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Traveling Waves for a Fractional Korteweg–De Vries and a Fractional Degasperis–Procesi Equation with an Inhomogeneous Symbol

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Abstract

We study two classes of equations: a fractional Korteweg–De Vries (fKdV) equation $u_t + uu_x + (\Lambda^{-s}u)_x = 0$ and a fractional Degasperis–Procesi (fDP) equation $u_t + uu_x + \frac{3}{2}(\Lambda^{-s}u^2)_x = 0$. The operator Λ^{-s} is a Fourier multiplier with symbol $(1+\xi^2)^{-s/2}$ and $s \in (0,1)$. For the fKdV equation, we prove that there exist local bifurcation branches emanating from the trivial solution, consisting of smooth and periodic traveling-wave solutions, and that the local branches extend to global solution curves. In the limit of such a curve we find a highest, cusped traveling-wave solution and prove its optimal s-Hölder regularity, attained in the cusp. For the fDP equation, we prove that local bifurcation branches of smooth and periodic traveling-wave solutions exist around a constant solution of the equation and that for sufficiently small periods global bifurcation occurs. Moreover, we discuss conditions under which a highest, cusped traveling-wave solution for the fDP equation exists, and its expected regularity. The theory is accompanied by numerical examples.

Sammendrag

Vi studerer to familier av ligninger: En fraksjonell Korteweg–De Vries-ligning (fKdV) gitt ved $u_t + uu_x + (\Lambda^{-s}u)_x = 0$ og en fraksjonell Degasperis–Procesi-ligning (fDP) gitt ved $u_t + uu_x + \frac{3}{2}(\Lambda^{-s}u^2)_x = 0$. Operatoren Λ^{-s} er en Fourier-multiplikator med symbol $(1+\xi^2)^{-s/2}$ og $s \in (0,1)$. For fKdV-ligningen beviser vi at det eksisterer lokale forgreninger av løsninger rundt den trivielle løsningen, bestående av glatte og periodiske reisende bølger, og at de lokale forgreningene eksisterer som globale løsningskurver. I grensen av en slik kurve finner vi en spiss reisende bølge med maksimal høyde og beviser dens optimale s-Hölder-regularitet, oppnådd i spissen. For fDP-ligningen viser vi at lokale løsningsforgreninger av glatte og periodiske reisende bølger eksisterer rundt en konstant løsning til ligningen, og at global forgrening forekommer for tilstrekkelig små perioder. Videre diskuterer vi betingelser for eksistensen av en spiss reisende bølge med maksimal høyde som løser fDP-ligningen, og dens forventede regularitet. Numeriske eksempler er gitt.

Preface

This is the final report for the course "TMA4900 - Master Thesis", spring semester 2021, at the Norwegian University of Science and Technology. The work was carried out under the supervision of Professor Mats Ehrnström, who also proposed the topic of the thesis.

The present work builds on my earlier project titled "Traveling Waves in a Whitham-Type Equation with a Bessel Potential Operator", written during the fall semester of 2020 as part of the course "TMA4500 - Specialization Project". Inspired by recent advances for the Whitham equation, the idea was to study the relationship between the regularity of traveling-wave solutions for a dispersive equation and the order of the dispersive operator present in the equation. In the project, local bifurcation theory for a fractional Korteweg–De Vries (or Whitham-type) equation with a parametrized dispersive operator was developed. The first part of this thesis addresses regularity and global bifurcation for this equation and thereby concludes the theory.

While studying our original model equation, I was also given the opportunity to explore generalizations in directions that I found interesting. This resulted in two additional topics which have been included in the thesis. Firstly, a section on characteristic features of the dispersive operator in the equation which permit the same traveling-wave phenomena that were shown for the original model, and the balance between dispersion and nonlinear effects in these equations. Secondly, a study of a fractional Degasperis–Procesi equation, where the main difference is that the dispersive term is nonlinear.

None of the proofs in this thesis have been copied directly from other sources, but many results are based on, or inspired by the works of others. All sources are referenced throughout, and those that are used repeatedly are provided at the beginning of each section.

I am sincerely grateful to Mats Ehrnström for introducing me to an interesting problem, giving me a large degree of freedom in my approach, and teaching me about mathematics.

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Magnus Christie Ørke

Contents

1 Introduction

The problems considered in this thesis are related to the mathematical study of water waves. We briefly review basic notions and relevant history of this subject. Then, we specify the problems that are investigated herein, and give a short overview of related research.

1.1 Background

The behavior of fluids, and the formation of waves, has long been a subject of mathematical research, and variations of the governing equations in hydrodynamics have been known for more than two centuries. An outline of early history is given in [8].

Pertaining to the topics in this thesis, a seminal event is J. S. Russel's observation of a wave-phenomena which he called the "wave of translation", on a canal in 1834. This was a smooth and solitary wave traveling without a change of shape, and was not predicted by the contemporary linear theory of water waves. With the report on this new discovery [25], the concept of a nonlinear solitary traveling wave was introduced to the mathematical community.

Another influential piece of history is G. G. Stokes article [27] from 1847, in which he argued that if there exists a singular wave solution with a steady profile to the free boundary problem for the Euler equations, then the wave must have an interior angle of $120°$ at the crest. Introducing thereby the idea of highest singular traveling waves, this later has become known as the Stokes conjecture. The existence of the Stokes wave was proved in [2], thus settling the conjecture. For a more detailed account of Stokes' work on water waves, we refer to [9].

Let us consider an infinitely wide fluid body with a fixed bottom and a free surface under the influence of gravity, as illustrated in Figure 1. The following discussion and notation are based on the monograph [20]. We assume that the fluid is homogeneous, inviscid, incompressible, and irrotational. Moreover, it is contained in a domain of bounded depth with a fixed bottom and a free surface, both of which can be parametrized as graphs. The fluid particles can not cross either the bottom or the surface. There is no surface tension present and the external pressure above the free surface is assumed to be constant. Furthermore, we assume that the fluid is at rest at infinity.

Let $\Omega_t \subset \mathbb{R}^2$ denote the two-dimensional domain occupied by the fluid at a time t. The velocity at a point (x, z) at time t is denoted by $U(x, z, t)$, and the pressure is $P(x, z, t)$. Furthermore, the bottom is located at a constant depth $-H_0$, and the free surface elevation is given by the function $\zeta(x, t)$.

The motion of the fluid can now famously be described by the free-surface Euler equations [20, pp. 2–3]. Introducing a velocity potential $\Phi(t, x, z)$, on the grounds of the fluid being irrotational, the Euler equations may be reformulated as the free surface

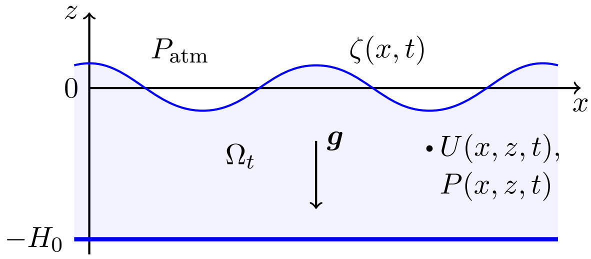


Figure 1: Illustration of a fluid body in a gravitational force.

Bernoulli equations. They are given by

$$\nabla\Phi = U \qquad \text{in} \quad \Omega_t, \tag{1.1a}$$

$$\Delta\Phi = 0 \qquad \text{in} \quad \Omega_t, \tag{1.1b}$$

$$\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + gz = -\frac{1}{\rho}(P - P_{\text{atm}}) \qquad \text{in} \quad \Omega_t, \tag{1.1c}$$

with boundary conditions

$$\begin{aligned} \Phi_z &= 0 && \text{on} \quad \{z = -H_0\}, \\ \zeta_t - \sqrt{1+\zeta_x^2}\partial_{\mathrm{n}}\Phi &= 0 && \text{on} \quad \{z = \zeta(x,t)\}, \\ P &= P_{\text{atm}} && \text{on} \quad \{z = \zeta(x,t)\}. \end{aligned}$$

Note that ∇ is the gradient operator with respect to spacial variables, and ∂_{n} denotes the outwards normal derivative. These equations are simply a mathematical restatement of the assumptions made on the fluid body.

It can be shown that if both the free surface elevation ζ, and the trace of the velocity potential $\psi = \Phi|_{z=\zeta}$ are known, then the potential Φ is uniquely determined. It may be recovered from ζ and ψ by solving the boundary value problem

$$\begin{cases} \Delta\Phi = 0 \qquad \text{in} \quad \Omega_t, \\ \Phi|_{z=\zeta} = \psi, \\ (\Phi_z)|_{z=-H_0} = 0. \end{cases} \tag{1.2}$$

This is known as the Zakharov–Craig–Sulem formulation of the water-wave problem.

In view of the Zakharov–Craig–Sulem formulation, it suffices to find a set of equations which determine ζ and ψ, in order to solve the water-wave problem. To this end, we introduce the Dirichlet–Neumann operator $\mathcal{G}[\zeta]$. It is defined as

$$\mathcal{G}[\zeta] \colon \psi \mapsto \sqrt{1+\zeta_x^2}\partial_{\mathrm{n}}\Phi|_{z=\zeta}, \tag{1.3}$$

where Φ solves the problem (1.2), with boundary conditions given by ζ and ψ. Hence, $\mathcal{G}[\zeta]$ maps Dirichlet boundary conditions to Neumann boundary conditions of the same problem, via the solution of the problem itself.

The utility of this definition becomes evident through the following calculations. Using the chain rule on the free surface boundary condition in (1.2), one obtains

$$(\Phi_t)|_{z=\zeta} = \psi_t - (\Phi_z)|_{z=\zeta},$$
$$(\Phi_x)|_{z=\zeta} = \psi_x - (\Phi_z)|_{z=\zeta}\ \psi_x.$$

Similarly, using the chain rule on the definition of the Dirichlet–Neumann operator yields

$$(\Phi_z)|_{z=\zeta} = \frac{\mathcal{G}[\zeta]\psi - \zeta_x\psi_x}{\sqrt{1+\zeta_x^2}}.$$

Since $P = P_{\text{atm}}$ at the surface, the right-hand side of the equation (1.1c) vanishes on $z = \zeta(x,t)$. Together with the free surface boundary condition, we arrive at the set of equations

$$\begin{cases} \zeta_t - \mathcal{G}[\zeta]\psi = 0, \\ \psi_t + g\zeta + \dfrac{1}{2}\psi_x^2 - \dfrac{(\mathcal{G}[\zeta]\psi + \zeta_x\psi_x)^2}{2(1+\zeta_x^2)} = 0 \end{cases} \tag{1.4}$$

for ζ and ψ. This system is called the water-wave equations.

Depending on the physical configuration of the problem, solutions to the water-wave equations may exhibit radically different qualitative properties. A diverse variety of behavior is contained within the equations, making them extremely difficult to solve in a unified way. For this reason, it is often useful to distinguish between different asymptotic regimes. An example of such a regime is shallow water theory, which refers to the situation when the ratio between the depth and characteristic length of the flow is small. If one assumes that the variation in surface elevation is small compared to the depth of the fluid, this constitutes what is usually called the small-amplitude regime.

It is shown in [20, Chapter 5] how, in the small-amplitude regime with shallow water, the water-wave equations can be justifiably reduced to different variations of simplified shallow water systems. A particular class of such systems is the Boussinesq equations, which, under the assumption that the free surface ζ takes the form of two counter-propagating waves, can be even further reduced to scalar models. One of the most studied such equations is the Korteweg–De Vries (KdV) equation [20, p. 179], which can be written as

$$u_t + uu_x + u_{xxx} = 0. \tag{1.5}$$

If one assumes medium-amplitude waves, equations such as the Degasperis–Procesi (DP) equation

$$u_t - u_{xxt} + 4uu_x - 3u_xu_{xx} - uu_{xxx} = 0 \tag{1.6}$$

are possible [20, p. 203]. Although we shall not be concerned with the details of these derivations here, it is essential to note that the variables have been scaled and nondimensionalized compared to the system (1.4) which is written in dimensional form.

We now consider the question: Is the dispersion relation of the full water-wave equations preserved through the reduction to the KdV equation? To answer this, we first give a short introduction to the notion of a dispersive equation.

The concept of dispersion originates from the study of linear differential equations that admit sinusoidal wave-train solutions of the form

$$u(x,t) = Ae^{\mathrm{i}\xi x - \mathrm{i}\omega t}. \tag{1.7}$$

Whenever the equation involves both space and time derivatives, the wavenumber ξ and the frequency ω has to satisfy a relationship $G(\xi, \omega) = 0$, where one often assumes an explicit relation $\omega = W(\xi)$. This is known as the dispersion relation of the equation. Observe that the phase velocity, defined as

$$c = \frac{\omega}{\xi} = \frac{W(\xi)}{\xi},$$

depends on the wave number ξ whenever $W(\xi)$ is not a constant multiple of ξ. The physical interpretation of this is that waves of different spacial frequency have different velocities. This phenomena is called dispersion. One can also talk about relative "magnitudes" of dispersion, and with weak dispersion we mean the situation when there are only small differences in velocity for different wave numbers.

The KdV equation can be written in dimensional form as

$$\zeta_t + c_0\zeta_x + \frac{3}{2}\frac{c_0}{H_0}\zeta\zeta_x + \frac{1}{6}c_0 H_0^2 \zeta_{xxx} = 0, \tag{1.8}$$

where $c_0 := \sqrt{gH_0}$ [33]. Linearizing the equation and inserting the generic solution (1.7), one finds that the phase velocity for these solutions has to be

$$c_{\mathrm{KdV}}(\xi) = c_0(1 - \frac{1}{6}H_0^2\xi^2).$$

For linear evolution equations with classical differential operators, the dispersion relations are always polynomials in ξ. In [33, p. 368], Whitham notes that inserting the wave-train solution (1.7) into the one-dimensional integro-differential equation

$$\zeta_t(x,t) + \int_{\mathbb{R}} K(x-y)\zeta_x(y,t)\, dy = 0, \tag{1.9}$$

one obtains the phase velocity

$$c(\xi) = \int_{\mathbb{R}} K(x)e^{-i\xi x}\, dx\,.$$

That is, the phase velocity corresponds to the Fourier transform of the convolution kernel appearing in (1.9) (see (2.2) for conventions for the Fourier transform). Therefore, by virtue of the Fourier inversion theorem [4, Theorem 1.19], any sufficiently integrable phase velocity $c = c(\xi)$ may be incorporated in equation (1.9) by choosing a convolution kernel

$$K(x) = \frac{1}{2\pi}\int_{\mathbb{R}} c(\xi)e^{i\xi x}\, d\xi\,.$$

The above can be used to deduce the dispersion relation of the original water-wave problem (1.4). The linearized equations around $(\zeta, \psi) = (0, 0)$ are given by

$$\begin{cases} \zeta_t - \mathcal{G}[0]\psi = 0, \\ \psi_t + g\zeta = 0, \end{cases} \tag{1.10}$$

where $\mathcal{G}[0]$ is defined as in (1.3). It turns out that the Dirichlet–Neumann operator has an explicit expression in this case. Indeed, taking the Fourier transform of (1.2) with respect to the horizontal variable x, one obtains

$$\begin{cases} -\xi^2\hat{\Phi}(\xi, z) + \hat{\Phi}_{zz}(\xi, z) = 0, \\ \hat{\Phi}(\xi, 0) = \hat{\psi}(\xi), \\ \hat{\Phi}_z(\xi, -H_0) = 0. \end{cases}$$

This is a second order ordinary differential equation with respect to z, and one can check that it has the unique solution

$$\hat{\Phi}(\xi, z) = \frac{\cosh((z + H_0)\xi)}{\cosh(H_0\xi)}\hat{\psi}(\xi).$$

Moreover, since

$$\hat{\Phi}_z(\xi, 0) = \xi \tanh(H_0\xi)\hat{\psi}(\xi),$$

the operator $\mathcal{G}[0]$ acts on ψ according to

$$\mathcal{G}[0]\psi = \mathscr{F}^{-1}(\xi \tanh(H_0\xi)\hat{\psi}(\xi)).$$

Note that this operation can equivalently be formulated as the convolution

$$\mathcal{G}[0]\psi = K * \psi = \int_{\mathbb{R}} K(x - y)\psi(y)\, dy,$$

where the convolution kernel K is given by the inverse Fourier transform of $\xi \tanh(H_0\xi)$. Eliminating ψ, the linearized system (1.10) may now be written as

$$\zeta_{tt} + g(K * \zeta) = 0. \tag{1.11}$$

Comparing this with the equation (1.9), we deduce that the dispersion relation of (1.11) is $\omega = \sqrt{g\xi \tanh(H_0\xi)}$, and solutions therefore has the phase velocity

$$c_{\mathrm{W}}(\xi) = \sqrt{\frac{g \tanh(H_0\xi)}{\xi}}. \tag{1.12}$$

This means that the dispersion relation of the KdV equation (1.8) is not the same as the dispersion relation for the original water-wave equations (1.4). There is a qualitative difference between these equations as well, mirroring the difference in the dispersion relations. While the KdV equation is a local differential equation with polynomial phase velocity, the phase velocity c_{W} is not a polynomial, and so the equation (1.11) is genuinely nonlocal. This mathematical observation can be thought of as a result of the coupling between the interior of the fluid and the free surface, where global information about the flow in the interior influences the motion at each point of the boundary.

The phase velocity c_{KdV} for the KdV equation is precisely equal to the two first terms in the Maclaurin expansion of c_{W}. This observation was also made by Whitham, who in 1967 in [32] proposed the improved model

$$\zeta_t + \frac{3}{2}\frac{c_0}{h_0}\zeta\zeta_x + (K_{\mathrm{W}} * \zeta) = 0, \tag{1.13}$$

where K_{W} is the inverse Fourier transform of c_{W}, having the exact dispersion relation of the original water-wave problem. This equation is presently known as the Whitham equation.

Physical considerations were also a motivating factor behind this improved model. As remarked by Whitham in [33, p. 476], nonlinear shallow water equations which neglect dispersion allow wave breaking, but not solitary and periodic traveling waves, while on the other hand, the KdV equation allows solitary and periodic waves, but not wave breaking. The dispersion in the Whitham equation (1.13) is much weaker than that of the KdV equation (1.8), suggesting perhaps a wider array of wave-phenomena than captured by either model on its own. This turned out to be correct. Both wave-breaking [18] and traveling waves [11, 31] have been proved for the Whitham equation.

In fact, in [11] it was shown that there exist cusped, periodic traveling-wave solutions to the Whitham equation, and that they have exact 1/2-Hölder regularity at crests. A natural question to ask is if similar results can be obtained for equations of the same form but with other dispersion relations. Modifications of the dispersion akin to that of (1.13) are possible in both the KdV equation (1.5) and the DP equation (1.6). This is the overarching theme of the present work.

1.2 Problem description

We consider two classes of equations: a fractional Korteweg–De Vries (fKdV) equation on the form

$$u_t + uu_x + (\Lambda^{-s}u)_x = 0, \qquad s \in (0,1), \tag{1.14}$$

and a fractional Degasperis–Procesi (fDP) equation given as

$$u_t + uu_x + \frac{3}{2}(\Lambda^{-s}u^2)_x = 0, \qquad s \in (0,1). \tag{1.15}$$

Here, $u(t,x)$ is a real-valued function on $\mathbb{R}^2$, and the operator Λ^{-s} is a Fourier multiplier operator defined as

$$\Lambda^{-s} \colon f \mapsto \mathscr{F}^{-1}\big(\langle\xi\rangle^{-s}\hat{f}(\xi)\big),$$

with symbol

$$\langle\xi\rangle^{-s} := (1+\xi^2)^{-\frac{s}{2}} \tag{1.16}$$

of order $-s$. The operator Λ^{-s} is frequently referred to as the Bessel potential operator, and it may equivalently be characterized as a convolution operator according to $\Lambda^{-s}u = K_s * u$, with convolution kernel given by

$$K_s(x) = \mathscr{F}^{-1}(\langle\xi\rangle^{-s})(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \langle\xi\rangle^{-s} e^{ix\xi}\, d\xi\,.$$

The fKdV and fDP equations are nonlocal and nonlinear evolution equations with weak dispersion. If one assumes traveling-wave solutions on the form $u(x,t) = \varphi(x-\mu t)$, where μ is interpreted as the wave-speed in the rightward direction, the fKdV equation reads

$$-\mu\varphi' + \frac{1}{2}(\varphi^2)' + (\Lambda^{-s}\varphi)' = 0.$$

Integrating yields the steady equation

$$-\mu\varphi + \frac{1}{2}\varphi^2 + \Lambda^{-s}\varphi = 0. \tag{1.17}$$

The right-hand side is assumed to be zero without loss of generality, due to the Galilean transformation

$$\varphi \mapsto \varphi + \gamma, \qquad \mu \mapsto \mu + \gamma,$$

with γ chosen such that $\gamma(1 - \mu - \frac{1}{2}\gamma)$ cancels the possible constant of integration.

The traveling-wave assumption for the fDP equation yields

$$-\mu\varphi + \frac{1}{2}\varphi^2 + \frac{3}{2}\Lambda^{-s}\varphi^2 = \kappa, \tag{1.18}$$

where it is not possible to obtain zero on the right-hand side with a transformation while at the same time preserving the structure of the equation. Therefore, we work with an arbitrary real constant κ on the right-hand side.

When referring to a traveling-wave solution of any of the two equations, we mean a real-valued continuous and bounded function φ satisfying the equation on $\mathbb{R}$.

The purpose of the present work is to study the existence and regularity of highest traveling waves for the fKdV and the fDP equations. The notion of highest waves stems from the observation that nonconstant solutions to both the fKdV and the fDP equations are smooth, except possibly at points where the wave-height equals the wave-speed μ (cf. Theorem 3.8, Theorem 6.8), and that this is the maximal height that can be attained by a family of solutions that bifurcate from the trivial solution to the equation. Accordingly, solutions φ that attain the height of μ are referred to as highest traveling waves.

We briefly review relevant research. As already noted, highest periodic traveling waves, and the regularity thereof, have been proved for the Whitham equation in [11]. The solitary case of (non-periodic) solutions φ with $\lim_{|x|\to\infty}\varphi(x) = 0$ has been studied in [31], where analogous results were obtained. The novelty in the present work lies in the parametrized dispersive operator Λ^{-s} of order $-s \in (-1,0)$. A partial result in this direction, for a class of generalized Whitham equations with a parametrized inhomogeneous symbol on the form (1.12) of order in $(-1,0)$, is given in [1]. In preparation is also a study of an fKdV equation with a homogeneous symbol of order in $(-1, 0)$ and a generalized nonlinearity [35].

The fKdV equation (1.17), with $s > 1$, has been studied in [21], where highest periodic traveling waves were proved to exist, and the waves were shown to be exactly Lipschitz continuous at the crests. The paper [6] considered the homogeneous counterpart of the fKdV equation with $s > 1$, and analogous results were obtained for this family of equations as well. However, note that in these cases, the equations incorporate strong enough dispersion to ensure that the solutions are at least Lipschitz continuous at the crests. As we will see, this does not hold in our case.

The Degasperis–Procesi equation (1.6) was first studied in [10], and is known to permit peaked traveling-wave solutions [22]. A nonlocal formulation of the equation corresponding to the fDP equation (1.15) with $s = 2$ was studied in [3], where the existence of highest periodic traveling waves of Lipschitz regularity at crests was proved.

Similar methods as used in this thesis have also been applied to other equations; see e.g. [13, 14] for a full-dispersion shallow water model and a capillary-gravity Whitham equation, respectively. We also mention that in [23] a dispersive equation similar to (1.14) with fixed nonlinearity and varied dispersion was studied in the context of well-posedness and blow-up.

1.3 Notation

Throughout, we use the notation $X \lesssim_p Y$ (for some mathematical objects X and Y) if there exists a positive constant C_p, depending on p, such that the inequality $X \leq C_p Y$ holds. The relation $X \lesssim Y \lesssim X$ is denoted $X \eqsim Y$, with the same convention for subscripts. We shall also occasionally employ the Landau notation $f(x) = O(g(x))$ whenever there exists a positive constant C with

$$|f(x)| \leq C|g(x)|$$

for all x in some domain, and use $O(g(x))$ as a placeholder for such functions. Furthermore, writing $X \ll Y$ signifies that X is "much smaller" than Y, that is, the inequality $X \leq cY$ holds for a sufficiently small positive constant c.

2 The Bessel potential operator

This section is a survey on the Bessel potential operator and serves as a prelude to the subsequent study of the fKdV and fDP equations. In Section 2.1, we examine the sign and asymptotic behavior of the convolution kernel K_s and its derivatives, and investigate how the operator Λ^{-s} acts on functions satisfying certain sign and parity conditions. In Section 2.2, we prove that Λ^{-s} is a smoothing operator on the scale of Hölder–Zygmund spaces.

2.1 The convolution kernel K_s

Throughout, we let $\mathbb{S}_P$ denote $\mathbb{R}/P\mathbb{Z}$, the compact interval $[-P/2, P/2]$ in $\mathbb{R}$ of length $P < \infty$ with coinciding endpoints. In the following, function spaces are defined on $\mathbb{R}$ for convenience, but they can be defined analogously on $\mathbb{S}_P$.

Let $C(\mathbb{R})$ denote the space of uniformly continuous and bounded functions over $\mathbb{R}$,

$$C(\mathbb{R}) := \{f \colon \mathbb{R} \to \mathbb{R}\,;\, f \text{ is bounded and uniformly continuous}\},$$

normed by $\|f\|_{C(\mathbb{R})} = \sup_{x\in\mathbb{R}} |f(x)|$. Characterizing functions that are k times continuously differentiable, we define

$$C^k(\mathbb{R}) := \{f \in C(\mathbb{R})\,;\, f^{(m)} \in C(\mathbb{R}) \text{ for } m = 0, 1, 2, ..., k\}, \tag{2.1}$$

and furnish the space with the usual norm $\|f\|_{C^k(\mathbb{R})} = \sum_{m=0}^{k} \|f^{(m)}\|_{C(\mathbb{R})}$. If a function f is contained in $C^k(\mathbb{R})$ for every $k \in \mathbb{N}$, then we say that the function f is smooth, and write $f \in C^\infty(\mathbb{R})$.

The Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$ is defined as

$$\mathcal{S}(\mathbb{R}) := \{f \in C^\infty(\mathbb{R})\,;\, \|f\|_{k,l} < \infty \text{ for every } k, l \in \mathbb{N}\},$$

where the semi-norms $\|\cdot\|_{k,l}$ are given by

$$\|f\|_{k,l} := \sup_{x\in\mathbb{R}} (1 + |x|)^k |f^{(l)}(x)|.$$

The dual space, comprising all linear and bounded complex-valued functionals over $\mathcal{S}(\mathbb{R})$, is denoted by $\mathcal{S}'(\mathbb{R})$. Similarly, let $\mathcal{D}(\mathbb{R})$ be the space of compactly supported smooth functions on $\mathbb{R}$, and $\mathcal{D}'(\mathbb{R})$ the collection of linear and bounded complex-valued functionals over $\mathcal{D}(\mathbb{R})$. The space $\mathcal{D}(\mathbb{R})$ is furnished with the usual countable family of semi-norms, transferred to $\mathcal{D}'(\mathbb{R})$ by duality. On the compact interval $\mathbb{S}_P$, the space $\mathcal{S}(\mathbb{S}_P)$ comprises smooth functions over $\mathbb{S}_P$, while $\mathcal{D}(\mathbb{S}_P)$ is the collection of smooth functions over $\mathbb{R}$ with support contained in $\mathbb{S}_P$. For details on how these spaces are defined, we refer to the monograph [4].

