}\phi_n\sin\theta_n\,\mathrm{d}\theta_n\,e^{-\varepsilon H(\Omega_n^\dagger,\Omega_{n-1})}{}_S\langle\Omega_n|\Omega_{n-1}\rangle_S\,,\end{aligned} \tag{3.14}$$

where we have introduced

$$H(\Omega_n^\dagger,\Omega_{n-1}) = \frac{{}_S\langle\Omega_n|H|\Omega_{n-1}\rangle_S}{{}_S\langle\Omega_n|\Omega_{n-1}\rangle_S}. \tag{3.15}$$

The way forward from here, again relies on going to the continuum limit. However, as we will discuss more in the next section, there are numerous problems associated with that process. The main problem is the assumption of continuity in the states $|\Omega\rangle_S$, and in general the continuum limit is only valid in the limit $S\to\infty$. We will see what happens in this limit now.

First of all, we write the states as a continuous function of imaginary time

$$|\Omega_n\rangle_{S\to\infty} = |\Omega(\tau)\rangle\,, \tag{3.16}$$

where $\tau = n\varepsilon$ and we drop the subscript denoting the spin. We call this the Hamiltonian function. We then investigate the overlap between states separated by ε in imaginary time, by Taylor expanding the state at $t+\varepsilon$ to first order in ε. The overlap can be written

$$\langle\Omega(\tau)|\Omega(\tau-\varepsilon)\rangle \approx \langle\Omega(\tau)|\left(|\Omega(\tau)\rangle - \varepsilon\,|\dot{\Omega}(\tau)\rangle\right) \approx e^{-\varepsilon\langle\Omega(\tau)|\dot{\Omega}(\tau)\rangle}, \tag{3.17}$$

where the dot denotes differentiation with respect to τ. Using eq. (2.26), we can can write the overlap between the state and its derivative as

$$\langle\Omega(\tau)|\dot{\Omega}(\tau)\rangle = Si(\cos\theta-1)\dot{\phi}.$$

Also, the Hamiltonian function is given by

$$H(\Omega_n^\dagger,\Omega_{n-1}) = S\,\mathbf{B}\cdot\mathbf{n}, \tag{3.18}$$

where $\mathbf{n}$ is a unit vector pointing in the direction parametrised by θ and ϕ on the unit sphere. We are already treating $|\Omega\rangle$ as a continuous variable, so we might as well replace the sum over imaginary time steps with an integral over the imaginary time. The expression we end up with is

$$\mathcal{Z} = \frac{2S+1}{4\pi} \int \mathcal{D}[\theta, \phi] e^{S \int \mathrm{d}\tau \left[i(1-\cos\theta)\dot{\phi} + \mathbf{B}\cdot\mathbf{n} \right]}. \tag{3.19}$$

The first term in the action of this expression is the so-called Berry phase [23]. The Berry phase is a geometrical object, and it can be shown to be related to the area enclosed by the path parametrised by $\theta(\tau)$ and $\phi(\tau)$, see chapter 10 of [24]. The second term in the action is related to the Hamiltonian of the system. Note that the Berry phase term appears even when the Hamiltonian is zero.

Chapter 4

The projection operator

In this chapter we will derive a projection operator that takes any state, and projects it down to an arbitrary subspace of the full Fock space. In particular we will be interested in using the projection operator to implement the Schwinger boson and Abrikosov fermion number constraints, and for the bosonic case we will see that the operator is the same as the projection operator of [5]. We will see how we can construct Schwinger boson and Abrikosov fermion coherent state path integrals, where their respective constraints are implemented by a projection operator. To check the validity of our approach, several simple systems are investigated.

4.1 Derivation of the projection operator

4.1.1 The bosonic case

The first step towards the projection operator we want is to find an operator, $\hat{P}_0$ that projects any state down to the vacuum state, because it is then simple to find a projection operator to any other state. Consider a system labelled by N different quantum numbers i. We will show that the operator

$$\hat{P}_0 =: e^{-\sum_i \hat{n_i}} : \tag{4.1}$$

projects any state down to the vacuum state. Here, the notation : : denotes normal ordering. We write the exponential in terms of the infinite Taylor series

$$\hat{P}_0 = \prod_{i=1}^{N} \sum_{l_i=0}^{\infty} \frac{(-1)^{l_i} (a_i^\dagger)^{l_i} a_i^{l_i}}{l_i!}, \tag{4.2}$$

To see that this is indeed the vacuum state operator, consider the matrix element $\langle m|\hat{P}_0|n\rangle$, where $|m\rangle$ and $|n\rangle$ are two states with fixed occupation numbers given by

$$\begin{aligned} |n\rangle &= |n_1, n_2, \ldots, n_N\rangle \\ |m\rangle &= |m_1, m_2, \ldots, m_N\rangle \,. \end{aligned}$$

Now, when we calculate the matrix element we notice that we have creation operators to some power acting to the left on $|m\rangle$, and annihilation operators to the same power acting to the right on $|n\rangle$. By using the definition of how creation and annihilation operators work on any state, we see that

$$\begin{aligned} a_i^{l_i} |n\rangle &= \sqrt{\frac{n_i!}{(n_i - l_i)!}} \, |n_1, n_2, \ldots, n_i - l_i, \ldots, n_N\rangle \\ \langle m| \, (a_i^\dagger)^{l_i} &= \langle m_1, m_2, \ldots, m_i - l_i, \ldots, m_N| \sqrt{\frac{m_i!}{(m_i - l_i)!}}. \end{aligned}$$

We then notice that this means that the matrix element will be proportional to $\delta_{n_1 m_1} \ldots \delta_{n_i m_i} \ldots \delta_{n_N m_N}$, where we have used that $\delta_{n-l,m-l} = \delta_{nm}$. Thus, we get zero if any of the quantum numbers are different in the two states. We can write down the expression for the matrix element

$$\langle m|\hat{P}_0|n\rangle = \prod_i \delta_{n_i m_i} \sum_{l_i=0}^{n_i} \frac{(-1)^{l_i} n_i!}{l_i!(n_i - l_i)!}. \tag{4.3}$$

Note that we have restricted the sum over l_i to only run up until n_i. This comes from the realization that if l_i is greater than n_i, the creation/annihilation operators acting l_i times on the states will give zero.

It is clear that if n_i is zero, the matrix element will be 1. This is true for all i. If n_i is anything else, we can invoke the binomial theorem (see Appendix A.4) to calculate the sum. Inside the sum, we can multiply with $1 = 1^{n_i - l_i}$, and by the binomial theorem the sum is equal to $(1-1)^{n_i} = 0$. Thus, we see that the matrix element of $\hat{P}_0$ is 0 unless all n_i and m_i are zero, and we see that $\hat{P}_0 = |0\rangle\langle 0|$, and does indeed project any state down to the vacuum state.

Now, we can generate a projection operator onto any state starting from eq. (4.1). Suppose we want to project down to an arbitrary state

$$|\psi\rangle = \prod_i \frac{\left(a_i^\dagger\right)^{l_i}}{\sqrt{l_i!}} |0\rangle , \tag{4.4}$$

where l_i is an arbitrary set of numbers. It is clear that we can create a projection operator down to this state by acting on the vacuum state projector operator by l_i creation operators from the left and l_i annihilation operators from the right. We can write

$$|\psi\rangle\langle\psi| = \prod_i \frac{1}{l_i!} (a_i^\dagger)^{l_i} : e^{-\hat{n}_i} : (a_i)^{l_i}. \tag{4.5}$$

One of the main motivation for developing the projection operator was to enforce the Schwinger boson constraint of eq. (2.8), and it should now be clear that the way we do this is sum over the projection operators down state with $l \uparrow$ bosons and $2S - l \downarrow$ bosons, where l is an integer $0 \leq l \leq 2S$. The projection operator can be written

$$\hat{P} = \sum_{l=0}^{2S} \frac{1}{l!(2S-l)!} \left(a_\uparrow^\dagger\right)^l \left(a_\downarrow^\dagger\right)^{2S-l} : e^{-\sum_\sigma \hat{n}_\sigma} : a_\uparrow^l a_\downarrow^{2S-l}. \tag{4.6}$$

Thus we have arrived at an operator that takes any state constructed by Schwinger boson operators, and projects it down to the physical subspace where the constraint is fulfilled.

4.1.2 The fermionic case

We will here see how to construct a projection operator for a fermionic system. In particular, as we are interested in spin systems, we will use the Abrikosov fermion representation. We write our states $|n_\uparrow, n_\downarrow\rangle$, where the two ns take values 0 or 1. A general state in the Fock space is given by

$$|\psi\rangle = a\,|0,0\rangle + b\,|1,0\rangle + c\,|0,1\rangle + d\,|1,1\rangle\,.$$

We will now see what happens when we act on this state with the operator $: e^{-\sum_\sigma \hat{n}_\sigma} :$. We get

$$: e^{-\sum_\sigma \hat{n}_\sigma} : |\psi\rangle = \left(1 - c_\uparrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\downarrow + c_\uparrow^\dagger c_\downarrow^\dagger c_\uparrow c_\downarrow\right)|\psi\rangle = a\,|0,0\rangle\,, \tag{4.7}$$

where we have used that $\hat{n}_\sigma^2 = \hat{n}_\sigma$. Thus, we see that the operator which looks exactly the same as in the bosonic case, indeed projects any state in the Fock space down to the vacuum state. We can then use this vacuum state projection operator to construct projection operators to any subspace of the full Fock space. In particular, we can create a projection operator down to the physical subspace in the Abrikosov fermion formalism, given by the constraint of eq. (2.10). This projection operator takes the form

$$\hat{P} = \sum_{l=0}^{1} \left(c_\uparrow^\dagger\right)^l \left(c_\downarrow^\dagger\right)^{1-l} : e^{-\sum_\sigma \hat{n}_\sigma} : c_\uparrow^l c_\downarrow^{1-l}. \tag{4.8}$$

4.2 Schwinger boson coherent state path integral using a projection operator

We will now see how we can use a projection operator instead of the delta function to enforce constraints in coherent state path integrals.Consider first a Hamiltonian that is quadratic in the Schwinger boson operators, and is given by

$$\hat{H} = \sum_{\sigma,\sigma'} c_\sigma^\dagger H_{\sigma\sigma'} c_{\sigma'}. \tag{4.9}$$

$H_{\sigma\sigma'}$ are the matrix elements of some Hermitian matrix H that is some linear combination of the Pauli matrices and the identity operator. The starting point of this derivation is eq. (3.6). We split the exponential operator and insert identity between each time step, but now we will also insert projection operators to project each state down to the physical subspace. We get

$$\begin{aligned}\mathcal{Z} &= \int \prod_{n=1}^{N} \mathrm{d}[z_n] \, \langle z_n | e^{-\varepsilon \hat{H}} \hat{P} | z_{n-1} \rangle \, e^{-z_n^\dagger z_n} \\ &= \int \prod_{n=1}^{N} \mathrm{d}[z_n, w_n] \, \langle z_n | e^{-\varepsilon \hat{H}} | w_n \rangle \, \langle w_n | \hat{P} | z_{n-1} \rangle \, e^{-z_n^\dagger z_n - w_n^\dagger w_n},\end{aligned} \tag{4.10}$$

where in the second line we have inserted another set of over completeness relations for a new set of coherent states called w_n at each time step. We have two types of matrix elements, the first being

$$\langle z_n | e^{-\varepsilon \hat{H}} | w_n \rangle = e^{-\varepsilon H(z_n^\dagger, w_n) + z_n^\dagger w_n}. \tag{4.11}$$

Next, we calculate the matrix elements of the projection operator

$$\begin{aligned}\langle w_n | \hat{P} | z_{n-1} \rangle &= \sum_{l=0}^{2S} \frac{1}{l!(2S-l)!} (w_{n\uparrow}^*)^l (w_{n\downarrow}^*)^{2S-l} (z_{n-1\uparrow})^l (z_{n-1\uparrow})^{2S-l} \\ &= \frac{1}{(2S)!} (w_n^\dagger z_{n-1})^{2S},\end{aligned} \tag{4.12}$$

where we have used the binomial theorem (see Appendix A.4) in the last equality. Note how the normal ordered exponential of the number operator cancels out the overlap between the two states. We now employ a notational trick we will use heavily throughout this thesis. When looking at eq. (4.12), it is clear that we can write this as

$$\frac{1}{(2S)!} (w_n^\dagger z_{n-1})^{2S} = \frac{\partial^{2S}}{\partial \eta_n^{2S}} \frac{1}{(2S)!} \, e^{\eta_n w_n^\dagger z_{n-1}} \Big|_{\eta_n = 0}. \tag{4.13}$$

