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Construction of Salem Sets

June 2021

Master's thesis

Master's thesis

2021

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Applied Physics and Mathematics
Submission date: June 2021
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Preface

This master's thesis is the final submission for the study programme Applied Physics and Mathematics at the Norwegian University of Science and Technology (NTNU). The thesis was written at the Department of Mathematical Sciences during the spring of 2021 under the supervision of Professor Eugenia Malinnikova, and with Associate Professor Sigrid Grepstad as co-supervisor.

I would like to thank my supervisor Professor Eugenia Malinnikova for her wonderful guidance throughout the last two semesters. A special thanks to Associate Professor Sigrid Grepstad is in order for her excellent feedback throughout this last semester.

Helge Jørgen Samuelsen
June 7, 2021
Sandnes, Norway

Abstract

In this thesis we study the concept of Salem sets. A Borel set is called a Salem set if the Hausdorff dimension coincides with the Fourier dimension. It is known that the Hausdorff dimension is bounded from below by the Fourier dimension, and that the inequality may be strict. This happens for instance with the Cantor set.

Our main focus will be on two explicit constructions of Salem sets on the unit interval. The first is that of random Cantor sets. This construction was introduced in the 1950s and later expanded upon in the 1990s. For a fixed $\alpha \in (0, 1)$, we will construct a random Cantor set where the Hausdorff dimension is bounded from above by α. By estimating the expected value of a probability measure supported on the random Cantor set, we are able to bound the Fourier dimension from below almost surely by α. It follows that the construction is almost surely a Salem set with dimension α, and that we are able to construct Salem sets of any dimension $\alpha \in (0, 1)$.

The second construction provides a deterministic construction of a Salem set. This method has its roots in number theory. We consider the set of α-well approximable numbers, denoted E_α. A known result by Jarník and Besicovitch is that E_α has Hausdorff dimension $2/(2 + \alpha)$. The main problem is therefore to estimate the Fourier dimension. We construct a subset $S_\alpha \subset E_\alpha$, and show that the Fourier dimension of S_α is bounded from below by $2/(2 + \alpha)$. This implies that both S_α and E_α are Salem set with dimension $2/(2 + \alpha)$.

Sammendrag

I denne oppgaven vil vi utforske salemmengder. En salemmengde er en borelmengde hvor hausdorffdimensjonen tilsvarer fourierdimensjonen. Det er kjent at hausdorffdimensjonen er begrenset nedenfra av fourierdimensjonen, og at ulikheten kan være streng. Dette er tilfellet for cantormengden.

Hovedfokuset vil være på to eksplisitte konstruksjoner av salemmengder på enhetsintervallet. Den første er av tilfeldige cantormengder. Denne konstruksjonen ble først introdusert på 1950-tallet, og senere videreutviklet på 1990-tallet. For en bestemt $\alpha \in (0,1)$, vil vi konstruere en tilfeldig cantormengde hvor hausdorffdimensjonen er begrenset ovenfra av α. Ved å estimere forventningsverdien til et sannsynlighetsmål med støtte på den tilfeldige cantormengden, klarer vi å begrense fourierdimensjonen nedenfra nesten helt sikkert med α. Dermed følger det at konstruksjon er nesten helt sikkert en salemmengde med dimensjon α, og at vi kan konstruere salemmengder med hvilken som helst dimensjon $\alpha \in (0,1)$.

Den andre konstruksjonen gir en deterministisk konstruksjon av salemmengder. Denne metoden har sine røtter i tallteori, hvor vi ser på mengden E_α av tall som kan α-approksimeres godt. Et kjent resultat av Jarník og Besicovitch er at mengden E_α har hausdorffdimensjonen $2/(2+\alpha)$. Hovedproblemet er dermed å estimere fourierdimensjonen. Vi konstruerer en undermengde $S_\alpha \subset E_\alpha$, og viser at fourierdimensjonen til S_α er begrenset nedenfra av $2/(2+\alpha)$. Dette fører til at både S_α og E_α vil være salemmengder med dimensjon $2/(2+\alpha)$.

Contents

1 Introduction

A famous concept, if not uniformly recognized, is that of dimensions. One way to think about the dimension of a set is by the number of coordinates needed to specify the points of the set. For instance, it is taught in courses on linear algebra that the dimension of a vector space is given by the smallest amount of linearly independent vectors needed to span the vector space. Even though this way of thinking about dimension is intuitive, it also has its downside of restricting the concept of dimension to the set of non-negative integers.

A key property of dimension can be observed through scaling. If we scale a two dimensional object with a factor of k, then the Lebesgue measure is scaled by a factor of k^2. On the other hand, for a three dimensional object the same scaling would result in a scaling of the Lebesgue measure by a factor of k^3. We see that what dictates the change in Lebesgue measure under scaling is the dimension.

Several definitions of dimension have been introduced in order to include non-integer dimensions. Two of these are the Hausdorff, and the Fourier dimension. The Hausdorff dimension can be though of as a covering dimension. Let us cover a set A by open balls of small radii, $r < \varepsilon$ for some fixed ε, and consider the sum $\sum r^s$. If we take the infimum of the sum $\sum r^s$ over all coverings of A by balls with radii $r < \varepsilon$, then one can show that there exists a value s_0 such that the infimum goes to ∞ as $\varepsilon \to 0$ for all $s < s_0$, while the infimum goes to 0 as $\varepsilon \to 0$ for all $s > s_0$. The value s_0 is referred to as the Hausdorff dimension of A. A precise definition is given in section 3. As the infimum is taken over all covers, the Hausdorff dimension is easier to bound from above than below.

The Hausdorff dimension turns out to be an excellent tool for classifying fractal sets, such as the Cantor set. On the other hand, the Hausdorff dimension coincides with the standard notion of dimension for sets of integer dimension. In fact, it can be shown that any open ball in $\mathbb{R}^n$ has Hausdorff dimension n, while any hypersurface in $\mathbb{R}^n$ has Hausdorff dimension $n-1$.

It turns out that the Hausdorff dimension is connected to an energy functional with respect to probability measures. For a probability measure μ, the energy functional is given by

$$I_s(\mu) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^{-s} d\mu(x) d\mu(y).$$

The Hausdorff dimension of a set A can be shown to coincide with the supremum over values of s where there exist a probability measure μ supported on A with $I_s(\mu) < \infty$.

Through the Fourier duality formula it is possible to write the energy functional in terms of the Fourier transform of the probability measure, that is

$$I_s(\mu) = c(s,n) \int_{\mathbb{R}^n} |\widehat{\mu}(\xi)|^2 |\xi|^{s-n} d\xi.$$

This leads us to the definition of Fourier dimension. The Fourier dimension of a set A is the supremum of s for which there exists a probability measure μ supported on A with $|\widehat{\mu}(\xi)| \le C|\xi|^{-s/2}$.

A crucial observation is that given a set A with Fourier dimension s, there exist a probability measure μ supported on A where $I_t(\mu) < \infty$ for all values $t < s$. This means that the Hausdorff dimension of A must be at least s, and so the Fourier dimension defines a lower bound for the Hausdorff dimension.

A set is called a Salem set if the Hausdorff and Fourier dimensions coincides. A simple calculation shows that any n-dimensional cube is a Salem set of dimension n in $\mathbb{R}^n$. However, it turns out that the Fourier dimension is dependent on the ambient space. Namely, given a probability measure μ supported on an $n-1$ dimensional cube in $\mathbb{R}^n$, there is a direction perpendicular to the cube which the Fourier transform of μ does not depend on. Thus, the Fourier transform of the measure will not have any decay in the perpendicular direction, and so the Fourier dimension has to be zero. On the other hand, the Hausdorff dimension does not depend on the ambient space. This means that an $n-1$ dimensional cube will be a Salem set of dimension $n-1$ in $\mathbb{R}^{n-1}$, but not in $\mathbb{R}^n$. In fact, it turns out to be rather difficult to find Salem sets with dimension $d \in (0,n)$ that are embedded in $\mathbb{R}^n$.

Let us look at the Cantor set once again. It can be shown that it has Hausdorff dimension $\log(2)/\log(3)$, yet there is too much structure for there to exist a probability measure supported on the Cantor set which decays to zero at infinity. For this reason, the Fourier dimension of the Cantor set is zero, and thus is not a Salem set.

The focus of this thesis will be the construction of non-trivial Salem sets on the unit interval. The first construction removes the structure of the Cantor set by introducing randomness. This was first done in 1950 by Salem in [9], and later in 1996 a slightly different method of using randomized translations was introduced by Bluhm in [1]. We will follow the method of Bluhm to show that for any $0 < \alpha < 1$, it is possible to construct a random Cantor set which is almost surely a Salem set with dimension α.

The second method has its roots in number theory. Here we consider the set of α-well approximable numbers on the unit interval. This was done in 1981 by Kaufmann in [6], before Bluhm proposed a slightly simpler example in his 1998 paper [2]. Again, we will follow the paper by Bluhm to give a more deterministic construction of Salem sets for any dimension $\alpha \in (0, 1)$.

Even though we only present one dimensional constructions, there do exist higher dimensional analogues. The 1996 article by Bluhm considers higher dimensional random Cantor sets, and was meant as an extension of the result of Salem. The construction by Kaufman has also been expanded upon in recent years. In 2019 Hambrook generalized the result further in [5] and was able to give an explicit construction of Salem sets in $\mathbb{R}^n$ for any dimension strictly between 1 and $n - 1$.

Section 3 is an introduction to the concepts of Hausdorff dimension, Fourier dimension and Salem sets. For this section we have mostly followed chapters 2 and 3 of [7], while chapter 11 of [4] and chapters 8 and 9 of [10] have been used as supplementary texts. We have included a few slightly more detailed proofs compared to those found in [7]. Other proofs have been omitted.

We have also included a few examples related to the material. For instance, examples presented in [7] have been worked through and presented here in greater detail. We have also included a calculation for an upper bound on Hausdorff dimension of the unit cube. We then extend the calculation to all of $\mathbb{R}^n$ in order to show that the Hausdorff dimension of a set $A \subset \mathbb{R}^n$ cannot be larger than n. The calculation of the unit cube, and the extension to $\mathbb{R}^n$ is not included in any of the references used for this section.

The first construction of a Salem set with dimension $\alpha \in (0, 1)$ is found in section 4. This section starts with a generalization of the Cantor set construction in terms of translations and contractions. Here we also include a few results, such as the representation of the set by translations. Moreover, we show the existence of a probability measure on the generalized Cantor set. The generalization and the representation of the set by translation is original work which, to the best of our knowledge, cannot be found in the references. The transition to a random Cantor set is straightforward by replacing translations by random variables.

When estimating the Fourier dimension of a random Cantor set, we have followed an outline of the proof presented in [1]. Unlike Bluhm, we restrict our attention to one dimension and uniformly distributed random variables. This gives a simpler and more detailed proof, but it is also less general. The last step of the proof consists of a transition from expected value to an almost surely bound, which was not written out in [1]. An example of such a transition was found in chapter 12 of [7], and after a minor modification we were able to use it to finish the proof.

In section 5 we follow a deterministic construction found in [2]. The construction is presented in detail, and we have divided the section into three parts. In the first part, the set E_α of α-well approximable numbers is introduced, and an upper bound on the Hausdorff dimension is provided. This is considered a classical result, and thus is not proven in [2]. The proof we present is inspired by a similar proof presented in chapter 9 of [10]. We also introduce a subset $S_\alpha \subset E_\alpha$. It is the set S_α we consider in the remaining parts.

In the second part, we estimate the Fourier dimension of S_α from below, and prove that both S_α and E_α are Salem sets of dimension $2/(2+\alpha)$. This is done by constructing a weakly convergent sequence of measures with the right decay properties. However, the weak convergence relies on proving that the sequence is Cauchy in the uniform norm on the Fourier side. Proving this is the focus in the third and final part of this section.

2 Preliminaries

In this section we cover the notation and basic background theory used throughout the thesis. As the thesis is heavily dependent on measure theory, probability theory and Fourier analysis, it is these topics which we will touch upon.

We denote the set of natural numbers by $\mathbb{N}$, the set of integers by $\mathbb{Z}$, the set of real numbers by $\mathbb{R}$, and the set of complex numbers by $\mathbb{C}$. For a complex number $z \in \mathbb{C}$, we denote the real part, $\Re(z)$, and the imaginary part, $\Im(z)$. Any real number $x \geq 0$, can be written

$$x = [x] + \{x\}.$$

Here $[x] \in \mathbb{Z}$ denotes the integer part of x, while $\{x\} \in [0, 1)$ denotes the factional part of x. The set of prime numbers is denoted $\mathbb{P} \subset \mathbb{N}$. Furthermore, the Euclidian inner product on $\mathbb{R}^n$ is defined by

$$x \cdot y := \sum_{j=1}^{n} x_j y_j,$$

for all $x, y \in \mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $r > 0$ we define the open ball of radius r centered at x by

$$B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}.$$

We have tried to follow standard notation with regards to spaces of continuous functions on a set $X \subseteq \mathbb{R}^n$. As is common, we denote the space of all continuous functions on X by $C(X)$, while the space of functions with continuous k derivatives is denoted $C^k(X)$. For the space of smooth functions on X, we write $C^\infty(X)$. We denote the space of continuous functions with compact support by $C_0(X)$, while

$$C_\infty(X) := \overline{C_0(X)}^{\|\cdot\|_u}$$

is the uniform closure of $C_0(X)$, and consists of continuous functions vanishing at infinity. It follows that $C_\infty(X) \subset C_b(X)$. Here $C_b(X)$ denotes the space of bounded continuous functions on X.

Another important space in this thesis is the space $C^{1,1}(X)$ of functions with Lipschitz continuous first derivatives. Since any Lipschitz continuous function is differentiable almost everywhere, it follows that for any $\phi \in C^{1,1}(X)$, the second derivative ϕ'' exists almost everywhere, and $\|\phi''\|_\infty < \infty$. Here $\|\cdot\|_\infty$ refers to the essential supremum with respect to the Lebesgue measure.

2.1 Measure and Probability Theory

We mainly consider the Borel σ-algebra, denoted $\mathscr{B}$, throughout the thesis. A set A is said to be a Borel set if $A \in \mathscr{B}$. By a Borel measure, we mean a non-negative countably subadditive set function $\mu : \mathscr{B} \to [0, \infty]$. The Lebesgue measure is always denoted by λ. A function $f : (X_1, \mathscr{A}_1) \to (X_2, \mathscr{A}_2)$ is said to be measurable if $f^{-1}(A) \in \mathscr{A}_1$ for all $A \in \mathscr{A}_2$. When integrating a measurable function f with respect to the Lebesgue measure, we write

$$\int_{\mathbb{R}^n} f(x) d\lambda(x) = \int_{\mathbb{R}^n} f(x) dx,$$

and for $1 \leq p < \infty$ we denote the L^p norm by

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}}.$$

Moreover, $L^p(\mathbb{R}^n)$ denotes the space of all measurable functions with finite L^p-norm.

There is a useful formula for integrating a measurable function f with respect to a Borel measure μ. The following result is found in chapter 6 of [4].

Theorem 2.1. *If $0 < p < \infty$, then*

$$\int_{\mathbb{R}^n} |f|^p d\mu = p \int_0^\infty s^{p-1} \mu \left(\{ x \in \mathbb{R}^n : |f(x)| > s \} \right) ds$$

Let μ be a Borel measure. Then for each $A \in \mathscr{B}$, we define the restriction of μ to A, denoted $\mu|_A$, by

$$\mu|_A(B) = \int_B \chi_A d\mu,$$

for every $B \in \mathscr{B}$. Here χ_A denotes the characteristic function of A, which is 1 for every element in A, and 0 otherwise.

Let $f : (X_1, \mathscr{A}_1) \to (X_2, \mathscr{A}_2)$ be a measurable function between two measurable spaces. Then given the measure space $(X_1, \mathscr{A}_1, \mu)$ we can define the pushforward measure $f_*\mu$ on $(X_2, \mathscr{A}_2)$ by $f_*\mu(B) = \mu(f^{-1}(B))$ for each $B \in \mathscr{A}_2$. When integrating a measurable function g with respect to the pushforward measure, the integral is given by

$$\int_{X_2} g d(f_*\mu) = \int_{f^{-1}(X_2)} g \circ f d\mu.$$

An n-dimensional Borel measure μ is said to be bounded if

$$\|\mu\| := \mu(\mathbb{R}^n) < \infty.$$

The set of all bounded Borel measures on $\mathbb{R}^n$ is denoted $\mathscr{M}(\mathbb{R}^n)$. More generally, given a Borel set A we denote the set of all bounded Borel measures with support in A by $\mathscr{M}(A)$.

Let μ be a bounded Borel measure. If $\mu(\mathbb{R}^n) = 1$, then μ is said to be a probability measure on $\mathbb{R}^n$. The set of all probability measures on $\mathbb{R}^n$ is denoted $\mathscr{P}(\mathbb{R}^n)$. Similarly, the set of all probability measures with support in A is denoted $\mathscr{P}(A)$.

We will now transition to some basic probability theory. Here we follow the presentation found in chapter 10 of [4]. A probability space $(\Omega, \mathscr{A}, P)$ is a measurable space $(\Omega, \mathscr{A})$ equipped with a probability measure P such that $P(\Omega) = 1$. A random variable $X : \Omega \to \mathbb{R}$ on a probability space $(\Omega, \mathscr{A}, P)$ is a real valued measurable function. The expected value of a random variable X is defined as

$$\mathbb{E}(X) := \int_\Omega X(\omega) dP(\omega),$$

and X is said to have finite expectation if

$$\mathbb{E}(|X|) := \int_\Omega |X(\omega)| dP(\omega) < \infty.$$

A given property is said to hold almost surely if it holds for all $\omega \in \Omega$, except possibly on a set of probability zero. This is the equivalent of a property holding almost everywhere in measure theory.

A collection of sets $\{E_\gamma\}_{\gamma \in \Gamma} \subset \mathscr{A}$ is independent if for every $N \in \mathbb{N}$ and all distinct $\gamma_1, \ldots, \gamma_N \in \Gamma$,

$$P\left(\bigcap_{i=1}^N E_{\gamma_i}\right) = \prod_{i=1}^N P(E_{\gamma_i}).$$

The concept of independence can be extended to random variables. A collection of random variables $\{X_v\}_{v \in \Upsilon}$ are independent, if the sets $X_v^{-1}(B_v)$ are independent for all $B_v \in \mathscr{B}$. A useful result for independent random variables is given by the next theorem.

Theorem 2.2. *Suppose that $X_1, \ldots, X_n$ are independent random variables with finite expectations. Then the random variable $\prod_{j=1}^n X_j$ will also have finite expectation and*

$$\mathbb{E}\left(\prod_{j=1}^n X_j\right) = \prod_{j=1}^n \mathbb{E}(X_j).$$

This theorem is presented and proven in the beginning of chapter 10 of [4]. We end the discussion on probability theory with a proposition which expresses the fact that functions of independent random variables are themselves independent, as shown in [4].

Proposition 2.3. *Let $\{X_n^j : 1 \leq j \leq d, 1 \leq n \leq N\}$ be independent random variables, and let $f_n : \mathbb{R}^d \to \mathbb{R}$ be a Borel measurable function for $1 \leq n \leq N$. Then the random variables $Y_n := f_n(X_n^1, \ldots X_n^d)$ for $1 \leq n \leq N$ are independent.*

2.2 Elements of Fourier Analysis

We start by defining the Fourier transform of bounded Borel measures.

Def. 2.1. Let $\mu \in \mathscr{M}(\mathbb{R}^n)$. Then the Fourier transform of μ is defined as

$$\mathscr{F}(\mu)(\xi) = \widehat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} d\mu(x),$$

and for a function $f \in L^1(\mathbb{R}^n)$ we define the Fourier transform of f as

$$\mathscr{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx.$$

For a measure $\mu \in \mathscr{M}(\mathbb{R}^n)$ it follows from the triangle inequality that $|\widehat{\mu}(\xi)| \leq \|\mu\|$, for every $\xi \in \mathbb{R}^n$. Moreover, by the dominated convergence theorem it can be shown that $\widehat{\mu}$ is a continuous function. Let us present some basic properties of the Fourier transform. Let $f_y(x) := f(x-y)$ denote translation by y, and define the function

$$e_y(x) := e^{-2\pi i y \cdot x}.$$

We then have

$$\widehat{f_y}(\xi) = e_y(\xi)\widehat{f}(\xi), \quad \widehat{e_y f}(\xi) = \widehat{f}(\xi + y).$$

These properties of the Fourier transform are easily proven by a change of variables.

