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Finiteness Obstruction in Model Categories

Master's thesis in Mathematical Sciences
Supervisor: Markus Szymik

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Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
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Kunnskap for en bedre verden

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Abstract

We show in this paper that there is a natural way to make sense of the finiteness of objects in some general setting model category. The operator $(Wa(-), w(-))$ proves to be a functor from any model category where we can make sense of cofibrant generation to the category of abelian groups. We also show how this behaves when applying products.

Acknowledgements

I would like to express my special thanks to my advisor prof. Markus Szymik at NTNU. Our weekly chats made the problem a lot easier to make sense of. Also the input on the layout of the thesis has been invaluable.

Chapter 0

Introduction

The general goal of this paper is to explore, and expand the theory of the finiteness obstruction constructed by Wolfgang Lück in his paper *the geometric finiteness obstruction*[Lü86]. Lück's invariant was built for equivariant CW-complexes. So our goal will be to extend the obstruction to the more general setting of model categories and I-cell complexes, and also try to make more sense of the structure of the groups that this construction creates.

Section 1 we give the definition of a model category, and the homotopy theory that we need to prove our results. While also trying to give examples to give some intuition.

Section 2 is dedicated to constructing and generalizing the concept of CW-complexes from topology, and also defining some relations between them.

Section 3 consists of defining the invariant groups, which we also prove are functors.

Section 4 will be dedicated to give a very quick introduction to some finiteness obstructions, in particular the Wall obstruction and Whitehead torsion, and then defining our own obstruction.

Section 5 has a quick proof of the universal property of our functor.

Section 6 looks at how the functor behaves with regard to other functors, in particular the product operation.

Section 7 has been dedicated to applying our functor and the general theory of finiteness to chain complexes.

Section 8 show the finiteness obstruction behaves when applying some "modifications" to our objects.

Chapter 1

Model Categories

1.1 Definition

In this section we will define most of the concepts needed to make sense of the matter at hand mostly regarding the theory of model categories. We will also take the time to state some basic results regarding these constructions (sometimes omitting proofs) to make sure we are on the same page:

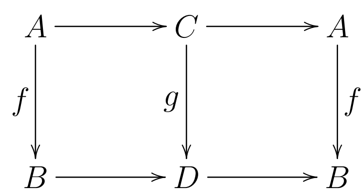
We will begin with defining the notion of a model category, which was introduced by Daniel G. Quillen in 1967. Model categories can provide a natural setting for homotopy theory: the category of topological spaces is a model category, with the homotopy corresponding to the usual theory. Similarly, objects that are thought of as spaces often admit a model category structure, such as the category of simplicial sets.

Another model category is the category of chain complexes of R-modules for a commutative ring R. Homotopy theory in this context is homological algebra. Homology can then be viewed as a type of homotopy, allowing generalizations of homology to other objects, such as groups and R-algebras, one of the first major applications of the theory. Because of the above example regarding homology, the study of closed model categories is sometimes thought of as homotopical algebra. The definition given by Quillen has proven to be somewhat cumbersome; so we will use the definitions provided by Hovey [Hov91].

Definition 1.1.1.
Let $\mathcal{C}$ be a category, then Map $\mathcal{C}$ is the category of morphisms of $\mathcal{C}$ whose morphisms are commutative squares.

Definition 1.1.2.
Suppose $\mathcal{C}$ is a category

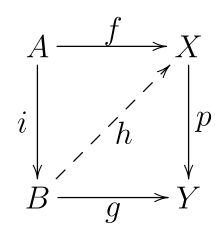
1. A map f in $\mathcal{C}$ is a *retract* of a map $g \in \mathcal{C}$ if f is a retract of g as objects in Map $\mathcal{C}$. That is, f is a retract of g if and only if there is a commutative diagram of the form



   where the horizontal compositions are identities

2. A *functorial factorization* is an ordered pair (α, β) of functors Map $\mathcal{C} \to$ Map $\mathcal{C}$ such that $f = \beta(f) \circ \alpha(f)$ for all $f \in$ Map $\mathcal{C}$. In particular, the domain of $\alpha(f)$ is the domain of f, the codomain of $\alpha(f)$ is the domain of $\beta(f)$ and the codomain of $\beta(f)$ is the codomain of f

Definition 1.1.3.
Suppose $i : A \to B$ and $p : X \to Y$ are maps in a category $\mathcal{C}$. Then i has the left lifting property with respect to p, and p has the right lifting property i if, for every commutative diagram



there is exists a lift $h : B \to X$ such that $hi = f$ and $ph = g$.

Definition 1.1.4.
A *model structure* on a category $\mathcal{C}$ is three subcategories of $\mathcal{C}$ called weak equivalences, cofibrations, and fibrations, and two factorizations (α, β) and (γ, δ) satisfying the following properties:

1. (2-out-of-3): If f and g are morphisms of $\mathcal{C}$ such that gf is defined and two of f, g and gf are weak equivalences, then so is the third.

2. (Retracts): If f and g are morphisms of $\mathcal{C}$ such that f is a retract of g and g is a weak equivalence, cofibration or fibration, then so if f.

3. (Lifting): Define a map to be a acyclic cofibration[1] if it is both a cofibration and a weak equivalence. Similarly, define a map the be a acyclic fibration if it is both a fibration and a weak equivalence. Then acyclic cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to acyclic fibrations.

4. (Factorization): For any morphism f, $\alpha(f)$ is a cofibration, $\beta(f)$ is a acyclic fibration, $\gamma(f)$ is a acyclic cofibration, and $\delta(f)$ is a fibration.

Definition 1.1.5.
A *model category* $\mathcal{M}$ is a category $\mathcal{C}$ with all small limits and colimits together with a model structure on $\mathcal{C}$.

Definition 1.1.6.
A model category $\mathcal{M}$ is *(left) right proper* if weak equivalences are preserved under (pushouts) pullbacks along (co)fibrations.

There are many examples of known categories that are model categories. The first that are natural to look at are the category of topological spaces **TOP**, the category of simplicial sets **sSet**, and the category of chain complexes $C(\mathcal{A})$ on some additive category $\mathcal{A}$.

For **TOP** the the weak equivalences play the role of homotopy equivalences or something a bit more general (such as weak homotopy equivalences). It is useful to say that two spaces have the same homotopy type if there is a map from one to the other that induces isomorphisms on homotopy groups for any choice of base-point in the first space. These maps are more general than homotopy equivalences, so they are called 'weak equivalences'. The fibrations play the role of nice surjections: A locally trivial fiber bundle is a fibration. More generally the fibrations here are the Serre fibrations. The cofibrations play the role of nice inclusions: an neighborhood retract pair is typically a cofibration.

For a chain-complex category $C(\mathcal{A})$ over some additive category $\mathcal{A}$ the weak equivalences takes the role of quasi-isomorphisms; Fibrations are the

[1] The acyclic prefix comes from the fact that the (co)fiber of the map will be acyclic, i.e. it would have trivial homology groups

morphisms that are epimorphisms in $\mathcal{A}$ in each positive degree; and cofibrations are degreewise monomorphisms with degreewise projective cokernel.

1.2 Homotopies

As stated earlier the motivation for using model categories is for the natural ways it lends itself to homotopy theory, so it would be beneficial to us to understand how homotopies are defined:

Definition 1.2.1.

1. A *cylinder object* of X is a factorization of the map

$$X \sqcup X \longrightarrow X$$

$$X \sqcup X \xhookrightarrow{i_0 \sqcup i_1} Cyl(X) \xrightarrow{\;\;p\;\;} X$$

2. A *left homotopy* from f to g consists of a cylinder object $Cyl(X)$ and a map $H : Cyl(X) \to Y$ such that $Hi_0 = f$ and $Hi_1 = g$. If there exists a left homotopy from f to g, then we say that f is left homotopic to g ($f \overset{l}{\sim} g$).

3. A *path object* of Y is factorization of the map

$$Y \longrightarrow Y \times Y$$

$$Y \xhookrightarrow{\;\;s\;\;} Path(Y) \xrightarrow{p_0 \times p_1} Y \times Y$$

4. A *right homotopy* from f to g consists of a path object $Path(Y)$ and a map $H : X \to Path(Y)$ such that $p_0H = f$ and $p_1H = g$. If there exists a right homotopy from f to g, then we say that f is right homotopic to g ($f \overset{r}{\sim} g$).

5. We say f is *homotopic* to g ($f \sim g$), if they are both left and right homotopic.

Furthermore homotopies defines an equivalence relation on the maps, but this does come with some caveats: Left homotopies only define an equivalence relation if the domain of the maps are cofibrant, and right homotopies only define an equivalence relation if the codomain of the maps are cofibrant. Hence to properly define a equivalence relation on homotopies on $Map(X,Y)$ we need X cofibrant and Y fibrant.

