Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Natural Sciences
Department of Physics

Snorre Bergan

Topological field theories of superconductor heterostructures

Master's thesis in Physics
Supervisor: Asle Sudbø
May 2021

NTNU
Norwegian University of
Science and Technology

Snorre Bergan

Topological field theories of superconductor heterostructures

Master's thesis in Physics
Supervisor: Asle Sudbø
May 2021

Norwegian University of Science and Technology
Faculty of Natural Sciences
Department of Physics

NTNU
Norwegian University of Science and Technology

Abstract

In topological materials, such as topological insulator heterostructures, the response of an externally applied electromagnetic field or some form of magnetization is manifested by the appearance of a Chern-Simons term. By coupling a Chern-Simons gauge field to the particles of a system, we can achieve interesting physical phenomena with possible desirable technological applications. Motivated by this, we consider a system of a superconductor proximate to a topological insulator and a layer of ferromagnetically aligned magnetic impurities. We anticipate that the effective field theory of this system contains a coupling between a Chern-Simons gauge field and the superconducting Cooper pairs, which in the language of quantum field theory are described by a complex scalar field. This theory would be comparable with the topological Abelian Higgs model, which in a recent study has shown to experience different forms of critical behavior depending on the magnitude of the Chern-Simons coefficient. Furthermore, we will also consider the interactions between the magnetic impurities and the surface fermions, which in turn will lead to additional magnetic interaction terms in the effective field theory, enabling us to compare this system with similar superconducting- and ferromagnetic heterostructures.

The effective topological field theory of the heterostructure is derived by integrating out the fermionic degrees of freedom in the partition function to second order in coupling constants and in the long wavelength limit. The resulting field theory contains the desired Chern-Simons term which couples to the complex scalar field of the superconductor. Additionally, the electric potential of the gauge field acquires a thermally induced mass, which leads to different kinds of effective potentials and screening effects. Furthermore, due to the presence of the magnetic impurities, we also get a Dzyaloshinskii-Moriya term, in addition to several magnetoelectric couplings. The magnitude of the corresponding Chern-Simons coefficient of the topological Abelian Higgs model is bounded from above such that it cannot be tuned between the critical regions of this model. Furthermore, we show that the Dzyaloshinskii-Moriya coefficient can alter the magnetic ordering of the system and thereby possibly tune between different kinds of superconducting- and ferromagnetic phases.

Sammendrag

I topologiske materialer, slik som topologisk isolator heterostrukturer, er responsen av et påført elektromagnetisk felt eller magnetisering manifestert ved et såkalt Chern-Simons ledd. Ved å koble et slikt Chern-Simons felt til andre partikler kan systemet fremvise interessante fysiske fenomener med mulige teknologiske anvendelser. Med dette som motivasjon ser vi på et system bestående av en superleder i nærheten av en topologisk isolator med et lag med ferromagnetiske urenheter imellom dem. Vi forventer at den effektive feltteorien til dette systemet innehar et Chern-Simons felt som er koblet til Cooper-parene i superlederen, som er beskrevet av et komplekst skalarfelt. En slik teori vil være sammenliknbar med den topologiske Abelske Higgs modellen, som i en tidligere studie har vist seg å kunne fremvise ulike former for kritiske fenomener avhengig av størrelsen på koeffisienten foran Chern-Simons leddet. I tillegg til dette tar vi også hensyn til vekselvirkningene mellom de magnetiske urenhetene og fermionene på den topologiske isolatoren. Slike ledd vil kunne lede til magnetiske vekselvirkninger som gjør det mulig for oss å sammenlikne dette systemet med liknende superledende- og ferromagnetiske heterostrukturer.

Den effektive topologiske feltteorien til denne heterostrukturen er utledet ved å integrere ut alle fermionske frihetsgrader i partisjonsfunksjonen til annen order in koblingskonstanter. I tillegg ser vi kun på langbølgefysikken til disse feltene. Den resulterende feltteorien inneholder det ønskede Chern-Simons leddet, som kobler til det komplekse skalarfeltet til superlederen. I tillegg så genereres det en termisk masse til det elektriske potensiale som resulterer i ulike former for effektivt potensiale mellom ladninger og skjermingseffekter. Grunnet vekselvirkningen med de magnetiske urenhetene får vi et såkalt Dzyaloshinskii-Moriya ledd, i tillegg til flere magneto-elektriske vekselvirkninger. Størrelsen på den effektive Chern-Simons koeffisienten i den topologiske Abelske Higgs modellen er øvrig begrenset, slik at systemet ikke kan justeres mellom de ulike kritiske fasene til denne modellen. Videre så vises det at Dzyaloshinskii-Moriya koeffisienten kan påvirke den magnetiske ordningen til systemet og på den måten muligens justere mellom ulike superledende- og ferromagnetiske faser.

Preface

This thesis is the result of a one-year study during my final year as a master's student at the Norwegian University of Science and Technology (NTNU) in Trondheim. The study was conducted at the Center for Quantum Spintronics (QuSpin) at the Department of Physics under the supervision of Prof. Asle Sudbø.

First and foremost, I would like to thank Prof. Sudbø for his excellent guidance, teaching, and cooperation throughout my master's studies and time as a physics student at NTNU. I would also like to thank Dr. Henning G. Hugdahl together with Prof. Sudbø and Dr. Flavio S. Nogueira for letting me participate in their research activities and helpful discussions regarding my contributions. Furthermore, I thank my fellow students at QuSpin, in particular Karl Kristian Ladegård Lockert and Niels Henrik Aase, for good conversations, help, and support during my stay at QuSpin. I would also like to show my appreciation to Fredrik Bakke for his mathematical consultation.

Then I would like to thank and express my gratitude to my family and friends for fulfilling my time as a student, especially my family in Tønsberg, my flatmates, the student organization Delta, and the Fathi family. Lastly, I would also like to show my appreciation to the people responsible for convincing me to pursue a career in the natural sciences.

Snorre Bergan
Trondheim, Norway
May, 2021

Physics Nobel Laureate Ivar Giæver and I at the Horten Science Prize ceremony warning me about the temptations of “Beer, ladies, and billiards” at Samfundet in Trondheim (2016).

Contents

Chapter 1

Introduction

> "Topology! The stratosphere of human thought! In the twenty-fourth century it might possibly be of use to someone."
>
> *Aleksandr Solzhenitsyn, 1968*

1.1 Motivation and background

Over the past decades, the mathematical study of topology has shown to be of great importance in condensed matter physics. Topology is the study of properties of geometrical objects and other mathematical spaces that are preserved under continuous transformations. In topological quantum matter theory, such properties are used to characterize different phases of matter and physical phenomena according to the topology of some geometry associated with a system. These phases are often manifested by emergent quantum states that are stable due to the non-trivial topology of the material in question. One of the most prominent examples of such systems is topological insulators. These are materials that are insulating in the bulk but host topologically- and symmetry-protected gapless states on the boundary of the system. The low energy description of these surface states has linear dispersion, meaning that these excitations are effective massless Dirac fermions [1]. The topological nature and the relativistic dispersion of these states make them interesting candidates for coupled condensed matter systems that could have possible technological applications.

By combining topological insulators with other quantum systems, we can generate new and interesting phenomena due to the interplay between the constituent systems. One way of achieving this is to place the materials physically on top of each other. The resulting system belongs to a class of systems commonly known as heterostructures. In the case of topological insulators, the relevant physics takes place on the interface formed by the different layers of the heterostructure where there are proximity-induced interactions between the surface fermions and the particles of the other materials.

In our study, we use a quantum field theoretical approach where we express the partition function of our heterostructure as a path integral in terms of quantum fields. To study the effects of the surface fermions, we integrate out the contributions from the corresponding Dirac fields to chosen order in perturbation theory. What we are left with is an effective field theory of the remaining systems on the boundary towards the topological insulator. Furthermore, by breaking time-reversal symmetry, we can introduce a gap in the spectrum of the surface states which corresponds to a mass term of these Dirac fields [1]. This in turn leads to the appearance of a so-called Chern-Simons coupling. This term gives rise to many interesting phenomena depending on the origin of the time-reversal symmetry breaking and what kind of fields it couples to. If this term contains a gauge field that is coupled to some form of matter fields, the resulting theory is an example of a topological field theory that supports several topologically protected physical properties [2].

In a recent study by F. Nogueira, J. van den Brink, and A. Sudbø they investigate the phase transitions and critical behavior of the topological Abelian Higgs model in $2+1$ spacetime dimensions. This theory contains a Chern-Simons gauge field coupled to a complex scalar field, which is essentially the field theory of a topological superconductor. They found that the phases of this model, its universality classes, and the type of critical behavior this model exhibits are highly dependent on the magnitude of the Chern-Simons coefficient [3]. This raises the question of whether it is possible to construct a quantum system from microscopic principles that supports these findings. Motivated by this, we want to investigate if it is possible to derive an effective field theory comparable with the topological Abelian Higgs model by considering a three-dimensional topological insulator proximate to a superconductor. The resulting Chern-Simons coupling would be a function of material parameters that in principle could be varied accordingly and possibly tune the effective theory between the different critical regions discussed in [3].

The time-reversal symmetry that protects the surface states of the topological insulator can be broken in several ways, for instance by coupling the fermions to some form of magnetic perturbation [1]. By adding a layer of ferromagnetically aligned magnetic impurities between the superconductor and the topological insulator, the fermions become massive if there is a magnetic ordering perpendicular to the boundary of this interface. Furthermore, if the spins of the magnetic impurities are fluctuating, we get additional spin-exchange interactions between the fermions and the quantum spins of the magnetic impurities. Motivated by this, we will also include these types of interactions in our study. The resulting effective field theory in terms of the superconductor, gauge field, and magnetic impurities allows for comparison with other ferromagnetic heterostructures, such as the ferromagnetic topological insulator heterostructures of [4], [5], and the superconductor proximate to a ferromagnetic material in [6].

1.2 Structure of the thesis

The content of this study is divided into four main parts. In chapters 2 to 5, we give a comprehensive graduate-level introduction and derive the models used in the main part of the thesis. In chapters 6 and 7, we present the results of this study including the most relevant calculations, followed by a discussion of the resulting physical models and some of their properties in chapter 8. Detailed calculations and relevant proofs can be found in the appendices.

In chapter 2, we start by giving a short introduction to the functional integral formalism of quantum condensed matter systems, which is the mathematical framework of this study. In chapter 3, we present some preliminary topology before discussing some central non-trivial topological effects in condensed matter theory and their realizations as topological field theories. In chapter 4, we derive the Heisenberg model from first principles in a general many-body theory followed by a derivation of its path integral description in addition to some relevant spin-interaction terms. In chapter 5, we end the theory section with a discussion on superconductivity in addition to a derivation of the Ginzburg-Landau theory of conventional superconductors and a discussion on the Higgs-Anderson mechanism.

In chapter 6, we derive the effective field theory of a superconductor proximity-coupled to a topological insulator by integrating out the fermionic degrees of freedom. In chapter 7, we consider the additional couplings due to the presence of a layer of ferromagnetically aligned magnetic impurities in the superconductor topological insulator heterostructure. In chapter 8, we present the resulting effective Lagrangian field theories before discussing some central aspects of these models.

1.3 Notations and conventions

Choice of units and physics notation

In this study, every quantity is expressed in natural units unless explicitly stated otherwise

$$\hbar = k_B = c = 1. \tag{1.1}$$

Hamiltonians and Lagrangian as denoted by uppercase letters H and L. Hamiltonian- and Lagrangian densities are denoted by calligraphic letters. In d dimensions, these can be expressed as follows

$$H = \int \mathrm{d}^d r \mathcal{H} \qquad L = \int \mathrm{d}^d r \mathcal{L}. \tag{1.2}$$

These functionals are expressed in terms of either canonical operators or quantum fields. In either case, if $\mathcal{H}$ is bilinear, we define the following notation

$$\mathcal{H} = \psi_\lambda^\dagger h \psi_{\lambda'}, \tag{1.3}$$

where ψ and $\psi^\dagger$ are fields or operators in terms of quantum numbers λ and λ'. Furthermore, in the first- and second quantization approach, we will denote Hamiltonians using a hat-notation ($\hat{H}$, etc.), to emphasize that they are operators and to distinguish them from corresponding Hamiltonians in the functional integral formalism.

Matrix- and vector notation

The Pauli matrices in their standard representation take the form [7]

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.4}$$

Besides these, vectors and matrices will not be denoted by any particular form of notation. The mathematical structure of the variables should be clear from the context. However, if the structure of the variable becomes too intricate or could cause confusion, we will sometimes use the bar notation to indicate spatial vectors, e.g.,

$$\bar{S} = S_i = (S_1, S_2, \ldots), \tag{1.5}$$

where i iterates over spatial dimensions and S_i can by any type of variable, typically an operator or a matrix. In the following, we will be using $\mathrm{tr}\,[\cdots]$ for matrix trace and $\mathrm{Tr}\,[\cdots]$ for quantum trace, i.e., the combination of the sum over quantum numbers and matrix trace. We will also be using the following equivalence relation

$$A \mathrel{\widehat{=}} B \iff \mathrm{tr}\,[A] = \mathrm{tr}\,[B]\,, \tag{1.6}$$

involving matrix trace exclusively.

Spacetime conventions

In Minkowski space, we will always use the following metric convention

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \mathrm{diag}(1, -1, -1, -1). \tag{1.7}$$

In both Euclidean and Minkowski spacetime, we will use Latin indices for spatial components and Greek indices for spacetime components, where the zeroth component is the temporal- or time component respectively, i.e.,

$$A_\mu = (A_0, A_i). \tag{1.8}$$

Contravariant vectors are defined as follows

$$A^\mu = \eta^{\mu\nu} A_\nu. \tag{1.9}$$

In Euclidean spacetime, the covariant and contravariant vectors coincide, $A^\mu = A_\mu$. However, we will keep using upper- and lower indices in inner-products for notational consistency.

Chapter 2

Quantum field theory approach to condensed matter systems

The functional integral formalism of quantum many-body physics is a mathematical framework where quantum many-body states are expressed using quantum fields rather than the canonical operators of the second quantization formalism. This formulation of quantum many-body physics expresses physical observables as a type of functional called path integrals, which we can treat systematically using different approximation schemes. In this chapter, we will first discuss this formalism for a general many-body system containing both fermionic- and bosonic excitations in addition to deriving the path integral description of the time-evolution operator and the partition function. Then we will discuss approximation schemes in the functional integral formalism, in particular the Hubbard-Stratonovich decoupling procedure.

2.1 Functional integral formalism of quantum many-body physics

2.1.1 Quantum fields of many-body states

Let $\{a_\lambda, a_\lambda^\dagger\}$ and $\{c_{\lambda'}, c_{\lambda'}^\dagger\}$ be the bosonic- and fermionic creation- and annihilation operators of some Fock space $\mathbb{F}$ with quantum numbers $\{\lambda\}$ and $\{\lambda'\}$. Assume that the fermionic and bosonic sectors of the Fock space separate, i.e., we can express any many-body state as $|\phi\rangle \otimes |\psi\rangle$, where $|\phi\rangle$ and $|\psi\rangle$ are purely bosonic- and fermionic many-body states, respectively. We define the coherent states as the eigenvectors of the annihilation operators

$$a_\lambda |\phi\rangle \otimes |\psi\rangle = \phi_\lambda |\phi\rangle \otimes |\psi\rangle \tag{2.1}$$

$$c_{\lambda'} |\phi\rangle \otimes |\psi\rangle = \psi_{\lambda'} |\phi\rangle \otimes |\psi\rangle , \tag{2.2}$$

where $\{\phi_\lambda, \psi_{\lambda'}\}$ are corresponding coherent states eigenvalues. These expressions define a bijective mapping between canonical operators $\{a_\lambda, a_\lambda^\dagger\}$ and $\{c_{\lambda'}, c_{\lambda'}^\dagger\}$ in the second quantized formalism onto the quantum fields $\{\phi_\lambda, \phi_\lambda^\dagger\}$ and $\{\psi_{\lambda'}, \psi_{\lambda'}^\dagger\}$ in the functional integral formalism. This gives us a way of computing quantum observables using quantum fields instead of operators. Since the eigenvalues of the coherent states must respect the commutational relations of their respective canonical operators, the bosonic eigenvalues $\{\phi_\lambda\}$ are complex scalar fields, whereas the fermionic eigenvalues $\{\psi_{\lambda'}\}$ are Grassmannian fields. The latter means that the fermionic fields map into the Grassmann algebra of anti-commuting numbers [8]. It can be shown that the coherent states can be represented as [7]

$$|\phi\rangle = \mathrm{e}^{\sum_\lambda \phi_\lambda a^\dagger_\lambda} |0\rangle \tag{2.3}$$

$$|\psi\rangle = \mathrm{e}^{\sum_\lambda \psi_{\lambda'} c^\dagger_{\lambda'}} |0\rangle \tag{2.4}$$

which in turn implies that the overlap of two coherent states can be written as

$$\langle \phi | \phi' \rangle = \mathrm{e}^{-\sum_{\lambda,\lambda'} \phi^*_\lambda \phi_{\lambda'}} \tag{2.5}$$

$$\langle \psi | \psi' \rangle = \mathrm{e}^{-\sum_{\lambda,\lambda'} \psi^*_\lambda \psi_{\lambda'}}. \tag{2.6}$$

Following the steps of [7], we can use the above result to derive the following resolutions of identity and trace relations[1]

$$\int \prod_\lambda \frac{\mathrm{d}\phi^*_\lambda \mathrm{d}\phi_\lambda}{2\pi i} \mathrm{e}^{-\sum_\lambda \phi^*_\lambda \phi_\lambda} |\phi\rangle \langle\phi| = 1 \tag{2.7}$$

$$\mathrm{Tr}\,[A] = \int \prod_\lambda \frac{\mathrm{d}\phi^*_\lambda \mathrm{d}\phi_\lambda}{2\pi i} \mathrm{e}^{-\sum_\lambda \phi^*_\lambda \phi_\lambda} \langle\phi| A |\phi\rangle \tag{2.8}$$

$$\int \prod_{\lambda'} \mathrm{d}\psi^\dagger_{\lambda'} \mathrm{d}\psi_{\lambda'} \mathrm{e}^{-\sum_{\lambda'} \psi^\dagger_{\lambda'} \psi_{\lambda'}} |\psi\rangle \langle\psi| = 1 \tag{2.9}$$

$$\mathrm{Tr}\,[A] = \int \prod_{\lambda'} \mathrm{d}\psi^\dagger_{\lambda'} \mathrm{d}\psi_{\lambda'} \mathrm{e}^{-\sum_{\lambda'} \psi^\dagger_{\lambda'} \psi_{\lambda'}} \langle -\psi| A |\psi\rangle\,, \tag{2.10}$$

where the minus sign in $\langle -\psi|$ comes from a permutation of the fermionic fields. Note that taking the adjoint of a complex field corresponds to complex conjugation, $\phi^* = \phi^\dagger$, whereas the adjoint of a Grassmann number cannot be similarly represented by another Grassmann number.

2.1.2 Path integrals and the partition function in the functional integral formalism

In our study, we are mainly interested in quantum observables related to the time-evolution operator and the partition function of a quantum many-body system. In particular, we are interested in deriving a quantum field theory description of the partition function using the coherent states defined in the previous section. The time-evolution operator in second quantized formalism takes the form

$$U = \langle \zeta(t_f)| \, \mathrm{e}^{i\hat{H}(t_f - t_i)} \, |\zeta(t_i)\rangle \tag{2.11}$$

where $|\zeta(t)\rangle$ is the many-body state at time t and $\hat{H} = \hat{H}(a_\lambda, a^\dagger_\lambda, c_{\lambda'}, c^\dagger_{\lambda'})$ is time-independent. Following the standard derivation of the path-integral, using the completeness relations of eq. (2.7) and eq. (2.9), we get the time-evolution operator expressed as a path integral in the functional integral formalism [7]

[1] This trace is the trace in Fock-space, ignoring any possible matrix structure of A.

$$U = \int_{\phi(t_i)}^{\phi(t_f)} \int_{\psi(t_i)}^{\psi(t_f)} \mathcal{D}\psi\mathcal{D}\psi^\dagger\mathcal{D}\phi\mathcal{D}\phi^* \mathrm{e}^{iS(\phi,\psi,\phi^*,\psi^\dagger)} \tag{2.12}$$

$$S(\phi,\psi,\phi^*,\psi^\dagger) = \sum_{\lambda,\lambda'} \int_{t_i}^{t_f} \mathrm{d}t \left(i\phi_\lambda(t)\frac{\partial\phi^*_\lambda(t)}{\partial t} + i\psi_{\lambda'}(t)\frac{\partial\psi^\dagger_{\lambda'}(t)}{\partial t} - \mathcal{H}(\phi^*_\lambda,\phi_\lambda,\psi^\dagger_{\lambda'},\psi_{\lambda'}) \right), \tag{2.13}$$

where the field integrals are over all paths connecting the fields at t_i and t_f. The Hamiltonian density $\mathcal{H}(\phi^*_\lambda,\phi_\lambda,\psi^\dagger_\lambda,\psi_\lambda)$ is the second quantized Hamiltonian density where the canonical operators have been replaced by the corresponding coherent state eigenvalues, $(a_\lambda, a^\dagger_\lambda, c_{\lambda'}, c^\dagger_{\lambda'}) \mapsto (\phi_\lambda, \phi^*_\lambda, \psi_{\lambda'}, \psi^\dagger_{\lambda'})$. The partition function can be written in the following basis-independent way

$$Z = \mathrm{Tr}\left[\mathrm{e}^{-\beta H}\right], \tag{2.14}$$

where Tr is a combination of the sum over quantum numbers and matrix trace

$$\mathrm{Tr}\left[\cdots\right] = \sum_{\lambda,\lambda'} \mathrm{tr}\left[\cdots\right], \tag{2.15}$$

where the sum over quantum numbers can be both discrete and continuous. Using the coherent states as our basis, we can use the trace relations of eq. (2.8) and eq. (2.10) to express the partition function as follows

$$Z = \int \prod_{\lambda,\lambda'} \mathrm{d}\psi^\dagger_{\lambda'}\mathrm{d}\psi_{\lambda'} \frac{\mathrm{d}\phi^*_\lambda\mathrm{d}\phi_\lambda}{2\pi i} \mathrm{e}^{-\sum_{\lambda'}\psi^\dagger_{\lambda'}\psi_{\lambda'}} \mathrm{e}^{-\sum_\lambda \phi^*_\lambda\phi_\lambda} \left\langle -\psi\right| \otimes \left\langle \phi\right| \mathrm{e}^{-\beta H} \left|\phi\right\rangle \otimes \left|\psi\right\rangle. \tag{2.16}$$

Writing $\beta = \int_0^\beta \mathrm{d}\tau$, we immediately see that this expression is a Wick rotated time-evolution operator where time t has been replaced by imaginary time $\tau = it$. We can therefore use eq. (2.12) directly to write the partition function as the following field theory

$$Z = \int_{\phi(0)=\phi(\beta)} \int_{\psi(0)=-\psi(\beta)} \mathcal{D}\psi\mathcal{D}\psi^\dagger\mathcal{D}\phi\mathcal{D}\phi^* \mathrm{e}^{S(\phi,\psi,\phi^*,\psi^\dagger)} \tag{2.17}$$

$$S = -\sum_{\lambda,\lambda'} \int_0^\beta \mathrm{d}\tau \left(\phi_\lambda(\tau)\frac{\partial\phi^*_\lambda(\tau)}{\partial\tau} + \psi_{\lambda'}(\tau)\frac{\partial\psi^\dagger_{\lambda'}(\tau)}{\partial\tau} + \mathcal{H}(\phi^*_\lambda,\phi_\lambda,\psi^\dagger_{\lambda'},\psi_{\lambda'}) \right), \tag{2.18}$$

where the field integrals are over all closed paths with periodic and anti-periodic boundary conditions for the bosonic and fermionic fields, respectively [7]. In the following, these conditions will not be stated explicitly for notational purposes.

2.2 Hubbard–Stratonovich decoupling and bosonization

Usually, the partition function field theory of eq. (2.17) cannot be solved exactly, particularly in the case of interacting fermions and bosons. In these situations, we need suitable approximation schemes to properly take into account the effects of the interaction terms. If the coupling is sufficiently strong, we will use a technique known as a Hubbard-Stratonovich decoupling, where one introduces a new free parameter into the theory to decouple complicated interaction terms. In this study, will formulate this procedure using real- and complex scalar fields as our parameters. However, most of the analysis of this section also applies in a more general situation. Assume a partition function in terms of the following action

$$S = S_0 + S_I \tag{2.19}$$

where S_0 describes a solvable non-interacting theory and S_I is the action of some non-bilinear interaction term, which makes the partition function unsolvable. In the case where S_I is small compared to S_0, we can treat eq. (2.19) perturbatively [8]. However, in the cases where S_I is large enough to cause quantitative changes compared to the free theory of S_0 (e.g., phase transitions [7], [9]), perturbation theory breaks down. In these cases, we can introduce auxiliary bosonic fields into the theory by using one of the following identities [2] [7]

$$\mathrm{e}^{J_i A_{ij} J_j^*} = \int \mathcal{D}\varphi\mathcal{D}\varphi^* \mathrm{e}^{-\varphi_i^* A^{-1}{}_{ij}\varphi_j + J_i\varphi_i^* + J_j^*\varphi_j} \tag{2.20}$$

$$\mathrm{e}^{J_i A_{ij} J_j} = \int \mathcal{D}\varphi \mathrm{e}^{-\varphi_i A^{-1}{}_{ij}\varphi_j + J_i\varphi_i} \tag{2.21}$$

for each quantum number of S[3]. The sources J_i and J_i^* are chosen appropriately depending on the structure of the interaction term. The former identity applies if S_I is in terms of complex scalar fields, whereas the latter applies if the fields are real-valued. In the functional integral formalism, we can write down the following general expression for an interacting fermionic theory

$$Z = \int \mathcal{D}\psi^\dagger\mathcal{D}\psi \mathrm{e}^{S}, \tag{2.22}$$

where S contains an interaction term which makes the theory unsolvable. By inserting eq. (2.20) into eq. (2.22), we get the following theory

$$Z = \int \mathcal{D}\psi\mathcal{D}\psi^\dagger\mathcal{D}\varphi\mathcal{D}\varphi^* \mathrm{e}^{S_{\text{eff}}}. \tag{2.23}$$

where S_{eff} is an effective action consisting of the non-interacting terms of S_0 and the action associated with the following Lagrangian

$$\mathcal{L} = -\varphi^* A^{-1}\varphi + J\varphi^* + J^*\varphi, \tag{2.24}$$

where $J = J(\psi, \psi^\dagger)$ and $J^* = J^*(\psi, \psi^\dagger)$ are functions in terms of the fermionic fields. Thus, we see that the interacting theory of eq. (2.22) has been transformed into a more comprehensible fermionic theory coupled to some auxiliary background field φ.

[2] These identities are equations modulo a constant energy contribution which factorizes out of the partition function in the end.

[3] Notice that the resulting bosonic field integral contains the inverse of the original coupling constants A_{ij}. Thus, the problem is mapped from a strong- to a weak coupling regime.

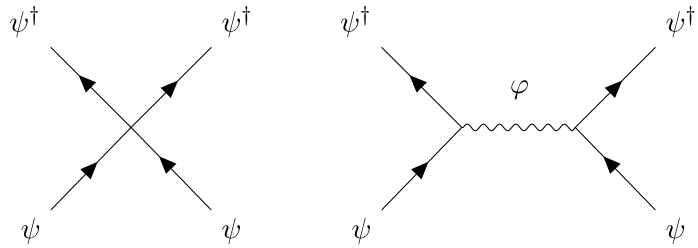

Figure 2.1: A Feynman diagram illustrating a typical Hubbard-Stratonovich decoupling procedure of a four-field vertex.

Most often the resulting theory of eq. (2.23) becomes bilinear in fermionic fields after the decoupling procedure. In these cases, the fermions can be eliminated from the problem by integrating them out of the partition function, leaving us with a theory solely in terms of the bosons φ. This effective bosonic action can be treated perturbatively, insensitive to the possibly singular perturbations of eq. (2.22). This procedure is an example of a bosonization [7].

Chapter 3

Topology and condensed matter physics

Over the last century, modern topics in mathematical sciences have proven to be of vital importance in most areas of theoretical physics. A prominent example of such a relationship is the characterization of physical structures and quantities using the language of abstract algebra. If a physical system is unaltered by a symmetry transformation, then the associated symmetry group is an intrinsic property of that system. In recent decades, the study of topology has also shown to be of great use in many areas of physics. As oppose to symmetry groups of a system, which describes what stays the same under symmetry transformations, the topology of a physical system concerns what stays the same when you continuously transform the geometries associated with that system. This in turn makes it possible to detect intrinsic properties of systems with possible physical consequences that would otherwise be insensitive to other mathematical characterizations. In this chapter, we will start by reviewing some preliminary results and concepts in topology. Then we will derive some basic results in topological condensed matter physics followed by a comprehensive discussion on the relevant topological systems used in this thesis.

3.1 What is topology?

Topology is the part of mathematics concerned with the study of properties of shapes, geometries, and other spaces that are preserved under continuous transformations, i.e., deformations, twisting, stretching, etc. In topology, two spaces are considered equivalent if one can be continuously transformed into the other. These continuous[1] mappings are called homeomorphism, analogously to the corresponding structure-preserving maps of abstract algebra, which are called isomorphisms. If there exist a homeomorphism between two (topological) spaces, then we say that they are homeomorphic. If two spaces are homeomorphic, then that means they share some characteristic properties that are unaltered or preserved by the homeomorphism. These properties are called topological invariants. These invariants can be very simple properties, e.g., the dimension of the spaces or the number of boundary points, etc., but they can also be more intricate, like e.g., homology- and homotopy groups[2], which are topologically invariant groups associated with these spaces. Such elaborate structures are studied in a sub-genre of topology

[1] The term “continuous” depends on the defined topology of the space in question, but we will only be using it whenever the standard topology of Euclidean spaces is applied, i.e., continuous in the usual $\delta - \epsilon$ sense.

[2] These groups do in fact also describe the simple kinds of topological invariants, but their applicability is far more general.

known as algebraic topology [10].

In some cases, it is also interesting to study the properties of spaces that are preserved under smooth transformations[3]. The spaces that possess these properties are known as manifolds. Roughly speaking, a manifold is a topological space where each subset looks and behaves like regular Euclidean space if the subspace is small enough. Similarly, we have that two manifolds are considered equivalent if there exists a smooth homeomorphism - a diffeomorphism - between the two spaces. Topological invariants that are preserved under diffeomorphisms are of particular interest in physics since most topological spaces associated with physical systems are manifolds [11].

3.2 Topological concepts in condensed matter physics

There are many concepts in physics that are labeled as "topological" even though the meaning of the term can be quite different. Sometimes topology is used as a way of distinguishing between different phases of matter where the Landau-paradigm of phase transitions[4] cannot be applied properly. In these cases, it is the low-energy effective field theory of the system which hosts the relevant topological invariants. These kinds of invariants will be discussed briefly at the end of this chapter. Other times, the term refers to the topology of the system itself, where the non-trivial topology is associated with the formation of so-called topological defects, e.g., domain walls. However, in most of the systems we will be working with, the term usually refers to the topology of an intrinsic geometrical structure of the system, which is the case in quantum Hall systems, topological insulators, and many other condensed matter systems [9], [12].

3.3 Topological insulators

Topological insulators are systems that are insulating in the bulk but host symmetry-protected gapless states on the boundary[5]. These boundary states are protected by the non-trivial topology of the bulk of the system, and they materialize due to a discontinuous change of this invariant on the interface between a trivial and a non-trivial material. In this section, we will first derive the necessary results needed to describe the topology of these systems. Then we discuss some systems experiencing these topological effects before deriving the Hamiltonian and corresponding quantum field theory of the surface states of a three-dimensional topological insulator.

3.3.1 The Berry phase

The Berry phase of a quantum state is one of the simplest topological invariants in all theoretical physics. The idea is that by adiabatically[6] transporting a quantum state in closed loops in some parameter space, the final state acquires a non-zero gauge-invariant phase due to the topology associated with the geometry of this space. Usually, the parameters are time-varying electric- or magnetic fields. However, the result is more general than that, as we will see later [4],

[3] i.e., mappings where the n-th derivative is continuous for every natural number n.

[4] This way of distinguishing between different phases of matter uses symmetry, which is insensitive to the topology of the system.

[5] There are other systems which has similar surface states which are not topological insulators, e.g., quantum Hall systems.

[6] In quantum physics, adiabatic means that the time-evolution of a system is slow compared to the relevant energy gaps.

[12]. Assume that you have a Hamiltonian $\hat{H}$ which depends on a parameter vector space V spanned by a set of vectors $\{p_i\}$. Furthermore, assume that these parameters are time-dependent, $p_i = p_i(t)$. At time $t = 0$, we assume that we have a prepared initial eigenstate $|\Psi\rangle$, which is the nth eigenvector in the set of instantaneous eigenvectors $\{|n(t)\rangle\}$ for $t = 0$. This set of eigenvectors satisfy

$$\hat{H}(t)\,|n(t)\rangle = E_n(t)\,|n(t)\rangle \tag{3.1}$$

for all t and eigenstates n [7]. Assume that we vary these parameters $\{p_i\}$ adiabatically, meaning that if we start out in the nth eigenstate at $t = 0$, we will remain in this eigenstate at $t = T$ according to eq. (3.1). This will form a path $\mathcal{C}$ in V. The time-evolution of the state $|\Psi(t)\rangle$ is given by the Schrödinger equation [12], [13]

$$i\hbar\partial_t\,|\Psi(t)\rangle = \hat{H}(t)\,|\Psi(t)\rangle \tag{3.2}$$

with the following solution

$$|\Psi(t)\rangle = \mathrm{e}^{i\theta(t)}\,|\Psi(0)\rangle \tag{3.3}$$

$$\theta(t) = \gamma(t) - \frac{1}{\hbar}\int_0^t E_n(t')\mathrm{d}t', \tag{3.4}$$

where we have defined

$$\gamma(t) = i\int_0^t \mathrm{d}t'\,\langle n(t')|\,\partial_{t'}\,|n(t')\rangle\,. \tag{3.5}$$

If $p_i(0) = p_i(T)$ for all i for some T, then $\gamma(T)$ is the Berry phase of this quantum state. It can also be written in the following way [12]

$$\gamma(T) \equiv \gamma = i\oint_{\mathcal{C}} \mathrm{d}p\,\langle n(p)|\,\boldsymbol{\nabla}_p\,|n(p)\rangle\,, \tag{3.6}$$

where $p \in V$ and $\mathcal{C}$ is a closed loop in V. Written in this form, we see that we can define the Berry connection

$$A(p) = -i\,\langle n(p)|\,\boldsymbol{\nabla}_p\,|n(p)\rangle \tag{3.7}$$

and correspondingly the Berry curvature

$$\mathcal{F}_{ij}(p) = \partial_i A_j(p) - \partial_j A_i(p) = \frac{\partial A_i(p)}{\partial p_i} - \frac{\partial A_j(p)}{\partial p_j}. \tag{3.8}$$

[7] We write $\hat{H}(t) = \hat{H}(V(t))$ etc. for notational purposes. The only time-dependence of the problem is via the parameter space V.

Using the generalized stokes theorem, we can write eq. (3.6) as follows [8]

$$\gamma = -\int_S \mathcal{F}_{ij} \mathrm{d}p_i \wedge \mathrm{d}p_j, \tag{3.9}$$

where $\partial S = \mathcal{C}$. Assuming that the Hamiltonian commutes with the parameters of V, the Hamiltonian is invariant under multiplication with an overall phase factor $|n(p)\rangle \mapsto \mathrm{e}^{i\phi(p)}\,|n(p)\rangle$. This is a gauge freedom of the system and the Berry connection in eq. (3.7) acts as a U(1) gauge field of this symmetry. Consequently, we see that eq. (3.9) is independent of the phase of $|\Psi\rangle$, as all physical quantities should be [4], [12].

The topological origin of the Berry phase relies on the mathematical framework of fiber bundles[9]. In this context, the fiber bundle is made up of the base space V and the gauge freedom described by the group U(1) at each point $p \in V$. The non-zero value of the Berry phase is a direct consequence of the fact that a topologically invariant group, called the Holonomy group, is non-trivial [10].

3.3.2 The Chern number

Now we move on to the special case where the parameter space V in the last section is a closed surface. In this case, the path $\mathcal{C}$ divides V into two distinct surfaces S_1 and S_2. Therefore, it does not matter which surface we use in eq. (3.9) to compute the Berry phase, at least up to an integer multiple of 2π. We can therefore write it as follows

$$-\gamma = \int_{S_1} \mathcal{F}_{ij} \mathrm{d}p_i \wedge \mathrm{d}p_j = 2\pi C - \int_{S_2} \mathcal{F}_{ij} \mathrm{d}p_i \wedge \mathrm{d}p_j \tag{3.10}$$

where there is a sign change in the last expression since $\mathcal{C}$ changes orientation in S_2 relative to S_1. Consequently, we get that the surface integral over the total parameter space $V = S_1 \cup S_2$ is an integer multiple of 2π

$$C = \frac{1}{2\pi} \int_V \mathcal{F}_{ij} \mathrm{d}p_i \wedge \mathrm{d}p_j \in \mathbb{Z}. \tag{3.11}$$

This is the Chern number of the parameter space V, or more precisely, the first Chern class of the manifold V [1], [4]. Its topological origin is related to the same mathematical constructions as the topological origin of the Berry phase. Nonetheless, the fact that it is an integer makes its robustness a bit more intuitive - one cannot transform an integer into another integer in a continuous fashion. These types of topological invariants are often referred to as topological indices in physics literature.

[8] This is a way of writing a generalized surface integral in higher dimensions. In three dimensions, the integral is just a regular surface integral over the curl of the field.

[9] The precise definition of a fiber bundle is quite cumbersome and irrelevant for this discussion. Superficially speaking, a fiber bundle is a topological space formed by attaching a topological space, e.g., a vector space, a group, or a discrete set, onto each point of the base space, which is also a topological space.

3.3.3 Topological band theory

The study of electronic band structures makes it possible to describe the electronic properties of different solid-state quantum systems. These bands are the possible energy levels for which the electrons can reside. By ordering these bands according to their energies, we can define the conduction band as the lowest unfilled energy band and the valence band as the highest filled energy band. If the Fermi level lies in between these two bands (i.e., no level crossing), then the system is in an insulating or semiconducting phase, depending on the size of the energy gap. In these cases, it will require a quantum of energy to excite a state into the conduction band. However, if the Fermi level crosses the valence band, any finite amount of energy can excite a state and the system is in a metallic phase [1], [14].

Assume a lattice system with discrete translational invariance. Under these circumstances, the system can be divided into unit cells with m internal degrees of freedom, which usually equals the number of distinguishable lattice atoms. The solutions to the Schröedingers equation of this system are of the form

$$|\Psi(r)\rangle = \mathrm{e}^{ik\cdot r} |u_n(k)\rangle \tag{3.12}$$

$$\hat{H}(k) |u_n(k)\rangle = E_n(k) |u_n(k)\rangle \tag{3.13}$$

where $\hat{H}(k)$ is known as the Bloch Hamiltonian, $|u_n(k)\rangle$ is the nth Bloch eigenstate and $E_n(k)$ is its corresponding energy-band. The Bloch Hamiltonian is m dimensional, meaning that there are m eigenfunctions describing the Bloch Hilbert Space [14]. In the following, we will only consider insulating planar systems, i.e., two-dimensional lattices with no level crossing between their corresponding energy bands. Furthermore, we will use periodic boundary conditions, meaning that the Brillouin zone is a Torus, $k \in \mathbb{T}^2$. By identifying $\mathbb{T}^2$ as the parameter space V of the previous section, we can define the Berry connection and -curvature using eq. (3.7) and eq. (3.8)

$$A_n = -i \langle u_n(k)| \boldsymbol{\nabla}_k |u_n(k)\rangle \tag{3.14}$$

$$\mathcal{F}_n = \boldsymbol{\nabla}_k \times A_n, \tag{3.15}$$

where $\boldsymbol{\nabla}_k = (\partial_{k_x}, \partial_{k_y})$. Since the Torus $\mathbb{T}^2$ is a closed surface, we can define the Chern number c_n for each band $E_k(k)$. Assuming that the bands $i = 1, ..., n$ are filled, we define the total Chern number of this Bloch system as follows

$$C = \frac{1}{2\pi} \sum_{i=1}^{n} c_n \tag{3.16}$$

$$c_n = \int_{\mathbb{T}^2} \mathrm{d}^2 k \mathcal{F}_n, \tag{3.17}$$

This topological index is invariant independent of the configuration of the bands as long as the gap between the valence- and the conduction band is present. This gives us a mathematically precise way of defining topological equivalence classes of systems described by Bloch Hamiltonians. Two insulating systems with translational invariance are topologically equivalent if their total Chern number is the same, i.e. if we can continuously deform the energy levels of the two systems into one another. For regular insulating systems, this index is zero. These systems constitute the topologically trivial systems. Topological insulators, on the other hand, belong to the equivalence class of non-trivial systems [1], [12].

3.3.4 The quantum Hall effect

Historically, the first non-trivial topological effect was the integer quantum Hall effect. Although many different systems are experiencing this phenomenon, we will derive it using a planar system in a perpendicular magnetic field. The first quantized Hamiltonian density of a free electron gas confined to a two-dimensional lattice in a magnetic field can be written as follows

$$\hat{\mathcal{H}} = \frac{1}{2m}\left(-i\boldsymbol{\nabla} - q\bar{A}\right)^2. \tag{3.18}$$

By choosing the Landau gauge, $\bar{A} = (-By, 0, 0)$, we get the following expression

$$\hat{\mathcal{H}} = -\frac{1}{2m}\boldsymbol{\nabla}^2 + \frac{ieB}{m}y\partial_x + \frac{e^2B^2}{2m}y^2. \tag{3.19}$$

This Hamiltonian commutes with translation in the x- and z-direction, meaning that its eigenstates are of the form

$$\psi(r) = \mathrm{e}^{ik_xx+ik_zz}\varphi(y). \tag{3.20}$$

Inserting this ansatz into eq. (3.19), we get the following equation for $\varphi(y)$

$$-\frac{1}{2m}\partial_y^2\varphi + \left(\frac{k_x^2}{2m} - \frac{eBk_x}{m}y + \frac{e^2B^2}{2m}y^2\right)\varphi + \frac{k_z^2}{2m}\varphi = E\varphi. \tag{3.21}$$

Completing the square on the left-hand side, we get

$$-\frac{1}{2m}\partial_y^2\varphi + \frac{1}{2}m\omega_c^2(y-y_0)^2\varphi + \frac{k_z^2}{2m}\varphi = E\varphi, \tag{3.22}$$

where $\omega_c = \frac{eB}{m}$ is the cyclotron frequency. The above expression is the equation of a harmonic oscillator with eigenvalues of the form

$$E_n(k) = E_n = \hbar\omega_c(n + \frac{1}{2}). \tag{3.23}$$

These energy levels are called Landau levels. They are the energies associated with the orbital motion of the electrons due to the magnetic field. Moreover, by applying an electric field to the system, these cyclotron orbits start to drift along the edges of the system, causing a discrete Hall current of the form

$$\sigma_{xy} = \frac{e^2}{h}N, \tag{3.24}$$

where N is the number of filled Landau levels [13]. By defining a re-scaled unit cell where the flux is zero (mod 2π), we can restore translational symmetry of the electron gas as if there were no magnetic field present [1]. In this sense, the above system is topologically equivalent to a Bloch system according to the classification scheme discussed in the previous sections. Consequently, we can define the total Chern number C of the filled Landau levels if the Fermi energy does not coincide with the valence band. A detailed analysis shows that this index is in fact the integer N, c.f. [12]. Consequently, the quantum Hall effect is a topological effect.