There exists an inverse Fourier transform as the next theorem shows.

Theorem 2.4. *Suppose that $f \in L^1(\mathbb{R}^n)$, and that $\widehat{f} \in L^1(\mathbb{R}^n)$ as well. Then for almost every $x \in \mathbb{R}^n$,*

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \widehat{f}(\xi) d\xi.$$

This result is known as the Fourier inversion theorem, and is a classical result in Fourier analysis. The proof can be found in chapter 3 of [10].

When working with the Fourier transform it is convenient to introduce the Schwartz space $\mathscr{S}(\mathbb{R}^n)$. The Schwartz space consists of all functions $\phi \in C^\infty(\mathbb{R}^n)$ such that

$$\|x^\alpha \partial^\beta \phi\|_\infty < \infty,$$

for all multi-indicies $\alpha, \beta \in \mathbb{N}_0^n$, where $\|\cdot\|_\infty$ denotes the essential supremum with respect to the Lebesgue measure. The Schwartz space is particularly useful as $\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ defines an isometric isomorphism from the Schwartz space to itself. Since $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ it is possible to extend the Fourier transform to a unitary isomorphism from L^2 to itself. This is known as Plancherel's theorem and can be found in chapter 3 of [10].

We will also consider Fourier series in dimension one. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be periodic if $f(x) = f(x+n)$ for all $n \in \mathbb{Z}$, and so can be identified with a function $f : \mathbb{T} = \mathbb{R}/\mathbb{Z} \to \mathbb{R}$. We will now define the Fourier series for periodic functions.

Def. 2.2. Let f be a periodic function. Then the Fourier series of f is defined as

$$\sum_{n\in\mathbb{Z}} \widehat{f}(n) e_{-n}(\xi) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{2\pi i n\xi},$$

where the Fourier coefficients $\widehat{f}$ are given by

$$\widehat{f}(n) = \int_0^1 e^{-2\pi i n x} f(x) dx.$$

Since the only integrable periodic function on $\mathbb{R}$ is the zero function, there will be no ambiguity surrounding the notation $\widehat{f}$ for Fourier transform and Fourier coefficients. It is possible to extend any function f supported on an interval of length 1 to a periodic function. Whence we can define a Fourier series for these functions. From the periodicity of the functions, the Fourier coefficients can be defined by integrating over the original interval instead of the unit interval.

To a measure $\mu \in \mathscr{M}([0,1])$ we can associate the Fourier series

$$\sum_{n\in\mathbb{Z}} \widehat{\mu}(n) e^{2\pi i n x},$$

where the Fourier coefficients are given by

$$\widehat{f}(n) = \int_0^1 e^{-2\pi i n x} d\mu(x).$$

It is known that the Fourier series may not converge pointwise for functions which are merely continuous. However, a useful theorem for ensuring uniform convergence of the Fourier series is the following.

Theorem 2.5. *If $f \in C^1(\mathbb{T})$, then the Fourier series of f converges uniformly to f.*

A slightly stronger version of this theorem is proven in chapter 1 of [8]. This means that for $f \in C^1(\mathbb{T})$ we can write

$$f(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2\pi i n x},$$

which is an analogue of the Fourier inversion theorem in $\mathbb{R}^n$ on the torus $\mathbb{T}$.

We end the discussion on Fourier analysis with Fourier duality and Parseval's identity. The next lemma is known as Fourier duality, and is a consequence of Fubini's theorem.

Lemma 2.6 (Fourier duality)**.** *Assume that $\mu, \nu \in \mathscr{M}(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} \widehat{\mu} d\nu = \int_{\mathbb{R}^n} \widehat{\nu} d\mu$$

Proof. By Fubini's theorem it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} \widehat{\mu}(\xi) d\nu(\xi) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} d\mu(x) d\nu(\xi) \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} d\nu(\xi) d\mu(x) = \int_{\mathbb{R}^n} \widehat{\nu}(x) d\mu(x).\end{aligned}$$

□

The next theorem is known as Parseval's identity. We will state it both for function in $L^2(\mathbb{R}^n)$ and periodic functions in $L^2(\mathbb{T})$.

Theorem 2.7 (Parseval's identity)**.** *Let $f, g \in L^2(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} f(x)\overline{g}(x) dx = \int_{\mathbb{R}^n} \widehat{f}(\xi)\overline{\widehat{g}}(\xi) d\xi.$$

If $f, g \in L^2(\mathbb{T})$, then

$$\int_0^1 f(x)\overline{g}(x) dx = \sum_{n \in \mathbb{Z}} \widehat{f}(n)\overline{\widehat{g}}(n).$$

The proof of Parseval's identity is found in [8]. For functions in $L^2(\mathbb{R}^n)$ we refer to chapter 4, while periodic functions are treated in chapter 1. There is a further generalization of Parseval's identity involving measures.

Theorem 2.8. *Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ and $\mu \in \mathscr{M}(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} \varphi(x) d\mu(x) = \int_{\mathbb{R}^n} \widehat{\mu}(\xi)\overline{\widehat{\varphi}}(\xi) d\xi.$$

If $\varphi \in \mathscr{S}([0,1])$ and $\mu \in \mathscr{M}([0,1])$, then

$$\int_0^1 \varphi(x) d\mu(x) = \sum_{n \in \mathbb{Z}} \widehat{\mu}(n)\overline{\widehat{\varphi}}(n).$$

The first result follows from the Fourier duality, while the periodic case is discussed towards the end of chapter 3 in [7].

2.3 Weak Convergence of Measures

One of the types of convergence we will consider in this thesis is weak convergence of measures. We only consider weak convergence of measures supported on a compact set, and so we define the weak convergence of measures in the following way,

Def. 2.3. Let $K \subset \mathbb{R}^n$ be a compact set, and let $\{\mu_k\} \subset \mathscr{M}(K)$ be a sequence of measures. Then the sequence $\{\mu_k\}$ is said to converge weakly to a measure μ if

$$\int_{\mathbb{R}^n} f d\mu_k \xrightarrow{k \to \infty} \int_{\mathbb{R}^n} f d\mu,$$

for all $f \in C(K)$.

We will use the notation $\mu_k \rightharpoonup \mu$ to denote that μ_k converges weakly to μ. The definition presented here is closer to the notion of weak$*$ convergence in functional analysis, than to weak convergence. However, we will not consider any other types of weak convergence, so there will be no room for confusion.

We include two main results on weak convergence of measures. The proofs are omitted.

Proposition 2.9. *Let $K \subset \mathbb{R}^n$ be a compact set, and $\{\mu_k\}_{k \in \mathbb{N}} \subset \mathscr{M}(K)$. If $\sup_{k \in \mathbb{N}} \|\mu_k\| < \infty$, then there exists a weakly convergent subsequence of $\{\mu_k\}_{k \in \mathbb{N}}$.*

This proposition can be found in chapter 2 of [7], and follows from a more general result on weak$*$ convergence in functional analysis.

A quite remarkable result which is proven towards the end of chapter 8 in [4], connects pointwise convergence on the Fourier side with weak convergence of measures.

Proposition 2.10. *Let $K \subset \mathbb{R}^n$ be a compact set. Suppose that $\{\mu_k\}_{k\in\mathbb{N}} \subset \mathscr{M}(K)$ and $\mu \in \mathscr{M}(K)$. If there exists a constant C such that $\|\mu_k\| \leq C < \infty$ for all k, and $\widehat{\mu}_k(\xi) \to \widehat{\mu}(\xi)$ pointwise for every $\xi \in \mathbb{R}^n$, then $\mu_k \rightharpoonup \mu$.*

Given a sequence $\{\mu_k\}_{k\in\mathbb{N}} \subset \mathscr{M}(K)$ such that $\sup_{k\in\mathbb{N}} \|\mu_k\| < \infty$, it is enough to show that the sequence is Cauchy on the Fourier side to conclude that there exists a $\mu \in \mathscr{M}(K)$ such that $\mu_k \rightharpoonup \mu$. To see why, we note that proposition 2.9 ensures that there exists a subsequence μ_{k_j} which converges weakly to some μ. However, weak convergence implies that the Fourier transform of the subsequence must converge pointwise to the Fourier transform of μ. If the original sequence is Cauchy on the Fourier side, then the whole sequence must converge pointwise to the Fourier transform of μ. Thus, it follows from proposition 2.10 that $\mu_k \rightharpoonup \mu$. In particular, for a sequence of probability measures it is enough to show that the sequence is Cauchy on the Fourier side to conclude that it converges weakly to some probability measure $\mu \in \mathscr{P}(K)$.

2.4 The Standard Cantor Set

The standard 1/3-Cantor set, denoted $\mathscr{C}$, will be used as an example throughout section 3. It also acts as an inspiration for the construction presented in section 4. For this reason, a brief introduction of the set $\mathscr{C}$ is in order.

When constructing $\mathscr{C}$, it is common to start by dividing the unit interval $[0,1]$ into three subintervals of equal length, and removing the middle interval. This process is then repeated with the remaining intervals. If we start with the set $C_0 = [0,1]$, then the next two sets are given by

$$C_1 = \left[0, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, 1\right], \quad C_2 = \left[0, \frac{1}{9}\right] \bigcup \left[\frac{2}{9}, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, \frac{7}{9}\right] \bigcup \left[\frac{8}{9}, 1\right].$$

By induction, it follows that the set C_k, for each $k \in \mathbb{N}$, can be written as

$$C_k = \bigcup_{j=1}^{2^k} I_{k,j},$$

where $I_{k,j}$ are disjoint closed intervals of length 3^{-k} for each $1 \leq j \leq 2^k$. Moreover, since each interval is divided into three subintervals, it follows that

$I_{k,j} \subset I_{k-1,i}$ for $j = 2i - 1$ and $j = 2i$. In fact, for integers $0 \leq m < n$ there are exactly 2^{n-m} intervals $I_{n,j} \subset I_{m,i}$ for each $1 \leq i \leq 2^m$. This implies that the sequence of sets $\{C_k\}_{k=0}^{\infty}$ is nested

$$C_0 \supset C_1 \supset C_2 \supset \dots .$$

The standard Cantor set is defined as the intersection of all C_k. That is

$$\mathscr{C} = \bigcap_{k=1}^{\infty} C_k := \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{2^k} I_{k,i}$$

Since $\mathscr{C}$ is an intersection of closed sets, it is itself a closed and hence compact set. The Cantor set is non-empty, as the sequence is nested. Moreover, for each point $x \in \mathscr{C}$ there exists a sequence $\{a_i\}_{i=1}^{\infty}$ such that,

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, \quad a_i \in \{0, 2\}.$$

3 Hausdorff Dimension and Salem Sets

In this section we introduce the concepts of Hausdorff and Fourier dimensions, which are needed to define Salem sets. Throughout the section, we will consider specific examples, most notably the standard Cantor set $\mathscr{C}$.

3.1 Hausdorff Dimension

In order to introduce the Hausdorff dimension, we first need the Hausdorff measures $\mathscr{H}^s$ for $s \geq 0$. Let $B(x, r)$ denote the open ball centered at x with radius r. Let $A \subset \mathbb{R}^n$ be a Borel set and fix $0 < \varepsilon \leq \infty$. We then define the set function

$$\mathscr{H}^s_\varepsilon(A) = \inf \left\{ \sum_{j \in J} r_j^s : A \subset \bigcup_{j \in J} B(x_j, r_j), r_j < \varepsilon \right\}, \tag{3.1}$$

where J is a countable index set.

There are a few different conventions for defining $\mathscr{H}^s_\varepsilon$. For instance, a normalization constant may be included. This is often done to ensure that $\mathscr{H}^n_\varepsilon$ coincides with the n-dimensional Lebesgue measure when $\varepsilon \to 0$. Another way to define $\mathscr{H}^s_\varepsilon$ is to use covers of Borel sets instead of open balls. For this definition, the radii of the open balls are replaced by the diameters of the Borel sets. This is done for instance in chapter 2 of [7]. As remarked by Mattila in [7], replacing open balls with Borel sets results in a scaled versions of $\mathscr{H}^s_\varepsilon$. Since the scaling of the function $\mathscr{H}^s_\varepsilon$ does not impact our results, we have chosen the simplest definition.

We note that $\mathscr{H}^s_\varepsilon$ is a non-increasing function of ε. This is because we take the infimum over coverings of A by balls with radii less than ε. As such, when allowing for a larger radius there will be more ways to cover A. This means the infimum will be taken over a larger set.

We now continue to the Hausdorff measure.

Def. 3.1. Let $s \geq 0$. We define the Hausdorff measure $\mathscr{H}^s$ as

$$\mathscr{H}^s(A) = \lim_{\varepsilon \to 0} \mathscr{H}^s_\varepsilon(A),$$

for each $A \subset \mathbb{R}^n$ in the Borel σ-algebra $\mathscr{B}$.

Let us make a few remarks regarding $\mathscr{H}^s$. First of all, if $\varepsilon < 1$ we see from (3.1) that $\mathscr{H}^\alpha_\varepsilon(A) > \mathscr{H}^\beta_\varepsilon(A)$ whenever $\alpha < \beta$ for any Borel set A. In particular, $\mathscr{H}^s(A)$ is a non-increasing function of s for fixed A. Furthermore,

$\mathscr{H}^s$ is countably subadditive on Borel sets, and thus defines a Borel measure. A proof that $\mathscr{H}^s$ defines a Borel measure can be found in chapter 11 of [4]. Moreover, if $s = n \in \mathbb{N}$, then there exists a constant $\gamma(n)$ such that $\gamma(n)\mathscr{H}^s$ is the Lebesgue measure on $\mathbb{R}^n$. Lastly, we have $\mathscr{H}^s(A) = 0$ for all Borel sets $A \subset \mathbb{R}^n$ whenever $s > n$.

To see the last statement, we consider first the unit cube $\mathbb{K} = [0,1]^n \subset \mathbb{R}^n$. For each $k \in \mathbb{N}$ we can divide $\mathbb{K}$ into k^n cubes with side lengths k^{-1}. We note that for each small cube, we can always find a ball with diameter equal to the diagonal of the cube which encloses the cube. This means that $r_j = 2^{-1}k^{-1}\sqrt{n}$ for each of these balls. Thus, for each $\varepsilon > 0$ we can find a $k_0 \in \mathbb{N}$ such that $1/k < \varepsilon$ for all $k > k_0$. In particular, we must have

$$\mathscr{H}^s_\varepsilon(\mathbb{K}) \leq \sum_{j=1}^{k^n} r_j^s = \left(\frac{\sqrt{n}}{2}\right)^s k^{n-s}, \tag{3.2}$$

for all $k > k_0$. So if $s > n$, it follows from (3.2) that

$$\mathscr{H}^s(\mathbb{K}) = 0.$$

Now for any $m \in \mathbb{Z}^n$, we let the translation of $\mathbb{K}$ be denoted

$$\mathbb{K} + m = [0 + m_1, 1 + m_1] \times \cdots \times [0 + m_n, 1 + m_n],$$

so that we can write

$$\mathbb{R}^n = \bigcup_{m \in \mathbb{Z}^n} \mathbb{K} + m.$$

By the same argument as above, $\mathscr{H}^s(\mathbb{K} + m) = 0$ for any $m \in \mathbb{Z}^n$, and thus by the countable subadditivity of the Hausdorff measure,

$$\mathscr{H}^s(\mathbb{R}^n) \leq \sum_{m \in \mathbb{Z}^n} \mathscr{H}^s(\mathbb{K} + m) = 0.$$

With the Hausdorff measure in mind, we are now ready to define the Hausdorff dimension. We start with a lemma, found in chapter 8 of [10].

Lemma 3.1. *Let $A \subset \mathbb{R}^n$ be a Borel set. Then there exists a unique number s_0 such that $\mathscr{H}^s(A) = \infty$ if $s < s_0$ and $\mathscr{H}^s(A) = 0$ if $s > s_0$.*

Proof. Let us first consider the case when $s = 0$. Then for each $\varepsilon > 0$, $\mathscr{H}^0_\varepsilon(A)$ will be the smallest number of balls with radius less than ε needed to cover the set A. We claim that $\mathscr{H}^0(A) < \infty$ if and only if A is a finite set. To see why, note that if A is a finite set, then for every $\varepsilon > 0$, we have

$$A \subset \bigcup_{a \in A} B(a, \varepsilon),$$

and so $\mathscr{H}^0(A) \leq \sum_{a \in A} 1 < \infty$. On the other hand, let us assume that A is an infinite set, and for a given $\varepsilon > 0$, there is a finite cover of A of N open balls of radius ε. Then since $\mathbb{R}^n$ is a Hausdorff space we have for any two points $a, b \in A$ such that $a \neq b$, there exist open sets V_a and V_b containing a and b, respectively, such that $V_a \cap V_b = \emptyset$. In particular, given $N+1$ points $a_i \in A$, we can find $0 < \varepsilon_1 < \varepsilon$ such that the open balls $B(a_i, \varepsilon_1)$ are disjoint. Thus, we need at least $N+1$ number of open balls with radii less than ε_1 to cover A. By repeating this argument, we can find a sequence $\varepsilon_n \to 0$, such that at least $N+n$ number of open balls with radii $r_j < \varepsilon_n$ is needed to cover A at step n. This shows that $\mathscr{H}^0(A) = \infty$ when A is not a finite set.

Let us still assume that A is a finite set. Then for $\varepsilon > 0$, let $\{B(a_j, r_j)\}_{j=1}^M$ be a finite covering of A by balls with radius $r_j < \varepsilon$, where $a_j \in A$ for $j \in \{1, \ldots, M\}$. Then for any $s > 0$

$$\sum_{j=1}^{M} r_j^s \leq \varepsilon^s \sum_{j=1}^{M} 1 = M\varepsilon^s,$$

which goes to zero as $\varepsilon \to 0$. Whence it follows that $\mathscr{H}^s(A) = 0$ for any $s > 0$, whenever A is a finite set.

Assume now that A is an infinite set, and let $s_0 = \sup\{s \geq 0 : \mathscr{H}^s(A) = \infty\}$. Since A is infinite, we know that $0 \in \{s \geq 0 : \mathscr{H}^s(A) = \infty\} \neq \emptyset$. Then since $\mathscr{H}^s(A)$ is a non-increasing function of s, we know that $\mathscr{H}^s(A) = \infty$ for all $s < s_0$. Suppose now that $s > s_0$. Then we can find some $\alpha \in (s_0, s)$, such that $\mathscr{H}^\alpha(A) < \infty$. Let $M := 1 + \mathscr{H}^\alpha(A) < \infty$, and observe that for $\varepsilon > 0$ we can find a covering $\{B(a_j, r_j)\}_{j \in J}$ of A by balls of radii $r_j < \varepsilon$, and $\sum_j r_j^\alpha \leq M$. This follows from the definition of $\mathscr{H}^\alpha$. In particular, we have

$$\sum_{j \in J} r_j^s = \sum_{j \in J} r_j^{s-\alpha+\alpha} \leq \varepsilon^{s-\alpha} \sum_{j \in J} r_j^\alpha \leq \varepsilon^{s-\alpha} M,$$

which tends to zero as $\varepsilon \to 0$ since $s > \alpha$. This shows that $\mathscr{H}^s(A) = 0$ for $s > s_0$. $\square$

We can apply lemma 3.1 to the following definition.

Def. 3.2. Let A be a Borel set. Then the Hausdorff dimension of A is defined as

$$\dim_{\mathscr{H}}(A) = \sup\{s \geq 0 : \mathscr{H}^s(A) = \infty\} = \inf\{s \geq 0 : \mathscr{H}^s(A) = 0\}. \quad (3.3)$$

Here we use the convention that $\sup \emptyset = 0$ and $\inf \emptyset = \infty$.