Let $\mathscr{F}$ denote the Fourier transform on $\mathcal{S}(\mathbb{R})$, extended to $\mathcal{S}'(\mathbb{R})$ via duality, and normalized as

$$(\mathscr{F} f)(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx, \qquad (\mathscr{F}^{-1} \hat{f})(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi}\, d\xi, \tag{2.2}$$

for $f \in \mathcal{S}(\mathbb{R})$. We shall sometimes write $\hat{f}$ for the Fourier transform of f.

The following proposition provides a basic understanding of the convolution kernel K_s, used throughout this thesis. It is based on [17, Proposition 1.2.5], but sharpened somewhat on the grounds of the restriction of $s \in (0, 1)$.

Proposition 2.1. *Let $s \in (0,1)$. Then*

(i) *K_s has the integral representation*

$$K_s(x) = \frac{1}{\sqrt{4\pi}\Gamma(\frac{s}{2})} \int_0^\infty e^{-t-\frac{x^2}{4t}}\, t^{\frac{s-3}{2}}\, dt, \tag{2.3}$$

(ii) *K_s is even and strictly positive,*

(iii) *K_s is smooth on $\mathbb{R} \setminus \{0\}$ and integrable with $\|K_s\|_{L^1(\mathbb{R})} = 1$,*

(iv) *we have*

$$K_s(x) \lesssim_s e^{-|x|} \tag{2.4}$$

for $|x| \geq 1$, and

$$K_s(x) = C_s|x|^{s-1} + H_s(x), \tag{2.5}$$

for $|x| < 1$ and $C_s > 0$, where

$$H_s(x) \eqsim_s 1 + O(|x|^{s+1})$$

and

$$|H_s'(x)| = O(|x|^s), \qquad |H_s''(x)| = O(|x|^{s-1}). \tag{2.6}$$

Proof. (i) For every complex number z with $\operatorname{Re} z > 0$, the gamma function is defined as

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt;$$

see e.g. [16, A.2]. Making the substitution $t \mapsto at$ for a positive real number a in the Gamma function evaluated in $s/2$ yields the identity

$$a^{-\frac{s}{2}} = \frac{1}{\Gamma(s/2)} \int_0^\infty e^{-at}\, t^{\frac{s}{2}-1}\, dt\,.$$

Setting $a = 1 + \xi^2$, one has

$$\langle \xi \rangle^{-s} = \frac{1}{\Gamma(s/2)} \int_0^\infty e^{-t(1+\xi^2)}\, t^{\frac{s}{2}-1}\, dt\,.$$

Note that both of the preceding integrals converge for every $s > 0$. Applying the inverse Fourier transform yields

$$\mathscr{F}^{-1}(\langle \xi \rangle^{-s}) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \left(\frac{1}{\Gamma(s/2)} \int_0^\infty e^{-t(1+\xi^2)}\, t^{\frac{s}{2}-1}\, dt \right) d\xi\,.$$

Changing the order of integration, and using the well known formula

$$\mathscr{F}^{-1}(e^{-t\xi^2})(x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$$

(a standard proof of which can be found in [4, p. 18]), one arrives at

$$K_s(x) = \frac{1}{\sqrt{4\pi}\ \Gamma(s/2)} \int_0^\infty e^{-t-\frac{x^2}{4t}}\ t^{\frac{s-3}{2}}\, dt\,.$$

This proves the formula (2.3).

(ii)-(iii) The representation (2.3) shows that $K_s \in C^\infty(\mathbb{R}\setminus\{0\})$, and that the kernel is even and strictly positive. The positivity of the kernel implies that

$$\|K_s\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} K_s(x)\,dx = (\mathscr{F}^{-1}\mathscr{F}\langle\xi\rangle^{-s})(0) = \langle 0\rangle^{-s} = 1,$$

where we have used the Fourier inversion theorem [16, Theorem 2.2.14].

(iii) We show the asymptotic bounds for K_s. Suppose $|x| \geq 1$. Then the inequality $t + \frac{2x^2-1}{8t} \geq |x|$ holds for all $t \in (0, \frac{1}{8})$. This means that $t + \frac{x^2}{4t} \geq \frac{1}{8t} + |x|$, and consequently

$$e^{-t-\frac{x^2}{4t}} \leq e^{-\frac{1}{8t}} e^{-|x|},$$

for all $t \in (0, \frac{1}{8})$. Note also that the inequality $t + \frac{x^2}{4t} \geq |x|$ holds for every $t > 0$. Indeed, the minimum of the function $t + \frac{x^2}{4t}$ is attained in $t = \frac{|x|}{2}$, where equality holds. Splitting the integral in the representation (2.3) on $t = \frac{1}{8}$, and using the inequalities above, yields

$$|K_s(x)| \leq \frac{e^{-|x|}}{\sqrt{4\pi}\ \Gamma(s/2)} \left(\int_0^{\frac{1}{8}} e^{-\frac{1}{8t}}\ t^{\frac{s-3}{2}}\, dt + \int_{\frac{1}{8}}^\infty t^{\frac{s-3}{2}}\, dt \right) \lesssim_s e^{-|x|}, \tag{2.7}$$

where we have used that both integrals converge to positive constants for every choice of $s \in (0,1)$. This proves (2.4).

Now suppose $|x| < 1$. The integral in (2.3) may be written as

$$K_s(x) = \frac{1}{\sqrt{4\pi}\ \Gamma(s/2)} \left(\int_0^{x^2} + \int_{x^2}^1 + \int_1^\infty \right) e^{-t-\frac{x^2}{4t}}\ t^{\frac{s-3}{2}}\, dt, \tag{2.8}$$

For the first term, the substitution $t \mapsto x^2 t$ gives

$$\begin{aligned}\int_0^{x^2} e^{-\frac{x^2}{4t}-t}\ t^{\frac{s-3}{2}}\, dt &= |x|^{s-1} \int_0^1 e^{-tx^2}\ e^{-\frac{1}{4t}}\ t^{\frac{s-3}{2}}\, dt \\ &= |x|^{s-1} \int_0^1 e^{-\frac{1}{4t}}\ t^{\frac{s-3}{2}}\, dt + O(|x|^{s+1}) \int_0^1 e^{-\frac{1}{4t}}\ t^{\frac{s-1}{2}}\, dt,\end{aligned}$$

where in the second step, we used that $e^{-tx^2} = 1 + O(tx^2)$ from the Taylor expansion of the exponential function. Note that this expansion also justifies the derivatives in (2.6). Both integrals in the last line converge to positive constants for every choice of $s \in (0,1)$. In the second term, we bound the exponential factor $e^{-t-\frac{x^2}{4t}}$ by positive constants above and below, and obtain the estimate

$$\int_{x^2}^1 e^{-t-\frac{x^2}{4t}}\ t^{\frac{s-3}{2}}\, dt \eqsim \int_{x^2}^1 t^{\frac{s-3}{2}}\, dt = \frac{2}{1-s}\big(|x|^{s-1} - 1\big).$$

Finally, for the third term we bound the exponential factor $e^{-\frac{|x|^2}{4t}}$ above and below, which simply gives

$$\int_1^\infty e^{-t-\frac{x^2}{4t}}\, t^{\frac{s-3}{2}}\, dt \approx \int_1^\infty e^{-t}\, t^{\frac{s-3}{2}}\, dt \approx_s 1.$$

Inserting the above estimates in (2.8) yields (2.5). □

The next result is a direct application of the positivity of the kernel K_s. A similar statement is given in [11, Lemma 3.5].

Corollary 2.2. *If f and g are functions belonging to $C(\mathbb{R})$ with $f \geq g$ and $f(x_0) > g(x_0)$ for some x_0, then*

$$\Lambda^{-s} f > \Lambda^{-s} g.$$

That is, the operator Λ^{-s} is strictly monotone on $C(\mathbb{R})$.

Proof. Let f and g be functions according to the assumptions. Since f and g are continuous, there exists a neighborhood of nonzero measure around x_0 on which $f > g$. Consequently,

$$(\Lambda^{-s} f)(x) - (\Lambda^{-s} g)(x) = \int_{\mathbb{R}} K_s(x-y)(f(y) - g(y))\, dy > 0,$$

since K_s is strictly positive. □

We now turn to an investigation of the signs of the derivatives of K_s. The arguments are based on the results in [11, Section 2], while a more detailed account of completely monotone functions and related topics can be found in [26]. We begin by introducing some notions that are useful in the proof of Proposition 2.3, where we shown that the kernel K_s is a completely monotone function.

A function $g\colon (0,\infty) \to \mathbb{R}$ is said to be completely monotone if it is smooth and satisfies

$$(-1)^n g^{(n)}(\lambda) \geq 0 \tag{2.9}$$

for all $n \in \mathbb{N}_0$ and all $\lambda > 0$, where $\mathbb{N}_0$ denotes the set of all nonnegative integers. This definition naturally extends to even functions $g\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, which are called completely monotone if they are completely monotone on $(0,\infty)$.

A subclass of the class of completely monotone functions are Stieltjes functions. A Stieltjes function is a function $g\colon (0,\infty) \to [0,\infty)$ that can be written in terms of the integral representation

$$g(\lambda) = \frac{a}{\lambda} + b + \int_{(0,\infty)} \frac{1}{\lambda + t}\, d\sigma(t),$$

where a and b are nonnegative constants, and σ is a Borel measure on $(0,\infty)$ satisfying

$$\int_{(0,\infty)} \frac{1}{1+t}\, d\sigma(t) < \infty.$$

It turns out that the class of Stieltjes functions can be completely characterized by analytic extensions. Precisely, by [26, Corollary 7.4], if g is a strictly positive function on $(0,\infty)$, then g is Stieltjes if and only if

$$\lim_{\lambda\searrow 0} g(\lambda)\in[0,\infty],$$

and g has an analytic extension to $\mathbb{C}\setminus(-\infty,0]$ with

$$\operatorname{Im}(z)\operatorname{Im}(g(z))\leq 0.$$

Note that, owing to [11, Lemma 2.12], if g is Stieltjes and $\alpha\in(0,1]$, then g^α is Stieltjes. Moreover, by [11, Proposition 2.20], if $f\colon\mathbb{R}\to\mathbb{R}$ and $g\colon(0,\infty)\to\mathbb{R}$ are two functions satisfying

$$f(\xi)=g(\xi^2)$$

for all $\xi\neq 0$, then f is the Fourier transform of an even, integrable, and completely monotone function if and only if g is Stieltjes with

$$\lim_{\lambda\searrow 0} g(\lambda)<\infty \qquad \text{and} \qquad \lim_{\lambda\to\infty} g(\lambda)=0.$$

We are now in the position to prove the complete monotonicity of K_s, or in other words, that all derivatives of K_s are strictly monotone with alternating sign.

Proposition 2.3. *For every $s\in(0,1)$, the convolution kernel K_s is completely monotone. In particular, it is strictly decreasing and strictly convex on $(0,\infty)$.*

Proof. Let

$$g\colon\lambda\mapsto(1+\lambda)^{-1}.$$

The function g extends analytically to $\mathbb{C}\setminus(-\infty,0]$, since the only singularity is in $\lambda=-1$. For every $z\in\mathbb{C}\setminus(-\infty,0]$, one has

$$\operatorname{Im}(g(z))=-\frac{\operatorname{Im}(z)}{(1+\operatorname{Re}(z))^2+\operatorname{Im}(z)^2},$$

and consequently

$$\operatorname{Im}(g(z))=-\frac{\operatorname{Im}(z)^2}{(1+\operatorname{Re}(z))^2+\operatorname{Im}(z)^2}\leq 0,$$

on $\mathbb{C}\setminus(-\infty,0]$. That is, g is a Stieltjes function. As we have seen, this implies that $g^{s/2}$ is Stieltjes. Furthermore,

$$\lim_{\lambda\searrow 0} g^{s/2}(\lambda)=1, \qquad \lim_{\lambda\to\infty} g^{s/2}(\lambda)=0$$

for every $s\in(0,1)$. Therefore, the function

$$f(\xi)=g^{\frac{s}{2}}(\xi^2)=(1+\xi^2)^{-\frac{s}{2}}=\langle\xi\rangle^{-s}$$

is the Fourier transform of an even, integrable and completely monotone function. But since

$$\mathscr{F}(K_s) = \langle \xi \rangle^{-s},$$

we conclude that K_s is completely monotone.

It remains to prove that K_s is strictly decreasing and strictly convex on $(0, \infty)$. However, it is noted in [26, Remark 1.5] that as a consequence of Bernstein's theorem [26, Theorem 1.4], if g is not identically constant, then (2.9) holds with strict inequality for every λ and every n. □

Towards analyzing periodic solutions of the fKdV and fDP equations, we now define the periodic convolution kernel

$$K_{P,s} := \sum_{n \in \mathbb{Z}} K_s(x + nP). \tag{2.10}$$

We mention that the same definition is made in both [11] and [3], and that it is motivated by the observation

$$\begin{aligned}
(K_s * f)(x) &= \int_{\mathbb{R}} K_s(x-y) f(y)\, dy \\
&= \sum_{n \in \mathbb{Z}} \int_{-\frac{P}{2}+Pn}^{\frac{P}{2}+Pn} K_s(x-y) f(y)\, dy \\
&= \int_{-\frac{P}{2}}^{\frac{P}{2}} K_{P,s}(x-y) f(y)\, dy,
\end{aligned}$$

for every P-periodic smooth function f. Owing to the exponential decay of K_s from Proposition 2.1, the periodic kernel can be bounded by

$$K_{P,s}(x) \eqsim_{P,s} |x|^{s-1}, \tag{2.11}$$

for $x \in (-P, P)$. In addition, we have the following properties of $K_{P,s}$. The proof of Proposition 2.4 is based on [11, Remark 3.4].

Proposition 2.4. *The periodic kernel $K_{P,s}$ is even, P-periodic and strictly increasing on $(-P/2, 0)$.*

Proof. By (2.10), the kernel $K_{P,s}$ is clearly P-periodic, and the evenness of $K_{P,s}$ follows from the evenness of K_s. Furthermore, the derivative of $K_{P,s}$ is

$$\begin{aligned}
K'_{P,s}(x) &= \sum_{n \in \mathbb{Z}} K'_s(x + nP) \\
&= \sum_{n=0}^{\infty} \big(K'_s(x + nP) + K'_s(x - (n+1)P) \big).
\end{aligned}$$

For $x \in (0, P/2)$ the inequality $|x+nP| < |x-(n+1)P|$ holds. Moreover, $K_s(x)$ is even and strictly convex on $(-P/2, 0)$, implying that $|K_s'(x+nP)| > |K_s'(x-(n+1)P|$ for all $n \in \mathbb{N}$ and all $x \in (0, P/2)$. Therefore, we must have

$$K_s'(x+nP) + K_s'(x-(n+1)P) < 0$$

for every $n \in \mathbb{N}_0$, implying that the periodic kernel $K_{P,s}$ is strictly decreasing on $(0, P/2)$. Hence, by evenness, it is strictly increasing on $(-P/2, 0)$. □

Having proved the necessary positivity and monotonicity of the periodic convolution kernel, we are now in the position to show how Λ^{-s} acts on periodic and odd functions which change sign only in the origin. This plays an important role in the bifurcation arguments constructed in Section 4 and 7.

Lemma 2.5. *Let f be a P-periodic, odd and continuous function with $f \geq 0$ on $(-P/2, 0)$ and $f(x_0) > 0$ for some $x_0 \in (-P/2, 0)$. Then*

$$\Lambda^{-s} f > 0$$

on $(-P/2, 0)$.

Proof. Let f be a function according to the assumptions above. Then

$$\begin{aligned}(\Lambda^{-s} f)(x) &= \int_{-P/2}^{P/2} K_{P,s}(x-y) f(y)\, dy \\ &= \int_{-P/2}^{0} (K_{P,s}(x-y) - K_{P,s}(x+y)) f(y)\, dy\,.\end{aligned}$$

Note that since f is nonnegative on $(-P/2, 0)$ and strictly positive on some domain of nonzero measure, it suffices to show that

$$K_{P,s}(x-y) - K_{P,s}(x+y) > 0 \tag{2.12}$$

for all $x, y \in (-P/2, 0)$. Firstly, by Proposition 2.4, the periodic kernel $K_{P,s}$ is strictly increasing on $(-P/2, 0)$ and strictly decreasing on $(0, P/2)$. Secondly,

$$\operatorname{dist}(x-y, 0) < \min\{\operatorname{dist}(x+y, 0), \operatorname{dist}(x+y, -P)\}. \tag{2.13}$$

for $x, y \in (-P/2, 0)$. Indeed,

$$|x-y| < |x| + |y| = |x+y|$$

for $x \neq y$ of same sign, and

$$|x-y| = \max\{x-y, y-x\} < P + x + y,$$

due to $-x < P + x$ and $-y < P + y$ for all $x, y \in (-P/2, 0)$. This means that (2.12) holds. □

2.2 Smoothing

Fourier multiplier operators are defined by multiplying the Fourier transform of a function with a given symbol function, thereby modifying its frequencies. Precisely, a Fourier multiplier $a(D)$ is defined as multiplication in frequency space with the symbol $a(\xi)$, and formally one has

$$a(D)f := \mathscr{F}^{-1}(a(\xi)\hat{f}(\xi)),$$

If the function a belongs to $\mathcal{S}(\mathbb{R})$, then the operator $a(D)$ is well defined and linear from $\mathcal{S}$ to itself, and that this holds more generally on the space of tempered distributions $\mathcal{S}'(\mathbb{R})$, via duality. Indeed, the space $\mathcal{S}(\mathbb{R})$ is a Banach algebra [16, Proposition 2.2.7], and the Fourier transform is an isomorphism from $\mathcal{S}(\mathbb{R})$ onto itself [16, Propositon 2.2.11]. Note also that since $a_1(D)a_2(D)f = (a_1 \circ a_2)(D)f$, for Fourier multiplier operators a_1 and a_2, the inverse of a multiplier $a(D)$ is given by a multiplier with the reciprocal symbol of a. Formally, we write

$$(a(D))^{-1} = a^{-1}(D). \tag{2.14}$$

We show how Fourier multipliers act on periodic functions. It is known that every smooth, P-periodic function f can be written as a uniformly convergent fourier series

$$f(x) = \sum_{k\in\mathbb{Z}} [f]_k e^{i\frac{2\pi k}{P}x}, \tag{2.15}$$

with Fourier coefficients

$$[f]_k := \frac{1}{P}\int_{-P}^{P} f(x)e^{-i\frac{2\pi k}{P}x}\,dx\,.$$

Fourier multipliers act on P-periodic, smooth functions by multiplying the Fourier coefficients of the function with the symbol of the operator. Precisely, for a multiplier $a(D)$, one has

$$a(D)f = \sum_{k\in\mathbb{Z}} a\big(\frac{2\pi k}{P}\big)[f]_k e^{i\frac{2\pi k}{P}x}. \tag{2.16}$$

This follows from the Fourier series representation of f from (2.15), and the calculation

$$\begin{aligned} a(D)e^{i\frac{2\pi k}{P}x} &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}a(\xi)\int_{\mathbb{R}} e^{i\frac{2\pi k}{P}y}e^{-iy\xi}\,dy\,d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\frac{2\pi k}{P}y}\int_{\mathbb{R}} a(\xi)e^{-i(y-x)\xi}\,d\xi\,dy \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{a}(y-x)e^{i\frac{2\pi k}{P}y}\,dy \\ &= a\big(\frac{2\pi k}{P}\big)e^{i\frac{2\pi k}{P}x}, \end{aligned}$$

where we have used Fubini's theorem [5, Theorem 4.5] to switch the order of integration, and in the last step the Fourier inversion theorem.

The operator Λ^{-s} is a Fourier multiplier with symbol $\langle\xi\rangle^{-s}$ as defined in (1.16). The symbol $\langle\xi\rangle^{-s}$ belongs to the space $\mathcal{S}(\mathbb{R})$, and the operation is well-defined on $\mathcal{S}'(\mathbb{R})$.

Since the operator Λ^{-s} increases the decay of the Fourier transform of the function on which it operates, a natural question to ask is what this means in terms of the regularity of the function itself. To answer this question, we introduce the Hölder and Zygmund spaces, which shall also be used extensively in later sections. An outline of these classes of functions, and generalizations thereof, can be found in [30].

The space of α-Hölder continuous functions on $\mathbb{R}$, with $\alpha \in (0,1)$, is defined as

$$C^{0,\alpha}(\mathbb{R}) := \{f \in C(\mathbb{R})\,;\, [f]_{C^{0,\alpha}(\mathbb{R})} < \infty\},$$

where $[\cdot]_{C^{0,\alpha}(\mathbb{R})}$ denotes the Hölder semi-norm

$$[f]_{C^{0,\alpha}(\mathbb{R})} := \sup_{\substack{x,y\in\mathbb{R}\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

We say that the function $f \in C(\mathbb{R})$ is α-Hölder continuous at the point $x \in \mathbb{R}$ if

$$[f]_{C^{0,\alpha}_x(\mathbb{R})} := \sup_{\substack{h\in\mathbb{R}\\ h\neq 0}} \frac{|f(x+h)-f(x)|}{|h|^\alpha} < \infty.$$

In analogy with (2.1), we define for every $\alpha \in (0,1)$ and any $k \in \mathbb{N}$ the space

$$C^{k,\alpha}(\mathbb{R}) := \{f \in C^k(\mathbb{R})\,;\, f^{(k)} \in C^{0,\alpha}(\mathbb{R})\},$$

containing all k times continuously differentiable functions on $\mathbb{R}$ with α-Hölder continuous k-th derivative. Moreover, we let $C^{k,\alpha}_{\text{even}}(\mathbb{S}_P)$ denote the closed subspace of $C^{k,\alpha}$, comprising even and P-periodic functions.

The Hölder spaces are defined using first-order differences of functions. The so-called Zygmund spaces can be defined in a similar way, using second- or higher-order differences. Let $\lfloor\alpha\rfloor$ and $\{\alpha\}$ denote the integer and fractional part of $\alpha > 0$, where we adopt the convention that $0 < \{\alpha\} \leq 1$. Furthermore let Δ_h be the first-order difference operator acting on a function f according to

$$(\Delta_h f)(x) = f(x+h) - f(x),$$

and let Δ_h^n be the nth-order iterated difference. That is,

$$(\Delta_h^2 f)(x) = (\Delta_h(\Delta_h f))(x) = f(x+2h) - 2f(x+h) + f(x),$$

and so forth. Then, we define for every $\alpha > 0$ the Zygmund spaces

$$\mathcal{C}^\alpha(\mathbb{R}) := \{f \in C^{\lfloor\alpha\rfloor}(\mathbb{R})\,;\, [f]_{\mathcal{C}^\alpha(\mathbb{R})} < \infty\},$$

where the Zygmund semi-norm $[f]_{\mathcal{C}^\alpha(\mathbb{R})}$ is given by

$$[f]_{\mathcal{C}^\alpha(\mathbb{R})} := \sup_{0\neq h\in\mathbb{R}} \frac{\|\Delta_h^2 f^{(\lfloor\alpha\rfloor)}\|_{C^0(\mathbb{R})}}{|h|^{\{\alpha\}}}.$$

Two important facts about the Hölder and Zygmund spaces, given in [30, Theorem 1.2.2], are used throughout this thesis. Firstly, for non-integer s, the Hölder space $C^{\lfloor s\rfloor,\{s\}}$ and the Zygmund space $\mathcal{C}^s$ coincide, in the sense of equivalent norms. It is in this context we sometimes refer to Hölder-Zygmund spaces, and the two are used interchangeably when there is no confusion. Secondly, the Zygmund space $\mathcal{C}^s$, with $s > 0$, is (norm-) equivalent to

$$\mathcal{C}^s = \left\{ f \in C^k \,;\, \sup_{0\neq h\in\mathbb{R}} \frac{\|\Delta_h^m f^{(k)}\|_{C^0}}{|h|^{s-k}} < \infty \right\}$$

for every choice of $k \in \mathbb{N}_0$ and $m \in \mathbb{N}$ with $k < s$ and $m > s - k$.

The following proposition shows how the Bessel potential operator Λ^{-s} changes the regularity of functions on which it acts, in the context of Hölder-Zygmund spaces. Working with differences, it is often useful to isolate the singularity $|x|^{s-1}$ from the kernel K_s. More precisely, we write

$$K_s(x) = C_s|x|^{s-1} + \widetilde{K}_s(x), \tag{2.17}$$

where we have

$$|\widetilde{K}_s'(x)| \lesssim_s (1 + |x|)^{s-2}, \tag{2.18}$$

and furthermore that

$$\begin{aligned} |\widetilde{K}_s''(x)| &= O(|x|^{s-1}), \qquad && |x| < 1, \\ |\widetilde{K}_s''(x)| &\lesssim_s (1 + |x|)^{s-3}, && |x| \geq 1, \end{aligned} \tag{2.19}$$

in view of the exponential decay of K_s and (2.5) from Proposition 2.1. Then by the mean value theorem

$$|\widetilde{K}_s(x+y) - \widetilde{K}_s(x)| \leq |y| \int_0^1 |\widetilde{K}_s'(x+ty)|\, dt$$

where we let $R_y^1(x)$ denote

$$R_y^1(x) = \int_0^1 |\widetilde{K}_s'(x+ty)|\, dt\,.$$

Similarly, we have

$$|\widetilde{K}_s(x+y) + \widetilde{K}_s(x-y) - 2\widetilde{K}_s(x)| \leq |y|^2 R_y^2(x), \tag{2.20}$$

with

$$R_y^2(x) = \int_0^1 \int_0^1 2t|\widetilde{K}_s''(x - ty + 2sty)|\, ds\, dt\,.$$

Note that the estimates (2.18) and (2.19) applies to R_y^1 and R_y^2, respectively.

Proposition 2.6. *For every $\alpha > 0$ and $s \in (0,1)$, the operator Λ^{-s} is linear and bounded from $\mathcal{C}^\alpha(\mathbb{R})$ to $\mathcal{C}^{\alpha+s}(\mathbb{R})$.*

Proof. Fix $\alpha > 0$ and $s \in (0,1)$. Due to the above norm equivalences for Zygmund spaces (based on [30, Theorem 1.2.2]), it suffices to prove that

$$\sup_{0\neq h\in\mathbb{R}} \frac{\|\Delta_h^m(\Lambda^{-s}f)^{(\lfloor\alpha\rfloor)}\|_{C^0(\mathbb{R})}}{|h|^{\{\alpha\}+s}} < \infty$$

for functions $f \in \mathcal{C}^\alpha$ and for some difference order $m \geq 2$. We shall do this for $m = 3$.