This means that we can make sense of notions like homotopy equivalences, by saying that two object X, Y are homotopy equivalent if there exists maps $f : X \to Y$, and $g : Y \to X$ such that $fg \sim id_Y$ and $gf \sim id_X$.

This also makes us able to define a localization of the model categories.

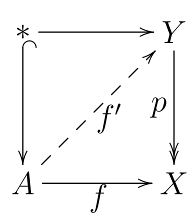
Definition 1.2.2.
Given a model category $\mathcal{M}$, and a class of maps J in $\mathcal{M}$, the a *localization* of $\mathcal{M}$ with respect to J is a category $L_W\mathcal{M}$ and a functor $F : \mathcal{M} \to L_W\mathcal{M}$ such that

- if $j \in J$, then $F(j)$ is an isomorphism.
- if $\mathcal{N}$ is a category, and $K : \mathcal{M} \to \mathcal{N}$ is a functor such that $K(j)$ is an isomorphism, for every $j \in J$, then there exists a unique functor $\partial : L_W\mathcal{M} \to \mathcal{N}$

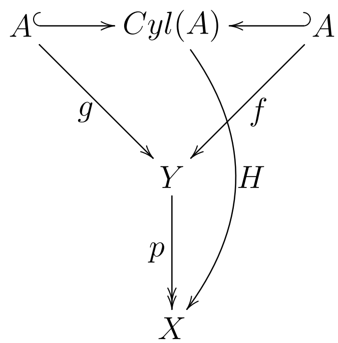
This give rise to the homotopy category of a model category, so if we let W be the class of weak equivalences in our category $\mathcal{M}$ then $L_W\mathcal{M}$ will be category such that the weak equivalences are isomorphisms. This is of course a generalization of the standard homotopy category $Ho(-)$. For example look at the homotopy category of topological spaces, where the new isomorphisms are the homotopy equivalences. There is also the homotopy category of chain complexes $K(-)$ where two chain maps $f^n, g^n : A^n \to B^n$ are homotopic if there is a collection of maps $h^n : A^n \to B^{n-1}$ such that $f^n - g^n = d_A h^n + h^{n+1} d_B$ where $d_{(-)}$ are the differentials.

Lemma 1.2.1.
Let A be cofibrant. Any acyclic fibration $p : X \to Y$ induces a bijection on the set of left homotopy equivalence classes of maps i.e. $\pi^l(A,Y) \cong \pi^l(A,X)$.

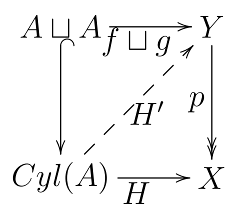
Proof. The map is well-defined by the fact that any homotopy $H : f \mapsto g$ gives a homotopy $pH : pf \mapsto pg$. So given any $f \in Hom(A,X)$ there is a lift



hence $pf' = f$, which also gives $[pf'] = [f]$, hence surjectivity. Now let $pf \sim pg$, which gives the following diagram



This produces the square



that contains the lift $H' : Cyl(A) \to Y$ which is a homotopy between f and g. $\square$

Theorem 1.2.2.
A morphism $f : A \to X$ between cofibrant-fibrant objects is a weak equivalence if and only of it is a homotopy equivalence

Proof. First suppose f is a weak equivalence. Factor f as

$$A \underset{\sim}{\overset{q}{\hookrightarrow}} M_f \overset{p}{\twoheadrightarrow} X$$

by the two out of three property, p is also a weak equivalence. Since q is a acyclic cofibration and A is fibrant, there is a left inverse $rq = 1$. By the previous lemma, q induces a bijection $\pi^l(C, C) \to \pi^l(A, C)$. Under this map, $[qr] \mapsto [qrq] \mapsto [q]$, but also $[1] \mapsto [q]$, hence $[qr] = [1]$. This shows q is a homotopy equivalence, and by a dual argument p is as well, hence f is a homotopy equivalence. $\square$

1.3 Quillen Functors

In this section we would like to study morphisms or more precisely functors between model categories, we will begin by defining the structure preserving functors called left and right Quillen functors.

Definition 1.3.1.
For categories $\mathcal{M}, \mathcal{N}$ a *Quillen adjunction* is a pair of adjoint functors $L : \mathcal{M} \leftrightarrows \mathcal{N} : R$ such that L preserves cofibrations and acyclic cofibrations, and R preserves fibrations and acyclic fibrations. L is then called a left Quillen functor and R a right Quillen functor

Lemma 1.3.1.
If (L, R) is a Quillen adjunction, then the left adjoint L preserves weak equivalences between cofibrant objects and the right adjoint R preserves weak equivalences between fibrant objects.

There are multiple examples of such functors, since we are working in a model categories we will of course assume that we have all limits and colimits hence one example of a Quillen functor is the product functor $\mathcal{M} \times -$

Definition 1.3.2.
Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, we can then form the product category $\mathcal{M} \times \mathcal{N}$ in the obvious way: Let $f \in hom(M)$ and $g \in hom(N)$, then (f, g) is an (acyclic) fibration if both f and g are (acyclic) fibrations, dually for cofibrations.

Another example is the geometric realization which send the cellular shape $[n]$ (the standard cellular globe, simplex or cube, respectively) to the corresponding standard topological shape with the obvious induced face and boundary maps. The right Quillen adjoint of this is the singular set functor we know and love from algebraic topology. Hence we get the Quillen equivalence pair

$$|-| : \mathbf{sSet} \rightleftarrows \mathbf{TOP} : Sing$$

Chapter 2

Cell Objects

In a model category there are ways of generating objects from other object. One of the ways to do this is by defining a class of maps such that we can generate objects by pushouts. The motivation is simply to generalize structures like the CW structure from the category of topological spaces. We recall the definition:

Definition 2.0.1.
We say a topological space X is a *CW-complex* if X is the colimit of a sequence

$$X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} X_2 \xrightarrow{i_2} \dots$$

such that the i_k's and X_i's are obtained (successively) by the pushout diagrams

$$\begin{array}{ccc} X_{k-1} & \xrightarrow{i_k} & X_k \\ \uparrow & & \uparrow \\ \coprod_{i=1} S^{k-1} & \longrightarrow & \coprod_{i=1} D^k \end{array}$$

This classical construction introduced by Whitehead can be generalized to a model categorical structure that we will be working with going forward which is the I-cell complexes:

Definition 2.0.2.
Let I be a class of maps in a category $\mathcal{C}$.

1. A map is I-injective if it has the right lifting property with respect to every map in I. The class of I-injective maps is denoted I-inj.

2. A map is I-projective if it has the left lifting property with respect to every map in I. The class of I-projective maps is denoted I-proj.

3. A map is an I-cofibration if it has the left lifting property with respect to every I-injective map. The class of I-cofibrations is the class (I-inj)-proj and is denoted I-cof.

4. A map is an I-fibration if it has the right lifting property with respect to every I-projective map. The class of I-fibrations is the class (I-proj)-inj and is denoted I-fib.

Note that when working in a model category we (usually) do away with indexing using the natural number and instead use ordinals so as to make the definition of transfinite composition well-defined. We recall that an ordinal is the well-ordered set of all smaller ordinals. Every ordinal λ has a successor ordinal $\lambda + 1$. We will often think of an ordinal as a category where there is a unique map $\alpha \to \beta$ if and only if $\alpha \leq \beta$. Furthermore we extend the definition of a sequence by defining a λ-sequence, which is a functor $X : \lambda \to \mathcal{C}$ commonly written as

$$* = X_0 \to X_1 \to \cdots \to X_\beta \to \dots$$

We refer to the map $X_0 \to colim_{\beta<\lambda} X_\beta$ as the *(transfinite) composition of X.*

Definition 2.0.3.
Let I be a set of maps in a category $\mathcal{C}$ containing all small colimits. A relative I-cell complex is a transfinite composition of pushouts of elements of I. That is, if $f : A \to B$ is a relative I-cell complex, then there is an ordinal λ and a λ-sequence $X : \lambda \to \mathcal{C}$ such that f is the composition of X and such that for each β such that $\beta + 1 < \lambda$ there is a pushout square:

$$\begin{array}{ccc} X_{\beta-1} & \xrightarrow{i_{\beta-1}} & X_{\beta} \\ \uparrow & & \uparrow \\ \coprod_{i=1} C_{\beta} & \xrightarrow[g_{\beta}]{} & \coprod_{i=1} D_{\beta} \end{array}$$

such that $g_\beta \in I$. We denote the collection of relative I-cell complexes by I-cell. We say that $A \in C$ is an I-cell complex if the transfinite composition $0 \to A$ is a relative I-cell complex.