Let us now consider the standard 1/3-Cantor set $\mathscr{C}$. We note that for each $k \in \mathbb{N}$ the Cantor set can be covered by 2^k closed intervals with length $2r_j = 3^{-k}$. So for any $\varepsilon > 0$ we can find k such that $3^{-k} < 2\varepsilon$. As such we get

$$\sum_{j=1}^{2^k} r_j^s = 2^k \left(\frac{3^{-k}}{2}\right)^s = \frac{1}{2^s}\left(\frac{2}{3^s}\right)^k,$$

which goes to zero as $k \to \infty$ as long as

$$\frac{2}{3^s} < 1 \Rightarrow s > \frac{\log(2)}{\log(3)}.$$

This implies that $\mathscr{H}^s(\mathscr{C}) = 0$ for $s > \log(2)/\log(3)$, and so from (3.3) we can conclude that $\dim_{\mathscr{H}}(\mathscr{C}) \leq \log(2)/\log(3)$. Even though the Hausdorff measure is defined using open sets, we can always cover a closed set by a slightly larger open set. Thus, this argument will still hold when considering closed sets.

To show that the upper bound for $\mathscr{C}$ is indeed the Hausdorff dimension, we will introduce a result by Frostman. The proof will be omitted, but can be found in chapter 2 of [7], as well as chapter 8 of [10]. Recall that the set of all probability measures supported on a Borel set A is denoted by $\mathscr{P}(A)$.

Theorem 3.2 (Frostman's Lemma)**.** *Let $0 \leq s \leq n$ and suppose that $A \subset \mathbb{R}^n$ is a compact Borel set. Then $\mathscr{H}^s(A) > 0$ if and only if there exists a $\mu \in \mathscr{P}(A)$ such that*

$$\mu(B(x,r)) \leq Cr^s, \quad \forall x \in \mathbb{R}^n,\ r > 0, \quad (3.4)$$

for a suitable constant C.

For a compact Borel set A, it follows from Frostman's Lemma, theorem 3.2, that if there exists a probability measure μ which satisfies (3.4) for s_0, then $\dim_{\mathscr{H}}(A) \geq s_0$. This means that a lower bound can be achieved for the Hausdorff dimension by considering probability measures supported on the set A.

Recall that the Cantor set is created by dividing the unit interval into three parts and removing the middle part, then repeating the process. After

k steps, we have 2^k number of disjoint intervals $I_{k,i}$ with length 3^{-k} where $i \in \{1, \dots, 2^k\}$. The Cantor set is given by

$$\mathscr{C} = \bigcap_{k\in\mathbb{N}} \bigcup_{i=1}^{2^k} I_{k,i}.$$

We now want to show that there exist a probability measure on $\mathscr{C}$, denoted $\mu_{\mathscr{C}}$, such that $\mu_{\mathscr{C}}(I_{k,i}) = 2^{-k}$ for each $k \in \mathbb{N}$ and $i \in \{1, \dots, 2^k\}$. This will be done by considering the weak limit of the sequence of measures

$$\mu_k = \left(\frac{2}{3}\right)^{-k} \sum_{i=1}^{2^k} \lambda|_{I_{k,i}}. \tag{3.5}$$

We first note that for any $k \in \mathbb{N}$

$$\|\mu_k\| = \mu_k(\mathbb{R}) = \left(\frac{2}{3}\right)^{-k} \sum_{i=1}^{2^k} 3^{-k} = 1 < \infty, \tag{3.6}$$

and so by proposition 2.9 the sequence has a weakly convergent subsequence. We now define $\mu_{\mathscr{C}}$ as the weak limit of this subsequence. For a fixed k and any $i \in \{1, \dots, 2^k\}$

$$\mu_k(I_{k,i}) = \left(\frac{2}{3}\right)^{-k} \lambda(I_{k,i}) = 2^{-k},$$

since the intervals $I_{k,i}$ are disjoint. Moreover, for any $m < k$, consider the interval $I_{m,j}$ for a fixed $j \in \{1, \dots, m\}$. There are exactly 2^{k-m} intervals I_{k,j_i} which are contained in $I_{m,j}$, and so

$$\mu_k(I_{m,j}) = \left(\frac{2}{3}\right)^{-k} \sum_{i=1}^{2^k} \lambda|_{I_{k,i}}(I_{m,j}) = \left(\frac{2}{3}\right)^{-k} 2^{k-m}3^{-k} = 2^{-m}. \tag{3.7}$$

Since property (3.7) holds for any element in the sequence $\{\mu_k\}_{k\in\mathbb{N}}$, it will also hold for the weak limit of the subsequence. This shows that there exists a probability measure $\mu_{\mathscr{C}}$ which is supported on the Cantor set $\mathscr{C}$, and has the property

$$\mu_{\mathscr{C}}(I_{k,i}) = 2^{-k} = 3^{-k\frac{\log(2)}{\log(3)}} = \lambda(I_{k,i})^{\frac{\log(2)}{\log(3)}}. \tag{3.8}$$

Note that it is possible to show that the entire sequence μ_k converges weakly to the measure $\mu_{\mathscr{C}}$. This is done by showing that $\int f d\mu_k$ is a Cauchy sequence for all $f \in C([0,1])$, and relies on the mean value theorem for integrals. However, the calculation is rather long, and not needed to find a lower bound on the Hausdorff dimension of $\mathscr{C}$.

We will now continue to establish a lower bound for the Hausdorff dimension through theorem 3.2. Recall that an open ball in dimension one is simply an open interval $B(x,r) = (x-r, x+r)$. We may therefore restrict our attention to open intervals, and show that $\mu_{\mathscr{C}}(B(x,r)) \leq Cr^s$. We also note that any set which does not intersect the unit interval is a $\mu_{\mathscr{C}}$-null set, which follows from the construction of the sequence μ_k. Furthermore, if $\mathscr{C} \cap B(x,r) = \emptyset$, then $\mu_{\mathscr{C}}(B(x,r)) = 0$. So it is enough to consider the case when $B(x,r) \subset [0,1]$ and $\mathscr{C} \cap B(x,r) \neq \emptyset$. Moreover, since for any $0 < r < 1/2$ we can always find a $k \in \mathbb{N}$ such that $3^{-k} \leq 2r < 3^{-k+1}$, there must exists at least one $I_{k,j}$ for some $j \in \{1, \ldots, 2^k\}$ such that $I_{k,j} \cap B(x,r) \neq \emptyset$ for this choice of k.

Fix $x \in (0,1)$ and let $0 < r < 1/2$ be such that $B(x,r) \subset [0,1]$. We claim that there are at most 3 intervals I_{k,j_i} where $i \in \{1,2,3\}$ and $j_i \in \{1, \ldots, 2^k\}$ which have a non-empty intersection with $B(x,r)$. To see this, recall that

$$\operatorname{dist}(I_{k,j}, I_{k,i}) \geq 3^{-k}, \quad i \neq j,$$

so if $x_i \in I_{k,i}$ and $x_j \in I_{k,j}$ for $i \neq j$, then $|x_i - x_j| \geq 3^{-k}$. Assume now that there are four such intervals I_{k,j_i} for $i \in \{1, \ldots, 4\}$. Then pick elements $x_i \in I_{k,j_i}$ such that $x_i \in B(x,r)$ for each $i \in \{1, \ldots, 4\}$. Relabel the elements, if necessary, such that $x_i < x_l$ for $i < l$. Then

$$3 \cdot 3^{-k} \leq (x_4 - x_3) + (x_3 - x_2) + (x_2 - x_1) = x_4 - x_1 < 2r,$$

from the assumption that $x_i \in B(x,r)$ for $i \in \{1, \ldots, 4\}$. However, this is a contradiction on our choice of k. This means that we have the cover

$$B(x,r) \cap \mathscr{C} \subset \bigcup_{i=1}^{3} I_{k,j_i},$$

and since $\mu_{\mathscr{C}}(B(x,r)) = \mu_{\mathscr{C}}(B(x,r) \cap \mathscr{C})$, we end up with

$$\mu_{\mathscr{C}}(B(x,r)) \leq \sum_{i=1}^{3} \mu_{\mathscr{C}}(I_{k,j_i}) = 3\lambda(I_{k,i})^{\frac{\log(2)}{\log(3)}} \leq 3\lambda(B(x,r))^{\frac{\log(2)}{\log(3)}} = \left(3 \cdot 2^{\frac{\log(2)}{\log(3)}}\right) r^{\frac{\log(2)}{\log(3)}},$$

where we used (3.8). It now follows from theorem 3.2 that $\dim_{\mathscr{H}}(\mathscr{C}) \geq \log(2)/\log(3)$. Thus, we conclude that

$$\dim_{\mathscr{H}}(\mathscr{C}) = \log(2)/\log(3).$$

3.2 Energy Integrals

One of the fundamental tools for studying the Hausdorff dimension is the s-energy integral for measures.

Def. 3.3. Let μ be a non-negative Borel measure. The s-energy integral of μ is defined as

$$I_s(\mu) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x-y|^{-s} d\mu(x) d\mu(y). \tag{3.9}$$

It is also possible to define the s-potential

$$V_s(\mu)(x) = \int_{\mathbb{R}^n} |x-y|^{-s} d\mu(y) = k_s * \mu(x),$$

which is the convolution of a measure μ with the Riesz kernel $k_s(x) = |x|^{-s}$. From (3.9), we can note that the s-energy is simply given by

$$I_s(\mu) = \int_{\mathbb{R}^n} V_s(\mu)(x) d\mu(x).$$

A key result on the connection between Hausdorff dimension and energy integrals, is given by the next theorem.

Theorem 3.3. *Let A be a compact Borel set. Then the Hausdorff dimension of A is given by*

$$\dim_{\mathscr{H}}(A) = \sup\{s : \exists \mu \in \mathscr{P}(A) \text{ such that } I_s(\mu) < \infty\}.$$

Proof. Let $s > 0$ be the Hausdorff dimension of A. By Frostman's lemma, theorem 3.2, there exists a probability measure $\mu \in \mathscr{P}(A)$ such that $\mu(B(x,r)) \leq Cr^s$. We now want to incorporate this fact into the s-potential. To do this, we start by writing the s-potential as

$$\begin{aligned} V_s(\mu)(x) =& \int_{\mathbb{R}^n} |x-y|^{-s} d\mu(y) \\ =& s \int_0^\infty t^{s-1} \mu\left(\{y : |x-y|^{-1} > t\}\right) dt. \end{aligned}$$

Using the change of variables $t = r^{-1}$, we arrive at

$$\begin{aligned} s \int_0^\infty t^{s-1} \mu\left(\{y : |x-y|^{-1} > t\}\right) dt =& - s \int_\infty^0 r^{1-2-s} \mu\left(\{y : |x-y| < r\}\right) dr \\ =& s \int_0^\infty \frac{\mu(B(x,r))}{r^{s+1}} dr. \end{aligned}$$

Whence the s-potential can simply be written as

$$V_s(\mu)(x) = s \int_0^\infty \frac{\mu(B(x,r))}{r^{s+1}} dr. \tag{3.10}$$

Since the measure μ is supported on the compact set A, it is enough to integrate the potential over the set A. Moreover, if $r \geq \operatorname{diam}(A) =: R$, then $A \subset B(x,r)$ for all $x \in A$. This means that $\mu(B(x,r)) = 1$ for all $r > R$ and $x \in A$. Thus, with help of Frostman's lemma, the t-energy can be estimated for any $t \in (0,s)$. Namely

$$\begin{aligned} I_t(\mu) =& \int_{\mathbb{R}^n} V_t(\mu)(x) d\mu(x) \\ =& t \int_A \int_0^\infty \frac{\mu(B(x,r))}{r^{t+1}} dr d\mu(x) \\ \leq& t \int_A \int_0^R C r^{s-t-1} dr d\mu(x) + t \int_A \int_R^\infty r^{-t-1} dr d\mu(x) \\ =& t \left(\frac{CR^{s-t}}{s-t} + \frac{R^{-t}}{t} \right) < \infty, \end{aligned}$$

which is finite since $0 < t < s$. Here we used the fact that $\mu(B(x,r)) \leq Cr^s$, as well as $\mu(B(x,r)) = 1$ whenever $r > R$ for all $x \in A$. This shows that there exists at least one $\mu \in \mathscr{P}(A)$ such that $I_t(\mu) < \infty$ for all $t < s = \dim_{\mathscr{H}}(A)$. It therefore follows that

$$\dim_{\mathscr{H}}(A) \leq \sup\{s : \exists \mu \in \mathscr{P}(A) \text{ such that } I_s(\mu) < \infty\}.$$

On the other hand, if $I_s(\mu) < \infty$ for some $\mu \in \mathscr{P}(A)$ and $s > 0$, then $V(\mu)(x) < \infty$ μ-almost everywhere. Thus, there must exist a constant $0 < M < \infty$ such that the set $C = \{x \in \mathbb{R}^n : V_s(\mu)(x) \leq M\}$ has positive μ measure. Now let $\mu|_C$ denote the restriction of μ to the set C. Then for any $x \in \mathbb{R}^n$, and any $r > 0$, we have

$$\begin{aligned} r^{-s} \mu|_C(B(x,r)) =& r^{-s} \int_{B(x,r)} d\mu|_C(y) \\ \leq& \int_{B(x,r)} |x-y|^{-s} d\mu|_C(y) \\ \leq& \int_{\mathbb{R}^n} |x-y|^{-s} d\mu|_C(y) \leq M, \end{aligned}$$

and so $\mu|_C(B(x,r)) \leq Mr^s$. Since the set C might not have full measure, the measure $\mu|_C$ can be scaled to make a probability measure. As such, we can define the probability measure $\tilde{\mu}|_C = \mu|_C/\mu(C) \in \mathscr{P}(A)$ which for any $x \in \mathbb{R}^n$ and any $r > 0$ satisfies

$$\tilde{\mu}|_C(B(x,r)) = \frac{\mu|_C(B(x,r))}{\mu(C)} \leq \frac{M}{\mu(C)} r^s.$$

Whence it follows from Frostman's lemma, theorem 3.2, that $\mathscr{H}^s(A) > 0$ and so $\dim_{\mathscr{H}}(A) \geq s$. In particular, since the same argument holds for any $\mu \in \mathscr{P}(A)$ and $s > 0$ where $I_s(\mu) < \infty$, it follows that

$$\dim_{\mathscr{H}}(A) \geq \sup\{s : \exists \mu \in \mathscr{P}(A) \text{ such that } I_s(\mu) < \infty\}.$$

□

3.3 Fourier Dimension and Salem Sets

We start by recalling that for a bounded Borel measure $\mu \in \mathscr{M}(\mathbb{R}^n)$ the Fourier transform is defined by

$$\widehat{\mu}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} d\mu(x).$$

By using Fourier duality, we can rewrite the s-energy integral from the previous section. A useful formula is the following.

Theorem 3.4. *Let μ be a positive measure with compact support and $0 < s < n$. Then*

$$I_s(\mu) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x - y|^{-s} d\mu(x) d\mu(y) = \frac{\Gamma\left(\frac{n-s}{2}\right) \pi^{s-\frac{n}{2}}}{\Gamma\left(\frac{s}{2}\right)} \int_{\mathbb{R}^n} |\widehat{\mu}(\xi)|^2 |\xi|^{s-n} d\xi.$$

The proof of this formula can be found in chapter 3 of [7] or chapter 8 of [10]. From theorem 3.4, we note that if $I_s(\mu) < \infty$, then we must have some decay property on the Fourier side of the measure μ. In particular, we expect the integrand $|\widehat{\mu}(\xi)|^2 |\xi|^{s-n}$ to decay faster than $|\xi|^{-n}$ as $|\xi|$ approaches infinity. As such, if the measure μ satisfies

$$|\widehat{\mu}(\xi)| \leq C|\xi|^{-\frac{s}{2}},$$

for some constant C, then the t-energy integral, $I_t(\mu)$, will converge for all $t < s$. With this in mind, let us now introduce the Fourier dimension of a Borel set.

Def. 3.4. Let $A \subset \mathbb{R}^n$ be a Borel set. The Fourier dimension of A is defined as

$$\dim_{\mathscr{F}}(A) := \sup\{s \in [0,n] : \exists \mu \in \mathscr{P}(A) \text{ such that } |\widehat{\mu}(\xi)| \leq C|\xi|^{-\frac{s}{2}}\}.$$

From the definition of the Fourier dimension, and theorem 3.4, we can see that the Hausdorff dimension and Fourier dimension are connected. In fact, it turns out that the Hausdorff dimension is bounded from below by the Fourier dimension, as the next theorem shows.

Theorem 3.5. *Let A be a compact Borel set. Then*

$$\dim_{\mathscr{F}}(A) \leq \dim_{\mathscr{H}}(A).$$

Proof. Let $0 < s < \dim_{\mathscr{F}}(A)$. Then there exists a probability measure $\mu \in \mathscr{P}(A)$ such that $|\widehat{\mu}(\xi)| \leq C|\xi|^{-\frac{s}{2}}$. We want to show that the integral

$$\int_{\mathbb{R}^n} |\widehat{\mu}(\xi)|^2 |\xi|^{t-n} d\xi < \infty, \tag{3.11}$$

for $0 < t < s$. Then the result will follow from theorem 3.4.

Let us consider the integral in (3.11) in the two regions $|\xi| < 1$ and $|\xi| \geq 1$, namely

$$\int_{\mathbb{R}^n} |\widehat{\mu}(\xi)|^2 |\xi|^{t-n} d\xi = \int_{|\xi|<1} |\widehat{\mu}(\xi)|^2 |\xi|^{t-n} d\xi + \int_{|\xi|\geq 1} |\widehat{\mu}(\xi)|^2 |\xi|^{t-n} d\xi.$$

Let $\omega(n)$ denote the surface measure of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Then in the first region, we can use the fact that $|\widehat{\mu}(\xi)| \leq \|\mu\|$, and use spherical coordinates to achieve the bound

$$\int_{|\xi|<1} |\widehat{\mu}(\xi)|^2 |\xi|^{t-n} d\xi \leq \omega(n)\|\mu\|^2 \int_0^1 r^{t-1} dr = \frac{\omega(n)\|\mu\|^2}{t} < \infty. \tag{3.12}$$

For the other region, $|\xi| \geq 1$, we use the fact that $|\widehat{\mu}(\xi)| \leq C|\xi|^{-\frac{s}{2}}$. Let $\varepsilon = s - t > 0$, then

$$\begin{aligned}\int_{|\xi|\geq 1} |\widehat{\mu}(\xi)|^2 |\xi|^{t-n} d\xi \leq & C \int_{|\xi|\geq 1} |\xi|^{t-s-n} d\xi \\ = & C\omega(n) \int_1^\infty r^{-(1+\varepsilon)} dr = \frac{C\omega(n)}{\varepsilon} < \infty. \end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), together with theorem 3.4, we have

$$I_t(\mu) < \infty$$

for all $t < s$. It then follows from theorem 3.3 that $\dim_{\mathscr{F}}(A) \leq \dim_{\mathscr{H}}(A)$. □

We are now ready to give the definition of a Salem set, before proceeding with a simple example of a Salem set in $\mathbb{R}$.