We claim that $\|\Delta_h K_s\|_{L^1(\mathbb{R})} \lesssim |h|^s$. Using the spitting of K_s into its regular and singular components given in (2.17), it is possible to write

$$\begin{aligned}\|\Delta_h K_s\|_{L^1(\mathbb{R})} &= \int_{\mathbb{R}} |K_s(x+h) - K_s(x)|\,dx \\ &\lesssim \int_{\mathbb{R}} \big||x+h|^{s-1} - |x|^{s-1}\big|\,dx + \int_{\mathbb{R}} |\widetilde{K}_s(x+h) - \widetilde{K}_s(x)|\,dx\,.\end{aligned} \tag{2.21}$$

The first integral in (2.21) can be estimated by

$$\int_{\mathbb{R}} \big||x+h|^{s-1} - |x|^{s-1}\big|\,dx = |h|^s \int_{\mathbb{R}} \big||t+1|^{s-1} - |t|^{s-1}\big|\,dt \lesssim |h|^s,$$

since

$$\big||t+1|^{s-1} - |t|^{s-1}\big| \lesssim |t|^{s-2}$$

for large t. For the second integral in (2.21), one has

$$\begin{aligned}\int_{\mathbb{R}} |\widetilde{K}_s(x+h) - \widetilde{K}_s(x)|\,dx &\lesssim |h| \int_{\mathbb{R}} R_h^1(x)\,dx \\ &\lesssim |h|,\end{aligned}$$

due to the characterization of the difference from (2.20). This proves $\|\Delta_h K_s\|_{L^1(\mathbb{R})} \lesssim |h|^s$. By Young's inequality [4, Lemma 1.4], we now have

$$\begin{aligned}\sup_{x\in\mathbb{R}} |\big(\Delta_h^3(\Lambda^{-s}f)^{(\lfloor\alpha\rfloor)}\big)(x)| &= \|\Delta_h^3(K_s * f^{(\lfloor\alpha\rfloor)})\|_{L^\infty} \\ &= \|(\Delta_h K) * (\Delta_h^2 f^{(\lfloor\alpha\rfloor)})\|_{L^\infty} \\ &\leq \|\Delta_h K_s\|_{L^1} \|\Delta_h^2 f\|_{L^\infty} \\ &\lesssim |h|^{\{\alpha\}+s}.\end{aligned}$$

The desired inequality is now obtained by dividing by $|h|^{\{\alpha\}+s}$ and passing to supremum with respect to h. □

Corollary 2.7. *For every $s \in (0,1)$, the operator Λ^{-s} is linear and bounded from $L^\infty(\mathbb{R})$ to $C^{0,s}(\mathbb{R})$.*

Proof. Recall that if $\alpha > 0$ is not an integer, the Hölder space $C^{\lfloor \alpha \rfloor, \{\alpha\}}(\mathbb{R})$ coincides (equivalent norms) with the Zygmund space $\mathcal{C}^{\alpha}(\mathbb{R})$. Now if $f \in L^{\infty}$, then $\|\Delta_h^2 f\|_{L^\infty} \lesssim 1$, and we have with the same reasoning as in the proof of Proposition 2.6 that

$$\begin{aligned} \sup_{x\in\mathbb{R}} |\big(\Delta_h^3(\Lambda^{-s} f\big)(x)| &\leq \|\Delta_h K_s\|_{L^1} \|\Delta_h^2 f\|_{L^\infty} \\ &\lesssim |h|^s. \end{aligned}$$

Passing to supremum yields $f \in C^{0,s}$. □

3 Traveling-wave solutions to the fKdV equation

We present properties pertaining to the sign and regularity of traveling-wave solutions to the fKdV equation. While this section is an a priori study of solutions to the fKdV equation, existence shall be established in Section 4. In Section 3.1, we recover information about the magnitude, and the sign of derivatives, of solutions that satisfy certain periodicity and parity conditions. In particular, Lemma 3.3 parallels the classical study of the nodal pattern of eigenfunctions to elliptic operators and will be of decisive importance in the subsequent bifurcation argument. Then, in Section 3.2, it is proved that all solutions which have an amplitude strictly smaller than the wave-speed μ are smooth, and in Section 3.3 that solutions which achieve the maximal amplitude of μ belongs to $C^{0,s}(\mathbb{R})$. The s-Hölder regularity is optimal and attained in the crest where $\varphi = \mu$.

Most of the methods in this section follow [11]. The main difference is that we here consider the parametrized operator Λ^{-s} with $s \in (0, 1)$, and we obtain new results on the relationship between the order of the operator and the optimal regularity of highest traveling waves. Note that in the following, the parameter s is considered to be fixed in $(0, 1)$, and the fKdV equation refers to equation (1.17) for this value of s.

3.1 Periodic traveling waves

Many properties of solutions φ to the fKdV equation can be inferred by analyzing the structure of the equation. We begin with a proposition giving bounds for the minima and maxima of solutions, making use of the Bessel potential operator being strictly monotone, and the observation that

$$\Lambda^{-s} c = K_s * c = c\|K_s\|_{L^1} = c$$

for every constant $c \in \mathbb{R}$.

Proposition 3.1. *If φ is a solution to the fKdV equation, then*

$$\begin{cases} 2(\mu - 1) \leq \min \varphi \leq 0 \leq \max \varphi \quad \text{or} \quad \varphi \equiv 2(\mu - 1) & \text{if } \mu \leq 1, \\ 0 \leq \min \varphi \leq 2(\mu - 1) \leq \max \varphi \quad \text{or} \quad \varphi \equiv 0 & \text{if } \mu > 1. \end{cases}$$

Proof. The equation can be written in the form

$$(\mu - \varphi)^2 = \mu^2 - 2\Lambda^{-s}\varphi.$$

Since $\Lambda^{-s}\varphi \geq \min \varphi$ and $\Lambda^{-s}\varphi \leq \max \varphi$, we have

$$(\mu - \varphi)^2 \leq \mu^2 - 2 \min \varphi,$$
$$(\mu - \varphi)^2 \geq \mu^2 - 2 \max \varphi.$$

In particular, this holds for $\min \varphi$ (resp. $\max \varphi$), which gives

$$\min \varphi \big(\frac{1}{2} \min \varphi - (\mu - 1) \big) \leq 0,$$
$$\max \varphi \big(\frac{1}{2} \max \varphi - (\mu - 1) \big) \geq 0.$$

Analyzing the sign of the factors on the left-hand sides above yields the claim. □

If a solution φ satisfies $\varphi(x) = 0$ at some point x, then evaluating the equation in $x = 0$ yields $\Lambda^{-s}\varphi = 0$. Therefore, since the convolution kernel K_s associated with Λ^{-s} is strictly positive, the solution φ must either be identically equal to zero, or it must change sign.

We now state a result regarding the L^2-integrability of periodic solutions of a finite period P.

Proposition 3.2. *Let $P < \infty$. Then every solution $\varphi \in L^1(\mathbb{S}_P)$ to the fKdV equation belongs to $L^2(\mathbb{S}_P)$. In particular,*

$$\|\varphi\|^2_{L^2(\mathbb{S}_P)} = 2(\mu - 1) \int_{\mathbb{S}_P} \varphi \, dx \,.$$

Proof. Integrating the equation $\varphi^2 = 2\mu\varphi - 2\Lambda^{-s}\varphi$ over $\mathbb{S}_P$ yields

$$\int_{\mathbb{S}_P} \varphi^2 \, dx = 2\mu \int_{\mathbb{S}_P} \varphi \, dx - 2 \int_{\mathbb{S}_P} \Lambda^{-s}\varphi \, dx = 2(\mu - \langle 0 \rangle^{-s}) \int_{\mathbb{S}_P} \varphi \, dx,$$

where we have used the formula (2.16) for Fourier multipliers on periodic functions. □

In the bifurcation preceedure in Section 4, we work with nonconstant, even and P-periodic solutions to the fKdV equation which are nondecreasing on $(-P/2, 0)$. The following proposition lists properties of such solutions.

Lemma 3.3. *Every P-periodic, nonconstant and even solution $\varphi \in C^1(\mathbb{R})$ to the fKdV equation which is nondecreasing on $(-P/2, 0)$ satisfies*

$$\varphi' > 0 \qquad \text{and} \qquad \varphi < \mu$$

on $(-P/2, 0)$. If in addition $\varphi \in C^2(\mathbb{R})$, then

$$\varphi''(0) < 0 \qquad \text{and} \qquad \varphi''(\pm P/2) > 0.$$

Proof. We have by assumption that φ' is odd, nontrivial and nonnegative on $(-P/2, 0)$. Hence, it satisfies the assumptions of Lemma 2.5, and we infer that $\Lambda^{-s}\varphi' > 0$ on $(-P/2, 0)$. Differentiating the fKdV equation, one has

$$(\mu - \varphi)\varphi' = \Lambda^{-s}\varphi' > 0$$

on $(-P/2, 0)$, and we conclude that $\varphi' > 0$ and $\varphi < \mu$ on $(-P/2, 0)$.

Now assume that $\varphi \in C^2(\mathbb{R})$. Differentiating twice, we get

$$(\mu - \varphi)\varphi'' = (\varphi')^2 + \Lambda^{-s}\varphi''.$$

Evaluating this equation at $x = 0$ yields

$$(\mu - \varphi(0))\varphi''(0) = (\Lambda^{-s}\varphi'')(0) = 2\int_0^{P/2} K_{P,s}(y)\varphi''(y)\,dy,$$

since $\varphi'(0) = 0$ by evenness and differentiability of φ, and because $K_{P,s}$ and φ'' are even functions. For some $\varepsilon > 0$, splitting the integral and using integration by parts, one obtains

$$\begin{aligned}\int_0^{P/2} K_{P,s}(y)\varphi''(y)\,dy &= \int_0^{\varepsilon} K_{P,s}(y)\varphi''(y)\,dy + \int_{\varepsilon}^{P/2} K_{P,s}(y)\varphi''(y)\,dy \\ &= \int_0^{\varepsilon} K_{P,s}(y)\varphi''(y)\,dy + \Big[K_{P,s}(y)\varphi'(y)\Big]_{y=\varepsilon}^{P/2} \\ &\quad - \int_{\varepsilon}^{P/2} K'_{P,s}(y)\varphi'(y)\,dy\end{aligned}$$

(recall that $K_{P,s}$ is smooth outside of the origin). The first term vanishes when $\varepsilon \searrow 0$, because

$$\lim_{\varepsilon\searrow 0}\Big|\int_0^{\varepsilon} K_{P,s}(y)\varphi''(y)\,dy\Big| \lesssim \|\varphi''\|_{C(\mathbb{R})}\lim_{\varepsilon\searrow 0}\int_0^{\varepsilon}|y|^{s-1}\,dy = 0,$$

where we have used (2.11) for the period kernel. The second term must also vanish in the limit, since $\varphi'(P/2) = 0$, and since $\varphi'(\varepsilon) \lesssim \varepsilon$ due to $\varphi'(0) = 0$ and the continuity of φ'. The last term is negative for each $\varepsilon > 0$, since we have proved both $\varphi' < 0$ and $K'_{P,s} < 0$ on $(-P/2, 0)$. Moreover, it is decreasing as $\varepsilon \searrow 0$, so passing to the limit we arrive at

$$(\mu - \varphi(0))\varphi''(0) = -2\lim_{\varepsilon\searrow 0}\int_{\varepsilon}^{P/2} K'_{P,s}(y)\varphi'(y)\,dy < 0.$$

In view of $\varphi < \mu$, we conclude that $\varphi''(0) < 0$.

We show that $\varphi''(\pm P/2) > 0$. Arguing similarly as above, one has

$$\begin{aligned}(\mu - \varphi(P/2))\varphi''(P/2) &= 2\int_0^{P/2} K_{P,s}(P/2 + y)\varphi''(y)\,dy \\ &= 2\Big(\int_0^{P/2-\varepsilon} + \int_{P/2-\varepsilon}^{P/2}\Big)K_{P,s}(P/2 + y)\varphi''(y)\,dy,\end{aligned}$$

where the second term vanishes when $\varepsilon \searrow 0$. For the first term, integration by parts yields

$$\begin{aligned}
&\int_0^{P/2-\varepsilon} K_{P,s}(P/2+y)\varphi''(y)\,dy \\
&= \Big[K_{P,s}(P/2+y)\varphi'(y)\Big]_{y=0}^{P/2-\varepsilon} - \int_0^{P/2-\varepsilon} K'_{P,s}(P/2+y)\varphi'(y)\,dy,
\end{aligned}$$

and passing to the limit we obtain

$$(\mu - \varphi(P/2))\varphi''(P/2) = -2\lim_{\varepsilon\searrow 0}\int_0^{P/2-\varepsilon} K'_{P,s}(P/2+y)\varphi'(y)\,dy > 0,$$

on account of $K'_{P,s}$ being P-periodic and strictly positive on $(-P/2, 0)$, and φ' strictly negative on $(0, P/2)$. Hence, $\varphi''(P/2) > 0$, and by evenness also $\varphi''(-P/2) > 0$. □

3.2 Regularity of solutions $\varphi < \mu$

One might ask whether the first- and second-order continuous differentiability assumptions of Lemma 3.3 are reasonable. The following proposition shows that all solutions which are strictly smaller than μ are smooth, and therefore that Lemma 3.3 applies to such solutions.

Lemma 3.4. *Let $\varphi \le \mu$ be a solution to the fKdV equation. Then φ is smooth on every open set where $\varphi < \mu$.*

Proof. Assume first that $\varphi < \mu$ uniformly on $\mathbb{R}$. We rewrite the fKdV equation to the form

$$\varphi = \mu - \sqrt{\mu^2 - 2\Lambda^{-s}\varphi}. \tag{3.1}$$

Note that if $f < \mu^2$ is a function which belongs to $\mathcal{C}^\alpha(\mathbb{R})$ for some $\alpha > 0$, then the mapping

$$f \mapsto \mu - \sqrt{\mu^2 - f}$$

takes f back into $\mathcal{C}^\alpha(\mathbb{R})$, since the function $x \mapsto \sqrt{x}$ is continuous for $x > 0$. Moreover, owing to Proposition 2.6 and Corollary 2.7, the operator Λ^{-s} is linear and bounded from $L^\infty(\mathbb{R})$ to $\mathcal{C}^s(\mathbb{R})$ and from $\mathcal{C}^\alpha(\mathbb{R})$ to $\mathcal{C}^{\alpha+s}(\mathbb{R})$. When $\varphi < \mu$, it is evident from the fKdV equation that $\Lambda^{-s}\varphi < \mu^2$, meaning that the right-hand side of (3.1) maps $L^\infty(\mathbb{R})$ to $\mathcal{C}^s(\mathbb{R})$ and $\mathcal{C}^\alpha(\mathbb{R})$ to $\mathcal{C}^{\alpha+s}(\mathbb{R})$. Bootstrapping now yields $\varphi \in C^\infty(\mathbb{R})$.

Now let U be an open set on which $\varphi < \mu$, and let $\varphi \in \mathcal{C}^s_{\text{loc}}(U)$, in the sense that $\psi\varphi \in \mathcal{C}^\alpha(\mathbb{R})$ for all $\psi \in \mathcal{D}(U)$. We claim that $\Lambda^{-s}\varphi \in \mathcal{C}^{\alpha+s}_{\text{loc}}(U)$, and that consequently the above iteration argument holds for $\varphi < \mu$ on every open set U. To see this, split φ according to

$$\psi\Lambda^{-s}\varphi = \psi\Lambda^{-s}(\rho\varphi) + \psi\Lambda^{-s}((1-\rho)\varphi),$$

where ψ and ρ belongs to $\mathcal{D}(U)$, and $\rho \equiv 1$ on a compact neighborhood of $\operatorname{supp}\psi$ in U. Since $\rho\varphi \in \mathcal{C}^\alpha(\mathbb{R})$, we have $\Lambda^{-s}(\rho\varphi) \in \mathcal{C}^{\alpha+s}(\mathbb{R})$. Furthermore, the second term

$$\psi\Lambda^{-s}((1-\rho)\varphi) = \int_{\mathbb{R}} K_s(x-y)\psi(x)(1-\rho(y))\varphi(y)\,dy$$

is smooth: the kernel K_s is smooth on $\mathbb{R} \setminus \{0\}$, and the integrand vanishes whenever x is sufficiently close to y, because either $\psi(x)$ is zero, or $1 - \rho(y)$ is zero when y approaches x. $\square$

3.3 Regularity of highest traveling waves

We now turn to an investigation of solutions that are allowed to attain the height of $\varphi = \mu$ in some but all points. Such solutions are referred to as highest traveling-wave solutions, and they exhibit different qualitative properties than the smooth solutions with $\varphi < \mu$ discussed in Section 3.2.

In Lemma 3.3 it was proved that solutions that are continuously differentiable are strictly increasing on $(-P/2, 0)$. The regularity assumption can be relaxed if φ does not exceed μ, as the following proposition shows. While the result can also be proven for the solitary case by a similar argument, we state it for periodic solutions.

Proposition 3.5. *Let φ be an even, P-periodic and nonconstant solution to the fKdV equation that is nondecreasing on $(-P/2, 0)$ with $\varphi \leq \mu$. Then φ is strictly increasing on $(-P/2, 0)$.*

Proof. Taking the difference of the fKdV equation evaluated in two points x and y, one obtains

$$(2\mu - \varphi(x) - \varphi(y))(\varphi(x) - \varphi(y)) = 2\big((\Lambda^{-s}\varphi)(x) - (\Lambda^{-s}\varphi)(y)\big). \tag{3.2}$$

Furthermore, for every $h \in (0, P/2)$, we have

$$\begin{aligned}
&(\Lambda^{-s}\varphi)(x+h) - (\Lambda^{-s}\varphi)(x-h) \\
&= \int_{-\infty}^{\infty} K_s(x+h-y)\varphi(y)\,dy - \int_{-\infty}^{\infty} K_s(x-h-y)\varphi(y)\,dy \\
&= \int_{-\infty}^{\infty} K_s(x-y)\varphi(y+h)\,dy - \int_{-\infty}^{\infty} K_s(x-y)\varphi(y-h)\,dy \\
&= \int_{-P/2}^{-P/2} K_{P,s}(x-y)\varphi(y+h)\,dy - \int_{-P/2}^{-P/2} K_{P,s}(x-y)\varphi(y-h)\,dy \\
&= \int_{-P/2}^{0} (K_{P,s}(x-y) - K_{P,s}(x+y))(\varphi(y+h) - \varphi(y-h))\,dy,
\end{aligned} \tag{3.3}$$

where we have used the evenness of $K_{P,s}$ and φ. Hence,

$$\begin{aligned}
&(2\mu - \varphi(x+h) - \varphi(x-h))(\varphi(x+h) - \varphi(x-h)) \\
&= 2\big((\Lambda^{-s}\varphi)(x+h) - (\Lambda^{-s}\varphi)(x-h)\big) \\
&= 2\int_{-P/2}^{0} (K_{P,s}(x-y) - K_{P,s}(x+y))(\varphi(y+h) - \varphi(y-h))\,dy\,.
\end{aligned}$$

The term $K_{P,s}(y-x) - K_{P,s}(y+x)$ in the above was shown in the proof of Lemma 2.5 to be strictly positive for every $x, y \in (-P/2, 0)$. Moreover, owing to (2.13), the factor

$(\varphi(y+h)-\varphi(y-h))$ is nonnegative, and larger than zero for some $y \in (-P/2, 0)$, since φ is assumed to be nonconstant and nondecreasing on $(-P/2, 0)$. Therefore

$$(2\mu - \varphi(x+h) - \varphi(x-h))(\varphi(x+h) - \varphi(x-h)) > 0$$

for every $x \in (-P/2, 0)$ and every $h \in (0, P/2)$, which implies that φ is strictly increasing on $(-P/2, 0)$. $\square$

Lemma 3.4 shows that every solution φ satisfying the assumptions of Proposition 3.5 is smooth on $\mathbb{S}_P \setminus \{0\}$. In the origin, the smoothness of the solution may break down if $\varphi(0) = \mu$. The following lemma shows that this is the case.

Lemma 3.6. *Let $P < \infty$, and let φ be an even, P-periodic and nonconstant solution to the fKdV equation that is nondecreasing on $(-P/2, 0)$ with $\varphi \leq \mu$. Then*

$$\mu - \varphi(P/2) \gtrsim_P 1. \tag{3.4}$$

Moreover, there exists $\varepsilon > 0$ such that

$$\mu - \varphi(x) \gtrsim_P |x|^s \tag{3.5}$$

uniformly for $|x| < \varepsilon$.

Remark 3.7. The estimate (3.5) is in fact uniform in P, when the period is assumed to be sufficiently large. Therefore, one can let $P \nearrow \infty$, and obtain the estimate in the case of solitary waves. This permits us to prove the subsequent Theorem 3.8 for periodic or solitary traveling-wave solutions to the fKdV equation in a unified way.

Proof. Since φ is smooth except possibly in $x = 0$, one has for $x \in (-P/2, 0)$ that

$$\begin{aligned}
(\mu - \varphi(x))\varphi'(x) &= (\Lambda^{-s}\varphi)'(x) \\
&= \lim_{h\to 0} \frac{((\Lambda^{-s}\varphi)(x+h) - (\Lambda^{-s}\varphi)(x-h))}{2h} \\
&\geq \liminf_{h\to 0} \frac{1}{2h} \int_{-P/2}^{0} (K_{P,s}(x-y) - K_{P,s}(x+y))(\varphi(y+h) - \varphi(y-h))\, dy \\
&\geq \int_{-P/2}^{0} (K_{P,s}(x-y) - K_{P,s}(x+y))\varphi'(y)\, dy\,.
\end{aligned}$$

In the third step we used the formula (3.3), and the last estimate, where differentiation is taken under the integral, is justified by Fatou's lemma [5, Lemma 4.1]. Fix $x_0 \in (-P/2, 0)$ and let $x \in [x_0, 0)$. Then, with $z \in [-P/2, x]$, we have

$$\begin{aligned}
(\mu - \varphi(z))\varphi'(x) &\geq (\mu - \varphi(x))\varphi'(x) \\
&\geq \int_{-P/2}^{0} (K_{P,s}(x-y) - K_{P,s}(x+y))\varphi'(y)\, dy \\
&\geq \int_{x_0/2}^{x_0/4} (K_{P,s}(x-y) - K_{P,s}(x+y))\varphi'(y)\, dy,
\end{aligned} \tag{3.6}$$

where we used that the integrand in the second step is strictly positive, since the difference $K_{P,s}(x-y)-K_{P,s}(x+y)$ is positive and $\varphi' > 0$ in $(-P/2, 0)$ owing to Proposition 3.5. Letting

$$C_P = \min\{K_{P,s}(x-y) - K_{P,s}(x+y)\,;\, x, y \in (\frac{x_0}{2}, \frac{x_0}{4})\} > 0,$$

we have

$$(\mu - \varphi(-P/2))\varphi'(x) \geq C_P(\varphi(\frac{x_0}{4}) - \varphi(\frac{x_0}{2})).$$

Integrating over $(\frac{x_0}{2}, \frac{x_0}{4})$ and dividing by the difference $\varphi(\frac{x_0}{4}) - \varphi(\frac{x_0}{2})$ proves (3.4).

Towards proving (3.5), we claim that there exists $\varepsilon > 0$ such that the estimate

$$K_{P,s}(x-y) - K_{P,s}(x+y) \gtrsim_P |x_0|^{s-1}$$

holds uniformly over $x \in [x_0, 0)$ and $y \in (x_0/2, x_0/4)$ with $|x_0| < \varepsilon$. Note first that for these ranges of x and y, we have $|x-y| < |x+y|$. Then, due to the estimate (2.11), we can pick a small enough x_0 and constants C_1 and C_1, depending on P and K_s, such that

$$K_{P,s}(x-y) \geq C_1|x-y|^{s-1} \qquad \text{and} \qquad K_{P,s}(x+y) \leq C_1|x+y|^{s-1}$$

hold for all $x \in [x_0, 0)$ and $y \in (x_0/2, x_0/4)$, and such that

$$\begin{aligned} K_{P,s}(x-y) - K_{P,s}(x+y) &\geq C_1|x-y|^{s-1} - C_2|x+y|^{s-1} \\ &\geq C_1\big(\frac{3}{4}\big)^{s-1}|x_0|^{s-1} - C_2\big(\frac{1}{4}\big)^{s-1}|x_0|^{s-1} \\ &\gtrsim_P |x_0|^{s-1}. \end{aligned}$$

Inserting the above estimate in (3.6) yields

$$(\mu - \varphi(z))\varphi'(x) \gtrsim_P |x_0|^{s-1}(\varphi(x_0/4) - \varphi(x_0/2)).$$

Integrating this inequality over $(x_0/2, x_0/4)$ with respect to x, dividing by the (positive) difference $(\varphi(x_0/4) - \varphi(x_0/2)$, and setting $z = x_0$, we obtain

$$(\mu - \varphi(x_0) \gtrsim_P (x_0/4 - x_0/2)|x_0|^{s-1} \gtrsim_P |x_0|^s,$$

and the inequality is uniform in x_0 for $|x_0| < \varepsilon$. The estimate (3.5) now follows by evenness of φ. □

Proposition 3.6 provides an upper bound for the regularity at the crests of periodic solutions which are allowed to touch μ from below in the origin. In Theorem 3.8, we also prove a global upper counterpart of the estimate (3.5), thereby establishing global s-Hölder regularity of solutions φ, attained at the crests. The method of the proof follows [11, Theorem 5.4], but with modified arguments for the global estimates related to the parametrized order $s \in (0, 1)$ of the Bessel potential operator.

Theorem 3.8. *Let $P \in (0,\infty]$, and let $\varphi \le \mu$ be an even and nonconstant solution to the fKdV equation which is nondecreasing on $(-P/2,0)$ with $\varphi(0)=\mu$. Then $\varphi \in C^{0,s}(\mathbb{R})$. Moreover,*

$$\mu - \varphi(x) \eqsim |x|^s \tag{3.7}$$

uniformly for $|x| \ll 1$.

Remark 3.9. Using the (nonperiodic) kernel K_s in the proof, and in view of Remark 3.7, the period P in Theorem 3.8 is allowed to be infinite.