This may seem more difficult than what we started with, but it will soon become clear why we do it. First, we write down the full expression for the partition function

$$\mathcal{Z} = \left[\prod_n \frac{1}{(2S)!}\frac{\partial^{2S}}{\partial\eta_n^{2S}}\right]\int \mathcal{D}[z,w]e^{-S[z^\dagger,z,w^\dagger,w,\eta]}\bigg|_{\eta=0}, \tag{4.14}$$

where by $|_{\eta=0}$ we mean that we let all η_n be zero after doing the differentiation. Now, the action takes the form

$$S[z^\dagger,z,w^\dagger,w,\eta] = \sum_{n=1}^{N} z_n^\dagger z_n + w_n^\dagger w_n - \varepsilon H(z_n^\dagger, w_n) - \eta_n w_n^\dagger z_{n-1}. \tag{4.15}$$

We would at this point like to integrate out all of the w variables. The way we do this is by rewriting the action in matrix form and use eq. (A.2) to do the w integrals. We define the vectors

$$\begin{aligned}
\mathbf{w}^\dagger &= \begin{pmatrix} w_{1\uparrow}^* & w_{1\downarrow}^* & \dots \end{pmatrix}\\
\mathbf{u}^\dagger &= \begin{pmatrix} z_{1\uparrow}(1-\varepsilon H_{\uparrow\uparrow}) - \varepsilon z_{1\downarrow}H_{\downarrow\uparrow} & z_{1\downarrow}(1-\varepsilon H_{\downarrow\downarrow}) - \varepsilon z_{1\uparrow}H_{\uparrow\downarrow}\dots \end{pmatrix}\\
\mathbf{v} &= \begin{pmatrix} \eta_1 z_{0\uparrow} & \eta_1 z_{0\downarrow} & \dots \end{pmatrix}^T.
\end{aligned} \tag{4.16}$$

We can then rewrite the w part of the integral, to take the form of eq. (A.2). We realise that the matrix coupling w to itself is just a $2N \times 2N$ identity matrix. We denote this matrix by M. Note that when completing the integral all the factors of π in the integration measure cancels. We get

$$\int \mathcal{D}[w]e^{-\mathbf{w}^\dagger M\mathbf{w}+\mathbf{u}^\dagger\mathbf{w}+\mathbf{w}^\dagger\mathbf{v}} = e^{\mathbf{u}^\dagger\mathbf{v}}. \tag{4.17}$$

Thus, we have successfully integrated out all w variables from the partition function, and what we are left with is the expression

$$\begin{aligned}
\mathcal{Z} &= \left[\prod_{n=1}^{N} \frac{1}{(2S)!}\frac{\partial^{2S}}{\partial\eta_n^{2S}}\right]\int \mathcal{D}[z]e^{-S[z^\dagger,z,\eta]}\bigg|_{\eta=0}\\
S &= \sum_{n=1}^{N}\left[z_n^\dagger z_n - \eta_n\Big(z_n^\dagger z_{n-1} - \varepsilon H(z_n^\dagger, z_{n-1})\Big)\right].
\end{aligned} \tag{4.18}$$

This is our expression for the single particle partition function. Note that there is no clear way to do a continuum version of this expression, due to the factor η_n in the action.

4.2.1 Equivalence to spin coherent state path integral

We will now show that the Schwinger boson coherent state path integral developed above is indeed equivalent to the spin coherent state path integral, before we do any approximations to the spin coherent states. It is obviously expected that the two ways of formulating the partition function are equivalent, and we will here provide a clear path for going from one to the other.

The first thing we do, is to do the η differentiations, which brings the term multiplied with η "down to the floor" of the expression. Now, if we write the z_ns as a vector of the $\uparrow$ and $\downarrow$ parts, $\mathbf{z}_n = \begin{pmatrix} z_{n\uparrow} & z_{n\downarrow} \end{pmatrix}^T$, the partition function takes the form

$$\mathcal{Z} = \left[\frac{1}{(2S)!} \right]^N \int \mathcal{D}[z] \prod_{n=1}^{N} e^{-\mathbf{z}_n^\dagger \mathbf{z}_n} \Big(\mathbf{z}_n^\dagger (I - \varepsilon H) \mathbf{z}_{n-1} \Big)^{2S}, \tag{4.19}$$

where I is the 2×2 identity matrix. We will now do a variable transformation from the four z variables we currently have at each time step in the following way

$$\mathbf{z}_n = r_n e^{i\gamma_n} \begin{pmatrix} \cos\frac{\theta_n}{2} \\ \sin\frac{\theta_n}{2} e^{i\phi_n} \end{pmatrix} = r_n e^{i\gamma_n} \tilde{\mathbf{z}}_n, \tag{4.20}$$

motivated by the fact that $\tilde{\mathbf{z}}_n$ clearly parametrise the unit sphere, and looks exactly like the spin-1/2 coherent state of eq. (2.25). By doing this transformation, the integration measure changes as

$$\frac{\mathrm{d}^2 z_n}{\pi} = \frac{1}{4\pi^2} \mathrm{d}r_n \, \mathrm{d}\gamma_n \, \mathrm{d}\phi_n \, \mathrm{d}\theta_n \, r_n^3 \sin\theta_n. \tag{4.21}$$

The goal is to integrate out r and γ, and thus being left with the θ and ϕ variables that parametrise the spin coherent state. By inserting our variable transformation into eq. (4.19), we notice that we can do the integrals over the different variables separably. First out is the integrals over r_n. At each n we get

$$\int \mathrm{d}r_n\, r_n^{3+4S} e^{-r_n^2} = \frac{1}{2}\Gamma(2S+2) = \frac{1}{2}(2S+1)!, \tag{4.22}$$

where we have used the definition of the Gamma function (see Appendix A.3). Next, we will look at the γ_n integrals. However, we notice that all the γs of the $z_{n\sigma}$ part cancels with the $z^*_{n\sigma}$ part giving integrals of 1 that all return 2π each. We are left with

$$\mathcal{Z} = \int \prod_{n=1}^{N} \frac{2S+1}{4\pi} \mathrm{d}\phi_n\, \mathrm{d}\theta_n \sin\theta_n \left[\tilde{\mathbf{z}}_n^\dagger (I - \varepsilon H)\tilde{\mathbf{z}}_{n-1}\right]^{2S}. \tag{4.23}$$

To realize that this is indeed equivalent to eq. (3.14) we first need to see that

$$\left[\tilde{\mathbf{z}}_n^\dagger (I - \varepsilon H)\tilde{\mathbf{z}}_{n-1}\right]^{2S} \approx e^{-2S\varepsilon H(\tilde{z}_n^\dagger, \tilde{z}_{n-1})} \left[\tilde{\mathbf{z}}_n^\dagger \tilde{\mathbf{z}}_{n-1}\right]^{2S}, \tag{4.24}$$

where

$$H(\tilde{z}_n^\dagger, \tilde{z}_{n-1}) = \frac{\tilde{\mathbf{z}}_n^\dagger H\, \tilde{\mathbf{z}}_{n-1}}{\tilde{\mathbf{z}}_n^\dagger \tilde{\mathbf{z}}_{n-1}}. \tag{4.25}$$

We notice that the term raised to the power of $2S$ is clearly equal to ${}_S\langle\Omega_n|\Omega_{n-1}\rangle_S$, and also that $2SH(\tilde{z}_n^\dagger, \tilde{z}_{n-1})$ is equal to the term in eq. (3.15). Thus, we see that the two formulations are indeed equivalent. However, as we will see next, working with the Schwinger boson coherent state path integral is easier, as we can then use the generalised Gaussian integral of Appendix A.1 to calculate the partition function. We will now explicitly calculate the partition function for some systems, and compare the results to "regular" statistical mechanics.

4.2.2 The zero energy case

First, we will see what happens in the zero energy case, where the Hamiltonian is simply equal to 0. Again, we will use eq. (A.2) to integrate out the z variables. This time, we only have to define two vectors

$$\mathbf{z}_\sigma^\dagger = \begin{pmatrix} z^*_{1\sigma} & z^*_{2\sigma} & \dots \end{pmatrix}, \tag{4.26}$$

for $\sigma =\uparrow,\downarrow$. Now we will let $M_\uparrow$ be the matrix connecting the up part of $z^*_{n\uparrow}$ with $z_{n\uparrow}$, and $M_\downarrow$ to be the same for the down parts. In this simple case, the matrices are equal, and take the form

$$M_\uparrow = M_\downarrow = \begin{pmatrix} 1 & -\eta_1 & & & \\ & 1 & -\eta_2 & & \\ & & \ddots & & \\ & & & 1 & -\eta_{N-1} \\ -\eta_N & & & & 1 \end{pmatrix}, \tag{4.27}$$

where the empty parts of the matrix is filled by zeros. We need to calculate the determinant of this matrix. We can transform this matrix to another matrix with the same determinant by first subtracting $-\eta_N$ times the top row from the bottom row. Next, we subtract $\eta_N\eta_1$ times the second row from the bottom row. Applying this technique for each row in the matrix, we are left with

$$\det M_\sigma = \begin{vmatrix} 1 & -\eta_1 & & & \\ & 1 & -\eta_2 & & \\ & & \ddots & & \\ & & & 1 & -\eta_{N-1} \\ & & & & 1-\prod_{n=1}^N \eta_n \end{vmatrix}, \tag{4.28}$$

and the determinant of the matrix is just going to be $1-\prod_{n=1}^N \eta_n$. Thus the partition function is reduced to

$$\mathcal{Z} = \left[\prod_{n=1}^N \frac{1}{(2S)!}\frac{\partial^{2S}}{\partial\eta_n^{2S}}\right] \left.\frac{1}{\left(1-\prod_{n=1}^N \eta_n\right)^2}\right|_{\eta=0}. \tag{4.29}$$

To evaluate the derivatives, we start by defining a constant $k=\prod_{n>1}\eta_n$, including all ηs except the first one. We can then differentiate with respect to η_1 $2S$ times. The first time we do it, we get a factor of $2k$ in the numerator, and the power in the denominator increase by 1. The next time we get a factor of $3k$ in the numerator, and again the power of the denominator increase by 1. After doing this $2S$ times, and then letting $\eta_1 \to 0$, we end up with

$$\mathcal{Z} = \left[\prod_{n=2}^{N} \frac{1}{(2S)!} \frac{\partial^{2S}}{\partial \eta_n^{2S}} \right] (2S+1)\, k^{2S} \Bigg|_{\eta=0} . \tag{4.30}$$

Now, the differentiation with respect to all the other ηs will produce one factor of $(2S)!$ each. We end up with the very simple answer for the partition function

$$\mathcal{Z} = 2S + 1. \tag{4.31}$$

By using regular statistical mechanics, we can check that this is the correct result. We have that

$$\mathcal{Z} = \operatorname{tr} I = 2S + 1, \tag{4.32}$$

where we have used that there are $2S+1$ available states for a spin S particle. Even though the computational procedure was a bit cumbersome, we see that we ended up with the correct result.