Def. 3.5. A Borel set A is called a Salem set if

$$\dim_{\mathscr{F}}(A) = \dim_{\mathscr{H}}(A).$$

We demonstrate a simple example of a Salem set, namely the interval $[-1,1] \subset \mathbb{R}$. If we calculate the Fourier transform of the Lebesgue measure restricted to this interval, we get

$$\mathscr{F}(2^{-1}\lambda|_{[-1,1]})(\xi) = 2^{-1}\int_{-1}^{1} e^{-2\pi i x\xi} dx = \frac{e^{2\pi i\xi} - e^{-2\pi i\xi}}{4\pi i\xi} = \frac{\sin(2\pi\xi)}{2\pi\xi},$$

and so it is clear that $\mu = 2^{-1}\lambda|_{[-1,1]}$ defines a probability measure on $[-1,1]$ with $|\widehat{\mu}(\xi)| \leq C|\xi|^{-1}$. From the definition of Fourier dimension, we have $\dim_{\mathscr{F}}([-1,1]) = \min\{1,2\} = 1$. Moreover, since the $[-1,1] \subset \mathbb{R}$, it follows that $\mathscr{H}^s([-1,1]) = 0$ for all $s > 1$. Thus, we must have $\dim_{\mathscr{H}}([-1,1]) \leq 1$. As such, by theorem 3.5 it follows that

$$1 = \dim_{\mathscr{F}}([-1,1]) \leq \dim_{\mathscr{H}}([-1,1]) \leq 1.$$

This shows that $[-1,1]$ is a Salem set with Fourier dimension 1 in $\mathbb{R}$. However, the interval is not a Salem set in $\mathbb{R}^2$. For $c \in \mathbb{R}$, let μ be a probability measure supported on $[-1,1] \times \{c\}$. Then for each $\xi \in \mathbb{R}^2$

$$e^{-2\pi i\xi\cdot x} = e^{-2\pi i c\xi_2}e^{-2\pi i\xi_1 x_1},$$

for all $x = (x_1, x_2) \in [-1,1] \times \{c\}$. This means that

$$\widehat{\mu}(\xi) = e^{-2\pi i c\xi_2}\widehat{\mu}(\xi_1).$$

Thus, for a fixed ξ_1 the value of $|\widehat{\mu}(\xi)| = |\widehat{\mu}(\xi_1)|$ remains constant whenever $|\xi_2| \to \infty$. This implies that the Fourier dimension is zero, and demonstrates that the Fourier dimension is dependent on the ambient space. On the other hand, for any $x \in \mathbb{R}^2$ and $r > 0$ such that the intersection of $B(x,r)$ with $[-1,1] \times \{c\}$ is non-empty, we can write

$$B(x,r) \cap ([-1,1] \times \{c\}) = I \times \{c\},$$

where $I \subset [-1,1]$ is an interval of length $\lambda(I) \leq 2r$. It then follows from theorem 3.2 that $\dim_{\mathscr{H}}([-1,1]\times\{c\}) \geq 1$. Thus, the interval $[-1,1]\times\{c\} \subset \mathbb{R}^2$ is not a Salem set.

The interval $[-1, 1] \subset \mathbb{R}$ is an example of a Salem set, where Hausdorff and Fourier dimension are equal. However, the Hausdorff and Fourier dimensions do not have to be equal, as seen by the set $[-1, 1] \times \{c\} \subset \mathbb{R}^2$. The next proposition will provide another example of a set where the Hausdorff and Fourier dimensions are different.

Proposition 3.6. *Let $\mathscr{C}$ denote the standard $1/3$-Cantor set. Then for any $\mu \in \mathscr{P}(\mathscr{C})$,*

$$\limsup_{|x|\to\infty} |\widehat{\mu}(x)| > 0.$$

Proof. Since $\mathscr{C} \subset [0, 1]$ we will consider the Fourier series of $\mu \in \mathscr{P}(\mathscr{C})$, and show that the coefficients $\widehat{\mu}(k)$ do not tend to zero for $k \in \mathbb{Z}$ as $|k| \to \infty$. Let us on the contrary assume there exists such a measure $\mu \in \mathscr{P}(\mathscr{C})$ for which the Fourier coefficients tend to zero as $|k| \to \infty$. Let $\varphi \in \mathscr{S}([0, 1])$ be a positive Schwartz function with $\operatorname{supp} \varphi \subset [1/3, 2/3]$ and

$$\int_0^1 \varphi(x) dx = \|\varphi\|_1 = 1.$$

Then for $j \in \mathbb{N}$ we define

$$\varphi_j(x) = \varphi(\{3^j x\}), \quad x \in [0, 1],$$

where $\{\cdot\}$ denotes the fractional part. We now claim that $\operatorname{supp}(\varphi_j) \cap \mathscr{C} = \emptyset$ for each $j \in \mathbb{N}$. To see this, we recall that we can expand any $x \in \mathscr{C}$, as

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, \quad a_i \in \{0, 2\}.$$

As such, for each fixed j,

$$3^j x = \sum_{i=1}^{\infty} 3^{j-i} a_i = \sum_{i=1}^{j} 3^{j-i} a_i + \sum_{i=j+1}^{\infty} 3^{j-i} a_i, \quad a_i \in \{0, 2\}.$$

when taking the fractional part, we simply end up with

$$\{3^j x\} = \sum_{i=1}^{\infty} 3^{-i} a_{j+i} \in \mathscr{C},$$

since each a_i is either 0 or 2, and since $\operatorname{supp}(\varphi) \cap \mathscr{C} = \emptyset$ we have the claimed result.

Now from the Fourier inversion formula,

$$\sum_{k\in\mathbb{Z}} \widehat{\varphi_j}(k)e^{2\pi ikx} = \varphi_j(x) = \varphi(\{3^j x\}) = \sum_{k\in\mathbb{Z}} \widehat{\varphi}(k)e^{2\pi ik3^j x}, \quad x \in [0,1]. \tag{3.14}$$

Here we used the fact that for any $x \in \mathbb{R}$, we can compose it into $x = [x] + \{x\}$, where $[x]$ is the integer part of x, and the 2π periodicity of the complex exponential. This results in

$$e^{2\pi ik3^j x} = e^{2\pi ik([3^j x]+\{3^j x\})} = e^{2\pi ik[3^j x]}e^{2\pi ik\{3^j x\}} = e^{2\pi ik\{3^j x\}}$$

From (3.14) we must have

$$\widehat{\varphi_j}(3^j k) = \widehat{\varphi}(k),$$

while the other coefficients of φ_j must vanish. Now for any $j \in \mathbb{N}$ and $m > 1$, we apply Parseval's identity together with the fact that $\operatorname{supp}(\varphi_j) \cap \mathscr{C} = \emptyset$. Namely, since $\mu \in \mathscr{P}(\mathscr{C})$ we have

$$\begin{aligned} 0 = \int_0^1 \varphi_j d\mu &= \sum_{k\in\mathbb{Z}} \overline{\widehat{\varphi_j}(k)}\widehat{\mu}(k) = \sum_{k\in\mathbb{Z}} \overline{\widehat{\varphi_j}(3^j k)}\widehat{\mu}(3^j k) = \sum_{k\in\mathbb{Z}} \overline{\widehat{\varphi}(k)}\widehat{\mu}(3^j k) \\ &= \overline{\widehat{\varphi_j}(0)}\widehat{\mu}(0) + \sum_{1\le|k|\le m} \overline{\widehat{\varphi}(k)}\widehat{\mu}(3^j k) + \sum_{|k|>m} \overline{\widehat{\varphi}(k)}\widehat{\mu}(3^j k). \end{aligned}$$

For the first term we simply have,

$$\overline{\widehat{\varphi_j}(0)}\widehat{\mu}(0) = \overline{\int_0^1 \varphi(x)dx} \int_0^1 d\mu = \mu(\mathscr{C}) = 1. \tag{3.15}$$

By choosing m large enough we can make the last term as small as we wish since $\varphi \in \mathscr{S}([0,1])$. In particular, we choose m such that $\sum_{|k|>m} |\widehat{\varphi}(k)| < 1/2$, which results in

$$\left| \sum_{|k|>m} \overline{\widehat{\varphi}(k)}\widehat{\mu}(3^j k) \right| \le \mu(\mathscr{C}) \sum_{|k|>m} |\widehat{\varphi}(k)| < \frac{1}{2}. \tag{3.16}$$

By (3.15) and (3.16), we must have

$$1 = \left| \sum_{|k|>m} \overline{\widehat{\varphi}(k)}\widehat{\mu}(3^j k) + \sum_{1\le|k|\le m} \overline{\widehat{\varphi}(k)}\widehat{\mu}(3^j k) \right| < \left| \sum_{1\le|k|\le m} \overline{\widehat{\varphi}(k)}\widehat{\mu}(3^j k) \right| + \frac{1}{2},$$

independent of how j is chosen. However, the last term can be estimated from above. Since m is fixed, we can choose j such that $\sup_{k\in\mathbb{Z},k\ge 3^j} |\widehat{\mu}(k)| < 1/(4m)$.

The choice of such a j follows from the assumption that $|\widehat{\mu}(k)| \to 0$ as $|k| \to \infty$. This means that the middle term has the upper bound

$$\left| \sum_{1 \le |k| \le m} \overline{\widehat{\varphi}(k)} \widehat{\mu}(3^j k) \right| \le \|\varphi\|_1 \sum_{1 \le |k| \le m} |\widehat{\mu}(3^j k)| \le m \sup_{k \in \mathbb{Z}, k \ge 3^j} |\widehat{\mu}(k)| < \frac{1}{4},$$

which leads to a contradiction. □

A consequence of proposition 3.6 is that the Fourier dimension of $\mathscr{C}$ must be zero. Namely, there cannot exist any probability measure supported on $\mathscr{C}$ where the Fourier transform goes to zero at infinity. As such, for any $\mu \in \mathscr{P}(\mathscr{C})$, it is not possible achieve the bound $|\widehat{\mu}(\xi)| \le C|\xi|^{-s}$ for any $s > 0$. From the definition of the Fourier dimension, it follows that

$$\dim_{\mathscr{F}}(\mathscr{C}) = 0.$$

Since we have already shown that $\dim_{\mathscr{H}}(\mathscr{C}) = \log(2)/\log(3)$, it follows that $\mathscr{C}$ is not a Salem set.

Even though $\mathscr{C}$ is not a Salem set, it serves its purpose as an inspiration on how to construct Salem sets. As it turns out, it is possible to construct random Cantor sets that are almost surely Salem sets. This was done by Salem in [9], and later by Bluhm in [1]. It is the construction of a random Cantor set we will consider in the next section.

4 Random Cantor Sets

A consequence of proposition 3.6 was that the standard 1/3-Cantor set, $\mathscr{C}$, could not be a Salem set. The proof relied on an exploit of the regularity of $\mathscr{C}$, where we could create a sequence of Schwartz functions which were not supported on $\mathscr{C}$. One way to overcome this, is to introduce some kind of randomness into the construction of the Cantor set. This was first done by Salem, in 1950, where random contractions were used at each step [9].

We have already seen that the interval $[-1, 1]$ is a Salem set with dimension 1 in $\mathbb{R}$. Since any probability measure supported on $[-1, 1]$ is also a probability measure on $\mathbb{R}$, we know that $\mathbb{R}$ is also a Salem set of dimension 1. We will see that through Salem's construction, it is possible to create a random Salem set of any dimension $\alpha \in (0, 1)$.

In the spirit of Salem's construction, in an article from 1996, by fixing $\alpha \in (0, n)$, Bluhm presented a recursive method of constructing a random Salem set in $\mathbb{R}^n$ with dimension α, [1]. Bluhm's random construction differs from Salem by using random translations rather than random contractions. It is the method by Bluhm, with random translations at each step, which we consider in this section.

In order to give a simple and clear presentation, we only consider the construction in dimension one. Moreover, all random variables will be uniformly distributed in the unit interval. This is done to avoid imposing any extra conditions on our random variables. For a more general construction in higher dimensions, we refer to Bluhm's original article, [1].

4.1 Facts about Cantor Sets

When defining $\mathscr{C}$, it is common to start with the unit interval and then recursively remove the middle 1/3 part of the remaining intervals at each step. Although this construction is quite easy to comprehend, it is not the construction we will consider in this section. In fact, the construction we will present may seem more complicated, yet has the benefit of generalizing easier. Moreover, the type of Cantor set construction presented below easily incorporates randomness when needed.

Let us now introduce a recursive construction of Cantor sets by translations and contractions. We start with a compact interval $C_0 = [0, c] \subset \mathbb{R}$ for some $c > 0$, and a fixed $\alpha \in (0, 1)$. At each step k, we denote by $N_k \geq 2$ the number of translations, while ρ_k denotes the contraction of the previous set at step k. Further, we impose the condition $N_k \rho_k^\alpha = 1$ at each step k. We will always let $N_0 = \rho_0 = 1$, as the zero'th step can be thought of as a single translation of the original set C_0 by 0. Further, let $\{X_j^k\}_{j=1}^{N_k}$ be the collection of non-negative translations at step k, which all satisfy $X_j^k \leq (1-\rho_k)c$. The condition on the translations is imposed to ensure that we cannot translate outside of the original set C_0. We now construct the Cantor set C, which depends on the choice of translations and contractions as follows

$$\begin{aligned}
C_0 =&[0,c],\\
C_1 =&\bigcup_{\nu_1=1}^{N_1}\left(X_{\nu_1}^1+\rho_1 C_0\right),\\
C_2 =&\bigcup_{\nu_2=1}^{N_2}\bigcup_{\nu_1=1}^{N_1}\left(X_{\nu_1}^1+\rho_1 X_{\nu_2}^2+\rho_2\rho_1 C_0\right),\\
&\vdots\\
C_k =&\bigcup_{\nu_k=1}^{N_k}\cdots\bigcup_{\nu_1=1}^{N_1}\left(\sum_{j=1}^{k}\left(\prod_{l=0}^{j-1}\rho_l\right)X_{\nu_j}^j+\left(\prod_{l=0}^{k}\rho_l\right)C_0\right),\\
C :=&\bigcap_{k=1}^{\infty}C_k.
\end{aligned}$$

Before continuing there are a few aspects of the construction we would like to emphasise. Firstly, the translation corresponding to step k, X_j^k, is contracted by a factor of $\prod_{l=1}^{k-1}\rho_l$ for each $j\in\{1,\ldots,N_k\}$. Since $X_j^k\leq(1-\rho_k)c$, this ensures that

$$X_j^k+\rho_k C_0\subset C_0 \Rightarrow \left(\prod_{l=1}^{k-1}\rho_l\right)X_j^k+\left(\prod_{l=1}^{k}\rho_l\right)C_0\subset\left(\prod_{l=1}^{k-1}\rho_l\right)C_0. \tag{4.1}$$

Thus, this condition is included to ensure that the translations remain inside of previously scaled sets.

Secondly, at each step k there is a translation, which has been properly scaled, from each of the previous steps present. This is done to make a nested sequence of sets. To see why, we note that for any choice of translations it follows from (4.1) that

$$\sum_{j=1}^{k}\left(\prod_{l=0}^{j-1}\rho_l\right)X_{\nu_j}^j+\left(\prod_{l=0}^{k}\rho_l\right)C_0\subset\sum_{j=1}^{k-1}\left(\prod_{l=0}^{j-1}\rho_l\right)X_{\nu_j}^j+\left(\prod_{l=0}^{k-1}\rho_l\right)C_0\subset C_{k-1}.$$

Whence, it follows that $C_k\subset C_{k-1}$ for all k. In particular, we have a nested sequence of sets,

$$C_0\supset C_1\supset C_2\supset\ldots,$$

which consequently ensures that $C \neq \emptyset$.

Finally, we note that no conditions of overlapping are imposed. It may happen that two translated sets will have some overlap, meaning that given two sequences $\nu^1 = (\nu_1^1, \nu_2^1, \dots \nu_k^1)$ and $\nu^2 = (\nu_1^2, \nu_2^2, \dots \nu_k^2)$, we might have

$$\left(\sum_{j=1}^{k}\left(\prod_{l=0}^{j-1}\rho_l\right)X^j_{\nu^1_j} + \left(\prod_{l=0}^{k}\rho_l\right)C_0\right)\bigcap\left(\sum_{j=1}^{k}\left(\prod_{l=0}^{j-1}\rho_l\right)X^j_{\nu^2_j} + \left(\prod_{l=0}^{k}\rho_l\right)C_0\right) \neq \emptyset.$$

This may happen if $|X_i^k - X_j^k| < \rho_k$ for $i \neq j$, or in the extreme case that all X_j^k are equal. For the latter case, all the sets making up C_k will overlap completely for each k. This will result in the Cantor set, C, consisting of only a single point.

Let us go back to the standard 1/3-Cantor set, and see how the construction above fits with $\mathscr{C}$. Here we have $N_k = 2$, where the translations are given by $X_1^k = 0$ and $X_2^k = 2/3$. The contractions are all given by $\rho_k = 1/3$, and so $\alpha = \log(2)/\log(3)$. Recall that α is nothing but the Hausdorff dimension of $\mathscr{C}$. In fact, as the next proposition shows, α is an upper bound for the Hausdorff dimension of the Cantor set C as defined above.

Proposition 4.1. *Let C be defined as above. Then* $\dim_{\mathscr{H}}(C) \leq \alpha$.

Proof. We note that C_k is a cover of C for any k. Moreover, we have $N_j \geq 2$, and $N_j\rho_j^\alpha = 1$. This means that

$$\rho_j = N_j^{-\frac{1}{\alpha}} \leq 2^{-\frac{1}{\alpha}} \leq 2^{-1},$$

for any $k \in \mathbb{N}$. At each step j, there are N_j new translations. This means that the set C_k consist of $\prod_{j=1}^k N_j$ translated versions of the set $\left(\prod_{l=0}^k \rho_l\right)[0, c]$. Now, for any $\varepsilon > 0$, we can find k_ε such that

$$c\prod_{l=0}^{k}\rho_l < \varepsilon,$$

for any $k > k_\varepsilon$. Hence, for any $s \geq 0$, we have

$$\mathscr{H}_\varepsilon^s(C) \leq \prod_{j=1}^{k} N_k \left(c\prod_{l=0}^{k}\rho_l\right)^s = \prod_{j=0}^{k}(N_j\rho_j^\alpha)\rho_j^{s-\alpha}c^s \leq 2^{k(\alpha-s)}c^s,$$

where we used that $N_0 = 1$, and that $N_j\rho_j^\alpha = 1$ for all j. However, when $s > \alpha$, we have

$$2^{k(\alpha-s)} \xrightarrow{k\to\infty} 0.$$

This gives an upper bound for the Hausdorff measure,

$$\mathscr{H}^s(C) \leq 0,$$

for all $s > \alpha$. From the definition of the Hausdorff dimension it follows that $\dim_{\mathscr{H}}(C) \leq \alpha$. □

Let us make a small comment on proposition 4.1. We noted that the construction allowed for overlap. In particular, the inequality in proposition 4.1 may be strict depending on how the translations are chosen. To see why, let $X_j^k = 0$ for all $k \in \mathbb{N}$ and $j \in \{1, \ldots, N_j\}$. Then $C = \{0\}$, which is finite. Since any finite set has Hausdorff dimension 0, we see that $\dim_{\mathscr{H}}(C) = 0 < \alpha$ for any $\alpha > 0$.

For simplicity, we define for each $k \in \mathbb{N}$ the path set,

$$\mathscr{D}_k = \prod_{j=1}^{k} \{1, \ldots, N_j\} = \{1, \ldots, N_1\} \times \ldots \times \{1, \ldots, N_k\}$$

where $\times$ denotes the Cartesian product. We note that each $\nu = (\nu_1, \ldots, \nu_k) \in \mathscr{D}_k$ gives rise to one of the sets making up C_k through the mapping

$$\nu \mapsto \sum_{j=1}^{k} \left(\prod_{l=0}^{j-1} \rho_l \right) X_{\nu_j}^j + \left(\prod_{l=0}^{k} \rho_l \right) C_0 \subset C_k.$$

Hence, we can think of each $\nu \in \mathscr{D}_k$ as a path down to a specific set in C_k. If for $\nu \in \mathscr{D}_k$ we define a new translation by

$$X_\nu = \sum_{j=1}^{k} \left(\prod_{l=0}^{j-1} \rho_l \right) X_{\nu_j}^j, \tag{4.2}$$

then we can write C_k in the more compact form

$$C_k = \bigcup_{\nu \in \mathscr{D}_k} \left(X_\nu + \left(\prod_{l=0}^{k} \rho_l \right) C_0 \right). \tag{4.3}$$

An interesting fact, is that any element $x \in C$ has a representation in terms of these translations.

Lemma 4.2. *Let C be a Cantor set as defined above. Then $x \in C$ if and only if it can be written on the form*

$$x = \lim_{k \to \infty} \sum_{j=1}^{k} \left(\prod_{l=0}^{j-1} \rho_l \right) X_{\nu_j(x)}^j,$$

for some sequence $\{\nu_j(x)\}_{j \in \mathbb{N}}$, where $\nu_j(x) \in \{1, \ldots, N_j\}$ for all $j \in \mathbb{N}$.

