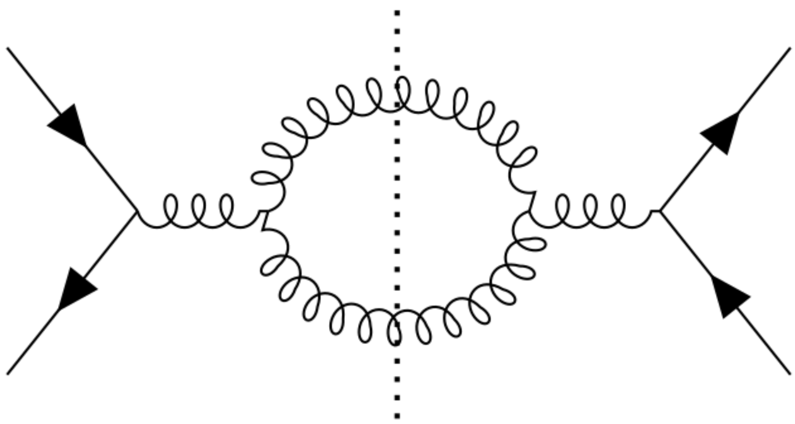
Magnus Nybråten Malmquist

External Gluons in QCD Scattering Amplitudes

Master's thesis in Physics
Supervisor: Michael Kachelrieß
May 2021



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Abstract

To ensure manifest Lorentz covariance, longitudinal and timelike polarizations have to be added to the sum over gauge boson polarizations. The resulting covariant sum is used as the propagator numerator in the Feynman-t' Hooft gauge. Unitarity together with the Cutkosky prescription implies that this numerator can be used also to sum over final states. This is the textbook approach to computing squared amplitudes in QED, given that we sum over polarizations. To apply the method to QCD the intermediate states with Faddeev-Popov ghosts must be considered as well, complicating the procedure. We consider tree level quark-annihilation into two gluons and gluon-gluon scattering and apply the method on up to four external gluons, reproducing the well know results. By treating in detail the Cutkosky prescription some subtle points of the application to QCD are elucidated. Also the generalization to any order in perturbation theory and with any number of external gluons is made clear. Emphasis is put on the Slavnov-Taylor identities that ensure the cancellation of unphysical degrees of freedom in general. These identities also quantify the large amount of redundancy in the amplitude as calculated from Feynman rules. The second part of this work considers the modern spinor-helicity approach to circumvent this redundancy and to obtain more directly gauge invariant on-shell amplitudes. This efficient technology is applied to several examples, and compared to the standard approach.

Sammendrag

Manifest Lorentz-kovarians krever at man legger til langsgående polarisasjonsvektorer i summen over polariseringer for justerbosoner. Det resulterende kovariante utrykket brukes også som teller i Feynman-t' Hooft propagatoren. Unitær symmetri sammen med Cutkosky-regelen impliserer at denne telleren kan brukes også til å summere over sluttilstander. Dette er lærebok-metoden for å beregne kvadrerte amplituder i QED, gitt at vi summerer over polariseringer. For å anvende metoden i QCD må mellomliggende tilstander med Faddeev-Popov spøkelser også tas i betraktning. Vi ser på kvark-annihilasjon og gluon-spredning ved laveste orden i perturbasjons-teori. Der anvender vi Feynman-t' Hooft telleren til å summere over sluttilstander og reproduserer de velkjente resultatene fra litteraturen. Ved å se på Cutkosky-regelen i detalj, belyser vi flere subtile aspekter ved annvendelsen i QCD. Generaliseringen til vilkårlig orden i perturbasjons-teori og med vilkårlig antall eksterne gluoner diskuteres også. Her vektlegges Slavnov-Taylor identitetene som tilrettelegger kansellasjonen av ufysiske frihetsgrader generelt. Disse identitetene kvantifiserer også den store graden av overflødighet i amplituder beregnet fra Feynman-regler. Den andre delen av dette arbeidet ser på den moderne helisitet-metoden, som unngår noe av denne overflødigheten. Denne effektive metoden anvendes på flere eksempler, og sammenlignes med lærebok-metoden.

Acknowledgments

This thesis is the result of work done during the final year of the Master Program in Physics at the Norwegian University of Science and Technology. I thank my supervisor Prof. Michael Kachelrieß for allowing me to work on such a interesting problem, and for excellent guidance along the way. Additionally I thank family, friends and Christina for their invaluable support.

Contents

Chapter 1

Introduction

To conveniently construct manifestly Poincaré invariant observables in quantum field theory, we limit our field content to fields transforming under some representation of the Poincaré Group. Massless spin-1 bosons have two internal degrees of freedom, often labeled by the helicity quantum number. However the two-component fields one obtains from the representation theory of the Poincaré group have fermionic statistics. Instead a vector-field A^μ is used to describe the spin-1 particles, and the additional degrees of freedom are removed by requiring a specific symmetry. This gauge symmetry is the defining feature of a gauge theory. It allows a manifestly Poincaré covariant theory of massless spin-1 particles, and constitutes a core part of the Standard Model of particle physics.

The gauge symmetry constitutes a redundancy. Two field configurations related by a gauge transformation are physically equivalent. Then in a physical observable the gauge symmetry must manifest itself as some identity guaranteeing that the redundant components do not contribute. For scattering amplitudes these are the generalized Ward identities, with the first identity of this type discovered by Ward.

If an initial state can evolve into a final state via some set of intermediate states, then the total probability amplitude is the sum of the amplitudes where each intermediate state is visited. This quantum mechanical fact underpins the method of Feynman diagrams. There the residue of a internal line at the physical momentum $p^2 = m^2$ is recognized as a sum over all possible states for the corresponding particle. In a gauge theory we include additional states in this sum over intermediate states in order to express it in a simple way, while the generalized Ward identities ensure that the addition sums up to zero. The physical requirement that our model includes all possible states, no more and no less, translates to the mathematical property of unitarity for the amplitudes. In this way the requirement of unitarity is directly tied to the existence of generalized Ward identities.

For a massless spin-1 particle like the gluon of quantum chromo-dynamics (QCD) the two physical degrees of freedom are the two transverse polarization states. In Chapter 2 we review the theory of polarization and how the amplitude for the evolution of a initial set of polarizations into a final one is constructed. How the residues of internal lines contain a sum over all states is made precise for the case of a massless spin-1 particle. Also we will see how additional states are added to express the sum in a covariant way. In the Feynman-t' Hooft gauge this covariant sum is used for the internal lines.

Chapter 3 considers unitarity, and how it relates the sum over intermediate states to a sum over final states via Cutkosky's rule. This is illustrated via two examples; the QED vacuum polarization and a four fermion process in QCD. Unitarity implies that the covariant propagator residue of the Feynman-t' Hooft gauge can be used to sum over final states. In quantum electro-dynamics (QED) this is the textbook way to sum a squared amplitude over polarization. At the level of the amplitude the Ward identity ensures that the additional states in the covariant sum do not contribute. Since it is based only on the basic physical property of unitarity the same procedure works also in QCD. There however we must consider that the sum over intermediate states includes also Faddeev-Popov ghosts. Using the covariant sum over polarization therefore requires considering also external ghosts. This corresponds to unitarity being enforced at the amplitude level by a more complex generalized Ward identity.

In Chapter 4 the examples of quark anti-quark annihilation into gluons and gluon-gluon scattering are used to illustrate the above procedure. For comparison we also apply the approach of using only the two physical states in the sum over polarizations, either by inserting these explicitly in a specific frame or by using a non-covariant gauge. The squared gluon-gluon scattering amplitude summed over polarization was first computed using the method of replacing the polarization sum by $-\eta^{\mu\nu}$ by Cutler and Sivers in Ref. [1]. Their result however differs from the now accepted result, first found the same year by Combridge, Kripfganz and Ranft using a different method [2]. Applying the method to QCD is also described briefly in Nachtmann's textbook [3]. However the method as described there does not work when applying it to more than two external gluons. We apply the procedure to obtain correct results for up to four external gluons and elucidate the subtle points with more than two gluons using the experience gained in Chapter 3. The last part of Chapter 4 considers the connection of the above to the generalized Ward identities of QCD; the Slavnov-Taylor identities. With the help of that connection we show how unitarity at the amplitude level is ensured in the general case, with any number of external gluons and at any order in perturbation theory.

The redundancy of the gauge symmetry becomes at the amplitude level the redundancy described by the generalized Ward identities. The end product, the on-shell scattering amplitude, is however physical and free of any redundancy. When deriving that on-shell amplitude from Feynman rules the intermediate calculations are complicated by additional noncontributing terms. The severity of that complication was put in the limelight as Parke, Taylor, Kleiss, Kuijf and others found surprising simplicity in the on-shell amplitudes with six and seven external partons [4–7]. The simplicity emerges in the latter case from cancellations between thousands of Feynman diagrams, each consisting of thousands of terms. In Chapter 5 we review the technique of writing amplitudes in terms of spinor products, and apply it to quark annihilation into two gluons. The technique will help us to rediscover some of the above mentioned simplicity.

The surprising simplicity of the on-shell amplitudes sparked an endeavor to see if it is possible to circumvent the complicated intermediate expressions of the Feynman rules, and to go more directly to the physical amplitude. Chapter 6 considers some of the techniques that have resulted from that approach. In particular the method of on-shell recursion is considered. That represents a computational method for tree-level amplitudes which is completely independent of Feynman rules. To illustrate the power of this new technique we use it to derive formulas for the n-gluon amplitude in special helicity configurations.

All computer code created for and referenced in this thesis can be found at `https://gitlab.com/magnunm/yang-mills-scattering-amplitudes`.

Chapter 2

Scattering Amplitudes in Yang-Mills Theory

The Lagrangian

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}, \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu, \tag{2.1}$$

describing a non-abelian gauge field A^a_μ was first studied by Yang and Mills in Ref. [8] for the gauge group SU(2). Its defining feature is its invariance under gauge transformations of the field $A_\mu = A^a_\mu T^a$,

$$A_\mu \to U A_\mu U^{-1} - \frac{\mathrm{i}}{g} U \partial_\mu U^{-1}, \tag{2.2}$$

where U is an element of a compact Lie group. Of these we will focus on the physically important special unitary groups SU(N). The general feature of these theories is that an infinite class of fields A_μ—those connected via a gauge transformation—describe the same physics. In this chapter we will review some of the fundamentals of computing scattering amplitudes in Yang-Mills theory.

2.1 Gauge Fixing

Despite of the great success of the path integral approach in quantizing electrodynamics, a similar application to Yang-Mills theory was for a long time out of reach. When constructing a path integral over $\exp(\mathrm{i}S_{\mathrm{YM}})$ the integrand is unchanged in the orbit generated by the gauge transformation (2.2). The integration over this orbit can thus be factored out. Since the latter involves an integral over the arbitrary U it renders the path integral ill-defined. Canceling the redundancy introduced by the gauge freedom is therefore required to describe the Yang-Mills field in a path integral formalism. We do this by adding gauge-fixing conditions $\delta(g^a(x))$ to the integrand. These conditions are brought into the action as the gauge-fixing term $\int \mathrm{d}^4x\, \mathcal{L}_{\mathrm{gf}}$. However the delta functions may also induce a change in the path integral measure. In the case of quantum electrodynamics (QED) this amounts to just a overall constant which does not affect the dynamics. In the non-abelian case on the other hand the new measure may be a function of the gauge field integration variable.

Faddeev and Popov [9] were the first to take also this new measure into the exponent as a term $\int \mathrm{d}^4x\, \mathcal{L}_{\mathrm{FP}}$. This allowed the derivation of Feynman rules for the Yang-Mills field, at the cost of introducing new anti-commuting auxiliary fields c and $\bar{c}$. We call the auxiliary fields Faddeev-Popov ghosts. With this addition we can define the generating functional for Yang-Mills theory as follows [10]

$$\begin{aligned} Z[J^\mu, \eta, \bar{\eta}] &= \int \mathcal{D}A_\mu \mathcal{D}c \mathcal{D}\bar{c} \exp\left[\mathrm{i}S + \mathrm{i} \int \mathrm{d}^4x\, J^{a\mu} A^a_\mu + \bar{\eta}^a c^a + \bar{c}^a \eta^a \right], \\ S &= \int \mathrm{d}^4x\, \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{FP}} + \mathcal{L}_{\mathrm{gf}}, \end{aligned} \tag{2.3}$$

where the path integral measure implicitly contains a product over all suppressed indices.

The gauge-fixing condition considered by Faddeev and Popov was the Lorenz gauge condition

$$\partial^\mu A^a_\mu = 0, \tag{2.4}$$

which leads to the gauge-fixing term

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2} \partial^\mu A^a_\mu \partial^\nu A^a_\nu.$$

In the Yang-Mills case we will call this choice the Feynman-t' Hooft gauge, a member of the R_ξ family of gauges,

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\xi} \partial^\mu A^a_\mu \partial^\nu A^a_\nu. \tag{2.5}$$

For the R_ξ-gauges the Faddeev-Popov ghosts are described by the Lagrangian

$$\mathcal{L}_{\mathrm{FP}} = \partial^\mu \bar{c}^a \partial_\mu c^a + g f^{abc} \partial^\mu \bar{c}^a c^b A^c_\mu, \tag{2.6}$$

leading to interactions between the ghost and gauge fields.

Such interactions can be avoided if one sacrifices manifest Lorentz covariance by choosing a non-covariant gauge-fixing condition. A set of gauge-fixing conditions of this type is

$$n^\mu A^a_\mu = 0, \tag{2.7}$$

for some fixed vector n^μ. These lead to the gauge-fixing term

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\kappa} n^\mu A^a_\mu n^\nu A^a_\nu, \tag{2.8}$$

and are known as the generalized axial gauges. In this case $\mathcal{L}_{\mathrm{FP}}$ contains just a kinetic term and can be absorbed in the normalization of the generating functional. Writing the $\mathcal{O}(g^0)$ part of $\mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{gf}}$ as $\frac{1}{2}A^{a\nu}G^{-1}_{ab\,\nu\mu}A^{b\mu}$ we find the propagator in the generalized axial gauges is

$$G^{\mu\nu}_{ab}(k) = \frac{-\mathrm{i}\delta_{ab}}{k^2+\mathrm{i}\epsilon}\left[\eta^{\mu\nu} + \frac{n^2-\kappa k^2}{(k\cdot n)^2}k^\mu k^\nu - \frac{k^\mu n^\nu + k^\nu n^\mu}{k\cdot n}\right]. \tag{2.9}$$

Of these we will use the light-cone gauge, in which $\kappa = 0$ and n^μ is lightlike.

2.2 Polarization

The gauge fields of a Yang-Mills theory are described by the Lorentz vector fields $A^\mu_a(x)$. Approximating these vector field solutions via the usual perturbative approach we start with the non-interacting—free—relativistic solutions. These are for a massless vector field the solutions to the Maxwell equations without sources

$$\partial_\mu\partial^\mu A^\nu_a - \partial^\nu\partial_\mu A^\mu_a = 0. \tag{2.10}$$

A particular solution of the above equation is the plane wave

$$A^\mu_a = \varepsilon^\mu_a \exp(-\mathrm{i}k^\nu x_\nu) + \mathrm{h.c.} \tag{2.11}$$

The 4-momentum k is lightlike $k^2 = 0$ and the 4 component object ε^μ_a is the polarization vector of the plane wave. It is a function of the momentum k. The Lorenz gauge condition (2.4) implies that the polarization vector satisfies $k_\mu\varepsilon^\mu_a = 0$. We label it here with the color index a to make explicit that it represents the polarization vector of the gauge field labeled by a. Since in this non-interacting case the different gauge fields decouple, the properties of ε^μ_a are independent of a. We will for this reason often omit the color index on the polarization vectors.

The second important property of the polarization vector we argue directly from gauge symmetry. Apply to the field A^a_μ an infinitesimal gauge transformation

$$\tilde{A}^a_\mu(x) = A^a_\mu(x) - D^{ac}_\mu\lambda^c(x) \tag{2.12}$$

$$\lambda^c(x) = \lambda^c \exp(-ik^\nu x_\nu). \tag{2.13}$$

We see this gauge transformation induces a change in the polarization vectors

$$\tilde{\varepsilon}^a_\mu = \varepsilon^a_\mu + \mathrm{i}\lambda^a k_\mu + g f^{abc}\lambda^c \exp(-\mathrm{i}k^\nu x_\nu)\varepsilon^b_\mu = \varepsilon^a_\mu + \mathrm{i}\lambda^a k_\mu, \tag{2.14}$$

where the second equality is setting $g = 0$ for the non-interacting case. By the appropriate choice of λ^a we can induce a part $\propto k_\mu$ in any of the polarization vectors. In other words the part of the polarization vector proportional to the 4-momentum is different within each equivalence class generated by the gauge symmetry. Since

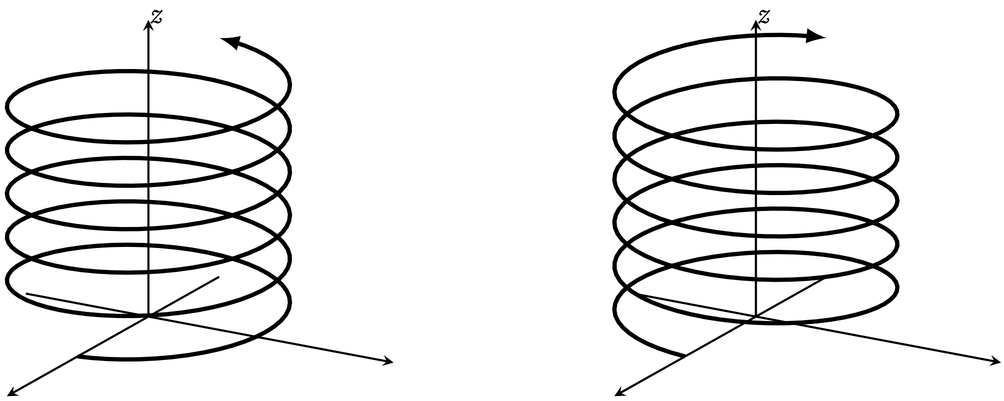


Figure 2.1: Right and left handed helices: the shape drawn out by the 3-vector of a vector plane wave with helicity +1 and -1 respectively.

any two states within such a equivalence class are physically identical, the aforementioned part of the polarization cannot contribute to any physical observable.

Consider now the specific 4-momentum $k^\mu = (\omega, 0, 0, \omega)$. A general 4-component object can be expressed in terms of 4 basis vectors. We choose these basis vectors as

$$\varepsilon_R^\mu = (0, 1, \mathrm{i}, 0)/\sqrt{2} \tag{2.15}$$

$$\varepsilon_L^\mu = (0, 1, -\mathrm{i}, 0)/\sqrt{2} \tag{2.16}$$

$$\varepsilon_+^\mu = (1, 0, 0, 1)/\sqrt{2} \tag{2.17}$$

$$\varepsilon_-^\mu = (1, 0, 0, -1)/\sqrt{2}, \tag{2.18}$$

for reasons that will become apparent. We see that the vector (2.18) does not satisfy $k^\mu \varepsilon_\mu = 0$ and it therefore does not contribute to the polarization vector. The second constraint that the polarization vector should not be proportional to k^μ implies that (2.17) does not contribute. The two constraints have reduced the number of degrees of freedom of the polarization vector from four to two. Any polarization vector can be written as a linear combination of the two basis vectors ε_R^μ and ε_L^μ. These are called the right- and left-handed transverse polarizations while the two excluded ones are called longitudinal.

Why the names right- and left-handed? Insert (2.15) and (2.16) in place of the polarization vector of (2.11) and rewrite the imaginary part as a phase shift between the x^1 and x^2 components. Visualizing the propagation of the resulting vector through 3-dimensional space it traces out the shape of a helix. In the case of ε_R^μ a right-handed helix and in the case of ε_L^μ a left-handed one, see Figure 2.1. We assign a number to express this geometric property of the plane waves, fittingly called the helicity. For a plane wave we define the helicity as the number h in the acquired phase $h\alpha$ of the plane wave when subject to a 3-dimensional rotation of angle α around its axis of propagation. For the present $\mathbf{k}$ this is a rotation of angle α around the z-axis, which we can express as the matrix $R^\mu{}_\nu(\alpha)$. A direct computation shows that $R^{\mu}{}_{\nu}(\alpha)\varepsilon_R^{\nu} = \exp(i\alpha)\varepsilon_R^{\mu}$ and $R^{\mu}{}_{\nu}(\alpha)\varepsilon_L^{\nu} = \exp(-i\alpha)\varepsilon_L^{\mu}$. Thus the right-handed polarization vector corresponds to helicity $+1$ and the left-handed polarization vector to helicity -1. Looking at the expressions for the left- and right-handed polarization vectors we see that the helicity can be reversed by complex conjugation. Using our definition of helicity or the geometric picture of right- and left-handed helices it is apparent that applying a parity transform $\mathbf{k} \to -\mathbf{k}$ while keeping ε^{μ} the same also reverses the helicity. In other words for a transverse polarization λ

$$\varepsilon_{\lambda}^{\mu}(-\mathbf{k}) = \varepsilon_{\lambda}^{\mu}(\mathbf{k})^{*}. \tag{2.19}$$

The decomposition into ε_R^{μ}, ε_L^{μ}, ε_+^{μ} and ε_-^{μ} is not Lorentz invariant. We can see this by applying a Lorentz boost along the x-direction to get $\tilde{k}^{\mu} = \Lambda^{\mu}{}_{\nu}k^{\nu} = (\gamma\omega, -\gamma v\omega, 0, \omega)$. If ε_R^{μ} is a 4-vector then $\tilde{\varepsilon}^{\mu} = \Lambda^{\mu}{}_{\nu}\varepsilon_R^{\nu}$ is the right-handed polarization vector of $\tilde{k}^{\mu}$ and so has helicity $+1$. A 3-dimensional rotation of angle α around the new axis of propagation $(-\gamma v, 0, 1)/\gamma^2$ is again given by a matrix $R^{\mu}{}_{\nu}(\alpha)$. We can check that $R^{\mu}{}_{\nu}(\alpha)\tilde{\varepsilon}^{\nu} \neq \exp(i\alpha)\tilde{\varepsilon}^{\mu}$, which implies that ε_R^{μ} is not a 4-vector.

The polarization vectors appear in scattering amplitudes. After squaring these amplitudes one often sums over the two polarization states ε_R^{μ} and ε_L^{μ}. This procedure is applicable when experimentally we are unable to probe the exact helicity states involved in a process. Thus computing the quantity

$$\mathcal{P}^{\mu\nu} = \sum_{\lambda} \varepsilon_{\lambda}^{\mu *}\varepsilon_{\lambda}^{\nu} = \varepsilon_R^{\mu *}\varepsilon_R^{\nu} + \varepsilon_L^{\mu *}\varepsilon_L^{\nu} \tag{2.20}$$

is important in calculating scattering processes. Do not let the notation confuse; since the physical polarization vectors are not Lorentz vectors the $\mathcal{P}^{\mu\nu}$ is not a Lorentz tensor of rank 2 even though it is written that way. Inserting the expressions (2.15), (2.16) valid for the momentum pointing along the z-axis we get

$$\mathcal{P}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.21}$$

We use this quantity to calculate Lorentz invariant observables. Then we should find some way to express the above in a arbitrary frame, preferably in terms of Lorentz tensors. We cannot express $\mathcal{P}^{\mu\nu}$ purely in terms of Lorentz tensors however, as this would imply the quantity itself transforms as a Lorentz tensor. Another way to see this is to note that no combination of the available tensors $\eta^{\mu\nu}$, $k^{\mu}k^{\nu}$, $k^{\nu}k^{\mu}$ reduce to (2.21) as we set $\mathbf{k}$ to point along the z-axis.

A trick to solving this is to add also a sum involving the longitudinal polarizations ε_+^{μ}, ε_-^{μ}. By inserting the explicit formulas (2.17) and (2.18) we see that

$$\mathcal{P}^{\mu\nu} - \left(\varepsilon_+^{\mu *}\varepsilon_-^{\nu} + \varepsilon_-^{\mu *}\varepsilon_+^{\nu}\right) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -\eta^{\mu\nu}. \tag{2.22}$$

In QED the Ward identity ensures that inserting the second term on the left in place of $\mathcal{P}^{\mu\nu}$ in a squared scattering amplitude gives zero. Then the replacement $\mathcal{P}^{\mu\nu} \rightarrow -\eta^{\mu\nu}$ can be made without changing the value of the squared amplitude. We will investigate whether this procedure is possible also in Yang-Mills theories in the next chapter.

A alternative way to write $\mathcal{P}^{\mu\nu}$ for a arbitrary frame is to introduce another 4-component object n^μ. We already have at our disposal the vector k^μ parallel to ε_+^μ and motivated by (2.22) we then only need a new vector $n^\mu \propto \varepsilon_-^\mu$ to again get $\eta^{\mu\nu}$. With this choice we compute

$$\mathcal{P}^{\mu\nu} = -\eta^{\mu\nu} + \frac{k^\mu n^\nu + n^\mu k^\nu}{k \cdot n}. \tag{2.23}$$

We will see in the next section that the sum over polarizations is connected to the numerator of the propagator. Then the above corresponds to the light-cone gauge, the $n^2 = \kappa = 0$ version of (2.9). The equation (2.23) is not a contradictory expression of $\mathcal{P}^{\mu\nu}$ in terms of Lorentz tensors because choosing the correct vector n^μ requires picking a Lorentz frame.

The description in this section does not consider interactions. In the abelian case the non-interacting theory is physically relevant and the above description is equivalent to the description of polarization in classical electrodynamics. In the non-abelian case (2.10) is no longer the correct equation of motion for a pure gauge field without sources. Furthermore the classical theory described by the Yang-Mills Lagrangian lacks many of the crucial observed properties that are believed to be predicted by the quantum theory. This inhibits a similar physical interpretation of the polarization vector in the Yang-Mills case. Yet we can understand their appearance in scattering amplitudes by looking at the role they play in our formalism for constructing the amplitudes in perturbation theory.

2.3 The LSZ Reduction Formula

In order to investigate the role of the polarization vectors above in computing S-matrix elements we aim in this section to write down the LSZ reduction formula for a Yang-Mills field. We will see how the LSZ reduction formula connects S-matrix elements to Feynman amplitudes. The latter we compute in perturbation theory via the usual approach, while the S-matrix elements are what we ultimately connect to scattering experiments.

Before writing down the reduction formula we derive some results that will be helpful in its interpretation. Firstly we use the particular solution (2.11) to the non-interacting equations of motion to write down a expression for the field operator in the non-interacting case.

$$A_a^\mu(x) = \sum_h \int \frac{\mathrm{d}^3 k}{\sqrt{(2\pi)^3 2\omega_{\mathbf{k}}}} \left[a_\lambda(\mathbf{k})\varepsilon^\mu(\mathbf{k}, \lambda) \exp(-\mathrm{i}(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x})) + \text{h.c.}\right]. \tag{2.24}$$

We have here used the shorthand $\lambda = (h,a)$ for the combination of the helicity and color state. Also we changed the notation for the color and helicity label of the polarization vectors. This to signal that in the following these labels are not used with the Einstein summation convention, while sub or superscripted indices are. The sum $\sum_h$ is over the two physical helicity states. In (2.24) the $a^\dagger_\lambda(\mathbf{k})$ and $a_\lambda(\mathbf{k})$ are one particle creation and annihilation operators satisfying the usual bosonic commutation relations

$$\left[a_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k}')\right] = \delta(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}, \tag{2.25}$$

$$\left[a_\lambda(\mathbf{k}), a_{\lambda'}(\mathbf{k}')\right] = \left[a^\dagger_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k}')\right] = 0, \tag{2.26}$$

and the factor $\sqrt{(2\pi)^3 2\omega_{\mathbf{k}}}$ is a convention dependent normalization factor. Using these relations we can compute the coefficient in the normalization of the vacuum-to-one particle matrix element. We denote this as the coefficient function

$$c^\mu_{ab}(\mathbf{p},h) \equiv \langle 0|A^\mu_a(0)|\mathbf{p},h,b\rangle = \langle 0|A^\mu_a(0)a^\dagger_{h,b}(\mathbf{p})|0\rangle = \frac{\varepsilon^\mu(\mathbf{p},h)\delta_{ab}}{\sqrt{(2\pi)^3 2\omega_{\mathbf{p}}}}. \tag{2.27}$$

Next we want to relate the propagator, the 2-point Green function, to the coefficient function. This will lead us to a connection between the residues of the propagator and the polarization sums of the previous section. The argument is inspired by a more general one found in Weinberg's book [11]. Consider the expression for the propagator in momentum space

$$G(q_1,q_2) = \int \mathrm{d}^4x_1\,\mathrm{d}^4x_2 \exp(\mathrm{i}q_1\cdot x_1)\exp(\mathrm{i}q_2\cdot x_2)\,\langle 0|TA(x_1)A(x_2)|0\rangle\,. \tag{2.28}$$

Here we suppressed the two Lorentz and color indices for brevity, since they are not important for the following discussion. We will show that G has a pole at $q_1^2 = 0$ and that the residue at this pole is given by the coefficient function. Of the 2 possible time orderings, look only at the one where $x_1^0 > x_2^0$. Then insert a complete set of states,

$$\begin{aligned} G(q_1,q_2) = &\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2 \exp(\mathrm{i}q_1\cdot x_1)\exp(\mathrm{i}q_2\cdot x_2)\theta(x_1^0 - x_2^0) \\ &\times \sum_\lambda \int \mathrm{d}^3p\; \langle 0|A(x_1)|\mathbf{p},\lambda\rangle\; \langle \mathbf{p},\lambda|A(x_2)|0\rangle + \ldots \end{aligned}$$

Here the ellipsis represents the other time ordering as well as any multi-particle states in the complete set of states. For non-interacting fields the latter does not contribute. Using

$$\langle 0|A(x)|\mathbf{p},r\rangle = \exp(-\mathrm{i}p\cdot x)\,\langle 0|A(0)|\mathbf{p},r\rangle \tag{2.29}$$

and the integral representation of the step-function

$$\theta(x_1^0 - x_2^0) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega \exp\left(\mathrm{i}\omega(x_1^0 - x_2^0)\right)}{\omega - \mathrm{i}\epsilon} \tag{2.30}$$

we see the x_1 and x_2 integrands are pure phases. The integrations can then be done, and yields delta functions

$$G(q_1,q_2) = -\mathrm{i}(2\pi)^7 \sum_\lambda \int \mathrm{d}^3p \int \frac{\mathrm{d}\omega}{\omega - \mathrm{i}\epsilon} \delta^{(3)}(\mathbf{p}-\mathbf{q}_1)\delta(q_1^0 - \omega_\mathbf{p} + \omega) \\ \times \delta^{(3)}(\mathbf{p}+\mathbf{q}_2)\delta(q_2^0 + \omega_\mathbf{p} - \omega) \langle 0|A(0)|\mathbf{p},\lambda\rangle \langle \mathbf{p},\lambda|A(0)|0\rangle + \ldots$$

Using the delta functions we can preform the remaining integrals

$$G(q_1,q_2) = -\mathrm{i}(2\pi)^7 \delta^{(4)}(q_1+q_2) \frac{1}{\omega_{\mathbf{q}_1} - q_1^0 - \mathrm{i}\epsilon} \sum_\lambda \langle 0|A(0)|\mathbf{q_1},\lambda\rangle \langle \mathbf{q_1},\lambda|A(0)|0\rangle + \ldots \tag{2.31}$$

