Master's thesis

NTNU
Norwegian University of Science and Technology
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Particle creation in vacuum states using quantum field theory

Master's thesis in physics
Supervisor: Jens Oluf Andersen
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Abstract

In this master thesis we introduce a quantized field theory, using canonical quantization as the method. The quantized field is used to show that particles are created in states that originally are vacuum states, i.e. states that contain no particles.

Einstein's field equations are derived using the principle of least action. This is done in two ways, first using Hilbert's variational principle, and later using Palatini's variational principle. In Palatini's approach some assumptions necessary in Hilbert's variational principle are omitted, meaning that Palatini's approach is a more general way of deriving Einstein's field equations. These field equations describe how the metric respond to energy and momentum densities in a similar way to the Maxwell equations that describe how the electric and magnetic fields respond to current and charge densities.

Observers moving with constant acceleration, often called Rindler observers, are studied. Minkowski vacuum states are states that all inertial observers agree are vacuum states. However, Rindler observers will observe particles in these states, meaning that particles seemingly have been created by the acceleration of an observer. Conversely, a Rindler vacuum state will not be observed to be a vacuum state for the inertial observers. The creation of particles in this case is called the Unruh effect, and it is relatively small, as acceleration would need to be as large as $\alpha \approx 10^{20}$ m/s^2 to reach a particle density temperature of $T \approx 1$K. We also show that particles are created by a continuous expansion of the universe. Observers in the initial, pre-expansion vacuum state will observe particles after the expansion, and these particles were created solely because of the expansion of the universe.

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Conventions

The following explains the notations and conventions that is extensively used throughout this thesis.

Units

Natural units is used, meaning that $c = \hbar = k_B = G = 1$, where c is the speed of light, $\hbar$ is the reduced Planck's constant, k_B is the Boltzmann constant and G is the universal gravitational constant.

Dimensions

We will consider Minkowski space in two and four dimensions, and in section 6 we consider curved space.

Metric

The Minkowski metric used is $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

Einstein's summation convention

Whenever a lower and an upper index is repeated in the same term, Einstein's summation convention applies. The term is summed over all possible values for the index. As an example, we have

$$a_\alpha b^\alpha = \sum_{\alpha=0}^{3} a_\alpha b^\alpha. \tag{0.1}$$

The convention is also that whenever Greek indices are used (e.g. α, β), the summation is taken from 0 to 3, and when Latin indices are used (e.g. a, b), the summation is taken from 1 to 3.

Contents

1 Introduction

General relativity, formulated by Albert Einstein in the early 1900's (Ref. [1]), is a generalization of special relativity, also formulated by Einstein. Before general relativity was formulated, the accepted theory of gravity was Newton's law of universal gravitation. Newton's theory simply states that gravity is a force that acts between two bodies, where the magnitude of the force *only* depends on the masses of the two bodies and the distance between them. Einstein saw however that Newton's law of gravitation was too simple, and formulated general relativity which was observed to be more accurate than Newton's theory.

Special relativity postulates that the laws of physics are invariant in all inertial frames, and that the speed of light in vacuum is the same for all observers. These postulates implies that time is a relative quantity, i.e. a quantity that depends on the observer. General relativity generalizes special relativity, and describes gravity as a property of spacetime, where the curvature of spacetime is directly related to energy and momentum densities. The relation between energy, momentum and the curvature of spacetime is specified by Einstein's field equations, which will be derived in this thesis.

Most of this thesis considers field theory, and the classical fields needs to be quantized in order to study quantum effects. The classical field theory is quantized using canonical quantization, and treats the classical field and the time-derivative of the classical field as dynamical variables. The field variable is called the canonical coordinate, and the time-derivative variable is called the canonical momentum. This procedure of quantizing a classical field is analogous to quantizing classical mechanics into quantum mechanics.

The quantum effects studied in this thesis are the creation of particles, and in the cases studied these particles are created in vacuum states, i.e. states originally containing no particles. These effects are the main topic of this thesis, and will be studied in detail in the following sections. The quantum effect of particles being created in vacuum states is very interesting to understand, since understanding the universe is interesting and important.

2 Canonical quantization

In this section we will quantize a free, real and relativistic scalar field $\phi(\boldsymbol{x}, t)$ in Minkowski space using canonical quantization. Some of the derivations follow from Ref. [2]. The Lagrangian density $\mathcal{L}$ and Hamiltonian H corresponding to the field $\phi(\boldsymbol{x}, t)$ is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2, \\ H &= \int d^3x \left[\frac{1}{2}\Pi^2(\boldsymbol{x}, t) + \frac{1}{2}\left(\nabla\phi(\boldsymbol{x}, t)\right)^2 + \frac{1}{2}m^2\phi^2(\boldsymbol{x}, t)\right], \end{aligned} \tag{2.1}$$

where $\Pi(\boldsymbol{x}, t)$ is the canonical momentum given by $\Pi(\boldsymbol{x}, t) = \partial\mathcal{L}/\partial\dot{\phi}$. The operators ϕ and Π obey the equal time commutation relation

$$\Big[\phi(\boldsymbol{x}, t),\ \Pi(\boldsymbol{y}, t)\Big] = i\delta(\boldsymbol{x} - \boldsymbol{y}). \tag{2.2}$$

We are currently working in the Heisenberg picture, which means that the operators Π and ϕ are time dependent while the states are time independent. The field operators satisfy the equations

$$\begin{aligned} i\partial_t\phi(\boldsymbol{x}, t) &= \Big[\phi(\boldsymbol{x}, t), H\Big], \\ i\partial_t\Pi(\boldsymbol{x}, t) &= \Big[\Pi(\boldsymbol{x}, t), H\Big]. \end{aligned} \tag{2.3}$$

The right hand side in the top equation in Eq. (2.3) is easily found by inserting the expression for the Hamiltonian H and using the commutation relation in Eq. (2.2), which gives $i\Pi(\boldsymbol{x}, t)$. The commutator between the canonical momentum and the Hamiltonian is calculated in the same manner, and we find

$$\begin{aligned} \partial_t\phi(\boldsymbol{x}, t) &= \Pi(\boldsymbol{x}, t), \\ \partial_t\Pi(\boldsymbol{x}, t) &= \nabla^2\phi(\boldsymbol{x}, t) - m^2\phi(\boldsymbol{x}, t). \end{aligned} \tag{2.4}$$

Substituting one into the other gives the field equation for the scalar field operator,

$$(\Box + m^2)\phi(\boldsymbol{x}, t) = 0, \tag{2.5}$$

where $\Box$ is the d'Alembertian operator, $\Box = \partial_t^2 - \nabla^2$. Equation (2.5) is also called the Klein-Gordon equation. We will solve this field equation using the Fourier transform of the scalar field operator,

$$\phi(\boldsymbol{x}, t) = \int \frac{d^3k}{(2\pi)^3}\phi(\boldsymbol{k}, t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{2.6}$$

This Fourier transform is inserted into the Klein-Gordon equation (2.5). In Fourier space we can simply exchange the spatial derivative with $(i\boldsymbol{k})^2$, as we are simply taking the derivative of an exponential function.

$$\begin{aligned}\int \frac{d^3k}{(2\pi)^3}\Big[\partial_t^2 - \nabla^2 + m^2\Big]\phi(\boldsymbol{k},t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = 0,\\ \int \frac{d^3k}{(2\pi)^3}\Big[\partial_t^2 + \boldsymbol{k}^2 + m^2\Big]\phi(\boldsymbol{k},t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = 0.\end{aligned} \tag{2.7}$$

For the above equation to hold we get the condition

$$\partial_t^2\phi(\boldsymbol{k},t) + (\boldsymbol{k}^2 + m^2)\phi(\boldsymbol{k},t) = 0. \tag{2.8}$$

The operator $\phi(\boldsymbol{x},t)$ is a real Hermitian field operator, $\phi^\dagger(\boldsymbol{x},t) = \phi(\boldsymbol{x},t)$. Substituting Eq. (2.6) into $\phi^\dagger(\boldsymbol{x},t) = \phi(\boldsymbol{x},t)$ gives a similar condition in momentum space.

$$\phi^\dagger(\boldsymbol{k},t) = \phi(-\boldsymbol{k},t). \tag{2.9}$$

We decompose the field operator in momentum space into two terms, with each of the terms split into a time component and a momentum component where the time component is simply a complex exponential function.

$$\phi(\boldsymbol{k},t) = \phi_+(\boldsymbol{k})e^{i\omega(\boldsymbol{k})t} + \phi_-(\boldsymbol{k})e^{-i\omega(\boldsymbol{k})t}. \tag{2.10}$$

Inserting the decomposition into Eq. (2.9) gives the two conditions

$$\begin{aligned}\phi_+(\boldsymbol{k}) = \phi_-^\dagger(-\boldsymbol{k}),\\ \phi_-(\boldsymbol{k}) = \phi_+^\dagger(-\boldsymbol{k}).\end{aligned} \tag{2.11}$$

Inserting the field decomposition into the Klein-Gordon equation in momentum space (2.8) gives the frequency $\omega(\boldsymbol{k})$.

$$\omega(\boldsymbol{k}) = \sqrt{\boldsymbol{k}^2 + m^2}. \tag{2.12}$$

2.1 Defining a creation-annihilation operator pair

Now we define a creation-annihilation operator pair $a^\dagger(\boldsymbol{k})$ and $a(\boldsymbol{k})$ that obeys the generalized creation-annihilation operator algebra.

$$\begin{aligned} a^\dagger(\boldsymbol{k}) = 2\omega(\boldsymbol{k})\phi_-^\dagger(\boldsymbol{k}), \quad a(\boldsymbol{k}) = 2\omega(\boldsymbol{k})\phi_-(\boldsymbol{k}), \\ \Big[a(\boldsymbol{k}), a^\dagger(\boldsymbol{k}')\Big] = (2\pi)^3 2\omega(\boldsymbol{k})\delta^3(\boldsymbol{k}-\boldsymbol{k}'). \end{aligned} \tag{2.13}$$

Rewriting the Fourier transform of the field operator $\phi(\boldsymbol{x})$ in Eq. (2.6) using the creation-annihilation operators we just defined gives

$$\phi(\boldsymbol{x}, t) = \int \frac{d^3k}{(2\pi)^3 2\omega(\boldsymbol{k})}\Big[a(\boldsymbol{k})e^{-i[\omega(\boldsymbol{k})t-\boldsymbol{k}\cdot\boldsymbol{x}]} + a^\dagger(\boldsymbol{k})e^{+i[\omega(\boldsymbol{k})t-\boldsymbol{k}\cdot\boldsymbol{x}]}\Big]. \tag{2.14}$$

The creation and annihilation operators are normalized with the $2\omega(\boldsymbol{k})$-factor so that the phase space factor becomes $d^3k/(2\omega(\boldsymbol{k}))$, which is Lorentz invariant. Taking the time-derivative of Eq. (2.14) gives the corresponding expansion of the canonical momentum:

$$\Pi(\boldsymbol{x}, t) = -i\int \frac{d^3k}{(2\pi)^3 2\omega(\boldsymbol{k})}\omega(\boldsymbol{k})\Big[a(\boldsymbol{k})e^{-i[\omega(\boldsymbol{k})t-\boldsymbol{k}\cdot\boldsymbol{x}]} + a^\dagger(\boldsymbol{k})e^{+i[\omega(\boldsymbol{k})t-\boldsymbol{k}\cdot\boldsymbol{x}]}\Big]. \tag{2.15}$$

Both the expansion of the field operator and the canonical momentum operator consists of terms with positive frequencies, and terms with negative frequencies. The positive frequency terms have creation operators in them, while the negative frequency terms have annihilation operators. This supports the previous decomposition $\phi(\boldsymbol{x}, t) = \phi_+(\boldsymbol{x}, t) + \phi_-(\boldsymbol{x}, t)$, where $\phi_+(\boldsymbol{x}, t)$ contains the positive frequency terms and $\phi_-(\boldsymbol{x}, t)$ contains the negative frequency terms.