3.3.5 Bulk-boundary correspondence

Assume that you have two topologically distinct systems on top of each other, e.g., two systems with different Chern-number, that share the same symmetries. The bulk-boundary correspondence tells us that the difference between right- and left moving chiral states on this boundary, N_R and N_L, is related to the topological indices of the bulks of the materials via the following relation[10]

$$N_R - N_L = \Delta n, \tag{3.25}$$

where Δn is the difference in topological indices. The proof of this relation is tedious and beyond the scope of this introduction. However, it can be explained by the following heuristic argument. In order to change the topological index at the interface, the Hamiltonian describing the non-trivial state must be transformed into a topologically trivial Hamiltonian. The only way this can be achieved is by closing the gap between the valence band and the Fermi level, otherwise the transformation would be continuous, and the Hamiltonian would still be in the same non-trivial equivalence class. This band will therefore host gapless states near this intersection with the Fermi level, which are the desired chiral boundary modes. More generally, the valence band must cross the Fermi level an odd number of times, where the orientation of the intersection corresponds to left- (N_L) and right (N_R) moving boundary modes [1], [4].

3.3.6 The quantum spin Hall effect and $\mathbb{Z}_2$ invariants

The main difference between integer quantum Hall systems and two-dimensional topological insulators is the fact that quantum Hall systems break time-reversal symmetry, whereas the gapless edge states of two-dimensional topological insulators are time-reversal symmetry-protected. As a first step towards the latter case, we introduce the Haldane model. This model exhibits the quantum Hall effect without any net flux through the lattice, as opposed to our previous case. Haldane's model is based on electrons moving on a two-dimensional honeycomb lattice placed in a periodic magnetic field. The periodicity of this field ensures that translational symmetry is preserved, implying that we can classify the energy bands of the Haldane model according to our topological classification scheme. We will first derive the integer quantum Hall effect of this model. Then we will show that by introducing a spin-orbit coupling to the problem, the resulting model will exhibit this effect even in the absence of any magnetic fields. In the second quantization approach, the Haldane can be written as

$$\hat{H} = t_1 \sum_{\langle i,j \rangle} c_i^\dagger c_j + t_2 \sum_{\langle\langle i,j \rangle\rangle} \mathrm{e}^{-i v_{ij} \phi} c_i^\dagger c_j + M \sum_i \epsilon_i c_i^\dagger c_i, \tag{3.26}$$

[10] In this context, chiral means that they propagate in one direction only.

where t_1 and t_2 are hopping amplitudes, M is a constant energy term and ϕ is a phase associated with the periodic field. We have also defined the following quantities

$$v_{ij} = \text{sign}(d_1 \times d_2)_z \tag{3.27}$$

$$\epsilon_i = \begin{cases} 1 & i \in A \\ -1 & i \in B \end{cases} \tag{3.28}$$

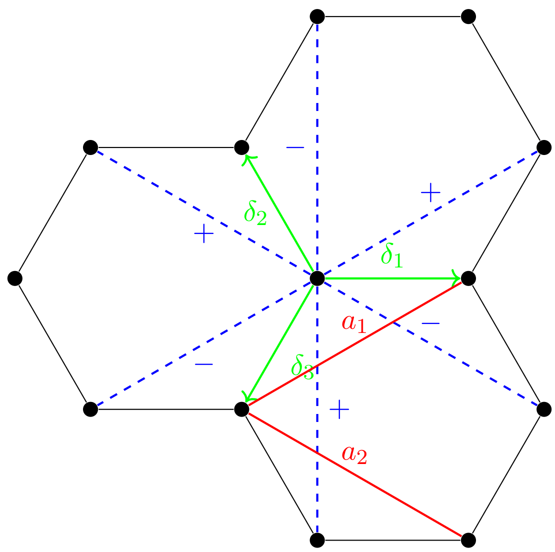

Figure 3.1: The different lattice vectors and signs associated with v_{ij} of the honeycomb lattice.

where d_1 and d_2 denote the first and second lattice vectors along the lattice towards one of the next-nearest neighbors and A and B refer to the two sub-lattices of the honeycomb lattice, see fig. 3.1 [12]. Here we see that the dynamical phases due to the magnetic field is only present in the next-nearest term, which preserves lattice translational symmetry. After Fourier transforming, following the steps of [12], we can express the Haldane model as the following Hamiltonian density

$$\hat{h}(k) = \epsilon(k) + d(k) \cdot \bar{\sigma}, \tag{3.29}$$

where we have defined

$$\epsilon(k) = 2t_2 \left(\cos\phi \cos k \cdot a_1 + \cos k \cdot a_1 + \cos k \cdot (a_1 - a_2)\right) \tag{3.30}$$

$$d_1(k) = \cos k \cdot a_1 + \cos k \cdot a_2 + 1 \tag{3.31}$$

$$d_2(k) = \sin k \cdot a_1 + \sin k \cdot a_2 \tag{3.32}$$

$$d_3(k) = M + 2t_2 \sin\phi \left(\sin k \cdot a_1 - \sin k \cdot a_2 - \sin k \cdot (a_1 - a_2)\right), \tag{3.33}$$

where a_i are the next-nearest neighbor lattice vectors. The Hamiltonian of eq. (3.29) describes a two-band Bloch system. At two points in the Brillouin zone, K and K', the gap closes and the dispersion around these points is approximately linear

$$\hat{h}(K+q) = -3t_2\cos\phi + \frac{3}{2}t_1(q_y\sigma^x - q_x\sigma^y) + \Big(M - 3\sqrt{3}t_2\sin\phi\Big)\,\sigma^z + \mathcal{O}\big(q^2\big) \tag{3.34}$$

$$\hat{h}(K'+q) = -3t_2\cos\phi - \frac{3}{2}t_1(q_y\sigma^x + q_x\sigma^y) + \Big(M + 3\sqrt{3}t_2\sin\phi\Big)\,\sigma^z + \mathcal{O}\big(q^2\big), \tag{3.35}$$

where we have set $|a_i|$ equal to unity. It can be shown that around these points, the Haldane model can experience three different phases depending on the values of M and ϕ: two Hall conducting phases with opposite orientation and one trivial phase. The former requires that M and ϕ take on values away from the time-reversal symmetric points $M = 0$ and $\phi = 0, \pi$[11] [12]. Thus, to obtain a quantum Hall current without breaking time-reversal symmetry, we need to introduce new couplings in eq. (3.29).

In the following calculations, we will only be considering the time-reversal invariant case where the Haldane phase vanishes, i.e., $\phi = 0$ and $M = 0$. Furthermore, we will rotate the honeycomb lattice by $\frac{\pi}{2}$, $q_x \to q_y$ and $q_y \to -q_x$, to be consistent with the literature. After this rotation, we can combine eq. (3.34) and eq. (3.35) into the following low-energy Hamiltonian density in the functional integral formalism

$$\mathcal{H}_0(q) = v_F\Psi^\dagger\left(\sigma^x\otimes\tau^z q_x + \sigma^y\otimes 1 q_y\right)\Psi, \tag{3.36}$$

$$= v_F\Psi^\dagger\begin{pmatrix} q_x\sigma^x + q_y\sigma^y & 0 \\ 0 & -q_x\sigma^x + q_y\sigma^y\end{pmatrix}\Psi, \tag{3.37}$$

where τ^z acts in K, K' space, σ^i acts in A, B sub-lattice space, and $v_F = -\frac{3t_1}{2}$. Here we have defined the four-component Dirac-spinors

$$\Psi = \begin{pmatrix}\psi_A(K) & \psi_B(K) & \psi_A(K') & \psi_B(K')\end{pmatrix}^T, \tag{3.38}$$

where $\psi_A(k)$ and $\psi_B(k)$ are quantum fields corresponding to the canonical operators associated with sub-lattice A and B respectively. In this notation, the only available mass terms are $m\sigma^z\otimes 1$ and $m\sigma^z\otimes\tau^z$. Both of these terms break time-reversal symmetry. However, by taking into account the spin of the fermions, we can transform eq. (3.37) into the following Hamiltonian density

$$\mathcal{H}(q) = \begin{pmatrix}\Psi_\uparrow^\dagger & \Psi_\downarrow^\dagger\end{pmatrix}\begin{pmatrix}\mathcal{H}_\uparrow(q) & 0 \\ 0 & \mathcal{H}_\downarrow(q)\end{pmatrix}\begin{pmatrix}\Psi_\uparrow \\ \Psi_\downarrow\end{pmatrix}, \tag{3.39}$$

where $H_\uparrow(q)$ and $H_\downarrow(q)$ are two copies of eq. (3.37), one for each spin. In this notation, we can add a spin-orbit coupling term of the form

$$\mathcal{H}_{SO} = \lambda_{SO}\begin{pmatrix}\Psi_\uparrow^\dagger & \Psi_\downarrow^\dagger\end{pmatrix}\sigma^z\otimes\tau^z\otimes s^z\begin{pmatrix}\Psi_\uparrow \\ \Psi_\downarrow\end{pmatrix} \tag{3.40}$$

$$= \lambda_{SO}\begin{pmatrix}\Psi_\uparrow^\dagger & \Psi_\downarrow^\dagger\end{pmatrix}\begin{pmatrix}\sigma^z\otimes\tau^z & 0 \\ 0 & -\sigma^z\otimes\tau^z\end{pmatrix}\begin{pmatrix}\Psi_\uparrow \\ \Psi_\downarrow\end{pmatrix} \tag{3.41}$$

[11] In this case, time-reversal symmetry requires that $d_3(k)$ is zero

where s^z acts in spin-space. This term is both time-reversal invariant and it also induces a gap in the spectrum of eq. (3.39). The latter is most visible by comparing $\mathcal{H}_0(q) + \mathcal{H}_{SO}$ with eq. (3.34) and eq. (3.34). For the up-spins, we can include the spin-orbit coupling in the following way

$$h(K+q) = -\frac{3}{2}t_1\left(q_x\sigma^x + q_y\sigma^y\right) + \lambda_{SO}\sigma^z \tag{3.42}$$

$$h(K'+q) = -\frac{3}{2}t_1\left(-q_x\sigma^x + q_y\sigma^y\right) - \lambda_{SO}\sigma^z \tag{3.43}$$

and correspondingly for the down-spins

$$h(K+q) = -\frac{3}{2}t_1\left(q_x\sigma^x + q_y\sigma^y\right) - \lambda_{SO}\sigma^z \tag{3.44}$$

$$h(K'+q) = -\frac{3}{2}t_1\left(-q_x\sigma^x + q_y\sigma^y\right) + \lambda_{SO}\sigma^z. \tag{3.45}$$

These equations correspond to the Haldane model in the absence of a periodic magnetic field with opposite mass terms for up- and down-spins. The spin-orbit coupling behaves as an effective magnetic field coupled to the fermions which cancel when you combine the two systems, making the system overall time-reversal invariant. In real space, these expressions can be rewritten as follows

$$\hat{H} = t_1 \sum_{\langle i,j\rangle} c^\dagger_{i,\sigma}c_{j,\sigma} + i\lambda_{SO}\sum_{\langle\!\langle i,j\rangle\!\rangle} v_{ij}c^\dagger_{i,\sigma}s^z_{\sigma,\sigma'}c_{j,\sigma'}, \tag{3.46}$$

which is a simplified version of the Kane-Mele model [12]. Due to the relative sign-change of the spin-orbit coupling term, they experience quantum Hall currents with opposite orientation. Thus, our spin-orbit coupled time-reversal symmetric Hamiltonian of eq. (3.39) induces a net flow of spin on the boundary. This effect is known as the quantum spin Hall effect, which is the defining feature of two-dimensional topological insulators [1], [12].

This derivation of the quantum spin Hall effect suggests that the robustness of these helical spin states is somehow related to the Chern numbers of the underlying Haldane models. However, this is not the case in either this model or any other topological insulator. It turns out that any pair of boundary states can collapse by an arbitrary weak disorder of the system due to backscattering. A single helical state, however, will persist. This even/odd classification suggests another type of topological index $\nu \in \mathbb{Z}_2$ independent of the Chern-number classification scheme [4], [12].

3.3.7 Three-dimensional topological insulators and the quantum field theory of the boundary states

One way of constructing three-dimensional topological insulators is by combining two-dimensional quantum spin Hall systems on top of each other. The resulting three-dimensional system will host topologically induced gapless states on a two-dimensional interface [1]. An effective theory due to Bernevig, Hughes, and Zhang (BHZ) gives a valid description of these insulators. The BHZ-model can be written as the following Hamiltonian density

$$\hat{h}_{\rm 3D}(k) = (C + D_1 k_z^2 + D_2) + \begin{pmatrix} M(k) & A_1 k_z & 0 & A_2 k_- \\ A_1 k_z & -M(k) & A_2 k_- & 0 \\ 0 & A_2 k_+ & M(k) & -A_1 k_z \\ A_2 k_+ & 0 & -A_1 k_z & -M(k) \end{pmatrix}, \tag{3.47}$$

where $k_\pm = k_x \pm i k_y$ and $M(k) = M - B_1 k_z^2 - B_2(k_x^2 + k_y^2)$. The parameters M, A_i, B_i, C, D_i are phenomenological constants that depend on specific material properties. These operators act on four-component canonical operators in terms of spin and chirality. It can be shown that by projecting this Hamiltonian onto the surface states, we get the following Hamiltonian [4]

$$\hat{h}_{\rm 3D}^{\rm surf}(k) = C + A_2\left(\sigma^x k_y - \sigma^y k_x\right) \tag{3.48}$$

In this reduced form, we can deduce that the constant C corresponds to the chemical potential of the system and A_2 corresponds to the velocity of the surface fermions v_F [4]. By rotating the lattice by $\frac{\pi}{2}$ and Fourier transforming into real space, we can write it as follows

$$\hat{h}_{\rm 3D}^{\rm surf} = i v_F \bar{\sigma} \cdot \boldsymbol{\nabla} + \mu, \tag{3.49}$$

where $\bar{\sigma} = (\sigma^x, \sigma^y)$ and $\boldsymbol{\nabla} = (\partial_x, \partial_y)$. We can convert this Hamiltonian into its corresponding Lagrangian field theory by mapping the canonical operators of eq. (3.49) into coherent states and substitute it into the Lagrangian density of eq. (2.13)

$$\mathcal{L} = \Psi^\dagger \partial_t \Psi - \Psi^\dagger h_{\rm 3D}^{\rm surf} \Psi \tag{3.50}$$

$$= \Psi^\dagger \left(i\partial_t - i v_F \bar{\sigma} \cdot \boldsymbol{\nabla} - \mu\right) \Psi, \tag{3.51}$$

where we have introduced the two-component spinor $\Psi = \begin{pmatrix} \psi_\uparrow & \psi_\downarrow \end{pmatrix}^T$. In two dimensions, we can gap the spectrum of the surface states by introducing the following term

$$\mathcal{L} = \Psi^\dagger \left(i\partial_t - i v_F \bar{\sigma} \cdot \boldsymbol{\nabla} - \mu + m\sigma^z\right) \Psi, \tag{3.52}$$

This term corresponds to a mass term of the fermions which breaks time-reversal symmetry [1]. Furthermore, we can also couple the fermions to an electromagnetic field by gauging the theory as follows

$$\mathcal{L}_{TI} = \Psi^\dagger \left(i\partial_t - e\phi + v_F\left(-i\boldsymbol{\nabla} - e\bar{A}\right) \cdot \bar{\sigma} - \mu + m\sigma^z\right) \Psi. \tag{3.53}$$

This expression is the quantum field theory of the surface states which we will employ later in this study.

3.4 Topological field theory and Chern-Simons theory

Spacetime is mathematically speaking a specific type of manifold known as a Riemannian manifold. By introducing a special kind of quantum field theory on these spaces, the resulting theory might host non-trivial invariants that are characteristic to the spacetime manifold and field theory in question. These topological invariants are gauge-invariant vacuum expectation values of observables, $\langle \mathcal{O} \rangle$, computed using eq. (2.14). Quantum field theories that produce such invariants are known as topological field theories. Furthermore, if these topological field theories appear in a physical context, then the associated invariants might have physical consequences [15]. In this section, we will discuss some basic results and terminology from topological field theory with an emphasis on Chern-Simons theories.

3.4.1 Schwarz–type TQFT's and the Chern-Simons action

Topological quantum field theories (TQFT) fall into two classes. These are Witten-type TQFT and Schwarz–type TQFT, where the latter is the only class that will be of interest in this study. In curved spacetime, the Minkowski tensor $\eta_{\mu\nu}$ is promoted to the more general metric tensor $g_{\mu\nu}$. This tensor defines the local structure of the spacetime manifold [8]. A quantum observable is considered topological if the following is true

$$\frac{\delta \langle \mathcal{O} \rangle}{\delta g_{\mu\nu}} = 0, \tag{3.54}$$

i.e., the vacuum expectation value of the observable is independent of the metric. This can be achieved by requiring that both $\mathcal{O}$ and the action S are independent of the metric tensor,

$$\frac{\delta \mathcal{O}}{\delta g_{\mu\nu}} = 0 \qquad \frac{\delta S}{\delta g_{\mu\nu}} = 0, \tag{3.55}$$

which is the case in Schwarz–type TQFT's. In $2+1$-dimensional spacetime, one of the simplest actions that fulfill the latter requirement is the following action

$$S_{CS} = \int_M \mathrm{Tr}\left(A \wedge \mathrm{d}A\right). \tag{3.56}$$

In flat spacetime, i.e. Euclidean- or Minkowski space, with gauge group U(1) we can write this trace as follows

$$S_{CS} = \int_M \mathrm{d}^{2+1}x \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \tag{3.57}$$

where $\varepsilon^{\mu\nu\lambda}$ is the Levi-Civita tensor[12]. This is the Chern-Simons action in terms of the gauge field A_μ. By performing a gauge transformation of eq. (3.57) we are left with the following surface term

$$\begin{aligned}
\int_M \mathrm{d}^{2+1}x \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \to & \int_M \mathrm{d}^{2+1}x \varepsilon^{\mu\nu\lambda} (A_\mu - \partial_\mu \Lambda) \partial_\nu (A_\lambda - \partial_\lambda \Lambda) \\
&= S_{CS} - \int_M \mathrm{d}^{2+1}x \varepsilon^{\mu\nu\lambda} \partial_\mu \Lambda \partial_\nu A_\lambda \\
&= S_{CS} - \int_M \mathrm{d}^{2+1}x \varepsilon^{\mu\nu\lambda} \partial_\mu \left(\Lambda \partial_\nu A_\lambda\right),
\end{aligned} \tag{3.58}$$

[12] Notice that this is the only anti-symmetric combination of gauge fields and derivatives available in $2+1$-dimensions [8].

where all symmetric tensors are put to zero due to the contraction with the anti-symmetric Levi-Civita tensor. Hence this action is only gauge-invariant using a particular choice of boundary conditions where this term vanishes [7]. An important class of topological observables in Chern-Simons theories is the Wilson loops W_γ, which in the case of eq. (3.57) can be expressed as

$$W_\gamma = \exp\left\{ i \oint_\gamma A_\mu \mathrm{d}x^\mu \right\}, \tag{3.59}$$

where γ is some closed loop in M. The topological invariants associated with these loops are mathematically intricate, but we can immediately compare them to the topology of Berry-phases discussed in previous sections [15].

3.4.2 Maxwell-Chern-Simons theory and coupling to matter fields

The Maxwell-Chern-Simons Lagrangian can be written as

$$\begin{aligned} \mathcal{L}_{MCS} &= \mathcal{L}_M + \mathcal{L}_{CS} \\ &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{2} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - A_\mu J^\mu, \end{aligned} \tag{3.60}$$

where the first term is the regular Maxwell Lagrangian, the second term is a Chern-Simons Lagrangian and the last term is a coupling to a source term $J_\mu = (\rho, \bar{J})$. The Euler-Lagrange equations of this theory are as follows

$$\partial_\mu F^{\mu\nu} + \frac{\kappa}{2} \varepsilon^{\nu\mu\lambda} F_{\mu\lambda} = J^\nu. \tag{3.61}$$

The first part of this expression, $\partial_\mu F^{\mu\nu} = J^\nu$, reproduces the familiar Maxwell's equations, whereas the middle term is an additional constraint due to the Chern-Simons Lagrangian. By using the pseudovector dual field $\tilde{F}^\nu = \frac{1}{2}\varepsilon^{\nu\mu\lambda} F_{\mu\lambda}$, this equation reduces to the Proca equation [8] [13]

$$(\partial_\mu \partial^\mu + m_{CS}^2) \tilde{F}^\nu = 0, \tag{3.62}$$

where $m_{CM} = \sqrt{\kappa}$ is the topological mass of the dual field. Ignoring the Maxwell Lagrangian in eq. (3.61) and writing it out in terms of electric- and magnetic fields, we get the following equations

$$\rho = \kappa B \tag{3.63}$$

$$J_i = \kappa \epsilon^{ij} E_j. \tag{3.64}$$

The first equation implies that the charge density is locally proportional to the magnetic flux, whereas the second equation ensures that this constraint is preserved under time evolution. Together these constraints imply that the Chern-Simons action attaches magnetic field lines to the charged constituents of the system. In the case of point particles, these magnetic field lines become quantized in units of $B = \frac{e}{\kappa}$ [2].

[13] A pseudovector field is a vector field which is odd under parity transformations. Notice that this field is gauge invariant.

3.4.3 Aharonov-Bohm interaction and anyons

As a result of this charge-flux coupling, the particles experience an effective interaction due to the Aharonov-Bohm effect. This effect arises if a wave function is adiabatically transported along a path $\mathcal{C}$ enclosing a magnetic field without being physically subjected to it anywhere along its trajectory. In a region where $B = 0$, the gauge vector potential can be written as

$$\bar{A} = \boldsymbol{\nabla}\lambda \implies \lambda(r) = \int_{r_i}^{r_f} \bar{A} \cdot \mathrm{d}r. \tag{3.65}$$

If the path $\mathcal{C}$ is closed, we can use Stokes' theorem to rewrite this expression as follows

$$\lambda(r) = \oint_{\mathcal{C}} \bar{A} \cdot \mathrm{d}r = \int_S B \cdot \mathrm{d}S. \tag{3.66}$$

This gauge-invariant quantity, which we will refer to as the Aharonov-Bohm phase, is finite for a non-zero magnetic field even though the wave function and the path $\mathcal{C}$ are completely isolated from the magnetic field [13].

Assume a system of two identical charged particles living in two dimensions minimally coupled to a gauge sector with a Chern-Simons action. By transporting one of the particles around the other in a closed loop, we get an Aharonov-Bohm phase since the particles carry a quantized magnetic flux

$$\lambda(r) = \oint_{\mathcal{C}} \bar{A} \cdot \mathrm{d}r = \int_S B \cdot \mathrm{d}S = \frac{e}{\kappa}, \tag{3.67}$$

where $\partial S = \mathcal{C}$. This contribution can be removed from the minimal coupling to the matter fields by performing a gauge transformation, $\bar{A} \to \bar{A} - \boldsymbol{\nabla}\lambda(r)$. However, as a consequence of this gauge transformation, the wave function of the transported particle acquires a non-zero phase

$$\psi \to \psi \exp\left\{-ie \oint \bar{A} \cdot \mathrm{d}r\right\} = \psi \exp\left\{-i\frac{e^2}{\kappa}\right\}. \tag{3.68}$$

In the corresponding two-body system, this factor becomes the relative phase shift due to a double interchange of two identical particles. Since the Aharonov-Bohm phase in this setting only depends on the number of enclosed particles, we can without loss of generality assume that $\mathcal{C}$ is a circular path symmetrically placed around the other particle. Thus, the relative phase shift due to a single interchange between two particles equals half the total Aharonov-Bohm phase, i.e.

$$\psi \to \psi' = \psi \exp\left\{-i\frac{e^2}{2\kappa}\right\}. \tag{3.69}$$

Since κ can be any real number, this phase shift need not be an integer multiple of π, meaning that $\psi' \neq \pm\psi$ after the permutation. Away from these values of κ, the charged particles in this Chern-Simons gauge theory are neither fermions nor bosons - they are anyons. This simple single-particle analysis can also be generalized to interacting quantum systems, resulting in similar anyonic excitations and vortex solutions in terms of the interacting particles and exotic topological quantum effects such as the fractional quantum Hall effect [2], [16].

3.4.4 Chern-Simons actions of topological insulators

In section 3.3.6 we briefly stated the physical reasoning behind the $\nu \in \mathbb{Z}_2$ invariant of topological insulators. A more detailed analysis using topological band theory can be applied to prove the topological origin of this index. However, this analysis is restricted to the non-interacting case [12]. It turns out that a three-dimensional topological insulator can be described using a descendent of the following topological field theory

$$S_{TI} = \frac{C_2}{24\pi^2} \int_M \mathrm{d}^{4+1} x \varepsilon^{\mu\nu\rho\sigma\tau} A_\mu \partial_\nu A_\rho \partial_\sigma A_\tau, \tag{3.70}$$

which is the $4+1$-dimensional analog of eq. (3.57) and $C_2 \in \mathbb{Z}$ [4]. This is the first spacetime dimension in which the Chern-Simons action respects time-reversal symmetry, which is to be required[14]. The manifold M is a form of flat spacetime where the x_4 coordinate is compactified ("rolled up" to a circle). Thus, we can integrate out the fourth component of the gauge field A_4, resulting in the following theory

$$S_\theta = \frac{\alpha}{32\pi^2} \int_{M'} \mathrm{d}^{3+1} x \theta(x) \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F^{\rho\sigma} \tag{3.71}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell field strength tensor, α is the fine structure constant and $\theta(x) = C_2 \phi$, where ϕ is the magnetic flux associated with the field A_4 and the closed integral over x_4[15]. For eq. (3.71) to maintain time-reversal symmetry, we must impose certain constraints on the field $\theta(x)$. Under time-reversal symmetry, the fourth component of the gauge transforms as $A_4 \to -A_4$, and hence ϕ and therefore $\theta(x)$ must be odd. Furthermore, physics must be unaltered after adding 2π to the flux ϕ. Consequently, we get that $\theta(x)$ can take on two values, one of them being $\theta(x) = 0$. In this case, the action S_θ vanishes, and the system is trivial, corresponding to $\nu = 0$. However, if θ is finite, the action in eq. (3.71) produces gapless edge states on the boundary of M', which corresponds to the topologically non-trivial case of $\nu = 1$ [4], [17]. By repeating this procedure, we arrive at the Chern-Simons action in eq. (3.57), which supports the edge states discussed in section 3.3.6 [16], and finally the one-dimensional case. The latter is always topologically trivial and hence there are no topological insulators in one dimension [4][16].

[14] A_0 transforms evenly and A_i transforms odd under time-reversal symmetry.

[15] This is in fact the action of a axion field, which is a hypothetical particle postulated in high-energy physics. In this context, the topological aspect of this field is associated with a non-trivial winding number of instantons in non-abelian Yang-Mills theories, in particular QCD [8].

[16] However, there are one-dimensional systems with topologically protected edge states, but they are not described by this topological classification scheme.

Chapter 4

Quantum magnetism and magnetic impurities

Magnetic phases of matter are associated with the ordering of magnetic moments or spins in a solid-state system. Although some magnetic effects can be derived from classical principles, a proper description of magnetism is purely quantum mechanical. Interactions between spins arise in the strong coupling regime of fermionic systems due to a combination of repulsive interactions between the fermions and local quantum effects due to the Pauli exclusion principle and quantum fluctuations. In this chapter, we will show that these interactions lead to the Heisenberg model, followed by a derivation of its spin path integral description. Then we will add a Dzyaloshinskii-Moriya interaction to our model and discuss some of its physical consequences.

4.1 The Heisenberg model

In the strong coupling regime, fermions can be described by the following generalized tight-binding model

$$\hat{H} = -\sum_{i,j,\sigma} t_{ij} c^{\dagger}_{i,\sigma} c_{j,\sigma} + \sum_{i,j,k,l,\sigma,\sigma'} V_{ijkl} c^{\dagger}_{i,\sigma} c^{\dagger}_{j,\sigma'} c_{k,\sigma'} c_{l,\sigma}, \tag{4.1}$$

where V_{ijkl} is a spin-independent coupling between fermions on sites i, j, k, and l with spins σ and σ'[1]. In the context of magnetism, we are mainly interested in local interactions that involve the spin of the fermions. We will therefore only consider nearest-neighbor hopping. Furthermore, we will only consider the terms where $i = j = k = l$ and $i = k$ and $j = l$. The former has a spin structure because of the Pauli exclusion principle, whereas the latter is non-diagonal in spin-space. Under these restrictions, eq. (4.1) can be written as

$$\hat{H} = -\sum_{\langle i,j \rangle} \sum_{\sigma} t_{ij} c^{\dagger}_{i,\sigma} c_{j,\sigma} + U \sum_{i} n_{i,\uparrow} n_{i,\downarrow} + \sum_{\langle i,j \rangle} \sum_{\sigma,\sigma'} V_{ij} c^{\dagger}_{i,\sigma} c_{i,\sigma'} c^{\dagger}_{j,\sigma'} c_{j,\sigma}, \tag{4.2}$$

where we have defined $U = 2V_{iiii}$ and $V_{ij} = -V_{ijij}$. The second term of eq. (4.2), which is known as the Hubbard term, is an on-site energy associated with an atomic orbital accom-

[1] There is only two spin degrees of freedom due to conservation of spin.

modating two fermions with opposite spins. The last term is a Coulomb interaction between neighboring lattice sites that involves a spin-flip. By adapting the following notation

$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{4.3}$$

we immediately get that the spin-flip operators have the following actions in spin-space

$$c^\dagger_{i,\uparrow}c_{i,\uparrow}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \qquad c^\dagger_{i,\uparrow}c_{i,\uparrow}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{4.4}$$

$$c^\dagger_{i,\downarrow}c_{i,\downarrow}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \qquad c^\dagger_{i,\downarrow}c_{i,\downarrow}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \tag{4.5}$$

$$c^\dagger_{i,\downarrow}c_{i,\uparrow}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \qquad c^\dagger_{i,\downarrow}c_{i,\uparrow}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{4.6}$$

$$c^\dagger_{i,\uparrow}c_{i,\downarrow}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \qquad c^\dagger_{i,\uparrow}c_{i,\downarrow}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}. \tag{4.7}$$

These relations imply that we can represent the spin-space part of these operators in the following way

$$c^\dagger_{i,\uparrow}c_{i,\uparrow} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = \frac{1}{2}\left(1+\sigma^z\right) \tag{4.8}$$

$$c^\dagger_{i,\downarrow}c_{i,\downarrow} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = \frac{1}{2}\left(1-\sigma^z\right) \tag{4.9}$$

$$c^\dagger_{i,\downarrow}c_{i,\uparrow} = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} = \frac{1}{2}\sigma^- \tag{4.10}$$

$$c^\dagger_{i,\uparrow}c_{i,\downarrow} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} = \frac{1}{2}\sigma^+, \tag{4.11}$$

where $\sigma^\pm = \sigma^x \pm i\sigma^y$. Inserting these expressions into eq. (4.2), we get

$$\sum_{\sigma,\sigma'} c^\dagger_{i,\sigma}c_{i,\sigma'}c^\dagger_{j,\sigma'}c_{j,\sigma} = \frac{1}{2}\left(1+\sigma^z\right)_i\frac{1}{2}\left(1+\sigma^z\right)_j + \frac{1}{2}\left(1-\sigma^z\right)_i\frac{1}{2}\left(1-\sigma^z\right)_j \tag{4.12}$$

$$+\frac{1}{2}\sigma_i^-\frac{1}{2}\sigma_j^+ + \frac{1}{2}\sigma_i^+\frac{1}{2}\sigma_j^- \tag{4.13}$$

$$= \frac{1}{2}\left(\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y + \sigma_i^z\sigma_j^z\right), \tag{4.14}$$

where the indices emphasizes that the operators are acting on their respective lattice sites. Using that the spin operators of spin-$\frac{1}{2}$ fermions can be written as $\bar{S} = \frac{1}{2}\bar{\sigma}$, we get the following expression

$$\sum_{\langle i,j\rangle}\sum_{\sigma,\sigma'} V_{ij}c^\dagger_{i,\sigma}c_{i,\sigma'}c^\dagger_{j,\sigma'}c_{j,\sigma} = -\sum_{\langle i,j\rangle} J_{ij}\bar{S}_i\cdot\bar{S}_j \tag{4.15}$$

$$J_{ij} = -2V_{ij} = 2V_{ijij} > 0, \tag{4.16}$$

where we have absorbed a constant energy term into the zero-point energy. Thus, we see that the last term of eq. (4.2) reduces to a spin-spin interaction that favors parallel alignment, i.e., ferromagnetism. The right-hand side of the above equation is known as the ferromagnetic quantum Heisenberg model.

Next, we include the effects of the hopping term of eq. (4.2). We do this by considering the limit $t \ll U$ at half-filling [2] and treat the first term of eq. (4.2) as a perturbation. A ground of the Hubbard Hamiltonian,

$$\hat{H}_0 = U \sum_i n_{i,\uparrow} n_{i,\downarrow} + \sum_{\langle i,j \rangle} \sum_{\sigma,\sigma'} V_{ij} c^\dagger_{i,\sigma} c_{i,\sigma'} c^\dagger_{j,\sigma'} c_{j,\sigma}, \tag{4.17}$$

is a state $|\Psi_0\rangle$ where there is one fermion per lattice site. Consequently, the first-order contribution vanishes [13],

$$\Delta E^{(1)} = - \sum_{\langle i,j \rangle} \sum_{\sigma} t_{ij} \langle \Psi_0 | c^\dagger_{i,\sigma} c_{j,\sigma} | \Psi_0 \rangle = 0, \tag{4.18}$$

since the state $|\Psi\rangle = c^\dagger_{i,\sigma} c_{j,\sigma} |\Psi_0\rangle$ is doubly occupied at lattice site i, making it orthogonal to $|\Psi_0\rangle$. In the second-order contribution, i.e.,

$$\Delta E^{(2)} = \sum_n \sum_{\langle i,j \rangle} \sum_{\langle k,l \rangle} \sum_{\sigma,\sigma'} t_{ij} t_{kl} \frac{\langle \Psi_0 | c^\dagger_{i,\sigma} c_{j,\sigma} | n \rangle \langle n | c^\dagger_{k,\sigma'} c_{l,\sigma'} | \Psi_0 \rangle}{E_0 - E_n}, \tag{4.19}$$

the only non-zero contributions are the once where $\langle \Psi_0 | c^\dagger_{i,\sigma} c_{j,\sigma} | n \rangle$ are non-zero, meaning that $|n\rangle$ is a state where sites j is doubly occupied and i is unoccupied. This implies that

$$E_n = U + E_0 \tag{4.20}$$

which means that we can write this contribution as

$$\begin{aligned} \Delta E^{(2)} &= -\frac{1}{U} \sum_n \sum_{\langle i,j \rangle} \sum_{\langle k,l \rangle} \sum_{\sigma,\sigma'} t_{ij} t_{kl} \langle \Psi_0 | c^\dagger_{i,\sigma} c_{j,\sigma} | n \rangle \langle n | c^\dagger_{k,\sigma'} c_{l,\sigma'} | \Psi_0 \rangle && (4.21) \\ &= -\frac{1}{U} \sum_{\langle i,j \rangle} \sum_{\langle k,l \rangle} \sum_{\sigma,\sigma'} t_{ij} t_{kl} \langle \Psi_0 | c^\dagger_{i,\sigma} c_{j,\sigma} c^\dagger_{k,\sigma'} c_{l,\sigma'} | \Psi_0 \rangle . && (4.22) \end{aligned}$$

which is non-zero if and only if the state $c^\dagger_{i,\sigma} c_{j,\sigma} c^\dagger_{k,\sigma'} c_{l,\sigma'} |\Psi_0\rangle$ is proportional to $|\Psi_0\rangle$. This is possible if the canonical operators $c^\dagger_{i,\sigma} c_{j,\sigma} c^\dagger_{k,\sigma'} c_{l,\sigma'}$ acts as an exchange of fermions between two neighboring lattice sites, i.e. if $i = k$ and $j = l$. In this case, eq. (4.22) is equivalent to a first-order contribution of the following effective Hamiltonian

[2] i.e. the number of fermions equals the number of lattice sites.

$$\hat{H}_{\text{eff}} = -\frac{|t_{ij}|^2}{U} \sum_{\langle i,j \rangle} \sum_{\langle k,l \rangle} c^{\dagger}_{i,\sigma} c_{j,\sigma} c^{\dagger}_{i,\sigma'} c_{j,\sigma'}$$
$$= \frac{|t_{ij}|^2}{U} \sum_{\langle i,j \rangle} \sum_{\langle k,l \rangle} c^{\dagger}_{i,\sigma} c_{i,\sigma'} c^{\dagger}_{j,\sigma'} c_{j,\sigma}, \tag{4.23}$$

where we have anti-commuted the operators and once again absorbed a constant energy contribution into the zero-point energy. Comparing this expression with our analysis of the third term in eq. (4.2), we immediately see that we can write eq. (4.23) as follows

$$\hat{H}_{\text{eff}} = -\sum_{\langle i,j \rangle} J_{ij} \bar{S}_i \cdot \bar{S}_j \tag{4.24}$$
$$J_{ij} = -\frac{2|t_{ij}|^2}{U} < 0. \tag{4.25}$$

Thus, we see that the hopping term of eq. (4.2) to second order can be written as a spin-exchange interaction with opposite sign compared to eq. (4.15). Eq. (4.24) is the anti-ferromagnetic quantum Heisenberg model. Combining eq. (4.15) and eq. (4.24), we get the following expression for eq. (4.2) at half-filling

$$\hat{H} = -\sum_{\langle i,j \rangle} J_{ij} \bar{S}_i \cdot \bar{S}_j, \tag{4.26}$$

where $J_{ij} = 2V_{ijij} - \frac{2|t_{ij}|}{U}$, which means that is a matter of detail whether eq. (4.2) supports ferromagnetism or anti-ferromagnetism. The absence of a hopping term means that eq. (4.26) describes processes with no net transport of charge. Systems described by the Heisenberg model are therefore often referred to as magnetic insulators [18].

4.2 Schwinger boson representation

The spin-operators in the quantum Heisenberg model obey the following commutational relation

$$\left[S^i, S^j\right] = i\varepsilon^{ijk} S^k, \tag{4.27}$$

where $i, j,$ and k run over spatial indices $\{x, y, z\}$[3]. Introducing the canonical operators a and b, it can be shown that a set of spin operators $\{S^x, S^y, S^z\}$ can be represented as

$$S^x + iS^y = a^{\dagger} b \tag{4.28}$$
$$S^x - iS^y = b^{\dagger} a \tag{4.29}$$
$$S^z = \frac{1}{2}\left(a^{\dagger} a - b^{\dagger} b\right) \tag{4.30}$$

[3]Not to be confused with lattice quantum numbers.

if a and b obey the usual bosonic commutational relations [18]. However, the Fock space generated by these operators is over-complete, in the sense that it accommodates more states than the corresponding spin operators. Thus, in order for a and b to be a proper representation of spin-operators, we need the additional constraint

$$a^\dagger a + b^\dagger b = 2S. \tag{4.31}$$

In this representation, we can write the spin-eigenstates as

$$|S, m\rangle = \frac{\left(a^\dagger\right)^{S+m}}{\sqrt{(S+m)!}} \frac{\left(b^\dagger\right)^{S-m}}{\sqrt{(S-m)!}} |0\rangle , \tag{4.32}$$

where S and m are the eigenvalues of $\bar{S}^2$ and S^z respectively [18].

4.3 Coherent states of spin operators

Similar to fermionic- and bosonic path integrals, we need a suitable basis in order to derive a spin path integral. The set of spin coherent states $\{|\Omega\rangle\}$ is defined by the following relations

$$|\Omega\rangle = g\, |S, S\rangle \tag{4.33}$$
$$g = \mathrm{e}^{-i\phi S^z} \mathrm{e}^{-i\theta S^y} \mathrm{e}^{-i\chi S^z} , \tag{4.34}$$

i.e. the set of spin states that can be achieved by acting on the maximally polarized spin state $|S, S\rangle$ with the operator g. These operators are members of SU(2) since the spin operators of eq. (4.27) are generators of the corresponding Lie algebra. In fact, the set $\{g\}$ is a representation of SU(2) known as the Euler angle representation [7]. Furthermore, since $|S, S\rangle$ is an eigenstate of S^z, the angle χ becomes a free parameter and the coherent states are uniquely defined by the spherical angles $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$[4]. Using eq. (4.27) and the definition above, we can define the following

$$\Omega = \langle\Omega|\, \bar{S}\, |\Omega\rangle = S\left(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta\right). \tag{4.35}$$

In order to derive the overlap of two arbitrary spin coherent states, we need to express the spin state $|S, S\rangle$ in the Schwinger boson representation and use the fact that the Schwinger bosons transform as vectors in SU(2), i.e.

$$\begin{aligned}\begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}' &= g \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix} \\ &= \mathrm{e}^{-i\phi\frac{\sigma^z}{2}} \mathrm{e}^{-i\theta\frac{\sigma^y}{2}} \mathrm{e}^{-i\chi\frac{\sigma^z}{2}} \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix} \\ &= \begin{pmatrix} u\mathrm{e}^{\frac{i\chi}{2}} & v\mathrm{e}^{\frac{i\chi}{2}} \\ -v^*\mathrm{e}^{-\frac{i\chi}{2}} & u^*\mathrm{e}^{\frac{-i\chi}{2}} \end{pmatrix} \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix} \end{aligned} \tag{4.36}$$

[4]This is related to the fact that SU(2) is isomorphic to $S^2 \times \mathrm{U}(1)$.

where we have used the Pauli matrices as a representation of the spin algebra and defined $u = \cos\frac{\theta}{2}\mathrm{e}^{\frac{i\phi}{2}}$ and $u = \sin\frac{\theta}{2}\mathrm{e}^{-\frac{i\phi}{2}}$. Consequently, the coherent states can be represented as

$$\begin{aligned}|\Omega\rangle = g\,|S,S\rangle &= \mathrm{e}^{iS\chi}\frac{\left(a^{\dagger\prime}\right)^{2S}}{\sqrt{(2S)!}}\,|0\rangle = \mathrm{e}^{iS\chi}\frac{\left(ua^{\dagger}+vb^{\dagger}\right)^{2S}}{\sqrt{(2S)!}}\,|0\rangle \\ &= \mathrm{e}^{iS\chi}\sqrt{(2S)!}\sum_{m}\frac{u^{S+m}v^{S-m}}{\sqrt{(S+m)!}\sqrt{(S-m)!}}\,|S,m\rangle\,, \end{aligned} \tag{4.37}$$

where we have used the binomial expansion in the last line. Following the steps of [18], we can use the above relations to derive the following

$$\langle\Omega|\Omega'\rangle = \left(\frac{1+\Omega\cdot\Omega'}{2}\right)^{S}\mathrm{e}^{-iS\psi} \tag{4.38}$$

$$\psi = 2\arctan\left[\tan\left(\frac{\phi-\phi'}{2}\right)\frac{\cos\frac{1}{2}(\theta+\theta')}{\cos\frac{1}{2}(\theta-\theta')}\right]+\chi-\chi', \tag{4.39}$$

where $\chi-\chi'$ can be chosen to be zero. Furthermore, using the Haar integral measure

$$\mathrm{d}\Omega = \mathrm{d}\theta\sin\theta\mathrm{d}\phi \tag{4.40}$$

we can use eq. (4.39) to derive the following resolution of identity

$$\frac{2S+1}{4\pi}\int\mathrm{d}\Omega\,|\Omega\rangle\langle\Omega| = 1 \tag{4.41}$$

For multiple spins, we define the following coherent many-body state

$$|\Omega\rangle = \prod_{i}|\Omega_i\rangle\,, \tag{4.42}$$

where i is a lattice index. The resolution of identity in this basis follows directly

$$\int\prod_{i}\left(\frac{2S+1}{4\pi}\mathrm{d}\Omega_i\right)|\Omega_i\rangle\langle\Omega_i|\,. \tag{4.43}$$

The trace of any operator can also be computed using eq. (4.43)

$$\mathrm{Tr}(A) = \int\prod_{i}\left(\frac{2S+1}{4\pi}\mathrm{d}\Omega_i\right)\langle\Omega_i|\,A\,|\Omega_i\rangle\,. \tag{4.44}$$

4.4 The path integral of a spin partition function

Using the results of the previous section, we can derive the partition function field theory of any spin systems given by a second quantized Hamiltonian $\hat{H}$ in terms of spin operators $\bar{S}_i$. Following a standard derivation of the path integral and using the completeness- and trace relations of eq. (4.43) and eq. (4.44), we get the following expression

$$Z = \int_{\Omega(0)=\Omega(\tau)} \mathcal{D}\Omega \mathrm{e}^{S} \tag{4.45}$$

$$S = i \sum_i \omega(\Omega) - \int_0^\beta \mathrm{d}\tau H(\omega), \tag{4.46}$$

where $H(\Omega) = \langle\Omega| \hat{H} |\Omega\rangle$. Compared to our previous case of eq. (2.17), the spin coherent state overlap of eq. (4.39) introduces a new time-derivative term which can be written as follows

$$\begin{aligned} \omega(\Omega) &= -\int_0^\beta \mathrm{d}\tau \partial_\tau \phi \cos\theta \\ &= -\int_{\phi_0}^{\phi_0} \mathrm{d}\phi \cos\theta(\phi). \end{aligned} \tag{4.47}$$

By the last equality, we see that this term is geometrical in the sense that it depends on the path taken by $\Omega = \Omega(\tau)$. Furthermore, it can also be shown that $\omega(\Omega)$ equals the area enclosed by this path. With this in mind, we can introduce a gauge field b via

$$i\omega(\Omega) = -\int_0^\beta \mathrm{d}\tau b \cdot \partial_\tau \Omega \tag{4.48}$$

$$= -\oint_C \mathrm{d}\Omega \cdot b(\Omega) \tag{4.49}$$

$$= -\int_S \varepsilon_{ijk} \frac{\partial b_k(\Omega)}{\partial \Omega_j} \Omega_i \mathrm{d}S \tag{4.50}$$

where i, j, and k are spatial indices. Thus, for eq. (4.48) to be true, we must have that eq. (4.50) equals S, which implies that

$$\varepsilon_{ijk} \frac{\partial b_k(\Omega)}{\partial \Omega_j} = \frac{\Omega_i}{\Omega^2}. \tag{4.51}$$

Due to the geometrical nature of this term, b is often referred to as the Berry phase of the spin path integral [18].