We see that G has a pole at $q_1^0 = \omega_{\mathbf{q}_1}$. Had we chosen the other time ordering we would have similarly found a pole at $q_2^0 = \omega_{\mathbf{q}_2}$. Close to the $q_1^0 = \omega_{\mathbf{q}_1}$ pole we may neglect the other time ordering, since it is not singular at that point. Also we can transform

$$\begin{aligned}\frac{1}{\omega_{\mathbf{q}_1} - q_1^0 - \mathrm{i}\epsilon} &= \frac{(\omega_{\mathbf{q}_1} + q_1^0 + \mathrm{i}\epsilon)}{(\omega_{\mathbf{q}_1} - q_1^0 - \mathrm{i}\epsilon)(\omega_{\mathbf{q}_1} + q_1^0 + \mathrm{i}\epsilon)} \\ &= -\frac{\omega_{\mathbf{q}_1} + q_1^0}{q_1^2 + \mathrm{i}\epsilon} \\ &\xrightarrow{q_1^2 \to 0} -\frac{2\omega_{\mathbf{q}_1}}{q_1^2 + \mathrm{i}\epsilon},\end{aligned}$$

where we re-scaled the arbitrary positive infinitesimal ϵ. Using this we obtain a simplified expression for the propagator close to the pole

$$G_{ab}^{\mu\nu}(q_1,q_2) = \mathrm{i}(2\pi)^4 \delta^{(4)}(q_1+q_2) \frac{2\omega_{\mathbf{q}_1}(2\pi)^3}{q_1^2 + \mathrm{i}\epsilon} \sum_h c_{ac}^\mu(\mathbf{q}_1,h) c_{cb}^\nu(\mathbf{q}_1,h)^*. \tag{2.32}$$

In the final expression we reinserted the Lorentz and color indices, and expressed the result in terms of the coefficient function. Often the variant integrated over q_2 is used. Inserting also the expression (2.27) for the coefficient function we get the expression most commonly used

$$G_{ab}^{\mu\nu}(q) = \mathrm{i}\delta_{ab} \frac{\sum_h \varepsilon^\mu(\mathbf{q},h)\varepsilon^\nu(\mathbf{q},h)^*}{q^2 + \mathrm{i}\epsilon}. \tag{2.33}$$

We are now ready to write down the LSZ formula in momentum space. We cover here the case where all external particles are gauge bosons. The corresponding expression for external spin-1/2 and spin-0 particles can be found e.g. in Greiner and Reinhardt's book [12]. Consider n incoming particles labeled by $\alpha_i = (\mathbf{p}_i, h_i, a_i)$ and m outgoing particles labeled by $\beta_i = (\mathbf{q}_i, \tilde{h}_i, b_i)$. The a_i and b_i are indices describing the color state. These make up the initial and final Fock state describing some scattering, and their inner product defines the S-matrix element [11]

$$\mathrm{S}_{\beta_1\ldots\beta_m;\alpha_1\ldots\alpha_n} = \langle \beta_1 \ldots \beta_m;\, \mathrm{out}|\alpha_1 \ldots \alpha_n;\, \mathrm{in}\rangle\,. \tag{2.34}$$

A common shorthand notation is to represent the lists $\beta_1 \ldots \beta_m$ and $\alpha_1 \ldots \alpha_n$ by single letters. We assume there exists a complete, orthonormal basis for the asymptotic Fock states, in the sense that

$$\sum_n |n;\, \mathrm{out}\rangle \langle n;\, \mathrm{out}| = 1 = \sum_{n'} \left|n';\, \mathrm{in}\right\rangle \left\langle n';\, \mathrm{in}\right|, \tag{2.35}$$

$$\delta_{n_1 n_2} = \langle n_1;\, \mathrm{out}|n_2;\, \mathrm{out}\rangle = \langle n_1;\, \mathrm{in}|n_2;\, \mathrm{in}\rangle\,. \tag{2.36}$$

The first relation is a statement that we include in our asymptotic states all possibilities, and the second that we normalize these properly. Using the completeness relation (2.35) it follows that $\left|n';\, \mathrm{in}\right\rangle = \sum_n \mathrm{S}_{nn'} |n;\, \mathrm{out}\rangle$. Then the orthonormality (2.36) implies that $\sum_n |\mathrm{S}_{nn'}|^2 = 1$. Interpreting the S-matrix element of two basis states now as the actual matrix element of some abstract operator S the preceding statement is exactly the statement of unitarity for this operator,

$$\mathrm{S}\mathrm{S}^\dagger = \mathbb{I}. \tag{2.37}$$

We look only at the connected part of the S-matrix, meaning that all the α_i, β_i are different. The connected part we denote by $i\mathrm{T} \equiv \mathrm{S}-\mathbb{I}$ where T is the transition operator. The reduction formula connects this quantity to the $N = n + m$ -point connected Green function $G(p_1, p_2, \ldots, p_N)$. It states that the connected Green function has simple poles when its external momenta satisfy the relativistic energy-momentum relation. Further the residue of the Green function as all these external momenta go on-shell is proportional to the connected part of the S-matrix element (2.34). The correct proportionality is given by the coefficient function (2.27) of all the initial and final Fock states. In the case of final one-particle states the complex conjugate of the coefficient function enters. With the convention for the Green function where all the external momenta flow inward,

$$\begin{aligned} G^{\mu_1\ldots\mu_n\,\nu_1\ldots\nu_m}_{a'_1\ldots a'_n\,b'_1\ldots b'_m}(p_1, \ldots, p_n, -q_1, \ldots, -q_m) \prod_{i=1}^{n} c^{a_i a'_i}_{\mu_i}(p_i, h_i) \prod_{j=1}^{m} c^{b_i b'_i}_{\nu_j}(q_j, \tilde{h}_j)^* \times \\ \left(\prod_{i=1}^{n} \frac{p_i^2}{-\mathrm{i}}\right)\left(\prod_{j=1}^{m} \frac{q_j^2}{-\mathrm{i}}\right) \xrightarrow[\text{on-shell}]{p_1\ldots p_n, q_1\ldots q_m} \mathrm{i}\,\mathrm{T}_{\beta_1\ldots\beta_m;\alpha_1\ldots\alpha_n}. \end{aligned} \tag{2.38}$$

To simplify this expression we define the amputated Green function $\mathcal{G}$ as the connected Green function stripped of its external propagators. Explicitly we have

$$\begin{aligned} &G^{\mu_1\ldots\mu_n\,\nu_1\ldots\nu_m}_{a'_1\ldots a'_n\,b'_1\ldots b'_m}(p_1, \ldots, p_n, -q_1, \ldots, -q_m) = \\ &\left(\prod_{i=1}^{n} G^{\mu_i\tilde{\mu}_i}_{a'_i\tilde{a}_i}(p_i)\right)\left(\prod_{j=1}^{m} G^{\nu_i\tilde{\nu}_i}_{b'_i\tilde{b}_i}(-q_i)\right)\mathcal{G}^{\tilde{a}_1\ldots\tilde{a}_n\,\tilde{b}_1\ldots\tilde{b}_m}_{\tilde{\mu}_1\ldots\tilde{\mu}_n\,\tilde{\nu}_1\ldots\tilde{\nu}_m}(p_1, \ldots, p_n, -q_1, \ldots, -q_m). \end{aligned}$$

In the limit of on-shell external particles the expression for the propagator close the pole (2.33) becomes exact. We may insert this expression therefore for the external propagators in the limit of (2.38). Then the contractions with the coefficient function factors prepending the connected Green function can be done explicitly. This allows us to construct a simplified reduction formula in terms of the amputated Green function $\mathcal{G}$. The basis of polarization vectors is normalized as

$$\varepsilon_\mu(\mathbf{p},h)^*\varepsilon^\mu(\mathbf{p},h') = -\delta_{hh'}. \tag{2.39}$$

Remembering also that the sum over physical polarizations is real, the contractions reduce to

$$\frac{p_i^2}{-\mathrm{i}} c_{\mu_i}^{a_i a_i'}(p_i,h_i) G_{a_i'\tilde{a}_i}^{\mu_i\tilde{\mu}_i}(p_i) = \frac{\delta^{a_i\tilde{a}_i}}{\sqrt{(2\pi)^3 2\omega_{\mathbf{p}_i}}}\varepsilon^{\tilde{\mu}_i}(\mathbf{p}_i,h_i), \tag{2.40}$$

$$\frac{q_j^2}{-\mathrm{i}} c_{\nu_i}^{b_j b_j'}(q_j,\tilde{h}_j)^* G_{b_j'\tilde{b}_j}^{\nu_j\tilde{\nu}_j}(-q_j) = \frac{\delta^{b_j\tilde{b}_j}}{\sqrt{(2\pi)^3 2\omega_{\mathbf{q}_j}}}\varepsilon^{\tilde{\nu}_j}(\mathbf{q}_j,\tilde{h}_j)^*. \tag{2.41}$$

The second line relies on the identity (2.19) of the polarization vectors. Inserting the contractions (2.40) and (2.41) into the reduction formula (2.38) we get the version of the reduction formula used in practical calculations,

$$\begin{aligned}
\mathrm{i}\mathrm{T}_{fi} =& \prod_{i=1}^{n}\left[(2\pi)^3 2\omega_{\mathbf{p}_i}\right]^{-1/2}\varepsilon^{\mu_i}(\mathbf{p}_i,r_i)\prod_{j=1}^{m}\left[(2\pi)^3 2\omega_{\mathbf{q}_j}\right]^{-1/2}\varepsilon^{\nu_j}(\mathbf{q}_j,\tilde{r}_j)^*\times \\
&\mathcal{G}^{a_1\ldots a_n b_1\ldots b_m}_{\mu_1\ldots\mu_n\nu_1\ldots\nu_m}(p_1,\ldots,p_n,-q_1,\ldots,-q_m) \\
=&(2\pi)^4\delta^{(4)}\Big(\sum_i p_i - \sum_j q_j\Big)\prod_{i=1}^{n}\left[(2\pi)^3 2\omega_{\mathbf{p}_i}\right]^{-1/2}\prod_{j=1}^{m}\left[(2\pi)^3 2\omega_{\mathbf{q}_j}\right]^{-1/2}\mathrm{i}\mathcal{A}.
\end{aligned} \tag{2.42}$$

The equality is for on-shell external momenta. In the final line we defined i times the Feynman amplitude $\mathcal{A}$ by shifting the overall momentum conservation factor out of the amputated Green function, and contracting it with the polarization vectors. It is a Lorentz scalar, and can be computed perturbatively by applying Feynman rules. The success of the reduction formula is that it makes connecting this computation to S-matrix elements easy by application of (2.42). The preceding discussion also makes clear the appearance of the polarization vectors ε^μ in Feynman amplitudes. They appear in our construction of non-interacting asymptotic states (2.24) required for the validity of the reduction formula.

2.4 Summing over Color

Experimentally we are not able to probe the actual color states involved in a QCD process. This phenomenon is called confinement. In computing matrix elements for these processes we therefore typically sum over the final color states and average over the initial color states. Also in the internal states of the Feynman diagrams we sum over color. We should therefore have a efficient way to compute sums over the color factors that appear in matrix elements. These color factors are expressed in terms of the generators of the continuous symmetry group on which the Yang-Mills theory is based. The appropriate groups are the compact Lie groups, of which we consider SU(N). We start the section therefore by reviewing some fundamental properties of these groups.

The defining matrix representation of SU(N) is the set of all $N \times N$ complex unitary matrices with determinant one. We call this the fundamental representation. It is a Lie group with N^2-1 generators which we denote by T^a. This means that for some set of real parameters ϑ^a we have

$$U = \exp(\mathrm{i}\vartheta^a T^a) \in \mathrm{SU}(N). \tag{2.43}$$

Note that both U and T^a are matrices and contain therefore two indices $U = U_{ij}$, that take N values, which we suppress in our notation. Directly from the definition we can derive some useful properties of the generators. From the unitarity of U it follows that the generators are Hermitian

$$1 = UU^\dagger = \exp\left(\mathrm{i}\vartheta^a T^a - \mathrm{i}\vartheta^a T^{a\dagger}\right). \tag{2.44}$$

From the constraint that the determinant is one we find that the generators are traceless,

$$1 = \det U = \exp(\mathrm{i}\vartheta^a \operatorname{tr} T^a). \tag{2.45}$$

Since there are also N^2-1 generators they form a basis for the $N \times N$ traceless Hermitian matrices. Like with any basis there is some ambiguity in our choice. By convention we fix the normalization by choosing the generators such that

$$\operatorname{tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}. \tag{2.46}$$

Of great practical importance for computing sums over expressions with SU(N) generators is the Fierz identity. It allows us to reduce long traces to shorter ones,

$$T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right). \tag{2.47}$$

The identity follows from the fact that the generators T^a together with the identity matrix form a basis for Hermitian matrices. Any Hermitian matrix A can then be written as

$$A = c_0\mathbb{I} + c_a T^a$$

for some coefficients c_i. Using the fact that the T^a are traceless and (2.46), we can write the coefficients in terms of the matrix entries of A and the generators as

$$A_{ij} = \frac{\operatorname{tr} A}{N}\delta_{ij} + 2\operatorname{tr}(AT^a)T^a_{ij}.$$

Factoring out the arbitrary Hermitian matrix A,

$$\begin{aligned} A_{kl}\delta_{ik}\delta_{jl} &= A_{kl}\frac{\delta_{lk}\delta_{ij}}{N} + 2A_{kl}T^a_{lk}T^a_{ij}, \\ \delta_{ik}\delta_{jl} &= \frac{\delta_{lk}\delta_{ij}}{N} + 2T^a_{lk}T^a_{ij}, \end{aligned}$$

the Fierz identity (2.47) follows.

Feynman rules are often expressed using the structure constants f^{abc} defined by

$$\mathrm{i}f^{abc}T^c = [T^a, T^b]. \tag{2.48}$$

We can rewrite any expression containing the structure constants f^{abc} as an expression containing the generators T^a using the relation

$$\mathrm{i}f^{abc} = 2\,\mathrm{tr}\big(T^aT^bT^c - T^bT^aT^c\big), \tag{2.49}$$

which follows from (2.48) by multiplying with T^d and taking the trace. This latter expression clearly shows the antisymmetry of the structure factors. By application of (2.49) we can always express our amplitudes in terms of the generators T^a and traces over the generators. In the pure gauge boson case there are no fundamental representation indices and the amplitude will only contain traces. With external quarks there will be free fundamental representation indices corresponding to the possible color states of the external quarks. When squaring the amplitude we sum over these indices due to confinement making the color states impossible to probe. Then again only traces over the generators result. We may focus our attention therefore on computing such color traces.

By using the Fierz identity (2.47) we can reduce long traces to shorter ones until we have traces that can be immediately evaluated. This approach of iteratively applying the Fierz identity straightforwardly translates to a algorithm applicable to an implementation in a symbolic language [13]. We have implemented such a algorithm in the program `color-traces.frm`, and apply it to compute some traces we will need later

$$\mathrm{tr}\, T^aT^aT^bT^b = \frac{(N^2-1)^2}{4N}, \tag{2.50}$$

$$\mathrm{tr}\, T^aT^bT^aT^b = -\frac{N^2-1}{4N}, \tag{2.51}$$

$$\mathrm{tr}\, T^aT^bT^cT^d\, \mathrm{tr}\, T^dT^cT^bT^a = \frac{N^6-4N^4+6N^2-3}{16N^2}, \tag{2.52}$$

$$\mathrm{tr}\, T^aT^bT^cT^d\, \mathrm{tr}\, T^dT^cT^aT^b = -\frac{N^4-4N^2+3}{16N^2}, \tag{2.53}$$

$$\mathrm{tr}\, T^aT^bT^cT^d\, \mathrm{tr}\, T^dT^aT^bT^c = \frac{N^4+2N^2-3}{16N^2}. \tag{2.54}$$

Chapter 3

Unitarity and the Cutting of Amplitudes

A direct consequence of the unitarity of the S-matrix (2.37) is the following relation for the matrix elements of the transition operator [10, 14]

$$\sum_{f} |\mathrm{T}_{fi}|^2 = 2\,\mathrm{Im}\{\mathrm{T}_{ii}\}. \tag{3.1}$$

This result is known as the optical theorem, and can be derived by inserting the definition of the transition operator into the unitarity relation (2.37). It relates for a given initial state the imaginary part of the forward scattering amplitude to the total probability of scattering into any possible final state. Note that the latter is not 1 because we removed the identity part from the S-matrix in defining T. Using the LSZ formula in the form (2.42) we can write down the optical theorem for Feynman amplitudes. We directly insert for the case of two final state particles in (3.1)

$$2\,\mathrm{Im}\{\mathcal{A}(i \to i)\} = \sum_{\xi_1 \xi_2} \int |\mathcal{A}(i \to \xi_1 \xi_2)|^2 \, \mathrm{d}\Phi^{(2)}\,, \tag{3.2}$$

where $\xi_1 \xi_2$ are the final state discrete quantum numbers and $\Phi^{(2)}$ is the two particle phase space. The full form of the optical theorem includes all possibilities for the number of final state particles. However when working to a given order in perturbation theory all amplitudes beyond some maximum number of final states are zero. We will in this chapter introduce the Cutkosky rule for finding the imaginary part and through that see how the optical theorem is realized in two specific examples; one from QED and one from QCD.

3.1 Cutkosky's Rule

What causes a Feynman graph to gain an imaginary part? It turns out the imaginary part can be traced to the appearance of branch cuts in the amplitude when considered as a complex function of its external momenta [10, 14]. For each of the possible final states in equation (3.2) there is some threshold energy $s_{\xi_1\xi_2}$ below which the process is kinematically disallowed. Letting s_0 be the smallest of these the right hand side of (3.2) is zero for $s < s_0$. It follows that below this value the forward scattering amplitude is real. Considering the amplitude as a complex function of s (which in turn is a function of the external momenta) we have

$$\mathcal{A}(s) = \mathcal{A}(s^*)^*.$$

Analytically continuing this relation into the entire complex plane we have close to the real axis for $s > s_0$ the relation

$$\mathrm{Im}\{\mathcal{A}(s + \mathrm{i}\epsilon)\} = -\mathrm{Im}\{\mathcal{A}(s - \mathrm{i}\epsilon)\}.$$

Thus starting from the point $s = s_0$ along the positive real axis there is a discontinuity in the complex function $\mathcal{A}(s)$ as we move across the real axis from negative to positive imaginary part. This is a branch cut, and when adding $\pm i\epsilon$ to the denominator of the propagator of internal lines we are choosing a particular side of the branch cut. The discontinuity gives the imaginary part of the Feynman graph,

$$2\mathrm{i}\,\mathrm{Im}\{\mathcal{A}(s + \mathrm{i}\epsilon)\} = \mathrm{Disc}\{\mathcal{A}(s)\}. \tag{3.3}$$

Using this we will uncover next a efficient tool for computing the imaginary part of Feynman amplitudes. Furthermore it will enable us to prove the optical theorem (3.2) for a given order in perturbation theory.

We write the amplitude corresponding to a single Feynman graph in the following way

$$\mathcal{A} = \int \prod_{i=1}^{L} \mathrm{d}^4 k_i \, \frac{P(\{k_i\}, \{p_i\})}{D_1 \ldots D_N}. \tag{3.4}$$

Here P is some polynomial function of the external momenta p_i and the L independent loop momenta k_i. The N denominators correspond to internal lines and are of the form $D_j = q_j^2 - M_j^2$ with q_j being some combination of p_i, k_i. With a Feynman parameter integral we can rewrite the denominator as

$$\frac{1}{D_1 \ldots D_N} = (N-1)! \int_0^1 \mathrm{d}\alpha_1 \ldots \int_0^1 \mathrm{d}\alpha_N \, \frac{\delta(\alpha_1 + \ldots + \alpha_N - 1)}{(\alpha_1 D_1 + \ldots + \alpha_N D_N)^N}.$$

Landau used this to show that the Feynman graph (3.4) has singularities when for each $i = 1, \ldots, N$

$$\alpha_i D_i = 0, \tag{3.5}$$

and for each closed loop,

$$\sum_i \alpha_i q_i = 0, \tag{3.6}$$

where the sum runs over the internal lines making up the loop [15].

Next Cutkosky considered the discontinuity across a branch cut starting from a singularity satisfying the Landau conditions above [16]. He found that given such a singularity where we order our indices such that $D_i = 0$ for $i \leq m$, the discontinuity across the corresponding branch cut is given by

$$\mathrm{Disc}(\mathcal{A}) = (2\pi \mathrm{i})^m \int \prod_{i=1}^{L} \mathrm{d}^4 k_i \, \frac{P(\{k_i\},\{p_i\})}{D_{m+1}\ldots D_N} \prod_{j=1}^{m} \delta^{(+)}(D_j). \tag{3.7}$$

The $\delta^{(+)}$ means we take only the positive root of $q_i^2 - M_i^2 = 0$. This replacement of a set of propagators with delta functions we will call a “Cutkosky cut”.

In order to use this result to compute the imaginary part of a graph we should investigate what kind of Cutkosky cut gives the discontinuity over the specific branch cut starting at $s = s_0$. Consider a graph where for some subset of the internal propagator momenta we have

$$(q_1 + \ldots + q_k)^2 = s,$$

and where there is a closed loop containing at least the propagators q_i for $i = 1, \ldots, k$. Further imagine a singularity satisfying the Landau conditions where $D_i = 0$ for $i = 1, \ldots, k$. The condition (3.5) implies $\alpha_i = 0$ for all other i. Applying the loop condition (3.6) to the closed loop containing at least the propagators q_i for $i = 1, \ldots k$ then gives the condition

$$\sum_{i=1}^{k} \alpha_i q_i = 0. \tag{3.8}$$

Multiplying this equation by q_k and solving for α_k using that $q_k^2 = M_k^2$ we obtain

$$M_k^2 \alpha_k = -\sum_{i=1}^{k-1} \alpha_i q_i \cdot q_k. \tag{3.9}$$

Next we multiply (3.8) by q_j and rearrange to

$$\alpha_k q_k \cdot q_j = -\sum_{i=1}^{k-1} \alpha_i q_i \cdot q_j.$$

Multiplying both sides by M_k^2 we can insert the relation (3.9) on the left hand side. Moving everything under one summation we get

$$\sum_{i=1}^{k-1} \left[(q_i \cdot q_k)(q_k \cdot q_j) - M_k^2 q_i \cdot q_j \right] \alpha_i = 0.$$

Since the α_i are free to vary independently between 0 and 1 for $i = 1, \ldots, k$ the expression in the square brackets must be zero for all i. Then looking at the term

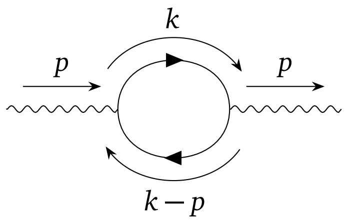


Figure 3.1: QED vacuum polarization diagram.

where $i = j$ we see that $(q_i \cdot q_k)^2 = M_k^2 M_i^2$. The ordering of the momenta is arbitrary and we could have ordered any q_j to be the last momentum q_k. Thus the relation $(q_i \cdot q_j)^2 = M_i^2 M_j^2$ holds for any i, j in $1, \ldots, k$. We can exclude the negative root of this relation by insertion into the expression in square brackets above, so it follows that at the singularity in question we have

$$q_i \cdot q_j = M_i M_j. \tag{3.10}$$

The argument above does not work for $M_k^2 = 0$, however in that case (3.9) gives $q_i \cdot q_k = 0$ immediately, so (3.10) holds in either case. We can also relax the condition that all the q_i momenta must share a loop. Clearly it is sufficient that every pair q_i, q_j with $i, j = 1, \ldots, k$ shares at least one closed loop in order for the identity (3.10) to hold at the singularity. By our choice of cut momenta and using (3.10) the singularity is then located at

$$s = (q_1 + \ldots + q_k)^2 = (M_1 + \ldots + M_k)^2, \tag{3.11}$$

the threshold energy for producing the particles corresponding to the cut propagators as final states. The branch cut starting at this kind of singularity is then of the type giving rise to a imaginary part of the amplitude via (3.3). Thus we have shown how the Cutkosky rule (3.7) can be used to compute the discontinuity across this branch cut, by determining the particular kind of Cutkosky cut required.

3.2 Example: QED Vacuum Polarization

In the remainder of this chapter we will consider the optical theorem (3.2) in concrete examples. We start with the relatively simple case of the QED vacuum polarization. This process is illustrated in Figure 3.1. The fermion in the loop is of Dirac type, with some mass m_f and charge Q_f in units of the elementary charge e. The amplitude for the diagram shown is

$$\mathrm{i}\mathcal{A} = -e^2 Q_f^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{tr}\big(\not{\epsilon}(p)^* (\not{k} + m_f) \not{\epsilon}(p) (\not{k} - \not{p} + m_f)\big)}{(k^2 - m_f^2)[(k-p)^2 - m_f^2]},$$

where the direction of the momenta is as in Figure 3.1. Recall that the fermion loop introduces a overall minus sign in the above.

Now we will apply the Cutkosky prescription to compute the imaginary part of the above amplitude. Remember that the Cutkosky cut giving the imaginary part of the amplitude satisfies $(q_1 + \ldots + q_k)^2 = s$ for the cut propagators, and that there is a closed loop containing at least these propagators. For the present case this corresponds to cutting the two fermion propagators. Combining (3.3) and (3.7) we have

$$2\mathrm{i}\,\mathrm{Im}(\mathcal{A}) = \mathrm{Disc}(\mathcal{A}) = -\,\mathrm{i}e^2Q_f^2 \int \frac{\mathrm{d}^4k}{(2\pi)^2}\delta^{(+)}(k^2 - m_f^2)\delta^{(+)}((k-p)^2 - m_f^2) \times \mathrm{tr}\Big(\not{\epsilon}(p)^*(\not{k} + m_f)\not{\epsilon}(p)(\not{k} - \not{p} + m_f)\Big).$$

The k^0 integral can now be canceled by using

$$\delta^{(+)}(k^2 - m_f^2) = \frac{\delta(k^0 - \omega_{\mathbf{k}})}{2\omega_{\mathbf{k}}}. \tag{3.12}$$

Expecting the loop momentum to become a final state momentum in the optical theorem we rename k to k_1. A new momentum k_2 can also be inserted, provided we cancel it by a delta function,

$$1 = \int \frac{\mathrm{d}^4k_2}{(2\pi)^4}(2\pi)^4\delta^{(4)}(k_2 + k_1 - p).$$

We chose the new momentum as $k_2 = p - k_1$ so that we can use the second $\delta^{(+)}$ to cancel the k_2^0 integral in the same way as in (3.12). The imaginary part can then be written as

$$2\,\mathrm{Im}(\mathcal{A}) = -\,e^2Q_f^2 \int \frac{\mathrm{d}^3k_1}{(2\pi)^3 2\omega_{\mathbf{k}_1}}\frac{\mathrm{d}^3k_2}{(2\pi)^3 2\omega_{\mathbf{k}_2}}(2\pi)^4\delta^{(4)}(k_2 + k_1 - p) \times \mathrm{tr}\Big(\not{\epsilon}(p)^*(\not{k}_1 + m_f)\not{\epsilon}(p)(-\not{k}_2 + m_f)\Big). \tag{3.13}$$

Next we turn our attention to the Dirac trace. Expecting the optical theorem to hold we want to split it into two parts we can recognize as coming from the two amplitudes that make up a square. The trick is to use the completeness relation for Dirac spinors. That relation is equation (C.11), derived in Appendix C. We let s denote the spin quantum number of the fermions. Writing out the matrix indices in the trace makes the transformation clearer,

$$\begin{aligned}\not{\epsilon}(p)^*_{ij}&(\not{k}_1 + m_f)_{jk}\not{\epsilon}(p)_{kl}(-\not{k}_2 + m_f)_{li} \\ &= \not{\epsilon}(p)^*_{ij}\Big(\sum_{s_1} u_{s_1}(k_1)\bar{u}_{s_1}(k_1)\Big)_{jk}\not{\epsilon}(p)_{kl}\Big(-\sum_{s_2} v_{s_2}(k_2)\bar{v}_{s_2}(k_2)\Big)_{li} \\ &= -\sum_{s_1,s_2}\Big[\bar{u}_{s_1}(k_1)_k\not{\epsilon}(p)_{kl}v_{s_2}(k_2)_l\Big]\Big[\bar{v}_{s_2}(k_2)_i\not{\epsilon}(p)^*_{ij}u_{s_1}(k_1)_j\Big].\end{aligned}$$

Noting also that the integrals and delta function on the first line of (3.13) is exactly the two particle phase space we have

$$2\,\mathrm{Im}(\mathcal{A}) = \sum_{s_1,s_2}\int \mathrm{d}\Phi^{(2)}\Big[eQ_f\bar{u}_{s_1}(k_1)\not{\epsilon}(p)v_{s_2}(k_2)\Big]\Big[eQ_f\bar{v}_{s_2}(k_2)\not{\epsilon}(p)^*u_{s_1}(k_1)\Big]. \tag{3.14}$$

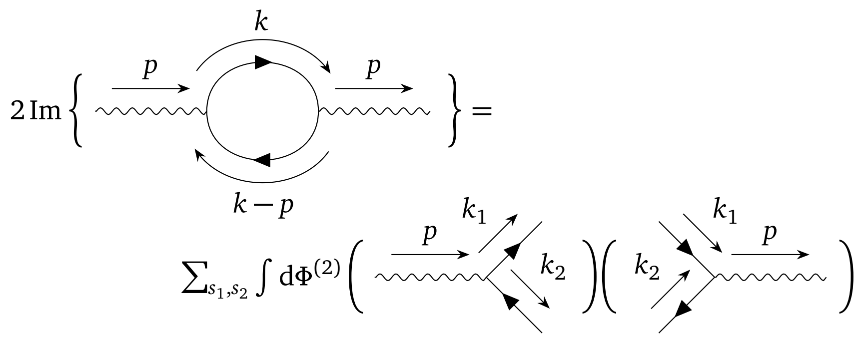


Figure 3.2: The imaginary part of the QED vacuum polarization diagram, as given by the Cutkosky prescription. Graphical representation of equation (3.14).

We recognize the expressions in the square brackets as the amplitude for a photon splitting into a fermion anti-fermion pair and the amplitude for that pair annihilating into a photon. The equation above then has a nice visual representation, shown in Figure 3.2. We see that the Cutkosky cuts correspond graphically to literally the kind of cuts one would make with a pair of scissors.

Equation (3.14) is not quite in the form of the optical theorem (3.2). For it to hold the second bracketed term must be the complex conjugate of the first, so to together produce a square. Recalling that $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ the complex conjugate of the first term is

$$\left[eQ_f\,\varepsilon(p)_\mu \bar{u}_{s_1}(k_1)\gamma^\mu v_{s_2}(k_2)\right]^* = eQ_f\,\varepsilon(p)^*_\mu \bar{v}_{s_2}(k_2)\gamma^\mu u_{s_1}(k_1),$$

exactly the second bracketed expression, so that the optical theorem holds.

Note that the transition from the cross-term of Figure 3.2 to the square of the optical theorem was facilitated by the spinor expression. Complex conjugating a Dirac chain,

$$(\bar{u}\gamma^{\mu_1}\ldots\gamma^{\mu_n}v)^* = \bar{v}\gamma^{\mu_n}\ldots\gamma^{\mu_1}u,$$

has two effects on the corresponding diagram; a outgoing spinor is turned into a incoming one and vice versa and the direction of the charge flow is reversed. Schematically we have $\mathcal{A}(\ldots \to \ldots \phi(p))^* = \mathcal{A}(\phi(p)\ldots \to \ldots)$, where ϕ can be both a photon and a fermion field. By looking at Figure 3.2 we can convince ourselves this is the relation needed for unitarity in general, with ϕ a arbitrary field. Assuming CPT symmetry, and therefore crossing symmetry, the above amplitude relation is equivalent to $\mathcal{A}(\phi^\dagger(-p)\ldots \to \ldots)^* = \mathcal{A}(\phi(p)\ldots \to \ldots)$. Each of these amplitudes can be written in terms of the vacuum expectation value of a time-ordered product (see Section 2.3 and (2.28) in particular). Then we see the amplitude relation holds given that $\langle 0|\ldots\phi^\dagger(x)\ldots|0\rangle^* = \langle 0|\ldots\phi(x)\ldots|0\rangle$, which is always true for a canonically quantized field. Thus CPT symmetry is sufficient for unitarity to hold, given that the s-channel discontinuity is $2\mathrm{i}\,\mathrm{Im}(\mathcal{A})$. Additionally the propagator numerators must give the correct sum over final states. This we will consider in detail in the next section.