The Hamiltonian can also be written in terms of the operators $a(\boldsymbol{k})$ and $a^\dagger(\boldsymbol{k})$,

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega(\boldsymbol{k})}\omega(\boldsymbol{k})\Big[a(\boldsymbol{k})a^\dagger(\boldsymbol{k}) + a^\dagger(\boldsymbol{k})a(\boldsymbol{k})\Big]. \tag{2.16}$$

This Hamiltonian can be modified so that the creation operator $a^\dagger(\boldsymbol{k})$ always is to the left of the annihilation operator $a(\boldsymbol{k})$. The result will be a *normal ordered* Hamiltonian. We define a *vacuum-state* $|0\rangle$ so that

$$a(\boldsymbol{k})\,|0\rangle = 0, \tag{2.17}$$

and we further split the Hamiltonian in two parts,

$$H = H' + E_0, \tag{2.18}$$

where H' is normal ordered relative to the defined vacuum state $|0\rangle$. This implies that that H' annihilates the vacuum state,

$$H'\,|0\rangle = 0. \tag{2.19}$$

E_0 is the ground state energy and is given by

$$E_0 = \int d^3k \frac{\omega(\boldsymbol{k})}{2}\delta(0), \tag{2.20}$$

where $\delta(0)$ is given by

$$\delta(0) = \lim_{\boldsymbol{k}\to 0} \delta^3(\boldsymbol{k}) = \lim_{\boldsymbol{k}\to 0} \int \frac{d^3x}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \frac{V}{(2\pi)^3}, \tag{2.21}$$

where V is the volume of space. The ground state energy is proportional to V and is thus extensive. It can be written in terms of a ground state energy density ϵ_0 and the volume, $E_0 = \epsilon_0 V$. The density is found by comparing the two equations for E_0

$$\epsilon_0 = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \omega(\boldsymbol{k}) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \sqrt{\boldsymbol{k}^2 + m^2}, \tag{2.22}$$

where we used the previously obtained $\omega(\boldsymbol{k}) = \sqrt{\boldsymbol{k}^2 + m^2}$ (2.12) in the last step. The ground state energy density is divergent since the integrand is large for large momentum. This is an ultraviolet divergence since it diverges for large momentum.

This divergence problem can be seen in two different ways. Since all experiments give information about the finite excited energies the ground state energy is not physically observable. Thus we can neglect the ground state energy by redefining the zero of the energy. This method is sufficient when we are considering free field theory, although divergences will appear when interactions are added, requiring additional terms to counter-act new divergences. The other method to deal with the divergences is to introduce a cutoff. This makes the theory finite, but we would have to show that the physics is independent of the introduced cutoff, and this is not a trivial task.

3 Einstein's field equations

In electromagnetism we have the Maxwell equations that describe how the electric and magnetic fields respond to current and charge densities. This is analogous to Einstein's field equations, where the equations describe how the metric responds to energy and momentum densities.

The Einstein field equations will be derived using two variational principles; Hilbert's and Palatini's variational principle. The latter is more general, as we make less assumptions in this approach. We will first consider Hilbert's variational principle (also called the Einstein-Hilbert variational principle), before considering the Palatini approach.

3.1 Hilbert's variational principle

We start with Hilbert's variational principle, and some of the derivations follow from Ref. [3]. Generally the action S is given by

$$S = \int d^4x \mathcal{L} \ , \tag{3.1}$$

where $\mathcal{L}$ is the Lagrangian and is integrated over a volume in 4 dimensions. We divide the Lagrangian into two parts; the Einstein-Hilbert Lagrangian for the gravitational field $\mathcal{L}_{EH}$ and the matter Lagrangian $\mathcal{L}_M$,

$$\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_M. \tag{3.2}$$

We consider the Einstein-Hilbert Lagrangian first, and move on to the second one at the end of the section.

In general relativity the field is the metric $g_{\mu\nu}$, and thus we need the Lagrangian to include the metric. The integrand of the action must be a scalar, so the metric can not be used by itself. One quantity that depends on the metric is the Riemann tensor, and the only independent scalar that can be constructed from the Riemann tensor is the Ricci scalar. With this information at hand, Hilbert suggested the following simple form for Lagrangian for general relativity,

$$L_{EH} = R + 2\Lambda, \tag{3.3}$$

while the corresponding action is

$$S_{EH} = \int d^4x L_{EH} \sqrt{-g} \ . \tag{3.4}$$

This action is often called the Einstein-Hilbert action. In this expression we have the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$, the metric $g^{\mu\nu}$, the Ricci tensor $R_{\mu\nu}$, the cosmological constant Λ and the quantity $g = \det(g_{\mu\nu})$, the determinant of the metric. The quantity $\sqrt{-g}$ is present in order to make the integrand invariant under any arbitrary coordinate transformations. The Ricci tensor is given by

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\alpha\nu} + \Gamma^\alpha_{\alpha\beta} \Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\mu} \Gamma^\beta_{\alpha\nu}, \tag{3.5}$$

where $\Gamma^{\alpha}_{\beta\gamma}$ are the connection coefficients. In the Hilbert variational principle the connection coefficients are the Christoffel symbols, given by

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\rho} \left(\partial_\gamma g_{\rho\beta} + \partial_\beta g_{\rho\gamma} - \partial_\rho g_{\beta\gamma} \right). \tag{3.6}$$

These connection coefficients are also called metric connection. The connection coefficients are chosen in this way in order to satisfy the properties

$$\begin{aligned} \Gamma^{\alpha}_{\beta\gamma} &= \Gamma^{\alpha}_{\gamma\beta}, \\ \nabla_\alpha g_{\beta\gamma} &= 0. \end{aligned} \tag{3.7}$$

Thus in Hilbert's variational principle the only free field is the metric $g^{\mu\nu}$. The difference between this approach and Palatini's approach is that in the latter both the metric and the connection coefficients are assumed to be free independent fields, and is thus more general.

To derive the Einstein field equations from Eq. (3.4) we calculate the variation of the action with respect to the metric and set it equal to zero by the action principle. Variation acts in a similar way to that of derivatives, so we can interchange them with integrals and derivatives. The chain and product rule are also in the same way as in derivatives.

$$\delta S_{EH} = \int d^4x \Big[(\delta R)\sqrt{-g} + (R + 2\Lambda)\delta(\sqrt{-g}) \Big]. \tag{3.8}$$

The variation in the last term is relatively simple and will be considered first. Using the chain rule we find

$$\delta(\sqrt{-g}) = -\frac{1}{2} \frac{\delta g}{\sqrt{-g}}. \tag{3.9}$$

To calculate the variation of the determinant of the metric (δg) we start with Jacobi's formula (Ref. [4]), given with a general matrix and derivative as

$$\frac{d}{dx} \det A(x) = \det A(x) \mathrm{Tr}\Big[A^{-1}(x) \frac{d}{dx} A(x) \Big], \tag{3.10}$$

where $A(x)$ is a square matrix and $\mathrm{Tr}[B]$ means the trace of the matrix B. Substituting variations in place of the derivatives and $A(x)$ with the metric $g_{\mu\nu}$ gives

$$\delta(\det g_{\mu\nu}) = \det g_{\mu\nu} \mathrm{Tr}\Big[g^{\mu\nu} \delta g_{\mu\nu} \Big]. \tag{3.11}$$

Since the product inside the trace is a scalar it can be taken outside the trace, effectively removing the trace from the equation. Thus we have shown that

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu}, \tag{3.12}$$

and thus that

$$\delta(\sqrt{-g}) = \frac{1}{2} \sqrt{-g}\; g^{\mu\nu} \delta g_{\mu\nu}. \tag{3.13}$$

We next consider δR. We know that $R = g^{\mu\nu} R_{\mu\nu}$, and so we get $\delta R = (\delta g^{\mu\nu}) R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}$. The last term is inspected first. Taking the variation of the Ricci tensor gives

$$\delta R_{\mu\nu} = \partial_\alpha \delta\Gamma^\alpha_{\mu\nu} - \partial_\mu \delta\Gamma^\alpha_{\alpha\nu} + \Gamma^\alpha_{\alpha\beta}\delta\Gamma^\beta_{\mu\nu} + \delta\Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\mu}\delta\Gamma^\beta_{\alpha\nu} - \delta\Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu}. \tag{3.14}$$

In order to proceed from this we calculate the covariant derivative (defined in Eq. (A.1)) of $\delta\Gamma^\alpha_{\mu\nu}$ and compare with Eq. (3.14). Note that we set the index in the covariant derivative equal to the upper index in the connection in order to find the same terms as in Eq. (3.14).

$$\nabla_\alpha \left[\delta\Gamma^\alpha_{\mu\nu}\right] = \partial_\alpha \delta\Gamma^\alpha_{\mu\nu} + \delta\Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\alpha} - \Gamma^\alpha_{\mu\beta}\delta\Gamma^\beta_{\alpha\nu} - \delta\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}. \tag{3.15}$$

The first three terms in this expression are the same as three of the terms on Eq. (3.14). Now we calculate a similar covariant derivative to retrieve the final three terms of Eq. (3.14).

$$\nabla_\mu \left[\delta\Gamma^\alpha_{\alpha\nu}\right] = \partial_\mu \delta\Gamma^\alpha_{\alpha\nu} + \delta\Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\delta\Gamma^\beta_{\alpha\nu} - \delta\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}. \tag{3.16}$$

Adding the first covariant derivative to the minus of the second one we end up with exactly Eq. (3.14). Note that two terms cancel in the addition of the covariant derivatives. Thus we have shown that

$$\delta R_{\mu\nu} = \nabla_\alpha \left[\delta\Gamma^\alpha_{\mu\nu}\right] - \nabla_\mu \left[\delta\Gamma^\alpha_{\alpha\nu}\right]. \tag{3.17}$$

The metric $g^{\mu\nu}$ is multiplied with the above variation, since that product appears in the variation for the Ricci scalar, and the inverse product rule of the covariant derivative is used.