Lemma 2.0.1.
Suppose λ is an ordinal and $X : \lambda \to \mathcal{C}$ is a λ sequence such that each map $X_\beta \to X_{\beta+1}$ is either a pushout of a map of I or an isomorphism. Then the transfinite composition of X is a relative I-cell.

Lemma 2.0.2.
Suppose $\mathcal{C}$ is a category with all small colimits, and I is a set of maps of C. Then any pushout of coproducts of maps of I is in I-cell.

We can immediately see that if we let

$$I = \{S^{n-1} \to D^n\}_{n \geq 1}$$

then this definition coincides with that of CW-complexes in the category of topological spaces.

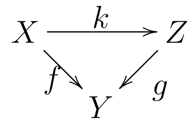
In topological spaces this gives the notion of the subcategory of CW-complexes. We can give such a subcategorical construction to the model categories as well:

Definition 2.0.4.
A model category is *(co)fibrantly generated* if there exists a proper set of (co)fibrations and one of acyclic (co)fibrations, such that all other (acyclic) (co)fibrations are generated from these.

So in the case of **TOP** the subcategory of CW-spaces is a cofibrantly generated subcategory.

Suppose $\mathcal{M}$ is a category containing all small colimits,and I is a set of maps in $\mathcal{M}$. Suppose the domains of the maps of I are small relative to I-cell. Then there is a functorial factorization (γ, δ) on $\mathcal{M}$ such that, for all morphisms f in $\mathcal{M}$, the map $\gamma(f)$ is in I-cell and the map $\delta(f)$ is in I-inj.

If we now look at the category of maps into some chosen I-cell object Y, denoted $\downarrow Y$, here the objects are of course maps $(-) \to Y$, and given objects $f : X \to Y$ and $g : Z \to Y$, a morphism between is a map $k : X \to Z$ such that the following diagram commute



For the following five lemmas let the category be $\downarrow Y$

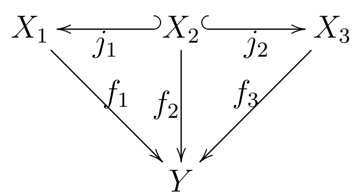
Lemma 2.0.3.
Given a diagram

$$X_1 \hookleftarrow X_2 \hookrightarrow X_3$$

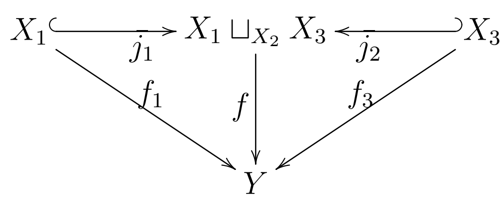
there exists an object W *in* $\downarrow Y$ *such that we get the diagram*

$$X_1 \hookrightarrow W \hookleftarrow X_3$$

Proof. This sequence corresponds to the diagram



if we then take the pushout along the horizontal maps we get the diagram



which gives the desired result. ∎

Lemma 2.0.4.
Given either diagram

$$X_1 \hookleftarrow X_2 \xrightarrow{\sim} X_3$$

$$X_1 \xleftarrow{\sim} X_2 \hookrightarrow X_3$$

there exists objects W and W' in $\downarrow Y$ such that we get the diagrams

$$X_1 \xrightarrow{\sim} W \hookleftarrow X_3$$

$$X_1 \hookleftarrow W' \xleftarrow{\sim} X_3$$

respectively

Proof. Take the pushout along the horizontal maps, here we would normally need to assume the properness axiom of the model category, but the weak equivalence is preserved by the assumption that we are working with cofibrant objects:

Proposition 2.0.5. *Let $\mathcal{M}$ be a model category then every pushout of a weak equivalence between cofibrant objects along a cofibration is again a weak equivalence;*

Proof. See proof of proposition 13.1.2 in [Hir91] □

□

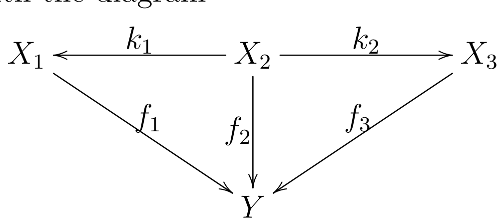
Lemma 2.0.6.
Given a diagram

$$X_1 \xleftarrow{\sim} X_2 \xrightarrow{\sim} X_3$$

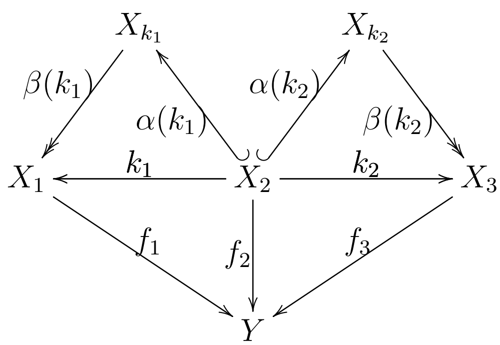
there exists an object W in $\downarrow Y$ such that we get the diagram

$$X_1 \xrightarrow{\sim} W \xleftarrow{\sim} X_3$$

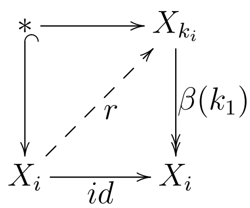
Proof. We begin with the diagram



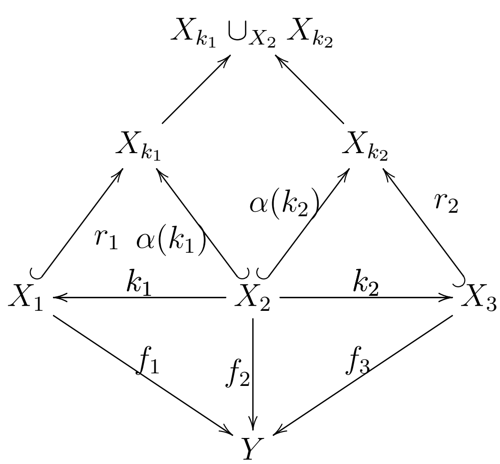
Let horizontal maps be weak equivalences. We factorize them by $k_i = \beta(k_i) \circ \alpha(k_i)$



By 2-out-of-3 we have that $k_i, \beta(k_i), \alpha(k_i)$ are all weak equivalences, which means we can find a left-inverses $r_i : X_i \to X_{k_i}$ such that $r_i \circ \beta(k_i) = id_{X_{k_i}}$. The map r_i is given as the lift of the diagram



Hence we get the diagram



With the map $X_{k_1} \cup_{X_2} X_{k_2} \to Y$ given by the universal property. □

Lemma 2.0.7.
Given either diagram

$$X_1 \xrightarrow{\sim} X_2 \hookrightarrow X_3$$

$$X_1 \hookleftarrow X_2 \xleftarrow{\sim} X_3$$

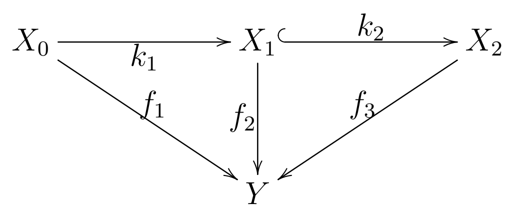
there exists objects $W_1, W_2, W_3, W_1', W_2', W_3'$ *such that we get the diagrams*

$$X_1 \xleftarrow{\sim} W_1 \hookrightarrow W_2 \xrightarrow{\sim} W_3 \xleftarrow{\sim} X_2$$

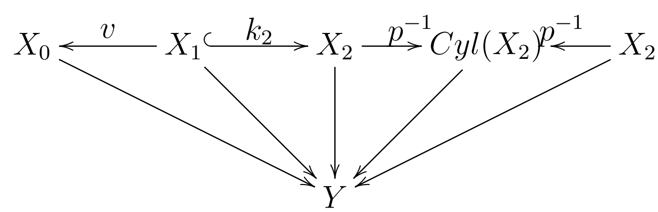
$$X_1 \xrightarrow{\sim} W_1' \xleftarrow{\sim} W_2' \hookleftarrow W_3' \xrightarrow{\sim} X_2$$

respectively.

Proof. Consider the diagram



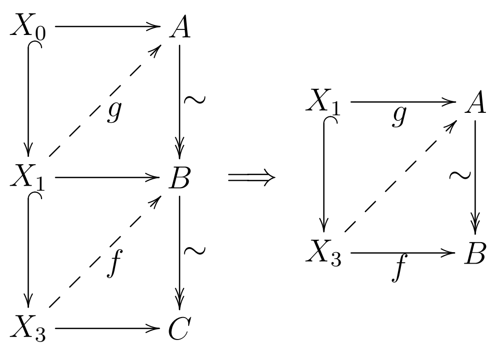
We can transform it to



by letting v be the section of k_1 and p^{-1} is the section of the map defined in definition 1.2.1. □

Lemma 2.0.8.
The composition of two weak equivalences is a weak equivalence, and the composition of two cofibrations are cofibrations.