3.3 Example: $q\bar{q} \to q\bar{q}$ at $\mathcal{O}(g^4)$

As a next example we will use the Cutkosky prescription to compute the imaginary part of a subset of the Feynman graphs for $q\bar{q} \to q\bar{q}$ at $\mathcal{O}(g^4)$ in QCD. The diagrams we will consider are illustrated in figure 3.3. For the diagrams in figures 3.3a–3.3d we see that the Cutkosky cut giving the imaginary part of the diagram is the one removing the two gluon propagators in the loop. Using the expression (2.33) for the propagator we see that after applying the Cutkosky rule the two gluon propagators will be replaced by a polarization sum and two delta functions. The two delta functions we can use to turn the integral d^4k of the free loop momentum into an integral over $\mathrm{d}\Phi^{(2)}$ for the two cut momenta, as in the previous section.

For forward scattering the spinor expression on the right-hand side of the diagrams in figures 3.3a–3.3d is simply the complex conjugate of the spinor expression on the left hand side. In the end the imaginary part of the diagram in figure 3.3a becomes the square of the s-channel diagram for $q\bar{q} \to gg$. The square is integrated over the two particle phase space of the two gluons and their helicity and color is summed. Similarly the imaginary part of figure 3.3b gives the square of the t- and u-channel diagrams of $q\bar{q} \to gg$. Finally the diagrams 3.3c and 3.3d produce the cross terms between the s and u- or t-channel diagrams for $q\bar{q} \to gg$. With a similar line of argumentation for the diagrams of figures 3.3e–3.3h we see that they produce the squares and cross-terms of the $\mathcal{O}(g^2)$ diagrams for $q\bar{q} \to q\bar{q}$. The role of the imaginary part of the final diagram, figure 3.3i, will be discussed in detail later.

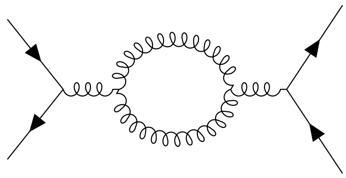
The above suggests applying the procedure outlined to all the diagrams of figure 3.3 we can prove the optical theorem (3.2) at $\mathcal{O}(g^4)$ in QCD with two quarks in the initial state. We repeat therefore the argument in more detail for the case of diagram 3.3a, keeping now track of all numerical factors. We label the external momenta and color indices counter-clockwise as $(1,i),(2,j),(3,k),(4,l)$. With $q = p_1 + p_2$ the amplitude corresponding to the diagram can be written

$$\mathrm{i}\mathcal{A}_{(a)}(12 \to 34) = \frac{C_{ijkl}}{(q^2+\mathrm{i}\epsilon)^2} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{S_{\mu\nu}P^{\mu\nu}(k,q)}{(k^2+i\epsilon)[(k-q)^2+i\epsilon]}. \tag{3.15}$$

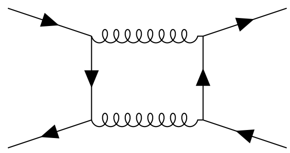
We collect all numerical and color-factors in the term C_{ijkl}. From the four gluon propagators, two quark-gluon vertices and two three-gluon vertices we get a numerical factor $(-\mathrm{i})^6(-1)^2g^4 = -g^4$. There are two ways to connect the two central vertices with two propagators, both giving a identical amplitude, so we should add to this the symmetry factor $1/2$. Thus we have

$$C_{ijkl} = \frac{-g^4}{2} f^{acd} f^{ced} T^a_{kl} T^e_{ij} = \frac{g^4}{2} f^{acd} f^{ecd} T^a_{kl} T^e_{ij}. \tag{3.16}$$

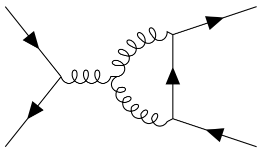
For the three-gluon vertex with all momenta incoming, except the factor $-gf^{abc}$, we will write $V^{\mu\nu\rho}(q_1,q_2,q_3)$. The gluon propagator numerator, except $-i\delta_{ab}$, we write as $N^{\mu\nu}(k)$. For simplicity we set $N^{\mu\nu} = \eta^{\mu\nu}$ for the two propagators not in the



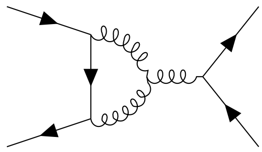
(a) Three gluon vertex loop.



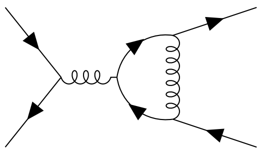
(b) First fermion and gluon loop.



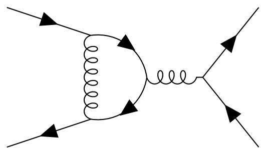
(c) Second fermion and gluon loop.



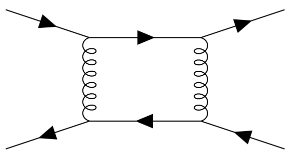
(d) Third fermion and gluon loop.



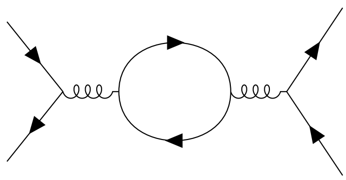
(e) Fourth fermion and gluon loop.



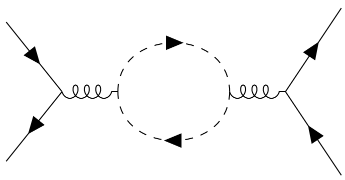
(f) Fifth fermion and gluon loop.



(g) Sixth fermion and gluon loop.



(h) Fermion loop.



(i) Ghost loop.

Figure 3.3: All Feynman graphs with a non-zero imaginary part for $q\bar{q} \to q\bar{q}$ at $\mathcal{O}(g^4)$.

loop, the calculation being essentially identical in the general case. We collect these factors in the tensor

$$\begin{aligned} P^{\mu\nu} &= V^{\nu\gamma\alpha}(q,k-q,-k)N^{\alpha\beta}(k)N^{\delta\gamma}(k-q)V^{\mu\beta\delta}(-q,k,-k+q) \\ &= V^{\nu\gamma\alpha}(q,k-q,-k)N^{\alpha\beta}(k)N^{\delta\gamma}(k-q)V^{\mu\delta\beta}(q,k-q,-k). \end{aligned} \tag{3.17}$$

Finally the spinor part is

$$S_{\mu\nu} = \bar{u}(4)\gamma_\mu v(3)\bar{v}(2)\gamma_\nu u(1). \tag{3.18}$$

We will now consider the forward scattering amplitude, and apply the Cutkosky rule (3.7) to compute the imaginary part. We get

$$\mathrm{Disc}\{\mathcal{A}_{(a)}(12\to 21)\} = \mathrm{i}\frac{C_{ijji}}{(q^2)^2}\int \frac{\mathrm{d}^4k}{(2\pi)^2} S_{\mu\nu}P^{\mu\nu}\delta^{(+)}(k^2)\delta^{(+)}((k-q)^2), \tag{3.19}$$

with $\delta^{(+)}$ taking only the positive root as before. We use the first delta-function to cancel the k^0-integral, by applying (3.12). Since the generators T^a are Hermitian we can write $T^a_{ji} = (T^a_{ij})^*$, giving

$$C_{ijji} = \frac{1}{2}\sum_{c,d}\left|g^2 f^{acd}T^a_{ij}\right|^2.$$

For the numerator of the two cut gluon propagators we will put a sum over polarizations, as in (2.33). Then we can write

$$N^{\alpha\beta}(k)N^{\delta\gamma}(k-q) = \sum_{\lambda\lambda'}\varepsilon^\alpha_\lambda(k)^*\varepsilon^\beta_\lambda(k)\varepsilon^\delta_{\lambda'}(q-k)\varepsilon^\gamma_{\lambda'}(q-k)^*. \tag{3.20}$$

The spinor part for the special case of forward scattering is

$$S_{\mu\nu} = [\bar{v}(2)\gamma_\mu u(1)]^*\bar{v}(2)\gamma_\nu u(1),$$

and combining this with (3.20) and (3.17) yields

$$S_{\mu\nu}P^{\mu\nu} = \sum_{\lambda,\lambda'}\left|\bar{v}(2)\gamma^\nu u(1)V_{\nu\gamma\alpha}(q,k-q,-k)\varepsilon^\gamma_{\lambda'}(q-k)^*\varepsilon^\alpha_\lambda(k)^*\right|^2.$$

Using (3.3) to go from the discontinuity to the imaginary part, and inserting the expressions of the previous paragraph, we find

$$\begin{aligned} 2\,\mathrm{Im}\{\mathcal{A}_{(a)}(12\to 21)\} = \frac{1}{2}\sum_{c,d}\sum_{\lambda,\lambda'}\int \frac{\mathrm{d}^3k}{(2\pi)^2 2\omega_{\mathbf{k}}}\delta^{(+)}((k-q)^2)\times \\ \left|g^2\frac{f^{acd}T^a_{ij}}{q^2}\bar{v}(2)\gamma^\nu u(1)V_{\nu\gamma\alpha}(q,k-q,-k)\varepsilon^\gamma_{\lambda'}(q-k)^*\varepsilon^\alpha_\lambda(k)^*\right|^2. \end{aligned} \tag{3.21}$$

Next we rename the integration momentum $k = p_4$ and insert a new momentum integral $\mathrm{d}^4 p_3$ that we also cancel by a delta-function $\delta^{(4)}(p_3+k-q) = \delta^{(4)}(p_3+p_4-p_1-p_2)$. These relabelings give an alternative expression for the imaginary part,

$$\begin{aligned}
&2\,\mathrm{Im}\{\mathcal{A}_{(a)}(12\to 21)\} \qquad (3.22)\\
=&\frac{1}{2}\sum_{c,d}\sum_{\lambda,\lambda'}\int \frac{\mathrm{d}^3 p_4}{(2\pi)^2 2\omega_{\mathbf{p}_4}}\int \mathrm{d}^4 p_3\,\delta^{(4)}(p_1+p_2-p_3-p_4)\delta^{(+)}(p_3^2)\times\\
&\left| g^2\frac{f^{acd}T^a_{ij}}{(p_1+p_2)^2}\bar{v}(2)\gamma^\nu u(1)V_{\nu\gamma\alpha}\,(p_1+p_2,-p_3,-p_4)\varepsilon^\gamma_{\lambda'}(3)^*\varepsilon^\alpha_\lambda(4)^*\right|^2\\
=&\frac{1}{2}\sum_{c,d}\sum_{\lambda,\lambda'}\int \frac{\mathrm{d}^3 p_4}{(2\pi)^3 2\omega_{\mathbf{p}_4}}\frac{\mathrm{d}^3 p_3}{(2\pi)^3 2\omega_{\mathbf{p}_3}}(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\,|\ldots|^2\\
=&\sum_{c,d}\sum_{\lambda,\lambda'}\int\,|\ldots|^2\,\mathrm{d}\Phi^{(2)},
\end{aligned}$$

where we used $\delta^{(+)}(p_3^2)$ to cancel the p_3^0 integral in the same way as in (3.12). In the phase space integral of the final line we integrate θ only from 0 to $\pi/2$ due to the identical particles in the final state. This is accounted for by the factor $1/2$ in the line above.

With the suggestive labeling of the integration momenta it is apparent that the expression inside the absolute value of (3.22) is the amplitude corresponding to the s-channel diagram for $\bar{q}q \to gg$. Thus we have as advertised

$$2\,\mathrm{Im}\{\mathcal{A}_{(a)}(12\to 21)\} = \sum_{c,d}\sum_{\lambda\lambda'}\int |\mathcal{A}_{s\text{-channel}}(\bar{q}q\to gg)|^2\,\mathrm{d}\Phi^{(2)}. \qquad (3.23)$$

3.3.1 Cutting the Ghost Loop

To obtain the result of the previous section we replaced the numerator of the cut gluon propagators with a sum over polarizations. However in computing the amplitude we are free to choose a covariant gauge like the Feynman-t' Hooft gauge for the internal lines. Then the numerator is $-\eta^{\mu\nu}$, which contains contributions also from longitudinal polarizations, as seen in (2.22). This signals trouble for the optical theorem. We can imagine repeating the procedure above for all the diagrams 3.3a–3.3d, all the while using a covariant gauge like the Feynman-t' Hooft gauge. This would give the sum over twice the imaginary parts as the square of the amplitude for $q\bar{q} \to gg$, summed over helicity by replacing the polarization sum by $\mathcal{P}^{\mu\nu} \to -\eta^{\mu\nu}$ and integrated $\mathrm{d}\Phi^{(2)}$. In QED the Ward identity ensures that this is equivalent to using a sum over the transverse polarizations only. However as we will see in the next chapter the Ward identity no longer holds in QCD. By the optical theorem (3.2) we would expect the sum over twice the imaginary parts to give the correct answer for $\sum\int|\mathcal{A}(q\bar{q}\to gg)|^2\,\mathrm{d}\Phi^{(2)}$. This is then not the case if we include only the diagrams 3.3a–3.3d, as the $\mathcal{P}^{\mu\nu}\to-\eta^{\mu\nu}$ replacement gives the wrong answer.

The above issue is resolved by considering that summing all the diagrams in Figure 3.3 we should get a gauge invariant amplitude. This means the gauge dependent unphysical parts of the $-\eta^{\mu\nu}$ gluon propagator numerator must be canceled in the final answer. Since the Ward identity no longer ensures this, diagrams with internal ghosts must be added to cancel the unphysical parts. For the same reason the inclusion of the diagram 3.3i also saves the optical theorem. Therefore we next compute the imaginary part of the ghost diagram explicitly.

The amplitude of the ghost loop diagram is the same as for the gluon loop diagram (3.15) except for a overall factor -1 due to the fermionic nature of the ghosts, a overall factor 2 since there is no longer a symmetry factor 1/2, and a different $P^{\mu\nu}$. Setting again $N^{\mu\nu}=\eta^{\mu\nu}$ for the two propagators not in the loop we have for the ghosts

$$P^{\mu\nu}=k^{\nu}(k-q)^{\mu}. \tag{3.24}$$

This gives for forward scattering

$$S_{\mu\nu}P^{\mu\nu}=\bar{v}(2)\not{k}u(1)\left[\bar{v}(2)(\not{k}-\not{q})u(1)\right]^{*}. \tag{3.25}$$

We repeat now the steps of the previous section, with the new numerical factors and (3.25). The analogue of equation (3.22) for the ghost loop is then

$$2\,\mathrm{Im}\{\mathcal{A}_{(i)}(12\to21)\}=$$
$$-2\int\left[g^2\frac{\mathrm{i}f^{acd}T^a_{ij}}{(p_1+p_2)^2}\bar{v}(2)\not{p}_4u(1)\right]\left[g^2\frac{\mathrm{i}f^{acd}T^a_{ij}}{(p_1+p_2)^2}\bar{v}(2)(-\not{p}_3)u(1)\right]^{*}\mathrm{d}\Phi^{(2)}.$$

The above is

$$2\,\mathrm{Im}\{\mathcal{A}_{(i)}(12\to21)\}=2\int\mathrm{tr}\,\mathcal{B}\mathcal{B}'^{*}\,\mathrm{d}\Phi^{(2)}, \tag{3.26}$$

where $\mathcal{B}$ is one of the two amplitudes for $q\bar{q}$ into two ghosts and $\mathcal{B}'$ is the other, where the charge flow is reversed. We will see in the next chapter that this is in fact exactly what we need to cancel the unphysical parts of the gluon diagram. The inclusion of ghosts is then required by the optical theorem.

Another consequence of this cancellation is that the $\mathcal{P}^{\mu\nu}\to-\eta^{\mu\nu}$ replacement can still be used in QCD to sum squared amplitudes over polarization, if we also include ghost terms like the one above. We see that in such a procedure the cross term between ghost amplitudes with two different charge flow directions should be used. Alternatively, using the Dirac equation on the $\mathcal{B}'$ term of (3.26),

$$\bar{v}(2)\not{p}_3u(1)=\bar{v}(2)(\not{p}_1+\not{p}_2-\not{p}_4)u(1)=-\bar{v}(2)\not{p}_4u(1),$$

we see that in the present example we can also use ordinary squares of ghost amplitudes, but then with a additional factor -1. This has lead some authors, like Nachtmann in Ref. [3], to believe that squared ghost amplitudes can be used in general, only modified with appropriate factors of -1. However this is a particularity of examples with only two cut propagators and we will see examples where it fails in the next chapter. In general the cross term between two opposite ghost charge flow directions should be used, and no additional factor -1 should be added.

Chapter 4

Ghosts, Ward Identities and Unitarity in QCD

The previous chapter argued based on unitarity that the $\mathcal{P}^{\mu\nu} \to -\eta^{\mu\nu}$ replacement could be used also in QCD. Diagrams with internal ghost lines cancel the gauge dependent part of gluon internal lines. Cutkosky's rule then gives us a way to cancel also the unphysical parts of the replacement $\mathcal{P}^{\mu\nu} \to -\eta^{\mu\nu}$ by using external ghosts. In this chapter we will support these statements with explicit calculations of four parton QCD processes. Additionally the examples will show us how the cancellation is facilitated at the amplitude level. This will lead us to generalizations of the Ward identity, that finally will enable us to understand the cancellation in the general case.

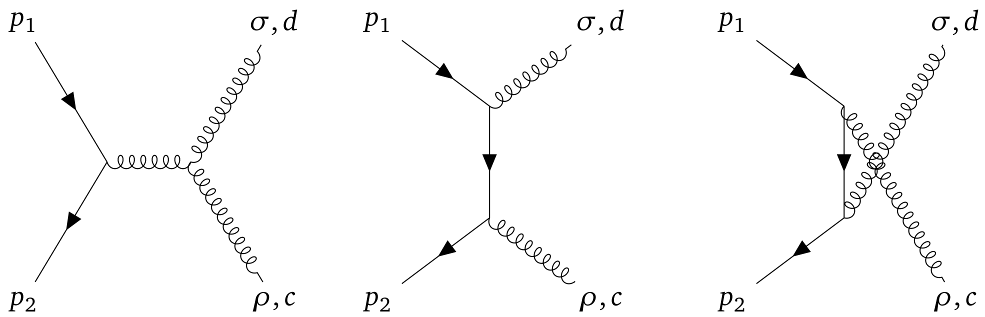
4.1 Example: $q\bar{q} \to gg$

First we consider quark-annihilation into two gluons. There are three diagrams that contribute to this process at tree level, illustrated in figure 4.1. We use the convention where all the momenta are incoming. The physically interesting cases of any two particles outgoing can be obtained by crossing. With the all-in momenta we define the Mandelstam variables as $s = (1+2)^2$, $t = (1+3)^2$ and $u = (1+4)^2$. This choice transforms to the correct expressions for the Mandelstam variables for $q\bar{q} \to gg$ when crossing the two gluons 3, 4 to be outgoing. As already introduced we will from now on use a notation where we let a number stand for its corresponding four-momentum. So $1 \cdot 2 = p_1 \cdot p_2 = \eta_{\mu\nu} p_1^\mu p_2^\nu$ and $\not{1} = \not{p}_1 = \gamma_\mu p_1^\mu$.

We use the relation (2.48) between the structure constants and the generators to see that the only color factors of the amplitude are $T^c T^d$ and $T^d T^c$. Then we can rearrange $\mathcal{A}$ according to these color factors

$$\mathcal{A} = \mathcal{A}_1 T^c T^d + \mathcal{A}_2 T^d T^c.$$

Writing out the expressions for the factors $\mathcal{A}_1$ and $\mathcal{A}_2$ we can see that exchanging $3 \longleftrightarrow 4$ in one of the expressions turns it into the other one. Then after computing



(a) s-channel diagram **(b)** u-channel diagram **(c)** t-channel diagram

Figure 4.1: Tree level Feynman diagrams for $q\bar{q} \to gg$

one of these expressions we can obtain the other by a simple exchange of momenta. To express this we use the suggestive notation

$$\mathcal{A} = A(1234)T^c T^d + A(1243)T^d T^c. \tag{4.1}$$

The amplitude $A(1234)$ is a color-ordered amplitude (CO amplitude), and it is by itself gauge invariant. For the present case of $q\bar{q} \to gg$ at tree level the CO amplitude is

$$\begin{aligned} \mathrm{i}A(1234) = -\mathrm{i}g^2\Big[&\bar{v}(2)\not{\varepsilon}(3)\frac{\not{1}+\not{4}+m}{u-m^2}\not{\varepsilon}(4)u(1) - \frac{\mathcal{N}_{\alpha\beta}}{s}\big(\varepsilon(3)\cdot\varepsilon(4)(3-4)^\beta \\ &+ \varepsilon^\beta(4)(4-1-2)\cdot\varepsilon(3) + \varepsilon^\beta(3)(1+2-3)\cdot\varepsilon(4)\big)\bar{v}(2)\gamma^\alpha u(1)\Big], \end{aligned} \tag{4.2}$$

where $\mathcal{N}^{\mu\nu}$ is the gauge dependent numerator of the gluon propagator. For the Feynman-'t Hooft gauge we have $\mathcal{N}^{\mu\nu} = -\eta^{\mu\nu}$.

Splitting a amplitude into gauge invariant CO components in this way is possible in general if we require that all polarization vectors are transverse. This we will discuss in detail in Chapter 6. The fact that we did not need the constraint of transverse polarizations presently is a peculiarity of our current example.

4.1.1 The Ward Identity

The usual textbook strategy for attacking a gauge invariant amplitude like (4.2) is to square it and sum over all spins and helicities. In this chapter we will use spin for the quark degrees of freedom and helicity for the gluon degrees of freedom. Using the completeness relations (C.11) of the spinors u and v the sum over spins is turned into a trace over gamma matrices. As already discussed one in QED then typically inserts for the photons $\mathcal{P}^{\mu\nu} \to -\eta^{\mu\nu}$. The right hand side is obtained by adding to the sum over polarizations also expressions containing the two longitudinal polarizations, as in (2.22). The QED Ward identity states that if we replace a polarization vector by its corresponding 4-momentum the amplitude must vanish. These two facts together mean we can replace the polarization sum by $-\eta^{\mu\nu}$ without changing the value of the amplitude.

We now investigate how this is changed in our present non-abelian case by checking the Ward identity explicitly for the $q\bar{q} \to gg$ amplitude. Replace $\varepsilon^{\mu}(3)$ by $\varepsilon^{\mu}_{+}(3) = 3^{\mu}/\sqrt{2}\omega_3$ in the color ordered amplitude (4.2). Set also $\mathcal{N}_{\alpha\beta} = -\eta_{\alpha\beta}$ in the numerator of the propagator. The first term in the square brackets in (4.2) can now be simplified using the Dirac equation $(\not{2}+m)v(2) = 0$,

$$\begin{aligned}\bar{v}(2)\not{3} &= \bar{v}(2)(-\not{1}-\not{2}-\not{4})\\ &= \bar{v}(2)(-\not{2}-m+m-\not{1}-\not{4})\\ &= \bar{v}(2)(-\not{1}-\not{4}+m).\end{aligned}$$

This gives $(-\not{1}-\not{4}+m)(\not{1}+\not{4}+m) = m^2-u$ in the numerator of the first term, canceling the denominator. Inserting and rearranging the terms we find that the prefactor of the $\not{\varepsilon}(4)$ term is $2\frac{4\cdot 3}{s}-1 = 0$. The remaining terms are then

$$A(1234) \xrightarrow{\varepsilon(3)\to\varepsilon_+(3)} \frac{g^2}{\sqrt{2}\omega_3}\bar{v}(2)\left[\frac{(3+4)\cdot\varepsilon(4)}{s}\not{3} + \frac{3\cdot\varepsilon(4)}{s}\not{4}\right]u(1).$$

Again using the Dirac equation to transform $\bar{v}(2)\not{4}u(1) = -\bar{v}(2)\not{3}u(1)$ we get

$$A(1234) \xrightarrow{\varepsilon(3)\to\varepsilon_+(3)} g^2\frac{4\cdot\varepsilon(4)}{\sqrt{2}\omega_3 s}\bar{v}(2)\not{3}u(1). \tag{4.3}$$

We see that unlike the QED case the amplitude for $\varepsilon(3)\to\varepsilon_+(3)$ only vanishes when $\varepsilon(4)\cdot 4$ is zero. The latter is true for transverse polarizations but not in general for the longitudinal ones. Looking at the expression for $-\eta_{\mu\nu}$ in terms of the polarization vectors (2.22) we see that inserting this for the sum over polarizations we would erroneously add an additional part to the squared amplitude. Inserting the longitudinal polarization $\varepsilon_-(4)$ for $\varepsilon(4)$ in the center-of-momentum (CoM) frame we have $\varepsilon(4)\cdot 4 = \sqrt{2}\omega_4 = \sqrt{2}\omega_3$. Then we get the Ward-identity violating part of the CO amplitude,

$$A(1234) \xrightarrow[\varepsilon(4)\to\varepsilon_-(4)]{\varepsilon(3)\to\varepsilon_+(3)} \frac{g^2}{s}\bar{v}(2)\not{3}u(1). \tag{4.4}$$

Repeating the calculation replacing $\varepsilon(3)\to\varepsilon_+(3)$ and $\varepsilon(4)\to\varepsilon_-(4)$ in $A(1243)$ we find the total Ward-identity violating part as

$$\mathcal{A} \to g^2(T^cT^d - T^dT^c)\frac{\bar{v}(2)\not{3}u(1)}{s}. \tag{4.5}$$

4.1.2 Inserting Explicit Polarizations

A way to get the correct amplitude without using the trick of replacing the sum over polarizations is to instead explicitly insert transverse polarizations into (4.2).

This requires choosing a particular Lorentz frame in order to have a explicit expression for the polarization vectors. We may however try to rewrite the resulting expression using only the frame-independent Mandelstam variables, thus regaining a manifestly Lorentz invariant expression for the squared amplitude.

In order to have a clearer physical picture we now cross the two gluon momenta to be outgoing $3 \to -3$, $4 \to -4$. Then we use also the usual expressions for the Mandelstam variables $u = (1-4)^2$, $t = (1-3)^2$. We choose the CoM frame defined by $\mathbf{p}_1 = -\mathbf{p}_2$. In this frame the two gluon 3-momenta are anti-parallel so with transverse polarizations we have $\varepsilon(3)\cdot 3 = \varepsilon(3)\cdot 4 = 0$ and conversely for $\varepsilon(4)$. Choosing also again $\mathcal{N}_{\alpha\beta} = -\eta_{\alpha\beta}$ we get

$$A(12-3-4) \overset{\text{CoM}}{=} -g^2\left[\bar{v}(2)\not{\varepsilon}(3)^*\frac{\not{1}-\not{4}+m}{u-m^2}\not{\varepsilon}(4)^*u(1) - \varepsilon(3)^*\cdot\varepsilon(4)^*\bar{v}(2)\frac{\not{3}+\not{4}}{s}u(1)\right]. \tag{4.6}$$

This expression is no longer Lorentz invariant as we allude to by the CoM text over the equals sign. Inserting expressions for the polarization vectors in another frame than the one we have chosen into (4.6) would give the wrong answer.

In the CoM frame we may define the z-axis such that

$$\begin{aligned} 3^\mu &= (\omega, 0, 0, \omega) \\ 4^\mu &= (\omega, 0, 0, -\omega) \\ \varepsilon_R^\mu(3) = \varepsilon_L^\mu(4) &= (0, 1, \mathrm{i}, 0)/\sqrt{2} \\ \varepsilon_L^\mu(3) = \varepsilon_R^\mu(4) &= (0, 1, -\mathrm{i}, 0)/\sqrt{2}. \end{aligned} \tag{4.7}$$

In total there are 4 possible combinations of gluon helicities L/R. However due to the vector current vertex the two initial quarks have opposite helicities and it follows that the two final state gluons must as well (we prove this statement in Chapter 5). This leaves only 2 options and we can compute directly that the final term in (4.6) vanishes due to the product $\varepsilon(3)^*\cdot\varepsilon(4)^*$ being zero,

$$A(12-3-4) \overset{\text{CoM}}{=} -g^2\bar{v}(2)\not{\varepsilon}(3)^*\frac{\not{1}-\not{4}+m}{u-m^2}\not{\varepsilon}(4)^*u(1). \tag{4.8}$$

It remains to compute the squared color-ordered amplitudes $|A(12-3-4)|^2$, $|A(12-4-3)|^2$ and the cross term $A(12-3-4)A(12-4-3)^*$. We define the zero of the azimuth angle $\phi = 0$ to be the $\mathbf{p}_1$ direction. With this definition, we have written the program `explicit-polarisations-spinor-trace.frm` that computes the squared amplitudes and cross term summed over spin in terms of $1\cdot\varepsilon(3)$ and the Mandelstam variables. Notationally it is convenient to introduce the quantity

$$\upsilon = 8(1\cdot\varepsilon(3))^2\left[s - 4(1\cdot\varepsilon(3))^2\right]. \tag{4.9}$$

The needed squares and the cross-term are then

$$\sum_{\text{spin}} |A(12-3-4)|^2 = \frac{g^4 v}{(u-m^2)^2}, \tag{4.10}$$

$$\sum_{\text{spin}} |A(12-4-3)|^2 = \frac{g^4 v}{(t-m^2)^2}, \tag{4.11}$$

$$\sum_{\text{spin}} A(12-3-4)A(12-4-3)^* = \frac{g^4 v}{(t-m^2)(u-m^2)}. \tag{4.12}$$

Now in order to correctly square (4.1) we should not get confused by the notation. The T^c and T^d are matrices in the fundamental representation of SU(N). Bringing back the indices the color factors are $T^c_{ij}T^d_{jk}$, a matrix multiplication leaving two free indices. Each of the two free indices corresponds to the three possible color states of the two initial quarks. The indices c, d take 8 possible values, corresponding to the 8 possible gluon color states. When squaring $\mathcal{A}$ we also sum over all these free group indices. For the quark indices we see that this is equivalent to taking the trace of $\mathcal{A}_{ij}\mathcal{A}^\dagger_{jk}$, and we therefore sometimes call this procedure tracing over color which we will denote simply by $\operatorname{tr}|\mathcal{A}|^2$. Squaring (4.1) we have

$$\begin{aligned}\sum_{\text{spin}} \operatorname{tr}|\mathcal{A}|^2 = &\operatorname{tr} T^c T^c T^d T^d \left(|A(12-3-4)|^2 + |A(12-4-3)|^2\right) \\ &+ 2\operatorname{tr} T^c T^d T^c T^d A(12-3-4)A(12-4-3)^*.\end{aligned} \tag{4.13}$$

Inserting the expressions for the traces, equations (2.50) and (2.51), we get

$$\sum_{\text{spin}} \operatorname{tr}|\mathcal{A}|^2 \overset{\text{CoM}}{=} \frac{4g^4 v}{3}\left[\frac{4}{(u-m^2)^2} + \frac{4}{(t-m^2)^2} - \frac{1}{(u-m^2)(t-m^2)}\right]. \tag{4.14}$$

To simplify the resulting expression we from now on set $m = 0$ for the two quarks. Then in the CoM frame we compute

$$1 \cdot \varepsilon(3)_R = 1 \cdot \varepsilon(3)_L = \frac{\omega \sin(\theta)}{\sqrt{2}} = \sqrt{\frac{tu}{2s}},$$

where θ is the scattering angle. In the CoM frame $\sin^2\theta = 4tu/s^2$ which we used to rewrite the expression in terms of the Mandelstam variables. Inserting this into (4.9) and (4.14) and expanding we get

$$\sum_{\text{spin}} \operatorname{tr}|\mathcal{A}|^2 = \frac{64g^4}{3}\left[\frac{t}{u} + \frac{u}{t} - \frac{1}{4} - 2\frac{t^2+u^2}{s^2} + \frac{1}{2}\frac{tu}{s^2}\right]. \tag{4.15}$$

The equations (4.6)–(4.14) relied on properties of the polarization vectors that only hold in our particular frame. Also the explicit calculation of v is only possible by choosing a Lorentz frame so that we can write down the polarization vectors explicitly like in (4.7). On the other hand we know (4.15) is correct in the CoM frame, and by writing it in terms of the Lorentz invariant Mandelstam variables the expression is also manifestly Lorentz invariant. Thus we have by expressing it in this manner regained a frame independent result, and the CoM text over the equals sign is removed. The expression is the same for the two possible gluon helicities, so the sum over helicities just gives a extra factor 2. Removing also the last term in (4.15) by using $tu/s^2 = 1/2-(t^2+u^2)/2s^2$ we obtain the final result

$$\sum_{\text{helicity}}\sum_{\text{spin}}\operatorname{tr}|\mathcal{A}|^2 = \frac{128g^4}{3}\left[\frac{t}{u}+\frac{u}{t}-\frac{9}{4}\frac{t^2+u^2}{s^2}\right]. \tag{4.16}$$

4.1.3 Using a Non-Covariant Gauge

In the previous section we had to calculate Lorentz products explicitly in a specific frame in order to compute the squared amplitude. This because we could no longer use the replacement $-\eta^{\mu\nu}$ for the sum over physical polarizations without introducing unphysical extra terms. Alternatively we can insert the expression for the polarization sum in the light cone gauge (2.23). As we saw this expression contains only the sum over physical, transverse polarizations. Thus inserting it for the polarization sum we do not introduce the erroneous extra terms and we should therefore re-derive the expression of the previous section.