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\left(\nabla_\alpha \left[\delta\Gamma^\alpha_{\mu\nu}\right] - \nabla_\mu \left[\delta\Gamma^\alpha_{\alpha\nu}\right]\right) = \nabla_\alpha\left[\delta\Gamma^\alpha_{\mu\nu}g^{\mu\nu}\right] - \cancelto{0}{\delta\Gamma^\alpha_{\mu\nu}\nabla_\alpha g^{\mu\nu}} - \nabla_\mu\left[\delta\Gamma^\alpha_{\alpha\nu}g^{\mu\nu}\right] + \cancelto{0}{\delta\Gamma^\alpha_{\alpha\nu}\nabla_\mu g^{\mu\nu}}. \tag{3.18}$$

The second and fourth terms vanish because of the second condition in Eq. (3.7). We simplify further by re-labeling the dummy indices.

$$\nabla_\alpha \left[\delta\Gamma^\alpha_{\mu\nu}g^{\mu\nu} - \delta\Gamma^\mu_{\mu\nu}g^{\alpha\nu}\right] \equiv \nabla_\alpha A^\alpha, \tag{3.19}$$

where A^α is defined to be the tensor inside the square brackets in order to simplify further. All components can now be inserted into the expression for the variation of the Einstein-Hilbert action from Eq. (3.8),

$$\delta S_H = \int d^4x \left[\sqrt{-g}\left(\nabla_\alpha A^\alpha + R_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{2}(R + 2\Lambda)g^{\mu\nu}\delta g_{\mu\nu}\right)\right]. \tag{3.20}$$

Using the divergence theorem on the first term gives us the vector A evaluated at the boundary. This boundary term is set to zero by making the variation zero at the boundary. To simplify further we use the property $g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu}$,

$$\delta S_H = \int d^4x \left[\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}(R + 2\Lambda)g_{\mu\nu}\right)\delta g^{\mu\nu}\right]. \tag{3.21}$$

This is zero by the action principle, and since $\delta g^{\mu\nu}$ is an arbitrary variation, we get Einstein's field equations in vacuum:

$$R_{\mu\nu} - \frac{1}{2}(R + 2\Lambda)g_{\mu\nu} = 0. \tag{3.22}$$

The calculations performed thus far holds in vacuum. When matter is considered, we need an additional term in the action, $S = S_{EH}/(16\pi) + S_M$, where we have included a normalization factor on S_{EH}. We find the full Einstein equations when calculating the variation with respect to the metric of the full action S,

$$\delta S = \frac{1}{16\pi}\delta S_{EH} + \delta S_M. \tag{3.23}$$

In addition to the calculations performed we know that the functional derivative of the action, where Φ^i are all fields being varied, satisfies (Ref. [3])

$$\delta S = \int d^4x \sum_i \left(\frac{\delta S}{\delta \Phi^i} \delta \Phi^i \right). \tag{3.24}$$

For our purpose we only have the varying field $g^{\mu\nu}$. Stationary points are those points where $\delta S/\delta\Phi^i = 0$ for all i. Substituting δS_M with Eq. (3.24) gives

$$\begin{aligned} \delta S = \frac{1}{16\pi}\delta S_{EH} + \delta S_M &= \int d^4x \left[\frac{\sqrt{-g}}{16\pi} \left(R_{\mu\nu} - \frac{1}{2}(R + 2\Lambda) g_{\mu\nu} \right) \delta g^{\mu\nu} + \frac{\delta S_M}{\delta g^{\mu\nu}} \delta g^{\mu\nu} \right] \\ &= \int d^4x \sqrt{-g} \left[\frac{1}{16\pi} \left(R_{\mu\nu} - \frac{1}{2}(R + 2\Lambda) g_{\mu\nu} \right) + \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu}. \end{aligned} \tag{3.25}$$

Now we define the energy-momentum tensor to be

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{3.26}$$

Inserting the defined energy-momentum tensor into $\delta S = 0$ gives Einstein's field equations in its complete form.

$$\boxed{R_{\mu\nu} - \frac{1}{2}(R + 2\Lambda) g_{\mu\nu} = 8\pi T_{\mu\nu}.} \tag{3.27}$$

Einstein's field equations describes how the metric respond to energy and momentum densities, in a similar way that Maxwell's equations describes how the electric and magnetic fields respond to current and charge densities.

3.2 Palatini's approach

In the previous section the only free field was the metric $g_{\mu\nu}$, while the connection coefficients $\Gamma^{\alpha}_{\mu\nu}$ were directly expressed in terms of the metric. In this section both the metric and the connection coefficients are viewed as independent fields . We will show that this method is equivalent to that in the previous section, even if we assume the fields to be independent. This work was originally published by Palatini in 1919 (Ref. [5]).

The assumption that the metric tensor is symmetric is used in this approach as well. In the previous section this meant that the connection coefficients were also symmetric in the lower indices (since those were given by the metric), but this is not immediately the case now that they are viewed as independent fields. We still need symmetry in the lower indices though, so this is an additional assumption. The following derivations follow from Ref. [6].

The Lagrangian as a function of the independent metric and connection coefficients is the same as in section 3.1,

$$\mathcal{L}_{EH} = R + 2\Lambda, \tag{3.28}$$

while the action is

$$S_{EH} = \int d^4x (R + 2\Lambda)\sqrt{-g} \ = \int d^4x (g^{\mu\nu} R_{\mu\nu} + 2\Lambda)\sqrt{-g} \ . \tag{3.29}$$

The metric dependency is found in $g^{\mu\nu}$ and $\sqrt{-g}$, while the connection dependency is found in $R_{\mu\nu}$. Next we vary the action (3.29) with respect to the connection coefficients $\Gamma^{\alpha}_{\mu\nu}$ and set it equal to zero by the action principle. Since the metric does not depend on the connection we find

$$\delta S_{EH} = \int d^4x g^{\mu\nu} \delta R_{\mu\nu} \sqrt{-g} \ = 0. \tag{3.30}$$

In section 3.1 we calculated the variation of the Ricci tensor (3.17), and multiplying with the metric $g^{\mu\nu}$ gave

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\alpha \left[\delta\Gamma^{\alpha}_{\mu\nu} g^{\mu\nu}\right] - \delta\Gamma^{\alpha}_{\mu\nu}\nabla_\alpha g^{\mu\nu} - \nabla_\mu \left[\delta\Gamma^{\alpha}_{\alpha\nu} g^{\mu\nu}\right] + \delta\Gamma^{\alpha}_{\alpha\nu}\nabla_\mu g^{\mu\nu}. \tag{3.31}$$

Previously the second and fourth terms vanished, as the property $\nabla_\alpha g_{\beta\gamma} = 0$ was used. This property does not generally hold in Palatini's approach, and thus we can not neglect these terms. The two other terms however is integrated and become surface terms that we set equal to zero in the same way as in the previous section. Thus we have

$$g^{\mu\nu}\delta R_{\mu\nu} = \delta\Gamma^{\alpha}_{\alpha\nu}\nabla_\mu g^{\mu\nu} - \delta\Gamma^{\alpha}_{\mu\nu}\nabla_\alpha g^{\mu\nu}. \tag{3.32}$$

By re-labeling the dummy indices of the first term we can factorize the variation of the connection to get

$$g^{\mu\nu}\delta R_{\mu\nu} = \Big[\delta^{\mu}_{\alpha}\nabla_\beta g^{\beta\nu} - \nabla_\alpha g^{\mu\nu}\Big]\delta\Gamma^{\alpha}_{\mu\nu}. \tag{3.33}$$

This is inserted back into Eq. (3.30),

$$\delta S_{EH} = \int d^4x \Big[\delta^{\mu}_{\alpha}\nabla_\beta g^{\beta\nu} - \nabla_\alpha g^{\mu\nu}\Big]\delta\Gamma^{\alpha}_{\mu\nu}\sqrt{-g} \ = 0. \tag{3.34}$$

Since the connection variation is arbitrary we get the following condition,

$$\delta^{\mu}_{\alpha}\nabla_{\beta}g^{\beta\nu} - \nabla_{\alpha}g^{\mu\nu} = 0, \tag{3.35}$$

which is only true for $\nabla_{\alpha}g^{\mu\nu} = 0$. Note that this is the property that was assumed in Hilbert's variational principle (section 3.1), which in addition to the symmetry of the metric gives the condition that the connection coefficients are the Christoffel symbols (3.6). Thus we have shown that by viewing the metric and the connection coefficients as independent fields, we still find that the connection coefficients are the metric connection. This makes the two methods, Hilbert's variational principle and Palatini's variational principle, equivalent, and thus we obtain the Einstein field equations in the same manner as in section 3.1.

4 Rindler space

In this section we will define Rindler space, which is a part of the four-dimensional Minkowski spacetime. Rindler space uses hyperbolic coordinates, and is very useful when looking at effects for observers with constant acceleration. One of these effects is the Unruh effect, and will be covered later in the thesis. Some of the derivations in this section follow from Ref. [3] and Ref. [7].

4.1 Defining Rindler space

We have an observer accelerating uniformly along the x-direction with a constant proper acceleration α. For simplicity we consider two-dimensional spacetime in this section. The Minkowski metric is given by

$$ds^2 = -dt^2 + dx^2. \tag{4.1}$$

The position of the observer is $x^\mu = (t, x)$. To find the corresponding proper velocity we evaluate the proper time derivative of the position, where the connection between the proper time τ and the coordinate-time t is given by

$$\gamma d\tau = dt,$$

where γ is the Lorentz factor which is given by

$$\gamma^2 = \frac{1}{1-v^2}. \tag{4.2}$$

Using the position of the observer, $x^\mu = (t, x)$, we find the velocity vector to be

$$u^\mu = \frac{d}{d\tau}x^\mu = \gamma\frac{d}{dt}x^\mu = \gamma(1, v), \tag{4.3}$$

where v is the spatial velocity along the x-direction. The acceleration is found by evaluating the proper time derivative of the velocity.

$$a^\mu = \frac{d}{d\tau}u^\mu = \gamma\frac{d}{dt}u^\mu = \gamma\left(\frac{d\gamma}{dt}, \frac{d(\gamma v)}{dt}\right). \tag{4.4}$$

Now we use that the proper acceleration is constant and equal to α.

$$a_\mu a^\mu = \gamma^2\left[\left(\frac{d\gamma}{dt}\right)^2 - \left(\frac{d(\gamma v)}{dt}\right)^2\right] \equiv -\alpha^2. \tag{4.5}$$

This can be rewritten to obtain the differential equation

$$\alpha = \frac{d(\gamma v)}{dt}. \tag{4.6}$$

This is a simple first order differential equation that is solved using the initial condition $v(t = 0) = 0$. Using the definition of the Lorentz factor (4.2) and solving for $v(t)$ gives

$$v(t) = \frac{\alpha t}{\sqrt{1 + (\alpha t)^2}}. \tag{4.7}$$

Using $v(t) = dx/dt$ we find another first order differential equation that we solve for $x(t)$ using $x(0) = 1/\alpha$.

$$x(t) = \frac{\sqrt{1 + (\alpha t)^2}}{\alpha}. \tag{4.8}$$

Now we have x as a function of time t, but we want to express the position as a function of proper time τ. To obtain this we first find $t(\tau)$ by integrating $\gamma d\tau = dt$ using the velocity in Eq. (4.7).

$$\tau = \int dt \sqrt{1 - v^2} = \int \frac{dt}{\sqrt{1 + (\alpha t)^2}} = \frac{1}{\alpha} \text{arsinh}(\alpha t). \tag{4.9}$$

After some rearranging we find time t as a function of proper time τ.

$$t(\tau) = \frac{1}{\alpha} \sinh(\alpha \tau). \tag{4.10}$$

Inserting into Eq. (4.8) we find position as a function of proper time.

$$x(\tau) = \frac{1}{\alpha} \cosh(\alpha \tau), \quad x(0) = \frac{1}{\alpha}. \tag{4.11}$$

From the trajectory $x(\tau)$ and $t(\tau)$ given in Eqs (4.10) and (4.11) we see that the following relation holds

$$x^2(\tau) - t^2(\tau) = \frac{1}{\alpha^2}. \tag{4.12}$$

This is the equation for a hyperbola with asymptotes at null paths $x = -t$ in the past and $x = t$ in the future. The observer moves from past null infinity to future null infinity, instead of timelike infinity that would be reached by geodesic observers.