Proof. Let us start by showing that the limit exists for any sequence $\{\nu_j\}_{j\in\mathbb{N}}$ where $\nu_j \in \{1, \dots, N_j\}$. Since $\rho_j = N_j^{-\frac{1}{\alpha}} \le 2^{-1}$, we have the estimate

$$\begin{aligned}
\lim_{k\to\infty} \left| \sum_{j=1}^{k} \left(\prod_{l=0}^{j-1} \rho_l \right) X^i_{\nu_j} \right| \le & \lim_{k\to\infty} \sum_{j=1}^{k} 2^{-(j-1)} |X^j_{\nu_j}| \\
< & c \lim_{k\to\infty} \sum_{j=0}^{k} 2^{-j} \\
= & 2c < \infty,
\end{aligned}$$

where we used that $(1-\rho_k) < 1$ for all k, as well as $\rho_0 = 1$. This shows that the sequence converges absolutely, since $c > 0$ is bounded. Thus, there exists a $z \in \mathbb{R}$ such that

$$z = \lim_{k\to\infty} \sum_{j=1}^{k} \left(\prod_{l=0}^{j-1} \rho_l \right) X^j_{\nu_j}. \tag{4.4}$$

Now assume that $x \in C$. From the definition of C, we know that $x \in C_k$ for all $k \in \mathbb{N}$. So for each $k \in \mathbb{N}$, there is a path $\nu(x) = (\nu_1(x), \dots, \nu_k(x)) \in \mathscr{D}_k$ such that

$$x \in \left(\sum_{j=1}^{k} \left(\prod_{l=0}^{j-1} \rho_l \right) X^j_{\nu_j(x)} + \left(\prod_{l=0}^{k} \rho_l \right) C_0 \right).$$

Since this holds for any $k \in \mathbb{N}$, x must also lie in the limit set as $k \to \infty$, and so,

$$x \in \lim_{k\to\infty} \left(\sum_{j=1}^{k} \left(\prod_{l=0}^{j-1} \rho_l \right) X^j_{\nu_j(x)} + \left(\prod_{l=0}^{k} \rho_l \right) C_0 \right).$$

However, since $C_0 = [0, c]$ for some $c > 0$, it follows that

$$\lim_{k\to\infty} \left(\prod_{l=0}^{k} \rho_l \right) C_0 = \{0\}.$$

To see why, let $y \in C_0$, and note that

$$\prod_{l=0}^{k} \rho_j y \le 2^{-k} y \xrightarrow{k\to\infty} 0.$$

What this means, is that

$$\begin{aligned} x &\in \lim_{k\to\infty}\left(\sum_{j=1}^{k}\left(\prod_{l=0}^{j-1}\rho_l\right)X^j_{\nu_j(x)} + \left(\prod_{l=0}^{k}\rho_l\right)C_0\right) \\ &= \lim_{k\to\infty}\left(\sum_{j=1}^{k}\left(\prod_{l=0}^{j-1}\rho_l\right)X^j_{\nu_j(x)} + \{0\}\right), \\ &= \left\{\lim_{k\to\infty}\sum_{j=1}^{k}\left(\prod_{l=0}^{j-1}\rho_l\right)X^j_{\nu_j(x)}\right\} \end{aligned}$$

which is a one-point set. Whence it follows that

$$x = \lim_{k\to\infty}\sum_{j=1}^{k}\left(\prod_{l=0}^{j-1}\rho_l\right)X^j_{\nu_j(x)}.$$

For the opposite claim, we need to show that for any path sequence $\nu = (\nu_1, \nu_2, \ldots)$,

$$z(\nu) = \lim_{k\to\infty}\sum_{j=1}^{k}\left(\prod_{l=0}^{j-1}\rho_l\right)X^j_{\nu_j} \in C_m,$$

for all $m \in \mathbb{N}$. Let $\nu = (\nu_1, \nu_2, \ldots)$ be an arbitrary path sequence. Then, since $C_0 = [0, c]$, we know that

$$\sum_{j=1}^{k}\left(\prod_{l=0}^{j-1}\rho_l\right)X^j_{\nu_j} + \left(\prod_{l=0}^{k}\rho_l\right)[0,c] \subset C_k,$$

for all k.

Now fix k. Then we want to show that $z(\nu) \in C_k$. Since $C_k \subset C_{k-1}$, for all $k \in \mathbb{N}$ by construction, it follows that $z(\nu)$ also lies in all C_m, for $m < k$. However, in order to show that the entire limit lies in C_k, we split the limit into two parts, one containing the elements up to k, and one for the remaining part. We then estimate the remaining part. Since we can write $z(\nu)$ as

$$z(\nu) = \lim_{m\to\infty}\left(\sum_{j=1}^{k}\left(\prod_{l=0}^{j-1}\rho_l\right)X^j_{\nu_j} + \sum_{j=k+1}^{m}\left(\prod_{l=0}^{j-1}\rho_l\right)X^j_{\nu_j}\right),$$

for any k, we want to estimate the remaining term and show that it will not translate outside of C_k. In particular, we need to show that the remaining term is bounded by $\left(\prod_{l=0}^{k}\rho_l\right)c$, as this ensures that $z(\nu)\in C_k$. By using the fact that $X_{\nu_i}^i \leq (1-\rho_i)c$, we can estimate the remaining term by a telescoping sum,

$$\begin{aligned}\left|\sum_{j=k+1}^{m}\left(\prod_{l=0}^{j-1}\rho_l\right)X_{\nu_j}^j\right| &\leq \sum_{j=k+1}^{m}\left(\prod_{l=0}^{j-1}\rho_l\right)(1-\rho_j)c \\ &= \left(\left(\prod_{l=0}^{k}\rho_l\right)-\left(\prod_{l=0}^{m}\rho_l\right)\right)c \\ &\leq \left(\prod_{l=0}^{k}\rho_l\right)c.\end{aligned}$$

This shows that

$$\lim_{m\to\infty}\left(\sum_{i=k+1}^{m}\left(\prod_{l=0}^{i-1}\rho_l\right)X_{\nu_i}^i\right)\in\left(\prod_{l=0}^{k}\rho_l\right)[0,c],$$

and so we must have $z(\nu)\in C_k$. Since k was arbitrary, the same argument holds for any other $k\in\mathbb{N}$. It therefore follows that

$$z(\nu)\in\bigcap_{k=1}^{\infty}C_k = C.$$

□

If we now go back to $\mathscr{C}$, the translations were given by $X_1^k = 0$, and $X_2^k = 2/3$. As such, we see that any $x\in\mathscr{C}$ can be written in the form

$$x=\sum_{j=1}^{\infty}\frac{a_j}{3^j},\quad a_j\in\{0,2\}.$$

By using lemma 4.2, we can define a sequence of probability measures $\{\mu_k\}_{k\in\mathbb{N}}$, where $\operatorname{supp}\mu_k\subset C_k$ for each $k\in\mathbb{N}$. This is done through a sum of Dirac δ-measures running over all possible paths up to step k. As such, we will use the path notation for the translations, as seen in (4.2). Furthermore, we divide the sum by $\prod_{i=1}^{k}N_i<\infty$, as this is the number of different paths in $\mathscr{D}_k$. Hence, we have a sequence of probability measures given by

$$\mu_k=\frac{1}{\prod_{j=1}^{k}N_j}\sum_{\nu\in\mathscr{D}_k}\delta_{X_\nu}=\frac{1}{\prod_{j=1}^{k}N_j}\sum_{\nu\in\mathscr{D}_k}\delta_{\sum_{m=1}^{k}\left(\prod_{l=0}^{m-1}\rho_l\right)X_{\nu_m}^m}.$$

The Fourier transform of the measure μ_k is simply given by

$$\widehat{\mu}_k(\xi) = \frac{1}{\prod_{j=1}^{k} N_j} \sum_{\nu \in \mathscr{D}_k} \int_{\mathbb{R}} e^{-2\pi i \xi x} d\delta_{X_\nu}(x) = \frac{1}{\prod_{j=1}^{k} N_j} \sum_{\nu \in \mathscr{D}_k} e^{-2\pi i \xi X_\nu},$$

for each $k \in \mathbb{N}$. We want to show that this sequence of probability measures converges weakly to a probability measure μ_α supported on the Cantor set. To help in our calculations, we first introduce a small lemma that simplifies the expression of $\widehat{\mu}_k$.

Lemma 4.3. *For each $k \in \mathbb{N}$, the Fourier transform of μ_k can be written as*

$$\widehat{\mu}_k(\xi) = \prod_{j=1}^{k} \left(\frac{1}{N_j} \sum_{\nu_j=1}^{N_j} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1} \rho_l\right) X_{\nu_j}^j} \right).$$

Proof. The proof follows from a simple calculation, where we multiply out the product on the right hand side,

$$\prod_{j=1}^{k} \left(\frac{1}{N_j} \sum_{\nu_j=1}^{N_j} e^{-2\pi i \xi \left(\prod_{l=0}^{i-1} \rho_l\right) X_{\nu_j}^j} \right) = \frac{\left(\sum_{\nu_1=1}^{N_1} e^{-2\pi i \xi X_{\nu_1}^1}\right) \cdots \left(\sum_{\nu_k=1}^{N_k} e^{-2\pi i \xi \left(\prod_{l=0}^{k-1} \rho_l\right) X_{\nu_k}^k}\right)}{\prod_{j=1}^{k} N_j}.$$

The result now follows from the distributive law, and some basic properties of the exponential function. Namely,

$$\begin{aligned}
&\frac{1}{\prod_{j=1}^{k} N_j} \left(\sum_{\nu_1=1}^{N_1} e^{-2\pi i \xi X_{\nu_1}^1} \right) \cdots \left(\sum_{\nu_k=1}^{N_k} e^{-2\pi i \xi \left(\prod_{l=0}^{k-1} \rho_l\right) X_{\nu_k}^k} \right) \\
&= \frac{1}{\prod_{j=1}^{k} N_j} \left(\sum_{\nu_1=1}^{N_1} \cdots \sum_{\nu_k=1}^{N_k} \prod_{j=1}^{k} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1}\right) X_{\nu_j}^j} \right) \\
&= \frac{1}{\prod_{j=1}^{k} N_j} \left(\sum_{\nu_1=1}^{N_1} \cdots \sum_{\nu_k=1}^{N_k} e^{-2\pi i \xi \sum_{j=1}^{k} \left(\prod_{l=0}^{j-1}\right) X_{\nu_j}^j} \right) \\
&= \frac{1}{\prod_{j=1}^{k} N_j} \sum_{\nu \in \mathscr{D}_k} e^{-2\pi i \xi X_\nu} = \widehat{\mu}_k(\xi).
\end{aligned}$$

□

One way to show that the sequence $\{\mu_k\}_{k \in \mathbb{N}}$ converges weakly, is by showing that the Fourier transform converges pointwise. Then the result follows from proposition 2.10. In order to show pointwise convergence, we first include a small lemma on the difference of complex exponential functions.

Lemma 4.4. *Assume that $x, y \in \mathbb{R}$. Then for any fixed $\xi \in \mathbb{R}$,*

$$|e^{-2\pi i\xi x} - e^{-2\pi i\xi y}| \leq |x-y||2\pi\xi|.$$

Proof. We will prove the lemma using the fact that $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$, and then use the mean value theorem for vector valued functions on $\mathbb{R}^2$. For simplicity, let us define $\theta := -2\pi\xi$. We will assume that $x > y$. Now, recall that

$$\begin{aligned}\Re(e^{i\theta x} - e^{i\theta y}) &= \cos(\theta x) - \cos(\theta y),\\ \Im(e^{i\theta x} - e^{i\theta y}) &= \sin(\theta x) - \sin(\theta y).\end{aligned}$$

So when estimating the difference, we are left with

$$\begin{aligned}|e^{-2\pi i\xi x} - e^{-2\pi i\xi y}|^2 &= (\cos(\theta x) - \cos(\theta y))^2 + (\sin(\theta x) - \sin(\theta y))^2\\ &= |f_\theta(x) - f_\theta(y)|^2\end{aligned}$$

where $f_\theta : \mathbb{R} \to \mathbb{R}^2$ is given by $f_\theta(x) = (\cos(\theta x), \sin(\theta x))$. By the mean value theorem, there exists a $c \in (x, y) \subset \mathbb{R}$ such that

$$\frac{|f_\theta(x) - f_\theta(y)|}{|x-y|} \leq |f'_\theta(c)|.$$

The derivative of f_θ is given by

$$f'_\theta(c) = \theta(-\sin(\theta c), \cos(\theta c)) \Rightarrow |f'_\theta(c)| = |\theta| = |2\pi\xi|.$$

It thus follows that

$$|e^{-2\pi i\xi x} - e^{-2\pi i\xi y}| = |f_\theta(x) - f_\theta(y)| \leq |x-y||2\pi\xi|.$$

□

Proposition 4.5. *Let the sequence of probability measures μ_k be defined by*

$$\mu_k = \frac{1}{\prod_{i=1}^k N_i} \sum_{\nu \in \mathscr{D}_k} \delta_{X_\nu}.$$

Then there exists a probability measure $\mu_\alpha \in \mathscr{P}(C)$, such that $\mu_k \rightharpoonup \mu_\alpha$.

Proof. For the weak convergence, it is enough to show that the Fourier transform of the sequence is a Cauchy sequence for each fixed ξ, and thus converges pointwise. Assume that ξ is fixed. Then by lemma 4.3 we have for $n > m$,

$$
\begin{aligned}
&|\widehat{\mu}_n(\xi) - \widehat{\mu}_m(\xi)| \\
=&\left|\prod_{j=1}^{n}\left(\frac{1}{N_j}\sum_{\nu_j=1}^{N_j} e^{-2\pi i\xi\left(\prod_{l=0}^{j-1}\rho_l\right)X_{\nu_j}^j}\right) - \prod_{j=1}^{m}\left(\frac{1}{N_j}\sum_{\nu_j=1}^{N_j} e^{-2\pi i\xi\left(\prod_{l=0}^{j-1}\rho_l\right)X_{\nu_j}^j}\right)\right| \\
=&\left|\prod_{j=1}^{m}\left(\frac{1}{N_j}\sum_{\nu_j=1}^{N_j} e^{-2\pi i\xi\left(\prod_{l=0}^{j-1}\rho_l\right)X_{\nu_j}^j}\right)\left(\prod_{j=m+1}^{n}\left(\frac{1}{N_j}\sum_{\nu_j=1}^{N_j} e^{-2\pi i\xi\left(\prod_{l=0}^{j-1}\rho_l\right)X_{\nu_j}^j}\right) - 1\right)\right| \\
\leq&\left|1 - \prod_{j=m+1}^{n}\left(\frac{1}{N_j}\sum_{\nu_j=1}^{N_i} e^{-2\pi i\xi\left(\prod_{l=0}^{j-1}\rho_l\right)X_{\nu_j}^j}\right)\right|,
\end{aligned}
$$

as the first term is simply $|\widehat{\mu}_m(\xi)| \leq 1$ for all $\xi \in \mathbb{R}$. If we multiply out the product, we are left with sum over all possible path combinations from $j = m+1$ to $j = n$. Since at each step j there are N_j different possibilities, we can rewrite the difference as,

$$
\begin{aligned}
&\left|1 - \prod_{j=m+1}^{n}\left(\frac{1}{N_j}\sum_{\nu_j=1}^{N_j} e^{-2\pi i\xi\left(\prod_{l=0}^{j-1}\rho_l\right)X_{\nu_j}^j}\right)\right| \\
=&\frac{1}{\prod_{j=m+1}^{n} N_j}\left|\sum_{\nu_{m+1}=1}^{N_{m+1}}\cdots\sum_{\nu_n=1}^{N_n}\left(1 - \prod_{j=m+1}^{n} e^{-2\pi i\xi\left(\prod_{l=0}^{j-1}\rho_l\right)X_{\nu_j}^j}\right)\right| \\
\leq&\frac{1}{\prod_{j=m+1}^{n} N_j}\sum_{\nu_{m+1}=1}^{N_{m+1}}\cdots\sum_{\nu_n=1}^{N_n}\left|e^{-2\pi i\xi\cdot 0} - e^{-2\pi i\xi\sum_{j=m+1}^{n}\left(\prod_{l=0}^{j-1}\rho_l\right)X_{\nu_j}^j}\right|.
\end{aligned}
$$

It now follows from lemma 4.4 that

$$
\begin{aligned}
\left|e^{-2\pi i\xi\cdot 0} - e^{-2\pi i\xi\sum_{j=m+1}^{n}\left(\prod_{l=0}^{j-1}\rho_l\right)X_{\nu_j}^j}\right| \leq&|2\pi\xi|\left|\sum_{j=m+1}^{n}\left(\prod_{l=0}^{j-1}\rho_l\right)X_{\nu_j}^j\right| \\
\leq&2^{-m}|2\pi\xi c|\sum_{i=1}^{n-m}2^{-i+1} \\
\leq&2^{-m+1}|2\pi\xi c|
\end{aligned}
$$

where we used the fact that $X_j^k \leq (1-\rho_k)c < c$ for all possible choices of j

and k, $\rho_0 = 1$, and that $\rho_l < 2^{-1}$ for all $l \geq 1$. This then implies that

$$\frac{1}{\prod_{j=m+1}^{n} N_j} \sum_{\nu_{m+1}=1}^{N_{m+1}} \cdots \sum_{\nu_n=1}^{N_n} \left| e^{-2\pi i \xi \cdot 0} - e^{-2\pi i \xi \sum_{j=m+1}^{n} \left(\prod_{l=0}^{j-1} \rho_l\right) X_{\nu_j}^j} \right|$$
$$\leq \frac{1}{\prod_{j=m+1}^{n} N_j} \sum_{\nu_{m+1}=1}^{N_{m+1}} \cdots \sum_{\nu_n=1}^{N_n} 2^{-m+1} |2\pi\xi c| = 2^{-m+1}|2\pi\xi c|,$$

and so for $n > m$, we have

$$|\widehat{\mu}_n(\xi) - \widehat{\mu}_m(\xi)| \leq 2^{-m+1}|2\pi\xi c|.$$

Now to show that the sequence is Cauchy, we let $\varepsilon > 0$. Since $\xi \in \mathbb{R}$ is fixed, there exists $N_\xi \in \mathbb{N}$, such that

$$2^{-N_\xi+1} < \frac{\varepsilon}{|2\pi\xi c|}.$$

Whence it follows that for any $n > m > N_\xi$, we have

$$|\widehat{\mu}_n(\xi) - \widehat{\mu}_m(\xi)| \leq 2^{-m+1}|2\pi\xi c| < \varepsilon,$$

and so the sequence is Cauchy in $\mathbb{R}$ for each fixed ξ. Since ξ was arbitrary and we have a sequence of probability measures $\{\mu_k\}_{k\in\mathbb{N}}$, it follows from proposition 2.9 that $\widehat{\mu}_k(\xi)$ converges pointwise to some $\widehat{\mu}_\alpha(\xi)$ for every $\xi \in \mathbb{R}$. This again means that μ_k converges weakly to μ_α by proposition 2.10. Moreover, lemma 4.2 ensures that supp $\mu_\alpha \subset C$. Finally, to see that μ_α actually is a probability measure, we let $\chi_{C_0} \in C(C_0)$ dentote the characteristic function on C_0, and note that

$$\mu_\alpha(\mathbb{R}) = \int_{\mathbb{R}} \chi_{C_0} d\mu_\alpha = \lim_{k\to\infty} \int_{\mathbb{R}} \chi_{C_0} d\mu_k = 1.$$

□

4.2 The Transition to a Random Cantor Set

Let us now take the construction in section 4.1, and replace the translations by random variables. Throughout this section we let $\alpha \in (0,1)$ be a fixed number. As in the case of the standard Cantor set, we will start with the unit interval $C_0 = [0,1]$. However, we will choose the sequence $N_k = k+1$ for the number of translations at step k. We still require $(k+1)\rho_k^\alpha = 1$, where ρ_k is the contraction at step k.

There is nothing special about the choice $N_k = k+1$; it simply makes for an easier construction. What is important in order to achieve an estimate for the Fourier dimension is to have a sequence, N_k, which grows to infinity, but not too fast. We will return to this matter later on. For now, let us focus on the definition of a random Cantor set.