Proof. Apply 2-out-of-3 for the weak equivalences, and the following diagrams



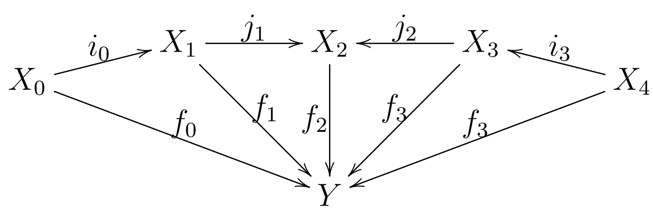
for the cofibrations for some acyclic fibrations $A \to B$ and $B \to C$

□

Chapter 3

Wall Groups

Let us now begin to introduce the groups that we will be working with. We begin by looking at the set of diagram which look like this



Here we have that j_1 and j_2 are weak equivalences, and i_0 and i_3 are relative I-cell maps from definition 2.0.3, which are constructed by attaching a finite number of I-cells. We would now like to use these diagrams to create a set of equivalence classes for the category, that is: Given some object Y, we say that $f_0 : X \to Y$ and $f_4 : X_4 \to Y$ are equivalent, $f_0 \sim f_4$, if a commutative diagram as above exists.

We now need to prove that such an equivalence relation is well-defined. So we need to check symmetry, reflexivity, and transitivity. The relation is both symmetric and reflexive by inspection, so the crux of the construction lies in proving the fact that it is transitive. This is where the end of the last section comes into play. Let us for simplicity's sake fix some object Y and look the model category of objects over Y. We need to prove that given a diagram of the form

$$X_0 \hookrightarrow X_1 \xrightarrow{\sim} X_3 \xleftarrow{\sim} X_4 \hookleftarrow X_5 \hookrightarrow X_6 \xrightarrow{\sim} X_7 \xleftarrow{\sim} X_8 \hookleftarrow X_9$$

there exists a diagram of the following form

$$X_0 \hookrightarrow W_0 \xrightarrow{\sim} W_1 \xleftarrow{\sim} W_2 \hookleftarrow X_9$$

So by applying the lemmas from the previous section we get the following sequence of diagrams:

$$\begin{gathered}
X_1 \hookrightarrow X_2 \xrightarrow{\sim} X_3 \xleftarrow{\sim} X_4 \hookleftarrow X_5 \hookrightarrow X_6 \xrightarrow{\sim} X_7 \xleftarrow{\sim} X_8 \hookleftarrow X_9 \\
\Downarrow \\
X_1 \hookrightarrow X_2 \xrightarrow{\sim} X_3 \xleftarrow{\sim} X_4 \hookrightarrow A \hookleftarrow X_6 \xrightarrow{\sim} X_7 \xleftarrow{\sim} X_8 \hookleftarrow X_9 \\
\Downarrow \\
X_1 \hookrightarrow X_2 \xrightarrow{\sim} X_3 \hookrightarrow B \xleftarrow{\sim} A \xrightarrow{\sim} C \hookleftarrow X_7 \xleftarrow{\sim} X_8 \hookleftarrow X_9 \\
\Downarrow \\
X_1 \hookrightarrow X_2 \xleftarrow{\sim} D \hookrightarrow E \xrightarrow{\sim} F \xleftarrow{\sim} B \xleftarrow{\sim} A \xrightarrow{\sim} C \xrightarrow{\sim} G \xleftarrow{\sim} H \hookleftarrow I \xrightarrow{\sim} X_8 \hookleftarrow X_9 \\
\Downarrow \\
X_1 \hookrightarrow X_2 \hookrightarrow J \xleftarrow{\sim} E \xrightarrow{\sim} F \xleftarrow{\sim} A \xrightarrow{\sim} G \xleftarrow{\sim} H \xrightarrow{\sim} I \hookleftarrow X_8 \hookleftarrow X_9 \\
\Downarrow \\
X_1 \hookrightarrow J \xrightarrow{\sim} E \xleftarrow{\sim} F \xleftarrow{\sim} A \xrightarrow{\sim} G \xrightarrow{\sim} H \xleftarrow{\sim} I \hookleftarrow X_9 \\
\Downarrow \\
X_1 \hookrightarrow J \xrightarrow{\sim} E \xleftarrow{\sim} A \xrightarrow{\sim} H \xleftarrow{\sim} I \hookleftarrow X_9 \\
\Downarrow \\
X_1 \hookrightarrow J \xrightarrow{\sim} E \xrightarrow{\sim} K \xleftarrow{\sim} H \xleftarrow{\sim} I \hookleftarrow X_9 \\
\Downarrow \\
X_1 \hookrightarrow J \xrightarrow{\sim} K \xleftarrow{\sim} I \hookleftarrow X_9
\end{gathered}$$

Hence the relation is transitive, and well-defined.

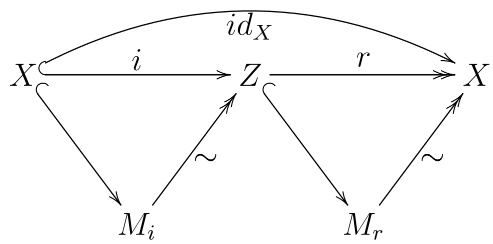
Definition 3.0.1.
Given some object X in some cofibrantly generated model category $\mathcal{M}$, then we define the following set

$$Wa(X) := \bigcup_{F \in \mathcal{M}} hom_{\mathcal{M}}(F, X) \big/ \sim$$

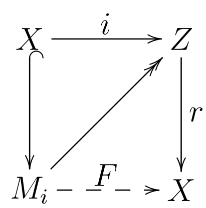
This set can be endowed with a group structure, to prove this we need to define addition and inverses. The addition on $Wa(Y)$ is defined by the coproduct:

$$[f : X_1 \to Y] + [g : X_2 \to Y] := [f \sqcup g : X_1 \sqcup X_2 \to Y]$$

with the identity element given by the initial object of the category. For the inverses let Z be a finite complex and define maps $r : Z \twoheadrightarrow X$ and $i : X \hookrightarrow Z$ such that $ri = id_X$.



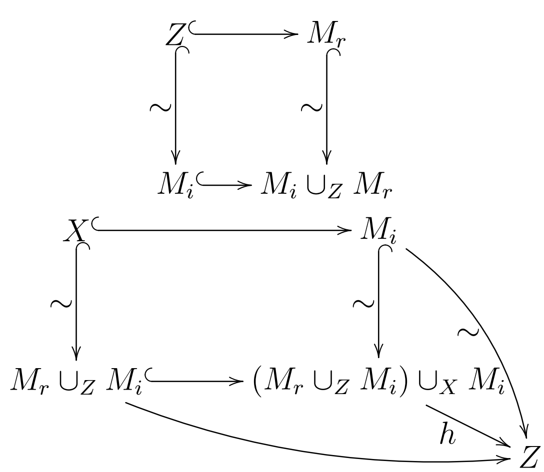
We then define the map $F : M_i \to Y$ by the diagram:



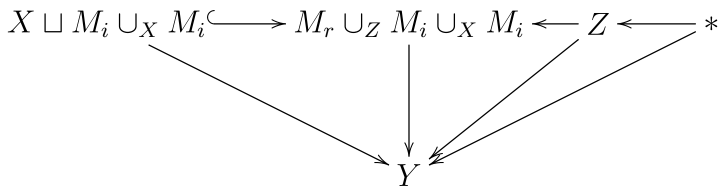
Thus our goal is to show

$$-[f] = [M_i \cup_X M_i \xrightarrow{F \cup_X F} X \xrightarrow{f} Y]$$

From the above diagram we can produce the following pushouts



Which furthermore yields the following diagram



Hence we get the relation $[X \to Y] = -[M_i \cup_X M_i \to Y]$ Moreover, if we let $f : X \to Z$ then $Wa(f) : Wa(X) \to Wa(Z)$ is defined by composition

$$\begin{aligned} Wa(f) : Wa(X) &\to Wa(Z) \\ [\phi : Y \to X] &\mapsto [f \circ \phi : Y \to Z] \end{aligned}$$

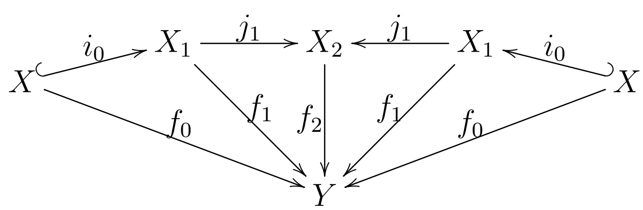
Hence $Wa(-)$ is not only a group, but also a functor.