4.2 Defining the hyperbolic coordinates

Now we define new coordinates (η, ξ) that is adapted to uniformly accelerated motion, and is defined in two-dimensional Minkowski space, where η is the spatial coordinate and ξ is the time coordinate. The coordinates are given implicitly by the relations

$$t = \frac{1}{a} e^{a\xi} \sinh(a\eta), \quad x = \frac{1}{a} e^{a\xi} \cosh(a\eta), \quad x > |t|, \tag{4.13}$$

where a is a positive constant. For a fixed ξ we get the hyperbolic equation

$$x^2 - t^2 = \frac{1}{a^2} e^{2a\xi} \tag{4.14}$$

Figure 1 shows how the coordinates η and ξ behaves in a (x, t) plot. Rindler space is the space where $x > |t|$, where the boundaries $x = t$ and $x = -t$ are shown as dashed lines.

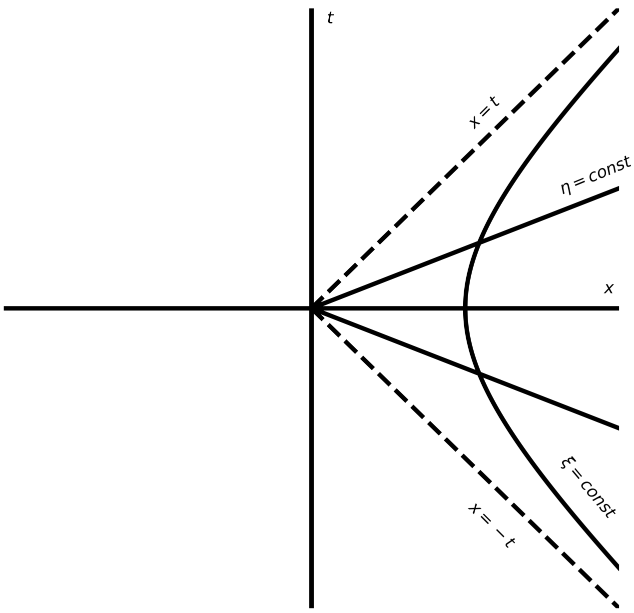

Figure 1: Rindler space lies inside the boundary lines $x = t$ and $x = -t$, shown as dashed lines.

The next step is to find explicit equations for $\eta(\tau)$ and $\xi(\tau)$. Inserting Eq. (4.13) into Eq. (4.12) and solving for ξ gives

$$\xi(\tau) = \frac{1}{a} \ln\left(\frac{a}{\alpha}\right) = \text{constant}. \tag{4.15}$$

Now that we have one of the coordinates, we equate the two equations for t in Eq. (4.10) and Eq.

(4.13) and solve for η using the $\xi(\tau)$ found above. That gives

$$\eta(\tau) = \frac{\alpha}{a}\tau. \tag{4.16}$$

What is evident from $\xi(\tau)$ and $\eta(\tau)$ is that the proper time τ is proportional to the coordinate η, while the spatial coordinate ξ is independent of τ. The metric in the new coordinates can be found using Eq. (A.2) in appendix A, which gives

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2). \tag{4.17}$$

The region with this metric and with $x > |t|$ is known as *Rindler space*, while a *Rindler observer* is one travelling along a trajectory with constant acceleration. Observe how the Rindler metric (4.17) resembles the Friedmann-Lemaître-Robertson-Walker (FRW) metric

$$ds^2 = a^2(t)(-dt^2 + dx^2), \tag{4.18}$$

where $a^2(t)$ is the scale factor, a parameter describing the relative expansion of the universe.

4.3 Calculating the Ricci scalar in Rindler space

The Ricci scalar R is a measure of the curvature of the space we are working with, and is thus an interesting quantity. This value is calculated using Eqs. (3.5) and (3.6), and all we need to proceed are the coordinates and the metric itself. The coordinates are $x^0 = \eta$ and $x^1 = \xi$, while the metric $g_{\mu\nu}$ (and it's inverse $g^{\mu\nu}$ that is needed to calculate R) is (4.17)

$$g_{\mu\nu} = e^{2a\xi}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad g^{\mu\nu} = e^{-2a\xi}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.19}$$

We start by evaluating the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$. There are $2^3 = 8$ combinations of indices, but the Christoffel symbols are symmetric in the lower indices because of our diagonal metric. This gives six independent calculations needed to find all Christoffel symbols.

First we let the top index α be zero, and since the metric is diagonal the dummy index ρ in Eq. (3.6) only gives a contribution for $\rho = 0$. Thus we have

$$\Gamma^0_{\beta\gamma} = -\frac{1}{2}e^{-2a\xi}\Big[\partial_\gamma g_{0\beta} + \partial_\beta g_{0\gamma} - \cancelto{0}{\partial_0 g_{\beta\gamma}}\Big]. \tag{4.20}$$

The last term vanishes because the metric does not have a $x^0 = \eta$ dependency. Inserting $\beta = \gamma = 0$ gives $\Gamma^0_{00} = 0$ by the same argument. The connection also vanishes for $\beta = \gamma = 1$, because that only uses non-diagonal elements of the metric, which are all zero. For $\beta = 0$ and $\gamma = 1$ (and $\beta = 1$ and $\gamma = 0$ because of symmetry) we find

$$\Gamma^0_{10} = \Gamma^0_{01} = -\frac{1}{2}e^{-2a\xi}\Big[\partial_1 g_{00} + \cancelto{0}{\partial_0 g_{01}}\Big] = -\frac{1}{2}e^{-2a\xi}\Big[-2ae^{2a\xi}\Big] = a. \tag{4.21}$$

Next we calculate the Christoffel symbols with the top index being $\alpha = 1$.

$$\Gamma^1_{\beta\gamma} = \frac{1}{2}e^{-2a\xi}\Big[\partial_\gamma g_{1\beta} + \partial_\beta g_{1\gamma} - \partial_1 g_{\beta}\gamma\Big]. \tag{4.22}$$

For $\beta = \gamma = 0$ we only get a contribution on the final term, and the result is

$$\Gamma^1_{00} = \frac{1}{2} e^{-2a\xi} \left[2a e^{2a\xi} \right] = a. \tag{4.23}$$

For $\beta = \gamma = 1$ all three terms in Eq. (4.22) are equal, and thus we find $\Gamma^1_{11} = a$. For the cross-terms $\beta = 0$ and $\gamma = 1$ (and opposite) all terms either include a derivative of the metric with respect to ξ, or include non-diagonal components of the metric. Thus we have $\Gamma^1_{10} = \Gamma^1_{01} = 0$. Summarized we find all Christoffel symbols to be

$$\Gamma^0_{00} = \Gamma^0_{11} = \Gamma^1_{01} = \Gamma^1_{10} = 0, \qquad \Gamma^0_{01} = \Gamma^0_{10} = \Gamma^1_{00} = \Gamma^1_{11} = a. \tag{4.24}$$

Next we use these components to find the Ricci curvature tensor $R_{\mu\nu}$ (3.5). Note that the two first terms vanish as the Christoffel symbols are all independent of the coordinates η and ξ.

$$R_{\mu\nu} = \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu}, \tag{4.25}$$

Note that we only need to do three independent calculations, as the symmetry $R_{\mu\nu} = R_{\nu\mu}$ follows from the symmetry of the Christoffel symbols. Setting $\mu = \nu = 0$ we find using the calculated Christoffel symbols (4.24)

$$R_{00} = \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{00} - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{\alpha 0} = 2a^2 - 2a^2 = 0. \tag{4.26}$$

The same is calculated for the other combinations of μ and ν, which gives

$$R_{00} = R_{01} = R_{10} = R_{11} = 0, \tag{4.27}$$

The Ricci scalar immediately follows as

$$R = R_{\mu\nu} g^{\mu\nu} = 0. \tag{4.28}$$

The Ricci scalar is zero for all ξ and η. The Ricci scalar is a measure of the curvature of the space, and since this is zero everywhere it would seem like Rindler space is a flat manifold. For it to be defined as a flat manifold however, the condition $R_{\mu\nu\alpha\beta} = 0$ must be satisfied, and $R = 0$ does not imply $R_{\mu\nu\alpha\beta} = 0$. Thus the calculations in this section is not sufficient to say that Rindler space is flat, although we know that it is flat because it is a part of the flat Minkowski space.

5 Massless scalar field in Rindler space and the Unruh effect

In this section we will look into the Unruh effect, first described by Fulling (Ref. [8]) in 1973 and Unruh (Ref. [9]) in 1976. If a state is observed to be a vacuum state by *one* inertial observer, all other inertial observers will agree that the state is a vacuum state. Observers with constant acceleration will disagree and observe a non-zero particle density where the inertial observers sees vacuum. The first part of the section will follow derivations from Ref. [6], while the latter part will follow Ref. [10] closely.