4.2.3 The Zeeman energy

Next, we move on to the case of the Zeeman energy. Here, we can write

$$H(z_n^\dagger, z_{n-1}) = \frac{B}{2} \Big(z_{n\uparrow}^* z_{n-1\uparrow} - z_{n\downarrow}^* z_{n-1\downarrow} \Big). \tag{4.33}$$

We do the calculation in a similar manner as before, but this time the matrix for the $\uparrow$ part is different than the $\downarrow$ part. We can, however, immediately write down the two matrices as

$$M_\uparrow = \begin{pmatrix} 1 & -\eta_1\Big(1-\varepsilon\frac{B}{2}\Big) & & & \\ & 1 & -\eta_2\Big(1-\varepsilon\frac{B}{2}\Big) & & \\ & & \ddots & & \\ & & & 1 & -\eta_{N-1}\Big(1-\varepsilon\frac{B}{2}\Big) \\ -\eta_N\Big(1-\varepsilon\frac{B}{2}\Big) & & & & 1 \end{pmatrix}$$

$$M_{\downarrow}=\begin{pmatrix} 1 & -\eta_1\left(1+\varepsilon\frac{B}{2}\right) & & & \\ & 1 & -\eta_2\left(1+\varepsilon\frac{B}{2}\right) & & \\ & & \ddots & & \\ & & & 1 & -\eta_{N-1}\left(1+\varepsilon\frac{B}{2}\right) \\ -\eta_N\left(1+\varepsilon\frac{B}{2}\right) & & & & 1 \end{pmatrix}.$$

In a similar fashion to what we did before, we can calculate the determinants of these matrices. The results are

$$\det M_{\uparrow}=1-\left(1-\varepsilon\frac{B}{2}\right)^N\prod_n\eta_n$$
$$\det M_{\downarrow}=1-\left(1+\varepsilon\frac{B}{2}\right)^N\prod_n\eta_n.$$

We are interested in the limit where $N\to\infty$. In this limit the determinants reduce to $1-e^{\pm\frac{\beta B}{2}}\prod_n\eta_n$, and we can write down a new expression for the partition function

$$\mathcal{Z}=\left[\prod_{n=1}^{N}\frac{1}{(2S)!}\frac{\partial^{2S}}{\partial\eta_n^{2S}}\right]\left.\frac{1}{\left(1-e^{\frac{\beta B}{2}}\prod_n\eta_n\right)\left(1-e^{-\frac{\beta B}{2}}\prod_n\eta_n\right)}\right|_{\eta=0}. \tag{4.34}$$

Now, we again have to do the differentiation with respect to the η variables. Again, we will focus first on the η_1 derivatives, and then the rest will follow. This time we define two new variables $k_1=e^{\frac{\beta B}{2}}\prod_{n>1}\eta_n$, and $k_2=e^{-\frac{\beta B}{2}}\prod_{n>1}\eta_n$. We have that

$$\left.\frac{\partial^m}{\partial x^m}\frac{1}{(1-k_1x)(1-k_2x)}\right|_{x=0}=m!\sum_{i=0}^{m}k_1^i\,k_2^{m-i}, \tag{4.35}$$

which can be shown in a similar fashion to the derivative of eq (4.29). Again we get that differentiating with respect to all η_n for $n>1$ gives a factor of $(2S)!$, and our partition function takes on the very simple form

$$\mathcal{Z}=\sum_{i=0}^{2S}e^{-i\beta\frac{B}{2}}e^{(2S-i)\beta\frac{B}{2}}=\sum_{n=-S}^{S}e^{n\beta B}. \tag{4.36}$$

That this is indeed the correct partition function can be checked easily by using regular statistical mechanics

$$\mathcal{Z} = \operatorname{tr} e^{-\beta \hat{H}} = \sum_{n=-S}^{S} e^{n\beta B}. \tag{4.37}$$

We see that even though our method for calculating the partition function is a bit cumbersome, it does yield the correct result.

4.3 Abrikosov fermion coherent state path integral using a projection operator

In this section we will calculate the single particle partition function of a system described by Abrikosov fermions in two different ways. The starting point of both is eq. (3.6), with $|\psi\rangle$ being Abrikosov fermionic coherent states. We will have to modify it slightly to make sure we only have a single particle. Therefore, we multiply by the projection operator down to the physical subspace

$$\hat{P} = \sum_{\sigma=\uparrow,\downarrow} c_\sigma^\dagger : e^{-\hat{n}} : c_\sigma. \tag{4.38}$$

Using this projection operator, we can write the partition function

$$\mathcal{Z} = \int \mathrm{d}[\psi]\ \langle\psi| e^{-\beta \hat{H}} \hat{P} |\psi\rangle\, e^{-\psi^\dagger \psi}. \tag{4.39}$$

We will now consider a simple two level system, where the particle has some energy E_σ depending on what state the particle is in. We write the Hamiltonian

$$\hat{H} = \sum_\sigma E_\sigma c_\sigma^\dagger c_\sigma. \tag{4.40}$$

We will now write the exponential and projection operator as a single, normal ordered operator. We expand the exponential, and multiply with the projection operator, and after normal ordering everything we end up with

$$e^{-\beta\hat{H}}\hat{P} = e^{-\beta E_\uparrow}\Big(c^\dagger_\uparrow c_\uparrow + c^\dagger_\uparrow c^\dagger_\downarrow c_\uparrow c_\downarrow\Big) + e^{-\beta E_\downarrow}\Big(c^\dagger_\downarrow c_\downarrow + c^\dagger_\uparrow c^\dagger_\downarrow c_\uparrow c_\downarrow\Big). \tag{4.41}$$

Now, we can take the matrix elements of this operator, and do the integral over the Grassmann variables, being very careful keeping track of minus signs. We end up with

$$\begin{aligned}\mathcal{Z} &= \int \mathrm{d}[\psi_\uparrow]\,\mathrm{d}[\psi_\downarrow]\left[\sum_\sigma \psi^*_\sigma\psi_\sigma e^{-\beta E_\sigma} - \psi^*_\uparrow\psi^*_\downarrow\psi_\uparrow\psi_\downarrow\Big(e^{-\beta E_\uparrow} + e^{-\beta E^\downarrow}\Big)\right] \\ &= e^{-\beta E_\uparrow} + e^{-\beta E_\downarrow}.\end{aligned} \tag{4.42}$$

This is indeed the correct result, as can easily be checked

$$\mathcal{Z} = \operatorname{tr} e^{-\beta\hat{H}} = \sum_\sigma e^{-\beta E_\sigma}. \tag{4.43}$$

4.3.1 A path integral expression for the partition function

Now, we will develop a path integral representation of the partition function using the projection operator. Here, we will consider a Hamiltonian, again written in terms of Abrikosov fermion operators as

$$\hat{H} = \sum_{\sigma,\sigma'} c^\dagger_\sigma H_{\sigma\sigma'} c_{\sigma'}. \tag{4.44}$$

Again, $H_{\sigma\sigma'}$ is the matrix elements of some matrix H. The starting point is again eq. (3.6), but we will now split the imaginary time interval into N parts of size $\varepsilon = \frac{\beta}{N}$. We insert the identity operator of eq. 2.23 between every time step, and every time we insert identity, we will use a projection operator to project the coherent states down to the physical subspace. We also insert another set of identities between the exponentials and the projection operators. Our expression for the partition function takes the form

$$\mathcal{Z} = \int \prod_{n=1}^{N} \mathrm{d}[\psi_n]\,\mathrm{d}[\xi_n]\ \langle\psi_n|e^{-\varepsilon\hat{H}}|\xi_n\rangle\ \langle\xi_n|\hat{P}|\psi_{n-1}\rangle\, e^{-\psi^\dagger_n\psi_n - \xi^\dagger_n\xi_n}. \tag{4.45}$$

We already know how to deal with the matrix elements of the exponential operator, and we get

$$\langle \psi_n | e^{-\varepsilon \hat{H}} | \xi_n \rangle = e^{-\varepsilon H(\psi_n^\dagger, \xi_n) + \psi_n^\dagger \xi_n} \tag{4.46}$$

where $H(\psi_n^\dagger, \xi_n)$ is what we get when we replace the creation and annihilation operators of the Hamiltonian with the corresponding ψ^* or ξ respectively.

The projection operator is supposed to project down to the single particle subspace, and is given by eq. (4.38). It is then easy to calculate the matrix elements of the projection operator

$$\langle \xi_n | \hat{P} | \psi_{n-1} \rangle = \xi_n^\dagger \psi_{n-1}. \tag{4.47}$$

Now, we would like to integrate out the ξ variables from the partition function. Multiplying everything out, using the exact Taylor expansion of the exponential, and also using that the integrals over ξ will only be non-zero for terms including one of each $\xi_{n\uparrow}^*$, $\xi_{n\uparrow}$, $\xi_{n\downarrow}^*$ and $\xi_{n\downarrow}$ due to the way Grassmann integration works. After some algebra we end up with an expression for the partition function that looks best written in matrix form

$$\begin{aligned} \mathcal{Z} = \int \prod_{n=1}^{N} \mathrm{d}[\psi_{n\uparrow}]\, \mathrm{d}[\psi_{n\downarrow}] \Bigg[&\begin{pmatrix} \psi_{n\uparrow}^* & \psi_{n\downarrow}^* \end{pmatrix} (I - \varepsilon H) \begin{pmatrix} \psi_{n-1\uparrow} \\ \psi_{n-1\downarrow} \end{pmatrix} \\ &- \begin{pmatrix} \psi_{n\downarrow}^* \psi_{n\downarrow} \psi_{n\uparrow}^* & \psi_{n\uparrow}^* \psi_{n\uparrow} \psi_{n\downarrow}^* \end{pmatrix} (I - \varepsilon H) \begin{pmatrix} \psi_{n-1\uparrow} \\ \psi_{n-1\downarrow} \end{pmatrix} \Bigg]. \end{aligned} \tag{4.48}$$

Our strategy will be to integrate out each ψ_n, one at a time. First we consider the integral over ψ_1. From when $n = 1$, we get a factor of

$$\begin{pmatrix} \psi_{1\uparrow}^* & \psi_{1\downarrow}^* \end{pmatrix} (I - \varepsilon H) \begin{pmatrix} \psi_{0\uparrow} \\ \psi_{0\downarrow} \end{pmatrix} - \begin{pmatrix} \psi_{1\downarrow}^* \psi_{1\downarrow} \psi_{1\uparrow}^* & \psi_{1\uparrow}^* \psi_{1\uparrow} \psi_{1\downarrow}^* \end{pmatrix} (I - \varepsilon H) \begin{pmatrix} \psi_{0\uparrow} \\ \psi_{0\downarrow} \end{pmatrix},$$

in the integrand, multiplied with a similar one for $n = 2$. Multiplying these factors together, integrating out ψ_1 is possible, and after some cleaning up we are left with

$$\begin{pmatrix}\psi^*_{2\uparrow} & \psi^*_{2\downarrow}\end{pmatrix}(I-\varepsilon H)^2\begin{pmatrix}\psi_{0\uparrow}\\ \psi_{0\downarrow}\end{pmatrix} - \begin{pmatrix}\psi^*_{2\downarrow}\psi_{2\downarrow}\psi^*_{2\uparrow} & \psi^*_{2\uparrow}\psi_{2\uparrow}\psi^*_{2\downarrow}\end{pmatrix}(I-\varepsilon H)^2\begin{pmatrix}\psi_{0\uparrow}\\ \psi_{0\downarrow}\end{pmatrix}.$$

We can take this, and multiply with the $n = 3$ factor, and then integrate out ψ_2, and we get

$$\begin{pmatrix}\psi^*_{3\uparrow} & \psi^*_{3\downarrow}\end{pmatrix}(I-\varepsilon H)^3\begin{pmatrix}\psi_{0\uparrow}\\ \psi_{0\downarrow}\end{pmatrix} - \begin{pmatrix}\psi^*_{3\downarrow}\psi_{3\downarrow}\psi^*_{3\uparrow} & \psi^*_{3\uparrow}\psi_{3\uparrow}\psi^*_{3\downarrow}\end{pmatrix}(I-\varepsilon H)^3\begin{pmatrix}\psi_{0\uparrow}\\ \psi_{0\downarrow}\end{pmatrix}.$$

We continue this way until we are left with only terms including ψ_0 and ψ_N, but remembering that we have $\psi_0 = -\psi_N$, we can do this integral as well. The last integral we end up with is the following

$$\begin{aligned}\mathcal{Z} = \int \mathrm{d}[\psi_{N\uparrow}]\,\mathrm{d}[\psi_{N\downarrow}]\Bigg[&-\begin{pmatrix}\psi^*_{N\uparrow} & \psi^*_{N\downarrow}\end{pmatrix}e^{-\beta H}\begin{pmatrix}\psi_{N\uparrow}\\ \psi_{N\downarrow}\end{pmatrix}\\ &+\begin{pmatrix}\psi^*_{N\downarrow}\psi_{N\downarrow}\psi^*_{N\uparrow} & \psi^*_{N\uparrow}\psi_{N\uparrow}\psi^*_{N\downarrow}\end{pmatrix}e^{-\beta H}\begin{pmatrix}\psi_{N\uparrow}\\ \psi_{N\downarrow}\end{pmatrix}\Bigg],\end{aligned} \tag{4.49}$$

where we have taken the limit $N \to \infty$ to write $(I - \varepsilon H)^N \approx e^{-\beta H}$. From this expression we can calculate the single particle partition function for any quadratic Hamiltonian.