Proof. Let $\varphi \in L^\infty(\mathbb{R})$ satisfy the assumptions above. We show first that, for every $\alpha < s$, the solution φ is α-Hölder continuous in 0. From (3.2) we obtain the formula

$$\begin{aligned}(\mu-\varphi(x))^2 &= 2\big((\Lambda^{-s}\varphi)(0) - (\Lambda^{-s}\varphi)(x)\big)\\ &= \int_{\mathbb{R}} (K_s(x+y)+K_s(x-y)-2K_s(y))(\varphi(0)-\varphi(y))\,dy\,.\end{aligned} \tag{3.8}$$

Splitting the kernel in the singular and regular parts, as shown in (2.17), gives

$$\begin{aligned}&\Big|\int_{\mathbb{R}} (K_s(x+y)+K_s(x-y)-2K_s(y))(\varphi(0)-\varphi(y))\,dy\,\Big|\\ &\lesssim \int_{\mathbb{R}} \big||x+y|^{s-1}+|x-y|^{s-1}-2|y|^{s-1}\big|(\varphi(0)-\varphi(y))\,dy\\ &\quad+\int_{\mathbb{R}} \big|\widetilde{K}_s(x+y)+\widetilde{K}_s(x-y)-2\widetilde{K}_s(y)\big|(\varphi(0)-\varphi(y))\,dy\end{aligned}$$

For the singular part one has

$$\begin{aligned}&\int_{\mathbb{R}} \big||x+y|^{s-1}+|x-y|^{s-1}-2|y|^{s-1}\big|(\varphi(0)-\varphi(y))\,dy\\ &\le 2\|\varphi\|_{L^\infty}|x|^s \int_{\mathbb{R}} \big||1+t|^{s-1}+|1-t|^{s-1}-2|t|^{s-1}\big|\,dt\\ &\lesssim |x|^s,\end{aligned} \tag{3.9}$$

where the integral in the last step converges every $s\in(0,1)$, owing to the inequality

$$\big||1+t|^{s-1}+|1-t|^{s-1}-2|t|^{s-1}\big| \lesssim |t|^{s-3} \tag{3.10}$$

for large t. The regular part can be estimated by

$$\begin{aligned}&\int_{\mathbb{R}} \big|\widetilde{K}_s(x+y)+\widetilde{K}_s(x-y)-2\widetilde{K}_s(y)\big|(\varphi(0)-\varphi(y))\,dy\\ &\lesssim \|\varphi\|_{L^\infty}|x|^2\int_{\mathbb{R}} R_x^2(y)\,dy\\ &\lesssim |x|^2,\end{aligned} \tag{3.11}$$

where we have used $R_x^2(y)$ from (2.20) and the estimates in (2.19) for it. Inserting (3.9) and (3.11) in (3.8) yields $(\mu-\varphi(x))^2 \lesssim |x|^s$. This implies that φ is at least $\frac{s}{2}$-Hölder continuous in 0. Using this information, the term $\varphi(0)-\varphi(y)$ can be bounded from above by $|y|^{\frac{s}{2}}$ in the estimate (3.9), giving $\frac{s/2+s}{2}$-Hölder continuity of φ in 0. Iteration of this argument proves that φ is α-Hölder regular in 0 for every $\alpha < s$.

We now show s-Hölder regularity in $x = 0$. To this end, we claim that there is a constant C, independent of α, such that

$$\int_{\mathbb{R}} \big|K_s(x+y) + K_s(x-y) - 2K_s(y)\big||y|^{\alpha}\,dy \le C|x|^{2\alpha}$$

for all $|x| \le 1$ and all $0 \le \alpha \le s$. Indeed, for the singular part we have

$$\begin{aligned}
&\int_{\mathbb{R}} \big||x+y|^{s-1} + |x-y|^{s-1} + 2|y|^{s-1}\big||y|^{\alpha}\,dy \\
&= |x|^{s+\alpha}\int_{\mathbb{R}} \big||1+t|^{s-1} + |1-t|^{s-1} - 2|t|^{s-1}\big||t|^{\alpha}\,dt \\
&\lesssim |x|^{s+\alpha} \\
&\le |x|^{2\alpha},
\end{aligned}$$

where the integral converges in view of (3.10), and in the last step it was used that $|x| \le 1$. Note that the estimate is uniform in $\alpha \in [0, s]$. Moreover, the regular part of the kernel can be bounded according to

$$\begin{aligned}
\int_{\mathbb{R}} \big|\widetilde{K}_s(x+y) + \widetilde{K}_s(x-y) - 2\widetilde{K}_s(y)\big||y|^{\alpha}\,dy &\lesssim |x|^2 \int_{\mathbb{R}} R_x^2(y)|y|^{\alpha}\,dy \\
&\lesssim |x|^2 \\
&\le |x|^{2\alpha},
\end{aligned}$$

for $|x| \le 1$. It was shown above that φ is α-Hölder continuous in the origin for every $\alpha \in [0, s)$. Hence,

$$\begin{aligned}
(\varphi(0)-\varphi(x))^2 &= \int_{\mathbb{R}} \big(K_s(x+y) + K_s(x-y) - 2K_s(y)\big)(\varphi(0)-\varphi(y))\,dy \\
&\le [\varphi]_{C_0^{0,\alpha}} \int_{\mathbb{R}} \big|K_s(x+y) + K_s(x-y) - 2K_s(y)\big||y|^{\alpha}\,dy \\
&\lesssim [\varphi]_{C_0^{0,\alpha}} |x|^{2\alpha}.
\end{aligned}$$

Dividing by $|x|^{2\alpha}$ and passing to supremum yields

$$[\varphi]_{C_0^{0,\alpha}} \lesssim 1$$

uniformly over $\alpha \in [0, s)$. We let $\alpha \nearrow s$, and obtain the estimate (3.7).

We now claim that for global α-Hölder continuity, with any $\alpha \in (0, 1)$, it suffices to prove that

$$\sup_{0<h<|x|<\delta} \frac{|\varphi(x+h)-\varphi(x-h)|}{h^{\alpha}} < \infty,$$

for some $\delta > 0$. This is equivalent to

$$\|\varphi\|_{C^{0,\alpha}(\mathbb{R})} \lesssim \max\left\{1, \sup_{0<h<|x|<\delta} \frac{|\varphi(x+h)-\varphi(x-h)|}{h^\alpha}\right\}. \tag{3.12}$$

Note first that $\varphi(x+y) - \varphi(x-y)$ is symmetric in x and y, which implies that

$$\|\varphi\|_{C^{0,\alpha}(\mathbb{R})} = \sup_{0<h<|x|} \frac{|\varphi(x+h)-\varphi(x-h)|}{h^\alpha},$$

where h denotes $\min\{|x|,|y|\}$. Now fix $\delta \ll 1$. If $h \geq \delta/2$, then

$$\frac{|\varphi(x+h)-\varphi(x-h)|}{h^\alpha} \leq 4\frac{\|\varphi\|_{L^\infty}}{\delta}.$$

On the other hand, if $|x| \geq \delta$ and $h \leq \delta/2$, the already established smoothness of φ outside of the origin implies that

$$\frac{|\varphi(x+h)-\varphi(x-h)|}{h^\alpha} \leq 2\Big(\frac{\delta}{2}\Big)^{1-\alpha}\|\varphi\|_{C^1(\delta/2,P/2)} \lesssim \|\varphi\|_{C^1(\delta/2,P/2)}.$$

This justifies the reduction (3.12).

We proceed to show that $\varphi \in C^{0,\alpha}(\mathbb{R})$ for every $\alpha < s$. Assume that $0 < h < x < \delta$ for some $\delta \ll 1$, in accordance with the reduction above, where x can be taken positive without loss of generality due to the evenness of φ. Since

$$\begin{aligned}
&(\varphi(x+h)-\varphi(x-h))^2 \\
&\leq \big|(2\mu - \varphi(x+h)-\varphi(x-h))(\varphi(x+h)-\varphi(x-h))\big| \\
&= \big|(\Lambda^{-s}\varphi)(x+h) - (\Lambda^{-s}\varphi)(x-h)\big|,
\end{aligned} \tag{3.13}$$

and Λ^{-s} maps L^∞ to $C^{0,s}$ and $\mathcal{C}^\alpha$ to $\mathcal{C}^{\alpha+s}$, we obtain that φ is at least α-Hölder regular for every $\alpha < s$ if $s \leq 1/2$ and $\alpha = 1/2$ if $s > 1/2$. Consequently, for $s > 1/2$ we need to pass the threshold $\alpha = 1/2$ in the iteration procedure of (3.13). So assume that $s > 1/2$ and that $\varphi \in C^{0,\alpha}$ with $\alpha + s > 1$. Note that for a function $f \in C^{1,\beta}$ with $\beta \in (0,1)$ and $f'(0) = 0$, one has

$$|f(x) - f(y)| = |x-y||f'(\xi) - f'(0)| \lesssim |x-y||\xi|^\beta \tag{3.14}$$

for $\xi \in (x,y)$. Hence,

$$\big|(\Lambda^{-s}\varphi)(x+h) - (\Lambda^{-s}\varphi)(x-h)\big| \lesssim h|\xi|^{\{\alpha+s\}},$$

with $\xi \in (x-h, x+h)$ and $\{\alpha+s\}$ being the fractional part of $\alpha+s$. Inserting this in (3.13) yields

$$\begin{aligned}
|\varphi(x+h)-\varphi(x-h)| &\lesssim \frac{h|\xi|^{\{\alpha+s\}}}{2\mu - \varphi(x+h) - \varphi(x-h)} \\
&\lesssim \frac{h|x+h|^{\{\alpha+s\}}}{|x+h|^s + |x-h|^s} \\
&\lesssim h|x+h|^{\alpha-1}
\end{aligned} \tag{3.15}$$

where we have used the estimate (3.5) from Lemma 3.6 in the second step, and in the last step that $\{\alpha + s\} - s = \alpha - 1$. Now we interpolate between (3.15) and the exact s-Hölder regularity in the origin. Precisely, with $\eta \in (0,1)$ one has

$$\begin{aligned}\frac{|\varphi(x+h) - \varphi(x-h)|}{h^\eta} &\leq \frac{|\varphi(x+h) - \varphi(x-h)|^\eta}{h^\eta} |\mu - \varphi(x+h)|^{1-\eta} \\ &\lesssim |x+h|^{\eta(\alpha-1)+(1-\eta)s}.\end{aligned}$$

This is bounded whenever

$$\eta \leq \frac{s}{1+s-\alpha},$$

and we choose the interpolation parameter η such that equality holds. Hence,

$$|\varphi(x+h) - \varphi(x-h)| \lesssim h^{\frac{s}{1+s-\alpha}}.$$

Iterating this argument, one obtains in each step for $\varphi \in C^{0,\alpha}$ that φ is $\frac{s}{1+s-\alpha}$-Hölder regular. The regularity is therefore increased in each iteration and tending to s, proving $\varphi \in C^{0,\alpha}(\mathbb{R})$ for every $\alpha < s$.

We now prove $\varphi \in C^{0,s}(\mathbb{R})$. To this end, note that the difference in the right-hand side of (3.2) can also be written as

$$\begin{aligned}&(\Lambda^{-s}\varphi)(x+h) - (\Lambda^{-s}\varphi)(x-h) \\ &= \int_{-\infty}^{\infty} K(x+h-y)\varphi(y)\,dy - \int_{-\infty}^{\infty} K(x-h-y)\varphi(y)\,dy \\ &= \int_{-\infty}^{\infty} K(y-h)\varphi(y+x)\,dy - \int_{-\infty}^{\infty} K(y+h)\varphi(y+x)\,dy \\ &= \int_{-\infty}^{0} (K_s(y+h) - K_s(y-h))(\varphi(y-x) - \varphi(y+x)\,dy\,.\end{aligned} \tag{3.16}$$

Let $0 < h < |x| < \delta$ for some $\delta \ll 1$, and assume that x is positive. Since

$$2\mu - \varphi(x+h) - \varphi(x-h) \geq \mu - \varphi(x+h) \geq \mu - \varphi(x),$$

we have with (3.2) and (3.16) that

$$\begin{aligned}&(\mu - \varphi(x))|\varphi(x+h) - \varphi(x-h)| \\ &\leq 2\int_{-\infty}^{0} |K_s(y+h) - K_s(y-h)||\varphi(y-x) - \varphi(y+x)|\,dy\,.\end{aligned} \tag{3.17}$$

To estimate the factor $|\varphi(y-x) - \varphi(y+x)|$, we interpolate between the sharp $C^{0,s}$-regularity in $x = 0$ and the global $C^{0,\alpha}$-regularity (for $\alpha < s$). That is,

$$|\varphi(y-x) - \varphi(y+x)| \lesssim \|\varphi\|_{C^{0,\alpha}} \min(|x|^\alpha, |y|^\alpha) \tag{3.18}$$

for every choice of $\alpha \in (0,s)$, and

$$|\varphi(y-x) - \varphi(y+x)| \lesssim [\varphi]_{C_0^{0,s}} \max(|x|^s, |y|^s), \tag{3.19}$$

which holds true in view of the s-Hölder regularity in $x = 0$ via

$$
\begin{aligned}
|\varphi(y-x) - \varphi(y+x)| &= |(\varphi(y-x) - \mu) + (\mu - \varphi(y+x))| \\
&\lesssim [\varphi]_{C_0^{0,s}} (|x-y|^s + |x+y|^s) \\
&\lesssim [\varphi]_{C_0^{0,s}} \max(|x|^s, |y|^s).
\end{aligned}
$$

Interpolation of (3.18) and (3.19) over a parameter η gives

$$
|\varphi(y-x) - \varphi(y+x)| \lesssim \|\varphi\|_{C^{0,\alpha}}^{\eta} \min(|x|,|y|)^{\alpha\eta} \max(|x|,|y|)^{s(1-\eta)}, \tag{3.20}
$$

with $(\alpha, \eta) \in (0, s) \times [0, 1]$. The integral in the right-hand side of (3.17) can be split in the singular and regular parts of the kernel K_s. Inserting (3.20) in the integral with the singular term yields

$$
\begin{aligned}
&\int_{-\infty}^{0} \big||y+h|^{s-1} - |y-h|^{s-1}\big| |\varphi(y-x) - \varphi(y+x)| \, dy \\
&\lesssim \|\varphi\|_{C^{0,\alpha}}^{\eta} \int_{-\infty}^{0} \big||y+h|^{s-1} - |y-h|^{s-1}\big| \min(|x|,|y|)^{\alpha\eta} \max(|x|,|y|)^{s(1-\eta)} \, dy \\
&= \|\varphi\|_{C^{0,\alpha}}^{\eta} |x|^{\alpha\eta} \int_{-\infty}^{-|x|} \big||y+h|^{s-1} - |y-h|^{s-1}\big| |y|^{s(1-\eta)} \, dy \\
&\quad + \|\varphi\|_{C^{0,\alpha}}^{\eta} |x|^{s(1-\eta)} \int_{-|x|}^{0} \big||y+h|^{s-1} - |y-h|^{s-1}\big| |y|^{\alpha\eta} \, dy \\
&\lesssim \|\varphi\|_{C^{0,\alpha}}^{\eta} |x|^{\alpha\eta} h^{s+s(1-\eta)} \int_{-\infty}^{0} \big||t+1|^{s-1} - |t-1|^{s-1}\big| |t|^{s(1-\eta)} \, dt \\
&\quad + \|\varphi\|_{C^{0,\alpha}}^{\eta} |x|^{s(1-\eta)} h^{s+\alpha\eta} \int_{-\delta}^{0} \big||t+1|^{s-1} - |t-1|^{s-1}\big| |t|^{\alpha\eta} \, dt \, .
\end{aligned} \tag{3.21}
$$

The integral in the last line clearly converges. For the difference in the second last line we have the identity

$$
|t+1|^{s-1} - |t-1|^{s-1} \lesssim |t|^{s-2}
$$

for large t. Thus, we need to choose η such that $s - 2 + s(1-\eta) < -1$ for convergence. But this is possible for every $s \in (0, 1)$ by requiring

$$
\eta > 2 - \frac{1}{s}. \tag{3.22}
$$

The regular part can be estimated by

$$
\begin{aligned}
&\int_{-\infty}^{0} |\widetilde{K}_s(y+h) - \widetilde{K}_s(y-h)||\varphi(y-x) - \varphi(y+x)|\,dy \\
&\lesssim \|\varphi\|_{C^{0,\alpha}}^{\eta} h \int_{-\infty}^{0} R_h^1(y) \min(|x|,|y|)^{\alpha\eta} \max(|x|,|y|)^{s(1-\eta)}\,dy \\
&\lesssim \|\varphi\|_{C^{0,\alpha}}^{\eta} h|x|^{\alpha\eta} \int_{-\infty}^{-|x|} R_h^1(y)|y|^{s(1-\eta)}\,dy \\
&\quad + \|\varphi\|_{C^{0,\alpha}}^{\eta} h|x|^{s(1-\eta)} \int_{-|x|}^{0} R_h^1(y)|y|^{\alpha\eta}\,dy \\
&\lesssim \|\varphi\|_{C^{0,\alpha}}^{\eta} |x|^{\alpha\eta} h^{1+s(1-\eta)} \int_{-\infty}^{0} R_h^1(th)|t|^{s(1-\eta)}\,dt \\
&\quad + \|\varphi\|_{C^{0,\alpha}}^{\eta} |x|^{s(1-\eta)} h^{1+\alpha\eta} \int_{-\delta}^{0} R_h^1(th)|t|^{\alpha\eta}\,dt,
\end{aligned} \tag{3.23}
$$

where both integrals converge. Note in particular that $s-2+s(1-\eta) < -1$ in the second last integral due to the choice of η given by (3.22) and the estimate (2.18) for R_h^1. Inserting (3.21) and (3.23) into (3.17) yields

$$
\begin{aligned}
&(\mu - \varphi(x))|\varphi(x+h) - \varphi(x-h)| \\
&\lesssim \|\varphi\|_{C^{0,\alpha}}^{\eta} \big(|x|^{\alpha\eta} h^{s+s(1-\eta)} + |x|^{s(1-\eta)} h^{s+\alpha\eta} + |x|^{\alpha\eta} h^{1+s(1-\eta)} + |x|^{s(1-\eta)} h^{1+\alpha\eta}\big) \\
&\lesssim \|\varphi\|_{C^{0,\alpha}}^{\eta} |x|^{\alpha\eta+s(1-\eta)} h^{s},
\end{aligned}
$$

where we have used $h < |x|$. Thus,

$$
\left(\frac{\mu - \varphi(x)}{|x|^{\alpha\eta+s(1-\eta)}}\right)\left(\frac{|\varphi(x+h) - \varphi(x-h)|}{h^s}\right) \lesssim \|\varphi\|_{C^{0,\alpha}}^{\eta},
$$

uniformly for $\alpha \in (0,s)$. Since $\mu - \varphi(x) \gtrsim |x|^s$ for small $|x|$ by Lemma 3.6, and $h < |x|$, this can be reduced to

$$
\frac{|\varphi(x+h) - \varphi(x-h)|}{h^{s-\eta(s-\alpha)}} \lesssim \|\varphi\|_{C^{0,\alpha}}^{\eta}.
$$

Splitting the estimate over η we arrive at

$$
\left(\frac{|\varphi(x+h) - \varphi(x-h)|}{h^{\alpha}}\right)^{\eta}\left(\frac{|\varphi(x+h) - \varphi(x-h)|}{h^{s}}\right)^{1-\eta} \lesssim \|\varphi\|_{C^{0,\alpha}}^{\eta}.
$$

Hence,

$$
\begin{aligned}
\|\varphi\|_{C^{0,\alpha}} &\lesssim \sup_{0<h<|x|<\delta} \left(\frac{|\varphi(x+h) - \varphi(x-h)|}{h^{\alpha}}\right) \\
&\leq \sup_{0<h<|x|<\delta} \left(\frac{|\varphi(x+h) - \varphi(x-h)|}{h^{\alpha}}\right)^{\eta}\left(\frac{|\varphi(x+h) - \varphi(x-h)|}{h^{s}}\right)^{1-\eta} \\
&\lesssim \|\varphi\|_{C^{0,\alpha}}^{\eta},
\end{aligned}
$$

which finally proves

$$\sup_{0<h<|x|<\delta} \left(\frac{|\varphi(x+h) - \varphi(x-h)|}{h^\alpha} \right)^{1-\eta} \lesssim 1$$

uniformly for $\alpha \in (0, s)$ with

$$\min(0, 2 - \frac{1}{s}) < \eta < 1$$

fixed. This justifies letting $\alpha \nearrow s$, thereby proving the claimed global s-Hölder regularity of the solution φ.

In view of of Lemma 3.6, we infer that the s-Hölder regularity is precisely attained in the crest, as claimed in (3.7). □

4 Analytic bifurcation for the fKdV equation

In this section, analytic bifurcation theory is applied to the fKdV equation to construct a global curve of even, periodic and smooth solutions which converge to a highest traveling wave. In Section 4.1, the existence of local bifurcation branches, composed of small-amplitude solutions, is proved. Then, in section 4.2, it is shown that the local branches can be extended to global analytic curves and that a highest traveling-wave solution can be found in the limit of this curve. By virtue of the theory from section 3, we show that the limiting wave is smooth outside the crests, and cusped with exactly s-Hölder regularity at the crests where the maximal height of μ is achieved.

The organization of the results in this section follows [11, Section 6]. The theory of analytic bifurcation is due to Buffoni, Dancer and Toland [7].

4.1 Local bifurcation

We consider the parameter $s \in (0, 1)$ appearing in the symbol $\langle \xi \rangle^s$ fixed, and set $\beta \in (s, 1)$. Let

$$F\colon (\varphi, \mu) \mapsto \mu\varphi - \frac{1}{2}\varphi^2 - \Lambda^{-s}\varphi, \tag{4.1}$$

where

$$F\colon C^{0,\beta}_{\text{even}}(\mathbb{S}_P) \times \mathbb{R} \to C^{0,\beta}_{\text{even}}(\mathbb{S}_P),$$

since Λ^{-s} maps $C^{0,\beta}_{\text{even}}(\mathbb{S}_P)$ onto itself, and $C^{0,\beta}_{\text{even}}(\mathbb{S}_P)$ is a Banach algebra.

Recall that if X and Y are Banach spaces, and $f\colon X \to Y$ is a function defined on these spaces, we say that f is Fréchet differentiable at a point $x_0 \in X$ if there exists a linear and bounded operator $A\colon X \to Y$ with

$$\lim_{0<\|h\|_X \to 0} \frac{\|f(x_0 + h) - f(x_0) - Ah\|_Y}{\|h\|_X} = 0.$$

If such an operator exists it is called the Fréchet derivative of f at x_0, denoted by $df[x_0]$, and it is unique. Moreover, the Fréchet derivative of a function $f\colon X \times Y \to Z$ at a point (x_0, y_0) with respect to the first argument is defined as the Fréchet derivative of $f(\cdot, y_0)$ at x_0, whenever this operator exists. It is denoted $\partial_x f[x_0, y_0]$, and is a linear and bounded operator from X to Z. The Fréchet derivative of the second argument is defined analogously. A more detailed account of calculus in Banach spaces can be found in [7, Chapter 3].

The function F defined in (4.1) is a polynomial in the variable φ. It follows that F is Fréchet differentiable with respect to φ and can be written as a convergent power series in $C^{0,\beta}_{\text{even}}(\mathbb{S}_P)$. That is, F is a real-analytic function.

Solutions to the equation

$$F(\varphi, \mu) = 0 \tag{4.2}$$

coincide with solutions to the fKdV equation, now with the additional requirement of evenness, P-periodicity and β-Hölder continuity of φ. Note that there are exactly two curves of constant solutions to (4.2),

$$\varphi \equiv 0 \qquad \text{and} \qquad \varphi \equiv 2(\mu - 1).$$

The implicit function theorem [7, Theorem 4.5.4] states that if (φ_0, μ_0) is a solution to (4.2) and the operator $\partial_\varphi F[\varphi_0, \mu_0]$ is a homeomorphism, then all solutions to the problem (4.2) in a neighborhood of (φ_0, μ_0) lie on a unique curve. Therefore, a necessary condition for a point (φ_0, μ_0) to be the origin of a bifurcation is that $\partial_\varphi F[\varphi_0, \mu_0]$ is not a homeomorphism. An important example of operators that contain such functions are the Fredholm operators.

We say that a linear bounded and operator $A\colon X \to Y$, on Banach spaces X and Y, is a Fredholm operator of index $p = n - r$ if one has

(*i*) $\dim \ker(A) = n < \infty$, and

(*ii*) $\operatorname{im}(A)$ is closed and $\operatorname{codim} \operatorname{im}(A) = r < \infty$.

Here, im and ker denote the image and the kernel of the operator A, and dim and codim denote (algebraic) dimension and codimension. We now characterize points along the trivial solution curve of (4.2) in which $\partial_\varphi F$ is a Fredholm operator.

Proposition 4.1. *For every finite period $P > 0$ and any $k \in \mathbb{N}$, there exists a unique number $\mu^*_{P,k} := \langle \frac{2\pi k}{P} \rangle^{-s}$ such that $\partial_\varphi F[0, \mu^*_{P,k}]$ is a Fredholm operator of index* 0 *with*

$$\dim \ker(\partial_\varphi F[0, \mu^*_{P,k}]) = \operatorname{codim} \operatorname{im}(\partial_\varphi F[0, \mu^*_{P,k}]) = 1.$$

Proof. The function F is Fréchet differentiable with respect to φ, and

$$\partial_\varphi F[0, \mu] = \mu \operatorname{id} - \Lambda^{-s},$$

where id is the identity operator on $C^{0,\beta}_{\text{even}}(\mathbb{S}_P)$. Note that Λ^{-s} is a compact operator on $C^{0,\beta}_{\text{even}}(\mathbb{S}_P)$. This is a consequence of the fact that Hölder spaces of a given exponent are compactly embedded in all Hölder spaces of strictly smaller exponents. More generally, the embedding

$$\mathcal{C}^{\beta+s}(\mathbb{S}_P) \hookrightarrow \mathcal{C}^{\beta}(\mathbb{S}_P) \tag{4.3}$$

is compact for every $s > 0$ and any finite $P > 0$; see [29, Chapter 13 (A.39)]. The notation $\hookrightarrow\hookrightarrow$ here signifies a compact embedding. Thus,

$$\Lambda^{-s} \colon \mathcal{C}^\beta(\mathbb{S}_P) \hookrightarrow \mathcal{C}^{\beta+s}(\mathbb{S}_P) \hookrightarrow\hookrightarrow \mathcal{C}^\beta(\mathbb{S}_P),$$

and this property is preserved by restricting the operation to the closed subspace of even functions. As a result of the Fredholm alternative theorem [7, Theorem 2.7.6], this implies that $\partial_\varphi F[0,\mu]$ is a Fredholm operator of index zero. Furthermore, for all $k \in \mathbb{N}$, the operator $\partial_\varphi F[0,\mu^*_{P,k}]$ maps the basis function $\varphi^*_{P,k} := \cos\left(\frac{2\pi k}{P}x\right)$ of $C^{0,\beta}_{\mathrm{even}}(\mathbb{S}_P)$ to zero while all others are multiplied by a positive constant. This is evident from (4.5) and the fact that $\langle\cdot\rangle^{-s}$ is strictly decreasing on $\mathbb{N}$. Hence, the dimension of the kernel and the codimension of the image of $\partial_\varphi F[0,\mu^*_{P,k}]$ is 1. $\square$

Fredholm operators play an important role in bifurcation theory, not only since they are examples of operators for which the implicit function breaks down, but because they allow certain classical bifurcation results. These include the Lyapunov–Schmidt reduction [7, Theorem 8.2.1], reducing the infinite-dimensional problem (4.2) in a neighborhood of a solution (φ_0,μ_0) to a finite-dimensional problem, and the Crandall–Rabinowitz theorem, providing the existence of local bifurcation branches emanating from the trivial solution curve at points in which the Fréchet derivative is Fredholm. Due to Proposition 4.1, we are now in the position to apply an analytic version of the Crandall–Rabinowitz theorem [7, Theorem 8.3.1] around the trivial solution curve of (4.2).