4.5 Quantum field theory of the ferromagnetic Heisenberg model

Using the ferromagnetic quantum Heisenberg model as our spin Hamiltonian,

$$\hat{H} = -\sum_{\langle i,j\rangle} J_{ij} \bar{S}_i \cdot \bar{S}_j, \tag{4.52}$$

our spin path integral takes the form

$$Z = \int \mathcal{D}\Omega \mathrm{e}^{S} \tag{4.53}$$

$$S = -\int_0^\beta \mathrm{d}\tau \left(\sum_i b \cdot \partial_\tau \Omega_i - \sum_{\langle i,j \rangle} J_{ij} \Omega_i \cdot \Omega_j \right), \tag{4.54}$$

where we have used that $\langle\Omega| \bar{S}_i \cdot \bar{S}_j |\Omega\rangle = \Omega_i \cdot \Omega_j$, which follows directly using eq. (4.35). By performing a discrete Fourier transform of the Ω_i fields in lattice space, i.e.

$$\Omega_i = \sum_q \Omega_q \mathrm{e}^{-iq\cdot r_i} \tag{4.55}$$

where r_i is the position of lattice index i, we can write the second term of eq. (4.54) as follows

$$\sum_{\langle i,j \rangle} J_{ij} \Omega_i \cdot \Omega_j = \sum_q \tilde{J}(q) \Omega_q \Omega_{-q} \tag{4.56}$$

$$\tilde{J}(q) = \sum_{\{\delta\}} J(\delta) \mathrm{e}^{iq\cdot\delta} \tag{4.57}$$

where $\delta = r_i - r_j$ are the nearest-neighbor lattice vectors. Assuming that the spin-spin coupling is symmetric, i.e., $J_{ij} = J_{ji}$, we get that $\tilde{J}(q) = \tilde{J}(-q)$. In the long-wavelength limit, we can therefore approximate this coupling as follows

$$\tilde{J}(q) = c_1 + c_2 q^2 + \mathcal{O}\big(q^3\big), \tag{4.58}$$

where $c_1 > 0$ since the Heisenberg model in eq. (4.52) favors ferromagnetic ordering. Inserting this expression into eq. (4.54), we get the following action

$$S = -\int_0^\beta \mathrm{d}\tau \sum_q \big(b \cdot \partial_\tau \Omega_q - c_1 \Omega_q \Omega_{-q} - q^2 c_2 \Omega_q \Omega_{-q} \big) \tag{4.59}$$

We can take the continuum limit of this model by treating q as a continuous variable. By doing this, we are transforming the discrete spin many-body states Ω_q into a continuous spin field $n(q)$. In this limit, our action becomes

$$S = -\int_0^\beta \mathrm{d}\tau \int \frac{\mathrm{d}^3 q}{(2\pi)^3} b \cdot \partial_\tau n(q) - \frac{\kappa}{2}(iq)(-iq) n(q) n(-q) - \frac{m^2}{2} n(q) n(-q) \tag{4.60}$$

where we have re-scaled the fields and defined the exchange coupling constants m and κ. Fourier transforming back into real space and performing a Wick rotation in eq. (4.54), we arrive at the following Lagrangian

$$\mathcal{L}_{FI} = b \cdot \partial_t n - \frac{\kappa}{2} (\boldsymbol{\nabla} n)^2 - \frac{m^2}{2} n^2. \tag{4.61}$$

This is the quantum field theory of the ferromagnetic Heisenberg model of eq. (4.15) [18].

4.6 Dzyaloshinskii-Moriya interactions

In certain magnetic systems, a lack of inversion symmetry allows for asymmetrical spin-exchange interactions between the spins of a magnetic insulator. In 1958, Igor Dzyaloshinskii proposed a general term for these types of interactions of the form

$$\hat{H}_{DM} = \sum_{i,j} D_{ij} \cdot \bar{S}_i \times \bar{S}_j \tag{4.62}$$

for spins at lattice sites i and j. There are many mechanisms responsible for these interactions. However, they are mostly due to spin-orbit coupling between the lattice atoms and the underlying fermions. In the spin coherent states basis of eq. (4.33), we can write the above Hamiltonian as

$$\langle\Omega| \hat{H}_{DM} |\Omega\rangle = \sum_{i,j} D_{ij} \cdot \Omega_i \times \Omega_j, \tag{4.63}$$

where we have used eq. (4.35). Restricting to nearest-neighbor interactions and performing a discrete Fourier transform using eq. (4.55), we can write this term as

$$\sum_{\langle i,j\rangle} D_{ij} \cdot \Omega_i \times \Omega_j = \sum_q D(q)\Omega_q \cdot \Omega_{-q} \tag{4.64}$$

$$D(q) = \sum_{\{\delta\}} D(\delta)\mathrm{e}^{iq\cdot\delta}. \tag{4.65}$$

In the following, we will assume that D_{ij} is anti-symmetric in lattice indices, which is the case of spin-orbit induced Dzyaloshinskii-Moriya interactions. This implies that $D(q)$ is anti-symmetric. In the long-wavelength limit, we can therefore rewrite eq. (4.65) as

$$\begin{aligned}
\langle\Omega| \hat{H}_{DM} |\Omega\rangle &= \lambda_{DM} \sum_q q \cdot \Omega_q \times \Omega_{-q} \\
&= \lambda_{DM} \sum_q q_i \varepsilon_{ijk} (\Omega_q)_j (\Omega_{-q})_k \\
&= i\lambda_{DM} \sum_q (\Omega_q)_j \varepsilon_{jki} (\Omega_{-q})_k (-iq_i) \\
&= -i\lambda_{DM} \sum_q (-iq) \times \Omega_{-q} \cdot \Omega_q.
\end{aligned} \tag{4.66}$$

Taking the continuum limit and Fourier transforming back into real space, we arrive at the following term

$$\mathcal{L}_{DM} = -i\lambda_{DM}(\boldsymbol{\nabla} \times n) \cdot n, \tag{4.67}$$

which is the field theory version of eq. (4.62) [19]. The spin-exchange coupling of eq. (4.62) energetically favors perpendicular spin alignment, in contrast to regular ferromagnetic and anti-ferromagnetic systems. This in turn makes it possible to achieve new kinds of spin configurations and magnetic phases compared to regular magnetic ordering and (anti-) ferromagnetism of systems described by the Heisenberg model [19], [20].

Chapter 5

Superconductivity

Superconductivity is a low-temperature phase of matter in which a system experiences zero electrical resistance and perfect diamagnetism. At low enough temperatures, the electrons near the Fermi surface become unstable and start to group into so-called Cooper-pairs. These pairs are composite fermionic condensates that form due to induced attractions between the fermions in the superconductor. The microscopic theory describing this mechanism is the Bardeen–Cooper–Schrieffer (BCS) theory[1], which is the topic in the first sections of this chapter. Then we will derive the Ginzburg-Landau theory of conventional superconductors followed by a discussion of the Higgs-Anderson mechanism.

5.1 The BCS Hamiltonian

A general fermionic many-body Hamiltonian in momentum space takes the form[2]

$$\hat{H} = \sum_{\sigma,k} (\varepsilon_k - \mu)\, c^{\dagger}_{\sigma,k} c_{\sigma,k} + \sum_{\sigma,\sigma',k,k',q} V^{\sigma,\sigma'}_{k,k',q} c^{\dagger}_{\sigma,k+q} c^{\dagger}_{\sigma',k'-q} c_{\sigma,k} c_{\sigma',k'}, \tag{5.1}$$

where ε_k is the single-particle energy, $V_{k,k',q}$ is a two-body interaction coupling and μ is the chemical potential. In the case of BCS superconductivity, we can use the following coupling

$$V_{k,k',q} = \frac{e^2}{4\pi\epsilon_0}\frac{1}{q^2} + |g_q^2|\frac{2\omega_q}{\omega^2 - \omega_q^2}, \tag{5.2}$$

where the first term is the Coulomb interaction in momentum space and the last term is an effective boson-mediated interaction to second order in g_q, which is the coupling between the fermions and the bosons of the problem[3]. The frequencies ω and ω_q are the energy transfer $\omega = \varepsilon_{k+q} - \varepsilon_k$ between the fermions and the energy dispersion of the intermediate bosons, respectively.

[1]Superconductors described by BCS theory are called conventional superconductors. Unconventional superconductivity will not be addressed in this introduction.

[2]Assuming that three-body interactions etc. are irrelevant or negligible.

[3]Although BCS theory is usually derived using an electron-phonon interaction, the intermediate boson can in principle be any kind of boson, e.g., magnons.

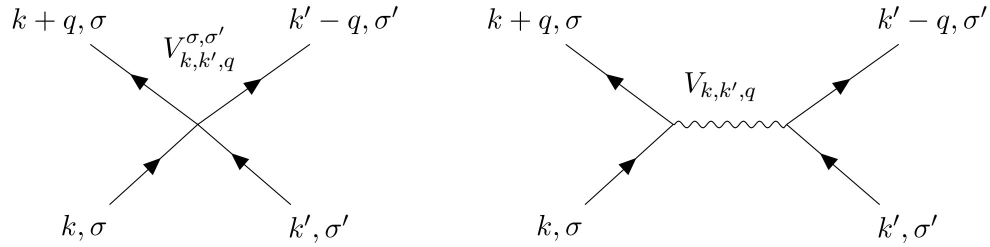

Figure 5.1: Feynman diagrams of two-body interactions with the generalized coupling (left) and boson-mediated effective coupling (right).

In most cases eq. (5.2) is positive, and the corresponding two-body interaction is repulsive. These processes are irrelevant in the context of Cooper-pair formation and can be treated separately using perturbation theory. However, when the energy transfer ω is smaller than but sufficiently close to the boson dispersion energy ω_q, the interaction term changes sign and becomes attractive. It is these interactions that participate in BCS-superconductivity.

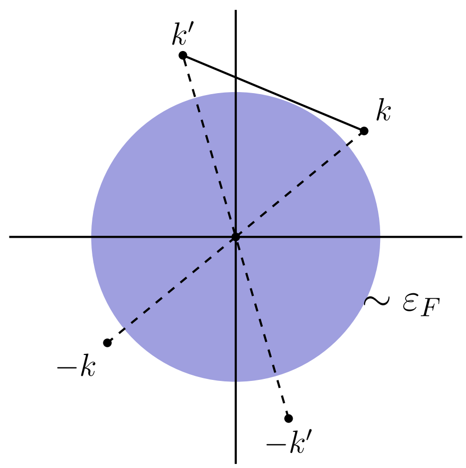

Figure 5.2: A typical scattering process close to the Fermi surface ε_F where two diametral fermions $(k,-k)$ are scattered into $(k',-k')$

In order to arrive at the BCS Hamiltonian, we need to make a series of assumptions and simplifications to the generalized model of eq. (5.1). First of all, we assume that the intermediate bosons interact with a characteristic frequency ω_0[4], which is typically small compared to the Fermi level ε_F. This means that attraction between fermions will mainly take place in a small region of width $\sim 2\omega_0$ around the Fermi surface, i.e.

$$|\varepsilon_{k+q}-\varepsilon_k| < \omega_0 \quad (k \leftrightarrow k', q \to -q). \tag{5.3}$$

The vectors k that fulfills these relations form a subspace Ω of the Brillouin zone. Furthermore, we will only retain the terms where $k'=-k$. This choice maximizes the scattering phase space of attractive interactions, as illustrated in fig. 5.2. Lastly, the spatial extent of these

[4]In the phonon case, this corresponds to some typical lattice vibration frequency, whereas in the magnon case it corresponds to some typical spin-precession frequency.

interactions are usually quite small, meaning that we can set $\sigma' = -\sigma$. Thus, we are left with the following Hamiltonian

$$\hat{H} = \sum_{\sigma,k} \varepsilon_k c^\dagger_{\sigma,k} c_{\sigma,k} + \sum_{\sigma,k,q} V_{k,k',q} c^\dagger_{\sigma,k+q} c^\dagger_{-\sigma,-k-q} c_{-\sigma,-k} c_{\sigma,k}, \tag{5.4}$$

where we have absorbed the chemical potential into the single-particle energy for notational purposes. By redefining variables $k \to k'$, $k+q \to k$ and $2V_{k,k',q} \to V_{k,k'}$, we arrive at

$$\hat{H}_{BCS} = \sum_{\sigma,k} \varepsilon_k c^\dagger_{\sigma,k} c_{\sigma,k} + \sum_{k,k' \in \Omega} V_{k,k'} c^\dagger_{\uparrow k} c^\dagger_{\downarrow -k} c_{\downarrow -k'} c_{\uparrow k'}, \tag{5.5}$$

which is the BCS Hamiltonian of superconductivity [21].

5.2 Second quantization approach to superconductivity

The Hamiltonian in eq. (5.5) cannot be treated exactly and hence we need to employ a suitable approximation scheme. Since the formation of Cooper-pairs is associated with the superconducting phase transition, we anticipate that the interaction term cannot be treated perturbatively [7], [9]. We will therefore approximate the problem in a non-perturbative manner by transforming the problem into a self-consistent one-particle problem using the following mean-field expressions [5]

$$\begin{aligned} c_{\downarrow -k} c_{\uparrow k} &= \langle c_{\downarrow -k} c_{\uparrow k} \rangle + c_{\downarrow -k} c_{\uparrow k} - \langle c_{\downarrow -k} c_{\uparrow k} \rangle \\ &= b_k + \delta b_k \end{aligned} \tag{5.6}$$

$$\begin{aligned} c^\dagger_{\uparrow k} c^\dagger_{\downarrow -k} &= \left\langle c^\dagger_{\uparrow k} c^\dagger_{\downarrow -k} \right\rangle + c^\dagger_{\uparrow k} c^\dagger_{\downarrow -k} - \left\langle c^\dagger_{\uparrow k} c^\dagger_{\downarrow -k} \right\rangle \\ &= b^\dagger_k + \delta b^\dagger_k \end{aligned} \tag{5.7}$$

$$b_k = \langle c_{\downarrow,-k} c_{\uparrow,k} \rangle \qquad b^\dagger_k = \left\langle c^\dagger_{\uparrow k} c^\dagger_{\downarrow -k} \right\rangle, \tag{5.8}$$

where δb_k and $\delta b^\dagger_k$ are deviations from the mean-field values b_k and $b^\dagger_k$. Inserting these expressions into eq. (5.5), we arrive at

$$\hat{H} = \sum_{\sigma,k} \varepsilon_k c^\dagger_{\sigma,k} c_{\sigma,k} - \sum_k \Delta_k c^\dagger_{\uparrow k} c^\dagger_{\downarrow -k} + \Delta^\dagger_k c_{\downarrow,-k} c_{\uparrow,k} + \sum_k \Delta_k b^\dagger_k + \mathcal{O}\Big(\delta b_k^2, \delta b_k^{\dagger\,2}\Big), \tag{5.9}$$

where we have defined

$$\Delta_k = -\sum_{k'} V_{k,k'} b_k \qquad \Delta^\dagger_k = -\sum_{k'} V_{k,k'} b^\dagger_{k'}. \tag{5.10}$$

This Hamiltonian describes a non-interacting fermion gas coupled to some background mean-field particle reservoir which can create and annihilate pairs of electrons. To diagonalize eq. (5.9), we need to perform a fermionic Bogoliubov transformation using the following operators

[5] A one-particle Hamiltonian is a Hamiltonian without any interaction terms.

$$\eta_k = u_k c_{\uparrow k} + v_k c^\dagger_{\downarrow -k} \tag{5.11}$$

$$\gamma_k = u_k c^\dagger_{\downarrow -k} - v_k c^\dagger_{\uparrow k}. \tag{5.12}$$

These are fermionic operators if we impose the following commutational relations

$$\left\{\eta^\dagger_k, \eta_{k'}\right\} = \delta_{k,k'} \quad \{\eta_k, \eta_{k'}\} = \left\{\eta^\dagger_k, \eta^\dagger_{k'}\right\} = 0 \quad (\gamma_k \leftrightarrow \eta_k) \tag{5.13}$$

$$\{\eta_k, \gamma_{k'}\} = \left\{\eta^\dagger_k, \gamma_{k'}\right\} = \left\{\eta_k, \gamma^\dagger_{k'}\right\} = 0. \tag{5.14}$$

which are satisfied if $u_k^2 + v_k^2 = 1$. Following the lines of [22], we can use eq. (5.11) and eq. (5.12) to rewrite eq. (5.9) in terms of η_k and γ_k and corresponding adjoint operators. The resulting Hamiltonian becomes diagonal if the parameters u_k and v_k satisfies

$$u_k^2 + v_k^2 = 1 \tag{5.15}$$

$$-4\varepsilon_k u_k v_k = (u_k^2 - v_k^2)(\Delta^\dagger_k + \Delta_k). \tag{5.16}$$

With these constraints, eq. (5.9) becomes

$$\begin{aligned}\hat{H} &= \sum_k E_k \eta^\dagger_k \eta_k - E_k \gamma^\dagger_k \gamma_k + \sum_k 2\varepsilon_k + \Delta_k b^\dagger_k \\ &= \sum_k E_k \left(\eta^\dagger_k \eta_k - \gamma^\dagger_k \gamma_k\right) + E_0 \end{aligned}\tag{5.17}$$

$$E_k = \varepsilon_k (u_k^2 - v_k^2) - u_k v_k (\Delta^\dagger_k + \Delta_k). \tag{5.18}$$

According to eq. (5.15), we can use the following parametrization

$$u_k = \cos\theta_k \quad v_k = \sin\theta_k \tag{5.19}$$

so that eq. (5.16) can be written as

$$\tan 2\theta_k = -\frac{\tilde{\Delta}_k}{\varepsilon_k}. \tag{5.20}$$

where $\tilde{\Delta}_k = \mathrm{Re}(\Delta_k)$. By choosing $\tilde{\Delta}_k > 0$, we have that

$$\cos 2\theta_k = \begin{cases} \frac{1}{\sqrt{1+\left(\frac{\tilde{\Delta}_k}{\varepsilon_k}\right)^2}} & \varepsilon_k > 0 \\ -\frac{1}{\sqrt{1+\left(\frac{\tilde{\Delta}_k}{\varepsilon_k}\right)^2}} & \varepsilon_k < 0 \end{cases} \tag{5.21}$$

Hence, we get the following

$$
\begin{aligned}
E_k &= \varepsilon_k(u_k^2 - v_k^2) - u_k v_k(\Delta_k^\dagger + \Delta_k) \\
&= \varepsilon_k \cos 2\theta_k - \tilde{\Delta}_k \sin 2\theta_k \\
&= \sqrt{\varepsilon_k^2 + \tilde{\Delta}_k^2}.
\end{aligned} \tag{5.22}
$$

Consequently, we see that the onset of Cooper-pair condensation creates a gap in the spectrum of the quasiparticles. These neutral, spin-$\frac{1}{2}$ fermions are known as Bogoliubov quasiparticles or Bogoliubons. They are the low-energy, long-lived single-particle excitations above the ground state of the BCS Hamiltonian, energetically close to the Cooper-pairs.

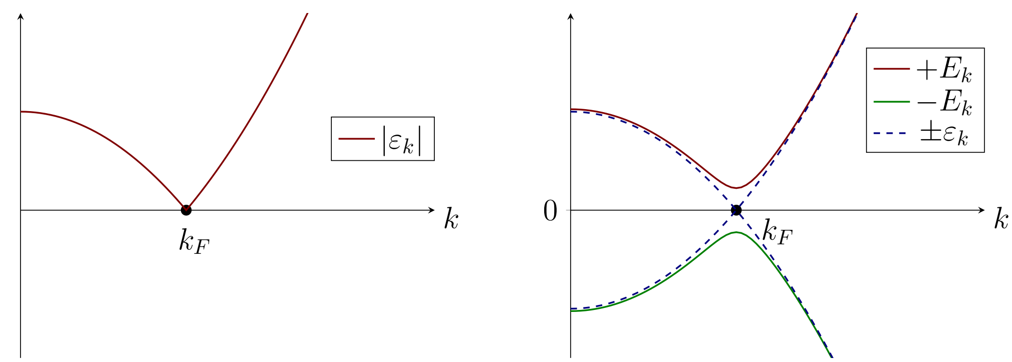

Figure 5.3: Energy spectrum of the fermions of the BCS Hamiltonian with (right) and without (left) finite energy gap $\tilde{\Delta}_k$.

The fact that the Bogoliubons are neutral particles is an important indication that electromagnetism in superconductors is radically different from that of regular conducting materials. In metals and semiconductors, charge is carried by renormalized quasiparticles that are qualitatively similar to free electrons. In superconductors, however, charge transport is a collective phenomenon. [22], [23].

5.3 The BCS gap-equation and critical temperature

In this mean-field treatment, the value of b_k (or equivalently Δ_k) must be self-consistently determined by minimizing the free energy of the system. The grand canonical ensemble of eq. (5.17) can be written as [22]

$$
Z_g = \mathrm{e}^{-\beta E_0} \prod_k \left(1 + \mathrm{e}^{-\beta E_k}\right)\left(1 + \mathrm{e}^{\beta E_k}\right) = \mathrm{e}^{-\beta F}, \tag{5.23}
$$

which gives us the following expression for the Helmholtz free energy

$$
F = E_0 - \frac{1}{\beta} \sum_k \left[\ln\left(1 + \mathrm{e}^{-\beta E_k}\right) + \ln\left(1 + \mathrm{e}^{\beta E_k}\right)\right]. \tag{5.24}
$$

We choose to minimize this energy with respect to Δ_k. Following the lines of [22], we get the following equation for b_k

$$\begin{aligned}\frac{\partial F}{\partial \tilde{\Delta}_k} &= 0 \\ \implies b_k &= \frac{\tilde{\Delta}_k}{\sqrt{\varepsilon_k^2 + \tilde{\Delta}_k^2}} \tanh \frac{\beta E_k}{2}. \end{aligned} \tag{5.25}$$

Inserting this expression into eq. (5.10), we get the following equation for Δ_k

$$\Delta_k = -\sum_{k'} V_{k,k'} \Delta_{k'} \frac{1}{\sqrt{\varepsilon_{k'}^2 + \Delta_{k'}^2}} \tanh \frac{\beta E_{k'}}{2}, \tag{5.26}$$

where we have written $\tilde{\Delta}_k = \Delta_k$ for notational purposes. This is the BCS-gap equation. We can solve it in the region constrained by eq. (5.3) by assuming that the two-body interaction is independent of k, i.e., $V_{k,k'} \approx -V$. Consequently, we get the following simplified expression

$$\Delta_k = V \sum_{k'} \Delta_{k'} \frac{1}{\sqrt{\varepsilon_{k'}^2 + \Delta_{k'}^2}} \tanh \frac{\beta E_{k'}}{2} \tag{5.27}$$

$$1 = V \sum_{k'} \frac{1}{\sqrt{\varepsilon_{k'}^2 + \Delta^2}} \tanh \frac{\beta E_{k'}}{2} \tag{5.28}$$

$$= V \int_{-\omega_0}^{\omega_0} \mathrm{d}\varepsilon \frac{N(\varepsilon)}{\sqrt{\varepsilon^2 + \Delta^2}} \tanh \frac{\beta \sqrt{\varepsilon^2 + \Delta^2}}{2}, \tag{5.29}$$

where we have converted the k'-sum into an energy integral by introducing the density of states

$$N(\varepsilon) = \sum_{k'} \delta\left(\varepsilon - \varepsilon_{k'}\right). \tag{5.30}$$

In this region, we can assume that $N(\varepsilon)$ is a slowly-varying function of ε, meaning that we can write $N(\varepsilon) \approx N(\varepsilon_F)$. Thus, we get

$$\begin{aligned} 1 &= V \int_{-\omega_0}^{\omega_0} \mathrm{d}\varepsilon \frac{N(\varepsilon)}{\sqrt{\varepsilon^2 + \Delta^2}} \tanh \frac{\beta \sqrt{\varepsilon^2 + \Delta^2}}{2} \\ &\approx V N(\varepsilon_F) \int_{-\omega_0}^{\omega_0} \mathrm{d}\varepsilon \frac{1}{\sqrt{\varepsilon^2 + \Delta^2}} \tanh \frac{\beta \sqrt{\varepsilon^2 + \Delta^2}}{2} \\ &\equiv \lambda \int_{-\omega_0}^{\omega_0} \mathrm{d}\varepsilon \frac{1}{\sqrt{\varepsilon^2 + \Delta^2}} \tanh \frac{\beta \sqrt{\varepsilon^2 + \Delta^2}}{2} \end{aligned} \tag{5.31}$$

At $T = 0$, we get the following equation for $\Delta(T = 0) \equiv \Delta_0$

$$1 = \lambda \int_0^{\omega_0} \mathrm{d}\varepsilon \frac{1}{\sqrt{\varepsilon^2 + \Delta_0^2}} \tag{5.32}$$

$$\implies \Delta_0 \underset{\lambda \ll 1}{\approx} 2\omega_0 \mathrm{e}^{-\frac{1}{\lambda}}. \tag{5.33}$$

This equation features an essential singularity as a function of $\lambda \sim V$. This implies that this result could not have been obtained to any finite order in perturbation theory. Note also that a non-zero density of states is required to have finite λ, meaning that superconductivity can only occur if there is a Fermi-surface involved. According to eq. (5.31), Δ decreases as temperature increases. Consequently, assuming that $\Delta(T)$ is a continuous function, we must have that $\Delta \to 0^+$ for some critical temperature $T \to T_c$. We obtain this temperature by considering the limit

$$1 = \lambda \int_0^{\omega_0} \mathrm{d}\varepsilon \frac{1}{\varepsilon} \tanh \frac{\beta_c \varepsilon}{2}. \tag{5.34}$$

which has the following solution in the limit $\lambda \ll 1$

$$k_B T_c = \frac{2}{\pi} \mathrm{e}^{\gamma} \omega_0 \mathrm{e}^{-\frac{1}{\lambda}} = \frac{\mathrm{e}^{\gamma}}{\pi} \Delta_0. \tag{5.35}$$

where γ is the Euler-Mascheroni constant.

5.4 Ginzburg-Landau field theory of conventional superconductors

The BCS Hamiltonian of eq. (5.5) can be modified into the following real-space continuum model [7]

$$\hat{H} = \int \mathrm{d}^d r \left(c_\sigma^\dagger(r) \left(\frac{1}{2m}(-i\boldsymbol{\nabla})^2 - \mu \right) c_\sigma(r) - g c_\uparrow^\dagger(r) c_\downarrow^\dagger(r) c_\downarrow(r) c_\uparrow(r) \right), \tag{5.36}$$

where a sum over repeated indices is assumed. By mapping this Hamiltonian into its corresponding quantum field theory using eq. (2.18), we get the following action and partition function in imaginary time formalism

$$\begin{aligned} S &= -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d r \left(\psi_\sigma^\dagger \partial_\tau \psi_\sigma + \psi_\sigma^\dagger \left(\frac{1}{2m}(-i\boldsymbol{\nabla})^2 - \mu \right) \psi_\sigma - g \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow \right) \\ &= -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d r \left(\psi_\sigma^\dagger \mathcal{G}_0^{-1} \psi_\sigma + g \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow \right) \qquad (5.37) \\ Z &= \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \mathrm{e}^S \qquad (5.38) \end{aligned}$$

where we have defined the following inverse fermionic Greens functions

$$\mathcal{G}_0^{-1} = (\partial_\tau + \frac{1}{2m}(-i\boldsymbol{\nabla})^2 - \mu) \tag{5.39}$$

Similarly to the case of eq. (5.5), the two-body interaction term in eq. (5.37) makes it impossible to solve this model exactly. However, we can solve it approximately and non-perturbatively in the vicinity of a superconducting phase transition by introducing the auxiliary complex scalar field φ using the following Hubbard-Stratonovich decoupling

$$
\mathrm{e}^{\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d r g \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow} = \int \mathcal{D}\varphi \mathcal{D}\varphi^* \mathrm{e}^{S_\varphi} \tag{5.40}
$$

$$
S_\varphi = -\int \mathrm{d}\tau \int \mathrm{d}^d r \left(\frac{1}{g}\varphi^2 - (\varphi^* \psi_\downarrow \psi_\uparrow + \varphi \psi_\uparrow^\dagger \psi_\downarrow^\dagger) \right), \tag{5.41}
$$

where $\varphi^2 = |\varphi|^2$. Comparing this action with the mean-field Hamiltonian of eq. (5.9), we immediately see that the bosonic field φ replaces the role of Δ_k in this field-theoretical treatment. Thus, φ is an effective bosonic field describing the Cooper-pairs, which we will refer to as the Cooper boson [6]. Inserting this expression into our partition function in eq. (5.38), we get

$$
Z = \int \mathcal{D}\varphi \mathcal{D}\varphi^* \mathcal{D}\Psi^\dagger \mathcal{D}\Psi \mathrm{e}^{S} \tag{5.42}
$$

$$
S = -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d r \left(\frac{1}{g}\varphi^2 + \Psi^\dagger \mathcal{G}^{-1} \Psi \right) \tag{5.43}
$$

$$
\mathcal{G}^{-1} = \begin{pmatrix} \mathcal{G}_0^{p^{-1}} & \varphi \\ \varphi^* & \mathcal{G}_0^{h^{-1}} \end{pmatrix} = \mathcal{G}_0^{-1} + \bar{\varphi} \tag{5.44}
$$

$$
\bar{\varphi} = \begin{pmatrix} 0 & \varphi \\ \varphi^* & 0 \end{pmatrix}, \tag{5.45}
$$

where we have changed to a two-component Nambu spinor basis $\Psi = \begin{pmatrix} \psi_\uparrow & \psi_\downarrow \end{pmatrix}^T$ in the fermionic sector and defined the following inverse particle- and hole propagators

$$
\mathcal{G}_0^{p^{-1}} = \partial_\tau + \frac{1}{2m}(-i\boldsymbol{\nabla})^2 - \mu \tag{5.46}
$$

$$
\mathcal{G}_0^{h^{-1}} = \partial_\tau - \frac{1}{2m}(-i\boldsymbol{\nabla})^2 + \mu. \tag{5.47}
$$

The fermionic sector of eq. (5.42) is bilinear and hence solvable. After integrating out the fermions, we are left with the following bosonic partition function and corresponding action

$$
Z_B = \int \mathcal{D}\varphi \mathcal{D}\varphi^* \mathrm{e}^{S_B} \tag{5.48}
$$

$$
S_B = -\int_0^\beta \int \mathrm{d}^d r \left(\frac{1}{g}\varphi^2 \right) + \mathrm{Tr}\left[\ln(\mathcal{G}^{-1}) \right]. \tag{5.49}
$$

Near a superconducting phase transition, the magnitude of φ is very small. Hence, we can treat $\bar{\varphi}$ in eq. (5.49) as a perturbation. By writing the Greens function as

$$
\mathcal{G}^{-1} = \mathcal{G}_0^{-1} + \bar{\varphi} = \mathcal{G}_0^{-1}(1 + \mathcal{G}_0 \bar{\varphi}) \tag{5.50}
$$

we see that the last factor becomes dimensionless. Consequently, we can perform a saddle-point approximation by expanding the last factor to fourth order in bosonic fields

[6] More precisely, φ is the conjugate bosonic field to the Cooper-pairs described by the operators $b_k^\dagger$, which are neither bosonic- nor fermionic operators [22]

$$Z_{\text{eff}} = \int \mathcal{D}\varphi\mathcal{D}\varphi^* \mathrm{e}^{S_{\text{eff}}} \tag{5.51}$$

$$S_{\text{eff}} = -\int \mathrm{d}\tau \int \mathrm{d}^d r \left(\frac{1}{g}\varphi^2\right) - \frac{1}{2}\,\mathrm{Tr}\left[(\mathcal{G}_0\bar{\varphi})^2\right] - \frac{1}{4}\,\mathrm{Tr}\left[(\mathcal{G}_0\bar{\varphi})^4\right] + \mathrm{Tr}\left[\ln\!\left(\mathcal{G}_0^{-1}\right)\right], \tag{5.52}$$

where odd powers of $\bar{\varphi}$ vanish since $\bar{\varphi}$ is off-diagonal. The last term in eq. (5.52) is the solution to an uncoupled fermionic sector, which factorizes out of the partition function. In order to solve the traces, we will use $\nu = (\omega_n, k)$ as our internal fermionic quantum numbers, where k is the momentum and $\omega_n = \frac{(2n+1)\pi}{\beta}$ are fermionic Matsubara frequencies. These frequencies automatically satisfy the anti-periodic boundary condition of eq. (2.17). Assuming that the order parameter is spacetime dependent, i.e., $\varphi = \varphi(\tau, r)$, we can Fourier transform it in the following way

$$\varphi(\tau, r) = \frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^d q}{(2\pi)^d}\varphi(\nu_l, q)\mathrm{e}^{iqr + i\nu_l\tau}. \tag{5.53}$$

Using as a complete basis the fermionic Matsubara wave functions [7],

$$\psi_{nk}(\tau, r) = \frac{1}{\sqrt{\beta}}\mathrm{e}^{ikr + i\omega_n\tau}, \tag{5.54}$$

and inserting the Fourier transformed expression of eq. (5.53) into eq. (5.52), we get the following second-order contribution

$$\begin{aligned} -\frac{1}{2}\,\mathrm{Tr}\left[(\mathcal{G}_0\bar{\varphi})^2\right] &= -\frac{1}{2}\int \frac{\mathrm{d}^d q}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{\beta^2}\sum_{n,l}\Big(\mathcal{G}_0^p(i\omega_n, k)\mathcal{G}_0^h(i\omega_n - i\nu_l, k - q) \\ &\quad + \mathcal{G}_0^h(i\omega_n, k)\mathcal{G}_0^p(i\omega_n - i\nu_l, k - q)\Big)\varphi(\nu_l, q)^2 \qquad (5.55) \\ &= -\int \frac{\mathrm{d}^d q}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{\beta^2}\sum_{n,l}\Big(\mathcal{G}_0^p(i\omega_n, k)\mathcal{G}_0^h(i\omega_n - i\nu_l, k - q)\Big)\varphi(\nu_l, q)^2 \\ &= -\int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{\beta}\sum_l \chi^{(2)}(i\nu_l, q)\varphi(\nu_l, q)^2, \qquad (5.56) \end{aligned}$$

where we have rotated all variables $k \to -k$ etc. in the second term. The Fourier transformed Greens functions reads

$$\mathcal{G}_0^p(i\omega_n, k) = \frac{1}{i\omega_n + \epsilon_k} \tag{5.57}$$

$$\mathcal{G}_0^h(i\omega_n, k) = \frac{1}{i\omega_n - \epsilon_k}. \tag{5.58}$$

where $\epsilon_k = \frac{k^2}{2m} - \mu$. Summing over fermionic Matsubara frequencies, we get [7]

$$\frac{1}{\beta}\sum_n \frac{1}{i\omega_n + \epsilon_k}\frac{1}{i\omega_n - i\nu_l - \epsilon_{k-q}} = -\frac{f(-\epsilon_k) + f(\epsilon_{k-q})}{i\nu_l + \epsilon_k + \epsilon_{k-q}} \tag{5.59}$$

where f is the Fermi-Dirac distribution. In the vicinity of a superconducting phase transition, the magnitude of the field φ is approximately constant. We therefore set $\nu_l \to 0$ and work to second order in spatial momenta q.

Using that $f(\epsilon_{k-q}) \approx f(\epsilon_k - \frac{k\cdot q}{m}) \approx f(\epsilon_k) + \frac{(k\cdot q)^2}{2m^2}\frac{\partial^2 f(\epsilon_k)}{\partial^2 \epsilon_k}$, we see that

$$\chi^{(2)}(i\nu_l, q) \sim c_1 - q^2 c_2 \tag{5.60}$$

where $c_1 < 0$ and $c_2 > 0$ are constants. Fourier transforming back into real space, the second-order contribution becomes

$$-\frac{1}{2}\,\mathrm{Tr}\left[(\mathcal{G}_0\bar{\varphi})^2\right] = -\int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{\beta}\sum_l \chi^{(2)}(i\nu_l, q)\varphi(q)^2 \tag{5.61}$$

$$= -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d r \left(c_2|-i\boldsymbol{\nabla}\varphi|^2 + c_1|\varphi|^2\right) \tag{5.62}$$

In the fourth-order contribution, we further simplify our calculations by assuming that the field φ is constant. Thus, we get [7]

$$-\frac{1}{4}\,\mathrm{Tr}\left[(\mathcal{G}_0\bar{\varphi})^4\right] \approx -\frac{|\varphi|^4}{4}\sum_l \int_{-\omega_D}^{\omega_D} \frac{\mathrm{d}\xi}{\omega_l^2 + \xi^2} = -\frac{|\varphi|^4}{2}\sum_l \int_0^{\omega_D} \frac{\mathrm{d}\xi}{\omega_l^2 + \xi^2} = -c_3|\varphi|^4 \tag{5.63}$$

where $c_3 > 0$ is a positive constant. Hence the fourth-order contribution can be written as

$$-\frac{1}{4}\,\mathrm{Tr}\left[(\mathcal{G}_0\bar{\varphi})^4\right] = -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d r c_3|\varphi|^4 \tag{5.64}$$

In total, we get the following action after re-scaling the fields

$$S_{\text{eff}} = -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d r \left(\frac{1}{2}|-i\boldsymbol{\nabla}\varphi|^2 + \alpha|\varphi|^2 + \beta|\varphi|^4\right), \tag{5.65}$$

where α and $\beta > 0$ are constants and

$$\alpha \sim c_1 + \frac{1}{g} \sim T - T_c. \tag{5.66}$$

This is the time-independent Ginzburg-Landau theory of a conventional superconductor with fluctuations. This theory is invariant under global U(1)-transformations, i.e.,

$$\varphi \to \mathrm{e}^{i\theta}\varphi \tag{5.67}$$

$$\varphi^* \to \mathrm{e}^{-i\theta}\varphi^*. \tag{5.68}$$

We can promote this global symmetry into a local U(1) gauge symmetry by minimally coupling the Cooper bosons to a gauge field $\bar{A}$. This gives us the following action

$$S_{\text{eff}} = -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d r \left(\frac{1}{2}\left|(-i\boldsymbol{\nabla} - e^*\bar{A})\varphi\right|^2 + \alpha|\varphi|^2 + \beta|\varphi|^4 + \frac{1}{2}\lambda F^{\mu\nu}F_{\mu\nu} \right), \tag{5.69}$$

where e^* is the charge of the Cooper bosons, $\frac{1}{2}\lambda F^{\mu\nu}F_{\mu\nu} = \lambda\left((\boldsymbol{\nabla}\phi)^2 + (\boldsymbol{\nabla}\times\bar{A})^2\right)$ is the Maxwell Lagrangian in Euclidean space and λ is a phenomenological constant. Note that have included the temporal component of the gauge field $A_0 = \phi$ in the Maxwell sector even though it does not couple to the Cooper bosons. This is because the gauge field fluctuates independently of the superconductor.

5.5 The Higgs-Anderson mechanism

At sufficiently low temperatures the Cooper bosons of eq. (5.69) acquires a finite expectation value. The resulting ground state does not share the same symmetries as its corresponding action[7], a phenomenon known as spontaneous symmetry breaking. In order to investigate this effect in the context of superconductivity, we will expand eq. (5.69) to low order in phase-, gauge and amplitude fluctuations above this ground state.

Below a certain critical temperature T_c, the mass-term α changes sign and becomes negative. This means that the Cooper boson potential

$$V(\varphi) = \alpha|\varphi|^2 + \beta|\varphi|^4 \tag{5.70}$$

is minimized by a finite value expectation value of the Cooper boson, $|\varphi_0| = \sqrt{\frac{-\alpha}{\beta}}$, as shown in fig. 5.4. In the low-energy limit below the critical temperature, we can therefore write φ as follows

$$\varphi = |\varphi|\mathrm{e}^{i\theta} = |\varphi_0 + \psi|\mathrm{e}^{i\theta}, \tag{5.71}$$

where θ and ψ are phase- and amplitude fluctuations above the ground state φ_0, respectively. Inserting this expression into eq. (5.69) and ignoring higher-order terms in θ, ψ, and $\bar{A}$, we get the following Lagrangian

$$\mathcal{L} = \frac{1}{2}|\varphi_0|^2\left|\boldsymbol{\nabla}\theta - e^*\bar{A}\right|^2 + \frac{1}{2}|\boldsymbol{\nabla}\psi|^2 + 2|\alpha||\psi|^2 + \frac{1}{2}\lambda F^{\mu\nu}F_{\mu\nu}, \tag{5.72}$$

where we have discarded a constant energy term. This theory describes a massless Goldstone boson θ coupled to a gauge field together with a massive neutral scalar field ψ[8]. We also have the following gauge-symmetry

$$A \to A + \boldsymbol{\nabla}\phi \tag{5.73}$$
$$\theta \to \theta + e^*\phi. \tag{5.74}$$

[7] Generally speaking, the ground state is symmetric under a subgroup of the original symmetry. In this case the subgroup is trivial, but the ground state manifold is also invariant under U(1) transformations. This should not be confused with the original symmetry.

[8] Goldstone's theorem states that field theories with a continuous symmetry which is not shared by the ground state of the system must host a massless scalar boson, which is called the Goldstone boson.

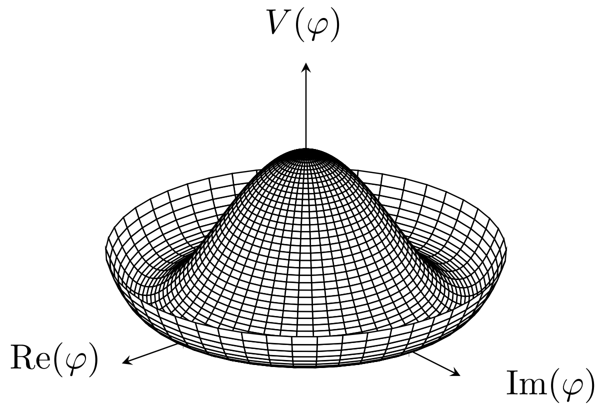

Figure 5.4: Qualitative illustration of the Mexican hat potential for a complex scalar field φ.