We are free to make a independent gauge choice for the internal lines and each of the external polarization vectors. We saw in Section 2.3 how internal propagators originate with the corresponding Green function derived from the path integral. Then all the internal propagators must be expressed in the same gauge; the one given by the gauge fixing term in the Lagrangian. For the case of the light cone gauge this means that the arbitrary vector n^μ must be the same for all internal propagators. On the other hand the external polarization vectors come from our constructed asymptotic states. They are by construction transverse, which is why the replacement $-\eta^{\mu\nu}$ is not a equality. The light cone gauge expression (2.23) gives however the correct sum over transverse polarizations with the proper choice of n^μ. This proper choice depends on the 4-momentum and must therefore be done independently for each external polarization vector.

We choose the Feynman-'t Hooft gauge for the internal propagator of (4.2), setting $\mathcal{N}^{\mu\nu} = -\eta^{\mu\nu}$. With this choice we have written the program `axial-gauge-compute-square.frm` to compute the squared amplitude summed over spin, helicity and color, using the light cone gauge expression (2.23) for the sum over polarization. We denote the fixed lightlike vector n_3^μ and n_4^μ for the gluons with momenta 3 and 4 respectively. The computation is done for the case of zero quark masses and is simplified by using $n_4 \cdot 3 = n_3 \cdot 4 = 0$. This latter fact can most easily be seen by expressing the n_3, n_4 vectors explicitly in the CoM frame. With some algebraic manipulation of the resulting expression, all relying on the $s+t+u=0$ identity of the Mandelstam variables, we find that all the n_3 and n_4 dependent terms cancel

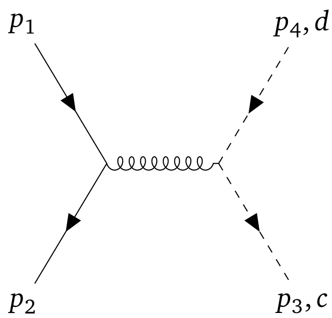


Figure 4.2: Tree -level Feynman diagram for $q\bar{q} \to$ ghosts. There is also a equivalent diagram with the ghost charge flow reversed.

out. The remaining expression is

$$\begin{aligned}\sum_{\text{helicity}} \sum_{\text{spin}} \operatorname{tr} |\mathcal{A}|^2 &= g^4 \left[-96 + \frac{128}{3} \left(\frac{t}{u} + \frac{u}{t} \right) + 192 \frac{tu}{s^2} \right] \qquad (4.17) \\ &= \frac{128 g^4}{3} \left[\frac{t}{u} + \frac{u}{t} - \frac{9}{4} \frac{t^2 + u^2}{s^2} \right].\end{aligned}$$

Which as we expected is again the correct expression from the previous section (4.16).

4.1.4 Using Faddeev-Popov Ghosts

Let us now investigate the possible amplitudes including also external Faddeev-Popov ghosts for the annihilation of a quark -anti-quark pair. We saw in the previous chapter that both gauge invariance and the optical theorem relies on these amplitudes canceling the unphysical contributions to gluon amplitudes. Specifically the square of Ward-identity violating contributions to the gluon amplitude like (4.5) should be canceled by a corresponding ghost term. The ghosts couple only to the gluons so for $q\bar{q}$ into ghosts only the s-channel diagram contributes, as shown in figure 4.2. The amplitude for the diagram shown is the one we denoted $\mathcal{B}'$ in Section 3.3.1. Using (2.48) to express it in terms of the SU(3) generators its value is

$$\mathrm{i}\mathcal{B}' = \mathrm{i}g^2 (T^c T^d - T^d T^c) \frac{\mathcal{N}_{\alpha\beta}}{s} 3^\beta \bar{v}(2) \gamma^\alpha u(1). \qquad (4.18)$$

There is also a corresponding diagram with the ghost charge flow reversed, which we denote by $\mathcal{B}$. It is related to the one above by replacing $3 \to 4$.

Notice that for the choice $\mathcal{N}_{\alpha\beta} = -\eta_{\alpha\beta}$ the expression (4.18) is exactly equal to minus the Ward-identity violating contribution (4.5). This suggests that the cancellation works in the way required by the optical theorem in Section 3.3.1. That means we should again be able to use the prescription of replacing the sum over physical polarizations by $-\eta^{\mu\nu}$, provided we use the ghosts to cancel the resulting contributions from unphysical degrees of freedom. We check this explicitly for our present example.

As seen in Section 3.3.1, the correct ghost term to include is twice the cross term between the two ghost amplitudes with different charge flow directions. That term we can compute straightforwardly with standard techniques. Alternatively we have written the script `ghosts-corresponding-amp.frm` that will compute it. The result summed over spin and traced over color is for zero quark masses

$$\sum_{\text{spin}} \operatorname{tr} \mathcal{B}\mathcal{B}'^{*} = -24 g^4 \frac{tu}{s^2}. \tag{4.19}$$

The other cross term amounts to interchanging $t \longleftrightarrow u$, or to taking the complex conjugate of the above, so we see it has the same value.

To compute the squared amplitude with gluons, where we replace the sum over polarizations by $-\eta^{\mu\nu}$, we have written the program `r-xi-compute-square.frm`. While the program gives the full expression we write down only the $m = 0$ case,

$$\begin{aligned}\sum_{\text{spin}} \sum_{\text{helicity}} \operatorname{tr} |\mathcal{A}|^2 &\xrightarrow{\mathcal{P}^{\mu\nu} \to -\eta^{\mu\nu}} \sum_{\text{spin}} \sum_{\text{helicity}} \operatorname{tr} |\tilde{\mathcal{A}}|^2 \\ &= g^4 \Big[-96 + \frac{128}{3} \Big(\frac{t}{u} + \frac{u}{t} \Big) + 240 \frac{tu}{s^2} \Big].\end{aligned} \tag{4.20}$$

Comparing this to the expression (4.17) from before we see that to regain the correct result we have to add twice the ghost cross term (4.19), or equivalently both ghost cross terms,

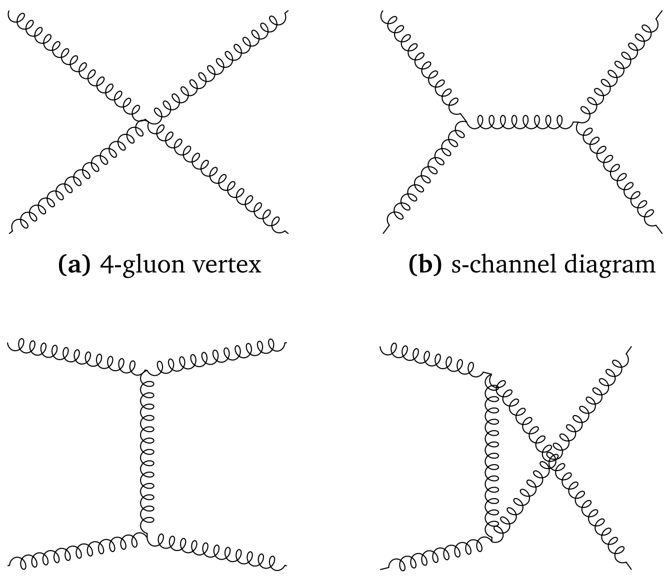
$$\sum_{\text{spin}} \sum_{\text{helicity}} \operatorname{tr} |\tilde{\mathcal{A}}|^2 + 2 \sum_{\text{spin}} \operatorname{tr} \mathcal{B}\mathcal{B}'^{*} = \frac{128 g^4}{3} \left[\frac{t}{u} + \frac{u}{t} - \frac{9}{4} \frac{t^2 + u^2}{s^2} \right]. \tag{4.21}$$

The unphysical extra terms introduced by replacing the polarization sum by $-\eta^{\mu\nu}$ are canceled exactly by the amplitudes with external Faddeev-Popov ghosts. Combining this result with the results of Section 3.3 proves the optical theorem for $q\bar{q} \to q\bar{q}$ at $\mathcal{O}(g^4)$.

4.2 Example: $gg \to gg$

Next we will consider gluon-gluon scattering at tree level. This will allow us to investigate how the procedure of the previous section generalizes to more than two external gluons. Feynman diagrams contributing to this process are shown in figure 4.3. The color factors are of the form

$$f^{abe} f^{cde} = -4 \operatorname{tr}\big(\big[T^a, T^b \big] T^e \big) \operatorname{tr}\big(\big[T^c, T^d \big] T^e \big),$$



(a) 4-gluon vertex **(b)** s-channel diagram

(c) u-channel diagram **(d)** t-channel diagram

Figure 4.3: Tree level Feynman diagrams for $gg \to gg$.

where we used (2.49) to express the structure constants in terms of the generators. Using the Fierz identity (2.47) it follows that for any two matrices A, B we have

$$\mathrm{tr}(AT^e)\,\mathrm{tr}(BT^e) = \frac{1}{2}\,\mathrm{tr}(AB) - \frac{1}{2N}\,\mathrm{tr}(A)\,\mathrm{tr}(B).$$

Then since the trace of a commutator is zero,

$$f^{abe} f^{cde} = -2\,\mathrm{tr}\big(\big[T^a, T^b\big]\big[T^c, T^d\big]\big).$$

Squaring and summing over color all the resulting contractions of two traces are of the form calculated in equations (2.52)–(2.54).

Choosing the Feynman-t' Hooft gauge for the internal lines is simplest. Doing the same for external lines however we need to consider also diagrams with external ghosts, as in the previous section. That procedure is now more complicated; with the Feynman-t' Hooft gauge on all external lines there are now twelve diagrams with two external ghosts and six with four external ghosts to consider. Using instead the non-covariant light-cone gauge we can avoid considering these extra diagrams. Its main drawback is the three terms in the polarization sum (2.23), which compared to the single term of the Feynman-t' Hooft gauge significantly increases the number of terms in intermediate expressions for the squared amplitude.

Since there are no diagrams with only one external ghost we can use the Feynman-t' Hooft gauge for one polarization sum without including any ghost diagrams. We choose the Feynman-t' Hooft gauge for $\varepsilon(4)$ and the light-cone gauge with lightlike vectors n_1, n_2, n_3 for the remaining polarization vectors. To calculate the square summed over polarization and traced over color with these choices we have written the program `1-feynman-3-light-cone-gauge.frm`. In the program the explicit relations $n_1 = p_2/\omega, n_2 = p_1/\omega$ and $n_3 = p_4/\omega$ valid in the CoM frame are used to simplify the resulting expression, here $\omega = \sqrt{s}/2$. The result is

$$\frac{1}{8^2}\frac{1}{2^2}\sum_{\text{pol.}} \operatorname{tr}|\mathcal{A}(gg \to gg)|^2 = \frac{9g^4}{2}\left(3 - \frac{tu}{s^2} - \frac{su}{t^2} - \frac{st}{u^2}\right), \tag{4.22}$$

where we added $1/(8^2 \cdot 2^2)$ to average over initial helicity and color.

Alternatively using the Feynman-t' Hooft gauge both for $\varepsilon(4)$ and $\varepsilon(3)$ we should include diagrams with ghosts in the p_4 and p_3 positions. There are three such diagrams, similar to the three final diagrams of figure 4.3. For $\varepsilon(1)$ and $\varepsilon(2)$ we again choose the light-cone gauge. To calculate the gluon and ghost contributions with these gauge choices we have created the script `2-feynman-2-light-cone-gauge.frm`. We adopt for convenience a notation where g_i stands for a gluon with momentum i which we sum over polarizations using the light-cone gauge while $\tilde{g}_i$ is a gluon where we replace the sum over polarizations by $-\eta^{\mu\nu}$. With c_i we denote a external ghost with the charge flow going in the same direction as the momentum, $\bar{c}_i$ is a ghost with the charge flow in the opposite direction as the momentum. Then we find

$$\sum_{\text{pol.}} \operatorname{tr} \mathcal{A}(g_1 g_2 c_3 \bar{c}_4)\mathcal{A}(g_1 g_2 \bar{c}_3 c_4)^* = -72g^4\left(1 - \frac{tu}{s^2}\right), \tag{4.23}$$

$$\sum_{\text{pol.}} \operatorname{tr}\left|\mathcal{A}(g_1 g_2 \tilde{g}_3 \tilde{g}_4)\right|^2 = \sum_{\text{pol.}} \operatorname{tr}\left|\mathcal{A}(g_1 g_2 g_3 g_4)\right|^2 + 144g^4\left(1 - \frac{tu}{s^2}\right). \tag{4.24}$$

As before we see we get the correct answer by adding the ghost cross term (4.23) and its complex conjugate.

As in sections 3.3.1 and 4.1.4 we could have alternatively used the square of $\mathcal{A}(g_1 g_2 c_3 \bar{c}_4)$ instead of (4.23), but then with a extra minus sign. For $q\bar{q} \to gg$ it was momentum conservation $3 = 1 + 2 - 4$ together the vanishing of $\not{1} + \not{2}$ between the two spinors that enabled this. Presently it is caused by the fact that the momentum combination $1 + 2$ will in the ghost diagrams be contracted either with $1-2$, $\varepsilon(1)$ or $\varepsilon(2)$. With $\varepsilon(1), \varepsilon(2)$ transverse all these contractions give zero so that $\mathcal{A}(g_1 g_2 \bar{c}_3 c_4) = -\mathcal{A}(g_1 g_2 c_3 \bar{c}_4)$. It is when we go to three or more external lines in the Feynman-t' Hooft gauge that using the cross terms over simple squares becomes a necessity.

The explicit calculations with three and four external lines in the Feynman-t' Hooft gauge are presented in our Mathematica script `gluon-gluon-scattering-ghost-cancellation.wl`. The script uses the FeynCalc [17–19] and FeynArts [20] packages in order to demonstrate the cancellations at hand in a concise way.

We obtain when replacing the sum over polarizations by $-\eta^{\mu\nu}$ on three external lines

$$\begin{aligned}\sum_{\text{pol.}} \operatorname{tr}\left|\mathcal{A}(g_1 \tilde{g}_2 \tilde{g}_3 \tilde{g}_4)\right|^2 = &\sum_{\text{pol.}} \operatorname{tr}\left|\mathcal{A}(g_1 g_2 g_3 g_4)\right|^2 \\ &+ 324g^4 - 288g^4\frac{st}{u^2} - 288g^4\frac{su}{t^2} - 432g^4\frac{tu}{s^2}.\end{aligned} \tag{4.25}$$

There are now three possible positions for a line with incoming ghost charge flow, and having fixed that there are two remaining positions for the outgoing ghost charge flow. In other words there are $3 \cdot 2 = 6$ possible ghost amplitudes. Of the six corresponding ghost cross terms only three are unique since the remaining three are related by a complex conjugation. The three ghost cross terms we need to consider are

$$\sum_{\text{pol.}} \operatorname{tr} \mathcal{A}(g_1 \tilde{g}_2 \bar{c}_3 c_4) \mathcal{A}(g_1 \tilde{g}_2 c_3 \bar{c}_4)^* = -90 g^4 + 216 g^4 \frac{tu}{s^2}, \tag{4.26}$$

$$\sum_{\text{pol.}} \operatorname{tr} \mathcal{A}(g_1 c_2 \tilde{g}_3 c_4) \mathcal{A}(g_1 \bar{c}_2 \tilde{g}_3 \bar{c}_4)^* = -18 g^4 + 36 g^4 \frac{t}{s} + 144 g^4 \frac{su}{t^2}, \tag{4.27}$$

$$\sum_{\text{pol.}} \operatorname{tr} \mathcal{A}(g_1 c_2 c_3 \tilde{g}_4) \mathcal{A}(g_1 \bar{c}_2 \bar{c}_3 \tilde{g}_4)^* = -54 g^4 - 36 g^4 \frac{t}{s} + 144 g^4 \frac{st}{u^2}. \tag{4.28}$$

Summing over all the six ghost cross terms gives minus the second line of (4.25). Comparing the square

$$\sum_{\text{pol.}} \operatorname{tr} \left| \mathcal{A}(g_1 \tilde{g}_2 \bar{c}_3 c_4) \right|^2 = 108 g^4 + 36 g^4 \frac{t}{s} - 216 g^4 \frac{tu}{s^2}, \tag{4.29}$$

to (4.26) we see that the cross terms and squares no longer have a simple relationship and we should use the cross terms only.

At this point we can make a interesting observation regarding crossing symmetry. Crossing symmetry states that the amplitude for a process involving a particle with momentum p in the initial state is equal to the amplitude for the otherwise identical process with an anti-particle with momentum $-p$ in the final state. Formally we would say that the amplitude involving the anti-particle is the analytic continuation to negative energies of the amplitude involving the corresponding particle [14]. Notice that the two amplitudes in (4.26) are related to the two in (4.27) by crossing 2 to be outgoing and 3 to be incoming, and then relabeling $2 \longleftrightarrow 3$. If crossing symmetry holds then the first two operations are equivalent to replacing $2 \to -2$ and $3 \to -3$. Then (4.26) is related to (4.27) by merely interchanging $s \longleftrightarrow t$. Looking at the expressions this is clearly not the case, so crossing symmetry is violated.

The above does not contradict that crossing symmetry holds in general for S-matrix elements. The external particles in (4.26)–(4.28) are not valid asymptotic states, and so they do not contribute to the S-matrix. Still we would like to understand why crossing symmetry fails in this case. Crossing symmetry fails at the amplitude level when we use $\varepsilon_+(k)$ or $\varepsilon_-(k)$ since in contrast to the transverse polarizations these do not satisfy $\varepsilon(-k) = \varepsilon(k)^*$, but rather $\varepsilon(-k) = -\varepsilon(k)$. However summing over polarizations $\varepsilon_+(k)$ and $\varepsilon_-(k)$ always appear in pairs, so the two negative signs cancel out. This is then not the cause of the violation observed between (4.26) and (4.27). Instead the answer does not come from a consideration of the unphysical external states as one might expect, but rather how we handle the remaining transverse gluon at p_1. To sum over transverse polarizations in (4.26) we use the light-cone gauge and the CoM frame to set $n_1 = 2/\omega$. Now imagine we interchange $s \longleftrightarrow t$ in the resulting expression. This is equivalent to $2 \to -3$, $3 \to -2$, which means that we effectively set $n_1 = -3/\omega$. However in (4.27) we still use $n_1 = 2/\omega$, leading to the discrepancy between (4.27) and the $s \longleftrightarrow t$ interchanged (4.26).

Clearly the above discrepancy is self afflicted. It would not occur had we simply kept the arbitrary vector n_1 around until the very end, instead of replacing it by $2/\omega$. However n_1 does not cancel out and would enter in (4.25)–(4.28), severely complicating those expressions. We might wonder at the explicit appearance of a vector whose whole premise is to not appear in the final answer. But this is exactly what we also determined in the previous paragraph; the amplitudes (4.25)–(4.28) depend on n_1. This is just another way to state that these amplitudes are not gauge invariant. Only when summing them all do we get a gauge invariant object. This means that if we want to keep n_1 around in one of the amplitudes, we must keep it around in all of them. When summing them all the n_1 will finally cancel out.

Remaining is the case of substituting $-\eta^{\mu\nu}$ for the sum over polarizations for all four external gluons. To cancel the contribution from unphysical polarizations we must now include also all amplitudes with four external ghosts. There are six such amplitudes and six corresponding cross terms. As before only three need to be computed, the remaining three being complex conjugates of the first. Now there are 12 amplitudes with two external ghosts. The six amplitudes in (4.26)–(4.28) plus an additional six where there is a ghost at position 1. The last six are however just given by relabelings of momenta in the first. The full equality required by unitarity is then

$$
\begin{aligned}
\sum_{\text{pol.}} \operatorname{tr}\left|\mathcal{A}(g_1 g_2 g_3 g_4)\right|^2 &= \sum_{\text{pol.}} \operatorname{tr}\left|\mathcal{A}(\tilde{g}_1 \tilde{g}_2 \tilde{g}_3 \tilde{g}_4)\right|^2 \qquad (4.30)\\
+4\sum_{\text{pol.}} \operatorname{tr}\mathcal{A}(\tilde{g}_1\tilde{g}_2 c_3 \bar{c}_4)\mathcal{A}(\tilde{g}_1\tilde{g}_2\bar{c}_3 c_4)^* &+ 4\sum_{\text{pol.}} \operatorname{tr}\mathcal{A}(\tilde{g}_1 c_2 \tilde{g}_3 c_4)\mathcal{A}(\tilde{g}_1\bar{c}_2\tilde{g}_3\bar{c}_4)^*\\
+4\sum_{\text{pol.}} \operatorname{tr}\mathcal{A}(\tilde{g}_1 c_2 c_3 \tilde{g}_4)\mathcal{A}(\tilde{g}_1\bar{c}_2\bar{c}_3\tilde{g}_4)^* &+ 2\sum_{\text{pol.}} \operatorname{tr}\mathcal{A}(c_1\bar{c}_2\bar{c}_3 c_4)\mathcal{A}(\bar{c}_1 c_2 c_3\bar{c}_4)^*\\
+2\sum_{\text{pol.}} \operatorname{tr}\mathcal{A}(c_1\bar{c}_2 c_3\bar{c}_4)\mathcal{A}(\bar{c}_1 c_2\bar{c}_3 c_4)^* &+ 2\sum_{\text{pol.}} \operatorname{tr}\mathcal{A}(c_1 c_2 c_3 c_4)\mathcal{A}(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4)^*.
\end{aligned}
$$

The 19 original terms on the right hand side have been reduced to seven. However we can do much better by considering that—since there are no light-cone gauge vectors n now—crossing again holds. This means we can get all the ghost terms above by crossing the expressions for $\mathcal{A}(\tilde{g}_1\tilde{g}_2 c_3\bar{c}_4)\mathcal{A}(\tilde{g}_1\tilde{g}_2\bar{c}_3 c_4)^*$ and $\mathcal{A}(c_1\bar{c}_2\bar{c}_3 c_4)\mathcal{A}(\bar{c}_1 c_2 c_3\bar{c}_4)^*$ into the appropriate channels. We have written the FORM script `4-feynman-gauge.frm` to calculate the three terms

$$\sum_{\text{pol.}} \operatorname{tr} \left| \mathcal{A}(\tilde{g}_1 \tilde{g}_2 \tilde{g}_3 \tilde{g}_4) \right|^2 = 72 g^4 \left(54 - 25 \frac{tu}{s^2} - 25 \frac{su}{t^2} - 25 \frac{st}{u^2} \right), \tag{4.31}$$

$$\sum_{\text{pol.}} \operatorname{tr} \mathcal{A}(\tilde{g}_1 \tilde{g}_2 c_3 \bar{c}_4) \mathcal{A}(\tilde{g}_1 \tilde{g}_2 \bar{c}_3 c_4)^* = -36 g^4 + 180 g^4 \frac{tu}{s^2}, \tag{4.32}$$

$$\sum_{\text{pol.}} \operatorname{tr} \mathcal{A}(c_1 \bar{c}_2 \bar{c}_3 c_4) \mathcal{A}(\bar{c}_1 c_2 c_3 \bar{c}_4)^* = -18 g^4 \frac{tu}{s^2} - 18 g^4 \frac{st}{u^2}. \tag{4.33}$$

Our Python script `4-feynman-gauge-crossings.py` takes these values and crosses them into the relevant channels to obtain again the correct result (4.22).

4.3 The Generalized Ward Identities; Unitarity to All Orders

In QCD we found that the Ward identity of QED was violated. As calculated in Section 4.1, the amplitude for $q\bar{q} \to gg$ did not vanish when replacing a polarization vector by its corresponding four-momentum. However we also found that the term violating the Ward identity was equal to a corresponding amplitude with external ghosts. More precisely the amplitude with unphysical polarization vectors $\varepsilon_+^\mu(p_3)$ and $\varepsilon_-^\mu(p_4)$ is equal to minus the amplitude with the external gluon lines replaced by a ghost line with the charge flow outgoing at p_3. This fact enabled the cancellation of Section 4.1.4. That cancellation in turn, as we saw in Section 3.3, is required for unitarity at $\mathcal{O}(g^4)$.

We may hope that there exists analogous connections between amplitudes with longitudinal external gluons and amplitudes with ghosts also for N external lines and at $\mathcal{O}(g^n)$. Such relations should facilitate the generalization of the cancellations of the previous two sections, ensuring unitarity at any order in perturbation theory. Examples of these generalized Ward identities were first found by t' Hooft in Ref. [21], and they play a central role also in the renormalizability of Yang-Mills theory. The derivation of t' Hooft is based on a combinatorial argument applied directly to Feynman diagrams. That approach was developed further by Taylor in Ref. [22], while Slavnov gave a alternative derivation using path integral methods [23]. The generalized Ward identities of Yang-Mills theory are therefore often called Slavnov-Taylor identities.

4.3.1 Slavnov-Taylor identities

We will focus on the Slavnov-Taylor identities needed to prove perturbative unitarity. To derive them we will use the path integral method of Slavnov. Apply to the generating functional (2.3) a infinitesimal gauge transformation,

$$A_\mu^a(x) \to A_\mu^a(x) - D_\mu^{ab}(x)\vartheta^b(x) = A_\mu^a(x) - \partial_\mu \vartheta^a(x) - g f^{abc} A_\mu^c(x) \vartheta^b(x). \tag{4.34}$$

The resulting variation of the generating functional is divided into the variation of the integrand and the change in the path integral measure. To compute the change in the path integral measure we should compute the Jacobian of the transformation (4.34),

$$\frac{\delta[A^a_\mu(x)-D^{ad}_\mu(x)\vartheta^d(x)]}{\delta A^b_\nu(y)} = \delta^\nu{}_\mu \delta(x-y)\big[\delta^{ab}+gf^{abd}\vartheta^d(x)\big].$$

Notice that the Jacobian determinant is of the form $\det(\mathbb{I}+\epsilon T)$ with ϵ infinitesimal and T a anti-symmetric matrix. This follows from the anti-symmetry of f^{abd}. Since the trace of a anti-symmetric operator is zero we have

$$\det(\mathbb{I}+\epsilon T) = \det(\exp(\epsilon T)) = \exp(\epsilon\, \mathrm{tr}(T)) = 1,$$

and the Jacobian determinant of the transformation (4.34) is unity.

Next we turn to the change in the integrand. By construction the pure Yang-Mills part S_{YM} is unchanged. By applying a partial integration to the Faddeev-Popov action (2.6) we can write it as

$$S_{\mathrm{FP}} = \int \mathrm{d}^4x\, \big[-\bar{c}^a(x)M^{ab}(x)c^b(x)\big], \tag{4.35}$$

with

$$M^{ab}(x) = \partial^\nu D^{ab}_\nu. \tag{4.36}$$

The change in the Faddeev-Popov action under (4.34) is then

$$\begin{aligned}\delta S_{\mathrm{FP}} &= -\int \mathrm{d}^4x\, \bar{c}^a(x)\delta M^{ab}(x)c^b(x)\\ &= gf^{abc}\int \mathrm{d}^4x\, \bar{c}^a(x)M^{cd}(x)\vartheta^d(x)c^b(x).\end{aligned}$$

This and later expressions are greatly simplified by a re-expression of the arbitrary function. We choose $\vartheta^d(x)$ such that

$$M^{cd}(x)\vartheta^d(x) = \chi^c(x), \tag{4.37}$$

for some new arbitrary function $\chi^c(x)$. Finding such a $\vartheta^d(x)$ can by done by the method of Green functions. Let $\Delta^{ab}(x,y)$ be the inverse of the differential operator $M^{ab}(x)$. Assuming this inverse exists the expression

$$\vartheta^c(x) = \int \mathrm{d}^4y\, \Delta^{cb}(x,y)\chi^b(y) \tag{4.38}$$

satisfies (4.37). With this choice the change in the Faddeev-Popov action is

$$\delta S_{\mathrm{FP}} = gf^{abc}\int \mathrm{d}^4x\, \bar{c}^a(x)\chi^c(x)c^b(x). \tag{4.39}$$

Recall that we can alternatively write the path integral over the ghost fields of $\exp(\mathrm{i}S_{\mathrm{FP}})$ as $\sim \det M$. Then we should be able to derive the change (4.39) also from the change of this determinant. Using the expression $\delta \det X = \det X \,\mathrm{tr} X^{-1}\delta X$ for the variation of a determinant we have

$$\det M \to \det M \left[1 + g f^{abc} \int \mathrm{d}^4x\, \mathrm{d}^4y\, \chi^c(x)\Delta^{ab}(x,y)\delta(x-y) \right].$$

Comparing this to (4.39) we see that inside the integrand of the path integral the replacement

$$\Delta^{ab}(x,y) \to -\mathrm{i}c^a(x)\bar{c}^b(y), \tag{4.40}$$

should be valid.

Out next goal is to remove the δS_{FP} contribution entirely by shifting also $c^a(x)$ in the generating functional. Consider the transformation of the ghost field

$$c^a(x) \to c^a(x) - \alpha g f^{dec} \int \mathrm{d}^4y\, \Delta^{ad}(x,y)\chi^c(y)c^e(y), \tag{4.41}$$

where α is some real parameter. Under this transformation the change δS_{FP} in the Faddeev-Popov action is α times the change (4.39) due to the infinitesimal gauge transformation. This follows directly from insertion into (4.35) and the definition of Δ^{ab}. If we set before the shift the ghost sources to zero the only additional change to the generating functional comes from the Jacobian. Again we utilize that the arbitrary function χ is infinitesimal to write the Jacobian determinant as

$$\begin{aligned}
\det\left[\frac{\delta c'^d(x)}{\delta c^a(z)}\right] &= \det\left[\delta^{da}\delta(x-z) - \alpha g f^{bac}\Delta^{db}(x,z)\chi^c(z)\right] \\
&= \exp\left[-\alpha g f^{bac} \int \mathrm{d}^4x\, \Delta^{ab}(x,x)\chi^c(x)\right] \\
&= 1 + \alpha g f^{abc} \int \mathrm{d}^4x\, \Delta^{ab}(x,x)\chi^c(x).
\end{aligned}$$

By using the replacement (4.40) this reveals itself as again $1 + \mathrm{i}\alpha\delta S_{\mathrm{FP}}$ where δS_{FP} is the change in the Faddeev-Popov action due to the infinitesimal gauge transformation. Thus setting $\alpha = -1/2$ the combined infinitesimal gauge transformation (4.34) and ghost shift (4.41) leaves S_{FP} unchanged.