Lemma 4.2. *For every finite period $P > 0$ and every $k \in \mathbb{N}$, the trivial solution curve of (4.2) has a bifurcation point at $(0,\mu^*_{P,k})$, and for each bifurcation point there exists $\varepsilon > 0$ and an analytic curve*

$$\mathcal{R}_{P,k} = \{(\varphi_{P,k}(t),\mu_{P,k}(t))\,;\, t \in (-\varepsilon,\varepsilon) \text{ and } (\varphi_{P,k}(0),\mu_{P,k}(0)) = (0,\mu^*_{P,k})\}$$

belonging to $C^{0,\beta}_{\mathrm{even}}(\mathbb{S}_P) \times \mathbb{R}$, such that $F(\varphi_{P,k}(t),\mu_{P,k}(t)) = 0$ for all $t \in (-\varepsilon,\varepsilon)$. Furthermore, all solutions to the equation (4.2) in a neighborhood of $(0,\mu^*_{P,k})$ lie either on $\mathcal{R}_{P,k}$ or on the trivial curve $\{(0,\mu)\,;\,\mu \in \mathbb{R}\}$.

Together with the transcritical bifurcation of constant solutions $\{(2(\mu-1),\mu)\,;\,\mu \in \mathbb{R}\}$ intersecting the trivial curve in $(0,1)$, the curves $\mathcal{R}_{P,k}$ constitute all nonzero solutions to (4.2) in $C^{0,\beta}_{\mathrm{even}}(\mathbb{S}_P) \times \mathbb{R}$ in a neighborhood of the trivial solution curve.

Proof. By Proposition 4.1, the operator $\partial_\varphi F[0,\mu^*_{P,k}]$ is a Fredholm operator for every $k \in \mathbb{N}$, and it was shown in the proof that $\ker(\partial_\varphi F[0,\mu^*_{P,k}])$ is one-dimensional. Furthermore,

$$\ker(\partial_\varphi F[0,\mu^*_{P,k}]) = \{\tau\varphi^*_{P,k}\,;\,\tau \in \mathbb{R}\} \qquad \text{and} \qquad \partial^2_{\varphi\mu}F[0,\mu^*_{P,k}](1,\varphi^*_{P,k}) = \varphi^*_{P,k}$$

for every $k \in \mathbb{N}$. This means that the transversality condition holds, that is,

$$\partial^2_{\mu\varphi}F[0,\mu^*_{P,k}](1,\varphi^*_{P,k}) \notin \mathrm{im}(\partial_\varphi F[0,\mu^*_{P,k}]),$$

and hence that the assumptions of [7, Theorem 8.3.1] are satisfied.

Since the kernel of $\partial_\varphi F[0,\mu]$ is trivial for all $\mu \neq \mu^*_{P,k}$, for every $k \in \mathbb{N}$ and $\mu \neq 1$, it follows from the implicit function theorem that the trivial solution is otherwise locally unique. □

Remark 4.3. The local bifurcation branches $\mathcal{R}_{P,k}$ from Lemma 4.2 can be uniquely determined by the quotient between k and P. Indeed, two solution branches with the same quotients $\frac{P_1}{k_1} = \frac{P_2}{k_2}$ have coinciding bifurcation points in $\langle \frac{2\pi k_1}{P_1} \rangle^{-s}$, and since $\frac{P_2}{k_2}$ is an integer multiple of P_1, the branches both belong to $C^{0,\beta}_{\text{even}}(\mathbb{S}_{P_1})$ where uniqueness is ensured by the lemma.

4.2 Global bifurcation

The Fréchet derivative of the function F at any point $(\varphi,\mu) \in C^{0,\beta}_{\text{even}}(\mathbb{S}_P) \times \mathbb{R}$ is given by

$$\partial_\varphi F[\varphi,\mu] = (\mu - \varphi)\,\mathrm{id} - \Lambda^{-s}. \tag{4.4}$$

Assuming that $\varphi < \mu$, we make the following observations. Firstly, the operator $(\mu-\varphi)\,\mathrm{id}$ is injective. Secondly, it is continuous, because $C^{0,\beta}_{\text{even}}(\mathbb{S}_P)$ is a Banach algebra. Thirdly, it is surjective, since for every $\vartheta \in C^{0,\beta}_{\text{even}}(\mathbb{S}_P)$ there exists $\frac{1}{\mu-\varphi}\vartheta \in C^{0,\beta}_{\text{even}}(\mathbb{S}_P)$. We conlude that for $\varphi < \mu$, the operator $(\mu-\varphi)\,\mathrm{id}$ is a linear homeomorphism on $C^{0,\beta}_{\text{even}}(\mathbb{S}_P)$. On the grounds of this, we define the set

$$U := \big\{(\varphi,\mu) \in C^{0,\beta}_{\text{even}}(\mathbb{S}_P) \times \mathbb{R}\,;\, \varphi < \mu\big\}.$$

Moreover, let

$$S := \big\{(\varphi,\mu) \in U\,;\, F(\varphi,\mu) = 0\big\},$$

and let S^1 denote the φ-component of S.

In the following, we consider only the first bifurcation point $(0,\mu^*_{P,1})$ and the corresponding one-dimensional basis $\varphi^*_{P,1} = \cos(\frac{2\pi}{P}x)$ for $\ker \partial_\varphi F[0,\mu^*_{P,1}]$. To simplify notation, let $(\varphi(t),\mu(t))$ denote the parametric curve $(\varphi_{P,1}(t),\mu_{P,1}(t))$ from Lemma 4.2, emanating from the point $(0,\mu^*_{P,1})$.

Our goal is now to invoke [7, Theorem 9.1.1], which gives conditions for the when the local bifurcation branch $(\varphi(t),\mu(t))$ can be extended to a global solution curve. Precisely, if $\partial_\varphi F[\varphi,\mu]$ is a Fredholm operator of index zero in S, all closed and bounded subsets of S are compact, and $\mu'(t) \not\equiv 0$ in a neighborhood of the bifurcation point, then there exists a continuous global curve which extends the local bifurcation branch. We prove the following propositions.

Proposition 4.4. *The operator $\partial_\varphi F[\varphi,\mu]$ is Fredholm of index zero for every $(\varphi,\mu) \in U$.*

Proof. The proposition is an application of [7, Theorem 2.7.6], which states that if K is a compact bounded and linear operator and T is a homeomorphism on Banach spaces, then $T + K$ is a Fredholm operator of index zero. By (4.4) the operator $\partial_\varphi F[\varphi,\mu]$ is equal to the sum of $(\mu-\varphi)\,\mathrm{id}$, which is a homeomorphism for every φ in U^1, and Λ^{-s}, which is compact on $C^{0,\beta}_{\text{even}}(\mathbb{S}_P)$. This implies that $\partial_\varphi F[\varphi,\mu]$ is a Fredholm operator of index zero. □

Proposition 4.5. *Every closed and bounded subset of S is compact.*

Proof. As in the proof of Lemma 3.4, the equation can be written in the form (3.1). In view of this, if K is a closed and bounded subset of S, then $K^1 = \{\varphi\,;\,(\varphi,\mu)\in K\}$ is a bounded subset of $\mathcal{C}^{\beta+s}_{\text{even}}(\mathbb{S}_P)$. Moreover, by (4.3), then K^1 is relatively compact in $C^{0,\beta}_{\text{even}}(\mathbb{S}_P)$. Since closed and bounded subsets of $\mathbb{R}$ are compact, every sequence in K has a convergent subsequence in the closure of K. But K is closed by assumption. □

Recall that for every even, P-periodic, smooth and real-valued function f, we define its Fourier cosine coefficients as

$$[f]_k = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\cos\big(\frac{2\pi k}{P}x\big)\,dx,$$

so that the Fourier series of f is

$$f(x) = \frac{[f]_0}{2} + \sum_{k=1}^{\infty}[f]_k\cos\big(\frac{2\pi k}{P}x\big).$$

As in (2.16), the action of Λ^{-s} on f is given by

$$\Lambda^{-s}f(x) = \frac{[f]_0}{2} + \sum_{k=1}^{\infty}\big\langle\frac{2\pi k}{P}\big\rangle^{-s}[f]_k\cos\big(\frac{2\pi k}{P}x\big). \tag{4.5}$$

It suffices now to prove that $\mu'(t)\not\equiv 0$ on $(-\varepsilon,\varepsilon)$ in order to extend the local curve from Lemma 4.2 to a global bifurcation curve. To this end, we parametrize $(\varphi(t),\mu(t))$ in such a way that $[\varphi(t)]_1 = t$ (this parametrization corresponds to the Lyapunov–Schmidt reduction). One can check that

$$[\varphi(\cdot + P/2)]_1 = -[\varphi]_1 = -t,$$

by using the identity $\cos(x-\pi) = -\cos(x)$, and the periodicity of φ. This implies that

$$(\varphi(t)(\cdot + P/2),\mu(t)) = (\varphi(-t),\mu(-t))$$

by the local uniqueness of the curve, and consequently $\mu(t) = \mu(-t)$. Since the curve $(\varphi(t),\mu(t))$ is analytic in a neighborhood of $(0,\mu^*_{P,1})$ by Lemma 4.2, we may expand $\varphi(t)$ and $\mu(t)$ in terms of

$$\varphi(t) = \sum_{n=1}^{\infty}\varphi_n t^n, \qquad \mu(t) = \sum_{n=0}^{\infty}\mu_{2n}t^{2n}.$$

Inserting this into the fKdV equation and extracting terms of equal power in t yields

$$\Lambda^{-s}\varphi_1 - \mu_0\varphi_1 = 0, \tag{4.6a}$$

$$\Lambda^{-s}\varphi_2 - \mu_0\varphi_2 = -\frac{1}{2}\varphi_1^2, \tag{4.6b}$$

$$\Lambda^{-s}\varphi_3 - \mu_0\varphi_3 = \mu_2\varphi_1 - \varphi_1\varphi_2, \tag{4.6c}$$

$$\Lambda^{-s}\varphi_4 - \mu_0\varphi_4 = \mu_2\varphi_2 - \varphi_1\varphi_3 - \frac{1}{2}\varphi_2^2, \tag{4.6d}$$

$$\Lambda^{-s}\varphi_5 - \mu_0\varphi_5 = \mu_4\varphi_1 + \mu_2\varphi_3 - \varphi_1\varphi_4 - \varphi_2\varphi_3. \tag{4.6e}$$

Let $m_j := \langle \frac{2\pi j}{P}\rangle^{-s}$ to lighten the notation in the subsequent calculations. Clearly, we have $\mu_0 = \mu_{P,1}^* = m_1$, which due to (4.6a) gives $\varphi_1(x) = \cos(\frac{2\pi}{P}x)$. Expanding φ_2 in a Fourier series, the equation (4.6b) implies that

$$\frac{[\varphi_2]_0}{2}(1-m_1) + \sum_{j=1}^{\infty}(m_j - m_1)[\varphi_2]_j \cos\big(\frac{2\pi j}{P}\big) = -\frac{1}{4}\cos\big(\frac{4\pi}{P}x\big) - \frac{1}{4}. \tag{4.7}$$

Hence,

$$\varphi_2(x) = -\frac{1}{4(1-m_1)} - \frac{1}{4(m_2-m_1)}\cos\big(\frac{4\pi}{P}x\big).$$

Now, the right-hand side of (4.6c) is

$$\left[\mu_2 - \frac{1}{4(m_1-1)} - \frac{1}{8(m_1-m_2)}\right]\cos\big(\frac{2\pi}{P}x\big) - \left[\frac{1}{8(m_1-m_2)}\right]\cos\big(\frac{6\pi}{P}x\big).$$

Since $\cos(\frac{2\pi}{P}x)$ is not in the image of the operator on the left-hand side of (4.6c), we infer that

$$\mu_2 = \frac{1}{4(m_1-1)} + \frac{1}{8(m_1-m_2)}. \tag{4.8}$$

Using the same principles for as in (4.7), one finds

$$\varphi_3(x) = \frac{1}{8(m_2-m_1)(m_3-m_1)}\cos\big(\frac{6\pi}{P}x\big).$$

Thus, the right-hand side of (4.6d) is given by

$$\begin{aligned}
&\frac{1}{32(1-m_1)^2} - \frac{1}{32(1-m_1)(m_1-m_2)} - \frac{1}{64(m_1-m_2)^2}\\
&\quad + \frac{1}{(m_1-m_2)}\left[\frac{1}{32(m_1-m_2)} + \frac{1}{16(m_3-m_1)}\right]\cos\big(\frac{4\pi}{P}x\big)\\
&\quad + \frac{1}{(m_1-m_2)}\left[\frac{1}{16(m_3-m_1)} - \frac{1}{64(m_1-m_2)}\right]\cos\big(\frac{8\pi}{P}x\big),
\end{aligned}$$

and one can check that

$$
\begin{aligned}
\varphi_4(x) = &\frac{1}{(1-m_1)}\left[\frac{1}{32(1-m_1)^2} - \frac{1}{32(1-m_1)(m_1-m_2)} - \frac{1}{64(m_1-m_2)^2}\right] \\
&- \frac{1}{(m_1-m_2)^2}\left[\frac{1}{32(m_1-m_2)} + \frac{1}{16(m_3-m_1)}\right]\cos\big(\frac{4\pi}{P}x\big) \\
&+ \frac{1}{(m_1-m_2)(m_4-m_1)}\left[\frac{1}{16(m_3-m_1)} - \frac{1}{64(m_1-m_2)}\right]\cos\big(\frac{8\pi}{P}x\big).
\end{aligned}
$$

The right-hand side of (4.6e) can not have a term with $\cos(\frac{2\pi}{P}x)$, as this function is not in the image of the operator on the left-hand side. Calculating the $\cos(\frac{2\pi}{P}x)$-part of the right-hand side of (4.6e) implies that

$$
\begin{aligned}
\mu_4 = &\frac{1}{32(1-m_1)^3} - \frac{1}{32(1-m_1)^2(m_1-m_2)} - \frac{1}{64(1-m_1)(m_1-m_2)^2} \\
&- \frac{1}{64(m_1-m_2)^3} - \frac{3}{64(m_1-m_2)^2(m_3-m_1)}
\end{aligned}
\tag{4.9}
$$

For every $s \in (0,1)$, one finds (see Remark 4.6) that μ_2 is nonzero for all but one unique value P_s^* of the period P, and moreover that μ_4 is nonzero for the value P_s^*. Therefore,

$$\mu' \not\equiv 0 \qquad \text{on} \qquad (-\varepsilon, \varepsilon).$$

In view of Proposition 4.4 and Proposition 4.5, the assumptions of [7, Theorem 9.1.1] are now satisfied. That is, the local bifurcation branch can be extended globally. We state the alternatives for the qualitative behavior of this extension given in the theorem.

Remark 4.6. We have not been able to establish analytically the uniqueness of P_s^* or that μ_4 is nonzero for this particular value of P. Numerical calculations supporting our conclusion are included in Appendix A.1.

Lemma 4.7. *The local bifurcation branch $t \mapsto (\varphi(t), \mu(t))$ extends to a global continuous curve $\mathfrak{R} := \{(\varphi(t), \mu(t))\,;\, t \in [0,\infty)\} \subset U$, and one of the following alternatives hold.*

(i) $\|(\varphi(t), \mu(t))\|_{C^{0,\beta}\times\mathbb{R}} \to \infty$ *as* $t \to \infty$*,*

(ii) $\operatorname{dist}(\mathfrak{R}, \partial U) = 0$*,*

(iii) $\mathfrak{R}$ *is a closed loop of finite period. That is, there exists $T > 0$ such that*

$$\mathfrak{R} = \{(\varphi(t), \mu(t))\,;\, 0 \le t \le T\},$$

where $(\varphi(T), \mu(T)) = (0, \mu^*_{P,1})$.

In addition to the above, we mention that [7, Theorem 9.1.1] ensures that $\mathfrak{R}$ has a local analytic re-parametrization at each point. Moreover, if $\mathfrak{R}(t_1) = \mathfrak{R}(t_2)$ for numbers $t_1 \neq t_2$, with

$$\ker \partial_\varphi F[\varphi(t_1), \mu(t_1)] = \{0\},$$

then alternative (iii) of Lemma 4.7 occurs, and $|t_1 - t_2|$ is an integer multiple of T.

We end this section with an observation about Galilean symmetry in the fKdV equation. One can check that if φ is a solution to then fKdV equation with wave-speed μ, then $\varphi + 2(1-\mu)$ solves the equation with wave-speed $2-\mu$. Therefore, the Galilean transformation

$$(\varphi, \mu) \mapsto (\varphi + 2(1-\mu), 2-\mu) \tag{4.10}$$

gives a one-to-one correspondence between solutions to (4.2) with $\mu \in (0,1)$ and solutions with $\mu \in (1,2)$. Precisely, the trivial curve $\{(0,\mu)\,;\, \mu \in (0,1)\}$ maps to the curve of constant solutions $\{(2(\mu-1),\mu)\,;\, \mu \in (1,2)\}$, and the curve of constant solutions $\{(2(\mu-1),\mu)\,;\, \mu \in (0,1)\}$ maps to the trivial curve $\{(0,\mu)\,;\, \mu \in (1,2)\}$. Consequently, the bifurcation points along the trivial curve with $\mu \in (0,1)$ are reflected to the curve of constant solutions $2(\mu-1)$ with $\mu \in (1,2)$, and they must therefore extend to global curves in the same way. Moreover, by Lemma 4.2, the trivial curve on $\mu > 1$ is locally unique, implying that the curve $\{(2(\mu-1),\mu)\,;\, \mu \in (0,1)\}$ must also be locally unique. This symmetry of solutions is illustrated in Figure 2.

4.3 Convergence to a highest traveling wave

Towards invoking [7, Theorem 9.2.2] and the exclusion of alternative (iii) in Lemma 4.7, we define the closed cone $\mathcal{K}$ (in the sense of [7, Definition 9.2.1]) as

$$\mathcal{K} := \{\varphi \in C^{0,\beta}_{\text{even}}(\mathbb{S}_P)\,;\, \varphi \text{ is nondecreasing on } (-P/2, 0)\}.$$

We begin by showing that nonconstant solutions to the fKdV equation which satisfies $\varphi \leq \mu$ and belongs to $\mathcal{K}$ cannot lie on the boundary of $\mathcal{K}$.

Proposition 4.8. *In S^1, every nonconstant function φ in $\mathfrak{R}^1 \cap \mathcal{K}$ belongs to the interior of $\mathcal{K}$.*

Proof. Let φ be a nonconstant solution to the fKdV equation in $\mathfrak{R}^1 \cap \mathcal{K}$, and let $\psi \in S^1$ with

$$\|\varphi - \psi\|_{C^{0,\beta}} < \delta,$$

for some $\delta > 0$. Both φ and ψ are smooth due to Lemma 3.4, and iteration of (3.1) shows

$$\|\varphi - \psi\|_{C^2} < \tilde{\delta}, \tag{4.11}$$

where $\tilde{\delta}$ can be made arbitrarily small at the expense of δ. Moreover, by Lemma 3.3 we have

$$\varphi' < 0 \text{ on } (-P/2, 0), \qquad \varphi''(0) < 0, \qquad \varphi''(\pm P/2) > 0. \tag{4.12}$$

Let a and b be points with $-P/2 < a < b < 0$, and such that

$$\sup_{x \in [a,b]} |\varphi'(x)| > \tilde{\delta}$$

and

$$\begin{aligned} \varphi'' &\geq \tilde{\delta} \qquad \text{on} \qquad (-P/2, a), \\ \varphi'' &\leq -\tilde{\delta} \qquad \text{on} \qquad (b, 0). \end{aligned}$$

This is possible if $\tilde{\delta}$ is sufficiently small, by virtue of (4.12). Then by (4.11), we have $\psi' > 0$ on $[a, b]$, which can be seen from

$$\begin{aligned} \psi'(x) &= \varphi'(x) - (\varphi'(x) - \psi'(x)) \\ &\geq \varphi'(x) - |\psi'(x) - \varphi'(x)| \\ &> \tilde{\delta} - \sup_{x \in [a,b]} |\psi'(x) - \varphi'(x)| > 0. \end{aligned}$$

It can be shown in the same manner that $\psi'' \leq 0$ on $(b, 0)$ and $\psi'' \geq 0$ on $(-P/2, a)$.

We claim that $\psi' \geq 0$ on $(b, 0)$. Assume on the contrary that $\psi'(x) < 0$ on $(b, 0)$. Then $\psi'(0) \leq \psi'(x) < 0$, since the second derivative is nonpositive. But this contradicts the evenness of ψ. With an analogous argument on $(-P/2, a)$, we arrive at $\psi' \geq 0$ on $(-P/2, 0)$. Thus, $\psi \in \mathcal{K}$, and φ belongs to the interior of $\mathcal{K}$. □

Having proved Proposition 4.8, the following corollary is now a direct application of [7, Theorem 9.2.2]. We include here the proof.

Corollary 4.9. *Alternative (iii) in Lemma 4.7 does not occur.*

Proof. We show that $\varphi(t) \in \mathcal{K} \setminus \{0\}$ for all $t > 0$. Assume by contradiction that there exists a finite number

$$\bar{t} := \sup\{t > 0\,;\, \varphi((0, t]) \subset \mathcal{K} \setminus \{0\}\}.$$

Since $\mathcal{K}$ is closed, the function $\varphi(\bar{t})$ belongs to $\mathcal{K}$. We claim that $\varphi(\bar{t})$ is constant. Indeed, if $\varphi(\bar{t})$ is not constant, then by Proposition 4.8 it lies in the interior of the cone $\mathcal{K}$; a contradiction.

We argue that $\varphi(\bar{t}) = 0$. All nonzero constant solutions are on the form $2(\mu - 1)$ with $\mu \in \mathbb{R} \setminus \{1\}$. By Proposition 3.2, we know that $\mathfrak{R}^1$ cannot cross the line $\mu = 1$ without at the same time approaching the zero solution. Moreover, by (4.10) the line $\{2(\mu - 1)\,;\, \mu \in (0, 1)\}$ is locally unique. Hence, $\varphi(\bar{t}) = 0$, and is therefore a local bifurcation point according to Lemma 4.2.

Thus, $\mu(\bar{t}) = \mu^*_{P,k}$ for some $k \in \mathbb{N}$. By the discussion below Lemma 4.7 (which is due to [7, Theorem 9.1.1]), one can choose a local, real-analytic re-parametrization of the curve $\mathfrak{R}$ around $\bar{t}$. We let j be the smallest number in $\mathbb{N}$ such that the j-th derivative of φ in $\bar{t}$ is nonzero. Then, in a neighborhood of $\bar{t}$, one can write

$$\varphi(t) = \frac{d^j \varphi[\bar{t}]}{j!} (t - \bar{t})^j + O(|t - \bar{t}|^{j+1}). \tag{4.13}$$

By definition of $\bar{t}$, one has $\varphi(t) \in \mathcal{K}$ for all $0 \leq t < \bar{t}$, and by (4.13) then

$$(-1)^j d^j \varphi[\bar{t}] \in \mathcal{K} \setminus \{0\}.$$

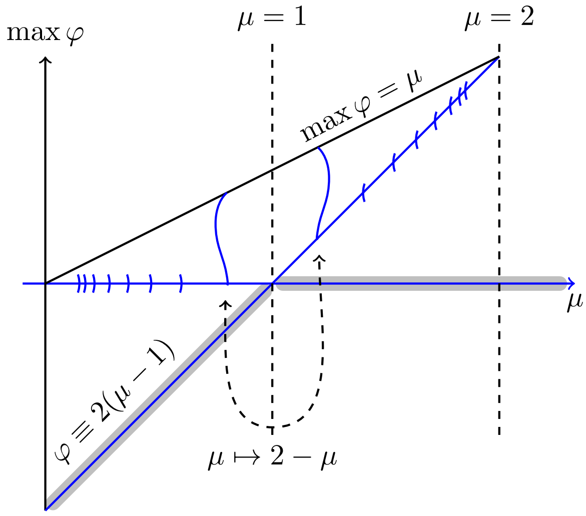


Figure 2: Bifurcation diagram for the FKdV equation (4.2). Local bifurcation curves (cf. Lemma 4.2) are spread along $\{(0, \mu)\,;\, \mu \in (0, 1)\}$ accumulating towards 0, and reflected to $\{(2(\mu - 1), \mu)\,;\, \mu \in (1, 2)\}$ via the transformation (4.10). The lines $\{2(\mu - 1, \mu)\,;\, \mu \in (0, 1)\}$ and $\{(0, \mu)\,;\, \mu > 1\}$ of constant solutions are locally unique. The local bifurcation branches extend to global curves (cf. Lemma 4.7), and a highest traveling-wave solution can be found at the end of the global curve (cf. Theorem 4.12). Note that the qualitative shape of the global bifurcation curve is not determined. The depiction here is in line with the numerical result given in Appendix B.1.

Differentiation j times of the equality $F(\varphi(t), \mu(t)) = 0$ with respect to t in the point $\bar{t}$ yields

$$(-1)^j \partial_\varphi F[0, \mu^*_{P,k}] d^j \varphi[\bar{t}] = 0.$$

We deduce that $(-1)^j d^j$ belongs to both $\ker(D_\varphi F[0, \mu^*_{P,k}])$ and $\mathcal{K} \setminus \{0\}$. Hence,

$$(-1)^j d^j \varphi[\bar{t}] = \tau \cos\big(\frac{2\pi k}{P} x\big),$$

for some $\tau \in \mathbb{R} \setminus \{0\}$ and some $k \in \mathbb{N}$. But since $\cos\big(\frac{2\pi k}{P} x\big)$ is in $\mathcal{K}$ if and only if $k = 1$, and $-\cos\big(\frac{2\pi}{P} x\big) \notin \mathcal{K}$, there is a segment of $\mathfrak{R}$, sufficiently close to $\bar{t}$, parametrized by $t < \bar{t}$, that coincides with the local bifurcation curve emanating from $(0, \mu^*_{P,1})$. However, then there exists countably many pairs $(t_{1,j}, t_{2,j})$ with $t_{1,j} \searrow 0$ and $t_{2,j} \nearrow \bar{t}$ and with $\mathfrak{R}(t_{1,j}) = \mathfrak{R}(t_{2,j})$. This implies that the bifurcation branch is an arbitrarily small loop, contradicting Lemma 4.2. We conclude that $\varphi(t) \in \mathcal{K} \setminus \{0\}$ for all $t > 0$, and consequently that alternative (iii) from Lemma 4.7 does not occur. $\square$

Having excluded the possibility of the global bifurcation curves forming closed loops, we now turn to prove that both alternative (i) and (ii) in Lemma 4.7 must occur simultaneously in the limit. In this direction, we first show that one can find a convergent subsequence along the curve and that it does not tend to the trivial solution with vanishing wave-speed.

Proposition 4.10. *Any sequence of solutions* $(\varphi_n, \mu_n)_{n\in\mathbb{N}} \subset S$ *to the fKdV equation with bounded* $(\mu_n)_{n\in\mathbb{N}}$ *has a subsequence that converges uniformly to a solution* φ.