The physical significance of the Goldstone bosons can be accounted for by integrating them out of the theory, which results in an effective action in terms of the gauge field $\bar{A}$ and the scalar field ψ [7]. Alternatively, we can define the gauge-invariant combination

$$\bar{A} \to \bar{A}' = \bar{A} + \frac{1}{e^*}\boldsymbol{\nabla}\theta \equiv \bar{A}, \tag{5.75}$$

which leaves the Maxwell sector unaltered, i.e., $F^{\mu\nu}(\bar{A}) = F^{\mu\nu}(\bar{A}')$ [8]. Hence, after substituting eq. (5.75) into eq. (5.72), we get the following Lagrangian

$$\mathcal{L} = \frac{1}{2}|\boldsymbol{\nabla}\psi|^2 + 2|\alpha||\psi|^2 + \frac{1}{2}m_A^2\bar{A}^2 + \frac{1}{2}\lambda F^{\mu\nu}F_{\mu\nu}. \tag{5.76}$$

where $m_A = e^*\phi_0$ is the mass of the gauge field. The Goldstone mode has been eliminated from the theory and as a consequence the gauge field has acquired a mass. The massless degree of freedom of the Goldstone boson has been transformed into a massive longitudinal degree of freedom of the gauge field sector. More intuitively, we say that the gauge boson has "absorbed" or "eaten" the Goldstone boson. This phenomenon is known as the Higgs-mechanism[9]. In the case of superconductivity, or more precisely in the case of abelian gauge theories, it is also known as the Higgs-Anderson mechanism [22].

The theory in eq. (5.69) is mathematically equivalent to the Higgs sector of the standard model of particle physics where φ corresponds to the Higgs-field and ψ corresponds to the field describing the Higgs boson[10]. The superconducting phase transition is therefore mathematically similar to the spontaneous symmetry breaking of the electroweak theory[11], which is the mechanism behind the massive gauge bosons and fermions of the standard model [8].

[9]The fact that there are no Goldstone bosons in gauge theories is a generic feature of general Yang-Mills theories [8].

[10]In this sense, the Bogoliubons we found using the second quantized approach are fermionic counterparts of the Higgs boson.

[11]The Glashow-Salam–Weinberg theory of electroweak unification describes the spontaneous symmetry breaking of a $\mathrm{SU}(2) \times \mathrm{U}(1)_{\mathrm{Y}}$ symmetric Lagrangian into a $\mathrm{U}(1)_{\mathrm{em}}$ symmetric Lagrangian[8].

5.6 Electromagnetism in superconductors

In order to derive the electromagnetic properties of superconductors, we need a constitutive relation for the current density $\bar{j}$. We can obtain this by deriving the linear response of the system described by eq. (5.76) by performing a functional derivative of the corresponding action without the Maxwell term with respect to the gauge field [7], i.e.

$$\bar{j} = -\frac{\delta S}{\delta \bar{A}} = m_A \bar{A}. \tag{5.77}$$

This is the (second) London equation. From this expression, we can derive the following two relations describing the systems repose to external magnetic- and electric fields[12] [7]

$$\nabla \times \bar{B} = \nabla \times (\nabla \times \bar{A}) = m_A \bar{A} \tag{5.78}$$
$$\partial_t \bar{j} = m_A \bar{E}. \tag{5.79}$$

Equation (5.78) has no static solutions, meaning that the superconductor cannot accommodate a constant magnetic field. More precisely, assuming an infinite interface, we obtain the following solution for the magnetic field

$$\bar{B} = \bar{B}_0 \mathrm{e}^{-\frac{x}{\lambda}}, \tag{5.80}$$

where $\lambda = \sqrt{\frac{1}{m_A}}$ is the London penetration length and x is the direction perpendicular to the interface of the superconductor. Consequently, magnetic fields are exponentially suppressed in the superconductor. This is known as the Meissner effect. The unbound linear increase of the current in eq. (5.79) is clearly an unphysical solution. However, this relation tells us is that the response of an electric field cannot be a corresponding increase of the current[13]. This in turn implies the dissipationless flow of charge and therefore complete loss of resistivity [7].

[12]This set of equations was obtained using a specific gauge and assuming that there are no constant terms in the vector potential. The time derivative was derived by performing a Wick rotation into real time [7].

[13]Which is the case is Ohmic materials, where $\bar{j} = \sigma \bar{E}$ [7]

Chapter 6

Quantum field theory of a superconductor coupled to a topological insulator

In this chapter, we derive an effective field theory of a superconductor topological insulator heterostructure in an electromagnetic field where we allow for proximity-induced interactions between the surface states of the topological insulator and the Cooper bosons of the superconductor.

6.1 Lagrangians and proximity couplings

The superconductor and the gauge field can be described using the Ginzburg-Landau theory of eq. (5.69)

$$\mathcal{L}_{SC} = \frac{1}{2}|(-i\boldsymbol{\nabla} - e^*\bar{A})\varphi|^2 + \alpha|\varphi|^2 + \beta|\varphi|^4 + \lambda\left((\boldsymbol{\nabla}\phi)^2 + (\boldsymbol{\nabla}\times\bar{A})^2\right), \tag{6.1}$$

where $\mathcal{L}_M = \lambda\left((\boldsymbol{\nabla}\varphi)^2 + (\boldsymbol{\nabla}\times A)^2\right)$ is the Maxwell sector with a phenomenological field-strength parameter λ and $e^* = 2e$ is the charge of the Cooper bosons. We can write the former contribution as

$$\mathcal{L}_M = \frac{1}{2}\lambda F^{\mu\nu}F_{\mu\nu}, \tag{6.2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu$ is the Maxwell field strength tensor in Euclidean space. In our effective surface theory, the motion of the relevant Cooper bosons is restricted to the plane, meaning that $\boldsymbol{\nabla} = (\partial_x, \partial_y, 0)$. We also assume that the magnetic field is perpendicular to this surface, and consequently $\bar{A} = (A_x, A_y, 0)$. The surface states of the topological insulator can be described using the Lagrangian of eq. (3.53)

$$\mathcal{L}_{TI} = \Psi^\dagger\left(i\partial_t - e\phi + v_F(-i\boldsymbol{\nabla} - e\bar{A})\cdot\bar{\sigma} - \mu + m\sigma^z\right)\Psi \tag{6.3}$$

where $\Psi = \begin{pmatrix}\psi_\uparrow & \psi_\downarrow\end{pmatrix}^T$ are fermionic fields written in two-component Nambu-formalism. The third term of this Lagrangian is a topological term ensuring spin-momentum locking of the surface states. The fourth term is a regular chemical potential term. The last term is a mass term of the fermions, which we assume is due to some form of magnetization perpendicular to the interface of the heterostructure. Due to the spatial proximity between the topological insulator and the superconductor, we assume that the Cooper bosons interact with the topological surface fermions in the following way

$$\mathcal{L}_{SC-TI} = g\varphi\psi_\uparrow^\dagger\psi_\downarrow^\dagger + g^*\varphi^*\psi_\downarrow\psi_\uparrow, \tag{6.4}$$

where g is a dimensionless coupling constant. These two terms describe the decay and the condensation of a Cooper boson on the superconductor, respectively. Combining eq. (6.3) and eq. (6.4), we get the following fermionic Lagrangian

$$\begin{aligned}\mathcal{L}_F &= \mathcal{L}_{TI} + \mathcal{L}_{SC-TI} \\ &= \Psi^\dagger \left(i\partial_t - e\phi + v_F(-i\boldsymbol{\nabla} - e\bar{A})\cdot\bar{\sigma} - \mu + m\sigma^z\right)\Psi + g\varphi\psi_\uparrow^\dagger\psi_\downarrow^\dagger + g^*\varphi^*\psi_\downarrow\psi_\uparrow.\end{aligned} \tag{6.5}$$

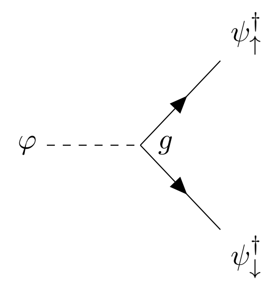

Figure 6.1: Feynman diagram illustrating the decay of a Cooper boson into two topological surface fermions.

6.2 Dirac Lagrangian of the topological fermions

The Lagrangian describing the topological surface states can be rewritten as a Dirac Lagrangian which closely resembles the field theory of (2+1)-dimensional quantum electrodynamics. Using the commutational relations of the Pauli matrices in eq. (A.2) and eq. (A.3), we can rewrite eq. (6.3) as follows

$$\begin{aligned}\mathcal{L}_{TI} &= \Psi^\dagger\left(i\partial_t - e\phi + v_F(-i\boldsymbol{\nabla} - e\bar{A})\cdot\bar{\sigma} - \mu + m\sigma^z\right)\Psi \\ &= \bar{\Psi}\left(i\partial_t\sigma^z - e\phi\sigma^z + v_F(-i\partial_i - eA_i)\sigma^z\sigma^i - \mu\sigma^z + m\right)\Psi \\ &= \bar{\Psi}\left(i\partial_t\sigma^z - e\phi\sigma^z + i\varepsilon_{zij}v_F(-i\partial_i - eA_i)\sigma^j - \mu\sigma^z + m\right)\Psi \\ &= \bar{\Psi}\left(i\partial_t\sigma^z - e\phi\sigma^z + v_F(\partial_x - ieA_x)\sigma^y - v_F(\partial_y - ieA_y)\sigma^x - \mu\sigma^z + m\right)\Psi \\ &= \bar{\Psi}\left(i(\partial_t + ie\phi + i\mu)\sigma^z - iv_F(\partial_x - ieA_x)(i\sigma^y) - iv_F(\partial_y - ieA_y)(-i\sigma^x) + m\right)\Psi.\end{aligned} \tag{6.6}$$

By combining the chemical potential and the time-derivative, we can define the following derivative operator

$$\partial_\mu = (\partial_t - \mu, v_F\boldsymbol{\nabla}),$$

Substituting this expression into the above Lagrangian, we can write eq. (6.6) in the following compact form

$$\mathcal{L}_{TI} = \bar{\Psi}(i\not{D} + m)\Psi, \tag{6.7}$$

where we have adapted the following notation

$$\not{D} = \gamma^\mu D_\mu = \eta^{\mu\nu}\gamma_\nu D_\mu \tag{6.8}$$
$$\gamma_\nu = (\gamma_0, \gamma_1, \gamma_2) = (\sigma^z, i\sigma^y, -i\sigma^x) \tag{6.9}$$
$$D_\mu = \partial_\mu + ieA_\mu \tag{6.10}$$
$$A_\mu = (\phi, -v_F\bar{A}) \tag{6.11}$$
$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1). \tag{6.12}$$

The γ-matrices of eq. (6.9) form a representation of the (2+1)-dimensional Clifford algebra and consequently eq. (6.7) is Lorenz invariant. Furthermore, due to the covariant derivative of eq. (6.10), the Lagrangian in eq. (6.7) is locally U(1) gauge invariant in terms of the gauge field A_μ. Consequently, the Lagrangian of the topological insulator is mathematically equivalent to the Lagrangian of quantum electrodynamics in (2+1)-dimensions [8].

6.3 Integrating out the fermions

By integrating out the fermionic fields, we are left with an effective partition function in terms of the Cooper bosons and the gauge field. We will do this to second order in coupling constants and in the low wavelength limit, which in our case means that we will be working to second order in spatial momentum and first order in temporal momentum. Combining eq. (6.7) with eq. (6.4), the action of the fermionic fields can be written as follows

$$\begin{aligned} S_{\mathrm{F}} &= i\int \mathrm{d}t \int \mathrm{d}^2 r \mathcal{L}_F \qquad (6.13)\\ \mathcal{L}_F &= \bar{\Psi}(i\not{D} + m)\Psi + \Delta\psi_\uparrow^\dagger\psi_\downarrow^\dagger + \Delta^*\psi_\downarrow\psi_\uparrow \\ &= \bar{\Psi}(i(\partial_t + i\mu + ie\phi)\gamma^0 - iD_i\gamma^i + m)\Psi + \Delta\psi_\uparrow^\dagger\psi_\downarrow^\dagger + \Delta^*\psi_\downarrow\psi_\uparrow, \end{aligned}$$

where we have defined the re-scaled Cooper bosons $\Delta = g\varphi$. Performing a Wick-rotation of eq. (6.13) using eq. (2.13) and eq. (2.18), we get the following imaginary time action

$$\begin{aligned} S_{\mathrm{F}} &= \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \mathcal{L}_F \qquad (6.14)\\ \mathcal{L}_F &= \bar{\Psi}(i(i\partial_\tau + i\mu + ie\phi)\gamma^0 - iD_i\gamma^i + m)\Psi + \Delta\psi_\uparrow^\dagger\psi_\downarrow^\dagger + \Delta^*\psi_\downarrow\psi_\uparrow \\ &= \bar{\Psi}(i(\partial_\tau + \mu + e\phi)(i\gamma^0) + iD_i(-\gamma^i) + m)\Psi + \Delta\psi_\uparrow^\dagger\psi_\downarrow^\dagger + \Delta^*\psi_\downarrow\psi_\uparrow \\ &= \bar{\Psi}(i(\partial_\tau + \mu + ie\phi)(i\gamma^0) + iD_i(-\gamma^i) + m)\Psi + \Delta\psi_\uparrow^\dagger\psi_\downarrow^\dagger + \Delta^*\psi_\downarrow\psi_\uparrow \\ &= \bar{\Psi}(i\not{D} + m)\Psi + \Delta\psi_\uparrow^\dagger\psi_\downarrow^\dagger + \Delta^*\psi_\downarrow\psi_\uparrow, \qquad (6.15) \end{aligned}$$

where we have re-scaled the scalar potential $-i\phi \to \phi$ in order to preserve gauge invariance, changed to Euclidean metric $\eta^{\mu\nu} = \delta^{\mu\nu}$ and defined the Euclidean γ-matrices $\gamma^\mu = (i\gamma_0, -\gamma_i) =$

$(i\sigma^z, -i\sigma^y, i\sigma^x)$. This matrix-algebra has the following (anti-) commutational relations (see section A.1 for details)

$$[\gamma_\mu, \gamma_\nu] = -2\varepsilon_{\mu\nu\lambda}\gamma^\lambda \quad \{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}. \tag{6.16}$$

The Lagrangian in eq. (6.15) cannot be expressed entirely in terms of two-component Nambu spinors due to the structure of the coupling terms. In order to write this expression on matrix form, we need to define the following four-component spinors

$$\Phi = \begin{pmatrix} \psi_\uparrow & \psi_\downarrow & \psi_\uparrow^\dagger & -\psi_\downarrow^\dagger \end{pmatrix}^T \tag{6.17}$$

$$\bar{\Phi} = \begin{pmatrix} \psi_\uparrow^\dagger & -\psi_\downarrow^\dagger & \psi_\uparrow & \psi_\downarrow \end{pmatrix}. \tag{6.18}$$

Using this basis, we can rewrite the action in eq. (6.14) as [7]

$$S_F = \frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \bar{\Phi}\mathcal{G}^{-1}\Phi, \tag{6.19}$$

where we have defined the following 4x4-matrices

$$\mathcal{G}^{-1} = \mathcal{G}_0^{-1} + \underline{\underline{\Delta}} + a = \mathcal{G}_0^{-1}(1 + \mathcal{G}_0(\underline{\underline{\Delta}} + a)) \tag{6.20}$$

$$\underline{\underline{\Delta}} = \begin{pmatrix} 0 & \Delta(-i\sigma^y) \\ \Delta^*(-i\sigma^y) & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Delta\gamma_1 \\ \Delta^*\gamma_1 & 0 \end{pmatrix} \tag{6.21}$$

$$a = \begin{pmatrix} -e\not{A} & 0 \\ 0 & e\not{A}^T \end{pmatrix} \tag{6.22}$$

$$\mathcal{G}_0^{-1} = \begin{pmatrix} i\not{\partial} + m + i\mu\gamma_0 & 0 \\ 0 & i\not{\partial}^T - m - i\mu\gamma_0 \end{pmatrix} = \begin{pmatrix} \mathcal{G}_0^{p^{-1}} & 0 \\ 0 & \mathcal{G}_0^{h^{-1}} \end{pmatrix}. \tag{6.23}$$

Notice that the chemical potential and the mass term changes sign in $\mathcal{G}_0^{h^{-1}}$ compared to $\mathcal{G}_0^{p^{-1}}$ due to a partial integration. In the following, we will absorb the chemical potential into the temporal parts of the above propagators, bearing in mind this relative sign change. It will be made explicit whenever a calculation involving it appears. The partition function of this theory can be expressed as

$$\begin{aligned} Z &= \int \mathcal{D}\varphi\mathcal{D}\varphi^*\mathcal{D}A_\mu\mathcal{D}\bar{\Phi}\mathcal{D}\Phi \mathrm{e}^S \\ &= \int \mathcal{D}\varphi\mathcal{D}\varphi^*\mathcal{D}A_\mu \mathrm{e}^{S_{SC}} \int \mathcal{D}\bar{\Phi}\mathcal{D}\Phi \mathrm{e}^{S_F} \end{aligned} \tag{6.24}$$

with the following actions

$$S_{SC} = -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \mathcal{L}_{SC} \tag{6.25}$$

$$S_F = \frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \bar{\Phi}\mathcal{G}^{-1}\Phi = -\frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \bar{\Phi}(-\mathcal{G}^{-1})\Phi \tag{6.26}$$

where we have separated the bosonic- and fermionic degrees of freedom. Integrating out the four-component Nambu fields, we get a purely bosonic partition function of the form

$$Z_B = \int \mathcal{D}\varphi \mathcal{D}\varphi^* \mathcal{D}A_\mu \mathrm{e}^{S_B} \tag{6.27}$$

$$\begin{aligned} S_B &= S_{SC} + \frac{1}{2}\,\mathrm{Tr}\left[\ln\left(-\mathcal{G}^{-1}\right)\right] \\ &= -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \left(\mathcal{L}_{SC} - \frac{1}{2}\sum_\nu \langle\nu|\,\mathrm{tr}\left[\ln\left(-\mathcal{G}^{-1}\right)\right]|\nu\rangle\right). \end{aligned} \tag{6.28}$$

By separating the terms in $\mathcal{G}^{-1}$ according to eq. (6.20), we can perform a series expansion of the leftmost factor

$$\begin{aligned} \mathrm{tr}\ln\left(-\mathcal{G}^{-1}\right) &= \mathrm{tr}\ln\left(-\mathcal{G}_0^{-1}(1+\mathcal{G}_0(\underline{\Delta}+a))\right) = \mathrm{tr}\left[\ln\left(-\mathcal{G}_0^{-1}\right)\right] + \mathrm{tr}\left[\ln(1+\mathcal{G}_0(\underline{\Delta}+a))\right] \\ &= \mathrm{tr}\left[\ln\left(-\mathcal{G}_0^{-1}\right)\right] + \mathrm{tr}\left[\mathcal{G}_0(\underline{\Delta}+a) - \frac{1}{2}(\mathcal{G}_0(\underline{\Delta}+a))^2\right], \end{aligned} \tag{6.29}$$

where we have truncated the expansion to second order in coupling constants. The linear contribution can be simplified as follows

$$\mathcal{G}_0(\underline{\Delta}+a) = \mathcal{G}_0\underline{\Delta} + \mathcal{G}_0 a \mathrel{\widehat{=}} \mathcal{G}_0 a, \tag{6.30}$$

where we have used the equivalence relation defined in eq. (1.6). The matrix $\mathcal{G}_0\underline{\Delta}$ is off-diagonal and hence it becomes zero after taking the trace. $\mathcal{G}_0 a$ also falls out of the partition function in the end, since it's linear in fields [7]. The second-order terms are

$$(\mathcal{G}_0(\underline{\Delta}+a))^2 = \mathcal{G}_0\underline{\Delta}\mathcal{G}_0\underline{\Delta} + \mathcal{G}_0\underline{\Delta}\mathcal{G}_0 a + \mathcal{G}_0 a\mathcal{G}_0\underline{\Delta} + \mathcal{G}_0 a\mathcal{G}_0 a \mathrel{\widehat{=}} \mathcal{G}_0\underline{\Delta}\mathcal{G}_0\underline{\Delta} + \mathcal{G}_0 a\mathcal{G}_0 a,. \tag{6.31}$$

where we have used that $\mathcal{G}_0\underline{\Delta}\mathcal{G}_0 a$ and $\mathcal{G}_0 a\mathcal{G}_0\underline{\Delta}$ are off-diagonal. Hence the effective bosonic action in eq. (6.28) to second order in coupling constants can be written as

$$S_{\text{eff}} = S_{SC} + \delta S_A + \delta S_\Delta + S_0 \tag{6.32}$$

$$S_0 = \frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \sum_\nu \langle\nu|\,\mathrm{tr}\left[\ln\left(-\mathcal{G}_0^{-1}\right)\right]|\nu\rangle = \frac{1}{2}\,\mathrm{Tr}\left[\ln\left(-\mathcal{G}_0^{-1}\right)\right] \tag{6.33}$$

$$\delta S_A = -\frac{1}{4}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \sum_\nu \langle\nu|\,\mathrm{tr}\left[\mathcal{G}_0 a\mathcal{G}_0 a\right]|\nu\rangle \tag{6.34}$$

$$\delta S_\Delta = -\frac{1}{4}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \sum_\nu \langle\nu|\,\mathrm{tr}\left[\mathcal{G}_0\underline{\Delta}\mathcal{G}_0\underline{\Delta}\right]|\nu\rangle \tag{6.35}$$

The action in eq. (6.33) corresponds to an uncoupled fermionic sector which factorizes from the rest of the partition function in eq. (6.27). We will therefore neglect this contribution. Computing the matrix products in eq. (6.34) and eq. (6.35), we get the following linear terms

$$\mathcal{G}_0 a = \begin{pmatrix} \mathcal{G}_0^p & 0 \\ 0 & \mathcal{G}_0^h \end{pmatrix} \begin{pmatrix} -e\not{A} & 0 \\ 0 & e\not{A}^T \end{pmatrix} = \begin{pmatrix} -e\mathcal{G}_0^p\not{A} & 0 \\ 0 & e\mathcal{G}_0^h\not{A}^T \end{pmatrix} \tag{6.36}$$

$$\mathcal{G}_0 \underline{\underline{\Delta}} = \begin{pmatrix} \mathcal{G}_0^p & 0 \\ 0 & \mathcal{G}_0^h \end{pmatrix} \begin{pmatrix} 0 & \Delta\gamma_1 \\ \Delta^*\gamma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{G}_0^p\Delta\gamma_1 \\ \mathcal{G}_0^h\Delta^*\gamma_1 & 0 \end{pmatrix} \tag{6.37}$$

and consequently, we get the following quadratic terms

$$\begin{aligned} \mathcal{G}_0 a \mathcal{G}_0 a &= \begin{pmatrix} -e\mathcal{G}_0^p\not{A} & 0 \\ 0 & e\mathcal{G}_0^h\not{A}^T \end{pmatrix}^2 = \begin{pmatrix} e^2\mathcal{G}_0^p\not{A}\mathcal{G}_0^p\not{A} & 0 \\ 0 & e^2\mathcal{G}_0^h\not{A}^T\mathcal{G}_0^h\not{A}^T \end{pmatrix} \\ &\mathrel{\widehat{=}} e^2\mathcal{G}_0^p\not{A}\mathcal{G}_0^p\not{A} + e^2\mathcal{G}_0^h\not{A}^T\mathcal{G}_0^h\not{A}^T \end{aligned} \tag{6.38}$$

$$\begin{aligned} \mathcal{G}_0\underline{\underline{\Delta}}\mathcal{G}_0\underline{\underline{\Delta}} &= \begin{pmatrix} 0 & \mathcal{G}_0^p\Delta\gamma_1 \\ \mathcal{G}_0^h\Delta^*\gamma_1 & 0 \end{pmatrix}^2 = \begin{pmatrix} \mathcal{G}_0^p\Delta\gamma_1\mathcal{G}_0^h\Delta^*\gamma_1 & 0 \\ 0 & \mathcal{G}_0^h\Delta^*\gamma_1\mathcal{G}_0^p\Delta\gamma_1 \end{pmatrix} \\ &\mathrel{\widehat{=}} \begin{pmatrix} \mathcal{G}_0^p\Delta\gamma_1\mathcal{G}_0^h\gamma_1\Delta^* & 0 \\ 0 & \gamma_1\mathcal{G}_0^h\gamma_1\Delta^*\mathcal{G}_0^p\Delta \end{pmatrix} \\ &\mathrel{\widehat{=}} \mathcal{G}_0^p\Delta\gamma_1\mathcal{G}_0^h\gamma_1\Delta^* + \gamma_1\mathcal{G}_0^h\gamma_1\Delta^*\mathcal{G}_0^p\Delta \end{aligned} \tag{6.39}$$

where we have performed a cyclic permutation of the γ_1 matrices in eq. (6.39). The propagators can be expressed as

$$\begin{aligned} \mathcal{G}_0^p &= (i\not{\partial} + m)^{-1} \\ (i\not{\partial} + m)\cdot\mathcal{G}_0^p &= 1 \\ (-\partial^2 + m^2)\mathcal{G}_0^p &= (-i\not{\partial} + m) \\ \mathcal{G}_0^p &= \frac{i\not{\partial} - m}{\partial^2 - m^2} \end{aligned}$$

$$\begin{aligned} \mathcal{G}_0^h &= (i\not{\partial}^T - m)^{-1} \\ (i\not{\partial}^T - m)\cdot\mathcal{G}_0^h &= 1 \\ (-\partial^2 + m^2)\mathcal{G}_0^h &= (-i\not{\partial}^T - m) \\ \mathcal{G}_0^h &= \frac{i\not{\partial}^T + m}{\partial^2 - m^2} \end{aligned}$$

where we have used that

$$i\not{\partial}(-i\not{\partial}) = \partial_\mu\partial_\nu\gamma^\mu\gamma^\nu = -\partial_\mu\partial_\mu(\gamma^\mu)^2 = -\partial^2 \tag{6.40}$$

$$i\not{\partial}^T(-i\not{\partial}^T) = \partial_\mu\partial_\nu(\gamma^\mu)^T(\gamma^\nu)^T = -\partial_\mu\partial_\nu\delta^{\mu\nu} = -\partial^2, \tag{6.41}$$

which can be readily verified using eq. (A.4). We evaluate the traces over quantum numbers ν in eq. (6.34) and eq. (6.35) using the following fermionic Matsubara wave functions

$$\psi_{nk}(\tau, x) = \frac{1}{\sqrt{\beta}}\mathrm{e}^{ikx + i\omega_n\tau} = \frac{1}{\sqrt{\beta}}\mathrm{e}^{i\kappa\cdot r} \tag{6.42}$$

$$\kappa = (\omega_n, k) \quad \omega_n = \frac{(2n+1)\pi}{\beta} \quad r = (\tau, x) \tag{6.43}$$

where ω_n are fermionic Matsubara frequencies and $\nu = (n, k)$. These wave functions automatically satisfy the fermionic boundary conditions of eq. (2.17). Since we are in Euclidean space, we will perform the Fourier transform of the bosonic fields using the following convention

$$A(x) = \frac{1}{\beta}\sum_l \int \frac{\mathrm{d}^2 q}{(2\pi)^2} A(\nu_l, q)\mathrm{e}^{iqr+i\nu_l\tau} = \frac{1}{\beta}\sum_l \int \frac{\mathrm{d}^2 q}{(2\pi)^2} A(\xi)\mathrm{e}^{i\xi\cdot x} \tag{6.44}$$

$$\Delta(x) = \frac{1}{\beta}\sum_l \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \Delta(q)\mathrm{e}^{iqr+i\nu_l\tau} = \frac{1}{\beta}\sum_l \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \Delta(\xi)\mathrm{e}^{i\xi\cdot x} \tag{6.45}$$

$$\xi = (\nu_l, q) \quad x = (\tau, r) \tag{6.46}$$

where $v_l = \frac{2l\pi}{\beta}$ are bosonic Matsubara frequencies. Inserting the Matsubara wave functions of eq. (6.42) into eq. (6.34), we get the following

$$\begin{aligned}
\delta S_A &= -\frac{1}{4}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \sum_\nu \langle\nu|\,\mathrm{tr}\left[\mathcal{G}_0 a \mathcal{G}_0 a\right]|\nu\rangle \\
&= -\frac{e^2}{4}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\mathrm{e}^{-i\kappa\cdot x}\,\mathrm{tr}\left[\mathcal{G}_0^p \not{A}\mathcal{G}_0^p \not{A} + \mathcal{G}_0^h \not{A}^T \mathcal{G}_0^h \not{A}^T\right]\mathrm{e}^{i\kappa\cdot x} \\
&= -\frac{e^2}{4}\int \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{1}{\beta^2}\sum_{n,l}\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\,\mathrm{tr}\Big[\mathcal{G}_0^p(\kappa - i\mu)\not{A}(\xi)\mathcal{G}_0^p(\kappa-\xi-i\mu)\not{A}(-\xi) \\
&+ \mathcal{G}_0^h(\kappa+i\mu)\not{A}^T(\xi)\mathcal{G}_0^h(\kappa-\xi+i\mu)\not{A}^T(-\xi)\Big]
\end{aligned} \tag{6.47}$$

where we have introduced the notation $\kappa \pm i\mu = (\omega_n \pm i\mu, k)$ and defined the Fourier transformed Green's functions

$$\mathcal{G}_0^p(\kappa) = \frac{\not{\kappa} + m}{\kappa^2 + m^2} \qquad \mathcal{G}_0^h(\kappa) = \frac{\not{\kappa}^T - m}{\kappa^2 + m^2}. \tag{6.48}$$

We can simplify this expression by noting that

$$\begin{aligned}
&\mathcal{G}_0^h(\kappa+i\mu)\not{A}^T(\xi)\mathcal{G}_0^h(\kappa-\xi+i\mu)\not{A}^T(-\xi) \\
&= \frac{(\not{\kappa}+i\mu)^T - m}{(\kappa+i\mu)^2+m^2}\not{A}^T(\xi)\frac{(\not{\kappa}-\not{\xi}+i\mu)^T - m}{(\kappa-\xi+i\mu)^2+m^2}\not{A}^T(-\xi) \\
&= \left(\not{A}(-\xi)\frac{(\not{\kappa}-\not{\xi}+i\mu) - m}{(\kappa-\xi+i\mu)^2+m^2}\not{A}(\xi)\frac{(\not{\kappa}+i\mu) - m}{(\kappa+i\mu)^2+m^2}\right)^T \\
&\widehat{=}\ \not{A}(-\xi)\frac{(\not{\kappa}-\not{\xi}+i\mu) - m}{(\kappa-\xi+i\mu)^2+m^2}\not{A}(\xi)\frac{(\not{\kappa}+i\mu) - m}{(\kappa+i\mu)^2+m^2} \\
&\widehat{=}\ \frac{(\not{\kappa}+i\mu) - m}{(\kappa+i\mu)^2+m^2}\not{A}(-\xi)\frac{(\not{\kappa}-\not{\xi}+i\mu) - m}{(\kappa-\xi+i\mu)^2+m^2}\not{A}(\xi) \\
&= \frac{(\not{\kappa}-i\mu) + m}{(\kappa-i\mu)^2+m^2}\not{A}(\xi)\frac{(\not{\kappa}-\not{\xi}-i\mu) + m}{(\kappa-\xi-i\mu)^2+m^2}\not{A}(-\xi) \\
&= \mathcal{G}_0^p(\kappa-i\mu)\not{A}(\xi)\mathcal{G}_0^p(\kappa-\xi-i\mu)\not{A}(-\xi),
\end{aligned} \tag{6.49}$$

where we have rotated the momentum variables according to $\kappa \to -\kappa$ and $\xi \to -\xi$ and used that the trace is invariant under transposing and cyclic permutations. Thus, we end up with the following simplified expression for eq. (6.47)

$$
\begin{aligned}
\delta S_A &= -\frac{e^2}{2}\int \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{1}{\beta^2}\sum_{n,l}\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\,\mathrm{tr}\left[\mathcal{G}_0^p(\kappa - i\mu)\not{A}(\xi)\mathcal{G}_0^p(\kappa-\xi-i\mu)\not{A}\right] \\
&= -\frac{e^2}{2}\int \frac{\mathrm{d}^2 q}{(2\pi)^2}\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\frac{1}{\beta^2}\sum_{n,l}\mathrm{tr}\left[\frac{\not{\kappa}+m}{\kappa^2+m^2}\not{A}(\xi)\frac{(\not{\kappa}-\not{\xi})+m}{(\kappa-\xi)^2+m^2}\not{A}(-\xi)\right].
\end{aligned}
\tag{6.50}
$$

Next, we simplify the traces in eq. (6.35)

$$
\begin{aligned}
\delta S_\Delta &= -\frac{1}{4}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \sum_\nu \langle \nu|\,\mathrm{tr}\left[\mathcal{G}_0\underline{\Delta}\mathcal{G}_0\underline{\Delta}\right]|\nu\rangle \\
&= -\frac{1}{4}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\mathrm{e}^{-i\kappa\cdot x}\,\mathrm{tr}\left[\mathcal{G}_0^p\Delta\gamma_1\mathcal{G}_0^h\gamma_1\Delta^* + \gamma_1\mathcal{G}_0^h\gamma_1\Delta^*\mathcal{G}_0^p\Delta\right]\mathrm{e}^{i\kappa\cdot x} \\
&= -\frac{1}{4}\int \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{1}{\beta^2}\sum_{n,l}\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\,\mathrm{tr}\Big[\mathcal{G}_0^p(\kappa-i\mu)\Delta\gamma_1\mathcal{G}_0^h(\kappa-\xi+i\mu)\gamma_1\Delta^* \\
&\quad + \gamma_1\mathcal{G}_0^h(\kappa+i\mu)\gamma_1\Delta^*\mathcal{G}_0^p(\kappa-\xi-i\mu)\Delta\Big].
\end{aligned}
\tag{6.51}
$$

Using the following γ-matrix relation

$$
\gamma_1\bar{\gamma}^T\gamma_1 = \bar{\gamma},
\tag{6.52}
$$

which can be readily verified using eq. (A.4), we can simplify the last term in eq. (6.51) accordingly

$$
\begin{aligned}
\gamma_1\mathcal{G}_0^h(\kappa+i\mu)\gamma_1 &= \gamma_1\frac{\kappa^\mu\gamma_\mu^T + i\mu\gamma_0 - m}{(\kappa+i\mu)^2+m^2}\gamma_1 \\
&= \frac{\kappa^\mu\gamma_\mu + i\mu\gamma_0 + m}{(\kappa+i\mu)^2+m^2} \\
&= \mathcal{G}_0^p(\kappa+i\mu).
\end{aligned}
\tag{6.53}
$$

where we have used that $\gamma_0^T = \gamma_0$ and that $\gamma_1^2 = -1$. Consequently, we get the following

$$
\begin{aligned}
&\mathcal{G}_0^p(\kappa-i\mu)\Delta\gamma_1\mathcal{G}_0^h(\kappa-\xi+i\mu)\gamma_1\Delta^* + \gamma_1\mathcal{G}_0^h(\kappa+i\mu)\gamma_1\Delta^*\mathcal{G}_0^p(\kappa-\xi-i\mu)\Delta \\
&= \mathcal{G}_0^p(\kappa-i\mu)\Delta\mathcal{G}_0^p(\kappa-\xi+i\mu)\Delta^* + \mathcal{G}_0^p(\kappa+i\mu)\Delta^*\mathcal{G}_0^p(\kappa-\xi-i\mu)\Delta \\
&= \mathcal{G}_0^p(\kappa-i\mu)\Delta\mathcal{G}_0^p(\kappa-\xi+i\mu)\Delta^* + \mathcal{G}_0^p(\kappa+i\mu)\Delta\mathcal{G}_0^p(\kappa-\xi-i\mu)\Delta^*
\end{aligned}
\tag{6.54}
$$

Thus, we can write the Cooper boson contribution as

$$
\begin{aligned}
\delta S_\Delta &= -\frac{1}{4}\int \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{1}{\beta^2}\sum_{n,l}\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\,\mathrm{tr}\left[\mathcal{G}_0^p(\kappa-i\mu)\Delta\mathcal{G}_0^p(\kappa-\xi+i\mu)\Delta^* + (\mu\leftrightarrow-\mu)\right] \\
&= -\frac{1}{4}\int \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{1}{\beta^2}\sum_{n,l}\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\,\mathrm{tr}\Big[\frac{(\not{\kappa}-i\mu)+m}{(\kappa-i\mu)^2+m^2}\Delta(\xi)\frac{(\not{\kappa}-\not{\xi}+i\mu)+m}{(\kappa-\xi+i\mu)^2+m^2}\Delta^* \\
&\quad + (\mu\leftrightarrow-\mu)\Big]
\end{aligned}
\tag{6.55}
$$

In order to calculate the matrix traces in eq. (6.50) and eq. (6.55), we use the trace relations derived in section A.2, i.e.,

$$\operatorname{tr}(\gamma_\mu\gamma_\nu) = -2\delta_{\mu\nu} \tag{6.56}$$
$$\operatorname{tr}(\gamma_\mu\gamma_\nu\gamma_\lambda) = -2\varepsilon_{\mu\nu\lambda} \tag{6.57}$$
$$\operatorname{tr}(\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho) = 2\delta_{\mu\nu}\delta_{\lambda\rho} - 2\delta_{\mu\lambda}\delta_{\nu\rho} + 2\delta_{\mu\rho}\delta_{\nu\lambda}. \tag{6.58}$$

Thus, the trace in the gauge field contribution in eq. (6.50) evaluates to

$$\begin{aligned}
&\operatorname{tr}\Big[(\kappa^\mu\gamma_\mu + m)A^\nu(\xi)\gamma_\nu((\kappa-\xi)^\lambda\gamma_\lambda + m)A^\rho(-\xi)\gamma_\rho\Big] \\
&= \kappa^\mu A^\nu(\kappa-\xi)^\lambda A^\rho \operatorname{tr}[\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho] + m\kappa^\mu A^\nu A^\rho \operatorname{tr}[\gamma_\mu\gamma_\nu\gamma_\rho] + mA^\nu(\kappa-\xi)^\lambda A^\rho \operatorname{tr}[\gamma_\nu\gamma_\lambda\gamma_\rho] \\
&+ m^2 A^\nu A^\rho \operatorname{tr}[\gamma_\nu\gamma_\rho] \\
&= \kappa^\mu A^\nu(\kappa-\xi)^\lambda A^\rho(2\delta_{\mu\nu}\delta_{\lambda\rho} - 2\delta_{\mu\lambda}\delta_{\nu\rho} + 2\delta_{\mu\rho}\delta_{\nu\lambda}) - 2m\kappa^\mu A^\nu A^\rho\varepsilon_{\mu\nu\rho} - 2mA^\nu(\kappa-\xi)^\lambda A^\rho\varepsilon_{\nu\lambda\rho} \\
&- 2m^2A^\nu A^\rho\delta_{\nu\rho} \\
&= A^\mu(\xi)\Big[2\kappa_\mu(\kappa-\xi)_\nu + 2\kappa_\nu(\kappa-\xi)_\mu + 2im\varepsilon_{\mu\lambda\nu}(-i\xi^\lambda) - 2\delta_{\mu\nu}(m^2 + \kappa^\lambda(\kappa-\xi)_\lambda)\Big] A^\nu(-\xi),
\end{aligned} \tag{6.59}$$

where we have omitted the chemical potential terms and denominators for notational convenience. In the last line, we permuted the indices in the leftmost Levi-Civita tensor, resulting in the following anti-symmetric contribution

$$\begin{aligned}
-2m\kappa^\mu A^\nu A^\rho\varepsilon_{\mu\nu\rho} - 2mA^\nu(\kappa-\xi)^\lambda A^\rho\varepsilon_{\nu\lambda\rho} &= -2mA^\nu\kappa^\mu A^\rho\varepsilon_{\nu\mu\rho} - 2mA^\nu(\kappa-\xi)^\lambda A^\rho\varepsilon_{\nu\lambda\rho} \\
&= 2mA^\nu\xi^\lambda A^\rho\varepsilon_{\nu\lambda\rho} \\
&= 2imA^\mu(-i\xi)^\lambda A^\nu\varepsilon_{\mu\lambda\nu}.
\end{aligned} \tag{6.60}$$

Similarly, the trace in the Cooper boson contribution in eq. (6.55) evaluates to

$$\begin{aligned}
&\operatorname{tr}\Big[((\kappa \pm i\mu)^\mu\gamma_\mu + m)\Delta(\xi)((\kappa-\xi\mp i\mu)^\lambda\gamma_\lambda + m)\Delta^*(\xi)\Big] \\
&= 2(\kappa\pm i\mu)^\mu(\kappa-\xi\mp i\mu)^\lambda \operatorname{tr}[\gamma_\mu\gamma_\lambda]\,\Delta^2(\xi) + 2m^2\operatorname{tr}[1]\Delta^2(\xi) \\
&= -2((\kappa\pm i\mu)^\mu(\kappa-\xi\mp i\mu)_\mu - m^2)\Delta^2(\xi),
\end{aligned} \tag{6.61}$$

where we have once again omitted the denominators for notational purposes. Inserting eq. (6.59) and eq. (6.61) into eq. (6.50) and eq. (6.55) respectively, we get the following simplified expressions for the gauge field sector

$$\delta S_A = -\int \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{1}{\beta} \sum_l A^\mu(\xi) \Pi_{\mu\nu}(\xi) A^\nu(-\xi) \tag{6.62}$$

$$\begin{aligned}\Pi_{\mu\nu}(\xi) &= \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\beta} \sum_n \frac{ime^2 \varepsilon_{\mu\lambda\nu}(-i\xi^\lambda) - e^2 \Sigma_{\mu\nu}}{((\kappa - i\mu)^2 + m^2)((\kappa - \xi - i\mu)^2 + m^2)} \\ &= \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\beta} \sum_n \frac{ime^2 \varepsilon_{\mu\lambda\nu}(-i\xi^\lambda) - e^2 \Sigma_{\mu\nu}}{(\kappa^2 + m^2)((\kappa - \xi)^2 + m^2)}\end{aligned} \tag{6.63}$$

$$\begin{aligned}\Sigma_{\mu\nu}(\xi) &= \delta_{\mu\nu}(m^2 + (\kappa - i\mu)^\lambda(\kappa - \xi - i\mu)_\lambda) - 2(\kappa - i\mu)_\mu(\kappa - i\mu)_\nu \\ &+ (\kappa - i\mu)_\mu \xi_\nu + (\kappa - i\mu)_\nu \xi_\mu \\ &= \delta_{\mu\nu}(m^2 + \kappa^\lambda(\kappa - \xi)_\lambda) - 2\kappa_\mu\kappa_\nu + \kappa_\mu\xi_\nu + \kappa_\nu\xi_\mu\end{aligned} \tag{6.64}$$

where we have absorbed the chemical potential term into κ. Similarly, we get the following simplified expression for the Cooper boson sector

$$\delta S_\Delta = -\int \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{1}{\beta} \sum_l \Delta(\xi)\Gamma(\xi)\Delta^*(\xi) \tag{6.65}$$

$$\Gamma(\xi) = -\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\beta} \sum_n \frac{1}{2} \frac{(\kappa - i\mu)^\mu(\kappa - \xi + i\mu)_\mu - m^2}{((\kappa - i\mu)^2 + m^2)((\kappa - \xi + i\mu)^2 + m^2)} + (\mu \leftrightarrow -\mu). \tag{6.66}$$

The physical processes described by eqs. (6.62) to (6.66) can be pictorially represented by the Feynman diagrams in fig. 6.2. To second order in coupling constants, the effect of the topological surface states reduces to a pair of one-loop renormalizations of the bosonic fields A_μ and Δ.

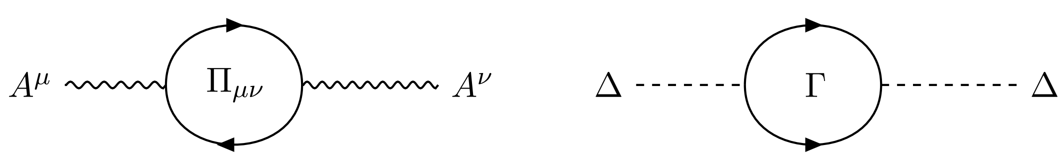

Figure 6.2: One-loop renormalization of the gauge field and scalar Cooper pairs.