It remains to compute the variation of the gauge fixing and source terms under the infinitesimal gauge transformation (4.34). The change of the gauge fixing action is

$$\delta S_{\mathrm{gf}} = \frac{1}{\xi} \int \mathrm{d}^4x\, \partial^\mu A^a_\mu(x) M^{ac}(x)\vartheta^c(x) = \frac{1}{\xi} \int \mathrm{d}^4x\, \partial^\mu A^a_\mu(x)\chi^a(x). \tag{4.42}$$

Using (4.38) we write also the variation of the source term as a function of χ,

$$\begin{aligned}\delta S_{\mathrm{s}} &= -\int \mathrm{d}^4x\, J^{\mu a}(x) D_\mu^{ac}(x) \int \mathrm{d}^4y\, \Delta^{cb}(x,y)\chi^b(y) \\ &= -\int \mathrm{d}^4x \int \mathrm{d}^4z\, J^{\mu b}(z) D_\mu^{bc}(z)\Delta^{ca}(z,x)\chi^a(x). \end{aligned} \tag{4.43}$$

The full change in the generating functional under the simultaneous transformations (4.34) and (4.41) is then finally

$$Z \to Z + \delta Z = \int \mathcal{D}A_\mu \mathcal{D}c\mathcal{D}\bar{c}\, \exp(\mathrm{i}S + \mathrm{i}S_s)\big(1 + \mathrm{i}\delta S_{\mathrm{gf}} + \mathrm{i}\delta S_{\mathrm{s}}\big).$$

However since this is simply a change in integration variables, the value of the path integral is unchanged. In other words $\delta Z = 0$. Since $\chi^a(x)$ is arbitrary this equation still holds if we drop χ and the integration over x. Inserting also the replacement (4.40) valid in the integrand of the path integral gives

$$\begin{aligned}\int \mathcal{D}A_\mu \mathcal{D}c\mathcal{D}\bar{c}\, \exp\!\left(\mathrm{i}S + \mathrm{i}\int \mathrm{d}^4x\, J^{a\mu}A_\mu^a\right)\!\Bigg[\frac{1}{\xi}\partial^\mu A_\mu^a(y) \\ + \mathrm{i}\bar{c}^a(y)\int \mathrm{d}^4z\, J^{b\mu}(z) D_\mu^{bc}(z) c^c(z)\Bigg] = 0.\end{aligned} \tag{4.44}$$

This equation will be our starting point for deriving a family of Slavnov-Taylor identities. Taking one functional derivative with respect to $J^{d\nu}(x_1)$ a factor $\mathrm{i}A_\nu^d(x_1)$ is brought down from the exponent. Applying the functional derivative to the term in the square brackets the d^4z is canceled, leaving $\bar{c}^a(y)D_\nu^{dc}(x_1)c^c(x_1)$. In total the equation (4.44) becomes

$$\begin{aligned}\int \mathcal{D}A_\mu^a \mathcal{D}c^a\mathcal{D}\bar{c}^a\, \exp\!\left(\mathrm{i}S + \mathrm{i}\int \mathrm{d}^4x\, J^{a\mu}A_\mu^a\right)\!\Bigg[\frac{1}{\xi}A_\nu^d(x_1)\partial^\mu A_\mu^a(y) \\ + \mathrm{i}A_\nu^d(x_1)\bar{c}^a(y)\int \mathrm{d}^4z\, J^{b\mu}(z) D_\mu^{bc}(z) c^c(z) + \bar{c}^a(y)D_\nu^{dc}(x_1)c^c(x_1)\Bigg] = 0.\end{aligned} \tag{4.45}$$

This equation is interesting in its own right. Setting J_μ^a to zero we can use it to show that there are no higher-order corrections to the longitudinal part of the gluon propagator (see e.g. Ref. [23]). The identity we will use to prove perturbative unitarity however is a relation between N-point functions. Thus to arrive at it we need to take more functional derivatives. Taking N functional derivatives with respect to

$J^{\mu_1 a_1}(x_1)\ldots J^{\mu_N a_N}(x_N)$ of the relation (4.44) we get

$$\int \mathcal{D}A^a_\mu \mathcal{D}c^a \mathcal{D}\bar{c}^a \exp\left(\mathrm{i}S + \mathrm{i}\int \mathrm{d}^4x\, J^{a\mu}A^a_\mu\right)\left[\frac{1}{\xi}\prod_{i=1}^{N} A^{a_i}_{\mu_i}(x_i)\partial^\mu A^a_\mu(y) \right. \tag{4.46}$$
$$+\mathrm{i}\prod_{i=1}^{N} A^{a_i}_{\mu_i}(x_i)\bar{c}^a(y)\int \mathrm{d}^4z\, J^{b\mu}(z)D^{bc}_\mu(z)c^c(z)$$
$$\left. +\,\bar{c}^a(y)\sum_{j=1}^{N}\prod_{\substack{i=1\\ i\neq j}}^{N} A^{a_i}_{\mu_i}(x_i)D^{a_jc}_{\mu_j}(x_j)c^c(x_j)\right] = 0.$$

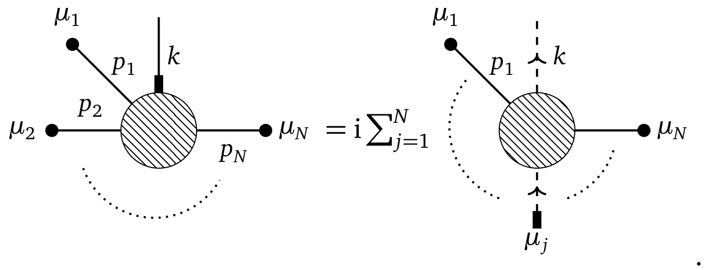
For $N = 1$ this agrees with (4.45). Taking another functional derivative of (4.46) we obtain the formula for $N+1$, thus proving it by induction. We want to rebrand this formula as a statement about N-point Green functions. Setting the external source J to zero and inserting the covariant derivative we can write it as

$$0 = \frac{1}{\xi}\partial^\mu_y \langle 0|\mathrm{T}A^a_\mu(y)\prod_{i=1}^{N} A^{a_i}_{\mu_i}(x_i)|0\rangle \tag{4.47}$$
$$+\sum_{j=1}^{N}\left[\partial^{x_j}_{\mu_j}\langle 0|\mathrm{T}\bar{c}^a(y)c^{a_j}(x_j)\prod_{\substack{i=1\\ i\neq j}}^{N} A^{a_i}_{\mu_i}(x_i)|0\rangle\right.$$
$$\left. +\,\mathrm{i}gf^{a_jcd}\langle 0|\mathrm{T}\bar{c}^a(y)c^c(x_j)A^d_{\mu_j}(x_j)\prod_{\substack{i=1\\ i\neq j}}^{N} A^{a_i}_{\mu_i}(x_i)|0\rangle\right],$$

where T stands for the time-ordered product.

By considering connected Green functions at a fixed order in perturbation theory $\mathcal{O}(g^n)$ we can simplify the equation (4.47). We argue that the identity then holds also without the term on the last line. The two first terms are Green functions with $N+1$ external legs while the last term is a Green function with $N+2$ external legs. Then for $n < N-1$ all the Green functions of (4.47) are zero, and the simplified identity holds vacuously. For $n = N-1$ the two Green functions with $N+1$ external legs become non-zero, while the one with $N+2$ does not. Then by (4.47) the simplified identity holds at order $n = N-1$. We can go to any order n now by induction. Assume the simplified identity holds at order $n-1$. Then equation (4.47) shows that the $\mathcal{O}(g^n)$ part of the final term is zero. Look next at the Green functions at order n. The final term must then be purely of order g^{n+1} and can be neglected. Thus the simplified identity holds at order g^n, which completes the inductive argument.

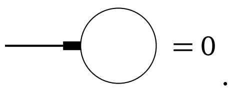
Next we transform to momentum space. The momentum corresponding to the spacetime coordinate x_i we denote p_i, while k corresponds to y. Also we specialize to the Feynman-t' Hooft gauge $\xi = 1$. With this choice, contracting the k^μ of the first term in (4.47) with a external propagator gives $-\mathrm{i}k^\nu$ in the numerator. The two external ghost propagators in the second term give $\mathrm{i}^2 = -1$. The version of the identity (4.47) without the last term can then be expressed graphically as



(4.48)

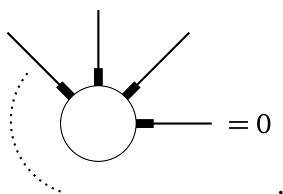
We used here a graphical notation where a rectangle at the end of a propagator (—▬ or --<--▬) represents the corresponding four-momentum k^μ. So on the left hand side of (4.48) the Green function is contracted with k^ν, while on the right hand side an extra $p_j^{\mu_j}$ is added to the ghost propagator at p_j. The latter fact makes the indices balance between the left and right hand sides, as we by a line with a circle mean a ordinary propagator with a free index. This graphical notation is inspired by the one used by t' Hooft [21].

We are here ultimately interested in relations between Feynman amplitudes, thus we let all external momenta now go on-shell. Then contracting the free indices μ_i with a transverse polarization all the terms on the right hand side vanish since $p_i \cdot \varepsilon_{R/L}(p_i) = 0$. To express the resulting identity in a concise way we will introduce a new graphical element. A circle without hatching we shall take to mean a Green function with an arbitrary number of external gluon lines taken on-shell and contracted with a transverse polarization vector. External lines not satisfying this will be expressed explicitly. Graphically we have then



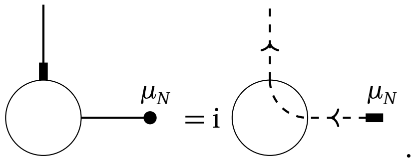
(4.49)

For any number of external lines in the previous argument we could have also contracted with the momentum $p_j^{\mu_j}$. Since $p_j^2 = 0$ the right hand side still vanishes with this choice and we have



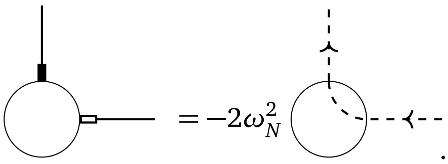
(4.50)

We are now ready to derive the generalization of the relation we saw in Section 4.1 between the amplitude with two unphysical polarizations and the amplitude with two external ghosts. Contracting in (4.48) a transverse polarization on all the external μ_j except where $j = N$ we find



(4.51)

The way this identity works on the level of Feynman amplitudes becomes clearer if we contract with $\tilde{p}_N^{\mu_N}$, where $\tilde{p}$ is the parity transform of p. On the right hand side we get then $p_N \cdot \tilde{p}_N = 2\omega_N^2$. The Feynman-t' Hooft gauge gives $-\mathrm{i}\tilde{p}_N^{\nu_N}$ on the left hand side. This contraction with the parity transformed momentum we express graphically by a open rectangle (——▭), so that

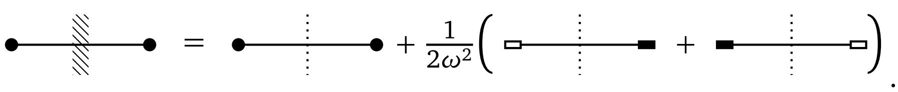


(4.52)

To hammer home the point that this is the identity of Section 4.1 consider that $k^\mu = \sqrt{2}\omega_k \varepsilon_+^\mu$ and $\tilde{k}^\mu = \sqrt{2}\omega_k \varepsilon_-^\mu$. Using polarization vectors on all external lines then, the two sides are identical except for a factor $-\omega_N/\omega_k$. The ratio was in Section 4.1 set to unity by our choice of the CoM frame.

4.3.2 Unitarity at $\mathcal{O}(g^n)$

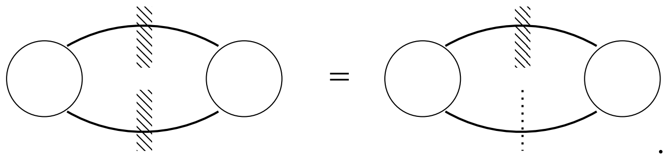
From Chapter 3 we know that the optical theorem, and by extension unitarity, relies upon the gauge invariance of the cut propagators in the Cutkosky cut giving the imaginary part of the amplitude. Cutting propagators in the Feynman-t' Hooft gauge, including all cuts over internal ghost lines, should be equivalent to cutting purely transverse propagators. Recall the relation (2.23) between the sum over transverse polarizations and $-\eta^{\mu\nu}$. Graphically we can write that relation as



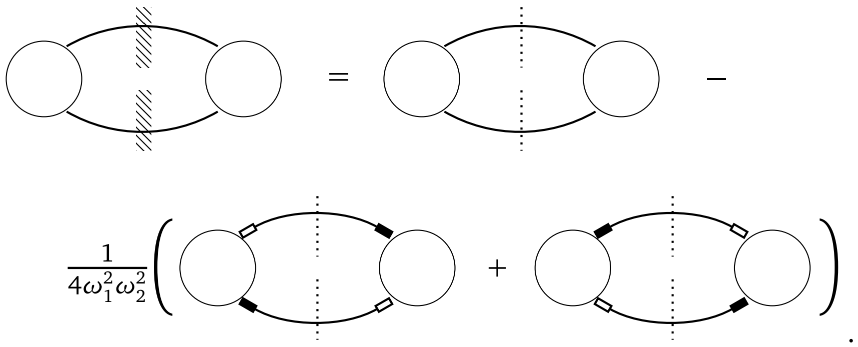
(4.53)

By the hatching on the left hand side we mean a Cutkosky cut involving only transverse polarizations. The first term on the right hand side is a ordinary Cutkosky cut of the propagator in the Feynman-t' Hooft gauge, while the two last terms are the $\tilde{k}^\mu k^\nu$ and $k^\mu \tilde{k}^\nu$ parts.

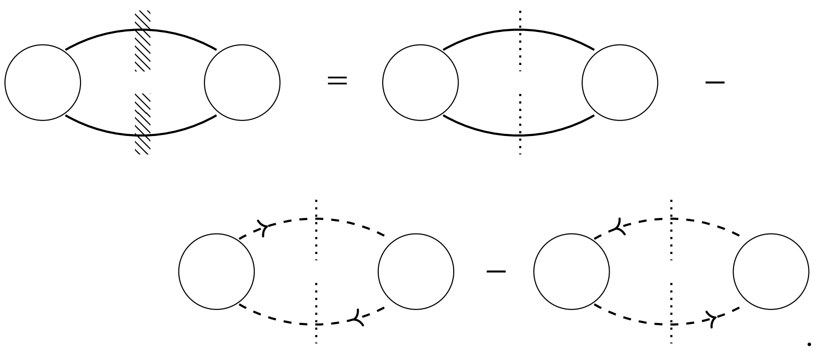
We start by considering the case of two cut propagators. We let the labels 1 and 2 represent the lower and upper cut propagator respectively. By inserting (4.53) in propagator 1 and using the identity (4.49) we have



Inserting (4.53) also in propagator 2 we get three terms on the right hand side. In the two last terms of the resulting expression we can again combine (4.53) with (4.49) in propagator 1 to find



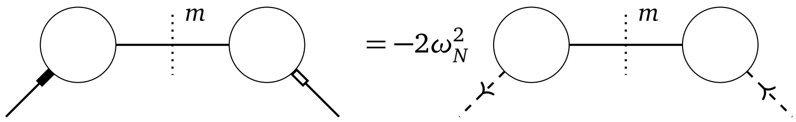
Looking at the expressions on each side of the cuts in the last two terms we see that we can apply the Slavnov-Taylor identity (4.52) a total of four times to obtain



In this notation the minus signs associated with the explicitly shown ghost loops are denoted explicitly. The two last terms are not topologically distinct and could be combined to give the relative symmetry factor two between the gluon and ghost loop diagrams, which is also explicit in the notation. Then the right hand side of the above equation is the sum over all relevant Cutkosky cuts with two cut propagators in the Feynman-t' Hooft gauge. The above equation is then exactly what we need to prove the optical theorem for the case of two final state particles.

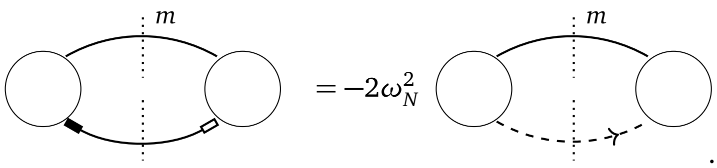
To go to the general case we could in principle have employed a similar argument to the one above, using then Slavnov-Taylor identities involving more than two external ghosts. The complexity of such identities however grows with the number of external ghosts considered. Instead we will reuse a variant of the identity (4.52) in a inductive argument.

Since the imaginary parts of each side of (4.52) are equal, so are the discontinuities over the s-channel branch cuts. The discontinuities of the amplitudes are in turn the sum of the discontinuities of their individual diagrams. In fact the sums over only the discontinuities corresponding to some fixed number of cut propagators are equal as well. Graphically we write this as



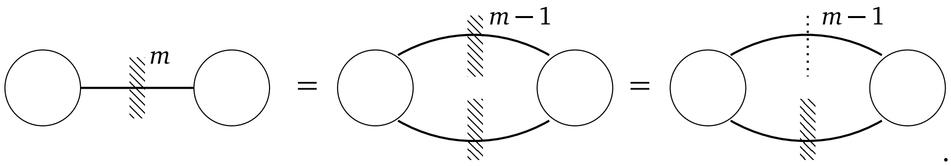
(4.54)

where the label m represents a sum over all Cutkosky cuts involving m cut propagators. A lucid proof of this equation can be found in Chapter 11 of Sterman's book [24]. Since we consider on-shell external lines a special case is

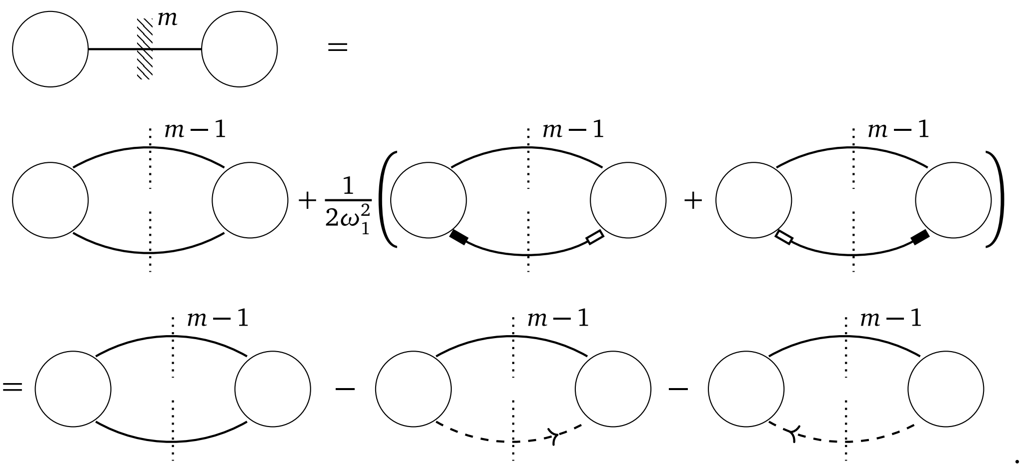


(4.55)

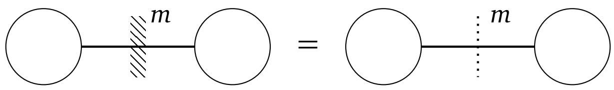
Next we proceed with the inductive argument. Assume the optical theorem holds for the case of $m-1$ final state particles, then we have



In the first equality there is no term with a ghost line at the bottom propagator because there are no amplitudes with just a single external ghost. The second equality uses the inductive hypothesis. As before we insert (4.53) in the bottom propagator,



In the second equality we used the identity (4.55) on the last two terms. In compact notation we have then



which proves the optical theorem by induction.

We have in this argument not considered fermions, either in external or internal lines. Since the representation of the fermions does not suffer from redundant degrees of freedom however, adding them to the argument adds no difficulty. Implicit in our argument is the assumption that the theory has been renormalized in such a way that the Slavnov-Taylor identities hold.

Chapter 5

The Spinor-Helicity Method

The previous chapter shows how the gauge redundancy of the Lagrangian becomes a complex redundancy at the level of the amplitude. Already with four external gluons in the Feynman-t' Hooft gauge do the ghost diagrams outnumber the diagrams with only gluons. We realize that the number of extra diagrams required grows out of hand if we attempt to use the Feynman-t' Hooft gauge with many external gluons, or at higher order in perturbation theory. Simultaneously the expressions corresponding to each diagram become longer. This means that also summing over polarizations using the light-cone gauge becomes unviable due to the length of the resulting expressions. The best option then is to insert explicit polarizations, which does not require us to square the amplitude first. Then we must pick a frame, and are faced with the problem of translating our result into a frame-independent one. In this chapter we consider a way of representing the explicit polarizations which makes the last step simple. The idea is to represent them using spinors, so we start by reviewing these. Next we consider some identities used to compute amplitudes with this method, before we apply them to quark-annihilation into two gluons.

5.1 Spinors

Considering the product of two Lorentz vectors,

$$x_\mu y^\mu = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3,$$

as a bilinear map $\mathbb{R}^{1,3} \times \mathbb{R}^{1,3} \to \mathbb{R}$ we define a Lorentz transformation to be a linear transformation which keeps this map invariant. The Lorentz group is the set of all such transformations. Of these the familiar Lorentz boosts and three-dimensional rotations make up the part continuously connected to the identity. We write $\mathrm{O}(1,3)$ for the Lorentz group, which is a Lie group [25]. To construct a quantum field theory with observables that are Lorentz covariant we use fields that transform according to the Lorentz group. At the same time the number of internal degrees of freedom of a field is connected to the spin of the corresponding particle. This leads to a connection between the different matrix representations of the Lorentz group and the theories describing relativistic particles of different spin [11]. In particular the concept of a spinor, our main object of interest in this chapter, emerges naturally from such a analysis.

A matrix representation of the Lie algebra automatically gives a matrix representation of the corresponding Lie group. Therefore a strategy for finding different matrix representations of a Lie group is to use its Lie algebra. In Appendix B we find the Lie algebra of the Lorentz group. It can be expressed as a direct sum of two SU(2) Lie algebras, and we denote the three basis elements of the two SU(2) Lie algebras J_+^i and J_-^i.

The representation theory of SU(2) is equivalent to the treatment of addition of angular momenta in non-relativistic quantum mechanics, which is covered in many introductory books, e.g. [26]. The fundamental representation of SU(2) is given by the generators $\sigma^i/2$ with σ^i the Pauli matrices. With this knowledge we can find the fundamental representation of the Lorentz group. Let the $\mathbf{J}_+$ subspace be represented by the fundamental representation while the $\mathbf{J}_-$ subspace is represented by the trivial representation. The trivial representation satisfies the defining commutation relation (B.14) by setting all generators to zero. Thus we have in this representation

$$J_-^i = 0 \qquad J_+^i = \frac{\sigma^i}{2}.$$

Solving these two equations for the generators of rotations J^i and Lorentz boosts K^i we find that $J^i = \sigma^i/2$ and $K^i = -\mathrm{i}\sigma^i/2$.

This representation of the Lorentz group acts on two-component objects instead of the four-component four-vectors. By definition of the Lie algebra (see Appendix B) the new two-component objects, called spinors, transform under a Lorentz transformation according to

$$\phi_R \to \exp[\mathrm{i}X]\phi_R = \exp\left[\frac{\mathrm{i}}{2}\theta^i\sigma^i + \frac{1}{2}\eta^i\sigma^i\right]\phi_R. \tag{5.1}$$

We could have equally well let the $\mathbf{J}_-$ subspace be represented by the fundamental representation, while representing the $\mathbf{J}_+$ subspace by the trivial representation. Repeating the argument above then leads to another class of two-component objects which now transform as

$$\phi_L \to \exp[\mathrm{i}X]\phi_L = \exp\left[\frac{\mathrm{i}}{2}\theta^i\sigma^i - \frac{1}{2}\eta^i\sigma^i\right]\phi_L. \tag{5.2}$$

The reason for the labeling R/L of the two classes of spinors will be explained shortly.

In order to construct a Lorentz invariant Lagrangian involving the spinors ϕ_R and ϕ_L we need to know how to construct Lorentz scalars out of them. Since the spinor is a complex two-component object a natural place to start is to look at the usual scalar product for complex vectors. For simplicity we will preform our calculations with infinitesimal parameters such that $\exp(\mathrm{i}tX) = 1 + \mathrm{i}tX$. We find that the scalar product of two right-handed spinors,

$$\begin{aligned}
\phi_R^\dagger \phi_R \to& \phi_R^\dagger \left(1 - \frac{\mathrm{i}}{2}\theta^i\sigma^i + \frac{1}{2}\eta^i\sigma^i\right)\left(1 + \frac{\mathrm{i}}{2}\theta^i\sigma^i + \frac{1}{2}\eta^i\sigma^i\right)\phi_R \\
=& \phi_R^\dagger \phi_R + \eta^i \phi_R^\dagger \sigma^i \phi_R,
\end{aligned}$$

transforms as the time-component of a 4-vector with spacial part $\phi_R^\dagger \sigma^i \phi_R$. Thus $\phi_R^\dagger \sigma^\mu \phi_R$ with $\sigma^\mu = (1, \sigma^i)$ might transform as a 4-vector if the spatial part $\phi_R^\dagger \sigma^i \phi_R$ transforms correctly. We verify this by a direct computation,

$$\begin{aligned}
\phi_R^\dagger \sigma^k \phi_R \to& \phi_R^\dagger \left(1 - \frac{\mathrm{i}}{2}\theta^i\sigma^i + \frac{1}{2}\eta^i\sigma^i\right)\sigma^k\left(1 + \frac{\mathrm{i}}{2}\theta^i\sigma^i + \frac{1}{2}\eta^i\sigma^i\right)\phi_R \\
=& \phi_R^\dagger \left(\sigma^k + \eta^i\sigma^i\sigma^k - \epsilon^{kij}\theta^i\sigma^j + \mathrm{i}\epsilon^{kij}\eta^i\sigma^j\right)\phi_R \\
=& \phi_R^\dagger \sigma^k \phi_R + \eta^k \phi_R^\dagger \phi_R + \epsilon^{kji}\theta^i \phi_R^\dagger \sigma^j \phi_R,
\end{aligned}$$

where we used $\left[\sigma^i, \sigma^j\right] = 2\mathrm{i}\epsilon^{ijk}\sigma^k$ to get to the second line and $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$ to get to the final line. By a similar calculation we find that for the other class of spinors, $\phi_L^\dagger \bar{\sigma}^\mu \phi_L$ transforms as a 4-vector, now with $\bar{\sigma}^\mu = (1, -\sigma^i)$.

With the knowledge that $\phi_R^\dagger \sigma^\mu \phi_R$ transforms as a 4-vector we can construct a Lorentz-invariant Lagrangian for the spinor ϕ_R as $\mathcal{L} \propto \phi_R^\dagger \partial_\mu \sigma^\mu \phi_R$, and similarly for ϕ_L using $\bar{\sigma}^\mu$. The corresponding Euler-Lagrange equations are

$$\partial_\mu \sigma^\mu \phi_R(x) = 0, \tag{5.3}$$

$$\partial_\mu \bar{\sigma}^\mu \phi_L(x) = 0, \tag{5.4}$$

which are known as the Weyl equations. Spinors satisfying the Weyl equation we will call Weyl spinors. In momentum space the equation reads

$$p_\mu \sigma^\mu \phi_R(p) = 0, \tag{5.5}$$

$$p_\mu \bar{\sigma}^\mu \phi_L(p) = 0. \tag{5.6}$$

Operating on the first equation with $p_\nu \bar{\sigma}^\nu$ we find that the momentum p^μ is lightlike,

$$p_\nu \bar{\sigma}^\nu p_\mu \sigma^\mu \phi_R(p) = [(p^0)^2 - p^i p^k \sigma^i \sigma^k]\phi_R(p) = p_\mu p^\mu \phi_R(p) = 0. \tag{5.7}$$

We are now ready to explain the label R/L on the Weyl spinor. Remember from Section 2.2 that we defined the helicity of a plane wave as the number h in the phase $\exp(\mathrm{i}h\alpha)$ acquired by the plane wave after a rotation of angle α around its axis of propagation. A positive helicity we called right-handed and a negative helicity left-handed, see figure 2.1. Using the transformation law (5.1) we can apply such a rotation around the axis of propagation $\mathbf{p}/|\mathbf{p}|$,

$$\phi_R(p) \to \exp\left[\mathrm{i}\frac{\alpha p^i}{2p^0}\sigma^i\right]\phi_R(p),$$

where $|\mathbf{p}| = p^0$ since the momentum p^μ is lightlike. From the Weyl equation (5.5) it follows that

$$\begin{aligned} p^i\sigma^i\phi_R(p) &= p^0\phi_R(p) \\ \frac{\alpha p^i}{2p^0}\sigma^i\phi_R(p) &= \frac{\alpha}{2}\phi_R(p) \\ \left(\frac{\alpha p^i}{2p^0}\sigma^i\right)^n \phi_R(p) &= \left(\frac{\alpha}{2}\right)^n \phi_R(p), \end{aligned}$$

which we can use to compute the matrix exponential. The result is that the rotation amounts to multiplying ϕ_R by the phase $\alpha/2$, in other words the helicity of ϕ_R is $+1/2$. Repeating the argument for the case of ϕ_L the Weyl equation (5.4) contains an additional relative factor -1, and the helicity is $-1/2$. A interesting sidenote is that this argument implies that to get back the same spinor we need to rotate a spinor 4π instead of the usual 2π for vectors. Looking at the above derivation it is clear that the helicity of both fields is also their eigenvalue with respect to the operator

$$h = \frac{p^i\sigma^i}{2|\mathbf{p}|}. \tag{5.8}$$

Two more identities involving the Weyl spinors will be of importance to us later. First we consider if there is some relationship between the two different kinds of Weyl spinors. Using the identity of the Pauli matrices

$$\sigma^2(\sigma^i)^* = -\sigma^i\sigma^2,$$

we compute that the transformation law of $-\mathrm{i}\sigma^2\phi_R^*$ is

$$\begin{aligned} -\mathrm{i}\sigma^2\phi_R^* \to &-\mathrm{i}\sigma^2\left[1 + \frac{\mathrm{i}}{2}\theta^i\sigma^i + \frac{1}{2}\eta^i\sigma^i\right]^* \phi_R^* \\ &= \left[1 + \frac{\mathrm{i}}{2}\theta^i\sigma^i - \frac{1}{2}\eta^i\sigma^i\right]\left(-\mathrm{i}\sigma^2\phi_R^*\right). \end{aligned}$$

In other words $\phi_R^c \equiv -\mathrm{i}\sigma^2\phi_R^*$ transforms as a left-handed spinor. By direct insertion and using the same property of the Pauli matrices we also find that ϕ_R^c solves the Weyl equation for ϕ_L given that ϕ_R solves the Weyl equation for ϕ_R. Thus for Weyl spinors

$$\phi_L = -\mathrm{i}\sigma^2\phi_R^*. \tag{5.9}$$

Doing the same calculation for $\phi_L^c \equiv \mathrm{i}\sigma^2\phi_L^*$ we see that it transforms as a right-handed spinor, and solves the Weyl equation for a right-handed spinor. That means we have also

$$\phi_R = \mathrm{i}\sigma^2\phi_L^*, \tag{5.10}$$

where we see the phase was chosen so that this equation is consistent with (5.9).