Proof. It follows directly from the fKdV equation that

$$\|\varphi\|^2_{L^\infty(\mathbb{R})} \leq 2\|\mu\varphi\|_{L^\infty(\mathbb{R})} + 2\|\Lambda^{-s}\varphi\|_{L^\infty(\mathbb{R})} \leq 2(|\mu| + 1)\|\varphi\|_{L^\infty(\mathbb{R})}$$

Thus, $(\varphi_n)_{n\in\mathbb{N}}$ is bounded insofar as $(\mu_n)_{n\in\mathbb{N}}$ is bounded. Furthermore, the sequence $(\Lambda^{-s}\varphi_n)_{n\in\mathbb{N}}$ is uniformly equicontinuous. Indeed, since K_s is integrable and continuous, then

$$\begin{aligned}\left|(\Lambda^{-s}\varphi_n)(x) - (\Lambda^{-s}\varphi_n)(y)\right| &= \left|\int_{\mathbb{R}} (K_s(x-\eta) - K_s(y-\eta))\varphi_n(\eta)\, d\eta\right| \\ &\leq \|\varphi_n\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \left|K_s(x-\eta) - K_s(y-\eta)\right| d\eta,\end{aligned}$$

which tends to zero when $|x-y| \to 0$. Then by the Arzela–Ascoli theorem [5, Theorem 4.25], the sequence $(\Lambda^{-s}\varphi_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence. Finally, owing to (3.1) one obtains the same conclusion for $(\varphi_n)_{n\in\mathbb{N}}$. □

Proposition 4.11. *For fixed period* $P > 0$*, one has*

$$\mu(t) \gtrsim 1$$

uniformly for $t \geq 0$ *along the global bifurcation curve from Lemma 4.7.*

Proof. Towards a contradiction, assume that there exists a sequence of wave-speeds $(\mu_n)_{n\in\mathbb{N}}$ with $\mu_n \to 0$ as $n \to 0$, and that the corresponding sequence $(\varphi_n)_{n\in\mathbb{N}}$ of solutions to the fKdV equation belongs to the global bifurcation branch from Lemma 4.7. Then by Proposition 4.10, there is a uniformly convergent subsequence of $(\varphi_n)_{n\in\mathbb{N}}$, converging to some φ_0, which is also a solution to the equation. But since $\varphi_n \leq \mu_n$ along the bifurcation branch, taking the limit one obtains $\varphi_0 \leq 0$. This means that $\max_x \varphi_0(x) = 0$ by Proposition 3.1, implying that $\varphi_0 \equiv 0$, by the discussion below Proposition 3.1. Using (3.4), we gather that

$$0 = \lim_{n\to\infty} (\mu_n - \varphi_n(P/2)) \gtrsim 1;$$

a contradiction. Hence, the wave-speed cannot not vanish identically along the global bifurcation branch. □

We are now in the position to conclude that a highest traveling-wave solution to the fKdV equation exists at the limit of the global bifurcation branch.

Theorem 4.12. *Both alternative (i) and (ii) in Lemma 4.7 occur. For every unbounded sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers, there exists a subsequence of $(\varphi(t_n), \mu(t_n))_{n\in\mathbb{N}}$ that converges to a solution (φ, μ) to the fKdV equation, with*

$$\varphi(0) = \mu \qquad \text{and} \qquad \varphi \in C^{0,s}(\mathbb{R}).$$

The limiting wave is even, P-periodic, strictly increasing on $(-P/2, 0)$, and is exactly s-Hölder continuous at $x \in P\mathbb{Z}$.

Proof. We show that alternative (i) and (ii) occur simultaneously. Assume first that (i) occurs, that is,

$$\|(\varphi(t), \mu(t))\|_{C^{0,\beta}\times\mathbb{R}} \to \infty$$

when $t \to \infty$. Since $\mu(t)$ is strictly bounded between 0 and 1, this can only happen if $\|\varphi(t)\|_{C^{0,\beta}}$ blows up. Aiming at a contradiction, suppose that there exists $\delta > 0$ with

$$\liminf_{t\to\infty} \inf_{x\in\mathbb{R}} (\mu(t) - \varphi(t)(x)) \geq \delta.$$

Then using (3.2), we have for every $x, y \in \mathbb{R}$ that

$$|\varphi(x) - \varphi(y)| = \frac{2|(\Lambda^{-s}\varphi)(x) - (\Lambda^{-s}\varphi)(y)|}{|2\mu - \varphi(x) - \varphi(y)|} \leq \frac{|(\Lambda^{-s}\varphi)(x) - (\Lambda^{-s}\varphi)(y)|}{\delta}.$$

Starting with bounded φ, iteration of $\Lambda^{-s}\colon L^\infty \to \mathcal{C}^s$ and $\Lambda^{-s}\colon \mathcal{C}^\beta \to \mathcal{C}^{\beta+s}$ yields $\varphi \in C^{0,\alpha}$ for some $\alpha > \beta$. But this contradicts $\|\varphi(t)\|_{C^{0,\beta}} \to \infty$.

Conversely, suppose (ii) occurs. That is, there exists a sequence $(\varphi_n, \mu_n)_{n\in\mathbb{N}}$ with $\varphi_n' \geq 0$ on $(-P/2, 0)$ and $\varphi_n < \mu_n$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} |\mu_n - \varphi_n(0)| = 0.$$

Towards a contradiction, assume that φ_n remains bounded in $C^{0,\beta}(\mathbb{R})$. Taking the limit of a subsequence in $C^{0,\beta'}(\mathbb{R})$ for $s < \beta' < \beta$, the limit must be exactly s-Hölder regular at the crest by (3.7), and we arrive at a contradiction to the boundedness of the sequence in $C^{0,\beta}(\mathbb{R})$.

We conclude that alternative (i) and (ii) from Lemma 4.7 both occur. By Proposition 4.10 there exists a subsequence converging to a highest periodic traveling-wave solution of the fKdV equation, with properties as given in Section 3. □

5 Generalizations of the fKdV equation

In this section, we consider a generalization of the fKdV equation on the form

$$-\mu\varphi + \frac{1}{2}\varphi^2 + L\varphi = 0, \tag{5.1}$$

where L is a Fourier multiplier operator with an inhomogeneous symbol $m(\xi)$, and a corresponding convolution kernel K. Note that taking $L = \Lambda^{-s}$ gives the fKdV equation studied in the previous sections. We discuss characteristic features of solutions to the equation (5.1), and examine conditions on the symbol $m(\xi)$ that promote traveling-wave phenomena similar to that of the fKdV equation proved in Section 3 and Section 4. In Section 5.1, we discuss the balance between dispersion and nonlinear effects in (5.1) and introduce a more general class of Fourier multipliers, which has smoothing properties comparable to the Bessel potential operator Λ^{-s}, and of which Λ^{-s} is a special case. Then, in Section 5.2, we summarize conditions for complete monotonicity of the convolution kernel K.

The material presented here is organized as a discussion around the equation (5.1), based on the framework which has been laid out in the previous sections. We focus on general concepts, and most technical proofs are omitted. Sources are mainly [4], for the theory of Fourier multipliers and Besov spaces, and [11], for completely monotone convolution kernels.

5.1 Fourier multiplier symbol classes

In studies of equations akin to (5.1), one often encounters the concept of balance between dispersion and nonlinear effects. The travelling-wave assumption $u(t,x) = \varphi(x - \mu t)$ in (1.14) restricts solutions to waves of a steady profile. There has to be a certain balance between the terms in the equation for it to allow such solutions, and the properties of the solutions will naturally reflect this balance.

The nonlinear term φ^2 is deregularizing with regards to solutions. As an example, consider the inviscid Burgers equation

$$u_t + uu_x = 0,$$

which integrates, under the traveling-wave assumption, to the two first terms of (5.1). This equation has no nonconstant traveling-wave solutions. On the other hand, as we have seen, adding a smoothing dispersive term $L\varphi$ the equation may admit nonconstant traveling waves.

The principle of this thesis is to fix the order of the nonlinear term in the equation (5.1) and to investigate the relationship between the strength of the dispersion and the regularity of traveling-wave solutions of the equation. We mention that one could also have fixed the dispersion and varied the nonlinearity [12, 15, 24], or let the orders of both be parametrized [35].

Theorem 4.12 shows that when the order of the symbol for the Bessel potential operator is $-s$, then there exist highest traveling-wave solutions to the fKdV equation that are cusped with s-Hölder regularity. In other words, weaker dispersion requires sharper crests of solutions. Let us briefly reflect on this result.

As we have seen, the operator Λ^{-s} is smoothing in the sense that it increases the regularity of functions on which it acts. Arguing heuristically, the smoothing also results in a certain leveling of functions. Indeed, the convolution of a function in a point is the average of the function weighted by the convolution kernel centered in that point. It is this leveling of solutions that counter the effects of the nonlinear term in the equation and permits traveling-wave solutions. But whereas the smoothing effects of Λ^{-s} for a given order $-s$ always increase the regularity of functions by the same amount, the effect of the leveling depends on the shape of the functions themselves. When the crest of a solution is sharper, leveling becomes more prominent, and vice versa. This gives an intuitive explanation for the observed relationship between dispersion and regularity of solutions: When the dispersion is weak, the crests of solutions are sharp, promoting sufficient leveling so that the equation is satisfied.

We ask which operators L in (5.1) have smoothing properties similar to that of Λ^{-s}. In the direction of understanding how Fourier multipliers act on Hölder–Zygmund spaces, we now introduce the more general Besov spaces. A crucial tool in this regard is the Littlewood–Paley decomposition which is a dyadic partition of unity used for spectral decomposition of functions.

Let $\varrho \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp} \varrho \subset [-2, 2]$ and $\varrho(\xi) \equiv 1$ on $|\xi| \leq 1$. Now define

$$\vartheta(\xi) := \varrho(\xi) - \varrho(2\xi)$$

with support on $1/2 \leq |\xi| \leq 2$. We set $\vartheta_0 = \varrho$ and $\vartheta_j(\xi) = \vartheta(2^{-j}\xi)$ for every $j \geq 1$, with support on $2^{j-1} \leq |\xi| \leq 2^{j+1}$. The collection $(\vartheta_j)_{j\in\mathbb{N}_0}$ norm form a partition of unity

$$\sum_{j=0}^{\infty} \vartheta_j(\xi) = 1, \qquad \xi \in \mathbb{R}.$$

A detailed explanation of this construction can be found in [4, pp. 59-61]. In view of the above, every function $f \in \mathcal{S}(\mathbb{R})$ can be written as

$$f = \sum_{j=0}^{\infty} \vartheta_j(D) f,$$

where we have used the Fourier multiplier notation from (2.14). This is called the Littlewood–Paley decomposition of f, and it naturally extends to the class of tempered distributions $\mathcal{S}'(\mathbb{R})$ via duality over $\mathcal{S}(\mathbb{R})$.

Following [4, Definition 2.68], the inhomogeneous Besov space $B^s_{p,q}$, with $s \in \mathbb{R}$ and $p, q \in [1, \infty]$, is defined as the collection of tempered distributions u with

$$\|u\|_{B^s_{p,q}} := \left\| (2^{js} \|\vartheta_j(D)u\|_{L^p})_{j\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} < \infty,$$

where $\|\cdot\|_{\ell^q(\mathbb{Z})}$ denotes the usual ℓ^q-norm of sequences. Although perhaps not immediately evident from the definition, p is an integrability parameter and s is a regularity parameter analogous to those of the usual Sobolev spaces. The parameter q gives additional information about the precise regularity of functions belonging to $B^s_{p,q}$, but we shall not further pursue the interpretation of this here. Important to us is the fact that the Zygmund space $\mathcal{C}^\alpha$ with $\alpha > 0$ coincides (equivalent norms) with the Besov space $B^\alpha_{\infty,\infty}$; see [30, Section 2.6.5] for justification of this, and a comprehensive review of related function spaces.

There is a convenient way of characterizing Fourier multiplier operators which are smoothing on the scale of Besov spaces. Indeed, it can be shown that if a is a smooth, real-valued function which, for some $r \in \mathbb{R}$, satisfies

$$|a(\xi)^{(k)}| \lesssim_k (1 + |\xi|)^{r-k} \tag{5.2}$$

for all $\xi \in \mathbb{R}$ and every $k \in \mathbb{N}_0$, then

$$a(D) : B^s_{p,q} \to B^{s-r}_{p,q}$$

is a linear and bounded operator [4, Proposition 2.78]. The collection of symbols which satisfy (5.2) is denoted by S^r.

Note that the symbol $\langle \xi \rangle^{-s}$ for the Bessel potential operator Λ^{-s} belongs to the symbol class S^{-s}. As such, it is a smoothing operator of order $-s$ in agreement with Proposition 2.6.

Let us now summarize the implications of the above regarding solutions to the equation (5.1). Assume that the symbol $m(\xi)$ of the operator L is inhomogeneous and belongs to some symbol class S^{-r} for $r \in (0, 1)$. Since then L is smoothing on the scale of Hölder–Zygmund spaces, it follows by bootstrapping the equation

$$\varphi = \mu - \sqrt{\mu^2 - 2L\varphi}$$

that all solutions that are strictly bounded from above by μ are smooth. This is the same argument that was used in Lemma 3.4 for the fKdV equation. The equation (5.1) has constant solutions

$$\varphi \equiv 0 \qquad \text{and} \qquad \varphi \equiv 2(\mu - m(0)),$$

and the Frechet derivative of the function

$$F : (\varphi, \mu) \mapsto \mu\varphi - \frac{1}{2}\varphi^2 - L\varphi,$$

which maps $C^{0,\beta}_{\text{even}}(\mathbb{S}_P) \times \mathbb{R}$ to $C^{0,\beta}_{\text{even}}$, is

$$\partial_\varphi F[\varphi, \mu] = (\mu - \varphi)\,\mathrm{id} - L.$$

Due to the compact embedding (4.3), this operator is Fredholm on every subset of $C^{0,\beta}_{\text{even}}(\mathbb{S}_P)$ with $\varphi < \mu$ and it has a one-dimensional kernel precisely when there exists a unique wave-speed μ for which $\mu = m(\frac{2\pi k}{P})$, for some $k \in \mathbb{N}$ and period P. Under the assumption that such a μ^* exists, we can apply [7, Theorem 8.3.1] and obtain local analytic bifurcation curves emanating from the trivial solution curve of (5.1) in the point $(0, \mu^*)$. This generalizes Lemma 4.2 for the fKdV equation, and solutions belonging to this curve are smooth.

Note that a sufficient condition for a one-dimensional kernel is that the symbol $m(\xi)$ is monotone. We mention that it is possible to bifurcate solutions even in the case of higher-dimensional kernels, and done in [14] for the capillary-gravity Whitham equation.

In the construction of a global solution curve which extends the local bifurcation branch, it was used in Lemma 4.7 that $\mu'(t) \not\equiv 0$ in a neighborhood around the point from which the local curve $(\varphi(t), \mu(t))$ emanates. The formulas (4.8) and (4.9) for the coefficients of the series expansion of $\mu(t)$ hold for general symbols in S^{-r}, and if they can be shown to be nonzero, a global bifurcation result holds in this case as well. Indeed, by [7, Theorem 9.1.1], the only condition left to verify is that every closed and bounded subset of S is compact, but this follows immediately by the proof Proposition 4.5 using the smoothing properties of L.

5.2 Properties of the convolution kernel

In the exclusion of the global bifurcation curve for the fKdV equation being a closed loop we used information about the signs of the first and second derivatives of solutions. This again depended on information about the convolution kernel of the operator present in the equation. We now investigate what can be said about the convolution kernel of more general smoothing operators.

First, note that whenever $m(\xi)$ belongs to some symbol class S^{-r}, then the convolution kernel of the corresponding Fourier multiplier operator is smooth outside of the origin [28, Proposition 2.1]. Indeed, let K be the convolution kernel associated with $m(\xi)$, given by

$$K(x) = \frac{1}{2\pi} \int_{\mathbb{R}} m(\xi) e^{ix\xi} \, d\xi \, .$$

Then, for every $j \in \mathbb{N}_0$ we can pick $k \in \mathbb{N}_0$ with $k > j - s + 1$ so that taking j derivatives with respect to x of the identity

$$x^k K(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} D_\xi^k m(\xi) \, d\xi,$$

and using the property (5.2) of $m(\xi)$, yields an absolutely convergent integral. Hence, $K \in C^\infty(\mathbb{R} \setminus \{0\})$.

Assume now that $m(\xi)$ is even and

$$\lim_{\lambda \searrow 0} g(\lambda) < \infty \qquad \text{and} \qquad \lim_{\lambda \to \infty} g(\lambda) = 0$$

holds for the function $g = m(\sqrt{\cdot})$, and that g has an analytic extension to $\mathbb{C} \setminus (-\infty, 0]$ with

$$\mathrm{Im}(z) \, \mathrm{Im}(g(z)) \leq 0.$$

Then, by the theory laid out in [11, Section 2], and more precisely by the argument of Proposition 2.3, the convolution kernel K associated with $m(\xi)$ is a completely monotone function.

In particular, this implies that both K, and K_P as defined as in (2.10), are strictly positive and strictly increasing for negative x. Then Lemma 2.5 holds also in this more general case. On the grounds of this, it can be shown that a nodal characterization of periodic solutions to the equation (5.1) holds, which means that we are also able to exclude loops in the global bifurcation branch, as was done for the fKdV equation in Corollary 4.9.

Remark 5.1. It would be sufficient to have nontrivial 2-monotonicity of the kernel, that is, the kernel is strictly positive, has a strictly negative derivative, and is strictly convex on $(0, \infty)$. However, there is to our knowledge no straightforward way to identify symbols of 2-monotone kernels. We mention that a characterization of the Laplace transform is known; see [34, Theorem 10].

We end this discussion with some remarks on the regularity and decay of the kernel K. Firstly, if $m(\xi)$ is analytic on a strip containing the real axis, then then the classical Paley–Wiener theorem [19, Chapter 7] applies, and one can conclude that K has exponential decay. Secondly, it is possible to estimate the singularity in the origin of the kernel. Indeed, due to [28, Propositon 2.2] we have

$$|K(x)| \lesssim |x|^{r-1}, \tag{5.3}$$

whenever $m(\xi)$ belongs to the symbol class S^{-r} with $r \in (0, 1)$. In order to obtain the precise regularity estimates for solutions similar to that of Lemma 3.6 and Theorem 3.8, and thereby to ensure the convergence of the global bifurcation branch to a highest wave, it would, in view of Proposition 2.1 and (5.3), be a sufficient condition that the symbol can be written as

$$m(\xi) = C\langle\xi\rangle^{-r} + \tilde{m}(\xi)$$

where C is a positive constant and $\tilde{m}(\xi)$ is a symbol which belongs to a symbol class $S^{-r+\varepsilon}$ for some $\varepsilon > 0$.

6 Traveling-wave solutions to the fDP equation

We now turn our attention to the fDP equation (1.15). Recall that it is given in (1.18) as

$$-\mu\varphi + \frac{1}{2}\varphi^2 + \frac{3}{2}\Lambda^{-s}\varphi^2 = \kappa,$$

for $s \in (0, 1)$ and $\kappa \in \mathbb{R}$. Compared to the fKdV equation, the main differences are that the nonlocal term is nonlinear and that we work with an arbitrary real constant κ on the right-hand side.

This section is a study of the regularity of traveling-wave solutions to the fDP equation. In Section 6.1, we give ranges for the parameter κ that permit nonconstant periodic solutions and show that such solutions must always cross the largest of the constant solutions to the equation. In Section 6.2, we prove that all solutions which are strictly smaller than μ are smooth, and that nonnegative solutions that attain the maximal height are exactly s-Hölder regular at the crest.

We mainly follow the framework that was used for the fKdV equation above. Details are sometimes omitted when proofs are the same as for the fKdV equation. The parameter $s \in (0, 1)$ is held fixed and the fDP equation refers to the equation (1.18) with this value of s.

6.1 Periodic traveling waves

We begin with an investigation of which role the integration constant κ on the right-hand side of the fDP equation plays regarding the existence of solutions. The following proposition and subsequent lemma are based on [3, Theorem 3.1]. Note that if φ solves the fDP equation with wave-speed μ, then $-\varphi(-x)$ is a solution to the equation with $-\mu$. Hence, it suffices to consider $\mu > 0$.

Proposition 6.1. *Let $\mu > 0$ and $P < \infty$. Then for the fDP equation with a parameter $\kappa \in \mathbb{R}$,*

(i) if $\kappa \leq 0$, all solutions are nonnegative, and for $\kappa < -\frac{\mu^2}{8}$ there are no real solutions,

(ii) if $\kappa \geq \mu^2$, there are no nonconstant P-periodic solutions.

Proof. (i) Let $\kappa \leq 0$. Then the left-hand side of (1.18) is also nonpositive, which is possible only if φ is nonnegative. If $\kappa = 0$, then $\varphi \equiv 0$ is a valid solution, otherwise φ is strictly negative by the monotonicity of Λ^{-s} (Corollary 2.2). Writing the equation as

$$(\mu - \varphi)^2 = 2\kappa + \mu^2 - 3\Lambda^{-s}\varphi^2, \tag{6.1}$$

and using $\min \varphi^2 \leq \Lambda^{-s}\varphi^2$, we obtain

$$(\mu - \varphi)^2 \leq 2\kappa + \mu^2 - 3\min(\varphi^2).$$

In particular,

$$\min \varphi(\min \varphi - \mu) \leq \kappa, \tag{6.2}$$

where we have used $\min(\varphi^2) = (\min \varphi)^2$, in view of φ being nonnegative. This equation has no real solutions when $\kappa < -\frac{\mu^2}{8}$, and when $\kappa = -\frac{\mu^2}{8}$ it has only the constant solution $\varphi \equiv \frac{\mu}{4}$.

(ii) First, we claim that

$$\max \varphi > |\min \varphi| \tag{6.3}$$

for P-periodic solutions whenever $\kappa > 0$. In the direction of a contradiction, assume that $\max \varphi \leq |\min \varphi|$. If φ is to take negative values on some intervals, it is smooth and bounded there (cf. Lemma 6.4). So there exists x_0 with $\varphi(x_0) = \min \varphi$. Observe now that the function $-\mu\varphi + \frac{1}{2}\varphi^2$ attains its maximum in x_0, since φ is minimal there, and $\max \varphi$ does not exceed $|\min \varphi|$. But as φ is a solution to the fDP equation with $\kappa > 0$, we deduce that $\Lambda^{-s}\varphi^2$ is minimized in x_0. Consequently, $\Lambda^{-s}\varphi^2$ is minimized in x_0 while at the same time φ^2 is maximized in x_0; a contradiction.

Let $\kappa \geq \mu^2$. Arguing as for (6.2), we obtain

$$\max \varphi(\max \varphi - \mu) \geq \kappa,$$

where we have used $\max(\varphi^2) = (\max \varphi)^2$ as a consequence of (6.3). If $\kappa = \mu^2$, then $\varphi \equiv \mu$ is a valid solution, otherwise φ must take values above μ. If $\varphi \geq \mu$ is a nonconstant solution, then the left-hand side of (6.1) attains its minimum in the minimum of φ, while the right-hand side attains its minimum where $\Lambda^{-s}\varphi^2$ is maximal. But φ^2 cannot have a minimum where $\Lambda^{-s}\varphi^2$ is maximal, meaning that such solutions do not exist. If on the other hand φ takes values both above and below μ, then for each interval on which φ is strictly larger than μ, the term $\Lambda^{-s}\varphi^2$ is minimal where φ^2 is maximal. As before, this is not possible. $\square$

We assume from now on that

$$-\frac{\mu^2}{8} < \kappa < \mu^2, \tag{6.4}$$

such that nonconstant periodic solutions to the fDP equation may exist. Such solutions intersect the value of the largest constant solution of the equation, as the following lemma shows.

Lemma 6.2. *Let $\mu > 0$ and $P < \infty$. Then every nonconstant and P-periodic solution φ to the fDP equation satisfies*

$$\min \varphi < \frac{\mu + \sqrt{\mu^2 + 8\kappa}}{4} < \max \varphi. \tag{6.5}$$

Proof. First, observe that the fDP equation, with parameters μ and κ satisfying (6.4), has exactly two constant solutions

$$\frac{\mu \pm \sqrt{\mu^2 + 8\kappa}}{4}. \tag{6.6}$$

Let φ be a solution satisfying the assumptions above. Since

$$\begin{aligned} \Lambda^{-s}\varphi^2 &> \Lambda^{-s}(\min(\varphi^2)) = \min(\varphi^2), \\ \Lambda^{-s}\varphi^2 &< \Lambda^{-s}(\max(\varphi^2)) = \max(\varphi^2), \end{aligned}$$

there exist both points in which $\varphi^2 > \Lambda^{-s}\varphi^2$ and points in which $\varphi^2 < \Lambda^{-s}\varphi^2$. Hence, the two terms must cross at some point; there exists x_0 such that $\varphi^2(x_0) = \Lambda^{-s}(\varphi^2)(x_0)$. Inserting this in the fDP equation one sees that $\varphi(x_0)$ takes the value of one of the constant solutions (6.6) of the equation. From the proof of Proposition 6.1, we know that $\max \varphi > |\min \varphi|$ for every choice of κ. This implies that the maximum of φ^2 is attained in the maximum of φ. Consequently, $\varphi(x_0)$ takes the value of the largest constant solution, and (6.5) holds. $\square$

We now prove a lemma concerning the nodal properties of solutions to the fDP equation. Note that nonnegativity of solutions is assumed: this is further commented on in Section 7.2.

Lemma 6.3. *Every P-periodic, nonconstant, nonnegative and even solution $\varphi \in C^1(\mathbb{R})$ to the fDP equation which is nondecreasing on $(-P/2, 0)$ satisfies*

$$\varphi' > 0 \qquad \text{and} \qquad \varphi < \mu$$

on $(-P/2, 0)$. If in addition $\varphi \in C^2(\mathbb{R})$, then

$$\varphi''(0) < 0 \qquad \text{and} \qquad \varphi''(\pm P/2) > 0.$$

Proof. By assumption, the solution φ' is odd, nontrivial and nonnegative on $(-P/2, 0)$. Then, since φ is even, nonconstant and nonnegative, the product $\varphi\varphi'$ satisfies the assumptions of Lemma 2.5. Differentiating the equation gives

$$(\mu - \varphi)\varphi' = 3\Lambda^{-s}(\varphi\varphi') > 0$$

on $(-P/2, 0)$, and we conclude that $\varphi' > 0$ and $\varphi < \mu$ on $(-P/2, 0)$.