Theorem 3.0.1.
Given $f, g : Z \to X$, if f and g are homotopic, then

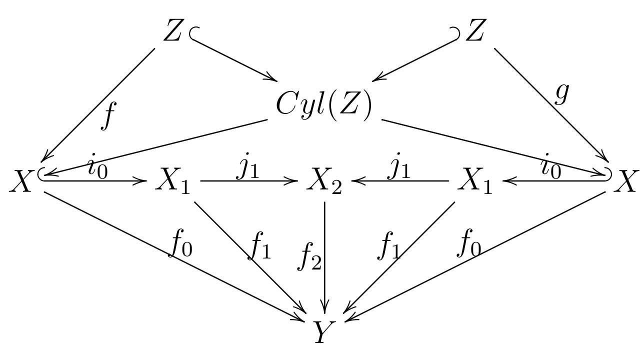
$$Wa(f) = Wa(g)$$

.

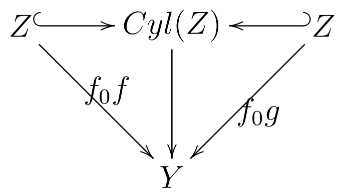
Proof. We start with diagrams of the form



By applying our map together with the definition of left homotopy we get the following diagram



By inspection we have

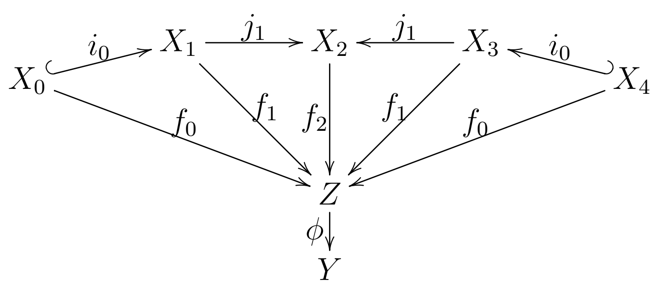


Hence $[f_0 f] = [f_0 g]$ which implies $Wa(f) = Wa(g)$ $\square$

Theorem 3.0.2.
Given a weak equivalence $\phi : Z \to Y$, then

$$Wa(X) \cong Wa(Y)$$

Proof. This is a direct consequence of theorem 1.2.2 and theorem 3.3, but we can also prove this directly: We look at the following diagram in



This implies $Wa(Z) \leqslant Wa(Y)$. Since Z and Y are cofibrant we can construct an inverse of ϕ. So we can "switch the places" of Z and Y, hence $Wa(Y) \leqslant Wa(Z)$ which further implies $Wa(X) \cong Wa(Y)$ $\square$

Chapter 4

Finiteness Obstruction

Obstruction theory is a sub-theory of many disciplines of mathematics, however the arguably most usual setting is that of homotopy theory. Where it is useful to distinguish a finite CW-complexes i.e. a finite dimensional, say dimension k, CW-complex with a finite number of n-cells for $n \leq k$ to some other space. We will look at some of these which have been developed, and then look at how the functor we defined earlier give us a similar invariant.

Definition 4.0.1.
We say an object X is *dominated* if there exists an I-cell object Y such that we have maps $r : Y \to X$ and $i : X \to Y$ with $ri \sim 1_X$. Furthermore, we say X is *finitely dominated* if it is dominated by a finite I-cell object.

4.1 Wall Obstruction

To begin we need to develop some K-theory, which in our case are functors K_i : **Rng** $\to$ **Ab** called algebraic K-theory (you can also develop the theory for categories other than **Rng**, eg. **TOP** called topological K-theory). We begin by choosing a ring R. From this ring we construct the set $P(R)$ which consists of finitely generated projective R-modules. Using the fact that direct-sum $\oplus$ is an commutative associative operation on this set, and that the 0 module is an identity element of this operation, we get that $P(R)$ is an abelian monoid. We then construct the completion $P(R)^{-1}P(R)$ of this monoid $P(R)$ by forming the free abelian group $F(M)$, and then quotient by the subgroup $R(M)$ generated by the relations $[x+y]-[x]-[y]$. We denote this completion by $K_0(R)$. Now let R be a commutative ring, then the tensor product of projective modules is again projective, and so tensor product induces a multiplication hence K_0 is a commutative ring with the class $[R]$ as identity. We write $H_0(R)$ for $C_0(Spec(R), \mathbb{Z})$, the ring of all continuous maps from $Spec(R)$ to $\mathbb{Z}$. There exists a map

$$rank : K_0(R) \to H_0(R)$$

the kernel of this map we define as $\tilde{K}_0(R)$, this is called the reduced K-theory of R. However in our case we would like to look at settings where $R = \mathbb{Z}[\pi_1(X)]$ for some X, and $\pi_1(-)$ is in general non-abelian. The map $n \mapsto n[R]$ determines a group homomorphism $\mathbb{Z} \to K_0(R)$. We let $\tilde{K}_0(R)$ denote the cokernel of this homomorphism.

We now give the following theorem on finiteness obstruction which is due to Wall

Theorem 4.1.1.

1. *A finitely dominated space X has a finiteness obstruction*

$$[X] \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$$

such that $[X] = 0$ if and only if X is homotopy equivalent to a finite CW-complex.

2. *If π is a finitely presented group then every element $\omega \in \tilde{K}_0(\mathbb{Z}[\pi])$ is the finiteness obstruction of a finitely dominated CW-complex X with $[X] = \omega$, $\pi_1(X) = \pi$.*

3. *A CW-complex X is finitely dominated if and only if $\pi_1(X)$ is finitely presented and the cellular $\mathbb{Z}[\pi_1(X)]$-module chain complex $C^*(\tilde{X})$ of the universal cover $\tilde{X}$ is chain homotopy equivalent to a finite chain complex P of finitely generated projective $\mathbb{Z}[\pi_1(X)]$-modules*

The actual finiteness obstruction $[X]$ is defined in the following way: Take any finitely generated projective $\mathbb{Z}[\pi_1(X)]$-module chain complex

$$\mathcal{P} : \cdots \to P_3 \to P_2 \to P_1 \to P_0$$

which is chain-equivalent to $C_*(\tilde{X})$. Then $[X]$ is defined as the alternating sum

$$[X] = \sum_{i=0}^{\infty} (-1)^i [P_i] \in \tilde{K}_0(\mathbb{Z}\pi_0(X))$$

If we say that X has the homotopy dimension n , then this class can also be obtained by choosing a CW complex Z of dimension $< n$, and an $(n-1)$-connected map $f : Z \to X$. The Wall finiteness obstruction of X is the class

$$w(X) = (-1)^n [H_n(X, Z; Z[\pi_1(X)])] \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$$

4.2 Whitehead Torsion

To make sense of this construction we begin by constructing the first K-group. Choose some ring R, and look at the following sequence of general linear groups over R

$$GL_0(R) \subset GL_1(R) \subset GL_2(R) \subset GL_3(R) \subset \dots$$

where each map which embeds $GL_n(R)$ in $GL_{n+1}(R)$ as the upper left block matrix. Denote the colimit of this as $GL(R)$, we call this the infinite general linear group over R. We then define $K_1(R)$ as the abelianization of $GL(R)$ i.e.

$$K_1(R) := GL(R)^{ab} = GL(R)/[GL(R), GL(R)]$$

where $[GL(R), GL(R)] = \{aba^{-1}b^{-1} | a, b \in GL(R)\}$

The Whitehead group of a manifold M is defined as to be $Wh(\pi_1(M))$. If we let G be a group, then the Whitehead group $Wh(G)$ is defined to be the cokernel of the map

$$G \times \{-1, 1\} \to K_1(\mathbb{Z}[G])$$

which sends $(g, \pm 1)$ to the invertible $(1, 1)$-matrix $(\pm g)$.

Let $C_\centerdot = (C_*, d_*)$ be a contractible finite chain complex of based left R-modules, i.e. free with a chosen finite basis. Select a chain contraction $s_* : C_* \to C_{*+1}$, which is a chain homotopy from id to 0; that is: $d \circ s + s \circ d = id$. The the algebraic torsion is well-defined by the formula

$$\tau(C_\centerdot) := [d + s : C_{even} \to C_{odd}]$$

with $C_{even} := C_0 \oplus C_0 \oplus \dots \oplus C_{2N}$ and $C_{odd} := C_1 \oplus C_3 \oplus \dots$ are finite based modules.