6.4 Gauge field sector

6.4.1 Chern-Simons term

We start by calculating the contribution from the first term in eq. (6.63)

$$S_{CS} = -\int \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{1}{\beta} \sum_l i\lambda_{CS} \varepsilon^{\mu\lambda\nu} A_\mu(\xi)(-i\xi_\lambda) A_\nu(-\xi) \tag{6.67}$$

$$\lambda_{CS} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\beta} \sum_n \frac{me^2}{(\kappa^2 + m^2)((\kappa - \xi)^2 + m^2)}. \tag{6.68}$$

To second order in momenta, we can drop the ξ dependence of the denominator, since the numerator of eq. (6.67) is linear in ξ. Hence, we are left with

$$\begin{aligned}
\lambda_{CS} &= \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\beta} \sum_n \frac{me^2}{(\kappa^2+m^2)^2} \\
&= \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\beta} \sum_n \frac{me^2}{(v_F^2 k^2 + (\omega_n - i\mu)^2 + m^2)^2} \\
&= \frac{me^2}{v_F^2} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\beta} \sum_n \frac{1}{(k^2 + (\omega_n - i\mu)^2 + m^2)^2}.
\end{aligned} \tag{6.69}$$

where we have reinstated the chemical potential and extracted the v_F factors by performing a linear change of variable. Using eq. (C.2), we can perform the k-integration directly

$$\begin{aligned}
\lambda_{CS} &= \frac{me^2}{4\pi v_F^2} \frac{1}{\beta} \sum_n \frac{1}{(\omega_n - i\mu)^2 + m^2} \\
&= \frac{me^2}{4\pi v_F^2} \frac{1}{\beta} \sum_n \frac{1}{-(i\omega_n + \mu)^2 + m^2}.
\end{aligned} \tag{6.70}$$

Using eq. (B.13), the sum over Matsubara frequencies evaluates to

$$\begin{aligned}
\lambda_{CS} &= \frac{me^2}{4\pi v_F^2} \frac{1}{2|m|} \frac{\sinh \beta|m|}{\cosh \beta|m| + \cosh \beta\mu} \\
&= \frac{e^2 \mathrm{sgn}(m)}{8\pi v_F^2} \frac{\sinh \beta|m|}{\cosh \beta|m| + \cosh \beta\mu}.
\end{aligned} \tag{6.71}$$

Inserting eq. (6.71) into eq. (6.67) and performing an inverse Fourier transform, we get the following contribution

$$S_{CS} = -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r i \lambda_{CS} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \tag{6.72}$$

6.4.2 Deriving the tensor structure of $\Pi_{\mu\nu}(\xi)$

The vacuum polarization amplitude of eq. (6.63) fulfills the Ward identity [24], i.e.,

$$\xi_\mu \Pi_{\mu\nu}(\xi) = 0. \tag{6.73}$$

This implies that the remaining contributions from eq. (6.63) can be expressed in terms of the following tensors [5], [24]

$$\Pi_{\mu\nu}(\xi) = F P^L_{\mu\nu} + G P^T_{\mu\nu} \tag{6.74}$$

$$P^T_{ij} = \left(\delta_{ij} - \frac{q_i q_j}{q^2}\right) \tag{6.75}$$

$$P^T_{0\nu} = P^T_{\mu 0} = P^T_{00} = 0 \tag{6.76}$$

$$P^L_{\mu\nu} = \left(\delta_{\mu\nu} - \frac{\xi_\mu \xi_\nu}{\xi^2}\right) - P^T_{\mu\nu}, \tag{6.77}$$

where F and G are constants of proportionality. In $(2+1)$-dimensions, we have that

$$\text{tr}\,P^T_{\mu\nu} = \text{tr}\left(\delta_{ij} - \frac{q_i q_j}{q^2}\right) = 1 \tag{6.78}$$

$$\text{tr}\,P^L_{\mu\nu} = \text{tr}\left(\delta_{\mu\nu} - \frac{\xi_\mu \xi_\nu}{\xi^2}\right) - \text{tr}\,P^T_{\mu\nu} = 1 \tag{6.79}$$

where we have used the following relations

$$\text{tr}(\xi_\mu \xi_\nu) = \xi^2 \tag{6.80}$$

$$\text{tr}(q_i q_j) = q^2, \tag{6.81}$$

which can be readily verified. Combining the above results, we get the following equations for F and G

$$\Pi_{\mu\mu}(\xi) = F + G \tag{6.82}$$

$$\Pi_{00}(\xi) = \left(1 - \frac{\nu_l^2}{\xi^2}\right) F = \frac{q^2}{\xi^2} F, \tag{6.83}$$

which in turn implies that

$$F = \frac{\xi^2}{q^2}\Pi_{00}(\xi) \tag{6.84}$$

$$G = \Pi_{\mu\mu}(\xi) - \frac{\xi^2}{q^2}\Pi_{00}(\xi). \tag{6.85}$$

Thus, in order to calculate the contributions from the remaining terms in eq. (6.63), we only need to consider $\Pi_{\mu\mu}(\xi)$ and $\Pi_{00}(\xi)$.

6.4.3 Evaluation of $\Pi_{\mu\mu}(\xi)$ and $\Pi_{00}(\xi)$

The trace of the vacuum polarization tensor in eq. (6.63) can be written as

$$\begin{aligned}\Pi_{\mu\mu}(\xi) &= -e^2 \frac{1}{\beta} \sum_n \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{\text{tr}\left[\delta_{\mu\nu}(m^2 + \kappa\cdot(\kappa - \xi)) - 2\kappa_\mu\kappa_\nu + \kappa_\nu\xi_\mu + \kappa_\mu\xi_\nu\right]}{(\kappa^2 + m^2)((\kappa-\xi)^2 + m^2)} \\ &= -e^2 \frac{1}{\beta} \sum_n \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{3m^2 + \kappa^2 - \kappa\xi}{(\kappa^2 + m^2)((\kappa-\xi)^2 + m^2)},\end{aligned} \tag{6.86}$$

where we have once again used eq. (6.80). By using the following relation

$$\kappa^2 - \kappa\xi = \frac{1}{2}[(\kappa^2 + m^2) + (\kappa - \xi)^2 + m^2] - m^2 - \frac{1}{2}\xi^2 \tag{6.87}$$

and performing a linear change of variables[1], we can write eq. (6.86) as follows

[1] A linear change of variables in terms of Matsubara frequencies is a non-trivial procedure which is discussed in section B.2

$$\Pi_{\mu\mu}(\xi) = -e^2\Big(2m^2 - \frac{\xi^2}{2}\Big)\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)}$$
$$- e^2\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\kappa^2+m^2}. \tag{6.88}$$

These integrals are treated in detail in section E.1. In the long wavelength limit, they evaluate to

$$\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)} = c_1 + q^2c_2 + \mathcal{O}\big(q^3\big) \tag{6.89}$$
$$\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\kappa^2+m^2} = c_3 \tag{6.90}$$

where c_1, c_2, and c_3 are defined in eq. (E.10), eq. (E.11), and eq. (E.19), respectively. Inserting eq. (6.89) and eq. (6.90) into eq. (6.88), we get

$$\Pi_{\mu\mu}(\xi) = -e^2\left(2m^2 - \frac{q^2}{2}\right)(c_1 + q^2c_2) - e^2c_3 = (-2m^2e^2c_1 - e^2c_3)$$
$$+ q^2\left(-2m^2e^2c_2 + \frac{e^2c_1}{2}\right) + \mathcal{O}\big(q^3, \nu_l^2\big), \tag{6.91}$$

The temporal component of eq. (6.63) can be written as

$$\Pi_{00}(\xi) = -e^2\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\big(m^2+\kappa^2-2(\omega_n-i\mu)^2+2(\omega_n-i\mu)v_l\big)}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)}$$
$$= -e^2\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\kappa^2+m^2} + 2e^2\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{(\omega_n-i\mu)^2}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)}$$
$$- 2e^2\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{(\omega_n-i\mu)v_l}{(\kappa^2+m^2)^2}, \tag{6.92}$$

where we have performed a linear change of variables in the first term and set $\xi \to 0$ in the denominator of the last term since the numerator is linear in ν_l. Here we immediately see that the first term corresponds to eq. (6.90). The last two integrals are treated in detail in section E.1 and evaluate to

$$2e^2\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{(\omega_n-i\mu)^2}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)} = -2m^2e^2(c_1+c_2q^2) + 2e^2c_4q^2 \tag{6.93}$$
$$-2e^2\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{(\omega_n-i\mu)\nu_l}{(\kappa^2+m^2)^2} = -\lambda_{\tau,\phi}i\nu_l, \tag{6.94}$$

where the constants c_1, c_2, and c_4 are defined in eq. (E.10), eq. (E.11) and eq. (E.26) respectively. The temporal correction coefficient of eq. (6.94) evaluates to

$$\lambda_{\tau,\phi} = \frac{e^2}{4\pi v_F^2}\frac{\sinh\beta\mu}{\cosh\beta\mu + \cosh\beta|m|}. \tag{6.95}$$

Hence, we get the following

$$\begin{aligned}\Pi_{00}(\xi) &= -e^2c_3 - 2e^2m^2(c_1 + c_2q^2) + 2e^2c_4q^2 - \lambda_{\tau,\phi}i\nu_l \\ &= \left(-e^2c_3 - 2m^2e^2c_1\right) + q^2\left(2e^2c_4 - 2m^2e^2c_2\right) - \lambda_{\tau,\phi}i\nu_l + \mathcal{O}\left(q^3, \nu_l^2\right)\end{aligned} \tag{6.96}$$

6.4.4 Maxwell sector renormalization

Inserting eq. (6.96) and eq. (6.91) into eq. (6.84) and eq. (6.85), we get the following expressions for F and G

$$\begin{aligned}F &= \frac{\xi^2}{q^2}\Pi_{00} = \frac{\xi^2}{q^2}\left(\left(-e^2c_3 - 2m^2e^2c_1\right) + q^2\left(2e^2c_4 - 2m^2e^2c_2\right) - \lambda_{\tau,\phi}i\nu_l\right) \\ &= \left(-e^2c_3 - 2m^2e^2c_1\right) + q^2\left(2e^2c_4 - 2m^2e^2c_2\right)\end{aligned} \tag{6.97}$$

$$\begin{aligned}G &= \Pi_{\mu\mu} - F = (-2m^2e^2c_1 - e^2c_3) + q^2\left(-2m^2e^2c_2 + \frac{e^2c_1}{2}\right) \\ &- \left(-e^2c_3 - 2m^2e^2c_1\right) - q^2\left(2e^2c_4 - 2m^2e^2c_2\right) \\ &= q^2\left(\frac{e^2c_1}{2} - 2e^2c_4\right).\end{aligned} \tag{6.98}$$

Hence the longitudinal part of eq. (6.74) in the long wavelength becomes

$$\begin{aligned}FP^L_{\mu\nu} &= (\left(-e^2c_3 - 2m^2e^2c_1\right) + q^2\left(2e^2c_4 - 2m^2e^2c_2\right))P^L_{\mu\nu} \\ &= \left(\left(-e^2c_3 - 2m^2e^2c_1\right) + q^2\left(2e^2c_4 - 2m^2e^2c_2\right)\right)\delta_{00}\end{aligned} \tag{6.99}$$

and similarly for the transversal part

$$\begin{aligned}GP^T_{\mu\nu} &= q^2\left(\frac{e^2c_1}{2} - 2e^2c_4\right)\left(\delta_{ij} - \frac{q_iq_j}{q^2}\right) \\ &= \left(\frac{e^2c_1}{2} - 2e^2c_4\right)\left(q^2\delta_{ij} - q_iq_j\right).\end{aligned} \tag{6.100}$$

Inserting these expressions into eq. (6.74) we get the following contribution from eq. (6.63)

$$S_M = -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2r \left(m_{\mathrm{el}}^2\phi^2 + \lambda_{\nabla,\mathrm{el}}(\boldsymbol{\nabla}\phi)^2 + \lambda_{\nabla,m}(\boldsymbol{\nabla}\times\bar{A})_z^2\right) \tag{6.101}$$

where we have used the following vector identity

$$(iq)(-iq)(\bar{A}(q)\cdot\bar{A}(-q)) - (iq\cdot\bar{A}(q))(-iq\cdot\bar{A}(-q)) = (iq\times\bar{A}(q))(-iq\times\bar{A}(-q)), \tag{6.102}$$

performed an inverse Fourier transform and defined the following coupling constants

$$
\begin{aligned}
m_{\text{el}}^2 &= \left(-e^2c_3 - 2m^2e^2c_1\right) = \frac{e^2}{4\pi\beta v_F^2}\ln(\cosh\beta|m| + \cosh\beta\mu) \\
&\quad - \frac{e^2|m|}{4\pi v_F^2}\frac{\sinh\beta|m|}{\cosh\beta\mu + \cosh\beta|m|} && (6.103) \\
\lambda_{\nabla,\text{el}} &= \left(2e^2c_4 - 2m^2e^2c_2\right) = \frac{-e^2}{48\pi|m|}\frac{\sinh\beta|m|}{\cosh\beta|m| + \cosh\beta\mu} \\
&\quad - \frac{\beta e^2}{192}\left(\frac{1}{\cosh^2\frac{\beta(|m|-\mu)}{2}} + \frac{1}{\cosh^2\frac{\beta(|m|+\mu)}{2}}\right) && (6.104) \\
\lambda_{\nabla,m} &= \left(\frac{e^2c_1}{2} - 2e^2c_4\right) = \frac{5e^2}{48\pi|m|}\frac{\sinh\beta|m|}{\cosh\beta|m| + \cosh\beta\mu}, && (6.105)
\end{aligned}
$$

where we have reinstated a factor v_F for each gradient.

6.4.5 Full theory of the gauge field sector

Combing eq. (6.72) and eq. (6.101) we can write the full gauge field contribution of eq. (6.62) as follows

$$
\begin{aligned}
\delta S_A &= S_{CS} + S_M \\
&= -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2r \Big(i\lambda_{CS}\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + m_{\text{el}}^2\phi^2 + \lambda_{\text{el}}(\boldsymbol{\nabla}\phi)^2 + \lambda_m(\boldsymbol{\nabla}\times\bar{A})_z^2\Big), && (6.106)
\end{aligned}
$$

where λ_{CS} is a Chern-Simons coefficient, $\lambda_{\text{el}} = \lambda + \lambda_{\nabla,\text{el}}$ and $\lambda_m = \lambda + \lambda_{\nabla,m}$ are the renormalized Maxwell sector coefficients and m_{el}^2 is a mass term of the electric potential ϕ.

6.5 Cooper boson sector

We start with $\Gamma(\xi)$ written in the following way

$$
\begin{aligned}
\Gamma(\xi) &= -\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta}\sum_n \frac{1}{2}\frac{(\kappa - i\mu)^\mu(\kappa - \xi + i\mu)_\mu - m^2}{((\kappa - i\mu)^2 + m^2)((\kappa - \xi - i\mu)^2 + m^2)} + (\mu \leftrightarrow -\mu) && (6.107) \\
&= -\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta}\sum_n \frac{(\kappa - i\mu)^\mu(\kappa - \xi + i\mu)_\mu - m^2}{((\kappa - i\mu)^2 + m^2)((\kappa - \xi - i\mu)^2 + m^2)}, && (6.108)
\end{aligned}
$$

where we have combined the two contributions by rotating the variables according to $\kappa \to -\kappa$ and $\xi \to -\xi$ in the second term. This in turn changes the sign of the momentum dependence of $\Delta(\xi)$ in eq. (6.66). However, the result is the same after performing the inverse Fourier transform. Since we are working to linear order in temporal momentum, cross terms between spatial- and temporal momentum components are always negligible. Hence, we can divide eq. (6.108) into the following contributions

$$\Gamma(0,q) = -\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta}\sum_n \frac{\omega_n^2 + v_F k(v_F k - q) - (m^2-\mu^2)}{(\omega_n^2 + v_F^2k^2 - 2i\mu\omega_n + m^2 - \mu^2)(\omega_n^2 + (v_F k - q)^2 + 2i\mu\omega_n + m^2 - \mu^2)} \tag{6.109}$$

$$\begin{aligned}\Gamma(\nu_l,0) &= -\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta}\sum_n \frac{-(i\omega_n+\mu)(i\omega_n - i\nu_l - \mu) + \epsilon_k^2 - 2m^2}{(i\omega_n+\mu)^2 + \epsilon_k^2)((i\omega_n - i\nu_l - \mu)^2 + \epsilon_k^2)} \\ &= -\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta}\sum_n \frac{-(i\omega_n+\mu)(i\omega_n - i\nu_l - \mu) + \epsilon_k^2 - 2m^2}{(i\omega_n+\mu+\epsilon_k)(i\omega_n+\mu-\epsilon_k)((i\omega_n - i\nu_l - \mu+\epsilon_k)(i\omega_n - i\nu_l - \mu-\epsilon_k)},\end{aligned} \tag{6.110}$$

where we have defined $\epsilon_k = \sqrt{v_F^2k^2 + m^2}$. The former expression corresponds to a renormalization of the gradient- and mass term of the Cooper boson sector, whereas the latter induces a time-dependent term in eq. (6.1).

6.5.1 Gradient- and mass-renormalization terms

Using the results derived in section E.2, we get the following contribution to the Cooper boson sector from eq. (6.109)

$$\begin{aligned}&-\int \frac{\mathrm{d}^2q}{(2\pi)^2}\frac{1}{\beta}\sum_l (\Theta_m - q^2\Theta_\nabla)\Delta^*(\xi)\Delta(\xi) && (6.111)\\ &= -\int \frac{\mathrm{d}^2q}{(2\pi)^2}\frac{1}{\beta}\sum_l (\Theta_m + (iq)^2\Theta_\nabla)\Delta^*(\xi)\Delta(\xi), && (6.112)\end{aligned}$$

where the coupling constants Θ_m and Θ_∇ are defined in eq. (E.41) and eq. (E.42), respectively. By performing an inverse Fourier transform, we can combine these contributions into the following action

$$\begin{aligned}S_q &= -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2r \left(\eta_\nabla|\boldsymbol{\nabla}\varphi|^2 + \eta_m|\varphi|^2\right) && (6.113)\\ \eta_\nabla &= v_F^2|g|^2\Theta_\nabla(\kappa) && (6.114)\\ \eta_m &= |g|^2\Theta_m(\kappa), && (6.115)\end{aligned}$$

where we have reinstated the coupling constant g and a factor v_F for each gradient.

6.5.2 Time-dependent terms

Because of the periodicity of the Fermi-Dirac distribution in terms of bosonic Matsubara frequencies, $f(i\omega_n + i\nu_l) = f(i\omega_n)$, it is important to perform the fermionic Matsubara sum and do an analytic continuation into real-time before performing any series expansion in terms of temporal momentum. Eq. (6.110) is evaluated in detail in section E.2 and we end up with the following first-order correction

$$-\int \frac{\mathrm{d}^2q}{(2\pi)^2}\int \frac{\mathrm{d}\omega}{2\pi}\tilde{\eta}_t\Delta^*\omega\Delta = -\int \frac{\mathrm{d}^2q}{(2\pi)^2}\int \frac{\mathrm{d}\omega}{2\pi}(-i\tilde{\eta}_t)\Delta^* i\omega\Delta \tag{6.116}$$

$$\begin{aligned}\tilde{\eta}_t = \frac{1}{4\pi v_F^2}\Big(\mathrm{P}\int_{|m|}^{\infty}\frac{\mathrm{d}\epsilon}{\epsilon}\Big[\Big(\frac{\epsilon^2-m^2}{4\epsilon_+^2}+\frac{m^2}{4\mu^2}\Big)\tanh\frac{\beta\epsilon_+}{2}+(\epsilon_+\leftrightarrow\epsilon_-)\Big]\\ + i\pi\beta\left[\Theta(\mu-|m|)-\Theta(-\mu-|m|)\right]\frac{\mu^2-m^2}{8\mu}\Big)\end{aligned} \tag{6.117}$$

where $\epsilon_\pm = \pm\epsilon - \mu$. This integral is superficially convergent and solvable under certain assumptions on μ. Fourier transforming back into real space, we get the following contribution to the action

$$S_\nu = i\int \mathrm{d}t\int \mathrm{d}^2r\,\eta_t\varphi^*\partial_t\varphi \tag{6.118}$$

$$\eta_t = -i|g|^2\tilde{\eta}_t \tag{6.119}$$

where we have reinstated the coupling constant g from the Cooper field Δ. Using eq. (2.13), we can Wick rotate this expression into imaginary time formalism as follows

$$S_\nu = -\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^2r\,\eta_\tau\varphi^*\partial_\tau\varphi \tag{6.120}$$

$$\eta_\tau = |g|^2\tilde{\eta}_t. \tag{6.121}$$

6.5.3 Full theory of the Cooper boson sector

Combining eq. (6.113) and eq. (6.120), we get the following expression for the Cooper boson contributions

$$\begin{aligned}\delta S_\Delta &= S_q + S_\nu\\ &= -\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^2r\left(\eta_\tau\varphi^*\partial_\tau\varphi + \eta_\nabla|\boldsymbol{\nabla}\varphi|^2+\eta_m|\varphi|^2\right),\end{aligned} \tag{6.122}$$

The first term is a dynamical term induced by the fermions. The second- and third term renormalize the gradient term and α-coefficient in eq. (6.1), respectively.

Chapter 7

Ferromagnetic impurities and their coupling to the fermionic sector

Thus far we have only considered the effects of magnetic perturbations in the form of a mass term due to some form of coupling between the fermions and a perpendicular magnetization. By adding a layer of ferromagnetically aligned magnetic impurities to the heterostructure, we get the desired fermionic mass term, in addition to several other exchange couplings between the fermions and the magnetic impurities. The magnetic impurities are described by the Lagrangian in eq. (4.61)

$$\mathcal{L}_{FI} = b \cdot \partial_t n - \frac{\kappa}{2} \left(\boldsymbol{\nabla} n\right)^2 - \frac{m^2}{2} n^2 \tag{7.1}$$

where $n = (n_x, n_y, n_z) = (\bar{n}, n_z)$ are vector fields, κ and m are exchange coupling constants, and $\bar{b}$ is a Berry phase with the defining property

$$\varepsilon_{ijk} \frac{\partial b_k}{\partial n_j} = \frac{n_i}{n^2} \tag{7.2}$$

7.1 Fermionic Lagrangian with spin-spin exchange couplings

The magnetic impurities interact with the surface fermions in eq. (6.3) via a spin-spin exchange interaction term, which in our notation takes the form

$$\mathcal{L}_{TI-FM} = \Psi^\dagger \left[J_\parallel \bar{n} \cdot \bar{\sigma} + J_\perp n_z \sigma^z \right] \Psi, \tag{7.3}$$

where $\bar{\sigma} = (\sigma^x, \sigma^y)$ and $J_\parallel$ and $J_\perp$ are coupling constants. If the spins of the magnetic impurities acquire a finite expectation value of the form $n_0 = (0, 0, \langle n_z \rangle)$, eq. (7.3) reduces to

$$\begin{aligned} \mathcal{L}_{TI-FM} &= \Psi^\dagger \left[J_\perp \left\langle n_z \right\rangle \sigma^z \right] \Psi \\ &\equiv \Psi^\dagger m \sigma^z \Psi. \end{aligned} \tag{7.4}$$

which corresponds to the mass term in eq. (6.3). In the following, we will assume that the spins of the magnetic impurities are aligned such that the mean-field contribution from the spin-spin exchange coupling is equal to eq. (7.4) and that the field n corresponds to fluctuations above this ground state. Furthermore, we will absorb the coupling constants of eq. (7.3) into the fields for notational purposes. Substituting eq. (7.3) into the fermionic sector of eq. (6.5), we get the following fermionic Lagrangian

$$\begin{aligned}\mathcal{L}_F = \Psi^\dagger \left(i\partial_t - e\phi + (-iv_F\boldsymbol{\nabla} - ev_F\bar{A} + \bar{n})\cdot\bar{\sigma} - \mu + (m+n_z)\sigma^z\right)\Psi \\ + g\varphi\psi_\uparrow^\dagger\psi_\downarrow^\dagger + g^*\varphi^*\psi_\downarrow\psi_\uparrow.\end{aligned} \tag{7.5}$$

Here we see that the in-plane fluctuations $\bar{n}$ can be absorbed into the gauge field A_μ, leaving us with the effective gauge field a_μ defined as follows

$$a_0 = \phi \quad \bar{a} = v_F\bar{A} - \frac{\bar{n}}{e} \tag{7.6}$$

$$a_\mu = (a_0, -\bar{a}). \tag{7.7}$$

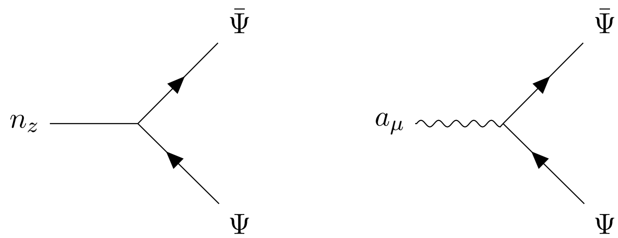

Figure 7.1: Feynman diagrams illustrating the effective interactions due to the spin-spin exchange term in eq. (7.3)

However, the perpendicular fluctuations n_z has to be treated as a separate scalar field. Substituting $m \to m + n_z$ and $A_\mu \to a_\mu$ into eq. (6.15), we immediately get the following Lagrangian in imaginary time formalism

$$\mathcal{L}_F = \bar{\Psi}(i\not{D} + m + n_z)\Psi + \Delta\psi_\uparrow^\dagger\psi_\downarrow^\dagger + \Delta^*\psi_\downarrow\psi_\uparrow. \tag{7.8}$$

where the covariant derivative is in terms of the effective gauge field a_μ instead of the pure gauge field A_μ.

7.2 Including the ferromagnetic impurities in the effective partition function

By including the magnetic sector in eq. (7.1) and the spin-spin exchange couplings in eq. (7.3) into our system, we get the following partition function

$$\begin{aligned}Z &= \int \mathcal{D}\varphi\mathcal{D}\varphi^*\mathcal{D}A_\mu\mathcal{D}n\mathcal{D}\bar{\Phi}\mathcal{D}\Phi \mathrm{e}^{S} \\ &= \int \mathcal{D}\varphi\mathcal{D}\varphi^*\mathcal{D}A_\mu\mathcal{D}n \mathrm{e}^{S_{SC}+S_{FI}} \int \mathcal{D}\bar{\Phi}\mathcal{D}\Phi \mathrm{e}^{S_F}\end{aligned} \tag{7.9}$$

with the following actions

$$S_{SC} = -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \mathcal{L}_{SC} \tag{7.10}$$

$$S_{FI} = -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \mathcal{L}_{FI} \tag{7.11}$$

$$S_F = \frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \bar{\Phi}\mathcal{G}^{-1}\Phi = -\frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \bar{\Phi}(-\mathcal{G}^{-1})\Phi. \tag{7.12}$$

In this case, the inverse Greens function $\mathcal{G}^{-1}$ is defined as follows

$$\mathcal{G}^{-1} = \mathcal{G}_0^{-1} + \underline{\Delta} + a + \underline{\mathrm{n}}_z = \mathcal{G}_0^{-1}\left(1 + \mathcal{G}_0(\underline{\Delta} + a + \underline{\mathrm{n}}_z)\right), \tag{7.13}$$

where the matrix a contains a_μ instead of A_μ compared to eq. (6.20). In terms of the four-component spinor basis defined in eq. (6.17) and eq. (6.18), the matrix $\underline{\mathrm{n}}_z$ takes the form

$$\underline{\mathrm{n}}_z = \begin{pmatrix} n_z & 0 \\ 0 & -n_z^T \end{pmatrix} = \begin{pmatrix} n_z & 0 \\ 0 & -n_z \end{pmatrix}. \tag{7.14}$$

After integrating out the fermions, we arrive at the following bosonic partition function

$$Z_B = \int \mathcal{D}\varphi\mathcal{D}\varphi^*\mathcal{D}A_\mu\mathcal{D}n \mathrm{e}^{S_B} \tag{7.15}$$

$$\begin{aligned} S_B &= S_{SC} + S_{FI} + \frac{1}{2}\operatorname{Tr}\ln\left[-\mathcal{G}^{-1}\right] \\ &= -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \left(\mathcal{L}_{SC} + \mathcal{L}_{FI} - \frac{1}{2}\sum_\nu \langle\nu|\operatorname{tr}\left[\ln\left(-\mathcal{G}^{-1}\right)\right]|\nu\rangle\right). \end{aligned} \tag{7.16}$$

By performing a series expansion of the last term in eq. (7.16) to second order in coupling constants, we get the following

$$\operatorname{tr}\ln\left(-\mathcal{G}^{-1}\right) = \operatorname{tr}\left[\ln\left(-\mathcal{G}_0^{-1}\right)\right] + \operatorname{tr}\left[-\frac{1}{2}(\mathcal{G}_0(\underline{\Delta} + a + \underline{\mathrm{n}}_z)^2)\right], \tag{7.17}$$

where we have once again discarded the linear contributions. Multiplying out the terms, we end up with

$$-\frac{1}{2}(\mathcal{G}_0(\underline{\Delta} + a + \underline{\mathrm{n}}_z))^2 \mathrel{\widehat{=}} -\frac{1}{2}\left(\mathcal{G}_0\underline{\Delta}\mathcal{G}_0\underline{\Delta} + \mathcal{G}_0 a\mathcal{G}_0 a + \mathcal{G}_0\underline{\mathrm{n}}_z\mathcal{G}_0 a + \mathcal{G}_0 a\mathcal{G}_0\underline{\mathrm{n}}_z + \mathcal{G}_0\underline{\mathrm{n}}_z\mathcal{G}_0\underline{\mathrm{n}}_z\right), \tag{7.18}$$

where we have used that odd powers of $\mathcal{G}_0\underline{\Delta}$ are off-diagonal matrices. Hence, we see that we get three new contributions to the effective action compared to eq. (6.32) due to the proximity-induced spin-spin exchange couplings. Inserting eq. (7.18) into eq. (7.16), we can write these as follows

$$\delta S_a = -\frac{1}{4}\int_0^\beta \int \mathrm{d}^2 r\, \mathrm{tr}\left[\mathcal{G}_0 a \mathcal{G}_0 a\right] \tag{7.19}$$

$$\delta S_{n_z,a} = -\frac{1}{4}\int_0^\beta \int \mathrm{d}^2 r\, \mathrm{tr}\left[\mathcal{G}_0 \underline{\mathrm{n}}_z \mathcal{G}_0 a\right] \tag{7.20}$$

$$\delta S_{a,n_z} = -\frac{1}{4}\int_0^\beta \int \mathrm{d}^2 r\, \mathrm{tr}\left[\mathcal{G}_0 a \mathcal{G}_0 \underline{\mathrm{n}}_z\right] \tag{7.21}$$

$$\delta S_{n_z} = -\frac{1}{4}\int_0^\beta \int \mathrm{d}^2 r\, \mathrm{tr}\left[\mathcal{G}_0 \underline{\mathrm{n}}_z \mathcal{G}_0 \underline{\mathrm{n}}_z\right] \tag{7.22}$$

The matrix product in eq. (7.19) can be treated completely analogous to eq. (6.34) and thus we can write it as follows

$$\mathcal{G}_0 a \mathcal{G}_0 a \mathrel{\widehat{=}} e^2 \mathcal{G}_0^p \not{a} \mathcal{G}_0^p \not{a} + e^2 \mathcal{G}_0^h \not{a}^T \mathcal{G}_0^h \not{a}^T. \tag{7.23}$$

Similarly, for the remaining three contributions, we get

$$\begin{aligned}
\mathcal{G}_0 a \mathcal{G}_0 \underline{\mathrm{n}}_z &= \begin{pmatrix} -e\mathcal{G}_0^p \not{a} & 0 \\ 0 & e\mathcal{G}_0^h \not{a}^T \end{pmatrix} \begin{pmatrix} \mathcal{G}_0^p n_z & 0 \\ 0 & -\mathcal{G}_0^h n_z^T \end{pmatrix} \\
&= \begin{pmatrix} -e\mathcal{G}_0^p \not{a} \mathcal{G}_0^p n_z & 0 \\ 0 & -e\mathcal{G}_0^h \not{a}^T \mathcal{G}_0^h n_z^T \end{pmatrix} \\
&\mathrel{\widehat{=}} -e\mathcal{G}_0^p \not{a} \mathcal{G}_0^p n_z - e\mathcal{G}_0^h \not{a}^T \mathcal{G}_0^h n_z^T \qquad (7.24)\\
\mathcal{G}_0 \underline{\mathrm{n}}_z \mathcal{G}_0 a &= \begin{pmatrix} \mathcal{G}_0^p n_z & 0 \\ 0 & -\mathcal{G}_0^h n_z^T \end{pmatrix} \begin{pmatrix} -e\mathcal{G}_0^p \not{a} & 0 \\ 0 & e\mathcal{G}_0^h \not{a}^T \end{pmatrix} \\
&\mathrel{\widehat{=}} -e\mathcal{G}_0^p n_z \mathcal{G}_0^p \not{a} - e\mathcal{G}_0^h n_z^T \mathcal{G}_0^h \not{a}^T \qquad (7.25)\\
\mathcal{G}_0 \underline{\mathrm{n}}_z \mathcal{G}_0 \underline{\mathrm{n}}_z &= \begin{pmatrix} \mathcal{G}_0^p n_z & 0 \\ 0 & -\mathcal{G}_0^h n_z^T \end{pmatrix}^2 \\
&\mathrel{\widehat{=}} \mathcal{G}_0^p n_z \mathcal{G}_0^p n_z + \mathcal{G}_0^h n_z^T \mathcal{G}_0^h n_z^T. \qquad (7.26)
\end{aligned}$$

Comparing eq.'s (7.23) to (7.26) with eq. (6.47), we see that they all have the same structure as the gauge field contribution from section 6.3. The only difference is that the field n_z does not carry a γ-matrix. By following the steps from eq. (6.47), we can immediately make the following simplifications

$$\delta S_a = -\frac{e^2}{2}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta^2}\sum_{n,l}\mathrm{tr}\left[\frac{\not\kappa+m}{\kappa^2+m^2}\not a(\xi)\frac{(\not\kappa-\not\xi)+m}{(\kappa-\xi)^2+m^2}\not a(-\xi)\right] \tag{7.27}$$

$$\begin{aligned}\delta S_{n_z,a} &= \frac{e}{4}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta^2}\sum_{n,l}\mathrm{tr}\Big[\frac{\not\kappa+m}{\kappa^2+m^2}n_z(\xi)\frac{(\not\kappa-\not\xi)+m}{(\kappa-\xi)^2+m^2}\not a(-\xi)\\ &\quad+\frac{\not\kappa+m}{\kappa^2+m^2}\not a(\xi)\frac{(\not\kappa-\not\xi)+m}{(\kappa-\xi)^2+m^2}n_z(-\xi)\Big] \end{aligned}\tag{7.28}$$

$$\begin{aligned}\delta S_{a,n_z} &= \frac{e}{4}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta^2}\sum_{n,l}\mathrm{tr}\Big[\frac{\not\kappa+m}{\kappa^2+m^2}\not a(\xi)\frac{(\not\kappa-\not\xi)+m}{(\kappa-\xi)^2+m^2}n_z(-\xi)\\ &\quad+\frac{\not\kappa+m}{\kappa^2+m^2}n_z(\xi)\frac{(\not\kappa-\not\xi)+m}{(\kappa-\xi)^2+m^2}\not a(-\xi)\Big] \end{aligned}\tag{7.29}$$

$$\delta S_{n_z} = -\frac{1}{4}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta^2}\sum_{n,l}\mathrm{tr}\left[\frac{\not\kappa+m}{\kappa^2+m^2}n_z(\xi)\frac{(\not\kappa-\not\xi)+m}{(\kappa-\xi)^2+m^2}n_z(-\xi),\right] \tag{7.30}$$

where the fields in the second term in eq. (7.28) and eq. (7.29) are permuted due to the cyclic permutation performed in eq. (6.49). Furthermore, we see that eq. (7.28) and eq. (7.29) combine. We will denote the sum of these contributions as follows

$$\begin{aligned}\delta S_{a,n_z} &\equiv \delta S_{a,n_z}+\delta S_{n_z,a}\\ &= \frac{e}{2}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta^2}\sum_{n,l}\mathrm{tr}\Big[\frac{\not\kappa+m}{\kappa^2+m^2}\not a(\xi)\frac{(\not\kappa-\not\xi)+m}{(\kappa-\xi)^2+m^2}n_z(-\xi)\\ &\quad+\frac{\not\kappa+m}{\kappa^2+m^2}n_z(\xi)\frac{(\not\kappa-\not\xi)+m}{(\kappa-\xi)^2+m^2}\not a(-\xi)\Big] \end{aligned}\tag{7.31}$$

By using the result obtained in eq. (6.62), we immediately get that the effective gauge field contribution of eq. (7.27) can be written as follows

$$\delta S_a = -\int\frac{\mathrm{d}^2q}{(2\pi)^2}\frac{1}{\beta}\sum_l a^\mu(\xi)\Pi_{\mu\nu}(\xi)a^\nu(-\xi), \tag{7.32}$$

where $\Pi_{\mu\nu}(\xi)$ is the same tensor as in eq. (6.63). The trace of the numerator of the first term in eq. (7.31) evaluate to

$$\begin{aligned}&\mathrm{tr}\Big[(\kappa^\mu\gamma_\mu+m)a^\nu(\xi)\gamma_\nu((\kappa-\xi)^\lambda\gamma_\lambda+m)n_z(-\xi)\Big]\\ &=\kappa^\mu a(\xi)^\nu(\kappa-\xi)^\lambda n_z(-\xi)\,\mathrm{tr}\left[\gamma_\mu\gamma_\nu\gamma_\lambda\right]+m\kappa^\mu a(\xi)^\nu n_z(-\xi)\,\mathrm{tr}\left[\gamma_\mu\gamma_\nu\right]\\ &+m(\kappa-\xi)^\lambda a(\xi)^\nu n_z(-\xi)\,\mathrm{tr}\left[\gamma_\nu\gamma_\lambda\right]\\ &=-2\kappa^\mu a(\xi)^\nu(\kappa-\xi)^\lambda n_z(-\xi)\varepsilon_{\mu\nu\lambda}-2m\kappa^\mu a(\xi)^\nu n_z(-\xi)\delta_{\mu\nu}-2m(\kappa-\xi)^\lambda a(\xi)^\nu n_z(-\xi)\delta_{\mu\lambda}\\ &=2\kappa^\mu a(\xi)^\nu\xi^\lambda n_z(-\xi)\varepsilon_{\mu\nu\lambda}-2m\kappa^\mu a(\xi)_\mu n_z(-\xi)-2m(\kappa-\xi)^\mu a(\xi)_\mu n_z(-\xi). \end{aligned}\tag{7.33}$$

where we have contracted all symmetric tensors with the anti-symmetric Levi-Civita tensor and put them to zero, i.e.,

$$\kappa^\mu \kappa^\nu \varepsilon_{\mu\nu\lambda} = 0. \tag{7.34}$$

We can rewrite the trace of the numerator in the second term as follows

$$\begin{aligned}
&\mathrm{tr}\left[(\kappa^\mu\gamma_\mu + m)n_z(\xi)((\kappa-\xi)^\lambda\gamma_\lambda + m)a^\nu(-\xi)\gamma_\nu\right] \\
&= \mathrm{tr}\left[((\kappa-\xi)^\lambda\gamma_\lambda + m)a^\nu(-\xi)\gamma_\nu(\kappa^\mu\gamma_\mu + m)n_z(\xi)\right] \\
&= \mathrm{tr}\left[(\kappa^\lambda\gamma_\lambda + m)a^\nu(-\xi)\gamma_\nu((\kappa+\xi)^\mu\gamma_\mu + m)n_z(\xi)\right] \\
&= \mathrm{tr}\left[(\kappa^\lambda\gamma_\lambda + m)a^\nu(\xi)\gamma_\nu((\kappa-\xi)^\mu\gamma_\mu + m)n_z(-\xi)\right],
\end{aligned} \tag{7.35}$$

where we have performed a linear change of variables in the third line and rotated the variable according to $\xi \to -\xi$ in the last line. Consequently, we see that the two terms in eq. (7.31) are equal and we get the following contribution

$$\begin{aligned}
\delta S_{a,n_z} &= -\int \frac{\mathrm{d}^2 q}{(2\pi)^2}\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\frac{1}{\beta^2}\sum_{n,l}\frac{2em\kappa^\nu + 2em(\kappa-\xi)^\nu - 2e\kappa^\mu\xi^\lambda\varepsilon_{\mu\nu\lambda}}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)}a(\xi)_\nu n_z(-\xi) \\
&= -\int \frac{\mathrm{d}^2 q}{(2\pi)^2}\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\frac{1}{\beta^2}\sum_{n,l}\frac{2em(2\kappa-\xi)^\nu - 2e\kappa^\mu\xi^\lambda\varepsilon_{\mu\nu\lambda}}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)}a(\xi)_\nu n_z(-\xi).
\end{aligned} \tag{7.36}$$

The trace of the numerator in eq. (7.30) evaluates to

$$\begin{aligned}
&\mathrm{tr}\left[(\kappa^\mu\gamma_\mu + m)n_z(\xi)((\kappa-\xi)^\nu\gamma_\nu + m)n_z(-\xi)\right] \\
&= \kappa^\mu(\kappa-\xi)^\nu\,\mathrm{tr}\left[\gamma_\mu\gamma_\nu\right] + m^2\,\mathrm{tr}\left[1\right]n_z(\xi)n_z(-\xi) \\
&= -2\kappa^\mu(\kappa-\xi)_\mu n_z(\xi)n_z(-\xi) + 2m^2 n_z(\xi)n_z(-\xi)
\end{aligned} \tag{7.37}$$

Thus, the contributions in eq. (7.30) can be written as follows

$$\delta S_{n_z} = \int \frac{\mathrm{d}^2 q}{(2\pi)^2}\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\frac{1}{\beta^2}\sum_{n,l}\frac{1}{2}\frac{(\kappa^\mu(\kappa-\xi)_\mu - m^2)n_z(\xi)n_z(-\xi)}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)}. \tag{7.38}$$

7.3 Effective gauge field sector and mixed terms

7.3.1 Effective gauge field sector with Chern-Simons coupling

The terms in eq. (7.32) can be divided into two contributions completely analogous to our previous case in section 6.4. The first contribution is the Chern-Simons term, which in this case takes the form

$$S_{CS} = -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r i\lambda_{CS}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda. \tag{7.39}$$

The second contribution from eq. (7.32) is the $\Sigma_{\mu\nu}$ tensor of eq. (6.64) in terms of the effective gauge field. Since the temporal component of the gauge field is unaffected by the magnetic impurities, the renormalization in this case is identical to eq. (6.101). However, in the spatial components, we get additional couplings in terms of the in-plane components of the magnetic fluctuations. Following the analysis in section 6.4.2, we get the following contribution

$$\begin{aligned} S_M &= -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \left(m_{\mathrm{el}}^2 \phi^2 + \lambda_{\nabla,el}(\boldsymbol{\nabla}\phi)^2 + \lambda_{\nabla,m}(\boldsymbol{\nabla}\times\bar{a})_z^2\right) \\ &= -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \big(m_{\mathrm{el}}^2 \phi^2 + \lambda_{\nabla,el}(\boldsymbol{\nabla}\phi)^2 + \lambda_{\nabla,m}(\boldsymbol{\nabla}\times\bar{A})_z^2 \\ &+ \zeta_{A,n}(\boldsymbol{\nabla}\times\bar{A})_z(\boldsymbol{\nabla}\times\bar{n})_z + \zeta_{\nabla,\bar{n}}(\boldsymbol{\nabla}\times\bar{n})_z^2\big). \end{aligned} \tag{7.40}$$

The magnetic coupling constants in the above equation are defined as

$$\zeta_{A,n} = \frac{-J_\parallel \lambda_{\nabla,m}}{e} \tag{7.41}$$

$$\zeta_{\nabla,\bar{n}} = \frac{J_\parallel^2 \lambda_{\nabla,m}}{e^2}, \tag{7.42}$$

where we have reinstated the appropriate spin-spin exchange coupling constants. If the in-plane fluctuations are divergence-free, we can write eq. (7.40) as follows

$$\begin{aligned} S_M = &-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \big(m_{\mathrm{el}}^2 \phi^2 + \lambda_{\nabla,el}(\boldsymbol{\nabla}\phi)^2 + \lambda_{\nabla,m}(\boldsymbol{\nabla}\times\bar{A})_z^2 \\ &+ \zeta_{A,n}(\boldsymbol{\nabla}\times\bar{A})_z(\boldsymbol{\nabla}\times\bar{n})_z + \zeta_{\nabla,\bar{n}}(\boldsymbol{\nabla}\bar{n})^2\big) \end{aligned} \tag{7.43}$$

Combining eq. (7.39) and eq. (7.43), we can write eq. (7.27) as follows

$$\begin{aligned} \delta S_a &= S_{CS} + S_M \\ &= -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \big(i\lambda_{CS}\varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + m_{\mathrm{el}}^2\phi^2 + \lambda_{\mathrm{el}}(\boldsymbol{\nabla}\phi)^2 + \lambda_m(\boldsymbol{\nabla}\times\bar{A})_z^2 \\ &+ \zeta_{A,n} B_z(\boldsymbol{\nabla}\times\bar{n})_z + \zeta_{\nabla,\bar{n}}(\boldsymbol{\nabla}\bar{n})^2\big), \end{aligned} \tag{7.44}$$

where we have used that $(\boldsymbol{\nabla}\times\bar{A})_z = B_z$. This contribution to the action replaces eq. (6.106) in the presence of magnetic impurities.