Now assume that $\varphi \in C^2(\mathbb{R})$. Differentiating twice we get

$$(\mu - \varphi)\varphi'' = 3\Lambda^{-s}[(\varphi\varphi')'] + (\varphi')^2.$$

Evaluating this equation in $x = 0$ yields

$$(\mu - \varphi(0))\varphi''(0) = 6\int_0^{P/2} K_{P,s}(y)[\varphi(y)\varphi'(y)]'\,dy,$$

For some $\varepsilon > 0$, splitting the integral and using integration by parts, we get

$$\begin{aligned}\int_0^{P/2} K_{P,s}(y)[\varphi(y)\varphi'(y)]'\,dy &= \left(\int_0^{\varepsilon} + \int_{\varepsilon}^{P/2}\right) K_{P,s}(y)[\varphi(y)\varphi'(y)]'\,dy \\ &= \int_0^{\varepsilon} K_{P,s}(y)[\varphi(y)\varphi'(y)]'\,dy + \Big[K_{P,s}(y)\varphi(y)\varphi'(y)\Big]_{y=\varepsilon}^{P/2} \\ &\quad - \int_{\varepsilon}^{P/2} K'_{P,s}(y)\varphi(y)\varphi'(y)\,dy\,.\end{aligned}$$

The first term vanishes,

$$\lim_{\varepsilon\searrow 0}\Big|\int_0^{\varepsilon} K_{P,s}(y)[\varphi(y)\varphi'(y)]'\,dy\Big| \lesssim \lim_{\varepsilon\searrow 0}\int_0^{\varepsilon} |y|^{s-1}\,dy = 0,$$

due to $\varphi \in C^2$ and (2.11) for the period kernel. The second also term vanishes since $\varphi'(P/2) = 0$, and since $\varphi'(\varepsilon) \lesssim \varepsilon$ due to $\varphi'(0) = 0$. The last term is negative for each $\varepsilon > 0$, since we have proved both $\varphi' < 0$ and $K'_{P,s} < 0$ on $(-P/2, 0)$, and φ is nonnegative. Moreover, it is decreasing as $\varepsilon \searrow 0$, so passing to the limit we obtain

$$(\mu - \varphi(0))\varphi''(0) = -6\lim_{\varepsilon\searrow 0}\int_{\varepsilon}^{P/2} K'_{P,s}(y)\varphi(y)\varphi'(y)\,dy < 0.$$

This implies $\varphi''(0) < 0$. The inequalities $\varphi''(\pm P/2) > 0$ can be proved in the same way as in Lemma 3.3, using the minor modifications above to deal with the nonlinear term $\Lambda^{-s}\varphi^2$. □

6.2 Regularity of traveling waves

The following lemma shows that solutions $\varphi < \mu$ to the fDP equation are smooth. This parallels Lemma 3.4 for the fKdV equation and relies on the same bootstrapping procedure set in the scale of Hölder-Zygmund spaces.

Lemma 6.4. *Assume that $\varphi \leq \mu$ is a solution to the fDP equation. Then φ is smooth on every open set where $\varphi < \mu$.*

Proof. The fDP equation can be written as

$$\varphi = \mu - \sqrt{\mu^2 + 2\kappa - 3\Lambda^{-s}\varphi^2}. \tag{6.7}$$

If $\varphi < \mu$ uniformly on $\mathbb{R}$ then $\mu^2 + 2\kappa > 3\Lambda^{-s}\varphi^2$. Therefore, the mapping (6.7) is continuous from L^∞ to $\mathcal{C}^s$ and from $\mathcal{C}^\alpha$ to $\mathcal{C}^{\alpha+s}$. Iteration of this map yields smoothness of φ. Proceeding as in Lemma 3.4 for an open interval U on which $\varphi < \mu$ completes the proof. □

Traveling-wave solutions to the fDP equation have similar features as the waves for the fKdV equation. Solutions which are strictly smaller than μ are smooth, but the smoothness may break down when the wave-height approaches μ. To show that a loss of regularity occurs for highest waves and to prove the exact regularity at the crest, we have to deal with the nonlinear term $\Lambda^{-s}\varphi^2$. To this end, we assume that solutions are nonnegative.

Proposition 6.5. *Let φ be an even, P-periodic, nonconstant and nonnegative solution to the fDP equation that is nondecreasing on $(-P/2, 0)$ with $\varphi \leq \mu$. Then φ is strictly increasing on $(-P/2, 0)$.*

Proof. Taking the difference of the fDP equation evaluated in two points x and y, one obtains

$$(2\mu - \varphi(x) - \varphi(y))(\varphi(x) - \varphi(y)) = 3\big((\Lambda^{-s}\varphi^2)(x) - (\Lambda^{-s}\varphi^2)(y)\big). \tag{6.8}$$

In the same way as in (3.3), it can be shown that

$$\begin{aligned} &(\Lambda^{-s}\varphi^2)(x+h) - (\Lambda^{-s}\varphi^2)(x-h) \\ &= \int_{-P/2}^{0} (K_{P,s}(x-y) - K_{P,s}(x+y))(\varphi^2(y+h) - \varphi^2(y-h))\, dy\,. \end{aligned} \tag{6.9}$$

Assume that $x \in (-P/2, 0)$ and $0 < h \ll 1$. Then integrand in (6.9) is nonnegative and positive on some set of nonzero measure. Indeed, the difference of the periodic kernels is positive as shown in the proof of Lemma 2.5, and due to φ being nonnegative and continuous it suffices to observe that both factors in

$$\varphi^2(y+h) - \varphi^2(y-h) = (\varphi(y+h) + \varphi(y-h))(\varphi(y+h) - \varphi(y-h))$$

must take positive values at the same time on some interval of nonzero measure. Thus,

$$(2\mu - \varphi(x+h) - \varphi(x-h))(\varphi(x+h) - \varphi(x-h)) > 0$$

for every $x \in (-P/2, 0)$, which implies that φ is strictly increasing on $(-P/2, 0)$. □

Proposition 6.5 implies that solutions satisfying the assumptions are smooth on $\mathbb{S}_P \setminus \{0\}$. The next lemma provides an upper bound for the regularity in the origin for solutions that are permitted to achieve the maximal wave-height of μ. This corresponds to Lemma 3.6 for the fKdV equation.

Lemma 6.6. *Let $P < \infty$, and let φ be an even, P-periodic, nonnegative and nonconstant solution to the fDP equation that is nondecreasing on $(-P/2, 0)$ with $\varphi \leq \mu$. Then*

$$\mu - \varphi(P/2) \gtrsim_P 1. \tag{6.10}$$

Moreover, there exists $\varepsilon > 0$ such that

$$\mu - \varphi(x) \gtrsim_P |x|^s \tag{6.11}$$

uniformly for $|x| < \varepsilon$.

Remark 6.7. As in Section 3, we state the lemma for the periodic case and remark that the estimate is uniform in P for large periods, so that the exact regularity at the crest given in Theorem 6.8 holds both in the periodic and in the solitary case.

Proof. We have the identity

$$\begin{aligned}
&(\Lambda^{-s}\varphi^2)(x+h) - (\Lambda^{-s}\varphi^2)(x-h) \\
&= \int_{-P/2}^{0} (K_{P,s}(x-y) - K_{P,s}(x+y))(\varphi^2(y+h) - \varphi^2(y-h))\, dy
\end{aligned}$$

for every $h \in (0, P/2)$. Let $x \in (-P/2, 0)$, and note that the integrand in the above is nonnegative. Differentiating the fDP equation now yields

$$\begin{aligned}
&(\mu - \varphi(x))\varphi'(x) = 3(\Lambda^{-s}\varphi^2)'(x) \\
&= 3 \lim_{h\to 0} \frac{((\Lambda^{-s}\varphi^2)(x+h) - (\Lambda^{-s}\varphi^2)(x-h))}{2h} \\
&\geq \liminf_{h\to 0} \frac{3}{2h} \int_{-P/2}^{0} (K_{P,s}(x-y) - K_{P,s}(x+y))(\varphi^2(y+h) - \varphi^2(y-h))\, dy \\
&\geq 6 \int_{-P/2}^{0} (K_{P,s}(x-y) - K_{P,s}(x+y))\varphi(y)\varphi'(y)\, dy,
\end{aligned}$$

where we appeal to Fatou's lemma in the last step. Fixing $x_0 \in (-P/2, 0)$ and letting $x \in [x_0, 0)$, one has for $z \in [-P/2, x]$ that

$$\begin{aligned}
(\mu - \varphi(z))\varphi'(x) &\geq (\mu - \varphi(x))\varphi'(x) \\
&\geq 6 \int_{-P/2}^{0} (K_{P,s}(x-y) - K_{P,s}(x+y))\varphi(y)\varphi'(y)\, dy \\
&\geq 6 \int_{x_0/2}^{x_0/4} (K_{P,s}(x-y) - K_{P,s}(x+y))\varphi(y)\varphi'(y)\, dy \\
&\gtrsim \int_{x_0/2}^{x_0/4} (K_{P,s}(x-y) - K_{P,s}(x+y))\varphi'(y)\, dy
\end{aligned}$$

where in the last step we have used that φ is strictly positive on $(x_0/2, x_0/4)$ due to φ being nonnegative and strictly increasing on $(-P/2, 0)$. The estimates (6.10) and (6.11) now follow by identical arguments to those in the proof of Lemma 3.6. □

Given the upper bound (6.11) for the regularity of solutions to the fDP equation with $\varphi(0) = \mu$ and satisfying the assumptions of Lemma 6.6, the question is now whether this bound is optimal. It turns out that it is and that solutions belong to $C^{0,s}(\mathbb{R})$. Although the proof of the following theorem is similar to the proof of Theorem 3.8, we write out most details for the sake of clarity.

Theorem 6.8. *Let $\varphi \leq \mu$ be an even and nonconstant solution to the fKdV equation that is nonnegative and nondecreasing on $(-P/2, 0)$ with $\varphi(0) = \mu$. Then $\varphi \in C^{0,s}(\mathbb{R})$. Moreover,*

$$\mu - \varphi(x) \eqsim |x|^s \tag{6.12}$$

uniformly for $|x| \ll 1$.

Proof. The assumption $\varphi(0) = \mu$ means that (6.8) can be rewritten to

$$\begin{aligned}(\mu - \varphi(x))^2 &= 3(\Lambda^{-s}\varphi^2)(0) - 3(\Lambda^{-s}\varphi^2)(x)\\ &= \frac{3}{2}\int_{\mathbb{R}} (K_s(x+y) + K_s(x-y) - 2K_s(y))(\varphi^2(0) - \varphi^2(y))\, dy\,.\end{aligned}$$

For the second factor in the integrand, note that

$$|\varphi^2(x) - \varphi^2(y)| \leq 2\|\varphi\|_{L^\infty}|\varphi(x) - \varphi(y)|,$$

so that for bounded φ we have

$$\begin{aligned}(\mu - \varphi(x))^2 &\lesssim \int_{\mathbb{R}} \big|K_s(x+y) + K_s(x-y) - 2K_s(y)\big|(\varphi(0) - \varphi(y))\, dy\\ &\lesssim \int_{\mathbb{R}} \big||x+y|^{s-1} + |x-y|^{s-1} - 2|y|^{s-1}\big|(\varphi(0) - \varphi(y))\, dy\\ &\quad + \int_{\mathbb{R}} \big|\widetilde{K}_s(x+y) + \widetilde{K}_s(x-y) - 2\widetilde{K}_s(y)\big|(\varphi(0) - \varphi(y))\, dy\\ &\lesssim |x|^s \int_{\mathbb{R}} \big||1+t|^{s-1} + |1-t|^{s-1} - 2|t|^{s-1}\big|\, dt\\ &\quad + |x|^2 \int_{\mathbb{R}} R_x^2(y)\, dy\\ &\lesssim |x|^s.\end{aligned} \tag{6.13}$$

Consequently, φ is at least $\frac{s}{2}$-Hölder regular in 0. Inserting $|\mu - \varphi(y)| \lesssim |y|^{\frac{s}{2}}$ in (6.13) yields $\frac{s/2+s}{2}$-Hölder regularity of φ in 0, and this procedure may be iterated to prove that φ is α-Hölder regular in 0 for every $\alpha < s$.

We now show s-Hölder regularity in $x = 0$. From the proof of Theorem 3.8 we know that there exists a constant C, independent of α, such that

$$\int_{\mathbb{R}} \big|K_s(x+y) + K_s(x-y) - 2K_s(y)\big||y|^\alpha \, dy \leq C|x|^{2\alpha}$$

for all $|x| \leq 1$ and all $0 \leq \alpha \leq s$. Hence,

$$\begin{aligned}(\varphi(0) - \varphi(x))^2 &= \frac{3}{2}\int_{\mathbb{R}} \big(K_s(x+y) + K_s(x-y) - 2K_s(y)\big)(\varphi^2(0) - \varphi^2(y))\, dy \\ &\lesssim \|\varphi\|_{L^\infty}[\varphi]_{C_0^{0,\alpha}} \int_{\mathbb{R}} \big|K_s(x+y) + K_s(x-y) - 2K_s(y)\big||y|^\alpha \, dy \\ &\lesssim [\varphi]_{C_0^{0,\alpha}}|x|^{2\alpha},\end{aligned}$$

where it was used that φ is α-Hölder continuous in the origin for every $\alpha \in [0, s)$ as shown above. Dividing by $|x|^{2\alpha}$ and passing to supremum yields

$$[\varphi]_{C_0^{0,\alpha}} \lesssim 1$$

uniformly over $\alpha \in [0, s)$. We let $\alpha \nearrow s$, and obtain the estimate (6.12).

We claim that $\varphi \in C^{0,\alpha}(\mathbb{R})$ for every $\alpha < s$. In view of the assumptions of the theorem and the smoothness of φ outside of the origin, it suffices to prove

$$\sup_{0<h<|x|<\delta} \frac{|\varphi(x+h) - \varphi(x-h)|}{h^\alpha} < \infty$$

for some $\delta > 0$, as shown in the proof of Theorem 3.8. So assume that $0 < h < |x| < \delta$ for some $\delta \ll 1$, where we let x be positive without loss of generality. We have

$$\begin{aligned}&(\varphi(x+h) - \varphi(x-h))^2 \\ &\leq \big|(2\mu - \varphi(x+h) - \varphi(x-h))(\varphi(x+h) - \varphi(x-h))\big| \\ &= 3\big|(\Lambda^{-s}\varphi^2)(x+h) - (\Lambda^{-s}\varphi^2)(x-h)\big|.\end{aligned} \tag{6.14}$$

The Hölder and Zygmund spaces are Banach algebras, and Λ^{-s} maps L^∞ to $C^{0,s}$ and $\mathcal{C}^\alpha$ to $\mathcal{C}^{\alpha+s}$. Thus, φ is at least α-Hölder regular for every $\alpha < s$ if $s \leq 1/2$ and $\alpha = 1/2$ if $s > 1/2$. So assume that $s > 1/2$ and that $\varphi \in C^{0,\alpha}$ with $\alpha + s > 1$. Then we have

$$\big|(\Lambda^{-s}\varphi^2)(x+h) - (\Lambda^{-s}\varphi^2)(x-h)\big| \lesssim h|\xi|^{\{\alpha+s\}},$$

with $\xi \in (x-h, x+h)$ due to (3.14). Inserting this in (6.14) yields

$$\begin{aligned}|\varphi(x+h) - \varphi(x-h)| &\lesssim \frac{h|\xi|^{\{\alpha+s\}}}{2\mu - \varphi(x+h) - \varphi(x-h)} \\ &\lesssim \frac{h|x+h|^{\{\alpha+s\}}}{|x+h|^s + |x-h|^s} \\ &\lesssim h|x+h|^{\alpha-1}\end{aligned} \tag{6.15}$$

where we have used the estimate (6.11) from Lemma 6.6 in the second step, in view of $h < x < \delta$ with δ at our disposal. Interpolation between (6.15) and the exact s-Hölder regularity in the origin over a parameter $\eta \in (0,1)$ yields

$$\begin{aligned}\frac{|\varphi(x+h)-\varphi(x-h)|}{h^\eta} &\leq \frac{|\varphi(x+h)-\varphi(x-h)|^\eta}{h^\eta}|\mu-\varphi(x+h)|^{1-\eta}\\ &\lesssim |x+h|^{\eta(\alpha-1)+(1-\eta)s},\end{aligned}$$

and choosing $\eta = \frac{s}{1+s-\alpha}$, this is bounded. Hence,

$$|\varphi(x+h)-\varphi(x-h)| \lesssim h^{\frac{s}{1+s-\alpha}},$$

and the argument may be iterated to obtain $\varphi \in C^{0,\alpha}(\mathbb{R})$ for every $\alpha < s$.

We now show that $\varphi \in C^{0,s}(\mathbb{R})$. To this end, note that

$$(\Lambda^{-s}\varphi^2)(x+h)-(\Lambda^{-s}\varphi^2)(x-h) = \int_{-\infty}^0 (K_s(y+h)-K_s(y-h))(\varphi^2(y-x)-\varphi^2(y+x))\,dy,$$

so that

$$\begin{aligned}&(\mu-\varphi(x))|\varphi(x+h)-\varphi(x-h)|\\ &\leq \big|(2\mu-\varphi(x+h)-\varphi(x-h))(\varphi(x+h)-\varphi(x-h))\big|\\ &= 3\big|(\Lambda^{-s}\varphi^2)(x+h)-(\Lambda^{-s}\varphi^2)(x-h)\big|\\ &\leq 3\int_{-\infty}^0 |K_s(y+h)-K_s(y-h)||\varphi^2(y-x)-\varphi^2(y+x)|\,dy\\ &\leq 6\|\varphi\|_{L^\infty}\int_{-\infty}^0 |K_s(y+h)-K_s(y-h)||\varphi(y-x)-\varphi(y+x)|\,dy\,.\end{aligned} \tag{6.16}$$

Let $0 < h < |x| < \delta$ for some $\delta \ll 1$. For the factor $|\varphi(y-x)-\varphi(y+x)|$ in the last line, we interpolate between the global α-Hölder regularity ($\alpha < s$)

$$|\varphi(y-x)-\varphi(y+x)| \lesssim \|\varphi\|_{C^{0,\alpha}}\min(|x|^\alpha,|y|^\alpha)$$

and the sharp s-Hölder regularity in the origin

$$|\varphi(y-x)-\varphi(y+x)| \lesssim [\varphi]_{C_0^{0,s}}\max(|x|^s,|y|^s).$$

This yields

$$|\varphi(y-x)-\varphi(y+x)| \lesssim \|\varphi\|_{C^{0,\alpha}}^\eta \min(|x|,|y|)^{\alpha\eta}\max(|x|,|y|)^{s(1-\eta)}, \tag{6.17}$$

with $(\alpha,\eta)\in(0,s)\times[0,1]$. Inserting (6.17) in (6.16), we have

$$\begin{aligned}
&(\mu-\varphi(x))|\varphi(x+h)-\varphi(x-h)| \\
&\lesssim \|\varphi\|_{C^{0,\alpha}}^{\eta}\int_{-\infty}^{0}|K_s(y+h)-K_s(y-h)|\min(|x|,|y|)^{\alpha\eta}\max(|x|,|y|)^{s(1-\eta)}\,dy \\
&\lesssim \|\varphi\|_{C^{0,\alpha}}^{\eta}|x|^{\alpha\eta}\int_{-\infty}^{-|x|}\big||y+h|^{s-1}-|y-h|^{s-1}\big||y|^{s(1-\eta)}\,dy \\
&\quad+\|\varphi\|_{C^{0,\alpha}}^{\eta}|x|^{s(1-\eta)}\int_{-|x|}^{0}\big||y+h|^{s-1}-|y-h|^{s-1}\big||y|^{\alpha\eta}\,dy \\
&\quad+\|\varphi\|_{C^{0,\alpha}}^{\eta}h|x|^{\alpha\eta}\int_{-\infty}^{-|x|}R_h^1(y)|y|^{s(1-\eta)}\,dy \\
&\quad+\|\varphi\|_{C^{0,\alpha}}^{\eta}h|x|^{s(1-\eta)}\int_{-|x|}^{0}R_h^1(y)|y|^{\alpha\eta}\,dy,
\end{aligned}$$

where we have split the kernel according to (2.17). Choosing the interpolation parameter η such that

$$\eta>2-\frac{1}{s}, \tag{6.18}$$

we obtain

$$\begin{aligned}
&(\mu-\varphi(x))|\varphi(x+h)-\varphi(x-h)| \\
&\lesssim \|\varphi\|_{C^{0,\alpha}}^{\eta}|x|^{\alpha\eta}h^{s+s(1-\eta)}\int_{-\infty}^{0}\big||t+1|^{s-1}-|t-1|^{s-1}\big||t|^{s(1-\eta)}\,dt \\
&\quad+\|\varphi\|_{C^{0,\alpha}}^{\eta}|x|^{s(1-\eta)}h^{s+\alpha\eta}\int_{-\delta}^{0}\big||t+1|^{s-1}-|t-1|^{s-1}\big||t|^{\alpha\eta}\,dt \\
&\quad+\|\varphi\|_{C^{0,\alpha}}^{\eta}|x|^{\alpha\eta}h^{1+s(1-\eta)}\int_{-\infty}^{0}R_h^1(th)|t|^{s(1-\eta)}\,dt \\
&\quad+\|\varphi\|_{C^{0,\alpha}}^{\eta}|x|^{s(1-\eta)}h^{1+\alpha\eta}\int_{-\delta}^{0}R_h^1(th)|t|^{\alpha\eta}\,dt \\
&\lesssim \|\varphi\|_{C^{0,\alpha}}^{\eta}|x|^{\alpha\eta+s(1-\eta)}h^{s}.
\end{aligned}$$

Hence,

$$\left(\frac{\mu-\varphi(x)}{|x|^{\alpha\eta+s(1-\eta)}}\right)\left(\frac{|\varphi(x+h)-\varphi(x-h)|}{h^s}\right)\lesssim\|\varphi\|_{C^{0,\alpha}}^{\eta}$$

uniformly for $\alpha\in(0,s)$. Furthermore, since $\mu-\varphi(x)\gtrsim|x|^s$ for $|x|\ll 1$ by Lemma 6.6 and $h<|x|$, this can be reduced to

$$\frac{|\varphi(x+h)-\varphi(x-h)|}{h^{s-\eta(s-\alpha)}}\lesssim\|\varphi\|_{C^{0,\alpha}}^{\eta}.$$

Splitting the estimate over η we arrive at

$$\left(\frac{|\varphi(x+h)-\varphi(x-h)|}{h^{\alpha}}\right)^{\eta}\left(\frac{|\varphi(x+h)-\varphi(x-h)|}{h^{s}}\right)^{1-\eta}\lesssim\|\varphi\|_{C^{0,\alpha}}^{\eta}$$

which proves

$$\sup_{0<h<|x|<\delta} \left(\frac{|\varphi(x+h) - \varphi(x-h)|}{h^\alpha} \right)^{1-\eta} \lesssim 1$$

uniformly for $\alpha \in (0, s)$, with

$$\min(0, 2 - \frac{1}{s}) < \eta < 1$$

fixed. We let $\alpha \nearrow s$ and deduce that φ belongs to $C^{0,s}(\mathbb{R})$.

Combined with Lemma 6.6, this also proves (6.12), that is, the s-Hölder regularity of φ is attained in the crest. □

7 Analytic bifurcation for the fDP equation

In this section, we develop bifurcation theory for the fDP equation (1.18). In Section 7.1, it is proved that there exist local analytic bifurcation branches consisting of small-amplitude, periodic and even solutions to the equation emanating from the largest curve of constant solutions to the equation. The local branches are in Section 7.2 shown to extend to global solution curves, and alternatives for the qualitative behavior of these curves are given. Moreover, we discuss the existence of a highest traveling-wave solution.

The organization of the results parallels Section 4 for the fKdV equation, and we rely on the theory of analytic bifurcation from the monograph [7]. Inspiration has also been taken from [3].

7.1 Local bifurcation

Consider the parameter $s \in (0, 1)$ fixed, set $\beta \in (s, 1)$, and assume that the period $P > 0$ is finite. Recall that we restrict our attention to $\mu > 0$, since antisymmetric solutions $-\varphi(-x)$ exist for $-\mu$ whenever solutions $\varphi(x)$ exist for the fDP equation with wave-speed μ.

Let the function

$$G\colon C^{0,\beta}_{\text{even}}(\mathbb{S}_P) \times \mathbb{R} \to C^{0,\beta}_{\text{even}}(\mathbb{S}_P)$$

be defined as

$$G\colon (\varphi, \mu) \mapsto \mu\varphi - \frac{1}{2}\varphi^2 - \frac{3}{2}\Lambda^{-s}\varphi^2 + \kappa.$$

We let $\gamma(\mu)$ denote one of the constant solutions to the fDP equation, and consider the function

$$\begin{aligned}\widetilde{G}(\phi, \mu) &:= G(\gamma(\mu) - \phi, \mu) \\ &= (\gamma(\mu) - \mu)\phi + 3\gamma(\mu)\Lambda^{-s}\phi - \frac{3}{2}\Lambda^{-s}\phi^2 - \frac{1}{2}\phi^2,\end{aligned}$$

where the relation

$$\phi = \gamma(\mu) - \varphi \tag{7.1}$$

implies that constant φ-solutions of the problem

$$G(\varphi, \mu) = 0 \tag{7.2}$$

maps one-to-one to trivial ϕ-solutions of the problem

$$\widetilde{G}(\phi, \mu) = 0. \tag{7.3}$$

Moreover, if (ϕ, μ) is a solution to (7.3), then (φ, μ) solves (7.2), and consequently also the fDP equation.

It is favorable to set

$$\gamma(\mu) := \frac{\mu + \sqrt{\mu^2 + 8\kappa}}{4},$$

seeing that nonconstant periodic solutions have to cross this branch of constant solutions as shown in Lemma 6.2. Moreover, the Fréchet derivative of $\widetilde{G}$ with respect to ϕ is

$$\partial_\phi \widetilde{G}[0, \mu] = (\gamma(\mu) - \mu)\,\mathrm{id} + 3\gamma(\mu)\Lambda^{-s}, \tag{7.4}$$

and in order to ensure the existence of bifurcation points along the trivial solution curve of (7.3), the kernel of $\partial_\phi \widetilde{G}[0, \mu]$ has to be nontrivial. That is, there must exist $k \in \mathbb{N}$ such that

$$\left\langle \frac{2\pi k}{P} \right\rangle^{-s} = \frac{1}{3} \frac{\mu - \gamma(\mu)}{\gamma(\mu)}. \tag{7.5}$$

This equation has solutions only if $\gamma(\mu)$ is chosen to be the largest constant solution to the fDP equation. The following lemma shows that in this case, local bifurcation curves do exist. We appeal to [7, Theorem 8.3.1], which is an analytic version of the Crandall–Rabinowitz theorem.

Lemma 7.1. *Assume that* $-\frac{\mu^2}{8} < \kappa < \mu^2$.

(i) If $\kappa < 0$, *then for every* $P > 0$ *and every* $k \in \mathbb{N}$ *with* $\frac{2\pi k}{P} < \sqrt{3^{2/s} - 1}$, *there exists a unique* $\mu^*_{P,k} \in (\sqrt{-8\kappa}, \infty)$,

(ii) if $\kappa > 0$, *then for every* $P > 0$ *and every* $k \in \mathbb{N}$ *with* $\frac{2\pi k}{P} > \sqrt{3^{2/s} - 1}$, *there exists a unique* $\mu^*_{P,k} \in (\sqrt{\kappa}, \infty)$

such that $(\gamma(\mu^*_{P,k}), \mu^*_{P,k})$ *is a bifurcation point for* G *in each case. For every such point, there exists* $\varepsilon > 0$ *and an analytic curve*

$$\mathcal{Q}_{P,k} = \{(\varphi_{P,k}(t), \mu_{P,k}(t))\,;\, t \in (-\varepsilon, \varepsilon)\} \subset C^{0,\beta}_{\mathrm{even}}(\mathbb{S}_P) \times \mathbb{R}$$

such that $G(\varphi_{P,k}(t), \mu_{P,k}(t)) = 0$ *for all* $t \in (-\varepsilon, \varepsilon)$ *and*

$$(\varphi_{P,k}(0), \mu_{P,k}(0)) = (\gamma(\mu^*_{P,k}), \mu^*_{P,k}).$$

For every $\kappa \neq 0$, *the curves* $\mathcal{Q}_{P,k}$ *constitute all nonconstant solutions of* (7.3) *in a neighborhood of the two constant solution curves of the equation.*

Proof. Since solutions of the problem (7.2) map one-to-one to solutions of (7.3), it suffices to establish the existence of local bifurcation curves $\widetilde{\mathcal{Q}}_{P,k}$ of $\widetilde{G}$. We check the assumptions of [7, Theorem 8.3.1], namely, that there exist $\mu^*_{P,k}$ such that $\partial_\phi \widetilde{G}[0, \mu^*_{P,k}]$ is a Fredholm operator of index zero with one-dimensional kernel spanned by $\phi^*_{P,k}$, and that the transversality condition

$$\partial^2_{\mu\phi} \widetilde{G}[0, \mu^*_{P,k}](\phi^*_{P,k}, 1) \notin \operatorname{im}(\partial_\phi \widetilde{G}[0, \mu^*_{P,k}])$$

holds.