Let $f : X \to Y$ be a homotopy equivalence of connected finite CW-complexes. We can then define the *Whitehead Torsion* $\tau(f)$ of f as follows:

Let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be the lift of f to the universal coverings. This will induces $\mathbb{Z}[\pi_1(Y)]$-chain homotopy equivalences $C_*(f) : C_*(\tilde{X}) \to C_*(\tilde{Y})$, where $C_*(-)$ is the standard chain-complex on a topological space. The torsion is then defined as

$$\tau(f) := [\tau(Cone(C_.(\tilde{f}))] \in Wh(\pi_1(Y))$$

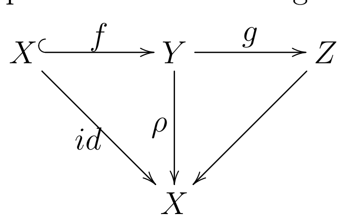
4.3 Generalized Geometric Obstruction

We want to show that the group that we have defined gives us much of the same information.

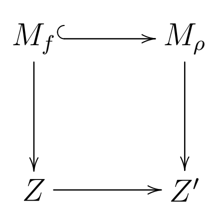
Definition 4.3.1.
Let X be in some model category, we define the *geometric finiteness obstruction* $w(X) \in Wa(X)$ of X as the class of the identity map id_X in $Wa(X)$.

Theorem 4.3.1.
Let X be an object in $\mathcal{M}$ of the homotopy type of a finitely dominated I-cell complex. Then X is weakly equivalent to a finite I-cell complex if and only if $w(X)$ vanishes.

Proof. Let X be an object such that $w(X) = 0$. Hence there exist object Y, Z such that $f : X \hookrightarrow Y$ is a cofibration, and $g : Y \xrightarrow{\sim} Z$ is a weak equivalence with Z being a finite complex and the following diagram commutes

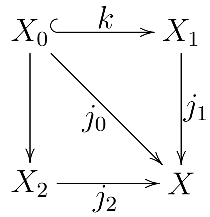


We factorize f into $X \hookrightarrow M_f \overset{\sim}{\twoheadrightarrow} Y$, and ρ into $Y \hookrightarrow M_\rho \overset{\sim}{\twoheadrightarrow} X$. Since Y is obtained from X by attaching a finite number of cells, we can do the same to M_f to obtain M_ρ. This enables us to produce the following pushout diagram:



which means that X is weakly equivalent to Z' i.e. a finite I-cell complex $\square$

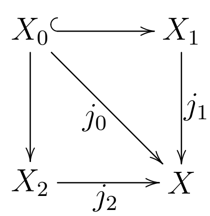
Theorem 4.3.2.
If the following diagram



of objects having the homotopy type of finitely dominated I-cell complexes is a push-out and k a cofibration then

$$w(X) = j_1^*(w(X_1)) + j_2^*(w(X_2)) - j_0^*(w(X_0))$$

Proof. The additivity claim is equivalent to the claim that given the following diagram



we get that

$$[j_0] + [j_1] - [j_0] = [id_X]$$

in $Wa(X)$. By doing the same type of construction that we did to define the inverses in section 4, we get the following diagram:

$$\begin{array}{ccccc} & & X_1 \cup M_r \cup_Z M_i \cup_X M_i \cup_Z M_r \cup_{X_0} X_2 & & \\ X_1 \sqcup M_i \cup_{X_0} M_i \sqcup X_2 & \hookrightarrow\!\nearrow & \downarrow & \nwarrow & X_1 \cup_{X_0} X_2 \\ & \searrow^{j_1 \sqcup j_0 \circ F \cup_X F \sqcup j_2} & & \swarrow_{j_1 \cup_{j_0} j_2} & \\ & & X & & \end{array}$$

Thus we are done. □

4.4 Correspondence

The previous section is a generalization of what Lück did in his paper *Geometric Finiteness Obstruction*[Lü86]. In this paper he constructed the functor Wa^G as follows: Let G be some Lie group, then Wa^G is a functor from the category of G-CW-complexes into the category of abelian groups, and an assignment w^G associating to a G-CW-complexes X having the homotopy type of a finitely dominated G-CW-complex (i.e. a homotopy retract of a finite G-CW-complex) an element $w^G(X)$ in $Wa^G(X)$. His construction relied on the homotopy equivalences between the spaces, where we instead looked at the weak equivalences.
Furthermore, the observant reader may have noticed the similarity in naming of the invariants, this is no coincidence. In Lück's original paper he proved that in the case of CW-complexes there is a relation between the Wall obstruction and his geometric obstruction. This is realized by defining a homomorphism

$$F(Y) : Wa(Y) \to \tilde{K}_0(\mathbb{Z}\pi_1(Y))$$
$$[f : X \to Y] \mapsto f^*([X])$$

This induces a natural equivalence, such that for any finitely dominated CW-complex Y, the relation $F(X)(w(Y)) = [X]$ holds. However this does prove to be problematic to generalize to our functor. For given any model category there are ways of defining what the corresponding K-theory would be by for example applying techniques found in Sagave's paper[Sav04]. Another way would be to try to define the homotopy groups naively, and then try to use those to compute the correspondence from there. Two possible ways of doing this could be by defining the groups in either of the following ways

1. $\pi_n^{\mathcal{M}}(X) := [A, \Omega^n X]$
2. $\pi_n^{\mathcal{M}}(X) := [S^n, X]$

In the first option the functor $\Omega : \mathbf{TOP} \to \mathbf{TOP}$ would be a generalization of the familiar loop-space. This could be realized by identifying the with the homotopy fiber of the map $path(X) \to X$.
In the second we would need to inductively construct the S^n, for this to make sense we need the category to be pointed. First take $S^0 := * \sqcup *$, this enables us to form the following sequence $S^0 \hookrightarrow D^1 \xrightarrow{\sim} *$, from which we can get $S^1 := {D^1}/{S^0}$. from this we can inductively define D^n and S^n for $n \in \mathbb{Z}$.
This is of course only conjecture, and proving such a correspondence would admittedly be outside the scope of this paper. The theory we laid out will of course agree with Lück given the proper categories, but we will not prove any more.

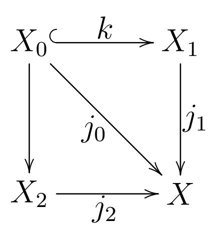
Chapter 5

Universal Property

What we in the last section describes is what is referred to as a functorial additive invariant.

Definition 5.0.1.
A *functorial additive invariant* of a category $\mathcal{C}$ is a pair (B, b) where $B : \mathcal{C} \to Ab$ is functor and $b(-) \in B(-)$ is an assignment, such that the following holds:

1. Homotopy invariance.
   (a) if $f : X \to Y$ is a homotopy equivalence in $\mathcal{C}$, then $f^* : B(X) \to B(Y)$ sends $b(X)$ to $b(Y)$.
   (b) If f and $g : X \to Y$ are homotopic, then $f^* = g^*$.
2. Additivity: If the following diagram is a push-out and k a cofibration



   then $b(X) = j_1^*(b(X_1)) + j_2^*(b(X_2)) - j_0^*(b(X_0))$
3. $b(\emptyset) = 0$

This is a generalization of the standard additive invariant, with the main difference being that with the additive invariant the functor F would be a constant functor. An example of an additive invariant is the Euler characteristic of a topological space X. The Euler characteristic which was first defined for polyhedra, is an assignment $\chi : \mathbf{TOP} \to \mathbb{Z}$. The classical definition for polyhedra was given as

$$\chi = V - E + F$$

i.e. #vertexes - #edges + #faces. In the modern definition it is given as the alternating sum of the Betti numbers b_i of the space:

$$\chi(X) = \sum_{i=0} (-1)^i b_i(X)$$

Where the i-th Betti number is the rank of the homology group $H_i(X)$. Another (albeit trivial) example is the cardinality of a set

Definition 5.0.2.
We say that an additive invariant (U, u) is *universal* if for any additive invariant (B, b) there exists a natural transformation $F_X : U(X) \to B(X)$ which is uniquely determined by $F_X(u(X)) = b(X)$

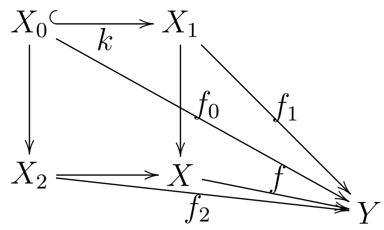
Proposition 5.0.1.

1. *There exists a universal functorial additive invariant unique up to natural equivalence.*

2. *There exists a universal additive invariant unique up to isomorphism. It is given by* $(\hat{U}, \hat{u}) = (U(*), U(- \to *)(u(-))$ *for the universal functorial additive invariant* (U, u).

Proof.