7.3.2 Mixing terms and the Dzyaloshinskii-Moriya coupling

We start by splitting eq. (7.36) into the following two terms

$$\begin{aligned} \delta S_{a,n_z} = &-\int \frac{\mathrm{d}^2 q}{(2\pi)^2} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\beta^2} \sum_{n,l} \frac{2em(2\kappa-\xi)^\nu}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)} a(\xi)_\nu n_z(-\xi) \\ &-\int \frac{\mathrm{d}^2 q}{(2\pi)^2} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\beta^2} \sum_{n,l} \frac{-2e\kappa^\mu \xi^\lambda \varepsilon_{\mu\nu\lambda}}{(\kappa^2+m^2)^2} a(\xi)_\nu n_z(-\xi) \end{aligned} \tag{7.45}$$

where we have omitted the factors of ξ in the denominator of the last term since the numerator is linear in ξ. Thus, we are left with the following two integrals

$$\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta}\sum_n \frac{2em(2\kappa-\xi)^\nu}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)} \tag{7.46}$$

$$\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta}\sum_n \frac{-2e\kappa^\mu\xi^\lambda\varepsilon_{\mu\nu\lambda}}{(\kappa^2+m^2)^2}. \tag{7.47}$$

These integrals are analyzed in detail in section E.3. Using eq. (E.60), the lowest-order contribution from eq. (7.46) equates to

$$-\frac{iem}{2\pi v_F^2}\frac{\sinh\beta\mu}{\cosh\beta|m|+\cosh\beta\mu}\phi(\xi)n_z(-\xi). \tag{7.48}$$

And using eq. (E.63), we get the following second-order contribution in eq. (7.46)

$$\begin{aligned}&-\frac{iem}{6\pi v_F^2}\Big(\frac{\beta}{16|m|}\Big[\frac{1}{\cosh^2\frac{\beta(|m|+\mu)}{2}}-\frac{1}{\cosh^2\frac{\beta(|m|-\mu)}{2}}\Big]\\&-\frac{1}{4|m|^2}\frac{\sinh\beta|m|}{\cosh\beta\mu+\cosh\beta|m|}\Big)(iq)^2\phi n_z(-\xi),\end{aligned} \tag{7.49}$$

Notice here that only the temporal part of the gauge field couples to the perpendicular fluctuations n_z since there is an odd factor of k in the numerator of eq. (7.46). Using eq. (E.65), we can write eq. (7.47) as

$$\frac{e}{4\pi v_F^2}\frac{\sinh\beta\mu}{\cosh\beta|m|+\cosh\beta\mu}\varepsilon_{0\lambda\nu}i\xi^\lambda a(\xi)_\nu n_z(-\xi). \tag{7.50}$$

Combining eq. (7.48), eq. (7.49) and eq. (7.50) and performing an inverse Fourier transform, we can write eq. (7.31) as follows

$$\delta S_{a,n_z}=-\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^2r\left[\zeta_{DM}\left((\boldsymbol{\nabla}\times\bar{n})_z-v_FeB_z\right)n_z+\zeta_{\nabla,\phi}\boldsymbol{\nabla}^2\phi\cdot n_z+\zeta_\phi\phi\cdot n_z\right], \tag{7.51}$$

where the coupling constants are defined as

$$\zeta_{DM}=-\frac{J_\parallel J_\perp}{4\pi v_F}\frac{\sinh\beta\mu}{\cosh\beta|m|+\cosh\beta\mu} \tag{7.52}$$

$$\zeta_\phi=-\frac{iemJ_\perp}{2\pi v_F^2}\frac{\sinh\beta\mu}{\cosh\beta|m|+\cosh\beta\mu} \tag{7.53}$$

$$\begin{aligned}\zeta_{\nabla,\phi}=&-\frac{iJ_\perp e\beta\,\mathrm{sgn}(m)}{96\pi}\Big[\frac{1}{\cosh^2\frac{\beta(|m|+\mu)}{2}}-\frac{1}{\cosh^2\frac{\beta(|m|-\mu)}{2}}\Big]\\&+\frac{iJ_\perp e}{24\pi|m|}\frac{\mathrm{sgn}(m)\sinh\beta\mu}{\cosh\beta\mu+\cosh\beta|m|},\end{aligned} \tag{7.54}$$

where we have reinstated a factor v_F for each gradient and the appropriate spin-spin exchange coupling constants. The first term of eq. (7.51) contains a Dzyaloshinskii-Moriya term in addition to a coupling between the magnetic fluctuations and a magnetic field. The two rightmost terms are magnetoelectric couplings[1] between the perpendicular fluctuations and the electric potential.

7.4 Renormalization of the perpendicular magnetic sector

The contributions to the perpendicular magnetic fluctuations in eq. (7.38) takes the form

$$\delta S_{n_z} = \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\beta^2} \sum_{n,l} \frac{1}{2} \frac{(\kappa^\mu(\kappa-\xi)_\mu - m^2) n_z(\xi) n_z(-\xi)}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)}. \tag{7.55}$$

By using the relation

$$\kappa^\mu(\kappa-\xi)_\mu - m^2 = \frac{1}{2}\left[(\kappa^2+m^2)+(\kappa-\xi)^2+m^2\right] - 2m^2 - \frac{1}{2}\xi^2 \tag{7.56}$$

we can rewrite the integrand of eq. (7.55) as follows

$$\begin{aligned} &\frac{1}{2}(-2m^2 - \frac{1}{2}\xi^2) \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\beta} \sum_n \frac{1}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)} n_z(\xi) n_z(-\xi) \\ &+ \frac{1}{2} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\beta} \sum_n \frac{1}{(\kappa^2+m^2)} n_z(\xi) n_z(-\xi). \end{aligned} \tag{7.57}$$

These are the same integrals as in section 6.4 which we computed in section E.1. Hence, we immediately get

$$\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)} = c_1 + q^2 c_2 \tag{7.58}$$

$$\frac{1}{2}\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\beta} \sum_n \frac{1}{(\kappa^2+m^2)} = \frac{c_3}{2}. \tag{7.59}$$

Thus, we can write eq. (7.57) as follows

$$\begin{aligned} &(-m^2 - \frac{q^2}{4})(c_1 + q^2 c_2) n_z(\xi) n_z(-\xi) + \frac{c_3}{2} n_z(\xi) n_z(-\xi) \\ &= (\frac{c_3}{2} - m^2 c_1) n_z(\xi) n_z(-\xi) + q^2(-m^2 c_2 - \frac{c_1}{4}) n_z(\xi) n_z(-\xi) \\ &= (\frac{c_3}{2} - m^2 c_1) n_z(\xi) n_z(-\xi) + (-m^2 c_2 - \frac{c_1}{4})(iq)(-iq) n_z(\xi) n_z(-\xi) \end{aligned} \tag{7.60}$$

By performing an inverse Fourier transform, we arrive at

[1] The term “magnetoelectric” refers to couplings between electric and magnetic degrees of freedom in a system. [4]

$$\delta S_{n_z} = -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \left[\zeta_{\nabla,n_z}(\boldsymbol{\nabla} n_z)^2 + \zeta_{m,n_z} n_z^2\right] \tag{7.61}$$

with the following coupling constants

$$\begin{aligned}
\zeta_{m,n_z} &= m^2 c_1 - \frac{c_3}{2} \\
&= \frac{J_\perp^2 |m|}{8\pi v_F^2} \frac{\sinh\beta|m|}{\cosh\beta\mu + \cosh\beta|m|} + \frac{J_\perp^2}{8\pi\beta v_F^2} \ln(\cosh\beta|m| + \cosh\beta\mu) && (7.62) \\
\zeta_{\nabla,n_z} &= m^2 c_2 + \frac{c_1}{4} \\
&= \frac{J_\perp^2}{48\pi|m|} \frac{\sinh\beta|m|}{\cosh\beta|m| + \cosh\beta\mu} + \frac{J_\perp^2 \beta}{384} \Big[\frac{1}{\cosh^2\frac{\beta(|m|-\mu)}{2}} + \frac{1}{\cosh^2\frac{\beta(|m|+\mu)}{2}}\Big], && (7.63)
\end{aligned}$$

where we have reinstated a factor v_F for each gradient and the spin-spin exchange coupling constants.

7.4.1 Full theory of the effective gauge field- and magnetic sector

By combining eq. (7.1), eq. (7.44), eq. (7.51) and eq. (7.61), we get the following effective action

$$\begin{aligned}
S_{\text{eff}} &= \delta S_a + \delta S_{a,n_z} + \delta S_{n_z} \\
&= -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 r \left(\mathcal{L}_a + \mathcal{L}_{a,n_z} + \mathcal{L}_{n_z} + \mathcal{L}_{FI}\right)
\end{aligned} \tag{7.64}$$

where we have defined the following Lagrangians

$$\begin{aligned}
\mathcal{L}_a &= i\lambda_{CS}\varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + m_{\text{el}}^2 \phi^2 + \lambda_{\text{el}}(\boldsymbol{\nabla}\phi)^2 + \lambda_m(\boldsymbol{\nabla}\times\bar{A})_z^2 \\
&\quad + \zeta_{A,n}(\boldsymbol{\nabla}\times\bar{A})_z(\boldsymbol{\nabla}\times\bar{n})_z + \zeta_{\nabla,\bar{n}}(\boldsymbol{\nabla}\bar{n})^2 && (7.65) \\
\mathcal{L}_{a,n_z} &= \zeta_{\nabla,\phi}\boldsymbol{\nabla}^2\phi\cdot n_z + \zeta_\phi\phi\cdot n_z + \zeta_{DM}\left[(\boldsymbol{\nabla}\times\bar{n})_z - v_F e B_z\right] n_z && (7.66) \\
\mathcal{L}_{n_z} &= \zeta_{\nabla,n_z}(\boldsymbol{\nabla} n_z)^2 + \zeta_{m,n_z} n_z^2 && (7.67) \\
\mathcal{L}_{FI} &= b\cdot\partial_t n - \frac{\kappa}{2}(\boldsymbol{\nabla} n)^2 - \frac{m^2}{2} n^2. && (7.68)
\end{aligned}$$

To this order in perturbations, the Cooper bosons are unaffected by the magnetic impurities. Thus, the contribution to the Cooper boson sector is identical to eq. (6.122).

Chapter 8

Chern-Simons-Ginzburg-Landau theory of the superconductor heterostructure

The effective bosonic field theory of our heterostructure after renormalization contains new terms that are not present in any of the original field theories of eq. (6.1), eq. (6.3), and eq. (7.1). Firstly, we will discuss the physical meaning and implications of these new coupling without the effects of the magnetic impurities, in light of the results found in [3]. Then we will include the magnetic terms, discuss some of their properties and elaborate on similar results found in comparable ferromagnetic heterostructure systems.

8.1 Tuning of the effective coupling constants

In this section, we look at the tunability of a selected set of coupling constants that are relevant in the discussions that follow. Each coupling is plotted as a function of normalized chemical potential $\mu/|m|$ for selected values of normalized temperature $\beta|m| = |m|/T$.

8.1.1 Electric mass term

In fig. 8.1 we see a plot of the electric mass coefficient defined in eq. (6.103). At low temperatures, the mass term is mostly present in the conduction phase of the surface fermions. However, for sufficiently large temperatures, the mass term becomes independent of gap size. At zero temperature, the mass becomes finite if the chemical potential is outside of the gap

$$\lim_{T\to 0} m_{\text{el}}^2 = \frac{e^2|\mu|}{4\pi v_F^2}\theta(|\mu| - |m|). \tag{8.1}$$

For temperatures above the Curie temperature, we get

$$\begin{aligned} m_{\text{el}}^2|_{T\gg|m|} &= \frac{e^2}{4\pi\beta v_F^2}\ln(\cosh\beta\mu + 1) \\ &= \frac{e^2}{4\pi\beta v_F^2}\ln\left(2\cosh^2\frac{\beta\mu}{2}\right). \end{aligned} \tag{8.2}$$

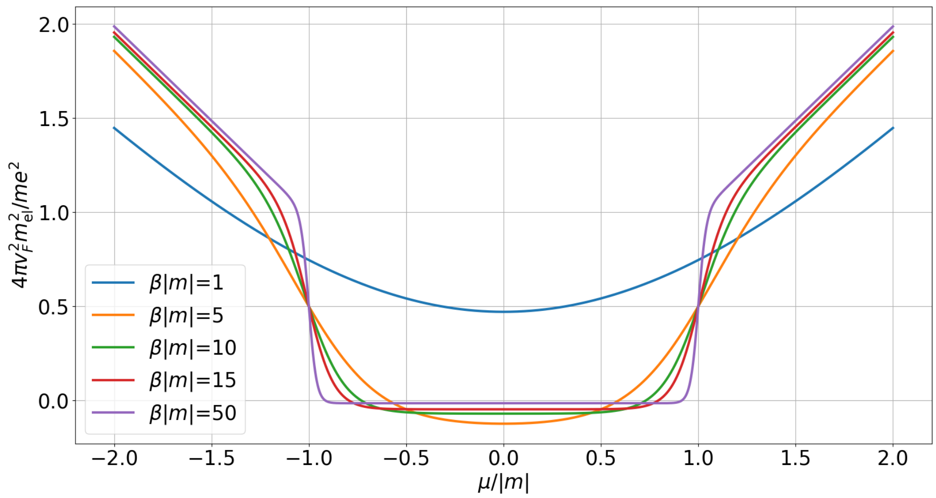

Figure 8.1: The mass term coefficient $m_{\rm el}^2$ in units of $\frac{me^2}{4\pi v_F^2}$ as a function of $\mu/|m|$

8.1.2 Chern-Simons coupling

In fig. 8.2 we see a plot of the Chern-Simons coefficient defined in eq. (6.71). The magnitude of the Chern-Simons coefficient increases rapidly as temperatures become small compared to the energy scale $|m|$ if the chemical potential lies in the energy gap. This means that this term is most prominent in the insulating phase of the surface fermions. However, in order to maintain superconductivity, the topological fermions in eq. (6.3) must have a Fermi surface. Thus, in order for the Chern-Simons action of eq. (6.72) to have any effect on the Cooper bosons, we must tune the magnitude of the chemical potential slightly larger than the gap $|m|$.

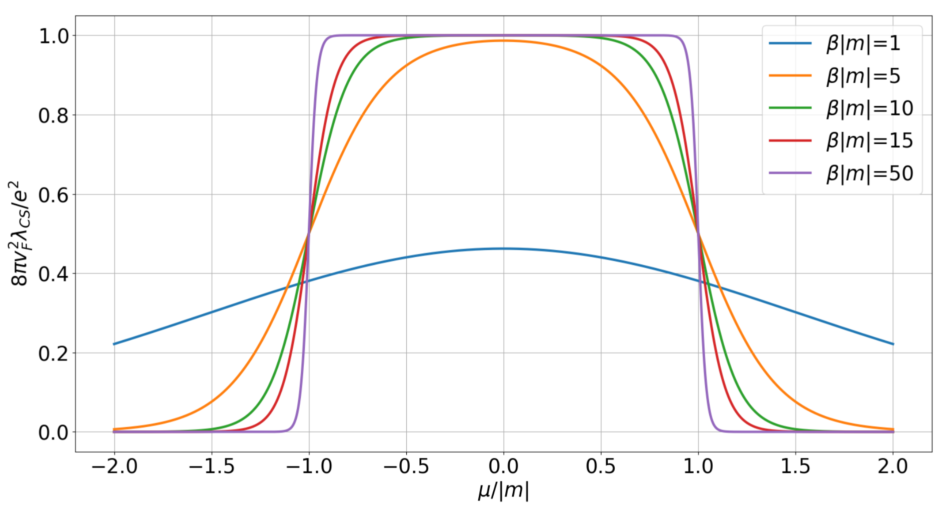

Figure 8.2: The Chern-Simons coefficient in units of $\frac{e^2}{8\pi v_F^2}$ as a function of $\mu/|m|$.

8.1.3 Dzyaloshinskii-Moriya coupling

In figure fig. 8.3 we see a plot of the Dzyaloshinskii-Moriya coupling defined in eq. (7.52). As opposed to the Chern-Simons coupling in fig. 8.2, we see that this coefficient is most prominent outside of the energy gap. However, both coefficients are finite at the special points $|\mu| \approx |m|$, which means that we can combine the effects of the two terms and still maintain superconductivity. For large values of the normalized chemical potential, the coupling constant can be approximated as

$$\begin{aligned}\zeta_{DM} &= -\frac{J_\perp J_\parallel}{4\pi v_F}\frac{\sinh\beta\mu}{\cosh\beta|m| + \cosh\beta\mu} \\ &\approx -\frac{J_\perp J_\parallel}{4\pi v_F}\tanh\beta\mu,\end{aligned} \tag{8.3}$$

which implies that ζ_{DM} ranges between $\pm\frac{J_\perp J_\parallel}{4\pi v_F}$.

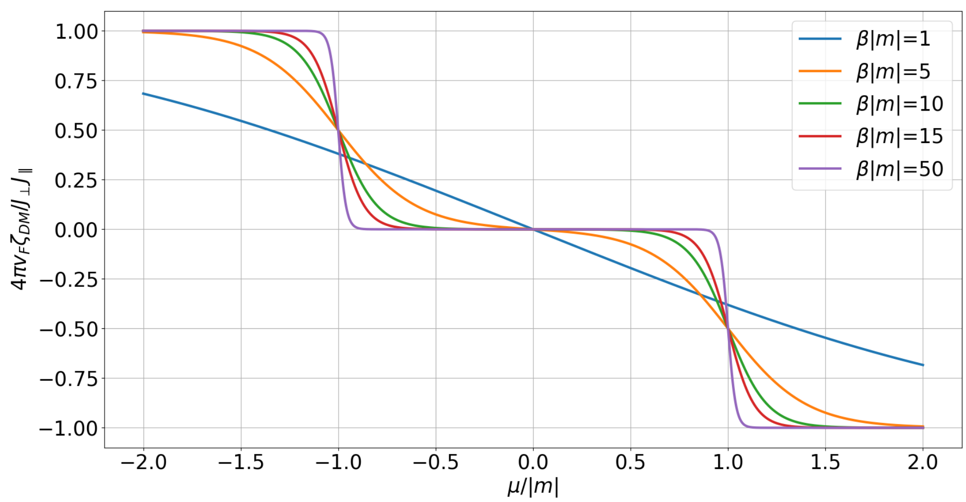

Figure 8.3: The Dzyaloshinskii-Moriya coefficient in units of $\frac{J_\perp J_\parallel}{4\pi v_F}$ as a function of $\mu/|m|$.

8.2 Lagrangian of effective bosonic theory

Combining eq. (6.1), eq. (6.106) and eq. (6.122), we can express the renormalized theory of the gauge field and the Cooper bosons as the following effective Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= i\lambda_{CS}\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + m_{\text{el}}^2\phi^2 + \lambda_{\text{el}}(\boldsymbol{\nabla}\phi)^2 + \lambda_m(\boldsymbol{\nabla}\times\bar{A})_z^2 \\ &\quad + \eta_\tau\varphi^*\partial_\tau\varphi + \frac{1}{2}\left|(-i\boldsymbol{\nabla} - e^*\bar{A})\varphi)\right|^2 + \tilde{\alpha}|\varphi|^2 + \beta\varphi^4 + \eta_\nabla|-i\boldsymbol{\nabla}\varphi|^2,\end{aligned} \tag{8.4}$$

where we have absorbed the gradient normalization constant of eq. (6.114) into α. The first term is a Chern-Simons term which couples to the Cooper bosons, making this theory an effective topological field theory of the superconductor. This modification to the electromagnetic sector is a general feature of topological materials [17], [25]. The resulting model is a

Chern-Simons-Ginzburg-Landau (CSGL) theory in terms of the Cooper bosons which therefore supports anyonic vortex solutions [2]. In the low energy regime, the Cooper boson sector reduces to eq. (5.72), which means that in this limit the Chern-Simons gauge field couples to the phase fluctuations of the superconducting Cooper bosons, reminiscent of the $(3+1)$-dimensional topological axionic field theory of [17].

The second term in eq. (8.4) is an electric mass of the electric potential ϕ. This term was also found in a ferromagnetic topological insulator heterostructure with a similar coupling constant in [5]. The derivation of this mass term also proves the absence of a corresponding magnetic mass associated with the spatial components of the gauge field. This is a generalization of the proof that the magnetic mass is zero in $2+1$-dimensions for finite energy gap and chemical potential, see e.g., [26]. The third and fourth term corresponds to the renormalized Maxwell sector. The couplings are asymmetrical due to the splitting of the vacuum polarization tensor in eq. (6.74).

The temporal- and spatial renormalizations of the Cooper bosons cannot be properly absorbed into the corresponding terms of the Cooper boson sector since these are not part of the gauge symmetry to this order in perturbations. The temporal correction induces a time-dependence in the Cooper boson sector, making the CSGL theory an intrinsic time-dependent Ginzburg-Landau theory. The spatial renormalization shifts the temperature dependence of α in eq. (5.65), which ultimately changes the critical temperature of the superconductor.

8.3 Including the effects of the magnetic impurities

In the presence of the magnetic impurities, the Lagrangian of eq. (8.4) is replaced by the following effective field theory

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= i\lambda_{CS}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + m_{\text{el}}^2\phi^2 + \lambda_{\text{el}}(\boldsymbol{\nabla}\phi)^2 + \lambda_m(\boldsymbol{\nabla}\times\bar{A})_z^2\\ &+ \zeta_{DM}\left((\boldsymbol{\nabla}\times\bar{n})_z - v_F e B_z\right)n_z + \zeta_{\nabla,\phi}\boldsymbol{\nabla}^2\phi\cdot n_z + \zeta_\phi\phi\cdot n_z + \zeta_{A,n}B_z(\boldsymbol{\nabla}\times\bar{n})_z\\ &+ \zeta_{\nabla,n_z}(\boldsymbol{\nabla}n_z)^2 + \zeta_{\nabla,\bar{n}}(\boldsymbol{\nabla}\bar{n})^2 + \zeta_{m,n_z}n_z^2 + b\cdot\partial_t n - \frac{\kappa}{2}\left[(\boldsymbol{\nabla}n)^2 + (\partial_z n)^2\right] - \frac{m^2}{2}n^2\\ &+ \eta_\tau\varphi^*\partial_\tau\varphi + \frac{1}{2}\left|(-i\boldsymbol{\nabla} - e^*\bar{A})\varphi\right|^2 + \tilde{\alpha}|\varphi|^2 + \beta|\varphi|^4 + \eta_\nabla|-i\boldsymbol{\nabla}\varphi|^2.\end{aligned} \tag{8.5}$$

In this case, the Chern-Simons coupling is in terms of the effective gauge field, which in turn introduces new types of interactions between the gauge field and in-plane fluctuations. We also get several magnetoelectric couplings between the electric potential, external magnetic field, and the magnetic fluctuations, in addition to a Dzyaloshinskii-Moriya coupling. Furthermore, we also get a renormalization of the magnetic sector of eq. (7.1). This renormalization induces an anisotropy in the magnetic sector due to the asymmetry between the in-plane and perpendicular couplings to the surface fermions.

Since this theory describes the interplay between superconductivity, gauge fields, and ferromagnetism, it allows for a comparison with the theory of a ferromagnetic superconductor coupled to a gauge field in [6]. In this model, the renormalization of the magnetic sector is isotropic, contrary to our results. Furthermore, this model contains neither a Chern-Simons term, a Dzyaloshinskii-Moriya term nor any of the magnetoelectric couplings found in eq. (8.5).

8.4 Thermal screening, renormalized Coulomb interaction, and negative mass term

8.4.1 Effective potential for positive mass term

Due to the presence of an electric mass term, we get modifications to the Coulomb potential on the interface of the heterostructure. In the long wavelength limit, the effective potential can be written as follows

$$\phi_{\mathrm{eff}}(q) = \frac{1}{\phi(q)^{-1} + \frac{1}{e^2}\Pi_{00}(0)} = \frac{2\pi e^2}{\epsilon(q+s)}, \tag{8.6}$$

where ϵ is the permittivity, $s = \frac{2\pi m_{\mathrm{el}}^2}{\epsilon}$, and $\phi(q) = \frac{2\pi e^2}{q}$ is the Fourier transform of the bare Coulomb potential $\phi(r) = \frac{e^2}{\epsilon r}$ [5]. By performing an inverse Fourier transform, we get the following effective potential in real space for $s > 0$

$$\phi_{\mathrm{eff}}(r) = \frac{e^2}{\epsilon r}\left(1 + \frac{\pi s r}{2}\left(Y_0(sr) - H_0(sr)\right)\right) \tag{8.7}$$

where $H_0(sr)$ is the zeroth Struve function and $Y_0(sr)$ is the zeroth Bessel function of second kind [5], [27]. For $r > \frac{1}{s}$, eq. (8.7) can be approximated by

$$\begin{aligned}\phi_{\mathrm{eff}}(r) &\approx \frac{e^2}{\epsilon r}\left(1 + \frac{\pi s r}{2}\left(-\frac{\left(\frac{sr}{2}\right)^{-1}}{\sqrt{\pi}\Gamma(\frac{1}{2})}\right)\right) + \mathcal{O}\left(\left(\frac{sr}{2}\right)^{-3}\right)\\ &= \mathcal{O}\left(\frac{1}{(sr)^3}\right).\end{aligned} \tag{8.8}$$

Thus, we get screening of charges at large distances compared to the scale $\frac{1}{s}$. In the opposite limit, eq. (8.7) reduces to the bare Coulomb potential [27]. The mass term in eq. (8.1) corresponds to a Thomas-Fermi screening of the system, whereas eq. (8.2) corresponds to a Debye screening [5]. In the Thomas-Fermi case, we get the following screening length

$$r_{\mathrm{TF}} \equiv \frac{1}{s} = \frac{\epsilon}{2\pi m_{\mathrm{el}}|^2_{T\to 0}} = \frac{4\pi\epsilon}{e^2}\frac{v_F^2}{2\pi\mu} = \frac{v_F^2}{2\pi\alpha|m|}, \tag{8.9}$$

where we have used that

$$\mu = E_F = \sqrt{k_F^2 + m^2} \approx |m|. \tag{8.10}$$

and that $\alpha = \frac{e^2}{4\pi\epsilon} \approx \frac{1}{137}$ in our units. Using that $v_F = \frac{v_F}{c} \approx 0.02$ and $|m| \sim J_\perp \approx 10\text{meV}$[1], we get the following numerical result

[1]Alternatively, one could insert the appropriate factors of $\hbar$ and c and use v_F in SI-units instead, which yields the same result.

$$r_{\mathrm{TF}} = \frac{v_F^2}{2\pi\alpha|m|} = \approx 1.1 \cdot 10^{-6}\mathrm{m} \tag{8.11}$$

where we have used that $\frac{1\mathrm{eV}}{\hbar c} = 1\mathrm{eV} \approx 8.065 \cdot 10^5\mathrm{m}^{-1}$ in our units. In the Debye case, we can use the following approximation for sufficiently high temperatures

$$m_{\mathrm{el}}^2|_{T\gg|m|} = \frac{e^2}{2\pi\beta v_F^2} \ln\left(2\cosh^2\frac{\beta\mu}{2}\right) \approx \frac{e^2\mu}{2\pi v_F^2} \tag{8.12}$$

and thus the Debye screening length is of the same order as the Thomas-Fermi screening length.

8.4.2 Negative electric mass

In fig. 8.1 we see that in the insulating phase of the surface fermions the value of s can be negative. According to fig. 8.4, the mass term reaches its minimum value for zero chemical potential at $T/|m| \approx 0.25$, which corresponds to

$$T_{\max} \approx \frac{|m|}{4k_B} \approx 30\mathrm{K}. \tag{8.13}$$

As $T/|m|$ approaches the value ≈ 0.5, the mass term eventually changes sign and becomes positive. This happens at

$$T_0 \approx \frac{|m|}{2k_B} \approx 60\mathrm{K}. \tag{8.14}$$

There are reasons as to why this negative mass term might be unphysical. First of all, we use dimensional regularization to evaluate the positive contributions to eq. (6.103) and thereby avoiding possible contributions from UV cut-off regularized terms. Additionally, there might be errors since we are computing the renormalization constants using a continuum limit of the sum over quantum numbers k and q. However, the mass term vanishes inside the mass gap in the Thomas-Fermi limit of eq. (8.1), which is to be expected.

8.4.3 Effective potential for negative mass term

Using eq. (8.6), we can write down the following expression for the effective potential for $s < 0$

$$\begin{aligned}
\phi_{\mathrm{eff}}^{s<0}(r) &= \frac{2\pi e^2}{\epsilon}\int \frac{\mathrm{d}^2q}{(2\pi)^2}\frac{1}{|q|-|s|}\mathrm{e}^{iq\cdot r} \\
&= \frac{e^2}{\epsilon}\int_0^\infty \mathrm{d}q\frac{q}{q-|s|}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\theta \mathrm{e}^{iqr\cos\theta} \\
&= \frac{e^2}{\epsilon}\int_0^\infty \mathrm{d}q\frac{q}{q-|s|}J_0(qr) \\
&= \frac{e^2}{\epsilon}\int_0^\infty \mathrm{d}q\frac{q-|s|+|s|}{q-|s|}J_0(qr) \\
&= \frac{e^2}{\epsilon}\int_0^\infty \mathrm{d}qJ_0(qr) + \frac{e^2|s|}{\epsilon}\int_0^\infty \mathrm{d}q\frac{1}{q-|s|}J_0(qr)
\end{aligned} \tag{8.15}$$

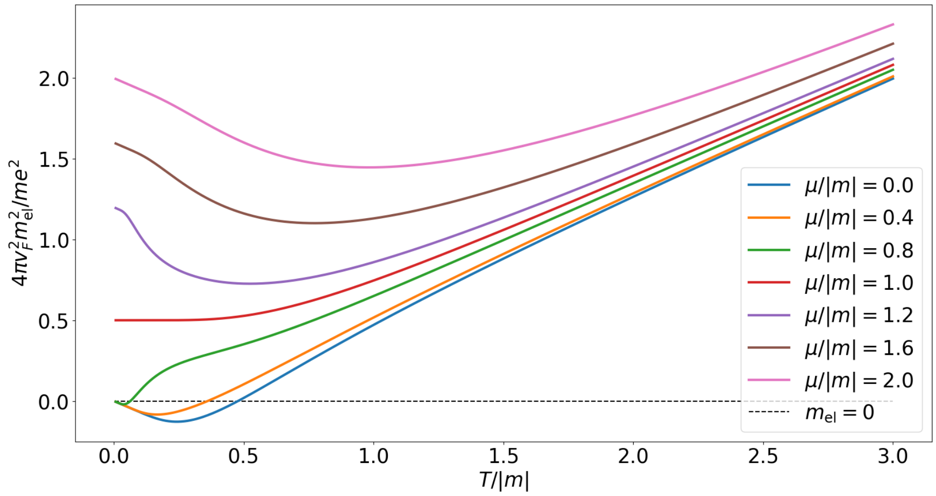

Figure 8.4: The mass term coefficient $m_{\rm el}^2$ in units of $\frac{|m|e^2}{4\pi v_F^2}$ as a function of $T/|m|$

where $J_0(qr)$ is the zeroth-order Bessel function of first kind. The first term of eq. (8.15) corresponds to the bare Coulomb potential without any mass terms. In order to evaluate the second term, we will use the Cauchy principal value theorem

$$\begin{aligned}\mathrm{P}\int_0^\infty \mathrm{d}q\frac{1}{q-|s|}J_0(qr) &= \mathrm{P}\int_0^\infty \frac{\mathrm{d}u}{r}\frac{1}{\frac{u}{r}-|s|}J_0(u) = \mathrm{P}\int_0^\infty \mathrm{d}u\frac{1}{u-r|s|}J_0(u)\\ &= \lim_{\epsilon\to 0}\left[\int_0^{r|s|-\epsilon}\mathrm{d}u\frac{J_0(u)}{u-r|s|}+\int_{r|s|+\epsilon}^\infty \mathrm{d}u\frac{J_0(u)}{u-r|s|}\right]. \end{aligned} \tag{8.16}$$

For small values of $r|s|$, we can solve the integrals as follows

$$\begin{aligned}\int_0^{r|s|-\epsilon}\mathrm{d}u\frac{J_0(u)}{u-r|s|} &\approx J_0(0)\int_0^{r|s|-\epsilon}\mathrm{d}u\frac{1}{u-r|s|}\\ &= \ln\epsilon - \ln r|s| \end{aligned} \tag{8.17}$$

$$\begin{aligned}\int_{r|s|+\epsilon}^\infty \mathrm{d}u\frac{J_0(u)}{u-r|s|} &= \int_\epsilon^\infty \mathrm{d}u\frac{J_0(u+r|s|)}{u}\\ &\approx \int_\epsilon^\infty \mathrm{d}u\frac{J_0(u)}{u}\\ &= -Ji_0(\epsilon)\end{aligned} \tag{8.18}$$

where $Ji_0(\epsilon)$ is the Bessel-integral function which evaluates to [28]

$$\begin{aligned}Ji_0(\epsilon) &= \gamma + \ln\frac{\epsilon}{2} + \sum_{s=1}^\infty \frac{(-1)^s(\frac{\epsilon}{2})^{2s}}{(s!)^2(2s)}\\ &= \gamma + \ln\epsilon - \ln 2 + \mathcal{O}\big(\epsilon^2\big)\end{aligned} \tag{8.19}$$

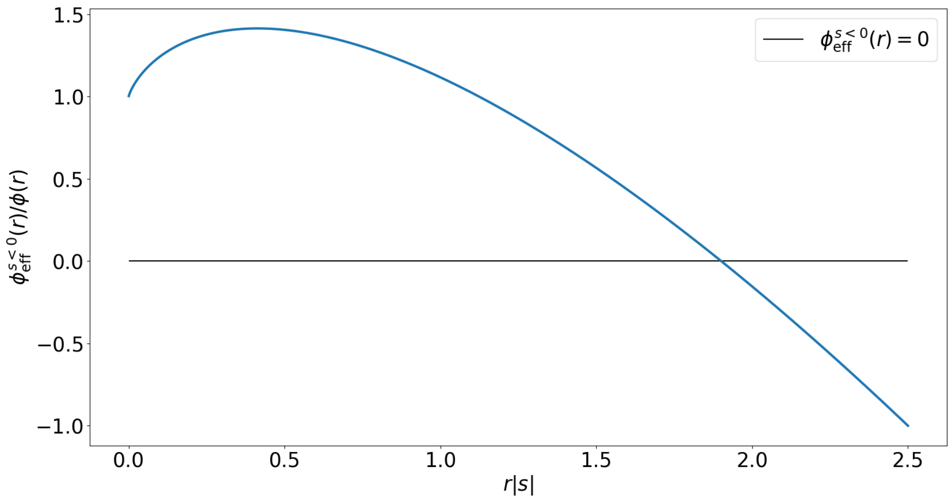

Figure 8.5: Effective potential $\phi_{\text{eff}}^{s<0}$ relative to the bare Coulomb potential $\phi(r)$.

where γ is the Euler-Mascheroni constant. Hence, we get the following

$$\begin{aligned} \mathrm{P}\int_0^\infty \mathrm{d}q \frac{1}{q-|s|} J_0(qr) &= \lim_{\epsilon\to 0}\left[\ln\epsilon - \ln r|s| - \gamma + \ln 2 - \ln\epsilon + \mathcal{O}(\epsilon^2)\right] \\ &= -\gamma + \ln 2 - \ln r|s| \end{aligned} \tag{8.20}$$

Inserting this into eq. (8.15), we get the following effective potential

$$\begin{aligned} \phi_{\text{eff}}^{s<0}(r) &= \frac{e^2}{\epsilon r} + \frac{e^2|s|}{\epsilon}\left(-\gamma + \ln 2 - \ln r|s|\right) \\ &= \phi(r)\left(1 + r|s|(\ln 2 - \gamma - \ln r|s|)\right). \end{aligned} \tag{8.21}$$

In fig. 8.5, we see that that eq. (8.21) changes sign when

$$r|s| \approx 1.9. \tag{8.22}$$

For $T \approx T_{\text{max}}$ the magnitude of s reaches its maximum value, which corresponds to the following length scale

$$r_{s<0} \equiv \frac{1.9}{s} = \frac{1.9\epsilon}{2\pi m_{\text{el}}^2} \approx \frac{19\epsilon v_F^2}{|m|e^2} = \frac{19}{2}\frac{4\pi\epsilon}{e^2}\frac{v_F^2}{2\pi|m|} = \frac{19}{2} r_{\text{TF}} \tag{8.23}$$

where we have used that $m_{\text{el}}^2 \approx 0.2$ in units of $\frac{|m|e^2}{4\pi v_F^2}$ at T_{max} in fig. 8.4. This implies that the effective potential becomes attractive at length scales comparable to the Thomas-Fermi screening for temperatures close to $T_{\text{max}} \approx 30K$ if the chemical potential is approximately zero.

8.4.4 Non-zero electric mass and Lorenz invariance

At $T = 0$, both the electric- and the magnetic mass are zero in $2+1$- and $3+1$-dimensional spacetime for arbitrary values of the fermion mass [26] and chemical potential [2]. Due to the Ward identity and Lorenz invariance, these two masses must be equal. The appearance of a mass term would therefore break gauge invariance [3] and consequently the electric mass must be zero due to Lorenz invariance.

At finite temperature, we must take into account the heat bath of the system, which picks out a specific rest frame. Thus, Lorenz invariance is broken at finite temperature and consequently, the Ward identity imposes fewer restrictions on the tensor structure of the vacuum polarization. The heat bath rest frame introduces a new tensor into the problem which is non-zero only in the temporal component. This in turn causes an asymmetry between the spatial- and temporal components in the tensor decomposition of eq. (6.63), which allows for a non-zero electric mass even though the magnetic mass is zero and therefore without breaking gauge invariance.

According to eq. (8.1), the mass term is finite even in the limit $T \to 0$ if the chemical potential is outside of the mass gap. This contrasts with the results obtained in section D.2, which implies that there is a singular behavior of the mass term in this limit due to the loss of Lorenz invariance.

8.5 Comparing the critical behavior of the topological Abelian Higgs model with the CSGL theory

8.5.1 Conformality and quantum criticality of the topological Abelian Higgs model

In [3] the phase transitions and critical behavior of the topological Abelian Higgs model, or equivalently a topological superconductor, is studied using the renormalization group equations (RGE) of the Higgs self-interaction to one-loop order. This model is given by the following Lagrangian in real space

$$\mathcal{L} = \frac{\kappa}{2}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + |(\partial_\mu - ia_\mu)\varphi|^2 - m_0^2|\varphi|^2 - \frac{u_0}{2}|\varphi|^4, \tag{8.24}$$

where κ is the Chern-Simons coupling constant and m_0 and u_0 are bare coupling of the Higgs sector, which is described by the complex scalar field φ. In the absence of a Chern-Simons coupling, or for sufficiently large values of κ, the Higgs field decouples from the gauge field. In this limit eq. (8.24) reduces to a regular complex scalar field theory with self-interactions, which belongs to the regular SO(2) $\cong$ U(1) universality class of the Landau-Ginzburg-Wilson paradigm of phase transitions [9]. However, for finite values of κ, the RGE takes on a different form due to the one-loop contributions from the gauge field. This in turn enables the model to experience more exotic critical behavior [3]. For a single massless Higgs field [4], the β-function to one-loop order takes the form

[2] See section D.2 and section D.4 for details.

[3] As opposed to the mass term due to the Higgs-Anderson mechanism, which is gauge invariant (c.f. eq. (5.75)).

[4] Which corresponds to $T \to T_c$ in the case of superconductivity.

$$\beta(g) = \frac{g_*}{2}\left[\frac{\kappa_c^2}{\kappa^2} - 1 + \left(\frac{g}{g_*} - 1\right)^2\right] \tag{8.25}$$
$$g(\mu) = \left[\left(1 + \frac{4}{3\pi|\kappa|}\right)\frac{u_0}{\mu} - \frac{5}{8}\frac{u_0^2}{\mu^2} + \frac{1}{\kappa^2}\right], \tag{8.26}$$

where $g_* = \frac{4}{5}$ and $\kappa_c^2 = \frac{2}{g_*} = \frac{5}{2}$. This structure of the RGE is precisely of the form studied in [29]. For values $\kappa^2 > \kappa_c^2$, eq. (8.25) has two non-trivial fixed points

$$g_\pm = g_*\left(1 \pm \sqrt{1 - \frac{\kappa_c^2}{\kappa^2}}\right). \tag{8.27}$$

where g_+ corresponds to the quantum critical point of the theory. At these values, eq. (8.24) is scale-invariant (i.e., the system has a conformal symmetry). However, as $\kappa \to \kappa_c$, the fixed points of eq. (8.27) merge and become complex-valued as $|\kappa| < \kappa_c^2$ where conformality is lost. For $\kappa \in [-\kappa_c, \kappa_c]$ and $g < g_*$, it is shown in [3] that the energy scale and therefore also the critical exponents become a function of κ. Furthermore, the system becomes critical as $\kappa \to \kappa_c$, featuring a scaling much like the BKT transition where T is replaced by $\frac{1}{\kappa}$. For $g > g_*$, the system features an essential singularity in the energy scale as $g \to g_*$. However, the energy scale in this case does not depend on κ and therefore the system does not become critical as $\kappa \to \kappa_c$.

For $\kappa^2 > \kappa_c^2$ the energy scale and critical exponents are also functions of κ. As $\kappa \to \kappa_c$, the scaling coincides with that of the $g > g_*$ case above. This implies that the critical point at $\kappa \to \kappa_c$ is associated with a conformal phase transition of the system [29].