Let $(\Omega, \mathscr{A}, P)$ denote a probability space. The translations will now be given by independent random variables $X_j^k : \Omega \to [0, 1-\rho_k]$, which are uniformly distributed on the interval $[0, 1-\rho_k]$. By uniformly distributed on an interval $[0, L]$, we mean that for any subinterval $(a, b) \subset [0, L]$,

$$P(\{\omega \in \Omega : X_j^k(\omega) \in (a, b)\}) = \frac{b-a}{L}. \tag{4.5}$$

Def. 4.1. Let $\left\{\{X_j^k : \Omega \to [0, 1-\rho_k]\}_{j=1}^{k+1}\right\}_{k=1}^{\infty}$ be a collection of independent uniformly distributed random variables. For each $\omega \in \Omega$, we define the random Cantor set by

$$\mathscr{C}_R(\omega) := \bigcap_{k=1}^{\infty} \bigcup_{\nu_k=1}^{k+1} \cdots \bigcup_{\nu_1=1}^{2} \left(\sum_{j=1}^{k} \left(\prod_{l=0}^{j-1} \rho_l \right) X_{\nu_j}^j(\omega) + \left(\prod_{l=0}^{k} \rho_l \right) [0,1] \right).$$

By using the path notation, given in (4.2), we can write the random Cantor set on the more compact form,

$$\mathscr{C}_R(\omega) = \bigcap_{k=1}^{\infty} \bigcup_{\nu \in \mathscr{D}_k} \left(X_\nu(\omega) + \left(\prod_{l=0}^{k} \rho_l \right) [0,1] \right).$$

By proposition 4.1, it follows that $\dim_{\mathscr{H}}(\mathscr{C}_R(\omega)) \leq \alpha$. Furthermore, it follows from proposition 4.5 that there exist a probability measure $\mu_\alpha(\omega)$ supported on $\mathscr{C}_R(\omega)$, which is the weak limit of the sequence

$$\mu_k(\omega) = \frac{1}{\prod_{i=1}^{k} N_k} \sum_{\nu \in \mathscr{D}_k} \delta_{X_\nu(\omega)}.$$

The method used to find a lower bound on the Fourier dimension of $\mathscr{C}_R(\omega)$ involves estimating the expected value of $\widehat{\mu}_k$. For this process we first need a few facts about the decay properties for the expected value of exponential functions.

Lemma 4.6. *Let $X : \Omega \to \mathbb{R}$ be a random variable uniformly distributed on the interval $[0, L]$. Then for every $\xi \in \mathbb{R}\backslash\{0\}$, we have the bound*

$$|\mathbb{E}(e_\xi \circ X)| = \left| \int_\Omega e^{-2\pi i \xi X(\omega)} dP(\omega) \right| \leq \frac{1}{\pi L |\xi|}.$$

Proof. In order to calculate the integral, we need to know what the push-forward measure X_*P is, as

$$\int_\Omega e^{-2\pi i\xi X(\omega)}dP(\omega) = \int_{X^{-1}(\Omega)} e^{-2\pi i\xi x}d(X_*P(x)).$$

Since X is uniformly distributed, it follows from (4.5) that for any subinterval $(a,b)\subset[0,L]$,

$$X_*P((a,b)) = P(X^{-1}(a,b)) = \frac{b-a}{L}.$$

This shows that the push-forward measure is nothing but the normalized Lebesgue measure on $[0,L]$. Whence it follows that

$$\int_\Omega e^{-2\pi i\xi X(\omega)}dP(\omega) = \int_0^L e^{-2\pi i\xi x}\frac{dx}{L} = \frac{1-e^{-2\pi iL\xi}}{2\pi iL\xi},$$

and using the triangle inequality, we have

$$\left|\int_\Omega e^{-2\pi i\xi X(\omega)}dP(\omega)\right| \le \frac{1}{\pi L|\xi|}.$$

□

Corollary 4.6.1. *Let $\xi\in\mathbb{R}\backslash\{0\}$. Then for each $k\in\mathbb{N}$, and each $j\in\{1,\dots,N_k\}$*

$$\left|\mathbb{E}\left(e_\xi\circ X_j^k\right)\right| = \left|\int_\Omega e^{-2\pi i\xi X_j^k(\omega)}dP(\omega)\right| \le \frac{2}{\pi|\xi|}.$$

Proof. Since $\rho_k\le 2^{-1}$ for each $k\in\mathbb{N}$, it follows that $1-\rho_k\ge 2^{-1}$. Thus, by lemma 4.6 it follows that

$$\left|\int_\Omega e^{-2\pi i\xi X_j^k(\omega)}dP(\omega)\right| \le \frac{1}{\pi(1-\rho_k)|\xi|} \le \frac{2}{\pi|\xi|}.$$

□

4.3 Decay of Expected Values

The goal of this section is to estimate the expected value of $\widehat{\mu}_\alpha$ raised to the power $2q$ for each $q\in\mathbb{N}$. This will give rise to an almost surely convergent series, which we use to estimate the Fourier dimension. However, estimating the expected value can only provide an almost surely lower bound for the Fourier dimension.

Using lemma 4.4, the Fourier transform of $\mu_k(\omega)$ can be written as

$$\widehat{\mu}_k(\omega,\xi) = \prod_{j=1}^{k} \left(\frac{1}{j+1} \sum_{\nu_j=1}^{j+1} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1} \rho_l\right) X^j_{\nu_j}(\omega)} \right).$$

Since the random variables $X^j_{\nu_j}$ are independent, it follows that

$$\mathbb{E}(|\widehat{\mu}_k(\cdot,\xi)|^{2q}) = \prod_{j=1}^{k} \mathbb{E} \left(\left| \frac{1}{j+1} \sum_{\nu_j=1}^{j+1} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1} \rho_l\right) X^j_{\nu_j}} \right|^{2q} \right).$$

We will prove the following lemma.

Proposition 4.7. *Let $q, j \in \mathbb{N}$, and assume that for $\xi \in \mathbb{R}$,*

$$\frac{2(j+1)^q}{q^q \pi} \leq \left(\prod_{l=0}^{j-1} \rho_l \right) |\xi|.$$

We then have the bound

$$\mathbb{E} \left(\left| \frac{1}{j+1} \sum_{\nu_j=1}^{j+1} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1} \rho_l\right) X^j_{\nu_j}} \right|^{2q} \right) \leq \frac{2q^q}{(j+1)^q}$$

Proof. Let us start by multiplying out the left hand side. This results in

$$\begin{aligned}
&\left| \frac{1}{j+1} \sum_{\nu_j=1}^{j+1} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1} \rho_l\right) X^j_{\nu_j}(\omega)} \right|^{2q} \\
=& \left(\frac{1}{j+1} \sum_{\nu_j=1}^{j+1} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1} \rho_l\right) X^j_{\nu_j}(\omega)} \right)^q \overline{\left(\frac{1}{j+1} \sum_{\nu_j=1}^{j+1} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1} \rho_l\right) X^j_{\nu_j}(\omega)} \right)^q} \\
=& \frac{1}{(j+1)^{2q}} \sum_{\nu_{j_{2q}}=1}^{j+1} \cdots \sum_{\nu_{j_1}=1}^{j+1} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1} \rho_l\right) \sum_{i=1}^{q} \left(X^j_{\nu_{j_i}}(\omega) - X^j_{\nu_{j_{q+i}}}(\omega) \right)}.
\end{aligned}$$

Thus, the expected value can be written as

$$\begin{aligned}
&\mathbb{E} \left(\left| \frac{1}{j+1} \sum_{\nu_j=1}^{j+1} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1} \rho_l\right) X^j_{\nu_j}} \right|^{2q} \right) \\
=& \frac{1}{(j+1)^{2q}} \sum_{\nu_{j_{2q}}=1}^{j+1} \cdots \sum_{\nu_{j_1}=1}^{j+1} \mathbb{E} \left(e_{\xi\left(\prod_{l=0}^{j-1} \rho_l\right)} \circ \sum_{i=1}^{q} \left(X^j_{\nu_{j_i}}(\omega) - X^j_{\nu_{j_{q+i}}}(\omega) \right) \right). \quad (4.6)
\end{aligned}$$

We note that given any choice of j_i and j_{q+i}, we can find numbers $h_m \in \mathbb{Z}$ with $0 \leq |h_m| \leq q$ such that

$$\sum_{i=1}^{q} \left(X^j_{\nu_{j_i}}(\omega) - X^j_{\nu_{j_{q+i}}}(\omega) \right) = \sum_{m=1}^{j+1} h_m X^j_m(\omega).$$

Furthermore, any permutation, σ, of $\{1, \dots, q\}$ does not change the summation, that is

$$\sum_{i=1}^{q} X^j_{\nu_{j_i}}(\omega) = \sum_{i=1}^{q} X^j_{\nu_{j_{\sigma(i)}}}(\omega).$$

Thus, if there is some permutation σ such that the equality $\nu_i = \nu_{q+\sigma(i)}$ holds for all $i \in \{1, \dots, q\}$, then $h_m = 0$ for all m. This follows from the equality

$$\sum_{i=1}^{q} \left(X^j_{\nu_{j_i}}(\omega) - X^j_{\nu_{j_{q+i}}}(\omega) \right) = \sum_{i=1}^{q} \left(X^j_{\nu_{j_i}}(\omega) - X^j_{\nu_{j_{q+\sigma(i)}}}(\omega) \right) = 0.$$

In particular, if it happens that all $h_m = 0$, then

$$\mathbb{E}\left(e_{\xi(\prod_{l=0}^{j-1} \rho_l)} \circ \sum_{i=1}^{q} \left(X^j_{\nu_{j_i}}(\omega) - X^j_{\nu_{j_{q+i}}}(\omega) \right) \right) = \mathbb{E}(1) = 1.$$

On the other hand, if there exists at least one i_0 such that $\nu_{j_{i_0}} \neq \nu_{j_{q+i}}$ for any choice of i, then there has to exist an h_{i_0} for which $|h_{i_0}| \geq 1$. From the independence of the random variables it follows that,

$$\begin{aligned}\mathbb{E}\left(e_{\xi \prod_{l=0}^{j-1} \rho_l} \circ \sum_{m=1}^{j+1} h_m X^j_m \right) =& \mathbb{E}\left(\prod_{m=1}^{j+1} e_{\xi \prod_{l=0}^{j-1} \rho_l} \circ h_m X^j_m \right) \\ =& \prod_{m=1}^{j+1} \mathbb{E}\left(e_{\xi \prod_{l=0}^{j-1} \rho_l} \circ h_m X^j_m \right).\end{aligned}$$

Moreover, since there is the trivial bound,

$$|\mathbb{E}(e_\xi \circ X)| \leq \int_\Omega \left| e^{-2\pi i \xi X(\omega)} \right| dP(\omega) = 1,$$

for any random variable X, it follows from corollary 4.6.1 that

$$\left| \mathbb{E}\left(e_{\xi \prod_{l=0}^{j-1} \rho_l} \circ \sum_{m=1}^{j+1} h_m X^j_m \right) \right| \leq \left| \mathbb{E}\left(e_{\xi \prod_{l=0}^{j-1} \rho_l} \circ h_{i_0} X^j_{i_0} \right) \right| \leq \frac{2|\xi|^{-1}}{\pi \prod_{l=0}^{j-1} \rho_l}. \quad (4.7)$$

When estimating the sum we can therefore consider the two different cases separately. Let us first consider the case when all $h_m = 0$, and try to bound the number of times this can happen. If for the first q elements we fix ν_j, then there are at most $q!$ different ways to choose the remaining ν_{q+j} such that all $h_m = 0$. On the other hand, there are $(j+1)^q$ different ways to fix the first q elements. This implies that there cannot be more than $q!(j+1)^q$ different ways to obtain

$$\sum_{i=1}^{q} \left(X^j_{\nu_{j_i}}(\omega) - X^j_{\nu_{j_{q+i}}}(\omega) \right) = 0. \tag{4.8}$$

Since (4.8) can hold for at most $q!(j+1)^q$ different terms, it can be combined with the bound given in (4.7) for the remaining terms. Thus, by (4.6) there is an upper bound given by

$$\mathbb{E}\left(\left| \frac{1}{N_j} \sum_{\nu_j=1}^{j+1} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1} \rho_l\right) X^j_{\nu_j}} \right|^{2q} \right) \leq \frac{q!}{(j+1)^q} + \frac{2|\xi|^{-1}}{\pi \prod_{l=0}^{j-1} \rho_l}.$$

Here we used the fact that summing over elements where (4.8) is not true, yields a fraction less than 1 when dividing by the total amount of elements.

From here, we can invoke the condition on ξ, which is equivalent to

$$\frac{2|\xi|^{-1}}{\pi \prod_{l=0}^{j-1} \rho_l} \leq \frac{q^q}{(j+1)^q}.$$

Hence, combining the condition on ξ together with the fact that $q! \leq q^q$, for all $q \in \mathbb{N}$, we are left with

$$\mathbb{E}\left(\left| \frac{1}{j+1} \sum_{\nu_j=1}^{j+1} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1} \rho_l\right) X^j_{\nu_j}} \right|^{2q} \right) \leq \frac{2q^q}{(j+1)^q}.$$

□

A result similar to proposition 4.7 can be proved for a general sequence N_j. If the sequence N_j happens to be bounded by some $M > 0$, then for any $q > M$ the result gives no further information than the trivial bound,

$$\mathbb{E}\left(\left| \frac{1}{N_j} \sum_{\nu_j=1}^{N_j} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1} \rho_l\right) X^j_{\nu_j}} \right|^{2q} \right) \leq 1.$$

As a consequence, a general sequence N_j must grow to infinity as $j \to \infty$ in order to give meaningful results. In particular, the approach for estimating the Fourier dimension presented here would not work for a bounded sequence.

Since the random variables are independent, the expected value of $\widehat{\mu}_k(\omega, \xi)$ can be written as a product of expected values of the form found in proposition 4.7. It is this property we exploit in proving the next proposition, which gives a decay property on the expected value of the random Cantor probability measure μ_α on the Fourier side.

Proposition 4.8. *For any $q \in \mathbb{N}$, and any $0 < \theta < 1$ there exists a number $\Theta = \Theta(\alpha, \theta, q) > 0$ such that*

$$\mathbb{E}(|\widehat{\mu_\alpha}(\cdot, \xi)|^{2q}) \leq |\xi|^{-\theta\alpha q},$$

for all $\xi \in \mathbb{R}$ with $|\xi| \geq \Theta$.

Proof. Throughout this proof, the integer $q \in \mathbb{N}$, and the number $0 < \theta < 1$ will remain fixed. Let us make a few remarks regarding the expected value of $\widehat{\mu}_k$. By lemma 4.3, we can, for each $k \in \mathbb{N}$, write the Fourier transform of μ_k as

$$\widehat{\mu}_k(\omega, \xi) = \prod_{j=1}^{k} \left(\frac{1}{j+1} \sum_{\nu_j=1}^{j+1} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1} \rho_l \right) X_{\nu_j}^j(\omega)} \right).$$

Since all $X_{\nu_j}^j$ are independent, we can write the expected value of the product as a product of expected values. That is,

$$\mathbb{E}(|\widehat{\mu}_k(\cdot, \xi)|^{2q}) = \prod_{j=1}^{k} \mathbb{E} \left(\left| \frac{1}{j+1} \sum_{\nu_j=1}^{j+1} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1} \rho_l \right) X_{\nu_j}^j} \right|^{2q} \right).$$

For any $j \in \mathbb{N}$ there is always the trivial bound

$$\left| \frac{1}{j+1} \sum_{\nu_j=1}^{j+1} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1} \rho_l \right) X_{\nu_j}^j(\omega)} \right|^{2q} \leq \left(\frac{1}{j+1} \sum_{\nu_j=1}^{j+1} 1 \right)^{2q} = 1.$$

Thus, it follows from monotonicity of the Lebesgue integral that for any $j \in \mathbb{N}$,

$$\mathbb{E} \left(\left| \frac{1}{j+1} \sum_{\nu_j=1}^{j+1} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1} \rho_l \right) X_{\nu_j}^j} \right|^{2q} \right) \leq 1.$$

Now, let $k_0 \in \mathbb{N}$ be fixed. For any $k \geq k_0$, we can achieve a bound in terms of k_0. That is for any $k \geq k_0$,

$$\begin{aligned}\mathbb{E}(|\widehat{\mu}_k(\cdot,\xi)|^{2q}) =& \prod_{j=1}^{k} \mathbb{E}\left(\left|\frac{1}{j+1}\sum_{\nu_j=1}^{j+1} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1}\rho_l\right) X^j_{\nu_j}}\right|^{2q}\right)\\ \leq& \prod_{j=1}^{k_0} \mathbb{E}\left(\left|\frac{1}{j+1}\sum_{\nu_j=1}^{j+1} e^{-2\pi i \xi \left(\prod_{l=0}^{j-1}\rho_l\right) X^j_{\nu_j}}\right|^{2q}\right)\\ =& \mathbb{E}(|\widehat{\mu}_{k_0}(\cdot,\xi)|^{2q}).\end{aligned}$$

When taking the limit as $k \to \infty$, it follows from the dominated convergence theorem that

$$\mathbb{E}(|\widehat{\mu}_\alpha(\cdot,\xi)|^{2q}) \leq \mathbb{E}(|\widehat{\mu}_{k_0}(\cdot,\xi)|^{2q}).$$

Thus, it is enough to prove the bound for some $k_0 \in \mathbb{N}$, which might depend on ξ.

Assume now that $\xi \in \mathbb{R}$ satisfies the condition

$$\frac{(k+1)^q}{q^q \pi} \leq \left(\prod_{l=0}^{k-1}\rho_l\right)|\xi|, \tag{4.9}$$

for some k. Then it also holds for any $m < k$. Thus, it follows from proposition 4.7, that

$$\mathbb{E}(|\widehat{\mu}_\alpha(\cdot,\xi)|^{2q}) \leq \mathbb{E}(|\widehat{\mu}_k(\cdot,\xi)|^{2q}) \leq \frac{(2q^q)^k}{((k+1)!)^q}. \tag{4.10}$$

The goal is to bound the right hand side of (4.10) by $|\xi|^{-\theta\alpha q}$ for some large value of ξ. In order to achieve such a bound, let us see what condition (4.9) implies. Recall that $(j+1)\rho_j^\alpha = 1$. As such, condition (4.9) can be rewritten as

$$(k!)^{\frac{1}{\alpha}} \frac{(k+1)^q}{q^q \pi} \leq |\xi|. \tag{4.11}$$

Note that the left hand side of (4.11) is a non-decreasing function of k. Thus, for a fixed ξ we can find $k_\xi \in \mathbb{N}$, such that

$$((k_\xi - 1)!)^{\frac{1}{\alpha}} \frac{k_\xi^q}{q^q\pi} \leq |\xi| \leq (k_\xi!)^{\frac{1}{\alpha}} \frac{(k_\xi+1)^q}{q^q\pi}. \tag{4.12}$$

By (4.12) it is enough to show that

$$\frac{(2q^q)^{(k_\xi - 1)}}{(k_\xi!)^q} \leq \left((k_\xi!)^{\frac{1}{\alpha}} \frac{(k_\xi+1)^q}{q^q\pi}\right)^{-\theta\alpha q}. \tag{4.13}$$

By rearranging terms, and taking the logarithm of (4.13), we get the inequality

$$(k_\xi - 1)\log(2q^q) + \theta\alpha q^2 \log(k_\xi + 1) - \theta\alpha q \log(q^q\pi) \le q(1-\theta)\log(k_\xi!). \quad (4.14)$$

Note that for any positive constants $C_1, C_2 \ge 0$, and $C_3 > 0$, we have

$$\frac{C_1 k + C_2 \log(k+1)}{C_3 k \log(k)} \xrightarrow{k\to\infty} 0.$$

Thus, there exists $K = K(C_1, C_2, C_3)$ such that

$$C_1 k + C_2 \log(k+1) \le C_3 k \log(k),$$

whenever $k \ge K$. By an appropriate choice of constants, namely

$$\begin{aligned} C_1 =& \log(2q^q) + q(1-\theta)\frac{\log(2)}{2}, \\ C_2 =& \theta\alpha q^q, \\ C_3 =& \frac{q(1-\theta)}{2}, \end{aligned}$$

the following inequality must hold for all $k \ge K(\theta, \alpha, q)$,

$$\begin{aligned} (k-1)\log(2q^q) + \theta\alpha q^2\log(k+1) - \theta\alpha q\log(q^q\pi) \le& k\log(2q^q) + \theta\alpha q^2 \log(k+1) \\ \le& q(1-\theta)\frac{k}{2}\left(\log(k) - \log(2)\right) \\ \le& q(1-\theta)\log(k!), \end{aligned}$$

where we used the fact that $k! \ge (k/2)^{k/2}$. This confirms (4.14) for sufficiently large k. We note that the condition $\theta < 1$ is crucial, otherwise (4.14) would not hold for large values of k_ξ, and consequently large values of $|\xi|$.