The Fréchet derivative of $\widetilde{G}$ given by (7.4) is a sum of the scaled identity and the scaled compact operator Λ^{-s}. As we have seen before, this implies that $\partial_\phi \widetilde{G}[0, \mu]$ is a Fredholm operator of index zero. For given P and k, using the characterization (2.16) of how Λ^{-s} acts on the basis functions $\cos(\frac{2\pi k}{P} x)$ of $C^{0,\beta}_{\text{even}}(\mathbb{S}_P)$, the kernel of $\partial_\phi \widetilde{G}[0, \mu]$ is one-dimensional precisely when there exists a unique μ such that the equation (7.5) is satisfied. That is, when there exists μ such that

$$\frac{2\pi k}{P} = \sqrt{\left(3 \frac{\gamma(\mu)}{\mu - \gamma(\mu)}\right)^{2/s} - 1}.$$

The right-hand side of this equation tends to $\sqrt{3^{2/s} - 1}$ when $\mu \to \infty$. When $\kappa < 0$, the right-hand side is always larger than $\sqrt{3^{2/s} - 1}$, when $\kappa > 0$, the right-hand side is always smaller than $\sqrt{3^{2/s} - 1}$, and equality holds if $\kappa = 0$. This shows that solutions μ to (7.5) are only possible for the ranges of P and k given in the lemma. For such values of P and k, solutions $\mu^*_{P,k}$ exist and are unique. Indeed, the function

$$\frac{1}{3} \frac{\mu - \gamma(\mu)}{\gamma(\mu)}$$

is continuous and monotone in μ for every κ, and takes values in $(1, \sqrt{3^{2/s} - 1})$ for $\kappa < 0$ and $(\sqrt{3^{2/s} - 1}, 0)$ for $\kappa > 0$, so that is must take the value $\langle \frac{2\pi k}{P} \rangle^{-s} \in (0, 1)$ for a unique μ in each case. Note that when $\kappa = 0$, the function is constant, and therefore only satisfied for a unique value of $\frac{2\pi k}{P}$.

For the points $(0, \mu^*_{P,k})$, the kernel is one-dimensional and given by

$$\ker \partial_\phi \widetilde{G}[0, \mu^*_{P,k}] = \{\tau \phi^*_{P,k} \,;\, \tau \in R\}$$

with $\phi^*_{P,k} := \cos(\frac{2\pi k}{P} x)$. Differentiating the operator $\partial_\phi \widetilde{G}[0, \mu^*_{P,k}]$ with respect to the bifurcation parameter μ, one can check that

$$\partial_{\mu\phi} \widetilde{G}[0, \mu^*_{P,k}](\phi^*_{P,k}, 1) = (\gamma'(\mu^*_{P,k}) - 1)\phi^*_{P,k} + 3\gamma'(\mu^*_{P,k})\Lambda^{-s}\phi^*_{P,k}.$$

This function belongs to the image of $\partial_\phi \widetilde{G}[0, \mu^*_{P,k}]$ if and only if

$$\gamma'(\mu^*_{P,k}) = \frac{\gamma(\mu^*_{P,k})}{\mu^*_{P,k}}.$$

It is easily verified that for $\kappa \neq 0$, this is not possible, and we conclude that the transversality condition holds. That is, we get nontrivial local bifurcation branches in this case. For $\kappa = 0$, the transversality condition does not hold, and we cannot conclude that bifurcation occurs. $\square$

In contrast to the fKdV equation, Lemma 7.1 shows that for given s, local bifurcation for the fDP equation can only happen if the fraction $\frac{2\pi k}{P}$ is either strictly smaller or strictly larger than $\sqrt{3^{2/s}-1}$, depending on the parameter κ. That is, we do not have complete freedom in choosing the period P of solutions. In particular, for $\kappa > 0$ and small s bifurcation only occurs for periods $P \ll 1$.

7.2 Global bifurcation

From this point on we assume $\kappa > 0$. This is a necessary assumption to ensure that global bifurcation branches do not converge to a constant solution; further comments are provided at the end of this section. To simplify the discussion, we consider the local bifurcation branch $\mathcal{Q}_{P,1}$ consisting of solutions of a fixed period

$$P < \frac{2\pi}{\sqrt{3^{2/s}-1}}$$

emanating from the trivial curve of (7.2) in $(\gamma(\mu^*), \mu^*)$. It is henceforth denoted by $(\varphi(t), \mu(t))$.

Let

$$V := \big\{(\varphi, \mu) \in C^{0,\beta}_{\text{even}}(\mathbb{S}_P) \times (\sqrt{\kappa}, \infty)\,;\, \varphi < \mu\big\},$$

and define

$$W := \big\{(\varphi, \mu) \in V\,;\, G(\varphi, \mu) = 0\big\}.$$

We follow the same conventions as in Section 4.2, and write V^1 and W^1 for the φ-components of V and W.

In the direction of global bifurcation for the fDP equation, we prove that $\partial_\varphi G[\varphi, \mu]$ is a Fredholm operator in V, that closed and bounded subsets of W are compact, and that $\mu'(t)$ does not vanish identically around the bifurcation point $(\gamma(\mu^*), \mu^*)$. Then we invoke [7, Theorem 9.1.1] in the same manner as in Section 4.

Proposition 7.2. *The operator $\partial_\varphi G[\varphi, \mu]$ is Fredholm of index zero for every $(\varphi, \mu) \in V$.*

Proof. The Fréchet derivative of G in (φ, μ) is

$$\partial_\varphi G[\varphi, \mu] = (\mu - \phi)\,\mathrm{id} - 3\Lambda^{-s}(\phi\,\cdot\,).$$

The first term is a homeomorphism in V, and the second term is a compact operator on $C^{0,\beta}_{\text{even}}(\mathbb{S}_P)$. The claim follows from [7, Theorem 2.7.6]. $\square$

Proposition 7.3. *Any closed and bounded subset of W is compact.*

Proof. Note that

$$\varphi = \mu - \sqrt{\mu^2 + 2\kappa - 3\Lambda^{-s}\varphi^2}$$

is a continuous map from $\mathcal{C}^\beta$ to $\mathcal{C}^{\beta+s}$ for $\varphi \in S$. Hence, if K is a closed and bounded subset of S, then K^1 must be a bounded subset of $\mathcal{C}^{\beta+s}_{\text{even}}(\mathbb{S}_P)$. That is, K^1 is relatively compact in $\mathcal{C}^{\beta}_{\text{even}}(\mathbb{S}_P)$ which coincides with $C^{0,\beta}_{\text{even}}(\mathbb{S}_P)$. Since K is closed by assumption, it is compact. □

If we let $\widetilde{V}$ and $\widetilde{W}$ denote the transformed sets V and W via (7.1), then both of the above propositions hold also for $\partial_\phi \widetilde{G}$ in $\widetilde{V}$, and $\widetilde{W}$. Thus, in order to to apply the global bifurcation result [7, Theorem 9.1.1] to the present situation, and extend the local bifurcation branches from Lemma 7.1 to global analytic solution curves, it suffices to show that $\mu(t) \not\equiv 0$ in a neighborhood of μ^*.

To this end, we choose a parametrization for curve $(\phi(t), \mu(t))$ (corresponding to the curve $(\varphi(t), \mu(t))$ via (7.1)) such that $[\phi(t)]_1 = t$, and expand

$$\phi(t) = \sum_{n=1}^{\infty} \phi_n t^n, \qquad \mu(t) = \sum_{n=0}^{\infty} \mu_{2n} t^{2n},$$

in view of the local branch being analytic. This is the same expansions which was used for the local bifurcation curves in Section 4.2. Inserting $\phi(t)$ and $\mu(t)$ into (7.3) yields

$$3\gamma(\mu_0)\Lambda^{-s}\phi_1 - (\mu_0 - \gamma(\mu_0))\phi_1 = 0, \tag{7.6a}$$

$$3\gamma(\mu_0)\Lambda^{-s}\phi_2 - (\mu_0 - \gamma(\mu_0))\phi_2 = \frac{1}{2}\phi_1^2 + \frac{3}{2}\Lambda^{-s}\phi_1^2, \tag{7.6b}$$

and

$$\begin{aligned} &3\gamma(\mu_0)\Lambda^{-s}\phi_3 - (\mu_0 - \gamma(\mu_0))\phi_3 \\ &= \phi_1\phi_2 + \mu_2(1 - \gamma'(\mu_0))\phi_1 + 3\Lambda^{-s}(\phi_1\phi_2) - 3\mu_2\gamma'(\mu_0)\Lambda^{-s}\phi_1, \end{aligned} \tag{7.6c}$$

for the first-, second-, and third-order terms in t. As before, we let m_j denote $\langle \frac{2\pi j}{P} \rangle^{-s}$. It is clear that $\mu_0 = \mu^*$, and since $3\gamma(\mu^*)m_1 = \mu^* - \gamma(\mu^*)$, the equation (7.6a) can be written as

$$3\gamma(\mu^*)(\Lambda^{-s}\phi_1 - m_1\phi_1) = 0.$$

Consequently we have $\phi_1 = \cos(\frac{2\pi}{P}x)$. Inserting this in the right-hand side of (7.6b) yields

$$3\gamma(\mu^*)(\Lambda^{-s}\phi_2 - m_1\phi_2) = 1 + \frac{1}{4}(1 + 3m_2)\cos\left(\frac{4\pi}{P}x\right),$$

which implies that

$$\phi_2 = \frac{1}{3\gamma(\mu^*)}\left(\frac{1}{1 - m_1} - \frac{1 + 3m_2}{4(m_1 - m_2)}\cos\left(\frac{4\pi}{P}x\right)\right)$$

The coefficients in front of the $\cos(\frac{2\pi}{P}x)$-terms on the right-hand side of (7.6c) can now be determined, and reads

$$\frac{1 + 3m_1}{3\gamma(\mu^*)}\left(\frac{1}{1 - m_1} - \frac{1 + 3m_2}{8(m_1 - m_2)}\right) + \mu_2(1 - \gamma'(\mu^*) - 3m_1\gamma'(\mu^*)).$$

However, since $\cos(\frac{2\pi}{P}x)$ is not in the image of the left-hand side of (7.6c), we have

$$\mu_2 = \frac{1+3m_1}{3\gamma(\mu^*)}\frac{1}{3m_1\gamma'(\mu^*)+\gamma'(\mu^*)-1}\left(\frac{1}{1-m_1}-\frac{1+3m_2}{8(m_1-m_2)}\right). \tag{7.7}$$

We find (see Remark 7.4) that there exist periods P such that $\mu_2 \neq 0$, implying that $\mu'(t) \not\equiv 0$ in a neighborhood of $t = 0$. Hence, the assumptions of [7, Theorem 9.1.1] are now satisfied. We state a lemma establishing global bifurcation for the fDP equation, and give qualitative alternatives for the curves.

Remark 7.4. Numerical calculations of the (exact) expression (7.7) supporting our conclusion are included in Appendix A.2.

Lemma 7.5. *There exists a period $P < 2\pi/\sqrt{3^{2/s}-1}$ such that the local bifurcation branch $t \mapsto (\varphi(t), \mu(t))$ of P-periodic solutions to the fDP equation from Lemma 7.1 extends to a global continuous curve $\mathfrak{Q} := \{(\varphi(t), \mu(t))\,;\, t \in [0,\infty)\} \subset V$, and one of the following alternatives hold.*

(i) $\|(\varphi(t), \mu(t))\|_{C^{0,\beta}\times\mathbb{R}} \to \infty$ *as* $t \to \infty$,

(ii) $\operatorname{dist}(\mathfrak{Q}, \partial V) = 0$,

(iii) $\mathfrak{Q}$ *is a closed loop of finite period. That is, there exists $T > 0$ such that*

$$\mathfrak{Q} = \{(\varphi(t), \mu(t))\,;\, 0 \le t \le T\},$$

where $(\varphi(T), \mu(T)) = (0, \mu^*)$.

Before we finish this section with a discussion of the behavior of the global bifurcation curves from Lemma 7.5, we state a proposition which shows that if $\mu(t)$ is bounded along the global solution curve, then one can find a limiting solution φ on that curve.

Proposition 7.6. *Any sequence of solutions $(\varphi_n, \mu_n)_{n\in\mathbb{N}} \subset W$ with bounded $(\mu_n)_{n\in\mathbb{N}}$ has a subsequence that converges uniformly to a solution φ.*

Proof. Assume that $(\mu_n)_{n\in\mathbb{N}}$ is bounded. Since

$$\frac{1}{2}\varphi^2 = \kappa + \mu\varphi - \frac{3}{2}\Lambda^{-s}\varphi^2 < \kappa + \mu\varphi,$$

we have

$$\|\varphi\|_{L^\infty}^2 \le 2\kappa + 2\mu\|\varphi\|_{L^\infty}.$$

Thus, the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is bounded. This implies that $(\Lambda^{-s}\varphi_n^2)_{n\in\mathbb{N}}$ is uniformly equicontinuous. Indeed, since K_s is integrable and continuous, one has

$$\begin{aligned}\left|(\Lambda^{-s}\varphi_n)(x) - (\Lambda^{-s}\varphi_n)(y)\right| &= \left|\int_{\mathbb{R}} (K_s(x-\eta) - K_s(y-\eta))\varphi_n(\eta)\,d\eta\right| \\ &\le \|\varphi_n\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}} \left|K_s(x-\eta) - K_s(y-\eta)\right| d\eta,\end{aligned}$$

which tends to zero when $|x-y| \to 0$, and it can be taken independently of n. Then $(\Lambda^{-s}\varphi_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence by the Arzela–Ascoli theorem. Owing to (6.7), one obtains the same conclusion for $(\varphi_n)_{n\in\mathbb{N}}$. □

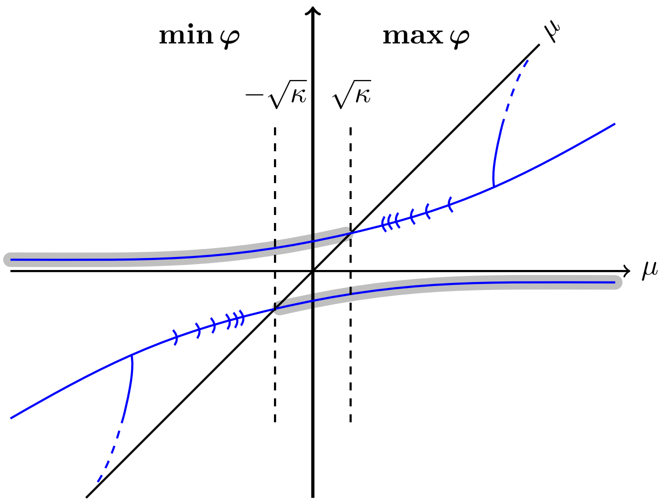


Figure 3: Bifurcation diagram for the fDP equation (1.18) with $\kappa > 0$. The diagram plots $\max\varphi$ for $\mu > 0$ and $\min\varphi$ for $\mu < 0$. Local bifurcation branches emanate from the largest constant solution of the equation for $\mu > \sqrt{\kappa}$ (cf. Lemma 7.1), and there exists a period P such that the first local branch can be extended to a global curve (cf. Lemma 7.5). The curves of constant solutions are otherwise locally unique. Numerical bifurcation suggests that the global bifurcation curve converges to a highest wave for the fDP equation for sufficiently small periods, here depicted as a dashed line.

Two key ingredients now lack before one can conclude that there exist highest periodic traveling-wave solutions to the fDP equation (1.18) that are cusped with $C^{0,s}$-regularity at the crests. Firstly, one needs to show that solutions are nonnegative along the main bifurcation branch from Lemma 7.5. Secondly, it must be established that alternative (i) does not happen by $\mu(t)$ approaching ∞.

If solutions are in fact nonnegative, then the nodal properties from Lemma 4.12 hold. This permits the exclusion of closed loops in the global bifurcation curve using the method of Corollary 4.9 from Section 4. Moreover, one can then show that $\mu(t)$ does not approach $\sqrt{\kappa}$ as $t \to \infty$ along the global bifurcation branch with the following argument, based on [3, Lemma 4.8].

Assume that there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ with $\mu_n \to \sqrt{\kappa}$ as $n \to \infty$. By Proposition 7.6 there is a convergent subsequence (φ_{n_k}) of the corresponding sequence of solutions to the fDP equation $(\varphi_n)_{n\in\mathbb{N}}$ that converges to a solution φ_0. For this subsequence we have

$$\max\varphi_{n_k} > \frac{\mu_{n_k} + \sqrt{\mu_{n_k}^2 + 8\kappa}}{4} > \sqrt{\kappa}$$

due to Lemma 6.2, while on the other hand $\max\varphi_{n_k} < \mu_{n_k} \to \sqrt{\kappa}$. Hence, $\max\varphi_0 = \sqrt{\kappa}$, and

$$\max\Lambda^{-s}\varphi_0^2 = \sqrt{\kappa}.$$

This can only happen if $\varphi_0 \equiv \sqrt{\kappa}$, and if the solution is nonnegative we arrive at a contradiction to (6.10) from Lemma 6.6. This also demonstrates why the assumption $\kappa > 0$ was made at the beginning of this section; otherwise one can not eliminate the possibility that the global bifurcation branch converges to a constant solution of the equation.

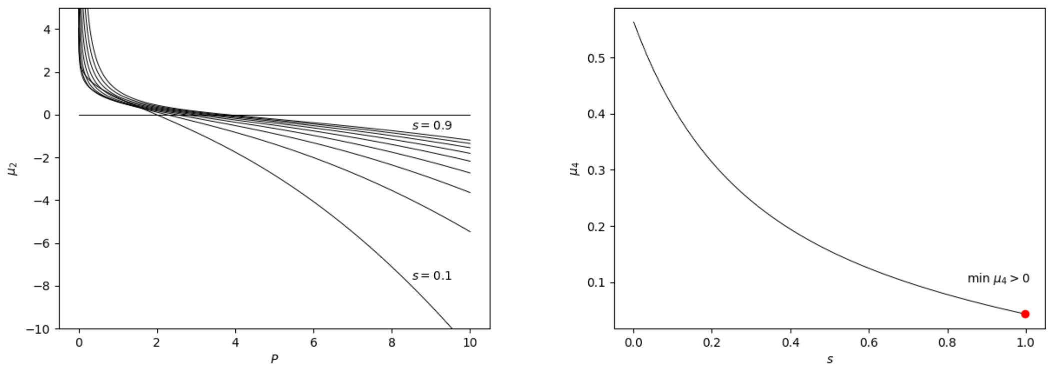
It is also essential to ensure that alternative (i) in Lemma 7.5 does not happen by $\mu(t) \to \infty$ while $\varphi(t)$ remains bounded in the $C^{0,\beta}$-norm. Numerical experiments suggest that this can be avoided by choosing a small enough period P. This claim is proved in [3, Proposition 4.10] for the nonlocal formulation of the Degasperis–Procesi equation. We have not been able to prove this for the fDP equation.

A Numerical computation of bifurcation coefficients

Appendix A.1 and A.2 reports numerical results for the coefficients in the expansions of $\mu(t)$ in the local bifurcation branches for the fKdV and the fDP equation. Note that while the calculations of the coefficients are done numerically, the expressions given in Section 4.2 and Section 7.2 are exact.

A.1 Coefficients for the fKdV equation

Numerical calculations show that μ_2, as given by (4.8), is strictly decreasing in P for every choice of $s \in (0, 1)$ and that there exists a unique P_s^* such that $\mu_2 = 0$. As an illustration, the function μ_2 is plotted in figure A.1a for the special cases $s \in \{0.1, \cdots, 0.9\}$. Calculating P_s^* for a discretized set $s \in (0, 1)$ and plotting μ_4 as given by (4.9) for these values of P_s^* yields the graph in figure A.1b. The values of μ_4 are strictly larger than 0 for all $s \in (0, 1)$.



(a) (b)

Figure A.1: (a) Plot of μ_2 for different periods. (b) Plot of μ_4 for P_s^*, for different values of s.

A.2 Coefficients for the fDP equation

From Section 7.2 we have that

$$\mu_2 = \frac{1+3m_1}{3\gamma(\mu^*)} \frac{1}{3m_1\gamma'(\mu^*) + \gamma'(\mu^*) - 1} \left(\frac{1}{1-m_1} - \frac{1+3m_2}{8(m_1-m_2)} \right)$$

in the expansion of $\mu(t)$ from the local bifurcation branch for the fDP equation with $\kappa > 0$. Thus μ_2 vanishes if and only if the last factor equals zero. Numerically, we find that this does not happen when $s < s^* \approx 0.76$, and that it happens for a unique P_s^* for $s > s^*$. In any case, one can pick small enough period P such that μ does not vanish.

A small set of illustrative sample plots are given in Figure A.2, where values of

$$9m_1 + 3m_1m_2 - 11m_2 - 1$$

from the third factor in the expression of μ_2 are plotted for all admissible periods

$$P < \frac{2\pi}{\sqrt{3^{2/s} - 1}},$$

for the special cases $s \in \{0.1, \cdots, 0.9\}$.

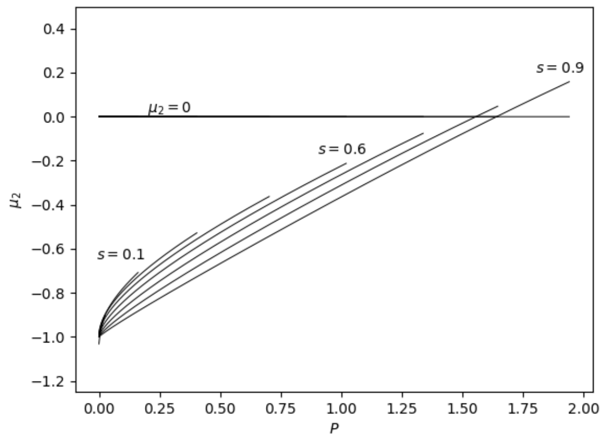


Figure A.2: Plot of μ_2 for different admissible periods.

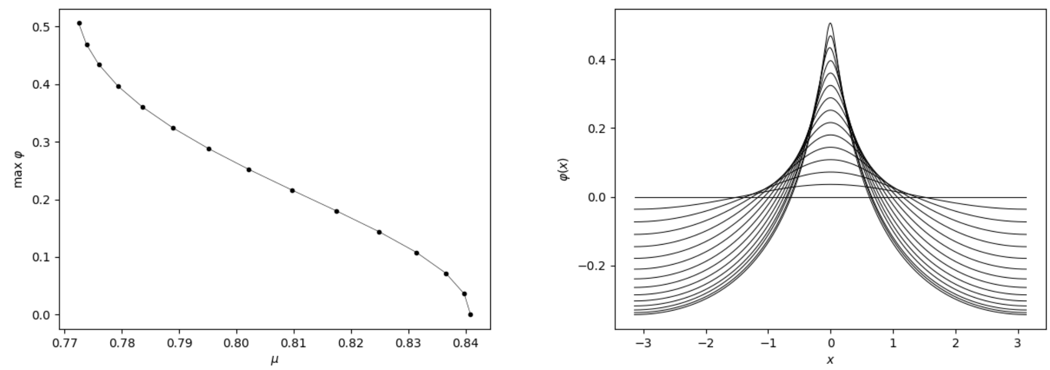
B Global bifurcation: a numerical example

We give here samples of numerical approximations of global bifurcation branches for the fKdV and the fDP equation, illustrating the results obtained in the previous sections.

A Fourier collocation method is used, and multiplication of nonlinear terms is carried out in physical space. For transformation between physical and frequency space we have used the discrete Fourier transform, and the approximation scheme utilizes a standard nonlinear root-finding algorithm. Details about the specific results are given below.

B.1 Bifurcation for the fKdV equation

Figure B.1 shows part of a numerical bifurcation branch for the fKdV equation. Here, $P = 2\pi$, and $s = 0.5$. The first solution is the trivial solution from which nontrivial, smooth solutions with consecutively higher amplitude arise. The numerical method breaks down as the crest becomes cusped in the limit.



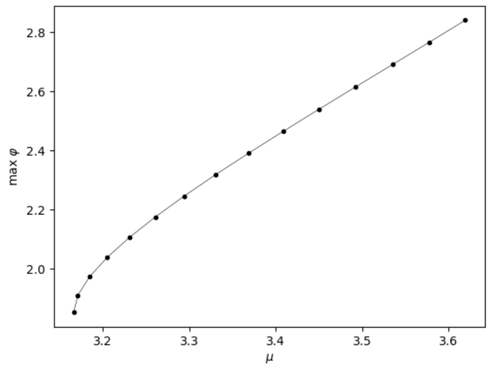
(a) (b)

Figure B.1: (a) Part of a numerical bifurcation branch for the fKdV equation. (b) Individual solutions along the bifurcation branch.

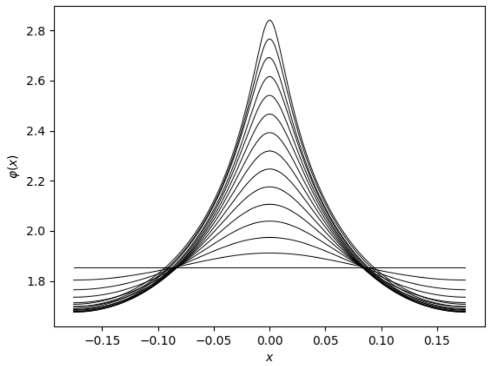
B.2 Bifurcation for the fDP equation

Figure B.2 shows part of a global bifurcation branch for the fDP equation with $\kappa = 1$ and $s = 0.5$. Here, the period is chosen as half the maximal possible period of $2\pi/\sqrt{3^{2/s} - 1}$, evaluating to $P \approx 0.35$.

We report two observations based on our numerical experiments. Firstly, all periodic solutions along the bifurcation branches are nonnegative for every $s \in (0, 1)$ and $\kappa > 0$. This suggests the exclusion of alternative (iii) in Lemma 7.5 thanks to Lemma 6.3. Secondly, it is possible that alternative (i) in Lemma 7.5 happens by $\mu(t) \to \infty$ while $\|\varphi\|_{C^{0,\beta}}$ remains bounded. This can be avoided however, by restricting solutions to sufficiently small periods. A similar bifurcation pattern has been proved for the case $s = 2$ in [3].



(a)



(b)

Figure B.2: (a) Part of a numerical bifurcation branch for the fDP equation. (b) Individual solutions along the bifurcation branch.

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