8.5.2 Transforming a modified CSGL theory into the topological Abelian Higgs model

The Cooper boson sectors in eq. (6.28) and eq. (7.16) to second order in coupling constants and in the long wavelength limit results in a model which is not equivalent to the topological Abelian Higgs model in eq. (8.24). However, by replacing the Ginzburg-Landau theory in eq. (6.1) with the corresponding time-dependent Ginzburg-Landau theory in [30], we get a Cooper boson sector which can be transformed into eq. (8.24)[5]. The resulting Lagrangian in real space takes the form

$$\mathcal{L}_\varphi = \lambda_{CS} v_F^2 \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \nu|(\partial_t - 2ie\phi)\varphi|^2 + \gamma\left|(\nabla - 2ie\bar{A})\varphi\right|^2 - \alpha|\varphi|^2 - \beta|\varphi|^2 \tag{8.28}$$

where we have extracted a factor v_F^2 using $A_\mu = (\phi, -v_F\bar{A})$ and $\partial_\mu = (\partial_t, v_F\nabla)$ in the Chern-Simons term and only included the terms in [30] that are directly comparable with the terms in eq. (8.24). In $2+1$-dimensions, the coupling constants ν and γ evaluates to

$$\nu = \frac{7D_0\zeta(3)}{16\pi^2 T_c^2} \tag{8.29}$$
$$\gamma = \frac{v_{F,S}^2}{2}\nu, \tag{8.30}$$

[5] This does not affect any of the other results obtained in this study, in particular the Chern-Simons coupling.

where $D_0 = \frac{m_e}{\pi}$ is the density of states of a two-dimensional electron gas, $\zeta(3)$ is the Riemann zeta function evaluated at $n = 3$, $v_{F,S}$ is the Fermi velocity of the superconductor and T_c is the critical temperature of the superconductor. In order to transform eq. (8.28) into the topological Abelian Higgs model, we must re-scale the fields and derivatives such that the factors in front of the second- and third term in eq. (8.28) becomes unity and the resulting Chern-Simons coefficient becomes dimensionless. By defining

$$\tilde{\varphi} = \frac{\sqrt{\gamma}}{v_{F,S}^{1/3}}\varphi \tag{8.31}$$

$$\tilde{\boldsymbol{\nabla}} = v_{F,S}^{1/3}\boldsymbol{\nabla} \tag{8.32}$$

$$\tilde{\bar{A}} = 2ev_{F,S}^{1/3}\bar{A} \tag{8.33}$$

$$\tilde{\partial}_t = v_{F,S}^{1/3}\sqrt{\frac{\nu}{\gamma}}\partial_t \tag{8.34}$$

$$\tilde{\phi} = v_{F,S}^{1/3}\sqrt{\frac{\nu}{\gamma}}\phi \tag{8.35}$$

we get the following re-scaled Lagrangian

$$\begin{aligned}\mathcal{L}_\varphi &= \frac{\lambda_{CS}v_F^2}{4e^2 v_{F,S}}\sqrt{\frac{\gamma}{\nu}}\varepsilon^{\mu\nu\lambda}\tilde{A}_\mu\tilde{\partial}_\mu\tilde{A}_\lambda + \left|(\tilde{\partial}_t - i\tilde{\phi})\tilde{\varphi}\right|^2 + \left|(\tilde{\boldsymbol{\nabla}} - i\tilde{\bar{A}})\tilde{\varphi}\right|^2 - \frac{\alpha v_{F,S}^{2/3}}{\gamma}|\tilde{\varphi}|^2 - \frac{\beta v_{F,S}^{4/3}}{\gamma^2}|\tilde{\varphi}|^4 \\ &= \frac{\kappa}{2}\varepsilon^{\mu\nu\lambda}\tilde{A}_\mu\tilde{\partial}_\mu\tilde{A}_\lambda + \left|(\tilde{\partial}_\mu - i\tilde{A}_\mu)\tilde{\varphi}\right|^2 - m_0|\tilde{\varphi}|^2 - \frac{u_0}{2}|\tilde{\varphi}|^4\end{aligned} \tag{8.36}$$

which is equivalent to the topological Abelian Higgs model in eq. (8.24).

8.5.3 Effective Chern-Simons coupling of the topological Abelian Higgs model

The dimensionless Chern-Simons coupling of eq. (8.36) can be written as

$$\begin{aligned}\kappa &= \frac{\lambda_{CS}v_F^2}{2e^2 v_{F,S}}\sqrt{\frac{\gamma}{\nu}} \\ &= \frac{v_F^2}{2e^2 v_{F,S}}\sqrt{\frac{\frac{v_{F,S}^2}{2}\nu}{\nu}}\frac{e^2\operatorname{sgn}(m)}{8\pi v_F^2}\frac{\sinh\beta|m|}{\cosh\beta|m| + \cosh\beta\mu} \\ &= \frac{\operatorname{sgn}(m)}{16\sqrt{2}\pi}\frac{\sinh\beta|m|}{\cosh\beta|m| + \cosh\beta\mu}.\end{aligned} \tag{8.37}$$

In fig. 8.2, we see that the maximum value of λ_{CS} is $\frac{e^2}{8\pi v_F^2}$ and that it quickly drops to 0 outside of the gap, which implies that

$$\kappa_{\max} = \frac{1}{16\sqrt{2}\pi} \approx 0.0141 \qquad \kappa_{\min} = 0 \tag{8.38}$$

Hence our effective Chern-Simons coupling of eq. (8.36) cannot be tuned above the critical value derived given in [3], i.e.,

$$\kappa_c = \sqrt{\frac{5}{2}} \approx 1.58, \tag{8.39}$$

which means that it is not possible to tune our system between the two regimes associated with $\kappa^2 > \kappa_c^2$ and $\kappa^2 < \kappa_c^2$. Furthermore, since we are using a planar superconductor in eq. (8.36) rather than a corresponding three-dimensional superconductor, we are likely to overestimate the proximity effects of our heterostructure and therefore also the effect of the Chern-Simons coupling to the superconductor. We have also excluded the Maxwell sector in eq. (8.36) since this term is assumed sub-dominant at long wavelengths near the critical points [3].

8.6 Topological magnetoelectric effect

It is shown in [31] that the Chern-Simons term in eq. (8.5) results in a renormalization of the in-plane components of the Berry phase and a so-called topological magnetoelectric coupling, in addition to the Chern-Simons term in eq. (8.4). These results were obtained in a ferromagnetic topological insulator heterostructure at zero temperature and chemical potential and hence our results are a generalization of these findings. Furthermore, a comparable Chern-Simons coupling at finite temperature in terms of a Coulomb potential and magnetic impurities was also found in a similar system in [5]. In our notation, the topological magnetoelectric coupling takes the form [31]

$$\begin{aligned}\mathcal{L}_{\text{TME}} &= \frac{2\lambda_{CS} v_F}{e} \int \mathrm{d}t \int \mathrm{d}^2 r \boldsymbol{\nabla}\phi \cdot \bar{n} \\ &= \lambda_{\text{TME}} \int \mathrm{d}t \int \mathrm{d}^2 r \bar{E} \cdot \bar{n}\end{aligned} \tag{8.40}$$

where we have performed a Wick-rotation into real-time. This magnetoelectric coupling is topological in the sense that the Chern-Simons coupling is topologically protected, which distinguishes it from the other magnetoelectric couplings of eq. (8.5). In [5] they solve the Landau-Lifshitz equations for a spin-system containing this term, which results in a topologically protected gap of the corresponding magnon spectrum for finite externally applied electric field. Furthermore, in [31], it is shown that eq. (8.40) can result in a non-local coupling of the form

$$\mathcal{L}_\rho(r) = \frac{1}{2}\rho(r) \int \mathrm{d}^2 r' \frac{\rho(r')}{|r - r'|}, \tag{8.41}$$

where $\rho(r) = \lambda_{\text{TME}} \boldsymbol{\nabla} \cdot \bar{n}$ are effective magnetic charges. However, these results require that we take into account the Coulomb interactions between the surface fermions, which amounts to adding the terms of eq. (12) in [31] into eq. (6.3).

8.7 Coexisting magnetism and superconductivity in the CSGL model

8.7.1 Fulde-Ferrell-Larkin-Ovchinnikov superconductivity

Superconductivity and ferromagnetism are mutually exclusive phases of matter in the sense that sufficiently strong magnetic ordering will polarize the spins of the fermions and thereby break the superconducting spin-singlet Cooper-pair condensates. In the case of uniform magnetization, a system in a superconducting phase is followed by a first-order transition to a ferromagnetic phase in which superconductivity vanishes. However, there are certain exceptions to this behavior in the case of non-uniform magnetization, where there is a co-existing superconducting phase of matter and magnetism in the case of e.g., oscillatory spiraling magnetic fields [6].

Superconductors under the influence of a constant uniform magnetization are predicted to experience a type of superconductivity between the pure superconducting and ferromagnetic phases [32], [33]. If the number of Cooper pairs broken by the magnetization is sufficiently small, then the BCS energy gap given by eq. (5.26) is unaltered. The resulting state has a higher energy than the BCS ground state. However, as the number of broken Cooper-pairs increases, the size of the energy gap is reduced, allowing for a new ground state of the ferromagnetic superconductor system known as the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) phase. In FFLO type superconductivity, the magnetic fields alter the momentum of the up- and down-spins which means that the Cooper-pairs obtain a non-zero center of mass momentum. This results in a modulation of the order parameter of the superconductor and therefore also a modulation between the superconducting state and the normal state. As a result, the electrodynamics of the superconductor is modified, restricting the supercurrent to be parallel to the induced center of mass momentum of the Cooper pairs [32], [34].

8.7.2 The possibility of an FFLO phase driven by the Dzyaloshinskii-Moriya term

Since the Dzyaloshinskii-Moriya coupling in eq. (8.5) favors perpendicular alignment of spins, it can alter the ferromagnetic ordering of the system, which in turn makes it easier for superconductivity and magnetism to coexist.

The presence of this term can induce so-called helimagnetism, which is a type of magnetic ordering where spins align in spiral patterns [20]. This effect could lead to a modulation of the ferromagnetic ordering of the system, which could support the coexistence of superconductivity and magnetism. The latter requires that the London penetration length of the superconductor and the spin-exchange couplings of the magnetic impurities take on specific values in accordance with the modulation of the helical ordering [6].

This ability to alter the ferromagnetic ordering of the system implies that the Dzyaloshinskii-Moriya term reduces the total magnetization of the system. This means that by changing the magnitude of the Dzyaloshinskii-Moriya coefficient, we could change the overall magnetization accordingly and thereby possibly change between ferromagnetism, superconductivity, and even tune the system into FFLO type superconductivity.

In a study by J. Rowland and S. Banerjee and M. Randeria, they investigate the phases of a magnetic Ginzburg Landau model [35]. This model contains a Dzyaloshinskii-Moriya coupling with coupling constant $D_{\parallel}$ due to a Dresselhaus spin-orbit coupling, which is directly comparable with the Dzyaloshinskii-Moriya term found in eq. (8.5). Furthermore, they assume a magnetic anisotropy in the z-direction with phenomenological coupling constant A and an external magnetic field H which couples linearly to the magnetic fluctuations. For easy-plane anisotropy ($A > 0$) in the absence of Rashba spin-orbit coupling, the magnetic system is dominated by a vertical cone phase where the spins are precessing with a fixed angle θ_0 relative to the z-axis.

This angle depends on the magnitude of the Dzyaloshinskii-Moriya coefficient according to the following equation

$$\cos\theta_0 = \frac{H}{2A + \frac{D_\parallel^2}{J}} = \frac{H}{2A\left(1 + \frac{D_\parallel^2}{AJ}\right)}, \tag{8.42}$$

Hence the ability to tune the ferromagnetic ordering of this system depends on the quantity $\eta = \frac{D_\parallel^2}{AJ}$, which in our notation can be written as

$$\eta = \frac{\zeta_{DM}^2}{\kappa A}, \tag{8.43}$$

where κ is the dynamical exchange coupling in eq. (7.1). Assuming that the ferromagnetic ordering of the magnetic impurities is mostly driven by an external field, we can ignore the mass term m^2 of eq. (7.1). In this case, the anisotropy of the magnetic sector is dominated by eq. (7.62), which corresponds to an easy-plane anisotropy of order $\frac{J_\perp^2|m|}{8\pi v_F^2}$. Hence, we get the following

$$\eta = \left(\frac{J_\parallel J_\perp}{4\pi v_F}\right)^2 \frac{8\pi v_F^2}{\kappa J_\perp^2|m|} = \frac{J_\parallel^2}{2\pi\kappa|m|}. \tag{8.44}$$

Thus, the tunability of the induced Dzyaloshinskii-Moriya coupling becomes considerable if the magnitude of the spin-spin exchange couplings $J_\parallel$ become sizeable compared to the dynamical exchange coupling κ and the magnetically induced Dirac mass.

Chapter 9

Conclusion and outlook

In this thesis, we have derived and studied the effective topological field theory of the interface between a superconductor proximate to a topological insulator coupled to a gauge field and ferromagnetically aligned magnetic impurities. The effects of the boundary modes of the topological insulator have been integrated out of the partition function to second order in proximity couplings to the respective materials. In addition to renormalizations of the bare couplings, the surface states of the topological insulator also induce a thermal mass of the electric potential, a Chern-Simons term, a Dzyaloshinskii-Moriya term in addition to several other magnetoelectric couplings not present in any of the individual systems. Some of these couplings are also new compared to similar systems and heterostructures.

The presence of a mass term in the effective field theory implies that the effective Coulomb potential is screened accordingly. At low temperatures, the system experiences a Thomas-Fermi type screening of the Coulomb repulsion, whereas at higher temperatures we get a Debye screening. In the insulating phase of the surface fermions, the mass term becomes negative in an intermediate temperature regime before the Debye screening is dominant. The effective potential in this case changes sign at length scales comparable with the Thomas-Fermi screening length, resulting in an attraction between charges. This analysis also proves the absence of a corresponding screening of the magnetic field in the more general situation of massive fermions at finite chemical potential. Furthermore, we have shown that the limit of zero temperature is singular if the chemical potential is outside of the mass gap of the surface fermions, which is due to a lack of Lorenz invariance.

Introducing a Chern-Simons term in the Ginzburg-Landau theory of the superconductor means that the resulting field theory is a topological field theory. Due to the matter-field coupling between the Cooper bosons and the Chern-Simons gauge field, the system can support exotic vortex- and anyonic excitations in terms of these fields, unlike a regular conventional superconductor. Furthermore, this coupling makes it possible to compare the resulting Chern-Simons-Ginzburg-Landau theory with the critical behavior of the topological Abelian Higgs model studied in [3]. Thus, our heterostructure system serves as a physical realization of this model with a tunable Chern-Simons coupling. However, it turns out that our Chern-Simons coupling can only be tuned below the critical value κ_c derived in [3], which implies that this system cannot be used to tune between the different types of critical behavior of the topological Abelian Higgs model. By including the effects of the magnetic impurities, the Chern-Simons coupling causes a modification to the Berry phase associated with the in-plane components of the magnetic fluctuations in addition to a topological magnetoelectric coupling. The latter results in e.g., a topologically protected gap in the magnon spectrum of the impurities and a non-local interaction term between effective magnetic charges. These couplings have also been found in comparable ferromagnetic topological insulator systems.

The anti-symmetric exchange interaction associated with the Dzyaloshinskii-Moriya term energetically favors perpendicular spin-alignments, which in turn can affect the magnetic ordering of the heterostructure. For sufficiently larges magnitudes of the coupling constant, the resulting system can experience e.g., helical magnetic ordering, which lowers the overall magnetization of the impurities. Since the Dzyaloshinskii-Moriya coupling is also a function of material parameters, it is therefore possible that the system can be tuned between a superconducting and a ferromagnetic phase, in particular the system might support tunable Fulde-Ferrell-Larkin-Ovchinnikov type superconductivity. Furthermore, it is shown that a tunable ferromagnetic ordering is possible if the spin-spin exchange couplings between the fermions and the magnetic fluctuations are comparable with the size of the dynamical exchange coupling of the magnetic impurities and the induced gap of the surface states.

To second order in coupling constants, this model features no direct couplings between the Cooper bosons and the magnetic impurities. Such terms might contribute to the coexistence and tunability of the superconducting and magnetic phases discussed in this thesis. Additionally, since the Chern-Simons action is in terms of an effective gauge field containing magnetic spins, we anticipate that these kinds of interactions might lead to interesting interplay between the magnetic sector and the topological field theory of the superconductor. Hence a higher-order expansion might complement these results and lead to more interesting phenomena. Furthermore, since some of the physical implications of this model have been discussed based on generalities, a thorough study of the model is of considerable interest.

Appendix A

Clifford Algebras and γ-matrix representations

A.1 Proof of the commutational relations of the 2+1-dimensional Euclidean γ-matrix algebra

In Euclidean space, the γ-matrices are

$$\bar{\gamma} = (\gamma^0, \gamma^1, \gamma^2) = (i\sigma^z, -i\sigma^y, i\sigma^x), \tag{A.1}$$

where $(\sigma^x, \sigma^y, \sigma^z)$ are the Pauli matrices which obey the following commutational relations [7]

$$[\sigma^\mu, \sigma^\nu] = 2i\varepsilon^{\mu\nu\lambda}\sigma^\lambda \tag{A.2}$$
$$\{\sigma^\mu, \sigma^\nu\} = 2\delta^{\mu\nu}, \tag{A.3}$$

where Greek indices iterates over Euclidean space. By using eq. (A.3), we immediately get the following

$$\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}. \tag{A.4}$$

Secondly, by using eq. (A.2), we get

$$\begin{aligned}
\gamma^0\gamma^1 &= (i\sigma^z)(-i\sigma^y) = \sigma^z\sigma^y = -\sigma^y\sigma^z = -i\sigma^x = -\gamma^2 \\
\gamma^1\gamma^2 &= (-i\sigma^y)(i\sigma^x) = \sigma^y\sigma^x = -\sigma^x\sigma^y = -i\sigma^z = -\gamma^0 \\
\gamma^2\gamma^0 &= (i\sigma^x)(i\sigma^z) = -\sigma^x\sigma^z = -i\sigma^y = -\gamma^1.
\end{aligned}$$

which can be compactly written as follows

$$[\gamma^\mu, \gamma^\nu] = -2\varepsilon^{\mu\nu\lambda}\gamma^\lambda. \tag{A.5}$$

A.2 Trace relations of the Euclidean 2+1-dimensional γ-matrices

Using eq. (A.4), eq. (A.5), and the fact that the Pauli-matrices are individually traceless, we can derive the following trace relations for the γ-matrices in section A.1

$$\mathrm{tr}(\gamma^\mu) = 0 \tag{A.6}$$

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) = \frac{1}{2}\,\mathrm{tr}(\gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\nu) = \frac{1}{2}\,\mathrm{tr}(\gamma^\mu\gamma^\nu + (\gamma^\nu\gamma^\mu - 2\delta^{\mu\nu})) = -2\delta^{\mu\nu} \tag{A.7}$$

$$\begin{aligned}\mathrm{tr}\Big(\gamma^\mu\gamma^\nu\gamma^\lambda\Big) &= \frac{1}{2}\,\mathrm{tr}\Big(\gamma^\mu\gamma^\nu\gamma^\lambda + (\gamma^\nu\gamma^\mu\gamma^\lambda - 2\varepsilon^{\mu\nu\lambda}(\gamma_\lambda\gamma^\lambda))\Big)\\ = \mathrm{tr}\Big(\gamma^\mu\gamma^\nu\gamma^\lambda\Big) &= \frac{1}{2}\,\mathrm{tr}\Big(\gamma^\mu\gamma^\nu\gamma^\lambda + ((-\gamma^\mu\gamma^\nu - 2\delta^{\mu\nu})\gamma^\lambda - 2\varepsilon^{\mu\nu\lambda})\Big) = -2\varepsilon^{\mu\nu\lambda}\end{aligned} \tag{A.8}$$

$$\begin{aligned}\mathrm{tr}\Big(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\Big) &= \frac{1}{2}\,\mathrm{tr}\Big(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\Big) + \frac{1}{2}\,\mathrm{tr}\Big(-\gamma^\nu\gamma^\mu\gamma^\lambda\gamma^\rho - 2\delta^{\mu\nu}\gamma^\lambda\gamma^\rho\Big)\\ &= 2\delta^{\mu\nu}\delta^{\lambda\rho} + \frac{1}{2}\,\mathrm{tr}\Big(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\Big) + \frac{1}{2}\,\mathrm{tr}\Big(-\gamma^\nu(-\gamma^\lambda\gamma^\mu - 2\delta^{\mu\lambda})\gamma^\rho\Big)\\ &= 2\delta^{\mu\nu}\delta^{\lambda\rho} - 2\delta^{\mu\lambda}\delta^{\nu\rho} + \frac{1}{2}\,\mathrm{tr}\Big(\gamma^\nu\gamma^\lambda(-\gamma^\rho\gamma^\mu - 2\delta^{\rho\mu})\Big) = 2\delta^{\mu\nu}\delta^{\lambda\rho} - 2\delta^{\mu\lambda}\delta^{\nu\rho} + 2\delta^{\mu\rho}\delta^{\nu\lambda}.\end{aligned} \tag{A.9}$$

A.3 Proof of the commutational relations of the 3+1-dimensional Euclidean γ-matrix algebra

In $3+1$-dimensional Euclidean space, we have the following γ-matrices

$$\gamma^\mu = (i\gamma^0, -\gamma^i), \tag{A.10}$$

where γ^0 and γ^i refer to the time- and spatial components of the usual γ-matrices in Minkowski space (c.f. [8]). Using this expression, we immediately get the following relations in Euclidean space

$$\{i\gamma^0, i\gamma^0\} = 2i^2 = -2 \tag{A.11}$$

$$\{i\gamma^0, -\gamma^i\} = -i\{\gamma^0, \gamma^i\} = 0 \tag{A.12}$$

$$\{-\gamma^i, -\gamma^j\} = \{\gamma^i, \gamma^j\} = -2\delta_{ij}. \tag{A.13}$$

This implies that in Euclidean space, we have the following commutational relations

$$\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu} \tag{A.14}$$

A.4 Trace relations of the Euclidean 3+1-dimensional γ-matrices

Comparing eq. (A.14) with eq. (A.4), we see that the γ-matrices in this case have the same anti-commutational relations as our previous Euclidean γ-matrices. Hence, the only adjustment we need to do in order to obtain the trace relations of even products of the $3+1$-dimensional γ-matrices is to multiply by a factor 2 since these are 4x4 matrices. Thus, we obtain the following

$$\operatorname{tr}\left(\gamma^\mu\gamma^\nu\right) = -4\delta^{\mu\nu} \tag{A.15}$$

$$\operatorname{tr}\left(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\right) = 4\left(\delta^{\mu\nu}\delta^{\lambda\rho} - \delta^{\mu\lambda}\delta^{\nu\rho} + \delta^{\mu\rho}\delta^{\nu\lambda}\right) \tag{A.16}$$

By using e.g., the Weyl representation [8] where the γ-matrices are off-diagonal, we immediately get that products containing an odd number of γ-matrices are off-diagonal, implying that

$$\operatorname{tr}\left(\gamma^\mu\gamma^\nu\gamma^\lambda\right) = 0, \tag{A.17}$$

which holds independently of the chosen representation.

Appendix B

Fermionic Matsubara sums

B.1 Evaluating convergent sums over meromorphic functions

Let $g(z)$ be a holomorphic complex valued function with isolated poles $\Sigma = \{z_1, \cdots z_m\}$ (i.e. a meromorphic function). Let $f(z) = \frac{1}{e^{\beta z}+1}$ be the Fermi-Dirac distribution with poles $\Omega = \{i\omega_n\}$, i.e. the set of fermionic Matsubara frequencies. If $\Sigma \not\subset \Omega$ and $\mathcal{C}$ is a contour enclosing all $z \in \Omega$, we have the following equality [7]

$$\oint_{\mathcal{C}} g(z)f(z) = 2\pi i \sum_n \mathrm{Res}_{z\to i\omega_n} g(z)f(z) = 2\pi i \sum_n g(i\omega_n)\,\mathrm{Res}_{z\to i\omega_n} f(z). \tag{B.1}$$

where we have used the residue theorem. In this setting, the function $f(z)$ is called a kernel function. By rewriting the Fermi-Dirac distribution as follows

$$\begin{aligned}\frac{1}{e^{\beta z}+1} &= \frac{1}{e^{\beta(z-i\omega_n+i\omega_n)}+1} = \frac{1}{1-e^{\beta(z-i\omega_n)}} \\ &= \frac{1}{1-(1+(z-i\omega_n)+\cdots)} = \frac{1}{-\beta(z-i\omega_n)(1+\frac{z-i\omega_n}{2}+\cdots)}\end{aligned} \tag{B.2}$$

we immediately see that the Matsubara frequencies are simple poles and that

$$\mathrm{Res}_{z\to i\omega_n} f(z) = \frac{-1}{\beta}, \tag{B.3}$$

which implies that

$$\sum_n g(i\omega_n) = \frac{-\beta}{2\pi i}\oint_{\mathcal{C}} g(z)f(z). \tag{B.4}$$

The path $\mathcal{C}$ can be (diffeomorphically) deformed into closed loops around all $z \in \Sigma$ plus some contribution at arbitrary radius in the complex plane. If the function $g(z)f(z)$ converges sufficiently fast to 0 when $z \to \infty$, then this contribution goes to zero, and we end up with the following relation [36] [7]

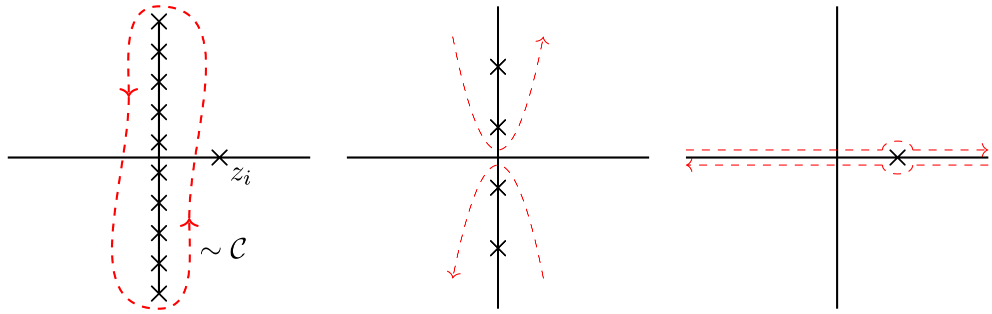

Figure B.1: Deformation of the path $\mathcal{C}$. Note the opposite orientation around the poles of $g(z)$.

$$\sum_n g(i\omega_n) = \frac{-\beta}{2\pi i}\oint_{\mathcal{C}} g(z)f(z) = \beta \sum_{z_i\in\Sigma} \mathrm{Res}_{z\to z_i}\, g(z)f(z). \tag{B.5}$$

The relative minus sign is because that the path we end up with is clockwise (see fig. B.1). This procedure works in the bosonic case as well, but then one must use the Bose-Einstein distribution instead of the Fermi-Dirac distribution in the residue integral.

B.2 Change of variables

Assume that you have a linear change of variables of the form $h(z) = z + i\nu_l$, where $\nu_l = \frac{2\pi l}{\beta}$ is a bosonic Matsubara frequency. Then, we get

$$\sum_n g(i\omega_n) = \frac{-\beta}{2\pi i}\oint_{\mathcal{C}} f(z)g(z)\mathrm{d}z = \frac{-\beta}{2\pi i}\oint_{\mathcal{C}'} f(h(z))g(h(z))h'(z)\mathrm{d}z, \tag{B.6}$$

where $\mathcal{C}'$ is a contour around the set $\Omega' = \{i\omega_n + i\nu_l\}$. Since $\mathcal{C}$ is a path around Ω, we immediately get that $\mathcal{C}' = \mathcal{C}$, since $\Omega' = \{i\omega_n + i\nu_l\} = \{i\omega_{n+l}\} = \Omega$. We also have that $f(z+i\nu_l) = f(z)$ and that the change of variables has unit Jacobian, $h'(z) = 1$. Hence, we arrive at

$$\frac{-\beta}{2\pi i}\oint_{\mathcal{C}'} f(z)g(h(z))\mathrm{d}z = \frac{-\beta}{2\pi i}\oint_{\mathcal{C}} f(z)g(h(z))\mathrm{d}z = \sum_n g(h(i\omega_n)) = \sum_n g(i\omega_n + i\nu_l). \tag{B.7}$$

Thus, sums over fermionic Matsubara frequencies are invariant under linear change of variables in terms of bosonic Matsubara frequencies.

B.3 Divergent sums and choice of kernel function

For simple poles, the sum over Matsubara frequencies diverges logarithmically,

$$\sum_n \frac{1}{i\omega_n - \epsilon} \sim \int \frac{\mathrm{d}\omega}{\omega - \epsilon} \sim \ln(\omega - \epsilon). \tag{B.8}$$

In these and similar situations, the choice of kernel function is non-trivial. Assuming that $g(z)$ is still a meromorphic function of z, we can resolve this issue by using the following kernels [7]

$$f(z) = \frac{1}{e^{\beta z} + 1} \tag{B.9}$$

if the pole resides in the rightmost half-plane, $\mathrm{Re}(z) > 0$, or

$$f(z) = \frac{-1}{e^{-\beta z} + 1} \tag{B.10}$$

if the pole resides in the leftmost half-plane, $\mathrm{Re}(z) < 0$.

B.4 One-loop sums

Assume that we have the following expression

$$\frac{1}{\beta}\sum_n \frac{1}{-(i\omega_n+\mu)^2+m^2}. \tag{B.11}$$

This convergent sum is over a meromorphic function $g(z) = \frac{1}{-(z+\mu)^2+m^2}$ and hence we can use the techniques established in section B.1.

$$\frac{1}{\beta}\sum_n \frac{1}{-(i\omega_n+\mu)^2+m^2} = \sum_{z_i\in\Sigma} \mathrm{Res}_{z\to z_i}\, \frac{1}{-(z+\mu)^2+m^2} f(z) \tag{B.12}$$

where $\Sigma = \{-\mu+|m|, -\mu-|m|\}$. These are simple poles and hence we get the following

$$\begin{aligned}
\sum_{z_i\in\Sigma} \mathrm{Res}_{z\to z_i}\, g(z)f(z) &= \lim_{z\to-\mu+|m|}(z-(-\mu+|m|))\frac{1}{(-z-\mu-|m|)(-z-\mu+|m|)}f(z) \\
&\quad + (-\mu+|m| \leftrightarrow -\mu-|m|) \\
&= -\frac{1}{2|m|}\frac{1}{e^{\beta(-\mu+m)}+1} + \frac{1}{2|m|}\frac{1}{e^{\beta(-\mu-|m|)}+1} \\
&= \frac{1}{2|m|}\frac{\sinh\beta|m|}{\cosh\beta\mu+\cosh\beta|m|}.
\end{aligned} \tag{B.13}$$

Now, assume that we have the following expression

$$-\frac{1}{\beta}\sum_n \frac{1}{(-(i\omega_n+\mu)^2+m^2)^2} \tag{B.14}$$

This is also a convergent sum over a meromorphic function, but this time the singularities are of second order. Thus, we need to evaluate the following

$$\sum_{z_i\in\Sigma}\mathrm{Res}_{z\to z_i}\frac{1}{((z+\mu)^2+m^2)^2}f(z)=-\sum_{z\in\Sigma}\mathrm{Res}_{z\to z_i}\frac{1}{(z-z_+)^2(z-z_-)^2}f(z) \tag{B.15}$$

where Σ is the same as in the previous example and $z_\pm=-\mu\pm|m|$. Evaluating the residues, we get

$$\begin{aligned}\mathrm{Res}_{z\to z_\pm}\frac{f(z)}{(z-z_+)^2(z-z_-)^2} &= \lim_{z\to z_\pm}\frac{\mathrm{d}}{\mathrm{d}z}\frac{f(z)}{(z-z_\mp)^2}\\ &= \lim_{z\to z_\pm}\Big[\frac{f'(z)}{(z-z_\mp)^2}-\frac{2f(z)}{(z-z_\mp)^3}\Big]\\ &= \frac{f'(z_\pm)}{4|m|^2}\mp\frac{f(z_\pm)}{4|m|^3}\end{aligned} \tag{B.16}$$

Consequently, we get that

$$\begin{aligned}-\frac{1}{\beta}\sum_n\frac{1}{(-(i\omega_n+\mu)^2+m^2)^2} &= -\left[\frac{f'(z_+)}{4|m|^2}-\frac{f(z_+)}{4|m|^3}+(-\leftrightarrow+)\right]\\ &= \frac{\beta}{16|m|^2}\Big[\frac{1}{\cosh^2\frac{\beta(|m|-\mu)}{2}}+\frac{1}{\cosh^2\frac{\beta(|m|+\mu)}{2}}\Big]\\ &-\frac{1}{4|m|^3}\frac{\sinh\beta|m|}{\cosh\beta\mu+\cosh\beta|m|}.\end{aligned} \tag{B.17}$$

Next, we evaluate the following sum

$$\frac{1}{\beta}\sum_n\frac{i\omega_n+\mu}{(i\omega_n+\mu)^2-m^2}. \tag{B.18}$$

By performing a partial fraction decomposition, we see that this sum is divergent, cf. section B.3

$$\frac{i\omega_n+\mu}{(i\omega_n+\mu)^2-m^2}=\frac{1}{2}\Big[\frac{1}{i\omega_n-\epsilon_+}+\frac{1}{i\omega_n-\epsilon_-}\Big], \tag{B.19}$$

where $\epsilon_\pm=-\mu\pm|m|$. Thus, we must use as kernels eq. (B.10) and eq. (B.9) for ϵ_- and ϵ_+ respectfully. Hence, we get the following

$$\begin{aligned}\frac{1}{\beta}\sum_n\frac{i\omega_n+\mu}{(i\omega_n+\mu)^2-m^2} &= \frac{1}{2}\Big[\frac{1}{\mathrm{e}^{\beta\epsilon_+}+1}-\frac{1}{\mathrm{e}^{-\beta\epsilon_-}+1}\Big]\\ &= \frac{1}{2}\frac{\sinh\beta\mu}{\cosh\beta\mu+\cosh\beta|m|}\end{aligned} \tag{B.20}$$

Appendix C

Dimensional regularization of Euclidean momentum integrals

C.1 One-loop integrals

Using as a starting point the integral in the Feynman amplitude appendix in [8], we can transform the integral

$$\int \frac{\mathrm{d}^{2\omega}\kappa}{(2\pi)^{2\omega}} \frac{1}{(\kappa^2 - m^2 + i\varepsilon)^\alpha} = i\frac{(-1)^\alpha}{(4\pi)^\omega} \frac{\Gamma(\alpha-\omega)}{\Gamma(\alpha)} \left[m^2 - i\varepsilon\right]^{\omega-\alpha} \tag{C.1}$$

into an integral over Euclidean space by performing a Wick rotation

$$\int \frac{\mathrm{d}^{2\omega}\kappa}{(2\pi)^{2\omega}} \frac{1}{(\kappa^2 + m^2)^\alpha} = \frac{1}{(4\pi)^\omega} \frac{\Gamma(\alpha-\omega)}{\Gamma(\alpha)} \left[m^2\right]^{\omega-\alpha}. \tag{C.2}$$

This integral is well defined as long as $\Gamma(\alpha - \omega)$ is well defined and the denominator has a non-zero imaginary part if it has a pole on the real axis (e.g. Matsubara frequency, convergence factor, etc.). If the integral is divergent, we introduce dimensional regularization by performing the integral in $2\omega = d - 2\epsilon$ dimensions, where the integral is assumed to be convergent, and then take the limit $\epsilon \to 0$ using the series expansion

$$\Gamma(z) = \frac{1}{z} - \gamma + \frac{1}{2}\left(\gamma^2 + \frac{\pi^2}{6}\right) z + \mathcal{O}(z^2), \tag{C.3}$$

from which we can extract a finite answer.

Appendix D

Euclidean vacuum polarization amplitude and mass terms at zero temperature

D.1 Vacuum polarization amplitude in $2+1$-dimensions

At zero temperature, the vacuum polarization tensor of eq. (6.63) can be written as follows

$$\delta S_A = -\int \frac{\mathrm{d}^3\xi}{(2\pi)^3} A^\mu(\xi)\Pi_{\mu\nu}(\xi)A^\nu(-\xi) \tag{D.1}$$

$$\Pi_{\mu\nu}(\xi) = \int \frac{\mathrm{d}^3\kappa}{(2\pi)^3} \frac{ime^2\varepsilon_{\mu\lambda\nu}(-i\xi^\lambda) - e^2\Sigma_{\mu\nu}}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)}, \tag{D.2}$$

where $\Sigma_{\mu\nu}$ is defined in eq. (6.64) and where we have used the following continuum limits

$$\frac{1}{\beta}\sum_n g(\omega_n) \to \int \frac{\mathrm{d}\omega}{2\pi} g(\omega) \qquad \frac{1}{\beta}\sum_l g(\nu_l) \to \int \frac{\mathrm{d}\nu}{2\pi} g(\nu), \tag{D.3}$$

where ω and ν are continuous energy variables. The chemical potential has been removed from the problem by performing a linear change of variables $\omega - i\mu \to \omega$. The Chern-Simons contribution is treated in detail in [4] and we neglect it in this discussion for the sake of argument. Hence, we are left with the following amplitude

$$\Pi_{\mu\nu}(\xi) = -e^2 \int \frac{\mathrm{d}^3\kappa}{(2\pi)^3} \frac{\delta_{\mu\nu}(m^2+\kappa(\kappa-\xi)) - 2\kappa^\mu\kappa^\nu + \kappa^\nu\xi^\mu + \xi^\mu\kappa^\nu}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)}. \tag{D.4}$$

Using the Ward identity,

$$\xi_\mu \Pi_{\mu\nu} = 0, \tag{D.5}$$

we can write eq. (D.4) as

$$\Pi_{\mu\nu}(\xi) = S(\xi)P_{\mu\nu}(\xi) \tag{D.6}$$

$$P_{\mu\nu} = \delta_{\mu\nu} - \frac{\xi_\mu\xi_\nu}{\xi^2}, \tag{D.7}$$

where $S(\xi)$ a scalar [4], [8]. By taking the trace of eq. (D.6), we get

$$\operatorname{tr}\Pi_{\mu\nu} = \Pi_{\mu\mu} = S(\xi)\operatorname{tr}P_{\mu\nu} = 2S(\xi), \tag{D.8}$$

where we have used that $\operatorname{tr}\xi_\mu\xi_\nu = \xi^2$. Hence, we get the following expression for $S(\xi)$

$$\begin{aligned} S(\xi) &= \frac{1}{2}\Pi_{\mu\mu} \\ &= -\frac{e^2}{2}\int\frac{\mathrm{d}^3\kappa}{(2\pi)^3}\frac{3(m^2+\kappa(\kappa-\xi)) - 2\kappa^2 + 2\kappa\cdot\xi}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)} \end{aligned} \tag{D.9}$$

Using the following relation

$$\kappa^2 - \kappa\xi = \frac{1}{2}[(\kappa^2+m^2) + (\kappa-\xi)^2 + m^2] - m^2 - \frac{1}{2}\xi^2 \tag{D.10}$$

and performing a linear change of variables, we can write $S(\xi)$ as follows

$$\begin{aligned} S(\xi) = &-e^2\Big(m^2 - \frac{\xi^2}{4}\Big)\int\frac{\mathrm{d}^3\kappa}{(2\pi)^3}\frac{1}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)} \\ &- \frac{e^2}{2}\int\frac{\mathrm{d}^3\kappa}{(2\pi)^3}\frac{1}{\kappa^2+m^2}. \end{aligned} \tag{D.11}$$

Using a Feynman parametrization, we can rewrite the first integral in eq. (D.11) as follows

$$\begin{aligned} \int\frac{\mathrm{d}^3\kappa}{(2\pi)^3}\frac{1}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)} &= \int\frac{\mathrm{d}^3\kappa}{(2\pi)^3}\int_0^1\mathrm{d}z\frac{1}{\big((\kappa^2+m^2)z + ((\kappa-\xi)^2+m^2)(1-z)\big)^2} \\ &= \int\frac{\mathrm{d}^3\kappa}{(2\pi)^3}\int_0^1\mathrm{d}z\frac{1}{\big(\kappa^2 + 2\kappa\xi(z-1) + m^2 + \xi^2(1-z)\big)^2} \\ &= \int\frac{\mathrm{d}^3\kappa}{(2\pi)^3}\int_0^1\mathrm{d}z\frac{1}{\big(\kappa^2 + m^2 + \xi^2 z(1-z)\big)^2}, \end{aligned} \tag{D.12}$$

where we have performed a linear change of variables in the last line. Using eq. (C.2), the κ-integral evaluates to

$$\begin{aligned} \int\frac{\mathrm{d}^3\kappa}{(2\pi)^3}\frac{1}{\big(\kappa^2+m^2+\xi^2 z(1-z)\big)^2} &= \frac{1}{(4\pi)^{\frac{3}{2}}}\frac{\Gamma\left(2-\frac{3}{2}\right)}{\Gamma(2)}\big(m^2+\xi^2 z(1-z)\big)^{\frac{3}{2}-2} \\ &= \frac{1}{8\pi}\frac{1}{m\sqrt{1+\frac{\xi^2 z(1-z)}{m^2}}} \\ &= \frac{1}{8\pi m}\left(1 - \frac{\xi^2 z(1-z)}{2m^2}\right) + \mathcal{O}(\xi^3), \end{aligned} \tag{D.13}$$

where we have expanded to second order in momenta and used that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. After performing the Feynman parameter integral, we are left with

$$\frac{1}{8\pi m} - \frac{\xi^2}{96\pi m^3}. \tag{D.14}$$

The second integral in eq. (D.11) evaluates to

$$\begin{aligned}\int \frac{\mathrm{d}^3\kappa}{(2\pi)^3}\frac{1}{\kappa^2+m^2} &= \frac{1}{(4\pi)^{\frac{3}{2}}}\frac{\Gamma\left(1-\frac{3}{2}\right)}{\Gamma(1)}\left(m^2\right)^{\frac{3}{2}-1} \\ &= -\frac{m}{4\pi},\end{aligned} \tag{D.15}$$

where we have used that $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$. Combining eq. (D.14) and eq. (D.15), we arrive at

$$\begin{aligned}S(\xi) &= -e^2\left(m^2 - \frac{\xi^2}{4}\right)\left(\frac{1}{8\pi m} - \frac{\xi^2}{96\pi m^3}\right) - \frac{e^2}{2}\left(-\frac{m}{4\pi}\right) \\ &= \frac{e^2\xi^2}{24\pi m}\end{aligned} \tag{D.16}$$

D.2 Possible mass terms in $2+1$ dimensions

A non-zero mass term requires that the vacuum polarization amplitude persists in the limit $\xi \to 0$. Using the results of the previous section, in particular eq. (D.16), we get that

$$\begin{aligned}\lim_{\xi\to 0}\Pi_{\mu\nu}(\xi) &= \lim_{\xi\to 0}\frac{e^2\xi^2}{24\pi m}\left(\delta_{\mu\nu} - \frac{\xi_\mu\xi_\nu}{\xi^2}\right) \\ &= 0\end{aligned} \tag{D.17}$$

and consequently, we get no mass term at $T=0$ in $2+1$ dimensions.

D.3 $\Pi^{\mu\nu}$ in general dimensions

In general $d+1$-dimensional spacetime at zero temperature, the Euclidean vacuum polarization amplitude takes the form

$$\Pi^{\mu\nu} \sim \int \frac{\mathrm{d}^{d+1}\kappa}{(2\pi)^{d+1}}\,\mathrm{tr}\left[\frac{\not{\kappa}-m}{\kappa^2+m^2}\not{A}(\xi)\frac{(\not{\kappa}-\not{\xi})-m}{(\kappa-\xi)^2+m^2}\not{A}(-\xi)\right] \tag{D.18}$$

where we have used the continuum limits of eq. (D.3).