Now consider, as an example, a particle that has some energy E no matter what configuration it is in, and that interacts with some external field that couples to the spin operator. We denote the external field by

$$\mathbf{B} = \begin{pmatrix}B_x & B_y & B_z\end{pmatrix}^T. \tag{4.50}$$

The Hamiltonian matrix then takes the form

$$H = \begin{pmatrix}E + \frac{B_z}{2} & \frac{B_x - iB_y}{2}\\ \frac{B_x + iB_y}{2} & E - \frac{B_z}{2}\end{pmatrix}. \tag{4.51}$$

Now, we may calculate the partition function for this system. One can immediately notice that the first term of eq. (4.49) will yield 0, due to the way Grassmann integrals work. Thus we are left with the second term. Upon multiplying out everything inside the integral, and removing anything where the same Grassmann number enters more than once, what remains is

$$\begin{aligned}\mathcal{Z} &= 2e^{-\beta E}\cosh\left(\frac{\beta|\mathbf{B}|}{2}\right)\int \mathrm{d}[\psi_{N\uparrow}]\,\mathrm{d}[\psi_{N\downarrow}]\,\psi^*_{N\downarrow}\psi_{N\downarrow}\psi^*_{N\uparrow}\psi_{N\uparrow} \\ &= 2e^{-\beta E}\cosh\left(\frac{\beta|\mathbf{B}|}{2}\right).\end{aligned} \tag{4.52}$$

That this is indeed the correct result can be checked. The partition function is given by the trace of $e^{-\beta\hat{H}}$. We can calculate this by diagonalizing the matrix H, and in the diagonal basis the trace is just the sum of the eigenvalues. Thus, the partition function is given by the sum of the eigenvalues of $e^{-\beta H}$, which is

$$\mathcal{Z} = 2e^{-\beta E}\cosh\left(\frac{\beta|\mathbf{B}|}{2}\right), \tag{4.53}$$

and our calculation is indeed correct.

Chapter 5

The Bruckmann-Urbina construction of the path integral

As mentioned in the introduction, the path integral of Bruckmann and Urbina was a proposition to fix the problems discussed by Wilson and Galitski[8, 12]. In their paper, Bruckmann and Urbina were able to circumvent the continuity problems of Wilson and Galitski by the use of a duality transform from variables living on time steps to variables living on time bonds in imaginary time. After some rather technical steps they derived a continuous-time path integral, that do not rely on the continuity of the paths. Instead, the trajectories were allowed to make discrete jumps in an infinitesimal time. Using this formalism, they showed that they got the correct result for the one site Bose-Hubbard model. They also extended their discussion to Schwinger boson systems, and got the correct result for a Hamiltonian proportional to $\hat{S}_z^n$ for an arbitrary n, and they did an expansion in $\beta\omega$ to second order for a Hamiltonian $\hat{H} = \omega\hat{S}_x$. However, in their derivation the Schwinger boson number constraint on the dual variables was inserted manually. In this chapter we will show how the Schwinger boson number constraint for the dual variables follows naturally when using the Schwinger boson projection operator in the construction of the path integral. To shed some light on how the construction can be used in calculations, we expand upon the calculation of Bruckmann and Urbina for the $\hat{S}_x$ Hamiltonian, doing the expansion to infinite order, thus recovering the exact partition function. Furthermore, we do a high temperature expansion of a system of two spins with a mutual Heisnberg interaction. We then extend the theory to many spins, and calculate the partition function of a Ising ring exactly for a longitudinal field, and do a high temperature expansion for a transversal field. Finally, we derive a real time propagator within the same framework. We verify that for a spin with a zero Hamiltonian the action still picks up a Berry phase, and for a spin coupled to an external field in the limit of $S \to \infty$ we recover the action of a classical spin in a magnetic field.

5.1 Constructing the path integral for a Schwinger boson model

The first step in constructing the Bruckmann and Urbina path integral for a single spin is to write the Hamiltonian in the so-called P-representation [25]. Suppose we have some Hamiltonian $\hat{H} = \hat{H}(a^\dagger, a)$ that is written in terms of Schwinger boson operators. The idea is to write this in the anti-normal order where all annihilation operators are to the left of creation operators. Inserting the identity operator of eq. (2.14), we then get

$$\hat{H} = \int \frac{\mathrm{d}^4 z}{\pi^2} e^{-z^\dagger z} h(z, z^\dagger) \left|z\right\rangle\!\left\langle z\right| , \tag{5.1}$$

where $h(z, z^\dagger)$ is the anti-normal ordered Hamiltonian where the operators $a_\sigma / a_\sigma^\dagger$ are replaced by the corresponding z_σ / z_σ^* and $\mathrm{d}^4 z = \prod_\sigma \mathrm{d}z_\sigma^* \,\mathrm{d}z_\sigma$. It is then clear that the relation

$$e^{-\varepsilon \hat{H}} \approx 1 - \varepsilon \hat{H} \approx \int \frac{\mathrm{d}^4 z}{\pi^2} e^{-z^\dagger z - \varepsilon h(z, z^\dagger)} \left|z\right\rangle\!\left\langle z\right| \tag{5.2}$$

holds. Consider eq. (3.6) with coherent states $\left|z\right\rangle$ that are eigenstates of the Schwinger boson annihilation operators. Now, we split the exponential into N parts and to make sure that the Schwinger boson number constraint is enforced we insert a projection operator at each time step. We get

$$\mathcal{Z} = \int \prod_n \frac{\mathrm{d}^4 z_n}{\pi^2} e^{-z_n^\dagger z_n - \varepsilon h(z_n^\dagger, z_n)} \langle z_n | \hat{P} | z_{n-1} \rangle \,. \tag{5.3}$$

Note that the Hamiltonian function h is diagonal in imaginary time. The matrix elements of the projection operator can be calculated, and using the same trick as before, we can write in terms of partial derivatives of an exponential. Our expression for the partition function takes the form

$$\mathcal{Z} = \partial_\eta \int \mathcal{D}[z] \exp \sum_{n=1}^{N} \left(-z_n^\dagger z_n - \varepsilon h(z_n^\dagger, z_n) + \eta_n z_n^\dagger z_{n-1} \right) \Big|_{\eta=0} \,, \tag{5.4}$$

where we have introduced the shorthand notation

$$\partial_\eta = \prod_n \frac{1}{(2S)!} \frac{\partial^{2S}}{\partial \eta_n^{2S}}, \tag{5.5}$$

and $\mathcal{D}[z] = \prod_{n=1}^{N} \frac{\mathrm{d}^4 z_n}{\pi^2}$. Now, similarly to Bruckmann and Urbina, we do a duality transform from variables living on time steps n to a set of dual variables living on the bond between time step n and $n+1$. The way we do this in practise, is to expand the only part of the partition function that in non-diagonal in imaginary time, namely the part responsible for the Schwinger boson number constraint. We also write the complex variables z in their polar form $z_n = \sqrt{r_n} e^{i\varphi_n}$, where $r_n \in \{0, \infty\}$ and $\varphi_n \in \{0, 2\pi\}$. Doing this transform changes the integration measure to $\frac{\mathrm{d}\varphi_n \mathrm{d}r_n}{2\pi}$. Expanding the non-diagonal part yields

$$e^{\sum_{n,\sigma} \eta_n z_{n\sigma}^* z_{n-1\sigma}} = \prod_{n\sigma} \sum_{m_{n\sigma}=0}^{\infty} \eta_n^{m_{n\sigma}} \frac{\left(r_{n\sigma} r_{n-1\sigma}\right)^{\frac{m_{n\sigma}}{2}}}{m_{n\sigma}!} e^{-i(\varphi_{n\sigma} - \varphi_{n-1\sigma}) m_{n\sigma}} \,. \tag{5.6}$$

Next, we would like to transform our expression with a sum over discrete ms to an expression with an integral over continuous ms. The way we will do this, is to use the concept of Poisson summation (see for instance [2] for how to use this in many different settings). The relation we will be using is

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{s=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}m\, f(m) e^{-i2\pi sm}. \tag{5.7}$$

In our case, $f(m)$ will be the right hand side of eq. (5.6), where we have to generalise the factorial to a Gamma function to accommodate continuous ms. Note that in our case the sum runs from 0 to ∞ rather than from $-\infty$. However, the only change in the Poisson summation formula coming from this, is that the integral runs from 0 to ∞. Putting all of this together into the partition function expression gives us

$$\begin{aligned} \mathcal{Z} = \partial_\eta \int \mathcal{D}[r,\varphi,m] \sum_{\mathbf{s}=-\infty}^{\infty} \prod_{n\sigma} \eta_n^{m_{n\sigma}} \frac{(r_{n\sigma} r_{n-1\sigma})^{\frac{m_{n\sigma}}{2}}}{\Gamma(m_{n\sigma}+1)} \\ \times \exp[-r_{n\sigma} - i(\varphi_{n\sigma} - \varphi_{n-1\sigma} + 2\pi s_{n\sigma}) m_{n\sigma} - \varepsilon h(r_{n\sigma}, \varphi_{n,\sigma})] \Big|_{\eta=0} . \end{aligned} \tag{5.8}$$

Here, the notation of $\mathbf{s}$ in the summation means that we sum over all the different $s_{n\sigma}$ from $-\infty$ to ∞ and the integration measure is

$$\mathcal{D}[r,\varphi,m] = \prod_{n=1}^{N} \frac{\mathrm{d}\varphi_n\, \mathrm{d}r_n}{2\pi}\, \mathrm{d}m_{n\uparrow}\, \mathrm{d}m_{n\downarrow} . \tag{5.9}$$

At this point we do the differentiation with respect to all of the η variables. It is clear that when differentiating we will be left with 0 unless $m_{n\uparrow} + m_{n\downarrow} = 2S$, and we see that the Schwinger boson number constraint has been carried over to the dual variables. When the constraint is fulfilled, the differentiation yields one factor of $(2S)!$ for each η_n, and this cancels out the $\frac{1}{(2S)!}$ factors in ∂_η.

Now, to keep our notation consistent with Bruckmann and Urbina we redefine all of our variables. First, we let

$$\begin{aligned} s_{n\sigma} &= q_{n\sigma} - q_{n+1\sigma} \\ s_{1\sigma} &= -q_{2\sigma} \\ q_{n+1\sigma} &= Q_\sigma, \end{aligned} \tag{5.10}$$

such that each φ is next to its corresponding q. Furthermore, we define a new set of variables, ϕ, that absorb the qs into φ the following way

$$\phi_{n\sigma} = \varphi_{n+1\sigma} + 2\pi q_{n+1\sigma}. \tag{5.11}$$

At the boundaries $n = N$ and $n = 0$, we have two special cases we have to treat with extra care

$$\begin{aligned} \phi_{0\sigma} &= \varphi_\sigma \\ \varphi_{N\sigma} &= \varphi_\sigma + 2\pi Q_\sigma. \end{aligned} \tag{5.12}$$

Note that the integrals over the new variables ϕ is taken over the entire real axis, as we "eat up" the sums over all of the qs.

Furthermore, following Bruckmann and Urbina, we rename $m_{n\sigma} = \rho_{n\sigma}$. However, as the ρ variables need to be the same at at $n = 0$ and $n = N$, we define the edge cases in a special way

$$\rho_{0\sigma} = \rho_{N\sigma} = \rho_\sigma. \tag{5.13}$$

We may then write our expression for the partition function in terms of these new variables

$$\begin{aligned} \mathcal{Z} = \sum_{\mathbf{Q}=-\infty}^{\infty} \int_0^\infty \mathrm{d}\boldsymbol{\rho} \int_0^{2\pi} \frac{\mathrm{d}\boldsymbol{\varphi}}{(2\pi)^2} \left[\prod_{n=1}^{N-1} \int_0^\infty \mathrm{d}\boldsymbol{\rho}_n \int_{-\infty}^\infty \frac{\mathrm{d}\boldsymbol{\phi}_n}{(2\pi)^2} \right] \\ \times \exp \sum_{n=0}^{N-1} [i\boldsymbol{\rho}_{n+1}(\boldsymbol{\phi}_{n+1} - \boldsymbol{\phi}_n) - \varepsilon\mathcal{H}(\boldsymbol{\rho}_{n+1}, \boldsymbol{\rho}_n; \boldsymbol{\phi}_n)], \end{aligned} \tag{5.14}$$

where the notation $\mathrm{d}\boldsymbol{\rho}$ means that we integrate over both $\rho_\uparrow$ and $\rho_\downarrow$. In eq. (5.14) we have also introduced $\mathcal{H}$ given by

$$\mathcal{H}(\boldsymbol{\rho}, \boldsymbol{\rho}'; \boldsymbol{\phi}) = \left[\prod_\sigma \int_0^\infty \mathrm{d}r_\sigma \, e^{-r_\sigma} \frac{r_\sigma^{\frac{\rho_\sigma + \rho'_\sigma}{2}}}{\sqrt{\Gamma(\rho_\sigma + 1)\Gamma(\rho'_\sigma + 1)}} \right] h(\mathbf{r}, \boldsymbol{\phi}). \tag{5.15}$$

$\mathcal{H}$ is called the "Laguerre" Hamiltonian because it looks similar to the Laguerre transformation of h.