5.1 Exponential redshift

We will first see how an inertial observer with constant velocity v and a Rindler observer with constant acceleration sees a monochromatic wave of a scalar massless field propagating in two-dimensional spacetime, before moving to an accelerating observer. We assume that the inertial observer moves towards the source of the wave, i.e. the observer has a velocity $v > 0$ away from the source of the wave. The trajectory for a inertial observer is

$$x^{\mu} = \big(t(\tau), x(\tau)\big) = (\gamma\tau, \gamma\tau v). \tag{5.1}$$

The solution of the field equation for a monochromatic wave is

$$\phi \propto \exp\big[-i\omega(t-x)\big], \tag{5.2}$$

and it is seen by the Minkowski observer as

$$\phi \propto \exp\big[-i\omega\gamma\tau(1-v)\big] = \exp\left[-i\omega\tau\sqrt{\frac{1-v}{1+v}}\right]. \tag{5.3}$$

We can see that the observed frequency for the Minkowski observer has shifted from the original frequency of the wave. The observed frequency is

$$\omega' = \sqrt{\frac{1-v}{1+v}}\omega, \tag{5.4}$$

and since we defined $v > 0$ we have $\omega' < \omega$. The frequency has decreased, meaning that the wave has been red-shifted. This effect is called the Doppler effect. Next we do a similar calculation for the accelerated Rindler observer. Recall the trajectory for a Rindler observer [(4.11) and (4.10)],

$$x(\tau) = \frac{1}{\alpha}\cosh(\alpha\tau), \quad t(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau). \tag{5.5}$$

Inserting into the wave equation for the monochromatic wave (5.2) gives

$$\phi \propto \exp\Big[-i\omega\frac{1}{\alpha}\big(\sinh(\alpha\tau)-\cosh(\alpha\tau)\big)\Big] = \exp\Big[\frac{i\omega}{\alpha}\exp(-\alpha\tau)\Big] = \exp[-i\theta], \quad \theta \equiv -\frac{\omega}{\alpha}\exp(-\alpha\tau) \tag{5.6}$$

Thus the accelerated observer does not see a monochromatic wave, but instead sees a superposition of plane waves with varying frequencies. The instantaneous frequency $\omega'(\tau)$ is defined as

$$\omega'(\tau) = \frac{d\theta}{d\tau} = \omega\exp(-\alpha\tau). \tag{5.7}$$

We see that the wave gets exponentially red-shifted as the proper time increases for the accelerated observer. As the next step we wish to determine the power spectrum $P(\nu) = |\phi(\nu)|^2$ measured by the accelerated observer. To proceed with that calculation we need the Fourier transform $\phi(\nu)$ of the wave $\phi(\tau)$. We will simply give the Fourier transform here, and the full calculation is shown in Ref. [6]. The result is

$$\phi(\nu) = \frac{1}{\alpha}\left(\frac{\omega}{\alpha}\right)^{i\nu/\alpha}\Gamma(-i\nu/\alpha)\exp\big[\pi\nu/(2\alpha)\big], \tag{5.8}$$

where Γ is the usual Γ-function. We will first consider the negative frequency components, and they are found as (Ref. [6])

$$\phi(-\nu) = \phi(\nu)\exp[-\pi\nu/\alpha] = \frac{1}{\alpha}\left(\frac{\omega}{\alpha}\right)^{i\nu/\alpha}\Gamma(-i\nu/\alpha)\exp\big[-\pi\nu/(2\alpha)\big]. \tag{5.9}$$

Now we proceed to calculate the power spectrum of the negative frequency components,

$$P(-\nu) = |\phi(-\nu)|^2 = \frac{1}{\alpha^2}\big|\Gamma(-i\nu/\alpha)\big|^2\exp\big[-\pi\nu/\alpha\big]. \tag{5.10}$$

To continue we use the Γ-function property from Eq. (A.3) to find

$$P(-\nu) = \frac{\pi}{\alpha^2}\frac{\exp\big[-\pi\nu/\alpha\big]}{(\nu/\alpha)\sinh(\pi\nu/\alpha)}. \tag{5.11}$$

Simplifying this expression further we find a familiar result,

$$P(-\nu) = \frac{\beta}{\nu}\frac{1}{\exp(\beta\nu)-1}, \quad \beta = \frac{2\pi}{\alpha}, \quad T = \frac{1}{\beta} = \frac{\alpha}{2\pi}. \tag{5.12}$$

This corresponds to a thermal Planck distribution with temperature $T = \alpha/(2\pi)$. Thus a uniformly accelerated detector will measure a thermal Planck spectrum with temperature T. This phenomenon is called the Unruh effect, and the temperature T is the Unruh temperature. Note that the temperature of the spectrum for non-accelerating observers ($\alpha = 0$) is zero, which means that they will not see a particle density unlike the accelerating observers. Recall that we used the negative frequency components in these calculations. The same calculations can be done on the positive frequency components, and after similar steps we find that it gives zero contribution to the thermal Planck spectrum (Ref. [6]). Thus the only interesting part is the negative frequency components.

5.2 Two-dimensional massless scalar field in Rindler space

Next we want to inspect this phenomenon on the quantum level, while still considering a massless scalar field in two dimensions. The rest of the section follows Ref. [10] closely. The corresponding action is

$$S = \frac{1}{2}\int dtdx\ \eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi = \frac{1}{2}\int d\xi d\eta\sqrt{-g}\ g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi, \tag{5.13}$$

where we have used $\sqrt{-g} = 1$ in the (t, x) coordinates and substituted ∂_α for ∇_α as the fields are scalars. The equations of motion can be derived by using the Euler-Lagrange equations with (5.13), which gives

$$\partial_t^2\phi - \partial_x^2\phi = \partial_\eta^2\phi - \partial_\xi^2\phi = 0. \tag{5.14}$$

Next we introduce light cone coordinates,

$$u = \eta - \xi, \quad v = \eta + \xi, \quad \tilde{u} = t - x, \quad \tilde{v} = t + x, \tag{5.15}$$

The relation between u and $\tilde{u}$ as well as v and $\tilde{v}$ can easily be found by inserting the coordinate definitions (4.13) into $\tilde{u}$ and $\tilde{v}$, with $a = \alpha$ being the acceleration of the observer,

$$\tilde{u} = -\frac{e^{-\alpha u}}{\alpha}, \quad \tilde{v} = \frac{e^{\alpha v}}{\alpha}. \tag{5.16}$$

Using the light cone coordinates the equations of motion (5.14) simplifies to

$$\frac{\partial^2\phi}{\partial u\partial v} = \frac{\partial^2\phi}{\partial\tilde{u}\partial\tilde{v}} = 0. \tag{5.17}$$

The solutions of the above equation are of the form

$$\phi(u, v) = f(u) + g(v), \quad \phi(\tilde{u}, \tilde{v}) = p(\tilde{u}) + q(\tilde{v}), \tag{5.18}$$

where f and g are arbitrary smooth functions specifying the wave form. In the region where Minkowski space and Rindler space overlap, $x > |t|$, we can quantize the field using Eq. (2.14), with the integration variable changed to spherical coordinates. Since the field is massless we have $\omega = \sqrt{k^2 + m^2} = |k|$, which gives

$$\phi(t, x) = \int_{-\infty}^{+\infty}\frac{dk}{\sqrt{(2\pi)2|k|}}\left[a_k e^{-i|k|t+ikx} + a_k^\dagger e^{i|k|t-ikx}\right], \tag{5.19}$$

$$\phi(\eta, \xi) = \int_{-\infty}^{+\infty}\frac{dk}{\sqrt{(2\pi)2|k|}}\left[b_k e^{-i|k|\eta+ik\xi} + b_k^\dagger e^{i|k|\eta-ik\xi}\right], \tag{5.20}$$

where a_k and b_k are annihilation operators, and $a_k^\dagger$ and $b_k^\dagger$ are creation operators. Note that the notation is changed from $a(k)$ to a_k for simplicity. The letters a and b distinguish between the coordinate sets, where a_k is used for the (t, x) coordinates and b_k is used for the (η, ξ) coordinates.

We define the Minkowski vacuum state $|0_M\rangle$ and Rindler vacuum state $|0_R\rangle$ as

$$a_k|0_M\rangle = 0, \quad b_k|0_R\rangle = 0, \tag{5.21}$$

where the two vacua differ from each other. A Rindler observer will see that the Minkowski vacuum state $|0_M\rangle$ has more energy than the Rindler vacuum $|0_R\rangle$. In other words, a particle detector at rest in the Rindler frame will detect a non-zero particle density when the scalar field is in the Minkowski vacuum $|0_M\rangle$. Conversely, an observer at rest in the lab frame will see the Rindler vacuum $|0_R\rangle$ as an excited state.

5.3 Introducing the Bogoliubov transformation

To show the Unruh effect we introduce a generalized Bogoliubov transformation. The transformation introduces Bogoliubov coefficients that relate the creation and annihilation operators in the Minkowski and Rindler coordinates. This will allow us to find the particle density observed in the accelerated frame. The particle density obtained will be shown to be on the form of a Bose-Einstein distribution at the Unruh temperature T.

First we find the light cone expansion of $\phi(\tilde{u}, \tilde{v})$ (5.18). To find this we split (5.19) in two parts, one positive k part and one negative k part,

$$\begin{aligned}\phi(t,x) = &\int_0^\infty \frac{dk}{\sqrt{(2\pi)2k}}\Big[a_k e^{-ikt+ikx} + a_k^\dagger e^{ikt-ikx}\Big] \\ &+ \int_{-\infty}^0 \frac{dk}{\sqrt{(2\pi)2|k|}}\Big[a_k e^{ikt+ikx} + a_k^\dagger e^{-ikt-ikx}\Big].\end{aligned} \tag{5.22}$$

Next we introduce the integration variable $\omega = |k|$. Substituting into (5.22) and using the light cone coordinate definitions (5.15) we find

$$\phi(\tilde{u}, \tilde{v}) = \int_0^\infty \frac{d\omega}{\sqrt{(2\pi)2\omega}}\Big[a_\omega e^{-i\omega\tilde{u}} + a_\omega^\dagger e^{i\omega\tilde{u}} + a_{-\omega} e^{-i\omega\tilde{v}} + a_{-\omega}^\dagger e^{i\omega\tilde{v}}\Big]. \tag{5.23}$$

Comparing (5.23) with the general solution of the field (5.18) we see that

$$\begin{aligned}p(\tilde{u}) &= \int_0^\infty \frac{d\omega}{\sqrt{(2\pi)2\omega}}\Big[a_\omega e^{-i\omega\tilde{u}} + a_\omega^\dagger e^{i\omega\tilde{u}}\Big], \\ q(\tilde{v}) &= \int_0^\infty \frac{d\omega}{\sqrt{(2\pi)2\omega}}\Big[a_{-\omega} e^{-i\omega\tilde{v}} + a_{-\omega}^\dagger e^{i\omega\tilde{v}}\Big].\end{aligned} \tag{5.24}$$

Making similar steps for the Rindler coordinates (η, ξ) and the corresponding light cone coordinates (u, v) gives

$$\begin{aligned}f(u) &= \int_0^\infty \frac{d\Omega}{\sqrt{(2\pi)2\Omega}}\Big[b_\Omega e^{-i\Omega u} + b_\Omega^\dagger e^{i\Omega u}\Big], \\ g(v) &= \int_0^\infty \frac{d\Omega}{\sqrt{(2\pi)2\Omega}}\Big[b_{-\Omega} e^{-i\Omega v} + b_{-\Omega}^\dagger e^{i\Omega v}\Big],\end{aligned} \tag{5.25}$$

where we use different integration variables for the two coordinate sets to make the distinction between them more clear. From Eq. (5.18) we know that $\phi = f(u) + g(v) = p(\tilde{u}) + q(\tilde{v})$, and since the light cone coordinate transformation (5.15) does not mix u's and v's, we can use

$$p(\tilde{u}(u)) = f(u), \quad q(\tilde{v}(v)) = g(v). \tag{5.26}$$

Note that the goal of these calculations is to express the operators a_ω and $a_\omega^\dagger$ as linear combinations of the operators b_Ω and $b_\Omega^\dagger$. To proceed with this we Fourier transform both sides on both equations on Eq. (5.26). To start we calculate the Fourier transform of $f(u)$,

$$\int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} e^{i\Omega u} f(u) = \int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} e^{i\Omega u} \int_0^\infty \frac{d\Omega'}{\sqrt{(2\pi)2\Omega'}} \Big[b_{\Omega'} e^{-i\Omega' u} + b_{\Omega'}^\dagger e^{i\Omega' u} \Big]. \tag{5.27}$$