1. The uniqueness is a direct consequence of the universal property. It remains to construct a universal functorial additive invariant (U, u). Given an object $Y \in \mathcal{M}$, define $U(Y)$ as the quotient of the free abelian group generated by the homotopy classes $[f]$ of maps $f : X \to Y$ in $\mathcal{M}$ and the subgroup generated by elements:
   $[f] \sim [g]$ if there exists a weak equivalence h with $fh = g$.
   $[f] - [f_1] - [f_2] + [f_0]$ if there exist representatives f, f_0, f_1, f_1 and a push-out with k a cofibration



A map $g : Y \to Z$ induces $U(g) : U(Y) \to U(Z)$ by composition. We assign to an object $X \in \mathcal{M}$ the element $u(X) \in U(X)$ represented by the identity.

2. Let (T, t) be any additive invariant, since (U, u) is universal there is a natural transformation $(T, t) \to (U, u)$, therefore we also have a natural transformation
$$(T, t) \to (U, u) \to (\hat{U}, \hat{u})$$

□

Chapter 6

Functoriality

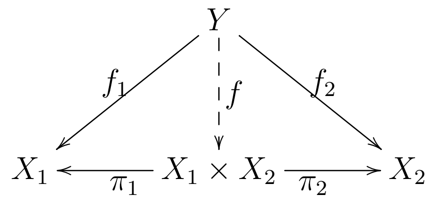
We would also like to see how the invariant behaves with regards to functors between model categories. We begin by looking at the Quillen adjoint functors $(\mathcal{L}, \mathcal{R}) : \mathcal{M} \leftrightarrows \mathcal{N}$. We know by definition that $\mathcal{L}$ preserves both cofibration and acyclic cofibrations, and as we saw earlier we also have that weak equivalences between cofibrant objects are preserved. Our construction of $Wa(-)$ was well-defined for the subcategory $\mathcal{N}$ consisting of I-cell complexes. Hence, we get a pair $(\mathcal{L}', l')$ such that $\mathcal{L}$ is the functor $\mathcal{L} : Ab \to Ab$ and l is an induced morphism that will make the following diagram commute:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow[\mathcal{L}]{} & \mathcal{N} \\ \downarrow & & \downarrow \\ (G, g) & \xrightarrow[(\mathcal{L}', l')]{} & (H, h) \end{array}$$

By the arguments above we get that the $\mathcal{L}$ restricted to the subcategory will induce isomorphisms on the $Wa(-)$ group.

Since we are working in a model categories we will of course assume that we have all limits and colimits. Having this available makes us able to define the product and coproduct of objects in our category. The natural question to ask is then: Does $Wa(X \times Y)$ depend on $Wa(X)$ and $Wa(Y)$ for X, Y in our model category, and if so, then how?

The standard categorical definition of the product is given by the universal property: For X_1, X_2 we have that for any Y and $f_i : Y \to X_i$ the following diagram exists and commutes:



So let $\mathcal{M}$ be a model category, and let $* \subset X_1 \subset \cdots \subset X_n$ and $* \subset Y_1 \subset \cdots \subset Y_m$ be in $\mathcal{M}$, we can then pass them to the product category $\mathcal{M} \times \mathcal{M}$ by constructing the product component-wise i.e.

$$(*, *) \subset (X_1, *) \subset \cdots \subset (X_n, *) \subset (X_n, Y_1) \subset \cdots \subset (X_n, Y_m)$$

The product functor does not necessarily preserve colimits, hence it is not a left Quillen functor and thus we cannot conclude $Wa(X \times Y) \ncong Wa(X) \times Wa(Y)$.

In his paper; Lück proved that the product will be determined by what he called a natural pairing, we will prove that this result can be extended to model categories:

Theorem 6.0.1.
Let X, Y be I-cell complexes in some model category $\mathcal{M}$, then we have a map

$$Wa(X) \otimes Wa(Y) \to Wa(X \times Y)$$

Proof. Let $(U, u), (V, v), (W, w)$ be the universal functorial additive invariants for $X, Y, - \times Y$ respectively. Denote $T(Y)$ as the abelian group of natural transformations $U(-) \to W(- \times Y)$, this is non-empty by universality assumption. Hence we have a functorial additive invariant (T, t) for X where t is the natural choice such that $t(Y)(X)$ sends $u(X)$ to $w(X)$. This is then interpreted as the natural pairing $U(X) \otimes V(Y) \to W(X \times Y)$, hence also $Wa(X) \otimes Wa(Y) \to Wa(X \times Y)$ □

Chapter 7

Finitesness of Chain complexes

A nice way of illustrating the theory we have developed is by looking at the category of chain complexes of the module category **RMod** of some ring R. To apply the theory we need to define a model structure on the category. It can be proven that there is a model category structure on non-negatively graded cochain complexes with the following properties: Let $f : A_* \to B_*$ for $A_*, B_* \in C(\mathbf{RMod})$, then:

- f is a cofibration iff it is injective, and $cok(f)_n$ is a free abelian group for all n.
- f is a fibration iff it is surjective.
- f is a weak equivalence iff it is a quasiisomorphism (i.e. $f_* : H_*(A) \to H_*(B)$ is an isomorphism)
- f is an acyclic cofibration iff it is a cofibration and a weak equivalence, or equivalently a monomorphism whose cokernel is acyclic and free in each degree.
- f is an acyclic fibration iff it is a fibration and a weak equivalence, or equivalently an epimorphism with acyclic kernel.

However this structure is not the only one that is available in the literature. So we can define another model structure called the *$\mathcal{G}$-model structure* using what is known as a descent structure. This construction does force us to look at commutative rings:

Definition 7.0.1.
Let R be a commutative ring, E a module in **RMod**, and $n \in \mathbb{Z}$. We then define the functors S^n and D^n from **RMod** to its category of complexes by $S^n E = E$ in degree n, and 0 elsewhere; and $D^n E = E$ in degree n and $n+1$, and 0 elsewhere.

Definition 7.0.2.
Let $\mathcal{G}$ be an essentially small set of objects of **RMod**. A morphism in $C(\mathbf{RMod})$ is called a $\mathcal{G}$-*cofibration* if it is contained in the smallest class of morphisms in $C(\mathbf{RMod})$ that is closed under pushouts, transfinite compositions and retracts, generated by the inclusions

$$[S^n E \hookrightarrow D^n E]$$

, for any integer n and any $E \in \mathcal{G}$. A complex C in $C(\mathbf{RMod})$ is called $\mathcal{G}$-cofibrant if the morphism $0 \to C$ is a $\mathcal{G}$-cofibration.

Definition 7.0.3.
A chain complex C in $C(\mathbf{RMod})$ is called $\mathcal{G}$-*local* if for all $E \in \mathcal{G}$ and $n \in \mathbf{Z}$, the canonical morphism

$$Hom_{K(\mathbf{RMod})}(E[n], C) \to Hom_{D(\mathbf{RMod})}(E[n], C)$$

is an isomorphism. Here $K(\mathbf{RMod})$ and $D(\mathbf{RMod})$ are the homotopy category of $C(\mathbf{RMod})$ and the derived category of $C(\mathbf{RMod})$ respectively.

Definition 7.0.4.
Let $\mathcal{H}$ be a small family of complexes in $C(\mathbf{RMod})$ An complex C in $C(\mathbf{RMod})$is called $\mathcal{H}$-*flasque* if for all $n \in \mathbb{Z}$ and $H \in \mathcal{H}$,

$$Hom_{K(\mathbf{RMod})}(H, C[n]) = 0$$

Definition 7.0.5.
A *descent structure* on $C(\mathbf{RMod})$ is a pair $(\mathcal{G}, \mathcal{H})$, where $\mathcal{G}$ is an essentially small set of generators of $C(\mathbf{RMod})$, and $\mathcal{H}$ is an essentially small set of $\mathcal{G}$-cofibrant acyclic complexes such that any $\mathcal{H}$-flasque complex is $\mathcal{G}$-local.

So after a lot of definitions we get to state the desired result.

Theorem 7.0.1.
*Let $(\mathcal{G}, \mathcal{H})$ be a descent structure on **RMod**. There is a proper cellular model structure on the category $C(\mathbf{RMod})$, where the weak equivalences are quasi-isomorphisms of complexes, and cofibrations are $\mathcal{G}$-cofibration.*

This result was originally proven for Grothendieck categories, for which the category of modules over a commutative ring is a special case of. The proof of the more general case of this result can be found in [CD09].

This makes us able to characterize all the finite complexes in the category:

Proposition 7.0.2.
An complex $C \in C(\boldsymbol{RMod})$ is made up of finite cells if and only if C is bounded above.