The only place Q_σ appears in eq. (5.14) is in a factor of $e^{2\pi Q_\sigma \rho_\sigma}$. Thus, preforming the sums over Q_σ quantize the corresponding ρ_σ to a non-negative integer n_σ, and the integral over ρ_σ reduce to a sum over n_σ. At this point, we can actually do a continuum limit of our path integral. By letting $\tau_n = n\varepsilon$, we can define paths $\boldsymbol{\rho}(\tau_n - \varepsilon/2) = \boldsymbol{\rho}_n$ and $\boldsymbol{\phi}(\tau_n) = \boldsymbol{\phi}_n$. Note that these paths are in general not continuous. The path integral representation of the partition function takes the form

$$\mathcal{Z} = \sum_{\mathbf{n}=0}^{\infty} \int_0^{2\pi} \frac{\mathrm{d}\boldsymbol{\varphi}}{(2\pi)^2} \oint \mathcal{D}[\rho(\tau), \phi(\tau)] e^{-S[\rho(\tau),\phi(\tau)]}, \tag{5.16}$$

where the action is given by

$$S[\rho(\tau), \phi(\tau)] = \int_0^\beta \mathrm{d}\tau \left[\mathcal{H}(\boldsymbol{\rho}(\tau+0), \boldsymbol{\rho}(\tau-0); \boldsymbol{\phi}(\tau)) - i\boldsymbol{\rho}(\tau)\dot{\boldsymbol{\phi}}(\tau)\right], \tag{5.17}$$

where we denote differentiation with respect to τ with a dot. The circle on the integration sign denotes that we have closed paths such that $\boldsymbol{\rho}(0-0) = \boldsymbol{\rho}(\beta - 0) = \mathbf{n}$ and $\boldsymbol{\phi}(0) = \boldsymbol{\phi}(\beta) = \boldsymbol{\varphi}$. It is also important to remember that we have the Schwinger boson number constraint on the components of $\boldsymbol{\rho}$ and $\mathbf{n}$ at each τ. Now we have been able to write the partition function in the continuum in a way that is not dependent on continuous paths, as is manifested in the $\tau \pm 0$ prescription in the Laguerre Hamiltonian.

5.2 Some toy models

We will here see explicitly how we work with the expression from eq. (5.16) by looking at some calculational examples.

5.2.1 A Hamiltonian proportional to $\hat{S}_x$

The first example we will consider is a single spin-1/2 particle with a Hamiltonian given by

$$\hat{H} = \omega \hat{S}_x = \frac{\omega}{2}(a_\uparrow^\dagger a_\downarrow + a_\downarrow^\dagger a_\uparrow). \tag{5.18}$$

The Hamiltonian function of the P-representation $h(z^\dagger, z)$ is calculated by anti-normal ordering the Hamiltonian, and replacing the operators by their corresponding z written in polar form. We get

$$h(z^\dagger, z) = \frac{\omega}{2}\Big(z_\downarrow z_\uparrow^* + z_\uparrow z_\downarrow^*\Big) = \omega\sqrt{r_\uparrow r_\downarrow}\cos(\phi_\uparrow - \phi_\downarrow), \tag{5.19}$$

where we used that we can write the exponentials as a cosine. The next step is then to calculate $\mathcal{H}$. Using the definition of the Gamma function we get

$$\begin{aligned}
\mathcal{H}(\boldsymbol{\rho}, \boldsymbol{\rho}', \boldsymbol{\phi}) &= \omega\cos(\phi_\uparrow - \phi_\downarrow)\prod_\sigma \int_0^\infty \mathrm{d}r_\sigma\, e^{-r_\sigma}\frac{r_\sigma^{\frac{\rho_\sigma+\rho'_\sigma+1}{2}}}{\sqrt{\Gamma(\rho_\sigma+1)\Gamma(\rho'_\sigma+1)}} \\
&= \omega\cos(\phi_\uparrow - \phi_\downarrow)\prod_\sigma \frac{\Gamma(\frac{\rho_\sigma+\rho'_\sigma}{2}+\frac{3}{2})}{\sqrt{\Gamma(\rho_\sigma+1)\Gamma(\rho'_\sigma+1)}} \\
&\equiv \omega\cos(\phi_\uparrow - \phi_\downarrow)\prod_\sigma \gamma(\rho_\sigma, \rho'_\sigma).
\end{aligned} \tag{5.20}$$

It is worth noting that $\gamma(\rho, \rho+1) = \sqrt{\rho+1}$ and $\gamma(\rho, \rho-1) = \sqrt{\rho}$. This property can easily be seen from the recursion relations for the Gamma function (see Appendix A.3). The way we proceed, is to expand the path integral expression in powers of the integral of $\mathcal{H}$ the following way

$$e^{-\int_0^\beta \mathrm{d}\tau \mathcal{H}(\tau)} = 1 - \int_0^\beta \mathrm{d}\tau_1\, \mathcal{H}(\tau_1) + \frac{1}{2}\int_0^\beta\int_0^\beta \mathrm{d}\tau_1\,\mathrm{d}\tau_2\, \mathcal{H}(\tau_1)\mathcal{H}(\tau_2) - \dots \tag{5.21}$$

where it is implicit that the τ dependence of $\mathcal{H}$ is through ρ and ϕ. We will go through the calculation for each term in this series one by one. To zeroth order we have

$$\mathcal{Z}^{(0)} = \sum_{\mathbf{n}=0}^\infty \int_0^{2\pi}\frac{\mathrm{d}\boldsymbol{\varphi}}{(2\pi)^2}\oint \mathcal{D}[\rho(\tau), \phi(\tau)]e^{i\int_0^\beta \phi(\tau)\dot{\rho}(\tau)}. \tag{5.22}$$

The first thing we will do, is the $\boldsymbol{\phi}$ integrals. If we take a step back to the discrete expression, we had that the exponentials at each time step look like $e^{i\boldsymbol{\phi}_n(\boldsymbol{\rho}_n - \boldsymbol{\rho}_{n-1})}$. Integrating over that particular $\boldsymbol{\phi}_n$ results in a Dirac delta function $\delta(\rho_{n\sigma} - \rho_{n-1\sigma})$. Thus, when integrating over the corresponding $\boldsymbol{\rho}$ variables, we end up fixing every $\boldsymbol{\rho}_n$ to be the same as the one at the last time bond. The integral over φ, that only appears at the boundary, yields a Kronecker Delta $\delta_{n_\sigma,\rho_\sigma(0+0)}$, thus fixing the start point of ρ to $\mathbf{n}$ and we see that $\boldsymbol{\rho}(\tau) = \mathbf{n}$ at all τ. Thus, integrating over all $\boldsymbol{\phi}$ and $\boldsymbol{\rho}$ simply reduce to 1. Remembering that the Schwinger boson number constraint applies to $\mathbf{n}$, we can write $n_\uparrow = n$ and $n_\downarrow = 1 - n$. The sum over $\mathbf{n}$, reduce to a sum over n, and upon doing the summation we see that the zeroth order contribution to the partition function is just

$$\mathcal{Z}^{(0)} = 2. \tag{5.23}$$

Now, we are ready to go to first order. The difference from before is that we now have the integral over $\mathcal{H}$ included

$$\mathcal{Z}^{(1)} = \sum_{\mathbf{n}=0}^{\infty} \int_0^{2\pi} \frac{\mathrm{d}\boldsymbol{\varphi}}{(2\pi)^2} \oint \mathcal{D}[\rho(\tau), \phi(\tau)] e^{i\int_0^\beta \phi(\tau)\dot{\rho}(\tau)} \int_0^\beta \mathrm{d}\tau_1\, \mathcal{H}(\tau_1). \tag{5.24}$$

By writing the cosines as sums of exponentials, we get factors of $e^{i\phi(\tau_1)(\rho(\tau_1+0)-\rho(\tau_1-0)\pm 1)}$ where before there were no ± 1. Thus when integrating over ϕ, we get unit jumps of ± 1 at τ_1. The result of doing the ϕ and ρ integrals is thus that $\rho(\tau_1 + 0) = \rho(\tau_1 - 0) \pm 1$. Note that ρ is a restricted to the interval from 0 to 1, and it has to be a periodic function such that if $\rho = 1$, then $\rho + 1 = 0$.

The first order term will not contribute to the partition function. The reason for this is that we should have $\rho(\tau)$ starting and ending in n. However, if we have one jump in ρ, there is no way to start and end at the same point. The same argument holds for all odd number terms in the expansion, they all give zero.

Now, we move on to the second order term. Doing the $\boldsymbol{\phi}$ and $\boldsymbol{\rho}$ integrals yield unit jumps again, however this time we get jumps at both τ_1 and τ_2. We write $\rho_\uparrow(\tau) = \rho(\tau)$, and $\rho_\downarrow(\tau) = 1 - \rho(\tau)$ We will only get contributions whenever ρ starts and ends up at the same value. Note that the contribution from a jump from 1 to $1 + 1$ will give zero because

$$\gamma(1, 1+1) = \sqrt{1+1} = 0, \tag{5.25}$$

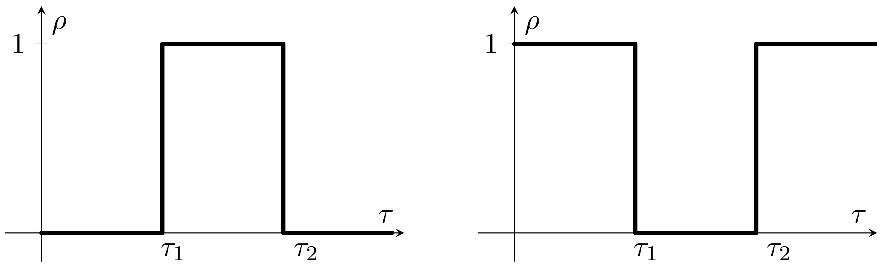


Figure 5.1: Legal paths for ρ in the second order of the high temperature expansion of the partition function for a Hamiltonian $\hat{H} = \omega \hat{S}_x$.

as the argument in the square root is supposed to be periodic with a period of 1. A similar argument shows that the jump from 0 to $0-1$ also gives a zero γ. Thus, the only times we get contributions is for the paths shown in Figure 5.1. In the case of the legal paths, we get

$$\mathcal{Z}^{(2)} = \frac{\omega^2}{4} \int_0^\beta \mathrm{d}\tau_1 \int_0^\beta \mathrm{d}\tau_2 \, [\gamma(0,1)\gamma(1,0)]^2 = \frac{\beta^2\omega^2}{4}. \tag{5.26}$$

We can then look at a general even order term. We we will get one factor 2 from the fact that there is two ways to start and end in the same point that gives non-zero contributions, by starting in either 0 or 1 and then have alternating jumps. The γ factors from this will all be 1. Thus, in general the nth order term will be

$$\mathcal{Z}^{(n)} = \begin{cases} 2\left(\frac{\beta\omega}{2}\right)^n \frac{1}{n!}, & n \text{ even} \\ 0, & n \text{ odd}. \end{cases} \tag{5.27}$$

Thus, we can finally find the full partition function

$$\mathcal{Z} = 2\sum_{n=0}^{\infty} \frac{(\beta\omega)^{2n}}{4^n (2n)!} = 2\cosh\left(\frac{\beta\omega}{2}\right), \tag{5.28}$$

which indeed is the correct result [12].