Next we move the left-most integral to the right, multiply the exponentials and use the delta-function inverse Fourier transform,

$$\delta(a) = \int_{-\infty}^{+\infty} \frac{du}{2\pi} e^{-iau}, \tag{5.28}$$

which in turn gives

$$\int_0^\infty \frac{d\Omega'}{\sqrt{2\Omega'}} \Big[b_{\Omega'} \delta(\Omega' - \Omega) + b_{\Omega'}^\dagger \delta(\Omega' + \Omega) \Big]. \tag{5.29}$$

Now we perform the integration, which will give two different results depending on the sign of Ω. The result is

$$\frac{b_\Omega}{\sqrt{2\Omega}} \text{ if } \Omega > 0, \qquad \frac{b_{|\Omega|}^\dagger}{\sqrt{2\Omega}} \text{ if } \Omega < 0. \tag{5.30}$$

This is the Fourier transform of $f(u)$. The Fourier transform of p($\tilde{u}$) is

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} e^{i\Omega u} p(\tilde{u}) &= \int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} e^{i\Omega u} \int_0^\infty \frac{d\omega}{\sqrt{(2\pi)2\omega}} \Big[a_\omega e^{-i\omega\tilde{u}} + a_\omega^\dagger e^{i\omega\tilde{u}} \Big] \\ &= \int_0^\infty \frac{d\omega}{\sqrt{(2\pi)2\omega}} \int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} \Big[a_\omega e^{i\Omega u - i\omega\tilde{u}} + a_\omega^\dagger e^{i\Omega u + i\omega\tilde{u}} \Big] \\ &= \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} \Big[a_\omega F(\omega, \Omega) + a_\omega^\dagger F(-\omega, \Omega) \Big], \end{aligned} \tag{5.31}$$

where we have defined the function $F(\omega, \Omega)$ as

$$F(\omega, \Omega) \equiv \int_{-\infty}^{+\infty} \frac{du}{2\pi} e^{i\Omega u - i\omega\tilde{u}} = \int_{-\infty}^{+\infty} \frac{du}{2\pi} \exp\Big[i\Omega u + i\frac{\omega}{\alpha} e^{-\alpha u} \Big], \tag{5.32}$$

and Eq. (5.16) has been used to express it using only u instead of a combination of u and $\tilde{u}$. Next we insert the Fourier transforms of $f(u)$ and $p(\tilde{u})$ into the Fourier transformed Eq. (5.26),

$$\begin{aligned} \Omega > 0: \quad \frac{b_\Omega}{\sqrt{2\Omega}} &= \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} \Big[a_\omega F(\omega, \Omega) + a_\omega^\dagger F(-\omega, \Omega) \Big], \\ b_\Omega &= \int_0^\infty d\omega \Big[\alpha_{\omega\Omega} a_\omega + \beta_{\omega\Omega} a_\omega^\dagger \Big], \end{aligned} \tag{5.33}$$

where we have defined $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$, the Bogoliubov coefficients, to be

$$\alpha_{\omega\Omega} \equiv \sqrt{\frac{\Omega}{\omega}} F(\omega, \Omega), \qquad \beta_{\omega\Omega} \equiv \sqrt{\frac{\Omega}{\omega}} F(-\omega, \Omega). \tag{5.34}$$

Now we have expressed the annihilation operator b_Ω as a function of the annihilation and creation operators a_ω and $a_\omega^\dagger$ as we wanted, and Eq. (5.33) is the Bogoliubov transformation. To find a similar expression for the creation operator $b_\Omega^\dagger$ we simply Hermitian conjugate Eq. (5.33), which gives

$$b_\Omega^\dagger = \int_0^\infty d\omega \Big[\alpha_{\omega\Omega}^* a_\omega^\dagger + \beta_{\omega\Omega}^* a_\omega \Big], \tag{5.35}$$

where the coefficients $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$ are simply complex conjugates because they are scalars. Note that the Bogoliubov transformation only express the relations between the annihilation and creation operators for $\Omega > 0$. To reach our result we used the left equation in Eq. (5.26). The exact same procedure on the right equation will give the Bogoliubov transformation valid for $\Omega < 0$.

Next we will derive the normalization condition for the Bogoliubov transformation. The commutation relations of the operators are as usual

$$\big[a_\omega, a_{\omega'}^\dagger\big] = \delta(\omega - \omega'), \qquad \big[b_\Omega, b_{\Omega'}^\dagger\big] = \delta(\Omega - \Omega'), \tag{5.36}$$

where $\delta(x)$ is the Dirac-delta function. The operator expressions, Eqs. (5.33) and (5.35), are inserted into the commutation relation for the b_Ω operator. This gives eight terms, where four of them cancel out and the other four can be manipulated using the commutation relation for the a_ω operator. We find

$$\delta(\Omega - \Omega') = \int_0^\infty \int_0^\infty d\omega d\omega' \Big[\alpha_{\omega\Omega} \alpha_{\omega'\Omega'}^* \delta(\omega - \omega') - \beta_{\omega\Omega} \beta_{\omega'\Omega'}^* \delta(\omega - \omega') \Big]. \tag{5.37}$$

Integrating over w' removes the delta-functions and sets $\omega' = \omega$, and the result is the normalization condition for the Bogoliubov transformation,

$$\delta(\Omega - \Omega') = \int_0^\infty d\omega \Big[\alpha_{\omega\Omega} \alpha_{\omega\Omega'}^* - \beta_{\omega\Omega} \beta_{\omega\Omega'}^* \Big]. \tag{5.38}$$

Thus we have shown that the Bogoliubov coefficients are normalized, as they should be. The normalization condition will be used in the following subsection to show the Unruh effect.

5.4 The Unruh effect

Now that the Bogoliubov transformation has been defined, we will inspect the mean number of particles in the Minkowski vacuum observed by the accelerated observer. It is given by the average of the number operator $N_\Omega = b_\Omega^\dagger b_\Omega$ in the Minkowski vacuum state $|0_M\rangle$,

$$\langle N_\Omega \rangle = \langle 0_M | N_\Omega |0_M\rangle = \langle 0_M | b_\Omega^\dagger b_\Omega |0_M\rangle . \tag{5.39}$$

Remember that the b-letter operators were defined for accelerated observers in Rindler space, while the a-letter operators were defined for inertial observers in Minkowski space. Inserting Eqs. (5.33) and (5.35) gives

$$\langle N_\Omega \rangle = \langle 0_M | \int_0^\infty \int_0^\infty d\omega d\omega' \left[\alpha_{\omega\Omega}^* a_\omega^\dagger + \beta_{\omega\Omega}^* a_\omega\right]\left[\alpha_{\omega'\Omega} a_{\omega'} + \beta_{\omega'\Omega} a_{\omega'}^\dagger\right] |0_M\rangle . \tag{5.40}$$

Expanding the brackets gives four terms. Two of the terms vanish because the right-most state is the vacuum state, and acting on it with an annihilation operator simply gives zero. One of the remaining two terms also vanish, since that term acts on the right-most state with two creation operators, meaning that we do not end up with the same state as the left-most state, and this also gives zero because of the orthogonality of the states. Thus we only have one non-zero term, which is

$$\langle N_\Omega \rangle = \langle 0_M | \int_0^\infty \int_0^\infty d\omega d\omega' \left[\beta_{\omega\Omega}^* \beta_{\omega'\Omega} a_\omega a_{\omega'}^\dagger\right] |0_M\rangle . \tag{5.41}$$

Next we integrate over ω', which only gives a non-zero contribution for $\omega' = \omega$. The other contributions are zero because the creation operator creates a state with momentum ω', and the annihilation operator removes a state with momentum ω. If ω and ω' are not equal, the result is simply zero. Thus we find

$$\langle N_\Omega \rangle = \langle 0_M | \int_0^\infty d\omega \left[\beta_{\omega\Omega}^* \beta_{\omega\Omega} a_\omega a_\omega^\dagger\right] |0_M\rangle . \tag{5.42}$$

We move $\langle 0_M|$ inside the integral and use the identity $\langle 0_M | a_\omega a_\omega^\dagger |0_M\rangle = 1$, which gives

$$\langle N_\Omega \rangle = \int_0^\infty d\omega \left|\beta_{\omega\Omega}\right|^2 = \int_0^\infty d\omega \frac{\Omega}{\omega} \left|F(-\omega,\Omega)\right|^2, \tag{5.43}$$

where we used Eq. (5.34) in the final step. Next we will inspect the function $F(\omega,\Omega)$ (5.32) more closely. To solve the integral in the function the substitution $x = e^{-\alpha u}$ is made. After the substitution the integral becomes

$$F(\omega,\Omega) = \frac{1}{2\pi\alpha} \int_0^\infty dx\ x^{s-1} e^{-bx}, \quad s = -i\frac{\Omega}{\alpha}, \quad b = -i\frac{\omega}{\alpha}. \tag{5.44}$$

In order to solve this integral we make use on an identity from the Γ-function, Ref. [10],

$$\int_0^\infty dx\ x^{s-1} e^{-bx} = e^{-s\ln b}\Gamma(s), \qquad \text{for } \mathrm{Re}(b) > 0 \text{ and } 1 > \mathrm{Re}(s) > 0, \tag{5.45}$$

where the logarithm is defined in the right complex half-plane as

$$\ln z = \ln(x+iy) = \ln r + i\theta = \ln|z| + i\arctan(y/x) = \ln|z| + i\ \text{sign}(y)\arctan(|y|/x), \qquad (5.46)$$

Note that we can not use this Γ-identity directly, as our coefficients s and b are purely imaginary. To avoid this problem we shift our coefficients by a very small, real parameter $\epsilon > 0$, taking the limit $\epsilon \to 0$ after the integral has been evaluated.

$$\begin{aligned} F(\omega,\Omega) &= \lim_{\epsilon\to 0}\frac{1}{2\pi\alpha}\int_0^\infty dx\ x^{(s+\epsilon)-1}e^{-(b+\epsilon)x} \\ &= \lim_{\epsilon\to 0}\frac{1}{2\pi\alpha}e^{-(s+\epsilon)\ln(b+\epsilon)}\Gamma(s+\epsilon). \end{aligned} \qquad (5.47)$$

Next we inspect the logarithm closely,

$$\ln(b+\epsilon) = \ln\left(\epsilon - i\frac{\omega}{\alpha}\right) = \ln\left|\epsilon - i\frac{\omega}{\alpha}\right| - i\ \text{sign}\Big(\frac{\omega}{\alpha}\Big)\arctan\left(\frac{\omega}{\alpha\epsilon}\right). \qquad (5.48)$$