To conclude the proof, we note that for any $\xi \in \mathbb{R}$, with

$$|\xi| \ge \Theta(\alpha, \theta, q) := ((K(\theta,\alpha,q) - 1)!)^{\frac{1}{\alpha}} \frac{(K(\theta,\alpha,q))^q}{q^q\pi}$$

we can find $k_\xi \ge K(\theta, \alpha, q)$ such that (4.12) holds. Since (4.12) holds, it follows that

$$\mathbb{E}\left(|\widehat{\mu}_\alpha(\cdot,\xi)|^{2q}\right) \le \mathbb{E}\left(\left|\widehat{\mu}_{k_\xi - 1}(\cdot,\xi)\right|^{2q}\right) \le \frac{(2q^q)^{k_\xi - 1}}{(k_\xi!)^q}.$$

By the choice of k_ξ, we know that (4.13) is true. Thus it follows that

$$\mathbb{E}\left(|\widehat{\mu}_\alpha(\cdot,\xi)|^{2q}\right) \le |\xi|^{-\theta\alpha q}.$$

□

Let us end this subsection with a few comments on proposition 4.8. First of all, the result holds for all $q \in \mathbb{N}$, and so it holds for $q = 1$. This means that

$$\mathbb{E}(|\widehat{\mu}_\alpha(\cdot, \xi)|^2) \leq |\xi|^{-\theta\alpha},$$

which is a good indication that almost surely $|\widehat{\mu}_\alpha(\omega, \xi)| \leq |\xi|^{-\theta\alpha/2}$ should hold whenever $|\xi|$ is sufficiently large. Unfortunately, the method used to transition to an almost surely bound will decrease the decay rate slightly. Even so, this will not affect the Fourier dimension estimate.

Secondly, we want to show that the Fourier dimension is bounded from below by α. For this reason, one ought to hope that $\theta = 1$. However, it follows from (4.14) that the case $\theta = 1$ cannot be true for large values of $|\xi|$. To see why, note that the right hand side would be zero, while the left hand side is a monotonically increasing function of k_ξ, and consequently also of $|\xi|$, which tends to infinity. On the other hand, since the Fourier dimension is defined through the use of the supremum, it is enough to show that the bound holds for any $\theta\alpha < \alpha$.

Lastly, as pointed out earlier, the construction of the random Cantor set could have been done using a more general sequence N_j instead of $j+1$. What will change in the proof, is the k_ξ would be replaced by $N_{k_\xi - 1}$. Thus, equation (4.14) reveals that the condition

$$\frac{\log(N_{k+1})}{\sum_{j=1}^{k} \log(N_j)} \xrightarrow{k\to\infty} 0,$$

is needed for the proof to work for a larger class of sequences. In particular, we could have chosen a sequence that grows to infinity while obeying the above growth condition. For a random Cantor set associated with such a sequence, we would be able to prove an analogue of proposition 4.8. It is these types of sequences Bluhm considers in his more general construction found in [1].

4.4 Transition from Expected Value to an Almost Surely Bound

Up until now, we have only found a bound for the expected value of the Fourier transform of the probability measure on our random Cantor set. In order to transition to an almost surely bound, we will utilize a method found in chapter 12 of [7] on how to create an almost surely absolutely convergent series. However, as the book considers a different bound on the expected value than we do, we have had to slightly modify the method.

Theorem 4.9. *Let $\beta > 0$ and assume that there is a probability measure $\mu(\omega)$ supported almost surely on a compact set such that for each $q \in \mathbb{N}$, there exist a constant $\Theta = \Theta(\beta, q)$ such that*

$$\mathbb{E}(|\widehat{\mu}(\cdot, \xi)|^{2q}) \leq |\xi|^{-\beta q}, \tag{4.15}$$

for all $|\xi| \geq \Theta$. Then for all $\gamma < \beta$ there exist constants $C = C(\omega, \beta, \gamma)$, and $\Xi = \Xi(\beta, \gamma) > 0$, such that almost surely

$$|\widehat{\mu}(\omega, \xi)| \leq C|\xi|^{-\frac{\gamma}{2}}.$$

for all $\xi > \Xi$.

Proof. We start by constructing a subset $Q_\beta \subset \mathbb{R}$ with some useful properties. For each $k \in \mathbb{N}$, define the set Q_k by

$$Q_k := \left\{2^{k-1} + i2^{-k\frac{\beta}{2}-1} : i \in \left\{0, \dots, \left[2^{k(\frac{\beta}{2}+1)}\right]\right\}\right\} \subset [2^{k-1}, 2^k],$$

where $[\cdot]$ denotes the integer part. We note that there are fewer than $2^{k(\beta/2+1)+1}$ points in each set Q_k. Taking the union of all $k \in \mathbb{N}$ results in the set

$$Q_\beta^+ := \bigcup_{k\in\mathbb{N}} Q_k.$$

Finally, we let

$$Q_\beta := -Q_\beta^+ \cup Q_\beta^+,$$

where we have included all points of Q_β^+ reflected around the origin. Note that for any point $\xi \in \mathbb{R}$ with $|\xi| \geq 1$, there exists k such that

$$|\xi| \in \left[2^{k-1}, 2^k\right].$$

In particular, we can find an $i \in \left\{0, \dots, \left[2^{k(\beta/2+1)}\right]\right\}$ such that

$$2^{k-1} + i2^{-k\frac{\beta}{2}-1} \leq |\xi| \leq 2^{k-1} + (i+1)2^{-k\frac{\beta}{2}-1}.$$

Whence it follows that whenever $|\xi| \geq 1$ there exists an element $z \in Q_\beta$ with $|z| \geq |\xi|$ such that

$$|z - \xi| \leq 2^{-k\frac{\beta}{2}-1} \leq 2^{-k\frac{\beta}{2}} \leq |\xi|^{-\frac{\beta}{2}},$$

where the last inequality follows from the fact that $|\xi| \leq 2^k$.

We now want to find an almost surely absolutely convergent series based on the set Q_β and our probability measure. Let us first consider the series,

$$\begin{aligned}\sum_{z\in Q_\beta} |z|^{-\frac{\beta}{2}-2} =& 2\sum_{k=1}^{\infty}\sum_{z\in Q_k} |z|^{-\frac{\beta}{2}-2}\\ \leq& 2\sum_{k=1}^{\infty} 2^{k\left(\frac{\beta}{2}+1\right)+1}2^{-(k-1)\left(\frac{\beta}{2}+2\right)}\\ =& 2^{\frac{\beta}{2}+4}\sum_{k=1}^{\infty}2^{-k} < \infty.\end{aligned}$$

We will now transition to finding an almost surely bound on the probability measure. Fix the integer $q \in \mathbb{Z}$, and consider the expected value of the series,

$$\begin{aligned}\mathbb{E}\left(\sum_{z\in Q_\beta, |z|\geq\Theta(\beta,q)} |z|^{-\frac{\beta}{2}-2}\frac{|\widehat{\mu}(\cdot,z)|^{2q}}{|z|^{-\beta q}}\right) &= \sum_{z\in Q_\beta, |z|\geq\Theta(\beta,q)} \mathbb{E}\left(|z|^{-\frac{\beta}{2}-2}\frac{|\widehat{\mu}(\cdot,z)|^{2q}}{|z|^{-\beta q}}\right)\\ &= \sum_{z\in Q_\beta, |z|\geq\Theta(\beta,q)} |z|^{-\frac{\beta}{2}-2}\frac{\mathbb{E}\left(|\widehat{\mu}(\cdot,z)|^{2q}\right)}{|z|^{-\beta q}}\\ &\leq \sum_{z\in Q_\beta} |z|^{-\frac{\beta}{2}-2} < \infty \qquad ,\end{aligned}$$

where we have used Fubini's theorem to interchange the summation and integration, as well as the condition (4.15). Since the expected value is bounded, the series has to be bounded almost surely as well. As such, there exist a constant $\tilde{C}(\omega,\beta,q)$ such that for every $z \in Q_\beta$ with $|z| > \Theta(\beta,q)$ we almost surely have the bound

$$|\widehat{\mu}(\omega,z)|^{2q} \leq \tilde{C}(\omega,\beta,q)|z|^{\frac{\beta+4}{2}}|z|^{-\beta q}. \tag{4.16}$$

Note that (4.16) is equivalent to

$$|\widehat{\mu}(\omega,z)| \leq K(\omega,\beta,q)|z|^{\frac{\beta+4}{4q}}|z|^{-\frac{\beta}{2}},$$

for some other constant $K(\omega,\beta,q)$.

In order to transition from z to an arbitrary ξ, we utilize lemma 4.4 to show that the Fourier transform is Lipschitz. So by lemma 4.4, and the fact that $\mu(\omega)$ is almost surely supported on a compact set, it follows that almost surely

$$
\begin{aligned}
|\widehat{\mu}(\omega,\xi)-\widehat{\mu}(\omega,z)| =& \left| \int_{\mathbb{R}} e^{-2\pi i\xi x} d(\mu(\omega))(x) - \int_{\mathbb{R}} e^{-2\pi i z x} d(\mu(\omega))(x) \right| \\
\leq& \int_{\mathbb{R}} |e^{-2\pi i\xi x} - e^{-2\pi i z x}| d(\mu(\omega))(x) \\
\leq& 2\pi|\xi - z| \int_{\mathbb{R}} |x| d(\mu(\omega))(x) \\
\leq& 2\pi M(\omega)|\xi - z|,
\end{aligned}
$$

where $M(\omega) = \sup_{x\in \text{supp } \mu(\omega)} |x| < \infty$. However, this means that for any $\xi \in \mathbb{R}$ with $|\xi| > \max\{\Theta(\beta, q), 1\}$ we can find $z \in Q_\beta$ with $|\xi - z| \leq |\xi|^{-\beta/2}$, and $|\xi| \leq |z| \leq 2|\xi|$, such that we almost surely have

$$
\begin{aligned}
|\widehat{\mu}(\omega,\xi)| \leq& |\widehat{\mu}(\omega,z)| + |\widehat{\mu}(\omega,\xi)-\widehat{\mu}(\omega,z)| \\
\leq& K(\omega,\beta,q)|z|^{\frac{\beta+4}{4q}}|z|^{-\frac{\beta}{2}} + 2\pi M(\omega)|\xi - z| \\
\leq& 2^{\frac{\beta+4}{4q}} K(\omega,\beta,q)|\xi|^{\frac{\beta+4}{4q}}|\xi|^{-\frac{\beta}{2}} + 2\pi M(\omega)|\xi|^{-\frac{\beta}{2}} \\
\leq& C(\omega,\beta,q)|\xi|^{\frac{\beta+4}{4q}}|\xi|^{-\frac{\beta}{2}}.
\end{aligned}
$$

To conclude the proof, we note that the argument holds for any $q \in \mathbb{N}$. Thus, for any $\gamma < \beta$ we can find $q_\gamma \in \mathbb{N}$, such that

$$
\frac{\gamma}{2} \leq \frac{\beta}{2} - \frac{\beta + 4}{4q_\gamma}.
$$

Thus, for each $|\xi| \geq \Xi(\beta, \gamma) := \max\{\Theta(\beta, q_\gamma), 1\}$ we have that

$$
|\widehat{\mu}(\omega,\xi)| \leq C(\omega,\beta,q_\gamma)|\xi|^{\frac{\beta+4}{4q_\gamma}}|\xi|^{-\frac{\beta}{2}} \leq C(\omega,\beta,\gamma)|\xi|^{-\frac{\gamma}{2}},
$$

holds almost surely. □

With the help of theorem 4.9, and proposition 4.8, we can show that almost surely there exists a lower bound on the Fourier dimension of our random Cantor set $\mathscr{C}_R(\omega)$. Namely, we have the following corollary, which ensures that $\mathscr{C}_R(\omega)$ is almost surely a Salem set.

Corollary 4.9.1. *The random Cantor set $\mathscr{C}_R(\omega)$ is almost surely a Salem set with dimension α.*

Proof. By proposition 4.1 we have seen that for any $\omega \in \Omega$ we have

$$\alpha \geq \dim_{\mathscr{H}}(\mathscr{C}_R(\omega)) \geq \dim_{\mathscr{F}}(\mathscr{C}_R(\omega)).$$

Thus, it only remains to find a lower bound on the Fourier dimension. However, this is done by combining proposition 4.8 with theorem 4.9. Namely, for each $0 < \theta < 1$ proposition 4.8 ensures that the conditions of theorem 4.9 are met. Whence it follows that for any $\gamma < \theta\alpha < \alpha$,

$$|\widehat{\mu}_\alpha(\omega, \xi)| \leq C(\omega, \alpha, \theta, \gamma)|\xi|^{-\frac{\gamma}{2}},$$

holds almost surely for all $|\xi| > \Xi(\alpha, \theta, \gamma)$. Thus, it follows from the definition of the Fourier dimension that almost surely

$$\alpha \leq \dim_{\mathscr{F}}(\mathscr{C}_R(\omega)) \leq \dim_{\mathscr{H}}(\mathscr{C}_R(\omega)) \leq \alpha.$$

This shows that $\mathscr{C}_R(\omega)$ is almost surely a Salem set with dimension α. □

5 Deterministic Construction of Salem Sets

In the previous section, we considered a construction of random Salem set on the unit interval. In this section, we will consider another construction of Salem sets on the unit interval. Unlike the random Cantor sets in section 4, we consider a deterministic construction. In particular, we look at the set of α-well-approximable numbers, E_α, for a given $\alpha > 0$. A known result by Jarník and Besicovitch is that E_α has Hausdorff dimension $2/(2+\alpha)$. Moreover, Kaufman gave a construction of a probability measure on E_α in 1981 which ensures that the Fourier dimension of E_α is at least $2/(2+\alpha)$. This result can be found in [6], as well as chapter 9 in [10]. We will not present the construction by Kaufman, but rather one by Bluhm, found in [2]. This enables us to construct a Salem set on the unit interval of any dimension between 0 and 1.

Using the construction by Bluhm, we will create a probability measure on a subset of E_α, denoted S_α. As a probability measure on S_α is also a probability measure on E_α, the result by Kaufman follows from Bluhm's result.

5.1 Definition of α-approximable Sets

In order to define the set E_α, we first need some notation. For a given point $x \in \mathbb{R}$, we define the seminorm

$$\|x\|_{\mathbb{Z}} := \min_{m\in\mathbb{Z}} |x-m|, \tag{5.1}$$

which is the distance from x to the nearest integer. For a given $\alpha > 0$, let E_α be the set

$$E_\alpha := \bigcap_{k=1}^{\infty} \bigcup_{q=k}^{\infty} \left\{ x \in [0,1] : \|qx\|_{\mathbb{Z}} < q^{-(1+\alpha)} \right\}. \tag{5.2}$$

We denote the set of all prime numbers $\mathbb{P}$, and define

$$\mathbb{P}_M = \mathbb{P} \cap [M, 2M]. \tag{5.3}$$

The cardinality of a set A is denoted $\#A$. We also define the prime-counting function $\pi(x)$ as the number of primes, $p \in \mathbb{P}$, such that $p \leq x$. The Prime Number Theorem states that

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)} = 1. \tag{5.4}$$

This is a classical result in number theory, which can be found in [3]. In particular, it follows from the Prime Number Theorem that

$$\lim_{M\to\infty} \frac{\#\mathbb{P}_M}{M/\log(M)} = 1. \tag{5.5}$$

Thus, given Λ large enough, we know that

$$\#\mathbb{P}_M \geq \frac{M}{2\log M}, \quad \forall M \geq \Lambda. \tag{5.6}$$

Let us choose a sequence of integers $\{M_k\}_{k\in\mathbb{N}}$ such that

$$\Lambda < M_1 < 2M_1 < M_2 < 2M_2 < \dots ,$$

and thus (5.6) holds for all elements of the sequence. For a fixed $\alpha > 0$, we define the set

$$S_\alpha := \bigcap_{k=1}^{\infty} \bigcup_{p\in\mathbb{P}_{M_k}} \left\{x \in [0,1] : \|px\|_{\mathbb{Z}} \leq p^{-(1+\alpha)}\right\}. \tag{5.7}$$

Note that S_α depends on the sequence $\{M_k\}_{k\in\mathbb{N}}$. We will show that it is possible to choose a sequence $\{M_k\}$ such that the resulting set S_α is a Salem set with dimension $2/(2+\alpha)$.

If we define the set

$$E_q(\alpha) := \left\{x : [0,1] : \|qx\|_{\mathbb{Z}} \leq q^{-(1+\alpha)}\right\}, \tag{5.8}$$

then it follows from (5.2) and (5.7), that

$$E_\alpha := \bigcap_{k=1}^{\infty} \bigcup_{q=k}^{\infty} E_q(\alpha), \quad S_\alpha := \bigcap_{k=1}^{\infty} \bigcup_{p\in\mathbb{P}_{M_k}} E_p(\alpha).$$

Moreover, for any $q \in \mathbb{N}$, it follows that

$$E_q(\alpha) = [0, q^{-(1+\alpha)-1}] \cup \bigcup_{m=1}^{q-1} \left[\frac{m}{q} - q^{-(1+\alpha)-1}, \frac{m}{q} + q^{-(1+\alpha)-1}\right] \cup [1-q^{-(1+\alpha)-1}, 1]. \tag{5.9}$$

Since $E_q(\alpha)$ is a finite union of closed sets, it is itself a closed set for each $q \in \mathbb{N}$. This in turn implies that

$$\bigcup_{p\in\mathbb{P}_{M_k}} E_p(\alpha),$$

is a closed set, since $\mathbb{P}_{M_k}$ is finite. As arbitrary intersections of closed sets are closed, it follows that

$$S_\alpha = \bigcap_{k=1}^{\infty} \bigcup_{p\in\mathbb{P}_{M_k}} E_p(\alpha),$$

is closed. On the other hand, we know that $S_\alpha \subset [0,1]$, and so S_α is actually a compact set. Furthermore, it follows from the way the sequence $\{M_k\}_{k=1}^{\infty}$ is defined that $M_k \geq k$. Whence it follows that

$$\bigcup_{p\in\mathbb{P}_{M_k}} E_p(\alpha) \subset \bigcup_{q=k}^{\infty} E_q(\alpha),$$

for each $k \in \mathbb{N}$. Consequently, we know that $S_\alpha \subset E_\alpha$. Moreover, from (5.9) it is clear that $0, 1 \in E_q(\alpha)$ for all $q \in \mathbb{N}$. Thus, we know that $0, 1 \in S_\alpha$, and both S_α and E_α are therefore non-empty.

We will now prove that $2/(2+\alpha)$ is an upper bound for the Hausdorff dimension of the set E_α. The proof presented here is inspired by the proof presented in chapter 9 of Wolff [10].