5.2.2 Two spins with a Heisenberg interaction

The next example we will consider is more complicated. We consider two spin-1/2 particles with a Heisenberg interaction between them. We then need to generalise the projection operator to include two sites

$$\hat{P} = \hat{P}_1 \hat{P}_2, \tag{5.29}$$

where $\hat{P}_i$ projects down to the physical subspace on site i. The derivation of the Bruckmann-Urbina path integral for this system follows exactly in the same way as above, and the path integral take the form

$$\mathcal{Z} = \sum_{\mathbf{n}=0}^{\infty} \int_0^{2\pi} \frac{\mathrm{d}\boldsymbol{\varphi}}{(2\pi)^4} \oint \mathcal{D}[\rho(\tau), \phi(\tau)] e^{-S[\rho(\tau), \phi(\tau)]}. \tag{5.30}$$

The only difference, is that now $\boldsymbol{\rho}$ and $\boldsymbol{\phi}$ have an extra index for site. The Hamiltonian of this system can be written as

$$\hat{H} = J \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2. \tag{5.31}$$

We can write the spin operator vector in terms of the Schwinger boson creation and annihilation operators, do the dot product and anti-normal order the resulting string of operators. We then replace the operators by the corresponding $z_{i\sigma}$, and then write the zs in polar coordinates. The result is the function h from the P-representation, and in this case it is given by

$$\begin{aligned} h(r, \phi) = J(S^2 - 1) + \frac{J}{2} \Bigg(& \sum_{i,\sigma} r_{i\sigma} - r_{1\uparrow} r_{2\downarrow} - r_{1\downarrow} r_{2\uparrow} \\ & + 2\sqrt{r_{1\uparrow} r_{2\uparrow} r_{1\downarrow} r_{2\downarrow}} \cos(\phi_{1\uparrow} - \phi_{1\downarrow} + \phi_{2\downarrow} - \phi_{2\uparrow}) \Bigg). \end{aligned} \tag{5.32}$$

Next, we will calculate $\mathcal{H}$. Note that since there is only one term in h that has any ϕ dependence, we can split $\mathcal{H}$ into two parts, one due to the spin exchange interaction $\mathcal{H}_{\mathrm{ex}}$, and one with the rest $\mathcal{H}_0$. All the ϕ dependence is contained in $\mathcal{H}_{\text{ex}}$. By using the definition of the Gamma function, we get the rather lengthy expression for the two $\mathcal{H}$ parts

$$\begin{aligned}
\mathcal{H}_0(\rho,\rho') = \frac{J}{2c}\Bigg[&-\frac{3}{2}\prod_{i\sigma}\Gamma\Big(\frac{\rho_{i\sigma}+\rho'_{i\sigma}}{2}+1\Big) \\
&+\sum_{i\sigma}\Gamma\Big(\frac{\rho_{i\sigma}+\rho'_{i\sigma}}{2}+2\Big)\prod_{i'\sigma'\neq i\sigma}\Gamma\Big(\frac{\rho_{i'\sigma'}+\rho'_{i'\sigma'}}{2}+1\Big) \\
&-\frac{J}{2c}\Gamma\Bigg(\frac{\rho_{1\uparrow}+\rho'_{1\uparrow}}{2}+2\Bigg)\Gamma\Bigg(\frac{\rho_{2\downarrow}+\rho'_{2\downarrow}}{2}+2\Bigg)\prod_{i\sigma\neq 1\uparrow,2\downarrow}\Gamma\Big(\frac{\rho_{i\sigma}+\rho'_{i\sigma}}{2}+1\Big) \\
&-\frac{J}{2c}\Gamma\Bigg(\frac{\rho_{1\downarrow}+\rho'_{1\downarrow}}{2}+2\Bigg)\Gamma\Bigg(\frac{\rho_{2\uparrow}+\rho'_{2\uparrow}}{2}+2\Bigg)\prod_{i\sigma\neq 1\downarrow,2\uparrow}\Gamma\Big(\frac{\rho_{i\sigma}+\rho'_{i\sigma}}{2}+1\Big), \\
\mathcal{H}_{\text{ex}}(\rho,\rho') = J\cos(&\phi_{1\uparrow}-\phi_{1\downarrow}+\phi_{2\downarrow}-\phi_{2\uparrow})\prod_{i\sigma}\gamma(\rho_{i\sigma},\rho'_{i\sigma}),
\end{aligned} \tag{5.33}$$

where we have defined

$$c=\prod_{i\sigma}\sqrt{\Gamma(\rho_{i\sigma}+1)\Gamma(\rho'_{i\sigma}+1)}. \tag{5.34}$$

All of the sums and products are taken over $i=1,2$ and $\sigma=\uparrow,\downarrow$. The plan now is similar to the previous example, we expand the exponential with the Hamiltonian in powers of $\int_0^\beta(\mathcal{H}_0+\mathcal{H}_{\text{ex}})$. To zeroth order, we get in a similar way to before, a lot of delta functions constraining $\boldsymbol{\rho}$ to $\mathbf{n}$. Again, due to the Schwinger boson number constraint at both sites, we may write $n_{i\uparrow}=n_i$ and $n_{i\downarrow}=1-n_i$, and the summation over $\mathbf{n}$ reduce to sums over n_1 and n_2. We end up with

$$\mathcal{Z}^{(0)}=\sum_{n_1=0}^{1}\sum_{n_2=0}^{1}1=4. \tag{5.35}$$

To first order, we get one integral over $\mathcal{H}$. The $\mathcal{H}_{\text{ex}}$ part will give zero, as we have jumps of ±1 in the $\boldsymbol{\rho}$ variables only once, and thus $\boldsymbol{\rho}$ at the start and end can not be the same. The $\mathcal{H}_0$ term also gives zero, but the reason is a bit more subtle. As there is no $\boldsymbol{\phi}$ dependence, $\boldsymbol{\rho}$ will just be equal to $\mathbf{n}$ the entire time. Thus, all of the Gamma functions reduce to factorials that cancels out the $\frac{1}{c}$ factor. We end up, after some algebra, with

$$\begin{aligned}\mathcal{Z}_0^{(1)} = \frac{\beta J}{2} \sum_{n_1,n_2=0}^{1} \Big[-\frac{3}{2} + (n_1+1) + (2-n_1) + (n_2+1) + (2-n_2) \\ - (n_1+1)(2-n_2) - (2-n_1)(n_2+1) \Big] = 0,\end{aligned} \tag{5.36}$$

where the last equality follows from the fact that all of the terms in parenthesis are periodic such that $1+1=0$ etc. To second order we consider the integral over $\mathcal{H}_0(\tau_1)\mathcal{H}_0(\tau_2)$, and similarly for the exchange term. The cross term will yield zero, due to the fact that we only get a single jump in each of the $\boldsymbol{\rho}$ variables. The $\mathcal{H}_0$ part gives

$$\begin{aligned}\mathcal{Z}_0^{(2)} &= \frac{\beta^2 J^2}{8} \sum_{n_{1,2}=0}^{1} \Bigg[(n_1+1) + (2-n_1) + (n_2+1) + (2-n_2) \\ &\qquad - (n_1+1)(2-n_2) - (2-n_1)(n_2+1) - \frac{3}{2} \Bigg]^2 \\ &= \frac{\beta^2 J^2}{8}.\end{aligned} \tag{5.37}$$

In the calculation above, we have to be careful to remember that the expressions in parentheses are periodic as discussed earlier. For the part with $\mathcal{H}_{\text{ex}}$, we get in a similar way to before, jumps in the ρs at times τ_1 and τ_2, and only the contributions starting starting and ending in n is included. The term takes the form

$$\mathcal{Z}_{\text{ex}}^{(2)} = \frac{J^2\beta^2}{4} (\gamma(0,1)\gamma(1,0)\gamma(0,1)\gamma(1,0))^2 = \frac{J^2\beta^2}{4}. \tag{5.38}$$

We can then add up the two terms, and to second order in $J\beta$, we get

$$\mathcal{Z} = 4 + \frac{3J^2\beta^2}{8} + \mathcal{O}\Big((\beta J)^4\Big). \tag{5.39}$$

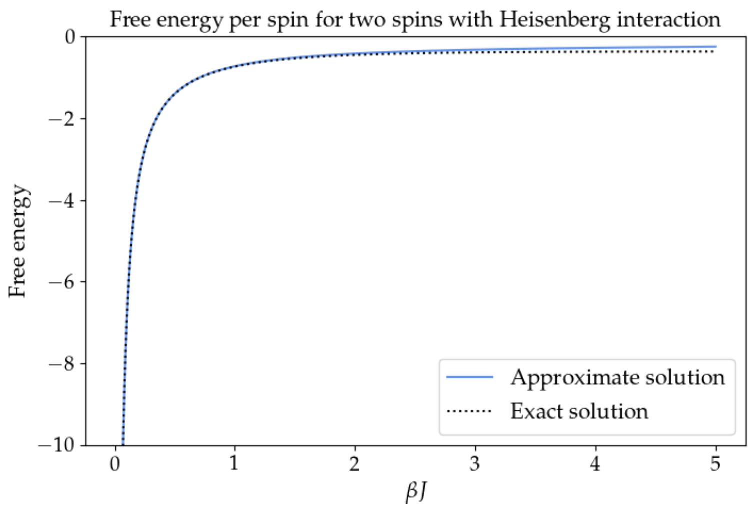


Figure 5.2: The exact and approximate free energy per spin for two spins with a mutual Heisenberg interaction.

We can check our calculation by remembering that the states $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$ and $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle)$ span the total Hilbert space, and we get

$$\mathcal{Z} = \operatorname{tr} e^{-\beta \hat{H}} = 3e^{-\frac{\beta J}{4}} + e^{3\frac{\beta J}{4}}. \tag{5.40}$$

To check how good of an approximation this is, we plot the free energy per spin for the approximate and exact solution, as can be seen in Figure 5.2. The free energy is given by $F = -\frac{1}{\beta} \ln \mathcal{Z}$ [26]. Going beyond second order is possible, but unlike the previous example, this will be calculational difficult due to the fact that there will be more and more cross terms contributing at each order.

5.3 1-dimensional Ising rings

5.3.1 Generalising to a lattice

We will here generalise the Bruckmann-Urbina path integral to a lattice of M spins. The first thing we need is the projection operator that projects any state down to the physical subspace at each lattice site. This is simply the product of the projection operators at each site, and we may write it as

$$\hat{P} = \prod_{i=1}^{M} \sum_{l_i=0}^{2S} \frac{1}{l_i!(2S-l_i)!} \left(a_{i\uparrow}^{\dagger}\right)^{l_i} \left(a_{i\downarrow}^{\dagger}\right)^{2S-l_i} : e^{-\sum_{i\sigma} \hat{n}_{i\sigma}} : a_{i\uparrow}^{l_i} a_{i\downarrow}^{2S-l_i}. \tag{5.41}$$

All of the steps in the derivation follow the single spin case, but there are a couple of generalisations. The matrix elements of the projection operator is now

$$\langle z_n|\hat{P}|z_{n-1}\rangle = \prod_{i=1}^{M} \frac{1}{(2S)!} \frac{\partial^{2S}}{\partial \eta_{ni}^{2S}} e^{\sum_i \eta_{ni} z_{ni}^* z_{n-1i}}. \tag{5.42}$$

We follow all the steps as before, do the duality transform and Poisson summation, and what we end up with is an expression completely equal to eq. (5.16), only that now the vectors $\boldsymbol{\rho}$ and $\boldsymbol{\phi}$ have an additional index for lattice site. Also, the Schwinger boson number constraint is present at each site, such that $\rho_{i\uparrow}(\tau) + \rho_{i\downarrow}(\tau) = 2S$ for all i.