In the limit $\epsilon \to 0$ the arctan-argument grows to infinity, and thus we find $\arctan(\omega/(\alpha\epsilon)) \to \pi/2$. The ϵ in the logarithm drops out in the limit as well, and the end result is

$$\lim_{\epsilon\to 0}\ln(b+\epsilon) = \ln\left|\frac{\omega}{\alpha}\right| - i\frac{\pi}{2}\text{sign}\Big(\frac{\omega}{\alpha}\Big). \qquad (5.49)$$

Inserting this back into (5.47), with the note that the rest of the ϵ's are simply cancelled out in the limit $\epsilon \to 0$, we find

$$F(\omega,\Omega) = \frac{1}{2\pi\alpha}\exp\left[i\frac{\Omega}{\alpha}\ln\left|\frac{\omega}{\alpha}\right| + \frac{\Omega\pi}{2\alpha}\text{sign}\Big(\frac{\omega}{\alpha}\Big)\right]\Gamma\Big(-i\frac{\Omega}{\alpha}\Big). \qquad (5.50)$$

Since the parameters ω and α are defined to be positive quantities, the sign function is $(+1)$ for $F(\omega,\Omega)$. Substituting $\omega \to -\omega$ leads to a (-1) factor in $F(-\omega,\Omega)$, and thus we have shown the property

$$F(\omega,\Omega) = F(-\omega,\Omega)\exp\left[\frac{\Omega\pi}{\alpha}\right]. \qquad (5.51)$$

Now that we have found expressions for $F(\omega,\Omega)$ and $F(-\omega,\Omega)$ we will go back to Eq. (5.38), inserting the $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$ from Eq. (5.34) and setting $\Omega = \Omega'$. This gives

$$\int_0^\infty d\omega\frac{\Omega}{w}\Big[\big|F(\omega,\Omega)\big|^2 - \big|F(-\omega,\Omega)\big|^2\Big] = \delta(0). \qquad (5.52)$$

Next we insert Eq. (5.51), and after some some algebra we find

$$\int_0^\infty d\omega\frac{\Omega}{\omega}\Big|F(-\omega,\Omega)\Big|^2 = \delta(0)\Big[\exp\Big(\frac{2\Omega\pi}{\alpha}\Big) - 1\Big]^{-1} \qquad (5.53)$$

The left side of the equation is the exact same as the average number operator (5.43). Thus we have found

$$\langle N_\Omega\rangle = \delta(0)\Big[\exp\Big(\frac{2\Omega\pi}{\alpha}\Big) - 1\Big]^{-1}. \qquad (5.54)$$

Dividing both sides by the volume factor $\delta(0)$ gives the number density n_Ω,

$$n_\Omega = \left[\exp\left(\frac{2\Omega\pi}{\alpha}\right) - 1\right]^{-1}, \quad \Omega > 0. \tag{5.55}$$

Recall that this calculation holds under the assumption $\Omega > 0$. A very similar calculation can be done assuming $\Omega < 0$, and the result is

$$n_\Omega = \left[\exp\left(\frac{2|\Omega|\pi}{\alpha}\right) - 1\right]^{-1}, \quad \Omega < 0. \tag{5.56}$$

For massless two-dimensional scalar fields, which is what was considered in this section, we have $|\Omega| \equiv E$. Thus the final result of the section is formulated as a Bose-Einstein distribution,

$$n(E) = \frac{1}{\exp(E/T) - 1}, \tag{5.57}$$

where $T = \alpha/(2\pi)$ is defined as the Unruh temperature. Note that the only parameter that determines the Unruh temperature is the acceleration α of the accelerated observer. Note also that the Unruh temperature is the same as the thermal Planck law temperature (5.12) we found previously.

We have seen that a massless scalar field in two-dimensional Minkowski space gives rise to the Unruh effect seen by accelerated observers, which means that in the Minkowski vacuum, the accelerated observer sees a thermal spectrum of particles with temperature directly given by the acceleration. The Unruh effect also generalizes to four-dimensional spacetime. Using SI-units it becomes clear that the Unruh effect is very small, as we would need an acceleration as big as $\alpha \approx 10^{20}\text{m/s}^2$ to reach a temperature of $T \approx 1\text{K}$ (Ref. [10]).

Figure 2 gives a visual perspective in how the particle density appears. Vacuum fluctuations are particle-antiparticle pairs that spontaneously appear, that annihilate each other very soon after they appear (so soon that they are not detectable). If these fluctuations happen on the Rindler horizon however, the Rindler observer only sees one particle in the particle-antiparticle pair appear at one point in spacetime, and disappear at another point in spacetime. This gives rise to the observed particle density.

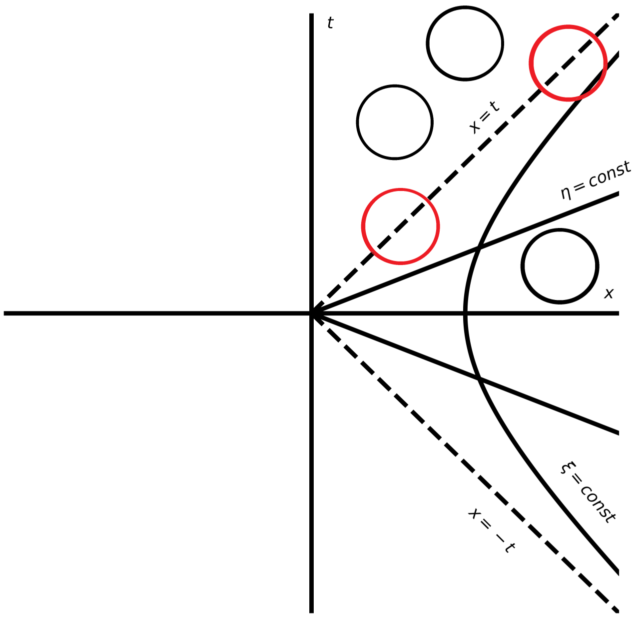

Figure 2: Vacuum fluctuations are shown as circles. The red vacuum fluctuations that cross the Rindler horizon are seen as real particles for an accelerated observer, while they are simply vacuum fluctuations for a stationary observer. The black circles are fluctuations that are not seen as particles, either because they are outside of the horizon and can not be observed, or inside the horizon and are seen as vacuum fluctuations to a Rindler observer as well.

6 Particle creation in an expanding universe

In this section we will consider a variable metric that is Minkowskian for $t = -\infty$ and $t = +\infty$ and is varying in between. As we will show, a vacuum state in $t = -\infty$ will be changed into a non-vacuum state for $t = +\infty$. This section will follow Ref. [11].

The metric that is considered in this section will be

$$ds^2 = dt^2 - a^2(t)dx^2, \tag{6.1}$$

where the function $a(t)$ determines the relative expansion of the universe, and is called the scale factor. We introduce a new time parameter η, called conformal time, via $d\eta = dt/a(t)$. Using the new parameter, the metric (6.1) becomes

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2) = C(\eta)(d\eta^2 - dx^2), \tag{6.2}$$

where $C(\eta) = a^2(\eta)$ is defined as the conformal scale factor. We will in this section choose the form of the conformal scale factor to be

$$C(\eta) = A + B\tanh(\rho\eta), \tag{6.3}$$

where A, B and ρ are constants. The conformal scale factor $C(\eta)$ is dimensionless since ds^2, dt^2 and dx^2 has the same dimensions. This implies that the constants A and B also are dimensionless. Conformal time η however has the same dimension as time t, which has dimensions $1/E$, with E being energy, since we are using natural units. The tanh-function is only defined for dimensionless quantities, and this implies that the constant ρ has dimensions of energy.

A conformal scale factor of this form is constant in the limit $\eta \to \pm\infty$ since

$$\lim_{\eta \to \pm\infty} C(\eta) = A \pm B. \tag{6.4}$$

The asymptotic behaviour of the conformal scale factor is also noticeable in a plot, shown in figure 3. The static universe corresponding to $\eta \to -\infty$ is called the *in*-region, while the static universe corresponding to $\eta \to \infty$ is called the *out*-region.

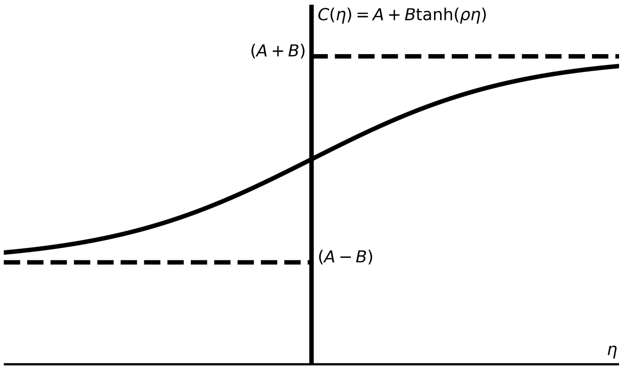

Figure 3: The conformal scale factor $C(\eta) = A + B\tanh(\rho\eta)$ approaches asymptotically $A+B$ as $\eta \to \infty$, and it approaches $A-B$ as $\eta \to -\infty$. This represents an asymptotically static universe that undergoes a period of smooth expansion.

6.1 Introducing a scalar field

The metric described thus far in this section is now applied to a scalar field. Recall from section 2 that we found the Klein-Gordon equation (2.5)

$$(\Box + m^2)\phi(\boldsymbol{x}, \eta) = 0. \tag{6.5}$$

Note that we substituted the standard time t used in section 2 with conformal time η used in this section. We introduce a complete set of orthonormal mode solutions $u_k(x,\eta)$ of (6.5) that obey the properties

$$(u_k, u_l) = \delta_{kl}, \quad (u_k^*, u_l^*) = -\delta_{kl}, \quad (u_k, u_l^*) = 0. \tag{6.6}$$

The scalar field $\phi(x,\eta)$ can be expanded by a linear combination of the orthonormal mode solutions as

$$\phi(x,\eta) = \sum_k \left[a_k u_k(x,\eta) + a_k^\dagger u_k^*(x,\eta)\right], \tag{6.7}$$

where a_k and $a_k^\dagger$ are annihilation and creation operators, respectfully.

The conformal scale factor $C(\eta)$ is not a function of the spatial coordinate x, and thus spatial translation invariance is still a symmetry in this spacetime. This means that we can separate the space and time variables in the scalar mode functions u_k as

$$u_k(x,\eta) = (2\pi)^{-1/2} e^{ikx} \chi_k(\eta) \tag{6.8}$$

Substituting the mode functions (6.8) into the scalar field equation (6.5) gives

$$\frac{d^2}{d\eta^2}\chi_k(\eta) + \left[k^2 + C(\eta)m^2\right]\chi_k(\eta) = 0, \tag{6.9}$$

which is an ordinary differential equation for $\chi_k(\eta)$. The differential equation can be solved in terms of hypergeometric functions, which are functions that can be defined in the form as a hypergeometric series. See appendix B for the definition of a hypergeometric function.