Next we multiply the Weyl equation for a right-handed spinor (5.5) by $\phi_R^\dagger$ from the right. This turns the Weyl equation into a statement that the product of the two 2 × 2 matrices $p_\mu \sigma^\mu$ and $\phi_R(p)\phi_R^\dagger(p)$ is the zero-matrix. Clearly the matrix $\phi_R(p)\phi_R^\dagger(p)$ is Hermitian. All Hermitian 2 × 2 matrices can be written as a linear combination of the Pauli matrices and the identity matrix, see e.g. Section 2.4. It follows that we can write

$$\phi_R(p)\phi_R^\dagger(p) = a_\mu \bar{\sigma}^\mu,$$

for some set of four parameters a_μ. Now looking at (5.7) it is clear that choosing $a_\mu \propto p_\mu$ we will satisfy the Weyl equation in the form $p_\mu \sigma^\mu \phi_R(p)\phi_R^\dagger(p) = 0$, since the momentum is lightlike. The proportionality constant depends on a normalization convention for the Weyl spinors. We set this constant to one such that

$$\phi_R(p)\phi_R^\dagger(p) = p_\mu \bar{\sigma}^\mu. \tag{5.11}$$

A completely equivalent argument for the case of $\phi_L(p)$ leads to the conclusion that

$$\phi_L(p)\phi_L^\dagger(p) = p_\mu \sigma^\mu. \tag{5.12}$$

What normalization of the Weyl spinors gave (5.11) and (5.12)? By computing $(\phi_R^\dagger \phi_R)^2 = \mathrm{tr}\{\phi_R \phi_R^\dagger \phi_R \phi_R^\dagger\}$, we find that the required normalization is $\phi_R^\dagger \phi_R = \phi_L^\dagger \phi_L = 2E$.

5.1.1 From Weyl to Dirac Spinors

The Weyl spinors describe spin-1/2 particles. We found the products $\phi_R^\dagger \phi_R$ and $\phi_L^\dagger \phi_L$ did not transform as scalars but as the time-like component of a 4-vector. Then we could construct the kinetic term of a Lagrangian describing the spinors but not a mass-term. Consequently the Euler-Lagrange equations described the propagation of massless particles. In nature however no massless spin-1/2 particles are known [27], and we would therefore like to add a mass-term to our Lagrangian.

A simple solution presents itself if we compute in the same way as in the previous section the transformation law of $\phi_R^\dagger \phi_L$. We find that it, along with $\phi_L^\dagger \phi_R$ transforms as a Lorentz scalar. Considering the kinetic term $\phi_R^\dagger \partial_\mu \sigma^\mu \phi_R$ we see the mass dimension of $\phi_{L/R}$ is 3/2. Then we can add the terms $m\phi_R^\dagger \phi_L$ and $m\phi_L^\dagger \phi_R$ to the Lagrangian with m a constant with mass dimension 1. We have

$$\mathcal{L} \propto \mathrm{i}\phi_R^\dagger \partial_\mu \sigma^\mu \phi_R + \mathrm{i}\phi_L^\dagger \bar{\partial}_\mu \sigma^\mu \phi_L - m\phi_R^\dagger \phi_L - m\phi_L^\dagger \phi_R, \tag{5.13}$$

with corresponding Euler-Lagrange equations

$$\mathrm{i}\partial_\mu \sigma^\mu \phi_R - m\phi_L = 0 \tag{5.14}$$
$$\mathrm{i}\partial_\mu \bar{\sigma}^\mu \phi_L - m\phi_R = 0. \tag{5.15}$$

Going to momentum-space and multiplying the first equation by $p_\nu \bar{\sigma}^\nu$ as in (5.7) we find $p_\nu \bar{\sigma}^\nu \phi_L = (p^2/m)\phi_R$. Inserting this into the second equation in momentum space we find $p^2 = m^2$, the relativistic dispersion relation. This is how we determined the correct relative numerical factor between the mass-term and the kinetic term in (5.13).

We see that with the introduction of the mass-term above the dynamics of ϕ_R and ϕ_L are now coupled. A propagating free particle will oscillate between the two states ϕ_R, ϕ_L. If we want the propagating particle to be described by a single field we need therefore to combine the two fields into one. We combine them into the 4-component spinor

$$\psi = \begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix}. \tag{5.16}$$

The two coupled equations (5.14) and (5.15) can then be rewritten as a single compact expression involving the new spinor,

$$(\mathrm{i}\partial_\mu \gamma^\mu - m)\psi(x) = 0, \tag{5.17}$$

by defining the gamma matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{5.18}$$

This is the Dirac equation and correspondingly the spinor ψ is a Dirac spinor. The general plane-wave solutions to the Dirac equation can be found in Appendix C.

A Dirac spinor having only the ϕ_L component we call left-chiral, while one having only the ϕ_R component we call right-chiral. These components are mixed by the mass-term so chirality is not conserved. The left and right chiral fields are the eigenvectors of $\mathrm{diag}(-1, 1)$ with eigenvalue -1 and $+1$ respectively. We find by a direct calculation that

$$\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus the eigenvectors of $\gamma^5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ may alternatively be used to define the chiral fields. The definition in terms of ϕ_R and ϕ_L combined with our experience from the previous section might mislead us to think the concept of chirality is completely equivalent to that of helicity. This is however not the case. From the previous section we have the transformation laws of the two spinors ϕ_L, ϕ_R in (5.2) and (5.1) respectively. The transformation law of the Dirac spinor ψ is then simply given by a block diagonal matrix with the matrix in (5.2) in the upper left corner and the matrix in (5.1) in the lower right corner. Then a rotation of angle α around the axis of propagation of the Dirac spinor is given by applying the matrix

$$\exp\left[\mathrm{i}\frac{\alpha p^i}{2|\mathbf{p}|}\sigma^i\right]$$

to the ϕ_L and ϕ_R component of the Dirac spinor simultaneously. Now for the Dirac spinor to have definite helicity this operation must be equivalent to multiplying the entire Dirac spinor by a phase $\exp(\mathrm{i}h\alpha)$. Using that the matrix exponential preserves block diagonal matrices this implies

$$\exp\begin{pmatrix} \mathrm{i}\frac{\alpha p^i}{2|\mathbf{p}|}\sigma^i & 0 \\ 0 & \mathrm{i}\frac{\alpha p^i}{2|\mathbf{p}|}\sigma^i \end{pmatrix}\psi = \exp(\mathrm{i}h\alpha)\psi,$$

which only holds if ψ is an eigenvector of the operator

$$h = \frac{p^i}{2|\mathbf{p}|}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \tag{5.19}$$

with eigenvalue h. This helicity operator is the generalization of the helicity operator for Weyl spinors (5.8). In contrast to chirality, helicity is a conserved quantity. However when considering massive particles it is not Lorentz invariant, as a boost can change the direction of $\mathbf{p}$.

5.2 Fermions in the Ultra-Relativistic Limit

We will in the remainder of this chapter consider processes at momentum scales large compared to the masses of the fermions involved. In this ultra-relativistic limit we may to a good approximation neglect the fermion masses. Taking the $m \to 0$ limit of the plane-wave solutions to the Dirac equation, equations (C.9) and (C.10) of Appendix C, we find

$$u(p) = \frac{1}{\sqrt{2E}}\begin{pmatrix} p_\mu \sigma^\mu \phi_1 \\ p_\mu \bar{\sigma}^\mu \phi_1 \end{pmatrix}$$
$$v(p) = \frac{1}{\sqrt{2E}}\begin{pmatrix} p_\mu \sigma^\mu \phi_2 \\ -p_\mu \bar{\sigma}^\mu \phi_2 \end{pmatrix}.$$

At the same time we know that in the $m \to 0$ limit the Dirac equation decouples to the Weyl equations (5.3) and (5.4). Accounting for the normalization $\phi_R^\dagger \phi_R = 2E$ of the Weyl spinors we may set $\phi_1 = \phi_R/\sqrt{2E}$. Inserting this choice of ϕ_1 into the above solution the upper component of $u(p)$ vanishes since ϕ_R is a solution to the Weyl equation. The lower component, by using (5.11), becomes $\phi_R^\dagger \phi_R \phi_R/\sqrt{2E} = \sqrt{2E}\phi_R$. Following the same argument for the choice $\phi_1 = \phi_L$ we find that these choices give the states of definite chirality

$$u_R(p) = \begin{pmatrix} 0 \\ \phi_R(p) \end{pmatrix} \qquad u_L(p) = \begin{pmatrix} \phi_L(p) \\ 0 \end{pmatrix}. \tag{5.20}$$

From our discussion of the Weyl spinors it follows that these states are also eigenstates of helicity. The spinor u_R has right-handed helicity and u_L has left-handed helicity. We see then that in the massless limit the expansion in states of definite chirality and helicity is the same. From now on we will when talking about handedness refer only to helicity unless otherwise stated.

Without the mass-term the two equations distinguishing the particle and anti-particle, (C.1) and (C.2), are identical. Therefore we can use the states (5.20) to describe also the anti-particle $v(p)$. In order to keep the property that crossing a incoming right-handed particle should give a outgoing left-handed anti-particle we set $v_L(p) = u_R(p)$ and vice versa $v_R(p) = u_L(p)$. The 4 possible spinors have then been reduced to two. We will in this chapter treat all particles as incoming. This means we will only encounter the spinors $u_{L/R}$ representing a incoming particle and $\bar{v}_{L/R} = \bar{u}_{R/L}$ representing a incoming anti-particle. Except that we here define the particles as incoming instead of outgoing we follow the notation of Peskin [28] and write

$$u_L(p) \equiv p] \quad \text{Incoming left-handed fermion,} \tag{5.21}$$

$$u_R(p) \equiv p\rangle \quad \text{Incoming right-handed fermion,} \tag{5.22}$$

$$\bar{u}_L(p) \equiv \langle p \quad \text{Incoming right-handed anti-fermion,} \tag{5.23}$$

$$\bar{u}_R(p) \equiv [p \quad \text{Incoming left-handed anti-fermion.} \tag{5.24}$$

We see immediately the reasoning behind the notation by computing that the two nonzero scalar products of spinors are expressed with a matching pair of brackets

$$\begin{aligned}
\langle pq] &= 0,\\
[pq\rangle &= 0,\\
\langle pq\rangle &= \phi_L^\dagger(p)\phi_R(q),\\
[pq] &= \phi_R^\dagger(p)\phi_L(q).
\end{aligned}$$

From the above expression it also follows that

$$\langle pq\rangle = [qp]^*. \tag{5.25}$$

This is the first in a series of identities containing the spinors (5.21)–(5.24) that we will derive in this section. These identities will enable us to rewrite complicated amplitudes involving spinors of massless fermions as simpler expressions in terms of the spinor products $\langle pq\rangle$ and $[pq]$. The second identity of this kind we get by considering

$$\langle pq\rangle[qp] = \phi_L^\dagger(p)\phi_R(q)\phi_R^\dagger(q)\phi_L(p) = \phi_L^\dagger(p)q_\nu\bar{\sigma}^\nu\phi_L(p),$$

where we inserted (5.11). If we look at the suppressed indices we can rewrite the above spinor product as a trace. Inserting also (5.12) we get

$$\begin{aligned}
\langle pq\rangle[qp] &= \mathrm{tr}\big\{q_\nu\bar{\sigma}^\nu\phi_L(p)\phi_L^\dagger(p)\big\}\\
&= \mathrm{tr}\big\{q_\nu\bar{\sigma}^\nu p_\mu\sigma^\mu\big\}\\
&= q_\nu p_\mu\,\mathrm{tr}\{\bar{\sigma}^\nu\sigma^\mu\} = q_\nu p_\mu(2\eta^{\nu\mu})\\
&= 2p\cdot q.
\end{aligned} \tag{5.26}$$

Or by combination with (5.25)

$$|\langle pq\rangle|^2 = |[pq]|^2 = 2p\cdot q. \tag{5.27}$$

Next we use the identity (5.10) to show that the spinor products are anti-symmetric,

$$\begin{aligned}\langle pq\rangle &= \phi_L^\dagger(p)\phi_R(q) = \phi_L^\dagger(p)\mathrm{i}\sigma^2\phi_L^*(q)\\ &= \mathrm{i}^2(\phi_R^*(p))^\dagger(\sigma^2)^\dagger\sigma^2\phi_L^*(q) = -\phi_L^\dagger(q)\phi_R(p)\\ &= -\langle qp\rangle,\end{aligned}$$

where we used $(\sigma^i)^\dagger = \sigma^i$ and $(\sigma^i)^2 = 1$. It follows that also $[pq] = -[qp]$. Repeating the above calculation for the case of $[q\gamma^\mu p\rangle$ we use instead $\sigma^2\sigma^\mu\sigma^2 = (\bar\sigma^\mu)^*$ to find that

$$[q\gamma^\mu p\rangle = \langle p\gamma^\mu q]. \tag{5.28}$$

Spinor products with a gamma matrix wedged between transform as four-vectors. Contracting two such products gives a scalar, which in turn should be expressible in terms of the available spinor products, the only scalars around. To find such a relation we recall that $T^i = \sigma^i/2$ are the generators of SU(2). This means that they satisfy the Fierz-identity (2.47), which in terms of the Pauli matrices becomes

$$\sigma^i_{ab}\sigma^i_{cd} = 2\delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd},$$

where $a, b, \ldots$ now are matrix indices. In terms of σ^μ this relation takes the simple form

$$(\bar\sigma^\mu)_{ab}(\sigma_\mu)_{cd} = 2\delta_{ad}\delta_{bc},$$

from which it follows that

$$\begin{aligned}\langle p\gamma^\mu q]\langle k\gamma_\mu l] &= \langle p\gamma^\mu q][l\gamma_\mu k\rangle\\ &= \phi_L^\dagger(p)\bar\sigma^\mu\phi_L(q)\phi_R^\dagger(l)\sigma_\mu\phi_R(k)\\ &= 2\phi_L^\dagger(p)\phi_R(k)\phi_R^\dagger(l)\phi_L(q)\\ &= 2\langle pk\rangle[lq].\end{aligned} \tag{5.29}$$

The final identity of the spinor products we will prove in this section is the Schouten identity. Consider the spinors $j\rangle, k\rangle, l\rangle$ as two-component vectors. With two components there cannot be more than two linearly independent vectors. Then assuming $j\rangle$ and $k\rangle$ are linearly independent we must have

$$l\rangle = c_1 j\rangle + c_2 k\rangle,$$

for some scalars c_1, c_2. Operating on this with $\langle k$ and using that $\langle kk\rangle = 0$, we find that $\langle kl\rangle = c_1\langle kj\rangle$. Since $k\rangle$ and $j\rangle$ are linearly independent, $\langle kj\rangle \neq 0$. Then we can solve for the scalar $c_1 = \langle kl\rangle/\langle kj\rangle$. Similarly by operating with $\langle j$ we find $c_2 = \langle jl\rangle/\langle jk\rangle$.

By inserting these relations into the expansion of $l\rangle$ and multiplying by $\langle kj\rangle = -\langle jk\rangle$ we find

$$l\rangle\langle kj\rangle = j\rangle\langle kl\rangle - k\rangle\langle jl\rangle.$$

Operating on this relation with $\langle i$—where i is some fourth four-momentum—and rearranging the terms we obtain the Schouten identity,

$$\langle ij\rangle\langle kl\rangle - \langle ik\rangle\langle jl\rangle - \langle il\rangle\langle kj\rangle = 0. \tag{5.30}$$

5.3 External Gauge Bosons

We would like to extend the formalism of the previous section to handle also external gauge bosons. Continuing to treat all particles as incoming we set for the polarization vectors [28, 29]

$$\varepsilon_R^\mu(k) = \frac{1}{\sqrt{2}}\frac{[r\gamma^\mu k\rangle}{[rk]} \qquad \varepsilon_L^\mu(k) = -\frac{1}{\sqrt{2}}\frac{\langle r\gamma^\mu k]}{\langle rk\rangle}, \tag{5.31}$$

where r is a arbitrary lightlike vector such that $r\cdot k \neq 0$. These polarization vectors satisfy the Lorenz gauge condition $k_\mu\varepsilon_\lambda^\mu = 0$ since $\not{k}k\rangle = 0 = \not{k}k]$ by the Dirac equation. Using the relation (5.29) we also see that these expressions satisfy the orthogonality $(\varepsilon_R^\mu)^*\varepsilon_{L\mu} = 0$ and normalization conditions $(\varepsilon_R^\mu)^*\varepsilon_{R\mu} = -1$ expected of polarization vectors.

In the frame where $k^\mu = (\omega, 0, 0, \omega)$ the Weyl equations (5.5) and (5.6) can be solved to give

$$\phi_R(k) = \sqrt{2\omega}\begin{pmatrix}1\\0\end{pmatrix} \qquad \phi_L(k) = \sqrt{2\omega}\begin{pmatrix}0\\1\end{pmatrix}.$$

The lightlike r satisfying $r\cdot k \neq 0$ must be of the form $r^\mu = (r, 0, 0, -r)$. Then we have $\phi_R(r) = \sqrt{r/\omega}\,\phi_L(k)$ and $\phi_L(r) = \sqrt{r/\omega}\,\phi_R(k)$. Using this we find by a direct computation that ε_R^μ of (5.31) becomes

$$\varepsilon_R^\mu(k) = \frac{1}{\sqrt{2}}(0, 1, \mathrm{i}, 0).$$

Repeating the calculation for $\varepsilon_L^\mu(k)$ we see that in this frame the polarization vectors (5.31) reduce to the familiar ε_R and ε_L of equations (2.15) and (2.16).

What is the role of the arbitrary vector r^μ? Its properties, being lightlike with $r\cdot k \neq 0$, suggest that it might be the same arbitrary vector we have already encountered as n^μ in the light-cone gauge. To check this we compute the polarization sum $\mathcal{P}^{\mu\nu}$ for the expressions (5.31). Using (5.25) we find

$$(\varepsilon_R^\mu)^*\varepsilon_R^\nu = \frac{1}{2}\frac{\langle r\gamma^\mu k][r\gamma^\nu k\rangle}{|\langle kr\rangle|^2}. \tag{5.32}$$

The denominator we have directly from (5.27) as $2k \cdot r$. To compute the numerator we will find an expression for the outer product $i\rangle[i$ of two spinors. That relation will be useful also in practical calculations later.

Writing the square and angle brackets in terms of Weyl spinors and using (5.11) we find,

$$i\rangle[i = \begin{pmatrix} 0 & 0 \\ \phi_R \phi_R^\dagger & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ k_\nu \bar{\sigma}^\nu & 0 \end{pmatrix} = k_\nu \gamma^\nu \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.33}$$

The matrix multiplying $\not{k}$ in this relation is simply $(1-\gamma^5)/2$; the projection operator on left-handed states. Since we work with helicity (and chirality) eigenstates the effect of this projection operator when wedged inside a spinor product will always be either $P_L i] = i]$ or $P_L i\rangle = 0$.

Then for the numerator of (5.32) we get

$$\langle r\gamma^\mu k][r\gamma^\nu k\rangle = [k\gamma^\mu r\rangle[r\gamma^\nu k\rangle = [k\gamma^\mu \not{r}\frac{1}{2}(1-\gamma^5)\gamma^\nu k\rangle.$$

Moving $(1-\gamma^5)/2$ to the right of γ^ν it is turned into $(1+\gamma^5)/2$, the projection operator on right-handed states. Since $k\rangle$ is right-handed the effect of the right-handed projection operator is that of the identity. Using this, and rewriting the spinor product as a trace, the numerator can be written as

$$\mathrm{tr}\{\gamma^\mu \not{r}\gamma^\nu k\rangle[k\} = \frac{1}{2}\,\mathrm{tr}\big\{\gamma^\mu \not{r}\gamma^\nu \not{k}(1-\gamma^5)\big\}.$$

Looking instead at $(\varepsilon_L^\mu)^*\varepsilon_L^\nu$ we get the same denominator as in (5.32), but the numerator is in this case

$$\langle k\gamma^\mu r]\langle r\gamma^\nu k] = \mathrm{tr}\{\gamma^\mu \not{r}\gamma^\nu k]\langle k\} = \frac{1}{2}\,\mathrm{tr}\big\{\gamma^\mu \not{r}\gamma^\nu \not{k}(1+\gamma^5)\big\}.$$

Combining the two expressions, and using the linearity of the trace,

$$(\varepsilon_R^\mu)^*\varepsilon_R^\nu + (\varepsilon_L^\mu)^*\varepsilon_L^\nu = \frac{\mathrm{tr}\{\gamma^\mu \not{r}\gamma^\nu \not{k}\}}{4k\cdot r} = -\eta^{\mu\nu} + \frac{k^\mu r^\nu + k^\nu r^\mu}{k\cdot r}, \tag{5.34}$$

where we also used trace identities of the gamma matrices. Comparing this to (2.23) our suspicion that the representation (5.31) corresponds to the light-cone gauge is confirmed. Our experience from Chapter 4 then tells us that we can choose a different vector r^μ for each external line.

5.4 Application to $q\bar{q} \to gg$

To illustrate how the technology of the previous two sections can be used to compute amplitudes we will apply it to quark-annihilation into two gluons, a process familiar from Chapter 4. With the Feynman-t' Hooft gauge for internal lines and massless fermions the color ordered (CO) amplitude for $\bar{q}q \to gg$, equation (4.2), becomes

$$\begin{aligned} A(1234) = &- g^2 \bar{v}(2)\Big\{ \not{\varepsilon}(3) \frac{\not{1} + \not{4}}{u} \not{\varepsilon}(4) + \\ &\frac{2}{s}\left[\varepsilon(3)\cdot\varepsilon(4)\not{3} + (4\cdot\varepsilon(3))\not{\varepsilon}(4) - (3\cdot\varepsilon(4))\not{\varepsilon}(3) \right] \Big\} u(1), \end{aligned} \tag{5.35}$$

where we used the Dirac equation to rewrite $\bar{v}(2)(\not{3}-\not{4})u(1) = 2\bar{v}(2)\not{3}u(1)$. Also, since we intend to use the representation (5.31), transverse polarization vectors were used. To express (5.35) with spinor products we need to choose some helicity for the two quarks and two gluons. This gives 16 different helicity combinations to consider. Thankfully, as we shall see, we only need to compute a few of these in practice.

The first simplification comes when we consider that the two quarks must have opposite helicities. This is a general fact of the vector current vertex, which we can now easily verify. Consider an odd number of gamma matrices between to spinors of the same helicity, for example $\langle i\gamma^{\mu_1}\dots\gamma^{\mu_{2n-1}} j\rangle$. The $\langle i$ is right-handed so inserting P_R, the projection operator on right-handed states, on the left does not change the spinor product. Since $P_R\gamma^\mu = \gamma^\mu P_L$ we can move this P_R all the way to the right, where it with an odd number of gamma-matrices becomes P_L. A right-handed state like $j\rangle$ is annihilated by P_L and thus $\langle i\gamma^{\mu_1}\dots\gamma^{\mu_{2n-1}} j\rangle = 0$. The same kind of argument shows that $[i\gamma^{\mu_1}\dots\gamma^{\mu_{2n-1}} j] = 0$. A tree level diagram with two external quarks and any number of external gluons has a single fermion line going through it. Since we are at tree level that fermion line has $N+1$ vertices given that it has N propagators. Thus it consists of $2N+1$ gamma matrices, an odd number. Then only when the two quark helicities are opposite is the amplitude non-zero.

Next we will show that also the two gluons must have opposite helicities. Consider $A(1_R 2_L 3_R 4_R)$, then the final term in (5.35) is proportional to

$$[2\not{\varepsilon}(3)1\rangle \frac{1}{\sqrt{2}} \frac{[2\gamma_\mu 1\rangle [r_3\gamma^\mu 3\rangle}{[r_3 3]}.$$

We denote the arbitrary lightlike vector of (5.31) belonging to the gluon with momentum i by r_i. Using (5.29) the numerator above becomes $2\langle 13\rangle[r_3 2]$, which vanishes by setting $r_3 = 2$. In a completely analogous way the second to last term vanishes by setting $r_4 = 2$. The identities of Section 5.2 and 5.3 will from now on be used without explicit mention if the identity used is clear from context. We have

$$\varepsilon_R(3)\cdot\varepsilon_R(4) = \frac{\langle 34\rangle[r_4 r_3]}{[r_3 3][r_4 4]},$$

which with our choice $r_3 = r_4 = 2$ is proportional to $[22] = 0$. Only the first term of (5.35) then remains for the case of $A(1_R 2_L 3_R 4_R)$. It is for our choice of reference vectors r_i given by

$$-\frac{g^2}{2u}[2\gamma_\mu(\not{1}+\not{4})\gamma_\nu 1\rangle \frac{[2\gamma^\mu 3\rangle[2\gamma^\nu 4\rangle}{[23][24]}.$$

A convenient "trick" for dealing with expressions like this is to consider that $(\not{1}+\not{4})\gamma_\nu 1\rangle$ is just another spinor $p\rangle$. Then we can transform

$$[2\gamma_\mu p\rangle[2\gamma^\mu 3\rangle = 2\langle p3\rangle[22] = 0,$$

which shows that the entire $A(1_R 2_L 3_R 4_R)$ CO amplitude is zero. A similar line of argumentation, now with the choices $r_3 = 1, r_4 = 1$ shows that $A(1_R 2_L 3_L 4_L)$ is zero as well.

The next fact that reduces the number of helicity combinations we need to compute considerably is that parity is a symmetry of a Yang-Mills theory coupled to fermions via the vector current vertex [14]. Since a parity transformation reverses the helicity of all particles involved, this halves the number of helicity combinations we need to compute. For example we have $A(1_L 2_R 3_L 4_L) = A(1_R 2_L 3_R 4_R) = 0$ and $A(1_L 2_R 3_R 4_R) = A(1_R 2_L 3_L 4_L) = 0$, which completes the check that in the non-zero CO amplitudes the two gluon helicities are opposite. Only four CO helicity amplitudes are non-zero and considering parity transformations we only need to calculate $A(1_R 2_L 3_R 4_L)$ and $A(1_R 2_L 3_L 4_R)$.

We calculate $A(1_R 2_L 3_R 4_L)$ next. By the same argument as before the final term of (5.35) is zero if we choose $r_3 = 2$. In the same way the second to last term vanishes when we choose $r_4 = 1$. For the first term we again have $\propto [2\gamma_\mu p\rangle[2\gamma^\mu 3\rangle = 0$. The only remaining part is

$$\begin{aligned} A(1_R 2_L 3_R 4_L) &= \frac{-2g^2}{s}[2\not{3}1\rangle \varepsilon_R(3)\cdot\varepsilon_L(4) = \frac{g^2}{s}[2\not{3}1\rangle \frac{[2\gamma^\mu 3\rangle\langle 1\gamma_\mu 4]}{[23]\langle 14\rangle} \\ &= \frac{2g^2}{s}[2\not{3}1\rangle \frac{\langle 31\rangle[42]}{[23]\langle 14\rangle} = \frac{2g^2}{s}[23]\langle 31\rangle \frac{\langle 31\rangle[42]}{[23]\langle 14\rangle}, \end{aligned} \tag{5.36}$$

where we in the final equality used the analogue of (5.33) for $i]\langle i$.

To simplify the expression for $A(1_R 2_L 3_R 4_L)$ further we can first of all insert $s = 2(1\cdot 2) = \langle 12\rangle[21]$, giving

$$A(1_R 2_L 3_R 4_L) = 2g^2 \frac{\langle 31\rangle^2[42]}{\langle 12\rangle\langle 14\rangle[21]}.$$

We will rewrite this expression only in terms of angle brackets since that leads to a particularly elegant expression. For four momenta i, j, k, l that satisfy $i+j+k+l=0$ we have

$$\frac{[ij]}{[ik]} = \frac{[ij]\langle jl\rangle\langle kl\rangle}{[ik]\langle kl\rangle\langle jl\rangle} = \frac{[i\not{j}l\rangle\langle kl\rangle}{[i\not{k}l\rangle\langle jl\rangle} = -\frac{[i\not{k}l\rangle\langle kl\rangle}{[i\not{k}l\rangle\langle jl\rangle} = -\frac{\langle kl\rangle}{\langle jl\rangle}. \tag{5.37}$$

Using this to remove all square brackets from $A(1_R 2_L 3_R 4_L)$ we get

$$A(1_R 2_L 3_R 4_L) = 2g^2 \frac{\langle 13\rangle^3}{\langle 12\rangle\langle 34\rangle\langle 41\rangle} = 2g^2 \frac{\langle 13\rangle^3\langle 23\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{5.38}$$

For $A(1_R 2_L 3_L 4_R)$ we can set $r_3 = 1$ to remove the final term of (5.35), and $r_4 = 2$

to remove the second to last. Apart from $-g^2/u$ the first term is

$$
\begin{aligned}
-\frac{1}{2}[2\gamma_\mu(\not{1}+\not{4})\gamma_\nu 1\rangle \frac{[2\gamma^\nu 4\rangle\langle 1\gamma^\mu 3]}{[24]\langle 13\rangle} &= -2\frac{\langle 14\rangle[2(\not{1}+\not{4})1\rangle[32]}{[24]\langle 13\rangle} \\
&= 2\frac{\langle 14\rangle[2\not{3}1\rangle[32]}{[24]\langle 13\rangle} = 2\frac{[23]^2\langle 14\rangle}{[24]}.
\end{aligned} \tag{5.39}
$$

Next we compute

$$
2[2\not{3}1\rangle\varepsilon_L(3)\cdot\varepsilon_R(4) = 2\frac{[23]^2\langle 14\rangle}{[24]},
$$

from which it follows that

$$
\begin{aligned}
A(1_R2_L3_L4_R) &= -2g^2\frac{[23]^2\langle 14\rangle}{[24]}\left(\frac{1}{u}+\frac{1}{s}\right) \\
&= -2g^2\frac{[23]^2\langle 14\rangle}{[24]}\left(\frac{1}{[23]\langle 32\rangle}+\frac{1}{[12]\langle 21\rangle}\right).
\end{aligned}
$$

The numerator of the expression in parenthesis is $s+u=-t=-[24]\langle 42\rangle$ when expressed with a common denominator. Using also the relation (5.37) to write the answer only in terms of angle brackets we have

$$
A(1_R2_L3_L4_R) = 2g^2\frac{\langle 14\rangle^2\langle 42\rangle}{\langle 21\rangle\langle 32\rangle\langle 34\rangle} = 2g^2\frac{\langle 14\rangle^3\langle 24\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{5.40}
$$

Notice that we were able to write the answer in a form intriguingly similar to (5.38).