5.3.2 Longitudinal field

We will here show how we can extend the theory of the Bruckmann and Urbina path integral to 1-dimensional Ising systems. We start by looking at a system of M spin-1/2 particles in a longitudinal field. The Hamiltonian is given by

$$\hat{H} = \sum_{i=1}^{N} \left(J\hat{S}_i^z \hat{S}_{i+1}^z + B\hat{S}_i^z\right), \tag{5.43}$$

where $\hat{S}_{M+1}^z = \hat{S}_1^z$, such that the spins form a ring. In a similar fashion to earlier in this chapter, we anti-normal order the Hamiltonian and replace the creation/annihilation operators by the corresponding z, and after writing this in the polar form, we get

$$\begin{aligned} h(r) = \sum_{i=1}^{N} \Bigg[& \frac{J}{4}(r_{i\uparrow}r_{i+1\uparrow} + r_{i\downarrow}r_{i+1\downarrow} - r_{i\uparrow}r_{i+1\downarrow} - r_{i\downarrow}r_{i+1\uparrow}) \\ & + \frac{B}{2}(r_{i\uparrow} + r_{i\downarrow}) \Bigg]. \end{aligned} \tag{5.44}$$

Note that there is no ϕ-dependence in h, which means that we can simply integrate out ϕ and ρ, and the result is simply that $\boldsymbol{\rho} = \mathbf{n}$ at all τ. The next step is to calculate the "Laguerre" Hamiltonian $\mathcal{H}$. We do the integrals over r, and by remembering the Schwinger boson number constraint $n_{i\uparrow} + n_{i\downarrow} = 1$ we have imposed, we can write $n_{i\uparrow} = n_i$ and $n_{i\downarrow} = 1 - n_i$. The Hamiltonian is then

$$\begin{aligned} \mathcal{H} &= \sum_{i=1}^{N} \Bigg[\frac{J}{4}(n_i n_{i+1} + (1-n_i)(1-n_{i+1}) - n_i(1-n_{i+1}) - (1-n_i)n_{i+1}) \\ &\qquad + \frac{B}{2}(2n_i - 1) \Bigg] \\ &= \sum_{i=1}^{N} \left[\frac{J}{4} s_i s_{i+1} + \frac{B}{2} s_i \right], \end{aligned} \tag{5.45}$$

where we have introduced a new variable $s_i = 2n_i - 1$. Note, that the sums over n_i is now replaced by sums over $s_i = \pm 1$. Inserting all that we have done here in the general expression for the Bruckmann-Urbina partition function of eq. (5.16), we get

$$\mathcal{Z} = \sum_{\mathbf{s}=\pm 1} e^{\beta \sum_i \left(\frac{J}{4} s_i s_{i+1} + B s_i \right)}. \tag{5.46}$$

Solving this is a standard exercise in undergraduate statistical mechanics, and can be done using the tranfer matrix method. We will not go through it here, but the interested reader can find the derivation in chapter 4.9 of [26]. We include the result for completeness, and we find in the limit of large M that the partition function takes the form

$$\mathcal{Z} = \left[e^{\beta J} + \sqrt{e^{-2\beta J} + e^{2\beta J}\sinh^2(\beta B)}\right]^M. \tag{5.47}$$

5.3.3 Transversal field

This time we study a Ising model with a field in the x-direction. The Hamiltonian changes slightly to

$$\hat{H} = \sum_{i=1}^{N}\Big(J\hat{S}_i^z\hat{S}_{i+1}^z + B\hat{S}_i^x\Big). \tag{5.48}$$

Furthermore, the P-representation Hamiltonian h, is now no longer independent of ϕ, due to the field being transversal. It instead takes the form

$$\begin{aligned} h(r,\phi) = \sum_{i=1}^{N}\Bigg[&\frac{J}{4}(r_{i\uparrow}r_{i+1\uparrow} + r_{i\downarrow}r_{i+1\downarrow} - r_{i\uparrow}r_{i+1\downarrow} - r_{i\downarrow}r_{i+1\uparrow}) \\ &+ \frac{B}{2}\sqrt{r_{i\uparrow}r_{i\downarrow}}\Big(e^{i(\phi_{i\uparrow}-\phi_{i\downarrow})} + e^{i(\phi_{i\downarrow}-\phi_{i\uparrow})}\Big)\Bigg]. \end{aligned} \tag{5.49}$$

This complicates our expression for $\mathcal{H}$, as we can no longer just do the $\boldsymbol{\phi}$ integrals to fix $\boldsymbol{\rho}$. Instead, we have to do as before, and expand in powers of the integral of $\mathcal{H}$. Similar to the two-spin Heisenberg problem, we split

$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{ex}}$ the following way

$$
\begin{aligned}
\mathcal{H}_0(\boldsymbol{\rho}, \boldsymbol{\rho}') = \sum_{i=1}^{M} \frac{\beta J}{4c} \Bigg[& \sum_{\sigma} \Gamma\left(\frac{\rho_{i\sigma} + \rho'_{i\sigma}}{2} + 2\right) \Gamma\left(\frac{\rho_{i+1\sigma} + \rho'_{i+1\sigma}}{2} + 2\right) \\
& \times \prod_{j\sigma' \neq i\sigma, i+1\sigma} \Gamma\left(\frac{\rho_{j\sigma'} + \rho'_{j\sigma'}}{2} + 1\right) \\
& - \sum_{\sigma} \Gamma\left(\frac{\rho_{i\sigma} + \rho'_{i\sigma}}{2} + 2\right) \Gamma\left(\frac{\rho_{i+1\bar{\sigma}} + \rho'_{i+1\bar{\sigma}}}{2} + 2\right) \\
& \times \prod_{j\sigma' \neq i\sigma, i+1\sigma} \Gamma\left(\frac{\rho_{j\sigma'} + \rho'_{j\sigma'}}{2} + 1\right) \Bigg] \\
\mathcal{H}_{\text{ex}}(\boldsymbol{\rho}, \boldsymbol{\rho}', \boldsymbol{\phi}) = \sum_{i=1}^{M} \frac{\beta B}{2} & \gamma(\rho_{i\uparrow}, \rho'_{i\uparrow}) \gamma(\rho_{i\downarrow}, \rho'_{i\downarrow}) \left[e^{i(\phi_{i\uparrow} - \phi_{i\downarrow})} + e^{i(\phi_{i\downarrow} - \phi_{i\uparrow})} \right] \\
& \times \prod_{j\sigma' \neq i\sigma, i+1\sigma} \Gamma\left(\frac{\rho_{j\sigma'} + \rho'_{j\sigma'}}{2} + 1\right).
\end{aligned}
$$

Here, we have defined c as in eq. (5.34), and $\bar{\sigma}$ is defined to be the opposite of σ. Now we do the Taylor expansion, and to zeroth order we get

$$
\mathcal{Z}^{(0)} = \sum_{\mathbf{n}=0}^{1} 1 = 2^M. \tag{5.50}
$$

To first order, $\mathcal{H}_{\text{ex}}$ will give one unit jump of ± 1, which can not contribute. The contribution from $\mathcal{H}_0$ is also zero. The calculation is similar to the one for the first order calculation for the two-spin Heisenberg case. To second order, we get one contribution from $\mathcal{H}_{\text{ex}}$ with legal jumps, and also from $\mathcal{H}_0$ without any jumps. The two contributions are given by

$$
\begin{aligned}
\mathcal{Z}_0^{(2)} &= \frac{\beta^2 J^2}{32} \sum_{\mathbf{n}=0}^{1} \left[\sum_i (2n_i - 1)^2 (2n_{i+1} - 1)^2 \right] = \frac{\beta^2 J^2 M}{16} 2^M \\
\mathcal{Z}_{\text{ex}}^{(2)} &= \frac{\beta^2 B^2}{8} \sum_{i=1}^{M} \left[2\gamma(1,0)\gamma(0,1)\gamma(0,1)\gamma(1,0) \right] = \frac{\beta^2 B^2 M}{8} 2^M.
\end{aligned} \tag{5.51}
$$

Thus, to second order in β, we get

$$\mathcal{Z} \approx 2^M \left(1 + \frac{\beta^2 J^2 M}{16} + \frac{\beta^2 B^2 M}{8}\right). \tag{5.52}$$

To check our calculation we will calculate the free energy per spin. For the transverse field Ising model we can actually calculate the free energy exactly, and the result is [27]

$$\begin{aligned} F &= -\frac{M}{\beta}\left(\ln 2 + \frac{1}{\pi}\int_0^\pi \mathrm{d}k \ln\cosh\left(\frac{1}{2}\beta B \Lambda(k)\right)\right) \\ \Lambda(k)^2 &= 1 + \left(\frac{J}{2B}\right)^2 + \frac{J}{B}\cos(k). \end{aligned} \tag{5.53}$$

In Figure 5.4, the approximate and exact free energy per spin is plotted as a function of inverse temperature for three different field strengths. We see that in all cases the high temperature behaviour is correct, however the approximation works best when the strength of the field is about the same as the interaction between the spins.

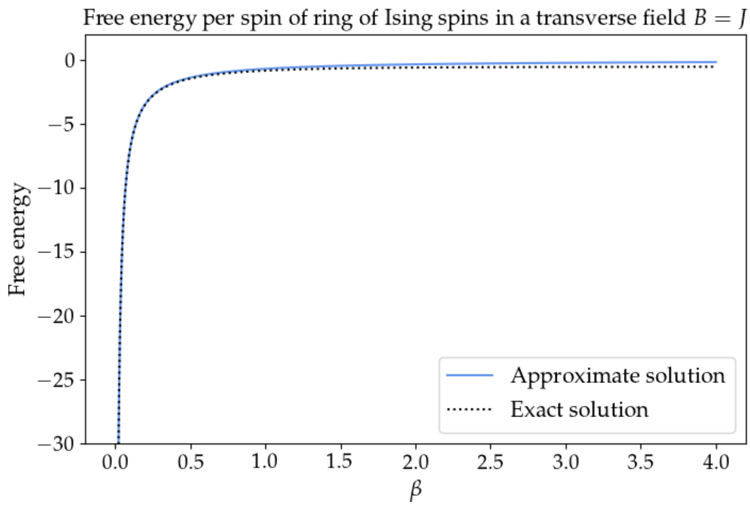
5.4 Real time propagator for a single spin

We will here explore the Bruckmann and Urbina construction of the path integral in a slightly different setting, namely we will write the path integral representation of the real time propagator of a single spin-S particle. We start from the general expression for the propagator

$$U(z_N, z_0) = \langle z_N | e^{-i\hat{H}t} | z_0 \rangle . \tag{5.54}$$

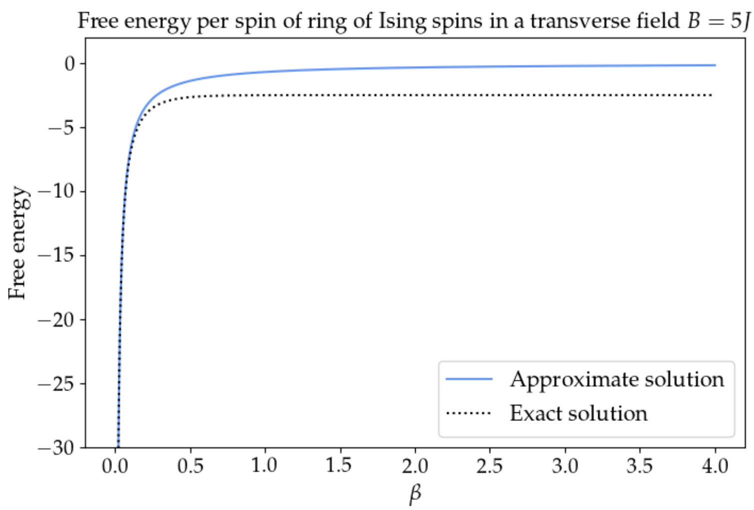
Now, by splitting up the exponential into N parts of size $\varepsilon = \frac{t}{N}$, and inserting a projection operator such that all of the states are projected down to the physical subspace, we get

$$\begin{aligned} U(z_N, z_0) &= \langle z_N | \hat{P} e^{-i\hat{H}\varepsilon} \hat{P} e^{-i\hat{H}\varepsilon} \dots e^{-i\hat{H}\varepsilon} \hat{P} | z_0 \rangle \\ &= \int \prod_{n=1}^{N-1} \frac{\mathrm{d}^4 z_n}{\pi^2} e^{-\sum_\sigma |z_{n\sigma}|^2 - i\varepsilon h(z_n^*, z_n)} \langle z_n | \hat{P} | z_{n-1} \rangle . \end{aligned} \tag{5.55}$$



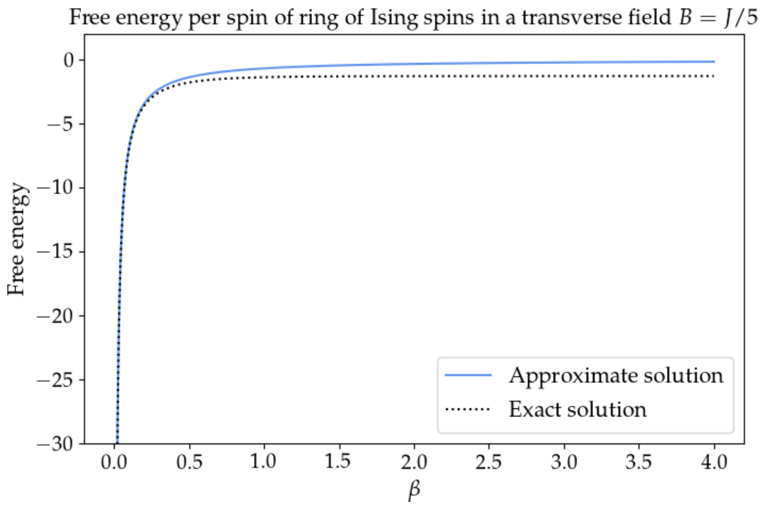
Free energy per spin of ring of Ising spins in a transverse field $B = J$
Free energy
0
−5
−10
−15
−20
−25
−30
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
β
Approximate solution
Exact solution

(a)



Free energy per spin of ring of Ising spins in a transverse field $B = 5J$
Free energy
0
−5
−10
−15
−20
−25
−30
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
β
Approximate solution
Exact solution

(b)



(c)

Figure 5.4: The exact and approximate free energy per spin for a ring of 1000 Ising spins in transverse fields (a) $B = J$, (b) $B = 5J$ and (c) $B = \frac{J}{5}$.