The mode solutions in the *in*-region ($\eta \to -\infty$), taken directly from Ref. [11], are

$$\lim_{\eta \to -\infty} u_k^{\text{in}}(x,\eta) = (4\pi\omega_{\text{in}})^{-1/2} \exp[ikx - i\omega_{\text{in}}\eta], \quad \omega_{\text{in}} = [k^2 + m^2(A-B)]^{1/2}. \tag{6.10}$$

Similarly, the mode solutions for the *out*-region are

$$\lim_{\eta \to -\infty} u_k^{\text{out}}(x,\eta) = (4\pi\omega_{\text{out}})^{-1/2} \exp[ikx - i\omega_{\text{out}}\eta], \quad \omega_{\text{out}} = [k^2 + m^2(A+B)]^{1/2}. \tag{6.11}$$

The two mode solutions in the different static regions are clearly different. It is possible to express $u_k^{\text{in}}(x,\eta)$ as a linear combination of the real and imaginary part of $u_k^{\text{out}}(x,\eta)$ in a similar way to what was shown in section 5.3,

$$u_k^{\text{in}}(x,\eta) = \alpha_k u_k^{\text{in}}(x,\eta) + \beta_k u_{-k}^{\text{out}*}(x,\eta). \tag{6.12}$$

To find the Bogoliubov coefficients α_k and β_k explicitly we make use of the linear transformation properties of hypergeometric functions (see Ref. [11] for details). Thus we find

$$\alpha_k = \Big(\frac{\omega_{\text{out}}}{\omega_{\text{in}}}\Big)^{1/2} \frac{\Gamma(1 - i\omega_{\text{in}}/\rho)\Gamma(-i\omega_{\text{out}}/\rho)}{\Gamma(-i\omega_+/\rho)\Gamma(1 - i\omega_+/\rho)} \tag{6.13}$$

and

$$\beta_k = \Big(\frac{\omega_{\text{out}}}{\omega_{\text{in}}}\Big)^{1/2} \frac{\Gamma(1 - i\omega_{\text{in}}/\rho)\Gamma(i\omega_{\text{out}}/\rho)}{\Gamma(i\omega_-/\rho)\Gamma(1 + i\omega_-/\rho)}, \tag{6.14}$$

where

$$\omega_\pm = \frac{1}{2}(\omega_{\text{in}} \pm \omega_{\text{out}}). \tag{6.15}$$

Using Eqs. (6.13) and (6.14) and the Γ-function properties from Eqs. (A.3) and (A.5) we find after some simplifications

$$|\alpha_k|^2 = \frac{\sinh^2(\pi\omega_+/\rho)}{\sinh(\pi\omega_{\text{in}}/\rho)\sinh(\pi\omega_{\text{out}}/\rho)} \tag{6.16}$$

and

$$|\beta_k|^2 = \frac{\sinh^2(\pi\omega_-/\rho)}{\sinh(\pi\omega_{\text{in}}/\rho)\sinh(\pi\omega_{\text{out}}/\rho)}. \tag{6.17}$$

Finally we will check if the normalization condition

$$|\alpha_k|^2 - |\beta_k|^2 = 1 \tag{6.18}$$

is met. To do this we first make use of the property in Eq. (A.7) that transform $\sinh^2$ into cosh, before using the property in Eq. (A.9). Inserting this into (6.18) gives

$$|\alpha_k|^2 - |\beta_k|^2 = \frac{\sinh(\pi\omega_{\text{in}}/\rho)\sinh(\pi\omega_{\text{out}}/\rho)}{\sinh(\pi\omega_{\text{in}}/\rho)\sinh(\pi\omega_{\text{out}}/\rho)} = 1, \tag{6.19}$$

which is exactly what we wanted. Next we consider the case where the field is in the vacuum state of the *in*-region, denoted by $|0_{\text{in}}\rangle$, defined by the modes u_k^{in}. In this region the spacetime is static, and all inertial particle detectors will see a vacuum, i.e. they will not detect any particles. Thus all unaccelerated observers will agree that this state is the vacuum state.

In the *out*-region the spacetime is also static, but the field is still in the same vacuum state $|0_{\text{in}}\rangle$. Unaccelerated observers will however not say that the state $|0_{\text{in}}\rangle$ is to be regarded as the physical vacuum state, as the state $|0_{\text{out}}\rangle$ would be the correct vacuum state in the *out*-region. Thus unaccelerated particle detectors will detect a particle density while still being in the initial vacuum state $|0_{\text{in}}\rangle$. The conclusion is that particles have been created solely as a consequence of the cosmic expansion, with the Bogoliubov coefficient $|\beta_k|^2$ in Eq. (6.17) determining the amount of particles in mode k being created, Ref. [11].

Note the special case where the scalar field is massless ($m = 0$). As seen in Eqs. (6.10) and (6.11), the consequence of a massless scalar field is that the frequency of the in-region is exactly the same as the frequency of the out-region. This implies that the exact same mode solutions exist on the out-region as the in-region, meaning that the vacuum states $|0_{\text{in}}\rangle$ and $|0_{\text{out}}\rangle$ are also identical. Thus the scalar field needs to have a non-zero mass in order for particles to be created as a consequence of the expansion of the universe.

This can also be seen by looking at the Bogoliubov coefficient $|\beta_k|^2$ in Eq. (6.17). If $m = 0$ is assumed, we get $\omega_- = (\omega_{\text{in}} - \omega_{\text{out}})/2 = 0$, which in turn means that $|\beta_k|^2 = 0$. Since $|\beta_k|^2$ determines the amount of particles in mode k being created, a massless scalar field creates no particles in any mode k.

7 Conclusions and outlook

In this master thesis we have introduced a quantized field theory that is quantized by canonical quantization, where the field itself is treated as a variable called the canonical coordinate, while the time-derivative of the field is a variable called canonical momentum. The quantized field is used to show that particles are created in states that originally are vacuum states.

Einstein's field equations have been derived using the principle of least action. This was done in two ways, first using Hilbert's variational principle, and later using Palatini's variational principle. In Palatini's approach some assumptions made in Hilbert's variational principle are omitted, meaning that Palatini's approach is a more general approach to deriving Einstein's field equations. These field equations describes how the metric respond to energy and momentum densities in a similar way to the Maxwell equations that describes how the electric and magnetic fields respond to current and charge densities.

Observers moving with constant acceleration, often called Rindler observers, have been studied. They move in Rindler space, which is one quarter of the full Minkowski space. These Rindler observers will observe particles in Minkowski vacuum states. These are states that all inertial observe will agree are vacuum states, and thus contain no particles. The Rindler observers however see a non-zero particle density in the same state. Conversely, a Rindler vacuum state will not be observed to be a vacuum state for the inertial observers. The creation of particles in this case is called the Unruh effect, and it is relatively small, as acceleration would need to be $\alpha \approx 10^{20}$ m/s^{s} to reach a particle density temperature of $T \approx 1$K. We have also shown that particles are created by a continuous expansion of the universe. Observers in the initial, pre-expansion vacuum state will observe particles after the expansion, and these particles were created solely because of the expansion of the universe.

The initial plan for the thesis was to study Hawking radiation, named by Stephen Hawking who developed a theoretical argument for it in 1974 (Ref [12]). Hawking radiation is black-body radiation that is released as a quantum effect near the event horizon of a black hole. In this case, as well as the cases studied in this thesis, the result is that particles are created because of quantum effects. If there were more time to work on the thesis, this would be studied in detail. Compared to the particle-creation cases studied in the thesis, Hawking radiation has the largest effect by a large margin, with the radiation of a black hole of the size of an atom having about the same power as the largest power plant in the world.

A Formulas and properties

A.1 Covariant derivative

The covariant derivative ∇_β of a tensor have varying number of terms depending on the shape of the tensor. For a tensor of the form $A^\alpha_{\mu\nu}$, the covariant derivative is given by

$$\nabla_\beta A^\alpha_{\mu\nu} = \partial_\beta A^\alpha_{\mu\nu} + \Gamma^\alpha_{\beta\gamma} A^\gamma_{\mu\nu} - \Gamma^\gamma_{\beta\mu} A^\alpha_{\gamma\nu} - \Gamma^\gamma_{\beta\nu} A^\alpha_{\mu\gamma}. \tag{A.1}$$

A.2 Metric after a change of coordinates

Given a coordinate change from x^μ with corresponding metric $g_{\mu\nu}$ to the new coordinates x'^λ, the metric transforms as

$$g'_{\lambda\delta} = \frac{\partial x^\mu}{\partial x'^\lambda} \frac{\partial x^\nu}{\partial x'^\delta} g_{\mu\nu}. \tag{A.2}$$

A.3 The gamma function

The Γ-function has some properties that simplifies calculations a lot. Some of those properties, taken from Eq. 8.332 in [13], are

$$\left|\Gamma(iy)\right|^2 = \frac{\pi}{y \sinh(\pi y)}, \tag{A.3}$$

$$\left|\Gamma(1/2 + iy)\right|^2 = \frac{\pi}{\cosh(\pi y)} \tag{A.4}$$

and

$$\Gamma(1 + iy)\Gamma(1 - iy) = \left|\Gamma(1 + iy)\right|^2 = \frac{\pi y}{\sinh(\pi y)}, \tag{A.5}$$

where y is a real quantity.

A.4 Hyperbolic functions

The hyperbolic functions are analogous to the trigonometric functions, but instead of tracing a circle, the hyperbolic functions trace a hyperbola. While the trigonometric functions change sign when derivated twice, the hyperbolic functions do not. The hyperbolic functions are defined using the exponential function e as

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2} \tag{A.6}$$

Some useful properties of the hyperbolic functions are

$$\sinh^2 x = \frac{1}{2}\big[\cosh(2x) - 1\big], \quad \cosh^2 x = \frac{1}{2}\big[\cosh(2x) + 1\big], \tag{A.7}$$

and

$$\sinh x - \sinh y = 2\cosh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right), \tag{A.8}$$

$$\cosh x - \cosh y = 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right). \tag{A.9}$$

B Hypergeometric functions

Hypergeometric functions are functions represented by a series that includes many special functions as limiting cases. The hypergeometric function is a solution of a second-order linear ordinary differential equation, and most second-order linear ordinary differential equations can be transformed into the hypergeometric function. The hypergeometric function is defined as [14]

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_s(b)_s}{(c)_s} \frac{z^s}{s!}, \quad |z| < 1 \tag{B.1} $$

where $(a)_s$ is the rising Pochhammer symbol, also called the rising factorial, defined by

$$ (a)_s = \begin{cases} 1 & s = 0 \\ a(a+1)(a+2)\cdots(a+s-1) & s > 0 \end{cases}. \tag{B.2} $$

The subscripts 2 and 1 in ${}_2F_1$ is used to describe that the hypergeometric function F has two parameters a and b before the semi-colon, and one parameter c after the first semi-colon. Two parameters before the semi-colon means that there are two rising factorials in the numerator, and one parameter after means that there are one rising factorial in the denominator. A hypergeometric function with those subscripts are often used to show definitions and properties of hypergeometric functions, but the subscripts can be any set of positive integers in general. The parameter after the last semi-colon is considered the variable in the function, and is usually labeled z.

A hypergeometric function can also be written using Γ-functions, as

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{s=0}^{\infty} \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)} \frac{z^s}{s!} \tag{B.3} $$

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