D.4 Possible mass terms in $3+1$-dimensions

In order to investigate possible mass term generation in this case, we start by solving the trace of eq. (D.18) using the relations derived in section A.4

$$
\begin{aligned}
&\operatorname{tr}\Big[(\kappa^\mu\gamma_\mu - m)A^\nu(\xi)\gamma_\nu((\kappa-\xi)^\lambda\gamma_\lambda - m)A^\rho(-\xi)\gamma_\rho\Big] \\
&= \kappa^\mu A^\nu(\kappa-\xi)^\lambda A^\rho \operatorname{tr}\left[\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\right] + m^2 A^\nu A^\rho \operatorname{tr}\left[\gamma_\nu\gamma_\rho\right] \\
&= 4\kappa^\mu A^\nu(\kappa-\xi)^\lambda A^\rho(\delta_{\mu\nu}\delta_{\lambda\rho} - \delta_{\mu\lambda}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\lambda}) - 4m^2 A^\nu A^\rho \delta_{\nu\rho} \\
&= 4A^\mu(\xi)\Big[\kappa_\mu(\kappa-\xi)_\nu + \kappa_\nu(\kappa-\xi)_\mu - \delta_{\mu\nu}(m^2 + \kappa^\lambda(\kappa-\xi)_\lambda)\Big] A^\nu(-\xi).
\end{aligned} \tag{D.19}
$$

And so, we get the following integrand up to a numerical constant

$$
\frac{\delta_{\mu\nu}(m^2 + \kappa\cdot(\kappa-\xi)) - 2\kappa_\mu\kappa_\nu + \kappa_\nu\xi_\mu + \kappa_\mu\xi_\nu}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)}. \tag{D.20}
$$

Taking the limit $\xi \to 0$, we get

$$
\begin{aligned}
\frac{\delta_{\mu\nu}(m^2+\kappa^2) - 2\kappa_\mu\kappa_\nu}{(\kappa^2+m^2)^2} &= \frac{\delta_{\mu\nu}(m^2+\kappa^2) - \frac{2}{4}\delta_{\mu\nu}\kappa^2}{(\kappa^2+m^2)^2} \\
&= \frac{1}{2}\Big[\frac{1}{\kappa^2+m^2} + \frac{m^2}{(\kappa^2+m^2)^2}\Big]\delta_{\mu\nu},
\end{aligned} \tag{D.21}
$$

where we have used that $\kappa_\mu\kappa_\nu = \frac{1}{d+1}\delta_{\mu\nu}\kappa^2$ for $d+1$-dimensional Euclidean space, which can be readily verified by contracting both sides with the metric tensor. Using eq. (C.2) we can evaluate the k-integrals using dimensional regularization

$$
\begin{aligned}
&\frac{1}{2}\int \frac{\mathrm{d}^4\kappa}{(2\pi)^4}\frac{1}{\kappa^2+m^2}\delta_{\mu\nu} + \frac{m^2}{2}\int\frac{\mathrm{d}^4\kappa}{(2\pi)^4}\frac{1}{(\kappa^2+m^2)^2}\delta_{\mu\nu} \\
&= \frac{1}{2}\frac{1}{(4\pi)^2}\Gamma(1-(2-\epsilon))\left[m^2\right]^{2-\epsilon-1} + \frac{m^2}{2}\frac{1}{(4\pi)^2}\Gamma(2-(2-\epsilon))\left[m^2\right]^{2-\epsilon-2} \\
&= \frac{\left[m^2\right]^{1-\epsilon}}{32\pi^2}\Big[\Gamma(-1+\epsilon) + \Gamma(\epsilon)\Big]
\end{aligned} \tag{D.22}
$$

By using the following identity

$$
\Gamma(-1+\epsilon) = -\frac{\Gamma(\epsilon)}{1-\epsilon}, \tag{D.23}
$$

we see that the two terms cancel in the limit $\epsilon \to 0$. Therefore, we can conclude that $\Pi_{\mu\nu}(\xi) = 0$ in the limit $\xi \to 0$, and therefore we get no mass terms at zero temperature.

Appendix E

Evaluation of loop integrals

E.1 Gauge field terms

E.1.1 Evaluation of $\Pi_{\mu\mu}$ terms

In section 6.4.2, we need to evaluate the following integrals in $\Pi_{\mu\mu}$

$$\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{(\kappa + m^2)((\kappa - \xi)^2 + m^2)} \tag{E.1}$$

$$\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\kappa^2 + m^2} \tag{E.2}$$

to first and second order in temporal- and spatial momenta respectively.

Evaluation of eq. (E.1)

We start by solving eq. (E.1). Using a Feynman parametrization, we can write it as [8]

$$\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{(\kappa + m^2)((\kappa - \xi)^2 + m^2)} \tag{E.3}$$

$$= \frac{1}{\beta}\sum_n \int_0^1 \mathrm{d}z \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{(\kappa^2 + m^2 + \xi^2 z(1-z))^2}, \tag{E.4}$$

where we have performed a linear change of variables $\kappa - z\xi \to \kappa$. Equation (E.4) is even in ξ and hence its series expansion is an even polynomial in ξ. Hence, since we are working to first order in temporal momenta, we can set $\nu_l \to 0$ in the following. Reinstating the chemical potential and substituting v_F factors out of the integral, we get the following

$$\frac{1}{\beta}\sum_n \int_0^1 \mathrm{d}z \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{(\kappa^2 + m^2 + \xi^2 z(1-z))^2}$$
$$= \frac{1}{\beta}\sum_n \int_0^1 \mathrm{d}z \frac{1}{v_F^2} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{(k^2 + (\omega_n - i\mu)^2 + m^2 + q^2 z(1-z))^2}. \tag{E.5}$$

Using eq. (C.2), the k-integral evaluates to

$$\frac{1}{4\pi v_F^2}\frac{1}{\beta}\sum_n\int_0^1 \mathrm{d}z\frac{1}{-(i\omega_n+\mu)^2+m^2+q^2z(1-z)} \tag{E.6}$$

$$=\frac{1}{4\pi v_F^2}\frac{1}{\beta}\sum_n\int_0^1 \mathrm{d}z\frac{1}{-(i\omega_n+\mu)^2+m^2}\frac{1}{1+\frac{q^2z(1-z)}{-(i\omega_n+\mu)^2+m^2}} \tag{E.7}$$

$$=\frac{1}{4\pi v_F^2}\frac{1}{\beta}\sum_n\int_0^1 \mathrm{d}z\Big[\frac{1}{-(i\omega_n+\mu)^2+m^2}-\frac{q^2z(1-z)}{(-(i\omega_n+\mu)^2+m^2)^2}\Big]+\mathcal{O}\big(q^3\big) \tag{E.8}$$

$$=\frac{1}{4\pi v_F^2}\frac{1}{\beta}\sum_n\Big[\frac{1}{-(i\omega_n+\mu)^2+m^2}-q^2\frac{1}{6}\frac{1}{(-(i\omega_n+\mu)^2+m^2)^2}\Big]+\mathcal{O}\big(q^3\big) \tag{E.9}$$

The first term, which we label by c_1, can be evaluated directly using eq. (B.11)

$$\begin{aligned}c_1&=\frac{1}{4\pi v_F^2}\frac{1}{\beta}\sum_n\frac{1}{-(i\omega_n+\mu)^2+m^2}\\&=\frac{1}{8\pi v_F^2|m|}\frac{\sinh\beta|m|}{\cosh\beta\mu+\cosh\beta|m|}.\end{aligned} \tag{E.10}$$

The second term, which we label by c_2q^2, can be evaluated using eq. (B.17)

$$\begin{aligned}c_2&=-\frac{1}{24\pi v_F^2}\frac{1}{\beta}\sum_n\frac{1}{(-(i\omega_n+\mu)^2+m^2)^2}\\&=-\frac{1}{96\pi v_F^2|m|^3}\frac{\sinh\beta\mu}{\cosh\beta|m|+\cosh\beta\mu}+\frac{\beta}{384v_F^2|m|^2}\Big[\frac{1}{\cosh^2\frac{\beta(|m|-\mu)}{2}}+\frac{1}{\cosh^2\frac{\beta(|m|+\mu)}{2}}\Big].\end{aligned} \tag{E.11}$$

Evaluation of eq. (E.2)

The second integral in eq. (6.86) is divergent. We will therefore regularize it using dimensional regularization (see section C.1 for details). We start by expressing the integral as follows

$$\frac{1}{\beta}\sum_n\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\kappa^2+m^2}=\frac{1}{\beta}\sum_n\frac{1}{v_F^2}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{k^2+(\omega_n-i\mu)^2+m^2} \tag{E.12}$$

where we have once again extracted the v_F factors. Using eq. (C.2), the integral evaluates to

$$\begin{aligned}c_3&=\frac{1}{\beta}\sum_n\frac{1}{(4\pi)^\omega v_F^2}\frac{\Gamma(1-\omega)}{\Gamma(1)}\left[(\omega_n-i\mu)^2+m^2\right]^{\omega-1}\\&=\frac{1}{4\pi v_F^2}\frac{1}{\beta}\sum_n\Gamma(\epsilon)\left[-(i\omega_n+\mu)^2+m^2\right]^{-\epsilon},\end{aligned} \tag{E.13}$$

where we have labeled the expression by c_3. In the limit $\epsilon\to 0$, we need to expand the two rightmost factors of eq. (E.13) in a Laurent series. Using eq. (C.3), we get the following

$$
\begin{aligned}
&\Gamma(\epsilon)\left[-(i\omega_n+\mu)^2+m^2\right]^{-\epsilon} \\
&=\left(\frac{1}{\epsilon}-\gamma+\frac{1}{2}\left(\gamma^2+\frac{\pi^2}{6}\right)\epsilon\right)\left(1-\ln\left(\frac{-(i\omega_n^2+\mu)^2+m^2}{\Lambda^2}\right)\epsilon\right) \\
&=\left(\frac{1}{\epsilon}-\gamma-\ln\left(\frac{-(i\omega_n+\mu)^2+m^2}{\Lambda^2}\right)\right)+\mathcal{O}(\epsilon) \\
&=\left(\frac{1}{\epsilon}-\ln\left(\frac{-(i\omega_n+\mu)^2+m^2}{\Lambda^2\mathrm{e}^{-\gamma}}\right)\right)+\mathcal{O}(\epsilon) \\
&=\left(\frac{1}{\epsilon}-\ln\big(-(i\omega_n+\mu)^2+m^2\big)+\ln\big(\|\Lambda^2\mathrm{e}^{-\gamma}\|\big)\right)+\mathcal{O}(\epsilon)
\end{aligned}
\tag{E.14}
$$

where γ is the Euler-Mascheroni constant, Λ is a renormalization parameter used to make the rightmost factor dimensionless and $\|\Lambda^2\mathrm{e}^{-\gamma}\|$ is the magnitude of the re-scaled renormalization parameter. In the limit $\epsilon \to 0$, we are left with the following finite contribution

$$
c_3=-\frac{1}{4\pi v_F^2}\frac{1}{\beta}\sum_n \ln\big(-(i\omega_n+\mu)^2+m^2\big) \tag{E.15}
$$

Taking a derivative with respect to m, we get [1]

$$
\begin{aligned}
\frac{\mathrm{d}c_3}{\mathrm{d}m}&=-\frac{1}{4\pi v_F^2}\frac{1}{\beta}\sum_n \frac{2m}{-(i\omega_n+\mu)^2+m^2} \\
&=\frac{-m}{2\pi v_F^2}\frac{1}{\beta}\sum_n \frac{1}{-(i\omega_n+\mu)^2+m^2}
\end{aligned}
\tag{E.16}
$$

Using eq. (B.11), we immediately get that

$$
\frac{\mathrm{d}c_3}{\mathrm{d}m}=-\frac{\mathrm{sgn}(m)}{4\pi v_F^2}\frac{\sinh\beta|m|}{\cosh\beta|m|+\cosh\beta\mu} \tag{E.17}
$$

$$
\begin{aligned}
c_3(m)-c_3(\epsilon)&=-\int_\epsilon^m \mathrm{d}m\frac{\mathrm{sgn}(m)}{4\pi v_F^2}\frac{\sinh\beta|m|}{\cosh\beta|m|+\cosh\beta\mu} \\
&=-\frac{1}{4\pi v_F^2}\int_\epsilon^m \mathrm{d}|m|\frac{\sinh\beta|m|}{\cosh\beta|m|+\cosh\beta\mu} \\
&=-\frac{1}{4\pi\beta v_F^2}\left[\ln(\cosh\beta|m|+\cosh\beta\mu)-\ln(\cosh\beta\epsilon+\cosh\beta\mu)\right],
\end{aligned}
\tag{E.18}
$$

Consequently, we get that

$$
c_3=-\frac{1}{4\pi\beta v_F^2}\ln(\cosh\beta|m|+\cosh\beta\mu). \tag{E.19}
$$

[1] We do this trick since the complex-valued logarithm has branch cuts rather than isolated poles, which makes it more cumbersome to work with. A more detailed analysis will give a similar result [7].

E.1.2 Evaluation of Π_{00} terms

In section 6.4, we need to evaluate the following integrals in Π_{00}

$$\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{(\omega_n - i\mu)^2}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)} \tag{E.20}$$

$$\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{(\omega_n - i\mu)v_l}{(\kappa^2+m^2)^2} \tag{E.21}$$

Evaluation of eq. (E.20)

To solve the first integral, we will follow the steps from eq. (E.1) and make the following simplifications

$$\begin{aligned}&\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{(\omega_n - i\mu)^2}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)}\\ &= \frac{1}{4\pi v_F^2}\frac{1}{\beta}\sum_n \int_0^1 \mathrm{d}z \frac{-(i\omega_n+\mu)^2}{-(i\omega_n+\mu)^2+m^2+q^2z(1-z)}\end{aligned} \tag{E.22}$$

where we have computed the k-integral using a Feynman parametrization. By defining the effective mass $m_q^2 = m^2 + q^2z(1-z)$, we can write eq. (E.22) as follows

$$\frac{1}{4\pi v_F^2}\frac{1}{\beta}\sum_n \int_0^1 \mathrm{d}z \frac{-(i\omega_n+\mu)^2}{-(i\omega_n+\mu)^2+m_q^2} \tag{E.23}$$

The poles of this numerator are $z_i = -\mu \pm |m_q|$, leaving us with a common factor of $-(\pm|m_q|)^2 = -m_q^2$ for each residue. Thus, we can make the following simplifications

$$\begin{aligned}&\frac{1}{4\pi v_F^2}\frac{1}{\beta}\sum_n \int_0^1 \mathrm{d}z \frac{-(i\omega_n+\mu)^2}{-(i\omega_n+\mu)^2+m_q^2}\\ &= \frac{-m_q^2}{4\pi v_F^2}\frac{1}{\beta}\sum_n \int_0^1 \mathrm{d}z \frac{1}{-(i\omega_n+\mu)^2+m_q^2}\\ &= \frac{-m^2}{4\pi v_F^2}\frac{1}{\beta}\sum_n \int_0^1 \mathrm{d}z \frac{1}{-(i\omega_n+\mu)^2+m_q^2} + \frac{q^2}{4\pi v_F^2}\frac{1}{\beta}\sum_n \int_0^1 \mathrm{d}z \frac{z(z-1)}{-(i\omega_n+\mu)^2+m},\end{aligned} \tag{E.24}$$

where we have set $q \to 0$ in the denominator of the last term since the numerator is quadratic in momenta. The first term is proportional to the contribution in eq. (E.6) and we can therefore immediately write it as follows

$$\frac{-m^2}{4\pi v_F^2}\frac{1}{\beta}\sum_n \int_0^1 \mathrm{d}z \frac{1}{-(i\omega_n+\mu)^2+m_q^2} = -m^2(c_1+q^2c_2) \tag{E.25}$$

where c_1 and c_2 are defined in eq. (E.10) and eq. (E.11) respectively. Using eq. (B.11) the latter expression in eq. (E.24) evaluates to

$$\frac{q^2}{4\pi v_F^2}\frac{1}{\beta}\sum_n\int_0^1 \mathrm{d}z\frac{z(z-1)}{-(i\omega_n+\mu)^2+m} = -\frac{q^2}{48\pi v_F^2|m|}\frac{\sinh\beta|m|}{\cosh\beta|m|+\cosh\beta\mu} = c_4q^2 \tag{E.26}$$

where we have defined the constant c_4. Hence, we can write eq. (E.20) as follows

$$\frac{1}{\beta}\sum_n\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{(\omega_n-i\mu)^2}{(\kappa^2+m^2)((\kappa-\xi)^2+m^2)} = -m^2(c_1+q^2c_2)+c_4q^2. \tag{E.27}$$

Evaluation of eq. (E.21)

The k-integral in eq. (E.21) can be evaluated by using eq. (C.2)

$$\begin{aligned}\frac{1}{\beta}\sum_n\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{(\omega_n-i\mu)v_l}{(\kappa^2+m^2)^2} &= \frac{1}{v_F^2}\frac{1}{\beta}\sum_n\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{(\omega_n-i\mu)v_l}{(k^2+(\omega_n-i\mu)^2+m^2)^2}\\ &= \frac{1}{4\pi v_F^2}\frac{1}{\beta}\sum_n\frac{(\omega_n-i\mu)\nu_l}{(\omega_n-i\mu)^2+m^2}\\ &= \frac{1}{4\pi v_F^2}\frac{1}{\beta}\sum_n\frac{-(i\omega_n+\mu)i\nu_l}{(i\omega_n+\mu)^2-m^2}.\end{aligned} \tag{E.28}$$

The Matsubara sum in eq. (E.28) can be evaluated directly using eq. (B.20)

$$\frac{1}{4\pi v_F^2}\frac{1}{\beta}\sum_n\frac{-(i\omega_n+\mu)i\nu_l}{(i\omega_n+\mu)^2-m^2} = \frac{1}{8\pi v_F^2}\frac{\sinh\beta\mu}{\cosh\beta\mu+\cosh\beta|m|}i\nu_l \tag{E.29}$$

E.2 Cooper boson terms

Evaluation of $\Gamma(0,q)$

In section 6.5, we defined the spatial contribution of eq. (6.66) in eq. (6.109). We simplify this expression by introducing a Feynman-parameter so that we can write the denominator in $\Gamma(0,q)$ as

$$\begin{aligned}&\int_0^1\mathrm{d}z\frac{1}{\left[((v_Fk-q)^2+m^2-\mu^2-2i\mu\omega_n)z+(\omega_n^2+v_F^2k^2+m^2-\mu^2+2i\mu\omega_n)(1-z)\right]^2}\\ &=\frac{1}{v_F^2}\int_0^1\mathrm{d}z\frac{1}{\left[k^2+M^2-2kqz+q^2z-2i\mu\omega_n(2z-1)\right]^2},\end{aligned} \tag{E.30}$$

where we have performed a change of variables, removing a factor v_F for each k and defined $M^2=\omega_n^2+m^2-\mu^2$. By performing a series expansion, we get the following

$$\frac{1}{v_F^2}\frac{D(\kappa,q)}{\left[k^2+M^2\right]^2} \tag{E.31}$$

where we have defined

$$D(\kappa, q) = \int_0^1 \mathrm{d}z \Big[1 - 2\frac{-2kqz + q^2z - 2i\mu\omega_n(2z-1)}{k^2+M^2} + 3\frac{(-2kqz + q^2z - 2i\mu\omega_n(2z-1))^2}{\left[k^2+M^2\right]^2}\Big] \tag{E.32}$$

Computing the Feynman-parameter integrals, we see that odd powers of $2i\mu\omega_n(2z-1)$ falls out of the equation. And so, we get

$$\int_0^1 \mathrm{d}z \left[-2kq + q^2z - 2i\mu\omega_n(2z-1)\right] = -kq + \frac{v_F^2}{2} \tag{E.33}$$

$$\int_0^1 \mathrm{d}z \left[-2kq + q^2z - 2i\mu\omega_n(2z-1)\right]^2 = \frac{4}{3}(k^2q^2 - \omega_n^2\mu^2) + \mathcal{O}(q^3) \tag{E.34}$$

$$D(\kappa, q) = 1 - 2\frac{-kq+q^2}{2}\frac{1}{k^2+M^2} + 3\frac{4}{3}(k^2q^2 - \omega_n^2\mu^2)\frac{1}{\left[k^2+M^2\right]^2} \tag{E.35}$$

$$\begin{aligned}\frac{1}{v_F^2}\frac{D(\kappa,q)}{\left[k^2+M^2\right]^2} &= \frac{1}{v_F^2}\frac{1}{\left[k^2+M^2\right]^2}\Big[1 - 2\frac{-kq+q^2}{2}\frac{1}{k^2+M^2} \\ &+ 3\frac{4}{3}(k^2q^2 - \omega_n^2\mu^2)\frac{1}{\left[k^2+M^2\right]^2}\Big] \qquad (E.36)\\ &= \frac{1}{v_F^2}\Big(\frac{1}{\left[k^2+M^2\right]^2} - \frac{4\omega_n^2\mu^2}{\left[k^2+M^2\right]^4}\Big) + \frac{q}{v_F^2}\Big(\frac{k}{\left[k^2+M^2\right]^3}\Big) \\ &+ \frac{q^2}{v_F^2}\Big(\frac{4k^2}{\left[k^2+M^2\right]^4} - \frac{1}{\left[k^2+M^2\right]^3}\Big) \qquad (E.37)\end{aligned}$$

(E.38)

Multiplying $D(\kappa, q)$ with the numerator, we get

$$\begin{aligned}&-(\omega_n^2 + k^2 - kq - (m^2-\mu^2))D(\kappa,q) \\ &= -(\omega_n^2 + k^2 - (m^2-\mu^2))\Big(\frac{1}{v_F^2\left[k^2+M^2\right]^2} - \frac{4\omega_n^2\mu^2}{v_F^2\left[k^2+M^2\right]^4}\Big) \\ &- \frac{q^2}{v_F^2}\Big((\omega_n^2 + k^2 - (m^2-\mu^2))\Big(\frac{4k^2}{\left[k^2+M^2\right]^4} - \frac{1}{\left[k^2+M^2\right]^3}\Big) - k\Big(\frac{k}{\left[k^2+M^2\right]^3}\Big)\Big) \qquad (E.39)\\ &= \Theta_m - q^2\Theta_\nabla \qquad (E.40)\end{aligned}$$

where we have omitted odd terms in k. From this, we can define the following coupling constants of eq. (6.109)

$$\Theta_m = -\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2k}{(2\pi)^2}(\omega_n^2 + k^2 - (m^2-\mu^2))\Big(\frac{1}{v_F^2\left[k^2+M^2\right]^2} - \frac{4\omega_n^2\mu^2}{v_F^2\left[k^2+M^2\right]^4}\Big) \tag{E.41}$$

$$\begin{aligned}q^2\Theta_\nabla = -\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^2k}{(2\pi)^2}\Big((\omega_n^2 + k^2 - (m^2-\mu^2))\Big(\frac{4k^2}{v_F^2\left[k^2+M^2\right]^4} - \frac{1}{v_F^2\left[k^2+M^2\right]^3}\Big) \\ - k\Big(\frac{k}{v_F^2\left[k^2+M^2\right]^3}\Big)\Big) \qquad (E.42)\end{aligned}$$

Evaluation of $\Gamma(\nu, 0)$

Using a partial fraction decomposition, we can rewrite eq. (6.110) as follows

$$\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\frac{1}{\beta}\sum_n \frac{-(i\omega_n+\mu)(i\omega_n-i\nu_l-\mu)+\epsilon_k^2-2m^2}{(i\omega_n+\mu+\epsilon_k)(i\omega_n+\mu-\epsilon_k)((i\omega_n-i\nu_l-\mu+\epsilon_k)(i\omega_n-i\nu_l-\mu-\epsilon_k)} \tag{E.43}$$

$$= \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\frac{1}{\beta}\sum_n \frac{\epsilon_k(i\nu_l+2\mu)-2m^2}{2(i\nu_l+2\mu)(i\nu_l+2\mu-2\epsilon_k)\epsilon_k}\Big[\frac{1}{i\omega_n+\mu-\epsilon_k}-\frac{1}{i\omega_n-\nu_l-\mu+\epsilon_k}\Big]$$
$$+\frac{\epsilon_k(i\nu_l+2\mu)+2m^2}{2(i\nu_l+2\mu)(i\nu_l+2\mu+2\epsilon_k)\epsilon_k}\Big[\frac{1}{i\omega_n+\mu+\epsilon_k}-\frac{1}{i\omega_n-i\nu_l-\mu-\epsilon_k}\Big] \tag{E.44}$$

Performing the fermionic Matsubara summation, we get

$$\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\frac{\epsilon_k(i\nu_l+2\mu)-2m^2}{2(i\nu_l+2\mu)(i\nu_l+2\mu-2\epsilon_k)\epsilon_k}\tanh\frac{\beta(\epsilon_k-\mu)}{2}$$
$$+\frac{\epsilon_k(i\nu_l+2\mu)+2m^2}{2(i\nu_l+2\mu)(i\nu_l+2\mu+2\epsilon_k)\epsilon_k}\tanh\frac{\beta(\epsilon_k+\mu)}{2}, \tag{E.45}$$

where we have used that $f(i\omega_n+iv_l)=f(i\omega_n)$. We can rewrite the integrand in this expression as follows,

$$\frac{m^2}{2\epsilon_k^2(i\nu_l+2\mu)}\Big[\tanh\frac{\beta(\epsilon_k-\mu)}{2}-\tanh\frac{\beta(\epsilon_k+\mu)}{2}\Big]+\frac{k^2}{2\epsilon_k^2(i\nu_l+2\mu-2\epsilon_k)}\tanh\frac{\beta(\epsilon_k-\mu)}{2}$$
$$-\frac{k^2}{2\epsilon_k^2(i\nu_l+2\mu+2\epsilon_k)}\tanh\frac{\beta(\epsilon_k+\mu)}{2}. \tag{E.46}$$

Changing coordinates using $\mathrm{d}\epsilon_k=\mathrm{d}\epsilon=\frac{1}{\epsilon_k}v_F^2 k\mathrm{d}k$, we get

$$\int_{|m|}^{\infty}\frac{\mathrm{d}\epsilon}{2\pi v_F^2}\Big(\frac{m^2}{2\epsilon(i\nu_l+2\mu)}\Big[\tanh\frac{\beta(\epsilon-\mu)}{2}-\tanh\frac{\beta(\epsilon+\mu)}{2}\Big]$$
$$+\frac{k^2}{2\epsilon(i\nu_l+2\mu-2\epsilon)}\tanh\frac{\beta(\epsilon-\mu)}{2}-\frac{k^2}{2\epsilon(i\nu_l+2\mu+2\epsilon)}\tanh\frac{\beta(\epsilon+\mu)}{2} \tag{E.47}$$

Next, we perform an analytic continuation where we replace $i\nu_l$ by $i\nu_l\to\omega+i\eta$, where ω is a continuous and real-valued frequency and $\eta=0^+$ is some convergence factor

$$\int_{|m|}^{\infty}\frac{\mathrm{d}\epsilon}{4\pi v_F^2}\epsilon\Gamma_\nu=\int_{|m|}^{\infty}\frac{\mathrm{d}\epsilon}{4\pi v_F^2}\Big(\frac{m^2}{\epsilon(\omega+i\eta+2\mu)}\Big[\tanh\frac{\beta(\epsilon-\mu)}{2}-\tanh\frac{\beta(\epsilon+\mu)}{2}\Big]$$
$$+\frac{k^2}{\epsilon(\omega+i\eta+2\mu-2\epsilon)}\tanh\frac{\beta(\epsilon-\mu)}{2}-\frac{k^2}{\epsilon(\omega+i\eta+2\mu+2\epsilon)}\tanh\frac{\beta(\epsilon+\mu)}{2} \tag{E.48}$$

In order to evaluate the ϵ-integral, we need to use the Dirac identity [7],

$$\frac{1}{x\pm i\eta}=\mathrm{P}\frac{1}{x}\mp i\pi\delta(x) \tag{E.49}$$

where P denotes the Cauchy principal value. Thus, we can rewrite the above expression as follows

$$\int_{|m|}^{\infty} \frac{\mathrm{d}\epsilon}{4\pi v_F^2 \epsilon} \Big(\frac{m^2}{(\omega + i\eta + 2\mu)} \Big[\tanh \frac{\beta(\epsilon - \mu)}{2} - \tanh \frac{\beta(\epsilon + \mu)}{2} \Big] + \frac{k^2}{(\omega + i\eta + 2\mu - 2\epsilon)} \tanh \frac{\beta(\epsilon - \mu)}{2} - \frac{k^2}{(\omega + i\eta + 2\mu + 2\epsilon)} \tanh \frac{\beta(\epsilon + \mu)}{2} \tag{E.50}$$

$$\begin{aligned} = \frac{1}{4\pi v_F^2} \Big(\int_{|m|}^{\infty} \frac{\mathrm{d}\epsilon}{\epsilon} \Big\{ \mathrm{P} \Big[& \frac{(\epsilon^2 - m^2) \tanh \frac{\beta(\epsilon-\mu)}{2}}{\omega + 2\mu - 2\epsilon} - \frac{(\epsilon^2 - m^2) \tanh \frac{\beta(\epsilon+\mu)}{2}}{\omega + 2\mu + 2\epsilon} \\ & + \frac{m^2}{\omega + 2\mu} \Big(\tanh \frac{\beta(\epsilon - \mu)}{2} - \tanh \frac{\beta(\epsilon + \mu)}{2} \Big) \Big] - i\pi\delta(\omega + 2\mu) m^2 \Big(\tanh \frac{\beta(\epsilon - \mu)}{2} - \tanh \frac{\beta(\epsilon + \mu)}{2} \Big) \\ & - i\pi\delta(\omega + 2\mu - 2\epsilon)(\epsilon^2 - m^2) \tanh \frac{\beta(\epsilon - \mu)}{2} + i\pi\delta(\omega + 2\mu + 2\epsilon)(\epsilon^2 - m^2) \tanh \frac{\beta(\epsilon + \mu)}{2} \Big\} \end{aligned} \tag{E.51}$$

Assuming that $\omega \ll |m|, \mu, |m \pm \mu|$, we can perform a series expansion in μ to first order in ω. Subtracting the zeroth-order term, we get

$$- \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \int \frac{\mathrm{d}\omega}{2\pi} \eta_t \Delta^* \omega \Delta = - \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \int \frac{\mathrm{d}\omega}{2\pi} (-i\eta_t) \Delta^* i\omega \Delta \tag{E.52}$$

$$\begin{aligned} \tilde{\eta}_t = \frac{1}{4\pi v_F^2} \Big(& \mathrm{P} \int_{|m|}^{\infty} \frac{\mathrm{d}\epsilon}{\epsilon} \Big[\Big(\frac{\epsilon^2 - m^2}{4\epsilon_+^2} + \frac{m^2}{4\mu^2} \Big) \tanh \frac{\beta\epsilon_+}{2} + (\epsilon_+ \leftrightarrow \epsilon_-) \Big] \\ & + i\pi\beta \left[\Theta(\mu - |m|) - \Theta(-\mu - |m|) \right] \frac{\mu^2 - m^2}{8\mu} \Big), \end{aligned} \tag{E.53}$$

where $\epsilon_\pm = \pm\epsilon - \mu$.

E.3 Magnetic impurities and mixed terms

In section 7.3.2 we arrive at the integrals of eq. (7.46) and eq. (7.47).

Evaluation of eq. (7.46)

We start by solving eq. (7.46). Using a Feynman parametrization, we can rewrite it as

$$\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\beta} \sum_n \int_0^1 \mathrm{d}z \frac{2em(2(\kappa^\nu + (1-z)\xi^\nu) - \xi^\nu)}{(\kappa^2 + m^2 + \xi^2 z(1-z))^2}, \tag{E.54}$$

where we have performed the linear change of variables $\kappa' = \kappa - (1-z)\xi$. We can divide this integral into two parts

$$\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\beta} \sum_n \int_0^1 \mathrm{d}z \frac{4em\kappa^\nu}{(\kappa^2 + m^2 + \xi^2 z(1-z))^2} \tag{E.55}$$

$$\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\beta} \sum_n \int_0^1 \mathrm{d}z \frac{2em(1-2z)\xi^\nu}{(\kappa^2 + m^2)^2}. \tag{E.56}$$

The first integral is odd in k and hence only the temporal parts survive after momentum integration. In the second integral, the ξ dependence drops out of the denominator since the numerator is linear in ξ. The resulting Feynman parameter integral is odd and hence this contribution vanishes. Thus, we are left with

$$\frac{1}{v_F^2}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta}\sum_n\int_0^1\mathrm{d}z\frac{4em(\omega_n-i\mu)}{(k^2+(\omega_n-i\mu)^2+m^2+q^2z(1-z))^2} \tag{E.57}$$

$$\tag{E.58}$$

where we have reinstated the factors v_F and substituted them out of the integral. By performing a series expansion, we get

$$\begin{aligned}&\frac{1}{v_F^2}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta}\sum_n\int_0^1\mathrm{d}z\frac{4em(\omega_n-i\mu)}{(k^2+(\omega_n-i\mu)^2+m^2)^2}\Big(1-2\frac{q^2z(1-z)}{(k^2+(\omega_n-i\mu)^2+m^2)}\Big)\\&=\frac{1}{v_F^2}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta}\sum_n\frac{4em(\omega_n-i\mu)}{(k^2+(\omega_n-i\mu)^2+m^2)^2}\Big(1-\frac{1}{3}\frac{q^2}{(k^2+(\omega_n-i\mu)^2+m^2)}\Big),\end{aligned} \tag{E.59}$$

where we have performed the Feynman-parameter integration. Using eq. (C.2), the lowest-order contribution becomes

$$\begin{aligned}&\frac{1}{v_F^2}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta}\sum_n\frac{4em(\omega_n-i\mu)}{(k^2+(\omega_n-i\mu)^2+m^2)^2}\\&=\frac{-iem}{\pi v_F^2}\frac{1}{\beta}\sum_n\frac{(i\omega_n+\mu)}{((i\omega_n+\mu)^2-m^2)}\\&=\frac{-iem}{2\pi v_F^2}\frac{\sinh\beta\mu}{\cosh\beta\mu+\cosh\beta|m|},\end{aligned} \tag{E.60}$$

where we have used eq. (B.20) to evaluate the Matsubara sum. Using eq. (C.2), the second-order contribution can be written as

$$\begin{aligned}\frac{4iem}{3v_F^2}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta}\sum_n\frac{i\omega_n+\mu}{(k^2+(\omega_n-i\mu)^2+m^2)^3}&=\frac{emi}{6\pi v_F^2}\frac{1}{\beta}\sum_n\frac{(i\omega_n+\mu)q^2}{((\omega_n-i\mu)^2+m^2)^2}\\&=\frac{iem}{6\pi v_F^2}\frac{1}{\beta}\sum_n\frac{(i\omega_n+\mu)q^2}{(-(i\omega_n+\mu)^2+m^2)^2}.\end{aligned} \tag{E.61}$$

The poles of this Matsubara sum are $z_\pm=-\mu\pm m$, which means that we get $\pm|m|$ in the numerator for each type of pole. Thus, we can use eq. (B.17) as follows

$$\begin{aligned}\frac{1}{\beta}\sum_n\frac{i\omega_n+\mu}{(-(i\omega_n+\mu)^2+m^2)^2}&=\Big[\frac{|m|f'(z_+)}{4|m|^2}-\frac{|m|f(z_+)}{4|m|^3}-|m|(+\to-)\Big]\\&=-\frac{\beta}{16|m|}\Big[\frac{1}{\cosh^2\frac{\beta(|m|+\mu)}{2}}-\frac{1}{\cosh^2\frac{\beta(|m|-\mu)}{2}}\Big]\\&+\frac{1}{4|m|^2}\frac{\sinh\beta\mu}{\cosh\beta\mu+\cosh\beta|m|}\end{aligned} \tag{E.62}$$

and hence, we get the following second-order contribution

$$\frac{iem}{6\pi v_F^2}q^2\left(\frac{\beta}{16|m|}\left[\frac{1}{\cosh^2\frac{\beta(|m|+\mu)}{2}}-\frac{1}{\cosh^2\frac{\beta(|m|-\mu)}{2}}\right]-\frac{1}{4|m|^2}\frac{\sinh\beta\mu}{\cosh\beta\mu+\cosh\beta|m|}\right). \tag{E.63}$$

Evaluation of eq. (7.47)

Eq. (7.47) is also odd in k and hence only the temporal part of κ^μ survive the integration. Thus, we are left with

$$\begin{aligned}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta}\sum_n\frac{-2e\kappa^\mu\xi^\lambda\varepsilon_{\mu\nu\lambda}}{(\kappa^2+m^2)^2}&=\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta}\sum_n\frac{-2e(\omega_n-i\mu)\xi^\lambda\varepsilon_{0\nu\lambda}}{(\kappa^2+m^2)^2}\\&=\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta}\sum_n\frac{2e(\omega_n-i\mu)\xi^\lambda\varepsilon_{0\lambda\nu}}{(v_F^2k^2+(\omega_n-i\mu)^2+m^2)^2}\\&=\frac{1}{v_F^2}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta}\sum_n\frac{-2e(i\omega_n+\mu)i\xi^\lambda\varepsilon_{0\lambda\nu}}{(k^2+(\omega_n-i\mu)^2+m^2)^2},\end{aligned} \tag{E.64}$$

where we have reinstated the factors v_F in the k-terms and substituted them out of the integral. Performing the k-integration using eq. (C.2), we get

$$-\frac{1}{v_F^2}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\beta}\sum_n\frac{2e(i\omega_n+\mu)i\xi^\lambda\varepsilon_{0\lambda\nu}}{(k^2+(\omega_n-i\mu)^2+m^2)^2}=\frac{e}{2\pi v_F^2}\frac{1}{\beta}\sum_n\frac{i\omega_n+\mu}{(i\omega_n+\mu)^2-m^2}(i\xi)^\lambda\varepsilon_{0\lambda\nu}$$

Thus, using eq. (B.20), we get the following expression

$$\frac{e}{4\pi v_F^2}\frac{\sinh\beta\mu}{\cosh\beta|m|+\cosh\beta\mu}\varepsilon_{0\lambda\nu}i\xi^\lambda \tag{E.65}$$

Bibliography

[1] M. Z. Hasan and C. Kane, "Colloquium: Topological insulators," *Reviews of modern physics*, vol. 82, 2010. DOI: `10.1103/RevModPhys.82.3045`.

[2] G. V. Dunne. (1998). Aspects of chern-simons theory, [Online]. Available: `https://arxiv.org/abs/hep-th/9902115`.

[3] F. S. Nogueira, J. van den Brink, and A. Sudbø, "Conformality loss and quantum criticality in topological higgs electrodynamics in 2 + 1 dimensions," *Physical Review D*, vol. 100, no. 085005, 2019. DOI: `10.1103/PhysRevD.100.085005`.

[4] S. Rex, *Electric and magnetic signatures of boundary states in topological insulators and superconductors*, eng, 1st ed. Trondheim, Norway: NTNU, 2014.

[5] F. S. Nogueira and I. Eremin, "Thermal screening at finite chemical potential on a topological surface and its interplay with proximity-induced ferromagnetism," *Physical Review B*, vol. 90, no. 014431, 2014. DOI: `10.1103/PhysRevB.90.014431`.

[6] E. I. Blount and C. M. Varma, "Electromagnetic effects near the superconductor-to-ferromagnet transition," *Physical Review Letters*, vol. 42, no. 16, 1979. DOI: `10.1103/PhysRevLett.43.1843`.

[7] A. Altland and B. Simons, *Condensed Matter Field Theory*, eng, 2nd ed. Cambridge University Press, 2006.

[8] M. Kachelriess, *Quantum Fields*, eng, 1st ed. Oxford university press, 2018.

[9] J. Binney, N. J. Dowrick, M. E. J. Newman, and A. J. Fisher, *The Theory of Critical Phenomena: An Introduction to the Renormalization Group*, eng, 1st ed. Oxford, UK: Oxford science publications, 1992, ISBN: 0198513933.

[10] M. Aromstrong, *Basic Topology*, eng, 2nd ed. New York, USA: Springer Science & Business Media, 2013, ISBN: 1475717938.

[11] V. Guillemin and A. Pollack, *Differential Topology*, eng, 1st ed. Rhode Island, USA: American Mathematical Society, 2010, ISBN: 0821851934.

[12] B. A. Bernevig and T. L. Hughes, *Topological Insulators and Topological Superconductors*, eng, 1st ed. New Jersey, USA: Princeton University Press, 2013, ISBN: 069115175X.

[13] J. Linder, *Intermediate Quantum Mechanics*, eng, 1st ed. Trondheim, Norway: Jacob Linder & Bookboon, 2017, ISBN: 978-87-403-1783-1.

[14] P. C. Hemmer, *Kvantemekanikk*, nor, 2nd ed. Trondheim, Norway: Tapir akademisk forlag, 2005, ISBN: 9788251920285.

[15] V. G. Ivancevic and T. T. Ivancevic. (2008). Undergraduate lecture notes in topological quantum field theory, [Online]. Available: https://arxiv.org/pdf/0810.0344.pdf.

[16] S. C. Zhang, "The chern-simons-landau-ginzburg theory of the fractional quantum hall effect," *International Journal of Modern Physics B*, vol. 6, no. 1, 1992.

[17] X.-L. Qi, E. Witten, and S.-C. Zhang, "Axion topological field theory of topological superconductors," *Physical Review B*, vol. 87, no. 134519, 2013. DOI: 10.1103/PhysRevB.87.134519.

[18] A. Auerbach, *Interacting Electrons and Quantum Magnetism*, eng, 1st ed. New York, USA: Springer, 1994, ISBN: 978-0-387-94286-5.

[19] B. van Dijk and R. Duine. (2014). Skyrmions and the dzyaloshinskii-moriya interaction, [Online]. Available: http://www.nanoer.net/d/img/2014vanDijk.pdf.

[20] A. Manchon, H. C. Koo, J. Nitta, S. M. Frolov, and R. A. Duine, "New perspectives for rashba spin–orbit coupling," *Nature materials*, vol. 14, no. 871, 2015. DOI: 10.1038/nmat4360.

[21] H. Bruus and K. Flensberg, *Many-Body Quantum Theory in Condensed Matter Physics: An Introduction*, eng, 1st ed. Niels Bohr Institute, Denmark: Oxford Graduate Texts, 2004, ISBN: 0198566336.

[22] A. Sudbø and K. Fossheim, *Superconductivity: Physics and Applications*, eng, 1st ed. Trondheim, Norway: Oxford Graduate Texts, 2004, ISBN: 9780470844526.

[23] S. A. Kivelson and D. S. Rokhsart, "Bogoliubov quasiparticles, spinons, and spin-charge decoupling in superconductors," *Physical Review B*, vol. 41, no. 16, 1990. DOI: 10.1103/PhysRevB.41.11693.

[24] M. L. Bellac, *Thermal field theory*, eng, 1st ed. Cambridge, UK: Cambridge University Press, 1996, ISBN: 0-521-46040-9.

[25] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, "Topological field theory of time-reversal invariant insulators," *Physical Review B*, vol. 78, no. 195424, 2008. DOI: 10.1103/PhysRevB.78.195424.

[26] P. Romatschke and M. Säppi, "Thermal free energy of large n qed in 2 + 1 dimensions from weak to strong coupling," *Physical Review D*, vol. 100, no. 0730095, 2019. DOI: 10.1103/PhysRevD.100.073009.

[27] F. Stern and W. E. Howard, "Properties of semiconductor surface inversion layers in the electric quantum limit," *Physical Review*, vol. 163, no. 3, 1967.

[28] P. Humbert, "Bessel-integral functions," *University of Montpellier*, 1933.

[29] D. B. Kaplan, J.-W. Lee, D. T. Son, and M. A. Stephanov, "Conformality lost," *Physical Review D*, vol. 80, no. 125005, 2009. DOI: 10.1103/PhysRevD.80.125005.

[30] A. M. J. Schakel. (1999). Time-dependent ginzburg-landau theory and duality, [Online]. Available: https://arxiv.org/abs/cond-mat/9904092.

[31] S. Rex, F. S. Nogueira, and A. Sudbø, "Nonlocal topological magnetoelectric effect by coulomb interaction at a topological insulator-ferromagnet interface," *Physical Review B*, vol. 93, no. 014404, 2016. DOI: 10.1103/PhysRevB.93.014404.

[32] P. Fulde and R. A. Ferrell, "Superconductivity in a strong spin-exchange field," *Physical Review*, vol. 135, no. 3A, 1964. DOI: 10.1103/PhysRev.135.A550.

[33] A. Larkin and Y. Ovchinnikov, "Nonuniform state of superconductors," *Soviet Physics JETP*, vol. 20, no. 3, 1965.

[34] A.I.Buzdin, "Proximity effects in superconductor-ferromagnet heterostructures," *Reviews of Modern Physics*, vol. 77, no. 935, 2005. DOI: 10.1103/RevModPhys.77.935.

[35] J. Rowland, S. Banerjee, and M. Randeria, "Skyrmions in chiral magnets with rashba and dresselhaus spin-orbit coupling," *Physical Review B*, vol. 93, no. 020404(R), 2016. DOI: 10.1103/PhysRevB.93.020404.

[36] A. Nieto. (1993). Evaluating sums over the matsubara frequencies, [Online]. Available: https://arxiv.org/pdf/hep-ph/9311210.pdf.

Snorre Bergan

NTNU
Norwegian University of
Science and Technology