The next thing we need to do is to calculate the matrix elements of the projection operator. Similar to what we have done earlier, we get

$$\langle z_n|\hat{P}|z_{n-1}\rangle = \frac{1}{(2S)!}\frac{\partial^{2S}}{\partial\eta_n^{2S}} e^{\eta_n \sum_\sigma z^*_{n\sigma} z_{n-1\sigma}}\bigg|_{\eta_n=0}. \tag{5.56}$$

We can then write down an expression for the propagator

$$\begin{aligned} U(z_N, z_0) &= \partial_\eta \int \mathcal{D}[z] e^{-S[z^\dagger,z,\eta]}\bigg|_{\eta_n=0} \\ S[z^\dagger, z, \eta] &= \sum_{n=1}^{N-1}\Big[z_n^\dagger z_n + i\varepsilon h(z_n^\dagger, z_n) - \eta_n z_n^\dagger z_{n-1}\Big], \end{aligned} \tag{5.57}$$

where ∂_η is defined in eq. (5.5). The derivation follows closely to the derivation of the partition function. We do the duality transform, Poisson summation and write $z_{n\sigma}$ in the polar form. After relabelling the variables in the same way as Bruckmann and Urbina, we get

$$U(z_N, z_0) = \int \mathcal{D}[\phi, \rho] e^{-i\int_0^T \mathrm{d}t\big(\boldsymbol{\rho}(t)\dot{\boldsymbol{\phi}}(t) + \mathcal{H}(\boldsymbol{\rho}(t-0), \boldsymbol{\rho}(t+0); \boldsymbol{\phi}(t))\big)}, \tag{5.58}$$

which is exact. Here, we have again introduced $\mathcal{H}$ as in eq. (5.15).

5.4.1 $\hat{H} = 0$ case

We start with a trivial example, where the Hamiltonian of the spin is just zero. In this case, we obviously have $\mathcal{H} = 0$. By using the Schwinger boson number constraint on $\boldsymbol{\rho}$, we may parametrise the components using a variable $\theta \in [0, \pi]$ as

$$\begin{aligned} \rho_\uparrow(t) &= 2S\cos^2(\theta(t)/2) \\ \rho_\downarrow(t) &= 2S\sin^2(\theta(t)/2). \end{aligned} \tag{5.59}$$

The action takes the form

$$\begin{aligned} S[\theta, \phi] &= -i\int_0^T \mathrm{d}t\, 2S(\cos^2(\theta/2)\dot{\phi}_\uparrow + \sin^2(\theta/2)\dot{\phi}_\downarrow) \\ &= i\int_0^T \mathrm{d}t\, 2S\Big(\frac{1}{2}\cos(\theta) - \frac{1}{2}\Big)\dot{\phi} \end{aligned} \tag{5.60}$$

Where we have defined $\phi_\uparrow - \phi_\downarrow = \phi$. Interestingly, we have a non-zero action even though the Hamiltonian is zero. This is the well-known Berry phase [23].

5.4.2 $\hat{H} = B \cdot \hat{S}$ case

Next we have the case of a spin-S particle in a magnetic field. We will consider a large S limit, i.e. the classical limit of the system, and show that what we get out is the action of a classical spin in a magnetic field. The function h takes the form

$$h(z^*, z) = \mathbf{B} \cdot \left[\begin{pmatrix} z_\uparrow^* \\ z_\downarrow^* \end{pmatrix} \boldsymbol{\sigma} \begin{pmatrix} z_\uparrow & z_\downarrow \end{pmatrix} \right] \tag{5.61}$$

We can then calculate $\mathcal{H}$, and we get

$$\begin{aligned} \mathcal{H}(\boldsymbol{\rho}, \boldsymbol{\rho}') = & \left[\prod_\sigma \frac{\Gamma\left(\frac{\rho_\sigma + \rho'_\sigma + 3}{2}\right)}{\sqrt{\Gamma(\rho_\sigma + 1)\Gamma(\rho'_\sigma + 1)}} \right] \\ & \times \left[e^{-i(\phi_\uparrow - \phi_\downarrow)} \frac{B_x + iB_y}{2} + e^{-i(\phi_\downarrow - \phi_\uparrow)} \frac{B_x - iB_y}{2} \right] \\ & + \frac{B_z}{2} \left[\frac{\Gamma\left(\frac{\rho_\uparrow + \rho'_\uparrow + 4}{2}\right)}{\sqrt{\Gamma(\rho_\uparrow + 1)\Gamma\left(\rho'_\uparrow + 1\right)}} \frac{\Gamma\left(\frac{\rho_\downarrow + \rho'_\downarrow + 2}{2}\right)}{\sqrt{\Gamma(\rho_\downarrow + 1)\Gamma\left(\rho'_\downarrow + 1\right)}} \right. \\ & \left. + \frac{\Gamma\left(\frac{\rho_\uparrow + \rho'_\uparrow + 2}{2}\right)}{\sqrt{\Gamma(\rho_\uparrow + 1)\Gamma\left(\rho'_\uparrow + 1\right)}} \frac{\Gamma\left(\frac{\rho_\downarrow + \rho'_\downarrow + 4}{2}\right)}{\sqrt{\Gamma(\rho_\downarrow + 1)\Gamma\left(\rho'_\downarrow + 1\right)}} \right] \end{aligned} \tag{5.62}$$

To make progress, we will do an expansion in $\int_0^T \mathrm{d}t \, i\mathcal{H}$. We do the same parametrisation of $\boldsymbol{\rho}$ as in eq. 5.59, and note that in the limit of large S, we have

$$\gamma(\rho_\uparrow, \rho_\uparrow + 1)\gamma(\rho_\downarrow, \rho_\downarrow - 1) \approx 2S \cos\frac{\theta}{2} \sin\frac{\theta}{2}. \tag{5.63}$$

The technique we will use is similar to what we have been doing the rest of this chapter. However, by defining a spin vector

$$\mathbf{S} = 2S\begin{pmatrix} e^{-i\phi}\cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix}\boldsymbol{\sigma}\begin{pmatrix} e^{i\phi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix},$$

we can actually see that the nth term in the series expansion of the integral of $\mathcal{H}$ is on the form $\frac{1}{n!}(\mathbf{B}\cdot\mathbf{S})^n$, as long as the approximation of eq. (5.63) is valid. Thus, the expression for the propagator takes the form

$$U = \int \mathcal{D}[\theta,\phi]e^{-S_s} \tag{5.64}$$

with the action

$$S_s = i\int \mathrm{d}t\, 2S\left(\frac{1}{2}\cos\theta + \frac{1}{2}\right)\dot{\phi} - \mathbf{B}\cdot\mathbf{S}. \tag{5.65}$$

This is indeed the action of a classical spin. A similar calculation can be done by using the path integral over spin coherent states, see chapter 2.3.4 of [21].

Chapter 6

Summary and outlook

In this thesis we have studied how a projection operator can be used to implement constraints in coherent state path integrals. After a brief overview of the theory of coherent states and the textbook approach to coherent state path integrals, we derived a bosonic and fermionic projection operator. We then saw how this could be used to calculate the partition function of a general quadratic Schwinger boson or Abrikosov fermion Hamiltonian. In particular, using the Schwinger boson formalism, we calculated the single spin function of a spin-S particle with a zero energy, and Zeeman energy Hamiltonian. Furthermore, using the Abrikosov fermion formalism, we calculated the single spin partition function for a two-level systems using two different methods.

We then showed how the path integral of Bruckmann and Urbina can be constructed for spin systems, using the projection operator to more rigorously implement the Schwinger boson constraint. The partition function of a Hamiltonian proportional to $\hat{S}_x$ was calculated exactly, and we showed how to make a high temperature expansion of the partition function by calculating the partition function of a system of two spins with a Heisenberg interaction. Furthermore, we generalised the formalism to a system of many spins, and showed how to calculate the partition function of a

Ising ring in a longitudinal field. The case of a transversal field was also studied, and a high temperature expansion of the partition function was calculated. Finally, we show how the framework of Bruckmann and Urbina can be extended to a real time propagator, and verify that even for a zero Hamiltonian the action picks up a Berry phase. We also verify that in the limit of $S \to \infty$, we recover the action of a classical spin in a magnetic field.

The most obvious extension of this work, is to extend the path integral of Chapter 4 to multiple spins. This has, however, proved difficult. For an n-spin system, the matrices M_σ acquire a block structure where each block is of size $n \times n$. In practise, this makes calculations almost impossible. However, there might still be some clever way of doing approximations using this formalism. It would also be interesting to find a continuous time version of this path integral.

The work of Bruckmann and Urbina have unfortunately not received much attention in the scientific community. Thus, the possibilities of their path integral construction have not been sufficiently investigated. In particular, it would be interesting to see if other approximation schemes, beyond the high temperature expansion explored in this work, is possible.

Appendix A

Useful mathematical relations

In this appendix we give some additional useful mathematical relations used throughout the main text without proof. Instead, references where more information can be found is included.

A.1 Gaussian integrals

The usual 1 dimensional Gausian integral

$$\int_{-\infty}^{\infty} e^{-\frac{a}{2}x^2}\,\mathrm{d}x = \sqrt{\frac{2\pi}{2}}, \tag{A.1}$$

holds true for any a with a real part greater than zero. We can generalise this to an integral over N complex variables the following way

$$\int \prod_{n=1}^{N} [\mathrm{d}\,\mathrm{Re}(z_n)\,\mathrm{d}\,\mathrm{Im}(z_n)] e^{-\mathbf{z}^\dagger M\mathbf{z}+\mathbf{u}^\dagger\mathbf{z}+\mathbf{z}^\dagger\mathbf{v}} = \pi^N \det M^{-1} e^{\mathbf{u}M^{-1}\mathbf{v}}, \tag{A.2}$$

where $\mathbf{z}$ is the vector containing all z_i, M is some $N \times N$ matrix and $\mathbf{u}$ and v are arbitrary N dimensional vectors, see for instance chapter 3.2 of [28].

A.2 Exponentials of operators

The exponential of an operator is defined by the series expansion

$$e^{\hat{A}} = \sum_{n=0}^{N} \frac{\hat{A}^n}{n!}. \tag{A.3}$$

The Suzuki-Trotter decomposition tells us we can write an exponential operators the following way

$$e^{\hat{A}+\hat{B}} = \lim_{n\to\infty} \left[e^{\frac{\hat{A}}{n}} e^{\frac{\hat{B}}{n}} \right]^n. \tag{A.4}$$

For any finite n this is only valid to $\mathcal{O}\left(\frac{1}{n^2}\right)$ [29].
The Baker-Campbell-Hausdorff formula states [30]

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]+\frac{1}{12}[\hat{A},[\hat{A},\hat{B}]]-\frac{1}{12}[\hat{B},[\hat{A},\hat{B}]]+\ldots} \tag{A.5}$$

A.3 The Γ function

The Γ function can be defined the following way [31]

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, \mathrm{d}t\,. \tag{A.6}$$

The fact that it obeys the relations $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1) = 1$, means that for any integer n

$$\Gamma(n+1) = n!, \tag{A.7}$$

and we will thus use the Γ function as a generalisation of the factorial.

A.4 Binomial and multinomial theorem

The binomial theorem theorem states that

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k}. \tag{A.8}$$

A generalisation of this result is the multinomial theorem that states[32]

$$\left(\sum_{i=1}^{q} x_i\right)^n = \sum_{k_1+k_2+\cdots+k_q=n} \frac{n!}{k_1!k_2!\ldots k_q!} \prod_{i=1}^{q} x_i^{k_i}. \tag{A.9}$$

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