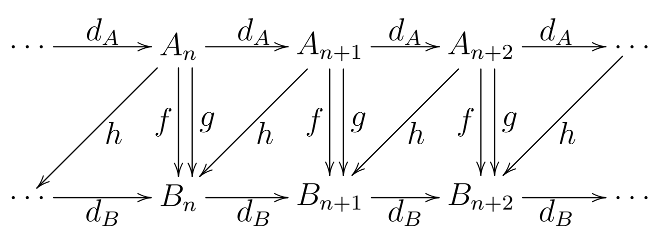
Proof. Let C be bounded above, this means there is some $n \in \mathbb{Z}$ such that $C_i = 0$ for $i > n$. We can then define the cells $[S^n C \to D^n C]$, these will obviously generate C by pushouts. The other direction is trivial. □

Now of course this is all well and fine, but this condition is solved very trivially so let us ease up on the condition and try to look at complexes that are equivalent in some way to a finite complex.

Some of the previous work done on this, like the work of Andrew Ranicki. Our case does however

We have previously done a lot of work to define a group that is invariant under homotopy equivalences so this would be a natural way to start. We begin by recalling the definition of chain homotopies that we briefly mentioned earlier:

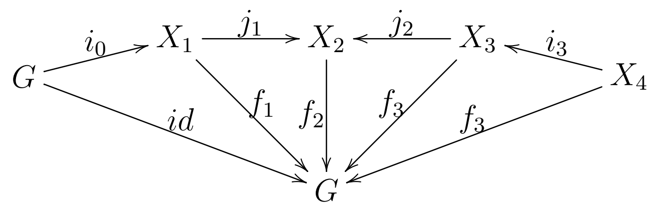
Definition 7.0.6.
$f^n, g^n : A^n \to B^n$ are homotopic $f \sim g$ if there is a collection of maps $h^k : A^k \to B^{k-1}$ such that $f^n - g^n = d_B h^n + h^{n+1} d_A$ where $d_{(-)}$ are the differentials



Furthermore we say that two chain complexes A, B are chain equivalent if there exists a chain homotopy $h : A \to B$ that admits a chain homotopy inverse, i.e. there exists maps $i : B \to A$ such that $ih \sim 1_A$ and $hi \sim 1_B$.

We can now try to apply the functor we constructed in the previous section.

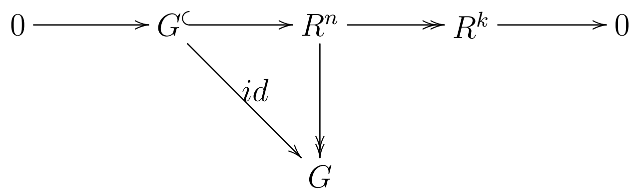
Let G be any complexes over some commutative ring, by applying $Wa(-)$ we get the following class of diagrams



We know X_1, X_2, X_3 are all finite complexes, and by the definition of cofibrant object we need $cok(* \to X_i)$ to be free abelian groups. Hence they will be of the form $X_i = X_i^1 \to X_i^2 \to X_i^3 \to \cdots \to X_i^{d_i}$ where the X_i^j are all free abelian and since $H_*(X_1) \cong H_*(X_2) \cong H_*(X_3)$ we have $d \in \mathbb{N}$.

So let G_* be made of finite cells i.e. finitely generated free abelian in every degree, and $G_i = 0$ for $i \leq n$ for some n. Then we obviously have a domination by any complex X_1 where $rank(G^i) \leq rank(X_1^i)$ for all $i \in \mathbb{N}$, hence we can easily find a X_2 with the appropriate properties.

Now let us see if we can classify all of the finitely dominated complexes over **RMod**. This ends up being an extension problem, since the dominated complex G_* would in each degree need to satisfy the following diagram



This is a left split short exact sequence, hence by the splitting lemma, we have that $R^n \cong G \oplus R^k$. This implies $G \cong R^{n-k}$ or $n = k$.

There has been some other work regarding finitely dominated chain complexes, for example the paper by Ranicki [Ran85]. Do note that their work did not have the restriction of having the "attaching maps" needing to have a free abelian cokernel.

Chapter 8

Killing groups

Since we have an abundance of invariants that give us a lot of data of our objects, whatever they may be, it is also interesting to look at how these groups behave after doing some slight "modifications" to our objects. In topology; given some space X, a natural way is to look at how these groups behave after gluing or removing some K from our chosen object. To make this more precise we can look at how we can produce topological spaces from other topological spaces such that the homotopy groups are either killed off or "reduced". Let X be a CW-complex, and let $[\alpha]$ be a class in $H^n(X, G)$. By the correspondence $H^n(X, G) \longleftrightarrow [X, K(G, n)]$ we get the following diagram

$$\begin{array}{ccccc} \Omega Y & \longrightarrow & E_\alpha & & \\ & & \downarrow & & \\ & & Y & \xrightarrow[\alpha]{} & K(G,n) \end{array}$$

where $E_\alpha \to Y$ is the fibration induced by α, i.e.

$$E_\alpha = \{(b, u) \in E \times K(G, n) | w(b) = u(l)\}$$

This also means that we can view $E_\alpha \to Y \to K(G, n)$ as a fibration. The homotopy sequence of that fibration shows that if α^* is surjective in dim n then $\pi_i(E_\alpha) \cong \pi_i(Y)$ and the sequence $0 \to \pi_n(E_\alpha) \to \pi_n(Y) \to G \to 0$ is exact. Hence if we assume that the order of the groups are finite, we get $|\pi_n(E_\alpha)| = {|\pi_n(Y)|}/{|G|}$.

Let us now form the trivial fibration $S^1 \to S^1 \times X \to X$, this gives us the following theorem:

Theorem 8.0.1.
Let X be topological space dominated by a finite CW-complex K. Then $S^1 \times X$ has the homotopy type of a finite CW-complex.

To see this geometrically if we replace the space X with the mapping cylinder of the map $K \to X$, which has the same homotopy type as X. Then the map $X \to K$ becomes a map $f : C \to C$ whose image lies in K embedded in X, and which is homotopic to the identity. We may suppose that $f|_K$ is cellular. Define the mapping torus $T(f)$ of f by taking $X \times I$ and identifying $(x, 1)$ with $(f(x), 0)$ for each $x \in X$. As with a mapping cone, $1 \sim f$ implies $T(1) \sim T(f)$. $T(1)$ is $X \times S^1$, so $X \times S^1 \cong T(f)$. Define a homotopy $h_t : T(f) \to T(f)$ by

$$h_t(x, s) = \begin{cases} (x, s + t), & \text{for } s + t \leq 1 \\ (x, s + t - 1), & \text{for } s + t \geq 1 \end{cases}$$

This homotopy is a weak retraction of $T(f)$ to $T(f|_K)$, naturally embedded in $T(f)$. Hence $X \times S^1 \cong T(f) \cong T(f|_K)$ which is a finite CW-complex.

We need to check if this agrees with the theory we have developed. Using the product formula we defined in theorem 6.0.1 we get the following map

$$U(S^1) \otimes U(X) \to U(S^1 \times X)$$

We know that this map is uniquely defined by the fact that $u(S^1) \otimes u(X)$ gets sent to $u(S^1 \times X)$. We also know that since since S^1 is a finite complex $u(S^1) \otimes u(X)$ vanishes, which means that $u(S^1 \times X)$ also vanishes.

This also agrees with the Wall and Whitehead torsion, since $K_i()$

This means that S^1 kills our invariant in the category of topological spaces, but does this generalize to other finite complexes? By our product formulae, it does! Simply take any finite complex F, this means that $w(F)$ vanishes which again implies that $w(F \times -)$ vanishes. This does of course generalize to any model category where this functor is well-defines. For example, take any chain-complex C and F in **RMod** such that C is finitely dominated and F is of finite type, then we get that $C \oplus F$ is of finite type. This is trivial to see if we were to use the cellular model structure we defined in theorem 7.0.1, but this also hold with the more standard model structure.

Algebraically this makes sense, but in higher dimensions there seems to not be any topological proof for the general case. We will not prove this, but encourage the reader to take a stab.

Chapter 9

Appendix

9.1 Notation

$=$ - Equal

$:=$ - Equal by definition

$\in$ - Element of

$\cong$ - Isomorphism

$\leqslant$ - Subgroup, or less than or equal

$H^n(-); H_n(-)$ - (co)Homology group of degree n

$\pi_n(-)$ - Homotopy group of degree n

$K_n(-)$ - K-theory of degree n

S^n - n-sphere

D^n - n-disc

$\sim$ - Equivalence relation

$[\ ]$ - Class in some equivalence relation

$\twoheadrightarrow$ - Fibration

$\hookrightarrow$ - Cofibration

$\xrightarrow{\sim}$ - Weak equivalence

$\times$ - Product

$\sqcup$ - Coproduct

$\mathcal{C}$	Category
$\mathcal{M}$	Model category
Grp	Category of groups
Ab	Category of abelian groups
Rng	Category of rings
TOP	Category of topological spaces
Set	Category of sets
sSet	Category of simplicial sets
RMod	Category of modules over a ring R

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