We know from Section 4.1.2 how to compute the squared amplitude traced over color from the CO amplitudes. We repeat the result (4.14) from there,

$$
\begin{aligned}
\operatorname{tr}|\mathcal{A}(1_R2_L3_R4_L)|^2 = &\operatorname{tr} T^cT^cT^dT^d\left(|A(1_R2_L3_R4_L)|^2+|A(1_R2_L4_L3_R)|^2\right) \\
&+2\operatorname{tr} T^cT^dT^cT^d A(1_R2_L3_R4_L)A(1_R2_L4_L3_R)^*.
\end{aligned}
$$

We have $A(1_R2_L3_R4_L)$ directly from (5.38). To get $A(1_R2_L4_L3_R)$ we can simply exchange $3\longleftrightarrow 4$ in (5.40). In the squares we insert immediately $s_{ij}\equiv 2(i\cdot j)=|\langle ij\rangle|^2$, while for the cross-term we have

$$
\begin{aligned}
A(1_R2_L3_R4_L)A(1_R2_L4_L3_R)^* &= 4g^4\frac{\langle 13\rangle^3\langle 23\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{[31]^3[32]}{[21][42][34][13]} \\
&= -4g^4\frac{s_{13}^2 s_{23}}{s_{12}s_{34}\langle 23\rangle[42]\langle 41\rangle[13]}.
\end{aligned}
$$

The spinor products in the denominator can be rewritten as vector products by transforming $\langle 23\rangle[42]\langle 41\rangle[13] = \langle 3\not{2}4]\langle 4\not{1}3] = -s_{13}s_{14}$. Inserting the cross-term and squares together with the evaluated traces (calculated in Section 2.4) gives

$$
\operatorname{tr}|\mathcal{A}(1_R2_L3_R4_L)|^2 = \frac{16}{3}\left(4g^4\frac{t^3}{s^2u}+4g^4\frac{tu}{s^2}\right)+2\left(-\frac{2}{3}\right)4g^4\frac{t^2}{s^2}.
$$

Applying the Mandelstam relation $t^3/(s^2u) = -tu/s^2 + t/u - 2t^2/s^2$ we get our final result for the squared helicity amplitude,

$$\mathrm{tr}\,|\mathcal{A}(1_R 2_L 3_R 4_L)|^2 = \frac{64}{3} g^4 \left[\frac{t}{u} - \frac{9}{4}\frac{t^2}{s^2} \right]. \tag{5.41}$$

The three other nonzero squared helicity amplitudes we obtain easily from (5.41). The $1_L 2_R 3_L 4_R$ combination we get by a parity transform, which does not change the value of the amplitude. To get the $1_R 2_L 3_L 4_R$ combination we can interchange $3 \longleftrightarrow 4$, or equivalently $t \longleftrightarrow u$. The final $1_L 2_R 3_R 4_L$ combination is a parity transform of $1_R 2_L 3_L 4_R$. Summing all the squared helicity amplitudes gives then

$$\sum_{\text{helicity}} \mathrm{tr}\,|\mathcal{A}(\bar{q}q \to gg)|^2 = \frac{128 g^4}{3} \left[\frac{t}{u} + \frac{u}{t} - \frac{9}{4}\frac{t^2 + u^2}{s^2} \right], \tag{5.42}$$

which we recognize as the correct answer (4.16).

Chapter 6

Efficient Techniques for Scattering Amplitudes

A tree level n-gluon amplitude $\mathcal{A}$ can be split into $(n-1)!$ color-ordered (CO) amplitudes A appended by color factors as follows

$$\mathcal{A}(1_{h_1}\dots n_{h_n}) = \sum_{\sigma\in S_{n-1}} A(1_{h_1}\sigma(2_{h_2}\dots n_{h_n}))\,\mathrm{tr}\,T^{a_1}T^{\sigma(a_2)}\dots T^{\sigma(a_n)}, \tag{6.1}$$

where σ is some permutation of $n-1$ elements. We will in this chapter consider a modern approach to calculating the CO amplitudes A. First we will discuss some identities satisfied by the CO amplitudes, and use these to give a concise calculation of the squared amplitude for gluon-gluon scattering. Next we will discuss on-shell recursion, and apply that to find the general n-gluon CO amplitudes for special helicity configurations. As we saw in Chapter 4 and used also in the previous chapter, a similar color decomposition exists with two external quarks.

6.1 Relations between CO amplitudes

Firstly we will consider in more detail the defining decomposition (6.1). Using (2.49) it is clear that the amplitude $\mathcal{A}$ can be written as a linear combination of the traces in (6.1) with purely kinematic prefactors. However it is not obvious that these prefactors should be related by permutations of the external legs in the way it is written in (6.1). To see why this is the case, consider that the full amplitude $\mathcal{A}$ does not care about the ordering of the external legs. Any permutation of external lines is a symmetry of $\mathcal{A}$. For this to hold the kinematic prefactors must transform into each other under permutations of the labels $\{1_{h_1},\dots n_{h_n}\}$ in the same way as the traces do under permutations of $\{a_1,\dots a_n\}$. The result is that they must be related as written in (6.1). Key to the preceding argument is the fact that the $(n-1)!$ color traces are linearly independent. Combining that fact also with the gauge invariance of $\mathcal{A}$ we find that the CO amplitudes are gauge invariant as well.

It is natural to wonder next if (6.1) is the shortest representation of the amplitude in terms of CO amplitudes or not. In other words, are the $(n-1)!$ CO amplitudes linearly independent? The answer is no, and we will investigate some of the resulting identities between the CO amplitudes. The simplest identities follow directly from the defining decomposition. Since the full amplitude $\mathcal{A}$ is invariant under cyclic permutations of all the external legs, and so is the trace, it follows that also the CO amplitudes must have cyclic symmetry,

$$A(\sigma(1,\dots n)) = A(1,\dots n), \qquad \sigma \in C(n).$$

We denote the set of cyclic permutations of n elements by $C(n)$, a set which includes the identity but otherwise does not allow permutations with fixed points.

Next consider extending the gauge group from $\mathrm{SU}(N)$ to $\mathrm{U}(N)$. This effectively means adding a non-interacting photon to the theory. The only difference between the newly added photon and the gluons is the color structure. Since the new generator $T^0 \propto \mathbb{I}$ commutes with all the other T^a any vertex with a photon is proportional to $f^{0ab} = 0$. The amplitude for $n-1$ gluons and one photon is therefore zero. On the other hand the kinematics are the same also with one gluon replaced by a photon, and therefore the CO amplitudes are the same. Inserting $T^{a_1} \propto \mathbb{I}$ in (6.1) then gives

$$0 = \sum_{\sigma\in S_{n-1}} A(1_{h_1}\sigma(2_{h_2}\dots n_{h_n}))\,\mathrm{tr}\,T^{\sigma(a_2)}\dots T^{\sigma(a_n)}.$$

Now use the cyclic property of the trace to collect all terms with a cyclic permutation of $2,\dots n$ together with the common color factor $\mathrm{tr}\,T^{a_2}\dots T^{a_n}$. That color factor then appears nowhere else in the sum above and it follows that its prefactor must vanish. This is known as the sub-cyclic property,

$$\sum_{\sigma\in C(n-1)} A(1,\sigma(2,\dots n)) = 0. \tag{6.2}$$

We can derive a related identity by replacing T^{a_2} with the $\mathrm{U}(1)$ generator. In this case any position of T^{a_2} gives the same trace, and again the prefactor of each unique trace must be zero,

$$A(1,2,3,\dots n) + A(1,3,2,\dots n) + \dots + A(1,3,\dots n,2) = 0. \tag{6.3}$$

We will refer to this as the U(1) decoupling equation. In the literature both names are used interchangeably for the two different identities [30, 31].

Already it is apparent that the description in terms of $(n-1)!$ CO amplitudes is far from minimal. In fact it is possible to express everything using only $(n-2)!$ CO amplitudes as follows

$$\mathcal{A}(1_{h_1}\dots n_{h_n}) = \frac{1}{2}\sum_{\sigma\in S_{n-2}} A(1_{h_1}\sigma(2_{h_2}\dots(n-1)_{h_{n-1}})n_{h_n})(F^{\sigma(a_2)}\dots F^{\sigma(a_{n-1})})_{a_1 a_n}. \tag{6.4}$$

Here we introduced $(F^a)_{bc} = \mathrm{i}f^{bac}$, and the above is a matrix product in the suppressed indices. This was first postulated, and checked up to $n = 8$ by Kleiss and

Kuijf in Ref. [7], while the general proof can be found in Ref. [32]. The powerful identity enabling this reduction is known as the Kleiss-Kuijf relation,

$$A(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta} \sum_{\sigma \in \mathrm{OP}(\{\alpha\},\{\beta\})} A(1,\{\sigma\},n). \tag{6.5}$$

Here $\mathrm{OP}(\{\alpha\},\{\beta\})$ stands for all ordered permutations of $\{\alpha\}\cup\{\beta\}$. That is, all permutations where the ordering of the sub-lists $\{\alpha\}$ and $\{\beta\}$ is preserved. For example $\{1,3,2\}$ and $\{3,1,2\}$ are in $\mathrm{OP}(\{1,2\},\{3\})$ while $\{2,3,1\}$ is not.

A corollary of the Kleiss-Kuijf relation we will use later is

$$\sum_{\sigma \in S_{n-1}} A(1,\sigma(2,\ldots n)) = 0, \tag{6.6}$$

where the sum now runs over all permutations of $n-1$ elements. To prove this we first sum both sides of the Kleiss-Kuijf relation (6.5) over all α,β for a fixed n_β. Summing $A(1,\{\sigma\},n)$ on the right hand side over all such α,β is clearly equivalent to summing it over all permutations σ of $n-2$ elements. Then every summand in the sum over the ordered permutations is the same, so summing just multiplies this value by the number of ordered permutations. The number of ordered permutations of $\{\alpha\},\{\beta\}$ is

$$\frac{(n_\alpha+n_\beta)!}{n_\alpha! n_\beta!} = \frac{(n-2)!}{(n-n_\beta-2)! n_\beta!} = \binom{n-2}{n_\beta},$$

from which we get

$$\sum_{\substack{\alpha,\beta \\ n_\beta \,\text{fixed}}} A(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta}\binom{n-2}{n_\beta} \sum_{\sigma\in S_{n-2}} A(1,\{\sigma\},n).$$

Summing this also over all possible n_β, the left hand side becomes just the sum over all possible permutations of the $n-1$ last elements, the left hand side of (6.6). On the right hand side the sum over $\sigma \in S_{n-2}$ does not depend on n_β, while the prefactor becomes by the binomial theorem

$$\sum_{n_\beta=0}^{n-2} (-1)^{n_\beta}\binom{n-2}{n_\beta} = (1+(-1))^{n-2} = 0.$$

6.2 Application to $gg \to gg$

We will now use the results of the previous chapter and section to efficiently calculate the squared amplitude for $gg \to gg$. First we argue that only amplitudes with two right-handed and two left-handed helicities are non-zero. The amplitude has mass dimension zero and is composed of diagrams with at most one propagator. This means that at most there are two powers of momenta in the numerator of each Feynman diagram. Since the numerator also must contain all four polarization vectors it has at least one contraction of two polarization vectors. With all helicities right-handed this contraction is $\varepsilon_R(i)\cdot\varepsilon_R(j) \propto [r_i r_j]$ where r_i is the reference vector of Section 5.3. Choosing then the same reference vector for all external lines the entire amplitude vanishes. With the polarization vector corresponding to momentum 1 left-handed we choose the reference vector $r = 1$ for all the other polarization vectors. Then $\varepsilon_R(i)\cdot\varepsilon_R(j) = 0$ for all $i, j \neq 1$ by the same argument as before. Since also $\varepsilon_L(1)\cdot\varepsilon_R(i) \propto [11] = 0$ the entire amplitude again vanishes. Parity implies that the $1_L 2_L 3_L 4_L$ and $1_R 2_L 3_L 4_L$ configurations also give zero. Thus only the amplitudes with two left-handed and two right-handed helicities are non-zero.

It is worth it to deviate briefly from the four gluon case to note that the argument above works equally well for the n-gluon case. In that case the mass dimension is $4-n$ and at tree-level each Feynman diagram has at most $n-3$ propagators. This means there are at most $n-2$ powers of momenta in the numerator of each Feynman diagram, but n polarization vectors, so we again get at least one factor $\varepsilon(i)\cdot\varepsilon(j)$. Now we can reuse the argument above to show that the amplitudes with all helicities the same or one helicity different are zero. The first non-zero amplitudes are therefore the ones where two helicities are different from the other. These amplitudes are called maximally helicity violating (MHV). The name comes from the fact that these are the non-zero amplitudes for which $\left|\sum_i^n h_i\right|$ deviates maximally from zero.

In the four gluon case the only non-zero amplitudes are the six MHV amplitudes:

$$\begin{array}{ll} \mathcal{A}(1_R 2_R 3_L 4_L) & \mathcal{A}(1_L 2_L 3_R 4_R) \\ \mathcal{A}(1_R 2_L 3_R 4_L) & \mathcal{A}(1_L 2_R 3_L 4_R) \\ \mathcal{A}(1_R 2_L 3_L 4_R) & \mathcal{A}(1_L 2_R 3_R 4_L). \end{array}$$

The amplitudes in the second column can be obtained from the first column via a parity transformation. The two last amplitudes of the first column are related to the first via $2 \longleftrightarrow 3$ and $2 \longleftrightarrow 4$ respectively. Thus we only need to calculate $\mathcal{A}(1_R 2_R 3_L 4_L)$.

The amplitude $\mathcal{A}(1_R 2_R 3_L 4_L)$ in turn depends on six CO amplitudes, one for each permutation of $\{2_R 3_L 4_L\}$ in accordance with (6.1). The relations of the previous section imply however that we only need to calculate two of these. We choose to calculate $A(1_R 2_R 3_L 4_L)$ and $A(1_R 3_L 2_R 4_L)$. As in Section 5.4 we insert for the polarization vectors the representation (5.31). Then we use the identities of Section 5.2 to write the answer only in terms of simple square brackets. The details of the calculation are mostly the same as in Section 5.4, so we do not repeat them here. In the end we get

$$A(1_R 2_R 3_L 4_L) = 4g^2 \frac{[34]^4}{[12][23][34][41]}, \tag{6.7}$$

$$A(1_R 3_L 2_R 4_L) = 4g^2 \frac{[34]^4}{[13][32][24][41]}. \tag{6.8}$$

The third CO amplitude $A(1_R 2_R 4_L 3_L)$ we can obtain from (6.7) by simply interchanging $3 \longleftrightarrow 4$. Similarly we can get $A(1_R 4_L 2_R 3_L)$ by interchanging 3 and 4 in

(6.8). Looking at the expression we see it is invariant under this interchange, so $A(1_R 4_L 2_R 3_L) = A(1_R 3_L 2_R 4_L)$. Next we can use cyclic symmetry to write

$$A(1_R 3_L 4_L 2_R) = A(2_R 1_R 3_L 4_L), \tag{6.9}$$

where the right hand side is the $1 \longleftrightarrow 2$ interchange of the known expression (6.7). This leaves only $A(1_R 4_L 3_L 2_R)$ which we again get by interchanging $3 \longleftrightarrow 4$ in the above.

If we were to compute cross sections numerically we would stop here. We can insert numerical values for the spinor products and evaluate the color-traces in (6.1) so to be left with just a complex number to square in the end. Appendix D gives a procedure for numerically evaluating spinor products. Presently however we would like to compute the square analytically in order to compare to our result from Section 4.2. Thankfully by using the trick of extending the gauge group to U(N) combined with our corollary to the Kleiss-Kuijf relation (6.6) we can simplify also that procedure considerably.

Since the corresponding additional photon decouples from the gluons, changing the gauge group to U(N) does not change our current amplitude. By choosing the additional U(1) generator as $T^0 = \mathbb{I}/\sqrt{2N}$ it follows directly from (2.47) that the Fierz identity for the U(N) generators is

$$T^a_{ij} T^a_{kl} = \frac{1}{2} \delta_{il} \delta_{jk}.$$

This makes the contraction of color traces simpler. When squaring $\mathcal{A}(1_R 2_R 3_L 4_L)$ and summing over color all squares of CO amplitudes get a color factor $N^4/2^4$, while cross terms get $N^2/2^4$. Squaring and summing over color gives then

$$\begin{aligned}
\operatorname{tr} |\mathcal{A}(1_R 2_R 3_L 4_L)|^2 =& \frac{N^4}{16} \sum_{\sigma \in S_{n-1}} |A(1_R, \sigma(2_R, 3_L, 4_L))|^2 \\
&+ \frac{N^2}{16} \sum_{\sigma \in S_{n-1}} \sum_{\sigma' \neq \sigma} A(1_R, \sigma(2_R, 3_L, 4_L)) A(1_R, \sigma'(2_R, 3_L, 4_L))^* \\
=& \frac{N^4 - N^2}{16} \sum_{\sigma \in S_{n-1}} |A(1_R, \sigma(2_R, 3_L, 4_L))|^2 \\
&+ \frac{N^2}{16} \left| \sum_{\sigma \in S_{n-1}} A(1_R, \sigma(2_R, 3_L, 4_L)) \right|^2 .
\end{aligned}$$

The second line of the final expression is zero by equation (6.6), reducing the number of terms we need to compute from 6^2 to 6.

All the equations (6.7)–(6.9) can be squared directly using (5.27). Specializing now to $N = 3$, the result when expressed in terms of Mandelstam variables is

$$\operatorname{tr} |\mathcal{A}(1_R 2_R 3_L 4_L)|^2 = 72 g^4 \left(2 \frac{s^2}{u^2} + 2 \frac{s^2}{t^2} + 2 \frac{s^4}{t^2 u^2} \right). \tag{6.10}$$

As argued earlier we can sum this over all helicities by multiplying by a factor two for the parity transformed amplitude and adding the two expressions with $2 \longleftrightarrow 3$ and $2 \longleftrightarrow 4$ exchanged. Simplifying the resulting expression we get the familiar result from Section 4.2,

$$\sum_{\text{helicity}} \operatorname{tr}|\mathcal{A}|^2 = 1152g^4\left(3 - \frac{tu}{s^2} - \frac{su}{t^2} - \frac{st}{u^2}\right).$$

6.3 On-Shell Recursion

Compared to the standard computational approach, presented in Chapter 4, the spinor-helicity formalism as presented so far is clearly efficient. But in essence it is just the method of inserting explicit polarizations, as in Section 4.1.2, only with the polarization vectors expressed in terms of Weyl spinors. This enables the calculation to be done at the amplitude level, while also expressing the result directly in terms of Lorentz scalars. However we first have to write down all the contributing diagrams as in the standard approach, meaning that amplitudes with a lot of external lines are still out of reach. Our next topic tackles this problem. On-shell recursion is a method for computing tree-level scattering amplitudes that does not rely on writing down the relevant diagrams. The first on-shell recursion relation was found and proved by Britto, Cachazo, Feng and Witten [33, 34]. That relation is the one we will discuss here.

We consider a n-parton tree level CO amplitude A with all momenta incoming. That amplitude can be written as a rational function of spinor products. We define therefore a function $A(z)$ of the complex variable z, which is analytic except at isolated points, by shifting two of the spinors in A as follows

$$\begin{aligned} \hat{1}\rangle &= 1\rangle & \hat{1}] &= 1] + z2] \\ \hat{2}\rangle &= 2\rangle - z1\rangle & \hat{2}] &= 2]. \end{aligned}$$

Since they are complex-valued we are free to shift the spinors by the complex variable z. However there are no real momenta $\hat{1}, \hat{2}$ for which the above holds. The above spinor shifts define the Feynman slashes of the momenta via $\hat{\not{1}} = \hat{1}\rangle[\hat{1} + \hat{1}]\langle\hat{1}$. This uniquely determines the momenta $\hat{1}, \hat{2}$, though they in general need to be complex. These unphysical momenta will appear only inside spinor products, where they are well-defined in terms of the above shifts. We can compute Lorentz products containing the unphysical momenta, by applying the spinor identity (5.26),

$$p \cdot \hat{1} = \frac{1}{2}\langle p\hat{1}\rangle[\hat{1}p].$$

From the this definition we see that the new momenta are on-shell $\hat{1}^2 = 0$. On the other hand we must take care not to use the identity (5.25) which no longer holds for the asymmetrically shifted spinors.

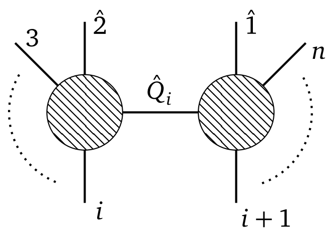


Figure 6.1: A factorization of the amplitude giving rise to a pole in $A(z)$.

Next consider the contour integral around the circle of radius R in the complex z-plane,

$$\oint_R \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{A(z)}{z}.$$

The residue theorem gives this integral as the sum over the residues of the poles of $A(z)/z$ inside the circle. We see immediately that there is a simple pole at $z = 0$. The residue at that pole is exactly $A(0)$, the CO amplitude we wish to compute. If $A(z) \to 0$ as $|z| \to \infty$ the integral above vanishes as $R \to \infty$. Then we get

$$A(0) = -\sum_{z_i} \mathrm{Res}\left(\frac{A(z)}{z}, z_i\right),$$

where the sum runs over all poles z_i of $A(z)/z$ except $z = 0$. We will return to the question of whether $A(z)$ vanishes as $|z| \to \infty$ later.

Any pole in $A(z)$ must come from a propagator denominator $\hat{Q}^2$ becoming zero. Since we are at tree level any such propagator factorizes the amplitude into two sub-amplitudes connected via the single propagator in question. We choose the direction of the momentum $\hat{Q}$ such that it is equal to the sum over all the momenta in the sub-amplitude which contains the $\hat{1}$ external line. By momentum conservation that is equal to minus the sum over the momenta in the other sub-amplitude. If the $\hat{2}$ external line is in the same sub-amplitude as $\hat{1}$ then $\hat{Q}$ does not depend on z since the other sub-amplitude contains no z. Poles in z can thus only come from propagators with $\hat{1}$ and $\hat{2}$ on opposite sides of the corresponding factorization.

We now consider a internal propagator which factorizes the amplitude as shown in Figure 6.1. Per the discussion above these are the propagators which lead to a pole in $A(z)$. There is one such propagator for every i from 3 to $n-1$ and we label the corresponding momentum $\hat{Q}_i$. We have

$$\begin{aligned}
\hat{Q}_i^2 &= (\hat{1} + \sum_{j=i+1}^{n} p_j)^2 \\
&= (\sum_{j=i+1}^{n} p_j)^2 + \sum_{j=i+1}^{n} \langle p_j \hat{1}\rangle[\hat{1} p_j] \\
&= Q_i^2 + z \sum_{j=i+1}^{n} \langle 1 p_j\rangle[p_j 2],
\end{aligned}$$

where we mean by no hat on Q_i that it uses the non-shifted ($z = 0$) momenta. We see that as a result $A(z)$ has a simple pole at

$$z_i = -\frac{Q_i^2}{\sum_{j=i+1}^{n} \langle 1p_j\rangle[p_j 2]}. \tag{6.11}$$

The residue of $A(z)/z$ at the pole $z = z_i$ can be found by expressing the amplitude in terms of the sub-amplitudes of the factorization. The propagator numerator contains two external line factors for the corresponding particle, summed over helicity. The propagator numerator we therefore include as part of the two sub-amplitudes, in a similar way as we did for the optical theorem in Chapter 3. We have then

$$\operatorname{Res}\left(\frac{A(z)}{z}, z_i\right) = N_{Q_i} \sum_h A(\hat{2}, \ldots, i, \hat{Q}_i^h) \frac{1}{Q_i^2} A(-\hat{Q}_i^{-h}, i+1, \ldots, n, \hat{1}), \tag{6.12}$$

where the sum over the helicity of $\hat{Q}$ comes from the propagator numerator. The helicity label of the other external particles is implicit. The shifted spinors are evaluated at the particular value $z = z_i$, given by (6.11), in each of the two sub amplitudes. Note that this is by definition the value of z such that $\hat{Q}^2 = 0$ so that $\hat{Q}$ is allowed as a external momentum in the on-shell sub-amplitudes. This is the essential trick that using the unphysical shifted momenta allows. Expressing the amplitude in terms of smaller *off-shell* sub-amplitudes is essentially the standard technique of Feynman rules. The trick of using the shifted spinors allows us to instead use on-shell sub-amplitudes. This is a significant simplification since the on-shell amplitudes are much simpler than their off-shell counterparts.

The second sub-amplitude in (6.12) is expressed as a all-in amplitude by utilizing crossing symmetry. The incoming $-\hat{Q}_i$ is equivalent to a outgoing $\hat{Q}_i$ with the opposite helicity. Spinor products of this negative momentum can be evaluated using the relation $\langle(-p)q\rangle = -\mathrm{i}\langle pq\rangle$ which we derive in Appendix D. One subtlety arises when the propagator is fermionic. Then the propagator numerator contains $\hat{Q}_i\rangle[\hat{Q}_i$ which by our definitions (5.21)–(5.24) gives a incoming fermion or anti-fermion in both the left and right sub-amplitude. Using $-\hat{Q}_i$ in the right sub-amplitude we must then compensate for the fact that one of the factors of $-\mathrm{i}$ is erroneous.

In addition to the above fermion correction, the overall factor N_{Q_i} in (6.12) accounts for the normalization of the CO amplitudes. This is easiest to understand by considering how a arbitrary normalization of the generators T^a affects the definition of the CO amplitudes. We parameterize the normalization by the constant C_R appearing in the Fierz identity,

$$T^a_{ij} T^a_{kl} = C_R\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right).$$

When squaring the expression (6.1) defining the CO amplitude and summing over color we apply the Fierz identity k times to reduce the contracted traces to simple numbers, where k is the number of external gluons. Since $\operatorname{tr}|\mathcal{A}|^2$ is independent on C_R it follows that the CO amplitude with k external gluons carries a factor $C_R^{-k/2}$.

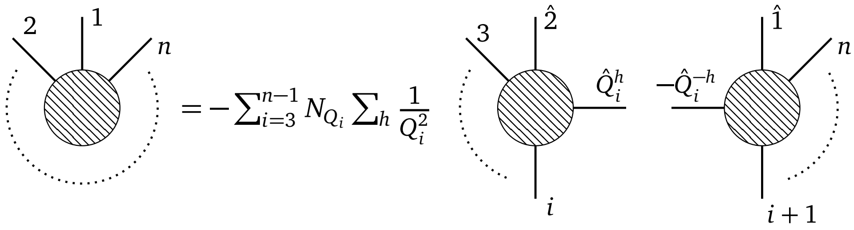


Figure 6.2: Graphical representation of the BCFW on-shell recursion relation (6.14).

The two CO amplitudes on the right hand side of (6.12) then together carry a factor $C_R^{-(k+2)/2} = C_R^{-k/2} C_R^{-1}$ if the Q_i propagator is a gluon line. To get the correct scaling $C_R^{-k/2}$ of the left hand side we must correct for the additional C_R^{-1} by multiplying the right hand side by C_R. If the Q_i propagator is a fermion line no such factor is needed. Thus

$$N_{Q_i} = \begin{cases} C_R & \text{if } Q_i \text{ is a gluon line,} \\ \mathrm{i} & \text{if } Q_i \text{ is a fermion line,} \end{cases} \tag{6.13}$$

where in our case $C_R = 1/2$ in accordance with (2.47).

Assuming $A(z) \to 0$ as $|z| \to \infty$ and using the above result for the residues of $A(z)/z$ we can write the n-parton CO amplitude as

$$A(1,\ldots,n) = -\sum_{i=3}^{n-1} N_{Q_i} \sum_h A(\hat{2},\ldots,i,\hat{Q}_i^h) \frac{1}{Q_i^2} A(-\hat{Q}_i^{-h}, i+1,\ldots,n,\hat{1}). \tag{6.14}$$

This expresses the amplitude solely in terms of on-shell amplitudes with fewer external legs. The relation is represented graphically in Figure 6.2. All tree level QCD amplitudes can be computed from the above formula, given the initial physical input of the three-point CO amplitudes.

We will now consider when (6.14) holds. That is, when does $A(z)$ vanish as $|z| \to \infty$. Factors of z in the denominator originate solely with the propagators we considered earlier. In any given diagram these form a unbroken line of propagators from $\hat{1}$ to $\hat{2}$. Consider the case where that line is purely composed of gluon propagators. The line has one more vertex than propagator, so since the propagators are $\mathcal{O}(z^{-1})$ and each vertex is at most linear in the connected momenta, the entire line is at most $\mathcal{O}(z)$. The remaining z-dependence is now in the external factors $\varepsilon(\hat{1})$ and $\varepsilon(\hat{2})$. Considering the helicity choice

$$\begin{aligned} \varepsilon_R^\mu(\hat{1}) &= \frac{1}{\sqrt{2}} \frac{[1\gamma^\mu \hat{1}\rangle}{[1\hat{1}]} = \frac{1}{z} \frac{[1\gamma^\mu 1\rangle}{\sqrt{2}[12]} \\ \varepsilon_L^\mu(\hat{2}) &= -\frac{1}{\sqrt{2}} \frac{\langle 2\gamma^\mu \hat{2}]}{\langle 2\hat{2}\rangle} = \frac{1}{z} \frac{\langle 2\gamma^\mu 2]}{\sqrt{2}\langle 21\rangle}, \end{aligned}$$

we see that this leads to $A(z)$ being $\mathcal{O}(z^{-1})$. In the all gluon case any nonzero amplitude has at least two helicities different from the rest. Combining this with the cyclic symmetry of the CO amplitude, we can always cycle the momenta such that $(h_1, h_2) = (R, L)$ and the relation (6.14) holds.

Other helicity combinations also lead to vanishing as $|z| \to \infty$. For a discussion of these and the case with external quarks see Ref. [28]. A physical interpretation of the large z limit and why it sometimes vanishes, as well as extensions to other theories can be found in Ref. [35].

6.4 The Parke-Taylor Formula

The simple form of the four gluon MHV amplitudes (6.7) and (6.8) was found to continue up to six external gluons by Parke and Taylor [6]. This lead them to postulate that it holds in general, so that for all but two gluons right-handed we have

$$A(1^R, \ldots, i^L, \ldots, j^L, \ldots n^R) = 2^{n/2}(g)^{n-2} \frac{[ij]^4}{[12][23]\ldots[(n-1)n][n1]}, \tag{6.15}$$

and with most gluons left-handed

$$A(1^L, \ldots, i^R, \ldots, j^R, \ldots n^L) = 2^{n/2}(-g)^{n-2} \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle (n-1)n\rangle\langle n1\rangle}. \tag{6.16}$$

These remarkable formulas where first proved in Ref. [36]. Armed with the recursion relation (6.14) of the previous section we can give a concise alternative proof, adapted from that used in Refs. [28] and [37].

First we need the $n = 3$ gluon scattering amplitudes. Since with lightlike momenta satisfying $i+j+k = 0$ we have $2(i \cdot j) = (i+j)^2 = k^2 = 0$, there are no Lorentz scalars for this amplitude to depend on. As a consequence the three-point scattering amplitudes of massless particles are zero. However with the shifted spinors of the previous section we can have $2(i \cdot \hat{j}) = \langle i\hat{j}\rangle[\hat{j}i] = 0$ by setting $\langle i\hat{j}\rangle = 0$ but keeping $[\hat{j}i] \neq 0$. This is possible only for the shifted spinors, where (5.25) no longer holds. In this way the analytic form of the three-point scattering amplitudes constitute the initial seed for on-shell recursion, even though the amplitudes are zero for any real momenta.

The CO amplitude for three incoming gluons is

$$\begin{aligned} A(1,2,3) = 2g\big[&\varepsilon(1)\cdot\varepsilon(2)(1-2)\cdot\varepsilon(3) \\ &+ \varepsilon(2)\cdot\varepsilon(3)(2-3)\cdot\varepsilon(1) + \varepsilon(3)\cdot\varepsilon(1)(3-1)\cdot\varepsilon(2)\big]. \end{aligned}$$

We will only need the amplitudes with one helicity different from the two others. Setting both 1 and 2 to be right-handed the first term of A vanishes with $r_2 = r_1$.

With $r_3 = 1$ we get $\varepsilon(3)_L \cdot \varepsilon(1)_R = 0$ so only the second term contributes. We find

$$\begin{aligned}
A(1^R, 2^R, 3^L) &= 2g\varepsilon_R(2)\cdot\varepsilon_L(3)(2-3)\cdot\varepsilon_R(1) \\
&= -2g\frac{\langle 12\rangle[r_1 3]}{\langle 13\rangle[r_1 2]}\frac{[r_1(\not{2}-\not{3})1\rangle}{\sqrt{2}[r_1 1]} \\
&= -2\sqrt{2}g\frac{\langle 12\rangle[r_1 3][r_1\not{2}1\rangle}{\langle 13\rangle[r_1 2][r_1 1]} \\
&= -2\sqrt{2}g\frac{\langle 12\rangle[r_1 3]\langle 21\rangle}{\langle 13\rangle[r_1 1]}.
\end{aligned}$$

To remove the r_1-dependent factors we multiply the numerator and denominator by $\langle 13\rangle$. Then we can use $[r_1 3]\langle 13\rangle = [r_1 2]\langle 21\rangle$ and $[r_1 1]\langle 13\rangle = -[r_1 2]\langle 23\rangle$ to arrive at

$$A(1^R, 2^R, 3^L) = -2^{3/2}g\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}. \tag{6.17}$$

Proceeding in the same way with 1 and 2 left-handed gives

$$A(1^L, 2^L, 3^R) = 2^{3/2}g\frac{[12]^4}{[12][23][31]}, \tag{6.18}$$

so the $n = 3$ amplitudes agree with the Parke-Taylor formulas (6.15) and (6.16).

Next we proceed via induction, assuming the formula (6.15) holds for $n-1$ external gluons. Due to the cyclic symmetry we can without loss of generality set 1 to be right-handed and 2 to be left-handed. Then $(h_1, h_2) = (R, L)$ so the recursion relation (6.14) holds,

$$\begin{aligned}
&A(1^R, 2^L \ldots, j^L, \ldots, n^R) = \\
&\qquad -\frac{1}{2}\sum_{i=3}^{n-1}\sum_h A(\hat{2}^L, \ldots, i^{h_i}, \hat{Q}_i^h)\frac{1}{Q_i^2}A(-\hat{Q}_i^{-h}, (i+1)^{h_{i+1}}, \ldots, n^{h_n}, \hat{1}^R).
\end{aligned}$$

It turns out most terms of this sum are zero due to the MHV result of Section 6.2. The two terms with $i = 3$ and $i = n-1$ are special since one of the sub-amplitudes is then a three-point amplitude, where the MHV result does not hold. The other terms in the sum we split into two cases:

- $i < j$: The two sub-amplitudes contain one of the left-handed helicities of 2 and j each. If the helicity h of $\hat{Q}$ is L in one sub-amplitude it is R in the other sub-amplitude. The other sub-amplitude is then a amplitude with four or more external gluons but only one left-handed, so it is zero.
- $i \geq j$: Both left-handed helicities of $2, j$ are in the left sub-amplitude. The right sub-amplitude does not contain the two left-handed helicities required to be non-zero, regardless of the helicity of $\hat{Q}$.

Only the two terms in the recursion containing three-point sub-amplitudes remain,

$$A(1^R,2^L\dots j^L\dots,n^R) = -\frac{1}{2}A(\hat{2}^L,3^R,\hat{Q}_3^R)\frac{1}{Q_3^2}A(-\hat{Q}_3^L,4^R,\dots j^L\dots,n^R,\hat{1}^R)$$
$$-\frac{1}{2}A(\hat{2}^L,\dots j^L\dots,(n-1)^R,\hat{Q}_{n-1}^R)\frac{1}{Q_{n-1}^2}A(-\hat{Q}_{n-1}^L,n^R,\hat{1}^R). \tag{6.19}$$

The sub-amplitudes that are not three-point ones are now MHV amplitudes with $n-1$ external legs, which by the inductive hypothesis are given by (6.15).

Using the inductive hypothesis and the three-point amplitude (6.17) the z_3-dependent part of the first term on the right hand side of (6.19) is

$$\frac{\langle 3\hat{Q}_3\rangle^4[(-\hat{Q}_3)j]^4}{\langle\hat{2}3\rangle\langle 3\hat{Q}_3\rangle\langle\hat{Q}_3 2\rangle[(-\hat{Q}_3)4][1(-\hat{Q}_3)]} = (-\mathrm{i})^2\frac{\langle 3\hat{Q}_3\rangle^4[\hat{Q}_3 j]^4}{\langle\hat{2}3\rangle\langle 3\hat{Q}_3\rangle\langle\hat{Q}_3 2\rangle[\hat{Q}_3 4][1\hat{Q}_3]}$$
$$= -\frac{(-\langle 3\hat{\not{2}}j])^4}{\langle\hat{2}3\rangle\langle 3\hat{\not{2}}4][1\not{3}\hat{2}\rangle} = \frac{[2j]^4\langle 3\hat{2}\rangle^4}{[24][13]\langle 3\hat{2}\rangle^3} = \frac{[2j]^4}{[24][13]}\langle 3\hat{2}\rangle,$$

where we used repeatedly that inside spinors products $\hat{Q}_3\rangle[\hat{Q}_3 = \hat{\not{Q}}_3 = -\hat{\not{2}} - \not{3}$ and $\not{3}3\rangle = \hat{\not{2}}\hat{2}\rangle = 0$. Then by computing

$$\langle 3\hat{2}\rangle = \langle 32\rangle - z_3\langle 31\rangle = \langle 32\rangle - \frac{Q_3^2}{\langle 1\not{3}2]}\langle 31\rangle = \langle 32\rangle - \frac{\langle 32\rangle[23]}{\langle 13\rangle[32]}\langle 31\rangle = 0,$$

we find that the first term on the right hand side of (6.19) is zero. Only the $i=n-1$ term contributes and we have

$$\begin{aligned}
A(1^R,2^L\dots,j^L,\dots,n^R) &= -\frac{1}{2}A(\hat{2}^L,\dots,j^L,\dots,(n-1)^R,\hat{Q}_{n-1}^R)\frac{1}{Q_{n-1}^2}A(-\hat{Q}_{n-1}^L,n^R,\hat{1}^R) \\
&= -\frac{1}{2}2^{(n-1)/2}g^{n-3}\frac{[\hat{2}j]^4}{[\hat{2}3][34]\dots[(n-1)\hat{Q}_{n-1}][\hat{Q}_{n-1}\hat{2}]} \\
&\quad\times\frac{1}{\langle n1\rangle[1n]}\times -2^{3/2}g\frac{\langle n\hat{1}\rangle^4}{\langle(-\hat{Q}_{n-1})n\rangle\langle n\hat{1}\rangle\langle\hat{1}(-\hat{Q}_{n-1})\rangle} \\
&= -2^{n/2}g^{n-2}\frac{[2j]^4\langle n1\rangle^2}{[23][34]\dots[(n-2)(n-1)][1n]} \\
&\quad\times\frac{1}{[(n-1)\hat{Q}_{n-1}][\hat{Q}_{n-1}2]\langle\hat{Q}_{n-1}n\rangle\langle 1\hat{Q}_{n-1}\rangle}.
\end{aligned}$$

We can remove the $\hat{Q}_{n-1}$ dependence of the last factor by using

$$[\hat{Q}_{n-1}2]\langle\hat{Q}_{n-1}n\rangle = -[2\hat{\not{Q}}_{n-1}n\rangle = -[12]\langle n1\rangle$$
$$[(n-1)\hat{Q}_{n-1}]\langle 1\hat{Q}_{n-1}\rangle = -[(n-1)\hat{\not{Q}}_{n-1}1\rangle = -[(n-1)n]\langle n1\rangle.$$

The factors of $\langle n1\rangle$ cancel and we are left with

$$A(1^R,2^L\dots,j^L,\dots,n^R) = 2^{n/2}g^{n-2}\frac{[2j]^4}{[12][23]\dots[(n-2)(n-1)][(n-1)n][n1]},$$

which is the Parke-Taylor formula (6.15). One can prove (6.16) in the same way.

Chapter 7

Conclusion

Guided by experience with unitarity and the Cutkosky procedure we have shown how to correctly apply the method of replacing a gluon polarization sum by $-\eta^{\mu\nu}$ in QCD. We have seen how unitarity follows from the gauge invariance of the propagator. Using Cutkosky's rule this means we can use the propagator in any gauge to sum over external polarizations, provided that we sum over the cuts of all relevant diagrams in that gauge. For the Feynman-t' Hooft gauge this means including also cuts over Faddeev-Popov ghosts. The latter cuts do not give squared amplitudes, but rather cross terms between ghost diagrams with two different charge flow directions. With just two external lines in the Feynman-t' Hooft gauge these cross terms are related to squares by a simple sign. This has mislead some to believe that squared ghost amplitudes can be used in general, see e.g. Ref. [3]. We demonstrated that this is not possible, and that the cross terms have to be used, without a additional sign.

A natural next step is to generalize the above discussion to massive vector bosons, like the $W^{\pm}$ and Z bosons of the electroweak theory. The extension of our proof of the optical theorem to this case is sketched in Ref. [38]. A alternative, somewhat more sophisticated, method for proving perturbative unitarity in the massive case can be found in the work of Becchi, Rouet and Stora [39]. The latter method is however not based on amplitude-level arguments and therefore is not suited to derive the cancellation needed for the $-\eta^{\mu\nu}$ replacement. Instead the same shift as used by BRS can be used in a argument similar to the one in Section 4.3. This is done in Ref. [24] and gives a alternative way to derive the central Slavnov-Taylor identity (4.47) as well as its generalization to any number of external ghosts.

The main disadvantage of the method of replacing the polarization sum in practical calculations is that it requires squaring the amplitude first. To avoid squaring we need to insert a explicit representation of the polarization vectors. That requires choosing a particular frame, and we should transform the result into a frame-independent one. In the second part of this work we considered the spinor-helicity formalism, which makes this process particularly convenient. We reviewed the basics of writing a amplitude in terms of spinor products and applied this to quark-annihilation into two gluons. In addition to being a efficient computational tool the new approach made it easier to uncover general structures and relationships between the amplitudes. This is particularly true when we also decompose the amplitude into color-ordered components. We considered that decomposition and used the resulting identities and relationships to give a concise calculation of the gluon-gluon scattering amplitude.

Next we considered on-shell recursion. We applied this technique to find general formulas for the n-gluon MHV amplitudes. On-shell recursion represents a method for computing tree-level scattering amplitudes that does not rely on writing down all topologically distinct diagrams. If we combine it with the fact that the lowest multiplicity scattering amplitudes can be deduced directly from basic physical assumptions, it represents a method independent of Feynman rules as a whole [29]. In principle it therefore does not even require a Lagrangian. This can be extended to loop level by the method of generalized unitarity, which constructs loop amplitudes out of tree-level amplitudes using the analytic properties of the former. At its core is the Cutkosky prescription presented in Chapter 3, which in generalized unitarity must be applied also to the branch cuts in other channels than the s-channel. A review of generalized unitarity can be found in Ref. [40]. These techniques have found success also outside the pure non-abelian gauge theories discussed here. The greatest progress have been made in supersymmetric theories, but recent years have seen advances also in gravity, effective field theories and even fluid mechanics [41–44].

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Appendix A

Notation and Conventions

The metric $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is used throughout, as well as units where $c = \hbar = 1$. Repeated indices are summed, unless context implies otherwise. We use Feynman rules as in Romão and Silva's resource [45], with sign conventions $\eta_s = +1, \eta_G = +1$, following their notation. In particular this means our covariant derivative is

$$D_\mu = \partial_\mu + \mathrm{i}gA_\mu, \tag{A.1}$$

when acting on spin-1/2 fields, and

$$D_\mu^{ab} = \delta^{ab}\partial_\mu + gf^{abc}A_\mu^c, \tag{A.2}$$

when acting on gauge or ghost fields. The Weyl representation is used for Dirac spinors and gamma matrices, and we write for the common contraction $\not{A} \equiv A_\mu \gamma^\mu$. We normalize Dirac spinors such that $\bar{u}(p)u(p) = 2m$ and Weyl spinors such that $\phi_R^\dagger(p)\phi_R(p) = \phi_L^\dagger(p)\phi_L(p) = 2E$. A number is sometimes used to represent its corresponding four-momentum, so that $2(1 \cdot 2) = 2p_1 \cdot p_2 = 2\eta_{\mu\nu}p_1^\mu p_2^\nu$ and $\not{1} = \not{p}_1 = \gamma_\mu p_1^\mu$.

We denote the full amplitude by $\mathcal{A}$ and color-ordered (CO) amplitudes by A. Our normalization for the SU(N) generators is $C_R = 1/2$ where

$$\mathrm{tr}\left(T^a T^b\right) = C_R \delta^{ab}.$$

The CO amplitudes A' for a arbitrary normalization C_R are related to the ones we compute by

$$A' = (2C_R)^{-n/2} A,$$

where n is the number of external gluons.

Appendix B

The Lie Algebra of the Lorentz Group

The aim of this appendix is to find a basis for the Lie algebra corresponding to the Lorentz group. Directly from the definition as a matrix that does not change the Lorentz product we for $\Lambda^{\mu}{}_{\nu}$ in the Lorentz group compute

$$\begin{aligned}\eta_{\nu\rho} x^{\nu} y^{\rho} &= x^{\mu} y_{\mu} \\ &= \Lambda^{\mu}{}_{\nu} \Lambda_{\mu\rho} x^{\nu} y^{\rho} \\ &= \Lambda^{\mu}{}_{\nu} \eta_{\mu\sigma} \Lambda^{\sigma}{}_{\rho} x^{\nu} y^{\rho} \\ &= (\Lambda^{T})_{\nu}{}^{\mu} \eta_{\mu\sigma} \Lambda^{\sigma}{}_{\rho} x^{\nu} y^{\rho},\end{aligned}$$

which as a matrix equation reads $\Lambda^{T} \eta \Lambda = \eta$. Operating on this from the right with Λ^{-1} and from the left with η^{-1} we find $\Lambda^{-1} = \eta^{-1} \Lambda^{T} \eta$, or in terms of components

$$(\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}. \tag{B.1}$$

Assume now $X^{\mu}{}_{\nu}$ is an element of the Lie algebra of the Lorentz group. By definition this means $\exp\bigl(i\vartheta X^{\mu}{}_{\nu}\bigr)$ is an element of the Lorentz group for any real number ϑ [25]. In particular the identity (B.1) should hold for any ϑ. Using that a similarity transform can be taken inside a exponential map it follows that

$$\exp\bigl(i\vartheta \eta^{-1} X^{T} \eta\bigr) = \exp(-i\vartheta X).$$

For this to hold for all ϑ we must have $\eta^{-1} X^{T} \eta = -X$. Looking at this relation for two covariant components we see that it is the statement of antisymmetry $X_{\mu\nu} = -X_{\nu\mu}$. The antisymmetric 4 by 4 matrices can all be constructed out of a basis of 6 elements.

We can choose as our basis the matrices

$$
\begin{aligned}
(\mathcal{J}^{01})_{\alpha\beta} &= \begin{pmatrix} 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (\mathcal{J}^{02})_{\alpha\beta} = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
(\mathcal{J}^{03})_{\alpha\beta} &= \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix} \quad (\mathcal{J}^{12})_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
(\mathcal{J}^{13})_{\alpha\beta} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix} \quad (\mathcal{J}^{23})_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \end{pmatrix}.
\end{aligned}
\tag{B.2}
$$

Since $\Lambda^{\mu}{}_{\nu}$ is real the elements of the Lie algebra must be purely imaginary, giving the factor i in the above definition. The curious labeling of the basis was introduced because it enables us to write all the basis elements in a compact expression,

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = \mathrm{i}(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}). \tag{B.3}$$

Here the μ, ν now label the different basis elements, while $\alpha\beta$ are the matrix indices. Being antisymmetric in $\mu\nu$ this expression still corresponds to only 6 different basis elements. To get a basis for the Lie algebra of the Lorentz group we should contract with the metric tensor to get one covariant and one contravariant index. This we can see means $(\mathcal{J}^{0i})^{\alpha}{}_{\beta}$ becomes symmetric while $(\mathcal{J}^{ij})^{\alpha}{}_{\beta}$ is multiplied by an overall -1 and remains antisymmetric. Any element X of the Lie algebra can be written as a linear combination of the basis elements,

$$X^{\alpha}{}_{\beta} = \omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta}, \tag{B.4}$$

where $\omega_{\mu\nu}$ are arbitrary real parameters.

Before continuing our investigation of the Lie algebra of the Lorentz group it is useful to see how the parameterization given by (B.4) is connected to the usual parameterization in terms of boosts and rotations. As an example let us set $\omega_{12} = -\omega_{21} = \alpha/2$ as the only non-zero elements of $\omega_{\mu\nu}$. Inserting this we get the Lorentz transformation

$$\Lambda^{\alpha}{}_{\beta} = \exp\left(\mathrm{i}X^{\alpha}{}_{\beta}\right) = \exp\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To compute the matrix exponential we can diagonalize the matrix argument using its eigenvectors. This gives $\mathrm{i}X = PDP^{-1}$ where $D = \mathrm{diag}(0, -i\alpha, i\alpha, 0)$ contains the eigenvalues of $\mathrm{i}X$ and P has the corresponding eigenvectors as its columns. Then using the property $\exp\big(PDP^{-1}\big) = P\exp(D)P^{-1}$ of the matrix exponential we have

$$\Lambda^{\alpha}{}_{\beta} = P\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\alpha} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\alpha} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}P^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{B.5}$$

which we recognize as a rotation of angle α around the 3-axis. We may say that the Lie algebra element $(\mathcal{J}^{12} - \mathcal{J}^{21})/2 = \mathcal{J}^{12}$ generates a rotation around the 3-axis. The natural generalization is that

$$J^i = \frac{1}{2}\epsilon^{ijk}\mathcal{J}^{jk} \tag{B.6}$$

generates a rotation around the i-axis. Here ϵ^{ijk} is the Levi-Civita symbol. By a similar calculation we can convince ourselves that the three elements

$$K^i = \mathcal{J}^{0i} \tag{B.7}$$

generate a boost in the i-direction. The J^i and K^i together make up all the 6 basis elements so we can alternatively write any Lie algebra element as

$$X = \theta^i J^i + \eta^i K^i, \tag{B.8}$$

where we suppressed the indices α, β. Then θ^i is the rotation angle for a rotation around the i-axis and η^i is the rapidity of a boost in the i-direction.

So far we have considered the Lorentz group in the standard way as consisting of 4 by 4 matrices which leave the Lorentz product unchanged. However our main motivation for computing the Lie algebra is to uncover new matrix representations of the Lorentz group. That means we should find a description of the Lie algebra that is representation independent. The commutation relations of a Lie algebra are exactly that kind of defining property, independent of representations [25]. In order to find new matrix representations our next goal is therefore to find the commutation relations for the basis J^i, K^i.

Using the explicit expression (B.3) we have a explicit expressions also for J^i and K^i. By a direct calculation we have then the commutation relations

$$\big[J^i, J^j\big] = \mathrm{i}\epsilon^{ijk}J^k \tag{B.9}$$

$$\big[J^i, K^j\big] = \mathrm{i}\epsilon^{ijk}K^k \tag{B.10}$$

$$\big[K^i, K^j\big] = -\mathrm{i}\epsilon^{ijk}J^k. \tag{B.11}$$

We see that while both sets of bases $\{J^i\}$, $\{K^i\}$ generate vector subspaces of the Lie algebra only the rotations are closed under multiplication. Instead we can introduce the basis elements

$$J^i_{\pm} = \frac{1}{2}(J^i \pm \mathrm{i}K^i), \tag{B.12}$$

which generate two subspaces that are also closed under multiplication. The commutation relations of the new operators are

$$\left[J_+^i, J_+^j\right] = \mathrm{i}\epsilon^{ijk} J_+^k \tag{B.13}$$

$$\left[J_-^i, J_-^j\right] = \mathrm{i}\epsilon^{ijk} J_-^k \tag{B.14}$$

$$\left[J_+^i, J_-^j\right] = 0, \tag{B.15}$$

which we recognize as the commutation relations of angular momentum. The commutation relations of angular momentum define the Lie algebra of SU(2). Thus the Lie algebra of the Lorentz group is the direct sum of two SU(2) Lie algebras.

Appendix C

Plane-Wave Solutions to the Dirac Equation

We look for plane wave solutions to the Dirac equation (5.17). Inserting the positive and negative frequency plane waves

$$\psi(x) = u(p)\exp(-\mathrm{i}p\cdot x) \qquad \psi(x) = v(p)\exp(\mathrm{i}p\cdot x),$$

into the Dirac equation (5.17) leads to the equations

$$(\not{p} - m)u(p) = 0 \tag{C.1}$$

$$(-\not{p} - m)v(p) = 0. \tag{C.2}$$

The aim of this appendix is to find the general solutions to these equations. In the rest frame where $p^\mu = (m, 0, 0, 0)$ the equations reduce to

$$(\gamma^0 - 1)u(0) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u(0) = 0$$

$$(\gamma^0 + 1)v(0) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v(0) = 0.$$

The general solutions of the above is seen to be

$$u(0) = \frac{\mathcal{N}}{\sqrt{2}}\begin{pmatrix} \phi_1 \\ \phi_1 \end{pmatrix} \qquad v(0) = \frac{\mathcal{N}}{\sqrt{2}}\begin{pmatrix} \phi_2 \\ -\phi_2 \end{pmatrix},$$

with some arbitrary two-component spinors ϕ_1, ϕ_2 which we normalize to $\phi_1^\dagger\phi_1 = 1 = \phi_2^\dagger\phi_2$ and a arbitrary constant $\mathcal{N}$ which we set equal for the two cases. The general plane wave solutions $u(p), v(p)$ can now be found by boosting to a arbitrary frame. Alternatively the relation

$$(\not{p} - m)(\not{p} + m) = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}p_\mu p_\nu - m^2 = p^2 - m^2 = 0,$$

implies that

$$u(p) = C_1(\not{p} + m)u(0) \qquad v(p) = C_2(\not{p} - m)v(0), \tag{C.3}$$

solves the equations (C.1) and (C.2) in a arbitrary frame. Here C_1, C_2 are (possibly momentum-dependent) scalars.

We can determine the factors C_1, C_2 by finding another frame-specific solution. Consider the frame where $p^\mu = (E, 0, 0, p)$, that is with the momentum pointing along the 3-axis. In this frame we the matrix equation $(\not{p} - m)u(p) = 0$ can be solved to give the general solution

$$u(p) = \begin{pmatrix} u_1 \\ u_2 \\ \frac{E+p}{m} u_1 \\ \frac{E-p}{m} u_2 \end{pmatrix}.$$

As before two degrees of freedom u_1 and u_2 remain. The quantity $\bar{u}(p)u(p)$ is the same in all Lorentz frames. Thus $\bar{u}(p)u(p) = \bar{u}(0)u(0) = \mathcal{N}^2$, or equivalently

$$2\frac{E+p}{m}|u_1|^2 + 2\frac{E-p}{m}|u_2|^2 = \mathcal{N}^2. \tag{C.4}$$

We use the two remaining degrees of freedom to expand now in helicity eigenstates. In the present frame the helicity operator (5.19) takes the form

$$h = \frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}.$$

Clearly then the state with helicity $+1/2$ corresponds to $u_2 = 0$. Finding also the value of u_1 by using (C.4) we get

$$u_{+1/2}(p) = \begin{pmatrix} \mathcal{N}\sqrt{\frac{m}{2(E+p)}} \\ 0 \\ \mathcal{N}\sqrt{\frac{E+p}{2m}} \\ 0 \end{pmatrix} = \begin{pmatrix} m/\sqrt{E+p} \\ 0 \\ \sqrt{E+p} \\ 0 \end{pmatrix}, \tag{C.5}$$

where we chose $\mathcal{N} = \sqrt{2m}$ in order to have a well defined limit $m \to 0$. The state with helicity $-1/2$ has instead $u_1 = 0$. Repeating the above calculation we then get

$$u_{-1/2}(p) = \begin{pmatrix} 0 \\ m/\sqrt{E-p} \\ 0 \\ \sqrt{E-p} \end{pmatrix}. \tag{C.6}$$

For the negative frequency solutions the matrix equation is $(\not{p} + m)v(p) = 0$. Solving this in the frame where $p^\mu = (E, 0, 0, p)$ and picking out the two helicity eigenstates as before gives

$$v_{+1/2}(p) = \begin{pmatrix} m/\sqrt{E+p} \\ 0 \\ -\sqrt{E+p} \\ 0 \end{pmatrix} \tag{C.7}$$

$$v_{-1/2}(p) = \begin{pmatrix} 0 \\ m/\sqrt{E-p} \\ 0 \\ -\sqrt{E-p} \end{pmatrix}. \tag{C.8}$$

To obtain the factors C_1 and C_2 in the general solution (C.3) we compare it to the specific solutions (C.5)–(C.8). Write $\phi_1 = (u'_1, u'_2)$ for the rest-frame solution $u(0)$. Computing then the general solution (C.3) in the frame where $p^\mu = (E, 0, 0, p)$ and requiring it to be equal to the specific solution of the previous paragraph amounts to the conditions

$$\begin{aligned} u_1 &= C_1\sqrt{m}(m+E-p)u'_1 \\ u_2 &= C_1\sqrt{m}(m+E+p)u'_2. \end{aligned}$$

The $u_2 = 0$ condition of the positive helicity state clearly requires also $u'_2 = 0$. The normalization $\phi_1^\dagger\phi_1 = 1$ for the rest frame solutions then implies $u'_1 = 1$. We get

$$C_1 = \frac{u_1}{\sqrt{m}(m+E-p)} = \sqrt{\frac{m}{E+p}}\frac{1}{m+E-p} = \frac{1}{\sqrt{2m(E+m)}},$$

where we inserted u_1 from the solution (C.5) and used $(m+E-p)^2 = 2(E+m)(E-p)$. Comparing in the same way the general solution for $v(p)$ in (C.3) with the specific solution (C.7) gives

$$C_2 = -\frac{1}{\sqrt{2m(E+m)}}.$$

Finally we have the general plane wave solutions of the Dirac equation by inserting the expressions for C_1, C_2 into (C.3),

$$u(p) = \frac{\not{p}+m}{\sqrt{2(E+m)}}\begin{pmatrix} \phi_1 \\ \phi_1 \end{pmatrix} \tag{C.9}$$

$$v(p) = \frac{-\not{p}+m}{\sqrt{2(E+m)}}\begin{pmatrix} \phi_2 \\ -\phi_2 \end{pmatrix}. \tag{C.10}$$

The two-component spinors ϕ_1, ϕ_2 express the two remaining degrees of freedom.

To expand $u(p)$ in a basis of two elements we may choose e.g. $\phi_+ = (1, 0)$ and $\phi_- = (0, 1)$. We found above that ϕ_+, ϕ_- correspond in the frame where $p^\mu = (E, 0, 0, p)$ to the solutions with helicity $+1/2$ and $-1/2$ respectively. Letting s label the states according to this quantum number, we aim to find a expression for the projection operator $\sum_s u_s \bar{u}_s$. The result will be a analogue of the equation (5.11) relating a spinor to its corresponding 4-momentum, now for the Dirac spinor. Firstly we have by a direct calculation with $\phi_1 = \phi_s$

$$u_s \bar{u}_s = \frac{\not{p}+m}{2(E+m)} \begin{pmatrix} \phi_s \phi_s^\dagger & \phi_s \phi_s^\dagger \\ \phi_s \phi_s^\dagger & \phi_s \phi_s^\dagger \end{pmatrix} (\not{p}+m).$$

The outer product $\phi_s \phi_s^\dagger$ is for ϕ_+ equal to diag$(1,0)$ and for ϕ_- equal to diag$(0,1)$. It follows that

$$\sum_s u_s \bar{u}_s = \frac{\not{p}+m}{2(E+m)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} (\not{p}+m).$$

The matrix product in the above relation we compute directly, using $p^i p^j \sigma^i \sigma^j = p^2$ and $E^2 = m^2 + p^2$,

$$\begin{aligned} (\not{p}+m)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}(\not{p}+m) &= \begin{pmatrix} m & E - p^i\sigma^i \\ E + p^i\sigma^i & m \end{pmatrix} \begin{pmatrix} E+m+p^i\sigma^i & E+m-p^i\sigma^i \\ E+m+p^i\sigma^i & E+m-p^i\sigma^i \end{pmatrix} \\ &= \begin{pmatrix} 2(E+m)m & 2(E+m)(E-p^i\sigma^i) \\ 2(E+m)(E+p^i\sigma^i) & 2(E+m)m \end{pmatrix} \\ &= 2(E+m)(\not{p}+m). \end{aligned}$$

Repeating also the calculation for $v(p)$ we obtain the two completeness relations in the convenient form

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m \qquad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m. \tag{C.11}$$

Even though we used ϕ_+, ϕ_- to derive this relation there was nothing special about this choice. In fact the relation $\sum_s \phi_s \phi_s^\dagger = 1$ holds for any complete, orthonormal set of basis vectors $\{\phi_s\}$. Thus the result (C.11) also holds for any such set. Choosing the basis as the eigenstates of some Hermitian operator we by the spectral theorem automatically get a complete and orthogonal set. Also we may choose the operator such that the eigenvalue is a quantum number we care about in the problem at hand. Some common choices are helicity, chirality and spin.

Appendix D

Numerically Evaluating Spinor Products

We will here consider how one can numerically compute $\mathrm{tr}\,|\mathcal{A}|^2$ from the scattering amplitudes of Chapter 6. The efficient techniques of that chapter rely on writing the amplitude in terms CO amplitudes, which in turn are written in terms of spinor products. In computing observables the cross section as a function of the external momenta is relevant, and we need a efficient procedure to connect the two. Using (5.27) one can evaluate squared CO amplitudes directly in terms of momenta. However cross terms of CO amplitudes enter in $|\mathcal{A}|^2$, which require non-trivial spinor simplifications to evaluate in terms of momenta using (5.27). It is much better to evaluate the spinor products directly in terms of momenta, as well as directly evaluating the color factors. Since this leaves only a complex number to square for $|\mathcal{A}|^2$ it is more performant.

To express a spinor product in terms of the momenta we can solve the momentum-space Weyl equation (5.5). The general solution for $p^3 \neq -E$ that also satisfies our normalization $\phi_R^\dagger \phi_R = 2E$ is

$$\phi_R = \frac{1}{\sqrt{E+p^3}} \begin{pmatrix} -p^1 + \mathrm{i}p^2 \\ E + p^3 \end{pmatrix}.$$

To obtain ϕ_L we use (5.9),

$$\phi_L = -\mathrm{i}\sigma^2 \phi_R^* = \frac{-1}{\sqrt{E+p^3}} \begin{pmatrix} E + p^3 \\ p^1 + \mathrm{i}p^2 \end{pmatrix}.$$

Combining these we have

$$\langle pq \rangle = \phi_L^\dagger(p)\phi_R(q) = -x(p^1 - \mathrm{i}p^2) + x^{-1}(q^1 - \mathrm{i}q^2) \qquad x = \sqrt{\frac{q^0+q^3}{p^0+p^3}}, \tag{D.1}$$

and $[pq] = \langle qp \rangle^*$. The expression above has singularities at $p^3 = -p^0$ and $q^3 = -q^0$ due to the special role given to the 3-axis. These can always be avoided by a change of reference frame.

Since amplitudes often are expressed with all momenta either incoming or outgoing we need to utilize crossing symmetry to go to the physically interesting case of $2 \to n$ scattering. Then we should have a convenient way to handle spinor products of analytically continued negative momenta. Using (D.1) we see that the effect is a overall phase,

$$\langle(-p)q\rangle = -\mathrm{i}\langle pq\rangle. \tag{D.2}$$

Using this avoids having to consider complex roots in the numerical procedure.

Magnus Nybråten Malmquist

External Gluons in QCD Scattering Amplitudes

NTNU
Norwegian University of
Science and Technology