Proposition 5.1. *The set E_α has Hausdorff dimension $\dim_{\mathscr{H}}(E_\alpha) \leq 2/(2+\alpha)$.*

Proof. We can cover E_α by

$$E_\alpha \subset \bigcup_{q=k}^{\infty} E_q(\alpha),$$

for any fixed $k \in \mathbb{N}$. We also know that $E_q(\alpha)$ consists of $q-1$ intervals with length $2q^{-(1+\alpha)-1}$, and 2 intervals of length $q^{-(1+\alpha)-1}$. Let $\varepsilon > 0$, and assume that $q^{-(2+\alpha)} < \varepsilon$. Then by a rough estimate, we have

$$\mathscr{H}^s_\varepsilon(E_q(\alpha)) \leq (q-1)(2q^{-(2+\alpha)})^s + 2q^{-s(2+\alpha)} \leq 2^s q^{-s(2+\alpha)+1}.$$

Therefore, if we let $k^{-(2+\alpha)} < \varepsilon$, we get

$$\mathscr{H}^s_\varepsilon(E_\alpha) \leq \sum_{q=k}^{\infty} 2^s q^{-s(2+\alpha)+1}.$$

By the integral test it follows that

$$\sum_{q=k}^{\infty} 2^s q^{-s(2+\alpha)+1} \leq Ck^{2-s(2+\alpha)},$$

which goes to zero as $k \to \infty$ if and only if $s > 2/(2+\alpha)$. This shows that

$$\mathscr{H}^s(E_\alpha) = 0,$$

whenever $s > 2/(2+\alpha)$. Thus, we can conclude that $\dim_{\mathscr{H}}(E_\alpha) \leq 2/(2+\alpha)$. □

There is also a similar result for the set S_α, where we can bound the Hausdorff dimension of S_α by $2/(2+\alpha)$.

Corollary 5.1.1. *The set S_a has Hausdorff dimension* $\dim_{\mathscr{H}}(S_\alpha) \leq 2/(2+\alpha)$.

Proof. This statement follows from the fact that $S_\alpha \subset E_\alpha$. In particular, from proposition 5.1 we have

$$\dim_{\mathscr{H}}(S_\alpha) \leq \dim_{\mathscr{H}}(E_\alpha) \leq \frac{2}{2+\alpha},$$

since any cover of E_α is also a cover of S_α. □

5.2 Construction of a Probability Measure on S_α

The goal of this section is to construct a probability measure $\mu_\alpha \in \mathscr{P}(S_\alpha)$, which satisfies the bound

$$|\widehat{\mu}_\alpha(\xi)| \leq C_\varepsilon |\xi|^{-\frac{1}{2+\alpha}+\varepsilon},$$

for every $\varepsilon > 0$. Given such a construction, it follows that S_α is a Salem set with dimension $2/(2+\alpha)$. Since any $\mu \in \mathscr{P}(S_\alpha)$ is a probability measure on E_α, it also follows that E_α is a Salem set with dimension $2/(2+\alpha)$. Hence, we have a deterministic construction of Salem sets for any dimension $\beta = 2/(2+\alpha) \in (0,1)$. However, this construction relies on a few preliminary results. It is these results we will now present before we end the section with the construction of a probability measure on S_α.

Let $M \in \mathbb{N}$ be such that

$$R := \frac{1}{(4M)^{1+\alpha}} < \frac{1}{2}. \tag{5.10}$$

Define the function F_M on $[-1/2, 1/2]$ by

$$F_M(x) = \begin{cases} \frac{15}{16} R^{-5}(R^2 - x^2)^2, & |x| \leq R, \\ 0, & R < |x| \leq \frac{1}{2}. \end{cases} \tag{5.11}$$

A simple calculation shows that

$$F''_M(x) = \begin{cases} \frac{15}{4}R^{-5}\left(3x^2 - R^2\right), & |x| \le R, \\ 0, & R < |x| \le \frac{1}{2} \end{cases}, \tag{5.12}$$

This means that F_M is twice differentiable, yet the second derivative is not continuous at $|x| = R$. On the other hand, the second derivative of F_M is bounded, and so $F_M \in C^{1,1}(\mathbb{R})$. Since F_M is supported on $[-1/2, 1/2]$, it is possible to extend F_M to a periodic function on $\mathbb{R}$ with period 1. We can therefore associate to F_M the Fourier series

$$\sum_{n\in\mathbb{Z}} a_n^{(M)} e^{2\pi i n x},$$

where the coefficients are given by

$$a_n^{(M)} = \int_{-\frac{1}{2}}^{\frac{1}{2}} F_M(t) e^{-2\pi i n t} dt. \tag{5.13}$$

It is possible to achieve some bounds on the Fourier coefficents found in (5.13). This leads us to the following lemma.

Lemma 5.2. *The Fourier coefficients for the function F_M satisfy*

$$a_0^{(M)} = 1, \quad |a_m^{(M)}| \le 1, \quad |a_m^{(M)}| \le m^{-2}R^{-2}, \tag{5.14}$$

for every $m \in \mathbb{Z}$.

Proof. For simplicity, we will use the abbreviation $a_m^{(M)} = a_m$, as M and thus F_M is fixed. Let us start by showing that $a_0 = 1$. This follows from a simple calculation using the definition of F_M in (5.11), as

$$\begin{aligned} a_0 = \frac{15R^{-5}}{16}\int_{-R}^{R}(R^2 - x^2)^2 dx =& \frac{15R^{-5}}{16}\left(R^4 x - \frac{2}{3}R^2x^3 + \frac{1}{5}x^5\right)\Big|_{x=-R}^{x=R} \\ =& \frac{15R^{-5}}{16}\left(\frac{8R^5}{15} - \frac{-8R^5}{15}\right) = 1. \end{aligned}$$

It is clear from (5.11) that $|F_M| = F_M$. As such, it follows from the triangle inequality and the non-negativity of F_M, that

$$|a_M| \le \int_{-\frac{1}{2}}^{\frac{1}{2}} |F_M(t)| dt = a_0 = 1.$$

For the final claim in (5.14), we note that F_M vanishes at the boundary $|x| = 1/2$, and so does the first derivative F'_M. The second derivative is given in (5.12) and so using integration by parts twice yields

$$a_m = \int_{-\frac{1}{2}}^{\frac{1}{2}} F_M(x)e^{-2\pi imx}dx = -\frac{1}{4\pi^2 m^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} F''_M(x)e^{-2\pi imx}dx.$$

Whence the final claim follows from the triangle inequality,

$$\begin{aligned}|a_m| \leq \frac{1}{4\pi^2 m^2}\int_{-R}^{R} |F''_M(x)|dx \leq& \frac{15R^{-5}}{16\pi^2 m^2}\int_{-R}^{R}(3x^2+R^2)dx \\ =& \frac{15}{4\pi^2}m^{-2}R^{-2} \leq m^{-2}R^{-2}.\end{aligned}$$

□

With the bounds given on the Fourier coefficients in lemma 5.2, it follows that the Fourier series of F_M converges uniformly to F_M. As such, we have

$$F_M(x) = \sum_{n\in\mathbb{Z}} a_n^{(M)} e^{2\pi inx}. \tag{5.15}$$

We now continue with the construction of μ_α. We begin by defining the new function

$$q_M(x) := \sum_{p\in\mathbb{P}_M} F_M(px) = \sum_{m\in\mathbb{Z}}\sum_{p\in\mathbb{P}_M} a_m^{(M)} e^{2\pi impx}. \tag{5.16}$$

Since $F_M \in C^{1,1}(\mathbb{R})$ and 1-periodic, it follows from (5.16) that $q_M \in C^{1,1}(\mathbb{R})$, and that q_M is a 1-periodic function. However, we want to scale q_M with a constant c_M such that $c_M\widehat{q}_M(0) = 1$. To find this constant, we use (5.16) to write the Fourier coefficients of q_M as

$$\widehat{q}_M(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} q_M(x)e^{-2\pi ikx}dx = \sum_{m\in\mathbb{Z}}\sum_{p\in\mathbb{P}_M} a_m^{(M)}\delta_{k,mp}, \tag{5.17}$$

where $\delta_{k,mp}$ is 1 when $k = mp$, and zero otherwise. Note that when $k = 0$, only terms with $m = 0$ will contribute. This results in

$$\widehat{q}_M(0) = \sum_{p\in\mathbb{P}_M} a_0^{(M)} = \#\mathbb{P}_M, \tag{5.18}$$

where we used lemma 5.2 to conclude that $a_0^{(M)} = 1$. Hence, the function

$$g_M(x) := \frac{q_M(x)}{\#\mathbb{P}_M} \tag{5.19}$$

is precisely the scaled version of q_M we wanted. Since F_M is a non-negative function, it follows that q_M and g_M are also non-negative functions.

Proposition 5.3. *If $g_M(x) > 0$, then there exists $p \in \mathbb{P}_M$ such that $\|px\|_{\mathbb{Z}} \leq p^{-(1+\alpha)}$.*

Proof. If $g_M(x) > 0$, then there is at least one $p \in \mathbb{P}_M$ such that $F_M(px) > 0$ by the definition of g_M. However, F_M is a 1-periodic function, and $F_M(px) > 0$ only when $|px - m| \leq R$ for some $m \in \mathbb{Z}$. This follows from (5.11). Using the definition of R, we can see that

$$|px - m| \leq R = \frac{1}{(4M)^{1+\alpha}} \leq \frac{1}{(2p)^{(1+\alpha)}} \leq p^{-(1+\alpha)}, \tag{5.20}$$

since $p \leq 2M$, whenever $p \in \mathbb{P}_M$. □

Our aim is to construct a measure μ_α supported on S_α by repeated multiplication of densities g_{M_k}, for a sequence of integers $\{M_k\}_{k\in\mathbb{N}}$. We introduce the function

$$\theta(x) := (1 + |x|)^{-\frac{1}{2+\alpha}} \log(e + |x|) \log\log(e^e + |x|),$$

for a cleaner presentation. Since $\theta \in C_\infty(\mathbb{R}) \subset C_b(\mathbb{R})$, we can also introduce the constant $\Theta := \sup_{x\in\mathbb{R}} \theta(x) < \infty$.

The following proposition is key in constructing our measure.

Theorem 5.4. *For every $\psi \in C_0^{1,1}(\mathbb{R})$, and $\delta > 0$, there exists a positive integer $M_0 = M_0(\psi, \delta) \geq \Lambda$ such that*

$$|\widehat{\psi g_M}(\xi) - \widehat{\psi}(\xi)| \leq \delta\theta(\xi), \tag{5.21}$$

for $\xi \in \mathbb{R}$, and all $M \geq M_0$.

The proof of theorem 5.4 is proven in the next subsection, as we will now proceed to construct a non-negative bounded measure on S_α. The construction relies on recursively finding a sequence $\{M_k\}_{k=1}^\infty$, which gives a Cauchy sequence in the uniform norm on the Fourier side.

We start by fixing a non-negative function $\phi \in C_0^{1,1}(\mathbb{R})$, with supp $\phi \subset [0,1]$ and $\phi > 0$ on $(0,1)$. Moreover, we require

$$\int_{\mathbb{R}} \phi(x)dx = 1.$$

Now, fix $0 < \delta < 1/2$. Then by theorem 5.4 we can find $M_1 \geq M_0(\phi, \delta 2^{-1})$ such that

$$\left|\widehat{\phi g_{M_1}}(\xi) - \widehat{\phi}(\xi)\right| \leq \delta 2^{-1}\theta(\xi).$$

Note that since $g_M \in C^{1,1}(\mathbb{R})$, we know that $\phi g_M \in C_0^{1,1}(\mathbb{R})$ whenever $\phi \in C_0^{1,1}(\mathbb{R})$. In particular, $\phi g_{M_1} \in C_0^{1,1}(\mathbb{R})$, and so we can continue the process. Namely, we can find $M_2 \geq M_0(\phi g_{M_1}, \delta 2^{-2})$, such that

$$\left|\widehat{\phi g_{M_1} g_{M_2}}(\xi) - \widehat{\phi g_{M_1}}(\xi)\right| \leq \delta 2^{-2}\theta(\xi).$$

This means that we can find a sequence $M_{k+1} \geq M_0(\phi \prod_{j=1}^{k} g_{M_j}, \delta 2^{-(k+1)})$, such that

$$\left|\left(\phi \prod_{j=1}^{k+1} g_{M_j}\right)^{\wedge}(\xi) - \left(\phi \prod_{j=1}^{k} g_{M_j}\right)^{\wedge}(\xi)\right| \leq \delta 2^{-(k+1)}\theta(\xi). \tag{5.22}$$

Now define the sequence of functions $G_k := \prod_{j=1}^{k} g_{M_j}$, where $G_0 = 1$. By (5.22), we have

$$|\widehat{\phi G_{k+1}}(\xi) - \widehat{\phi G_k}(\xi)| \leq \delta 2^{-(k+1)}\theta(\xi) \leq \Theta \delta 2^{-(k+1)},$$

where $\Theta = \sup_{\xi\in\mathbb{R}} \theta(\xi)$. This shows that the sequence $\{\widehat{\phi G_k}\}_{k=0}^{\infty}$ is Cauchy in the uniform norm.

For each $k \in \mathbb{N}$, let us define the non-negative measure $d\mu_k = \phi G_k dx$, where dx refers to the usual Lebesgue measure. Then the sequence $\{\widehat{\mu}_k\}_{k\in\mathbb{N}}$ is Cauchy in the uniform norm. The non-negativity follows the fact that ϕ and g_{M_k} are non-negative functions for any $k \in \mathbb{N}$. Moreover, since any Cauchy sequence is bounded and $\phi G_k \geq 0$ for all $k \in \mathbb{N}$, it follows that

$$\sup_{k\in\mathbb{N}} \mu_k(\mathbb{R}) = \sup_{k\in\mathbb{N}} \widehat{\phi G_k}(0) < \infty.$$

Thus, by proposition 2.9 there exists a subsequence converging weakly to a measure $\mu_\alpha \in \mathscr{M}([0,1])$. However, since the sequence $\{\widehat{\phi G_k}\}_{k=0}^{\infty}$ is Cauchy in the uniform norm, the entire sequence must converge pointwise to $\widehat{\mu}_\alpha$. It therefore follows from proposition 2.10 that $\mu_k \rightharpoonup \mu_\alpha$.

It remains to show that $\mu_\alpha \neq 0$. For this, we use that $G_0 = 1$, and use a telescoping sum. Thus, for any $k \in \mathbb{N}$

$$|\mu_k(\mathbb{R}) - 1| = |\widehat{\phi G_k}(0) - \widehat{\phi G_0}(0)| \leq \sum_{i=1}^{k} |\widehat{\phi G_i}(0) - \widehat{\phi G_{i-1}}(0)| \leq \delta \sum_{i=1}^{k} 2^{-i}\theta(0).$$

Since the sum is bounded by 1 for any k, and $\theta(0) = 1$, it follows that for any $k \in \mathbb{N}$,

$$\frac{3}{2} > 1 + \delta \geq \mu_k(\mathbb{R}) \geq 1 - \delta > \frac{1}{2}.$$

In particular, we can show that $\mu_\alpha \neq 0$ since

$$\mu_\alpha(\mathbb{R}) = \int_0^1 d\mu_\alpha = \lim_{k\to\infty} \int_0^1 d\mu_k = \lim_{k\to\infty} \mu_k(\mathbb{R}) > \frac{1}{2}.$$

Thus, by the same argument, we know that $1/2 < \mu_\alpha(\mathbb{R}) < 3/2$.

Let us now show that μ_α is supported on S_α. This follows from proposition 5.3. To see why, we note that if $\phi(x)G_k(x) > 0$, then $g_{M_j}(x) > 0$ for all $j \in \{1, \ldots, k\}$. By proposition 5.3, we can conclude that supp $g_{M_j} \subset \bigcup_{p\in\mathbb{P}_{M_j}} E_p(\alpha)$. This means that

$$\operatorname{supp} \mu_k \subset \bigcap_{j=1}^{k} \bigcup_{p\in\mathbb{P}_{M_j}} E_p(\alpha).$$

Whence, it follows that supp $\mu_\alpha \subset S_\alpha$.

For the decay of μ_α on the Fourier side, we note that there is some $\xi_0 > 0$ such that $(1+|\xi|)^{-2} < \theta(\xi)$ whenever $|\xi| > \xi_0$. This follows from the fact that $(1+|\xi|)^{-2}$ tends to zero faster than $\theta(\xi)$ as $|\xi| \to \infty$. Thus, we can estimate the decay

$$\begin{aligned}
|\widehat{\mu}_a(\xi)| = \lim_{k\to\infty} |\widehat{\mu}_k(\xi)| &\leq \lim_{k\to\infty} \sum_{j=1}^{k} |\widehat{\mu}_j(\xi) - \widehat{\mu}_{j-1}(\xi)| + |\widehat{\mu}_0(\xi)| \\
&\leq \lim_{k\to\infty} \sum_{j=1}^{k} \delta 2^{-j}\theta(\xi) + C_1(1+|\xi|)^{-2} \leq C\theta(\xi),
\end{aligned}$$

whenever $|\xi| > \xi_0$. This shows that for any $\varepsilon > 0$,

$$|\widehat{\mu}_a(\xi)| \leq C|\xi|^{-\frac{1}{2+\alpha}+\varepsilon}, \quad |\xi| > \xi_0.$$

Theorem 5.5. *The set S_α is a Salem set with dimension $2/(2+\alpha)$.*

Proof. By using corollary 5.1.1, we know that

$$\dim_{\mathscr{H}}(S_\alpha) \leq 2/(2+a).$$

Moreover, define the measure P_α by

$$P_\alpha = \frac{\mu_\alpha}{\mu_\alpha(\mathbb{R})} \in \mathscr{P}(S_\alpha).$$

Then for $|\xi| > \xi_0$,

$$|\widehat{P}_a(\xi)| \leq \frac{C}{\mu_\alpha(\mathbb{R})}|\xi|^{-\frac{1}{2+\alpha}+\varepsilon},$$

for any $\varepsilon > 0$. We can therefore conclude that $2/(2+\alpha) \leq \dim_{\mathscr{F}}(S_\alpha)$. Whence it follows that S_α is a Salem set with dimension $2/(2+\alpha)$. □

Corollary 5.5.1. *The set E_α is a Salem set with dimension $2/(2+\alpha)$.*

Proof. We know that any probability measure on S_α is a probability measure on E_α since $S_\alpha \subset E_\alpha$. Thus, the result follows directly from theorem 5.5 and proposition 5.1. □

5.3 Proof of Theorem 5.4

We will now start the process of proving theorem 5.4. For this, we first need a few lemmas.

Lemma 5.6. *Let $M \in \mathbb{N}$. Then given any integer $k \in \mathbb{N}$, we have*

$$\#\{\textit{Prime factors of } k \textit{ inside } \mathbb{P}_M\} \leq \frac{\log k}{\log M}$$

Proof. Since k has a unique prime factorization, we can write

$$k = \prod_{p\in\mathbb{P}} p^{\alpha_p} = \prod_{p\in\mathbb{P}_M} p^{\alpha_p} \prod_{p\in\mathbb{P}\setminus\mathbb{P}_M} p^{\alpha_p},$$

where $\alpha_p \in \mathbb{N}\cup\{0\}$. The number of prime factors in $\mathbb{P}_M$ is therefore given by

$$\#\{\text{Prime factors of } k \text{ inside } \mathbb{P}_M\} = \sum_{p\in\mathbb{P}_M} \alpha_p.$$

Since for each $p \in \mathbb{P}_M$ we have $p \geq M$, we get

$$k \geq \prod_{p \in \mathbb{P}_M} p^{\alpha_p} \geq \prod_{p \in \mathbb{P}_M} M^{\alpha_p}.$$

Taking the logarithm on both sides yields

$$\log k \geq \sum_{p \in \mathbb{P}_M} \alpha_p \log M \Rightarrow \frac{\log k}{\log M} \geq \sum_{p \in \mathbb{P}_M} \alpha_p.$$

□

The next lemma gives a bound on the coefficients $\widehat{g}_M(k)$.

Lemma 5.7. *There exists a constant $A = A(\alpha) > 0$ such that for all $M \geq \max\{4, \Lambda\}$, we have*

$$\begin{aligned} |\widehat{g}_M(k)| &\leq A M^{-1} \log M, \quad \forall k \in \mathbb{Z} \backslash \{0\} \\ |\widehat{g}_M(k)| &\leq A |k|^{-\frac{1}{2+\alpha}} \log |k|, \quad \forall k \in \mathbb{Z} \text{ with } |k| > (4M)^{2+\alpha} \end{aligned}$$

Proof. We start by recalling the definition of q_M, given in (5.17). Combining the definition of q_M with lemma 5.2, where we use that $|a_m^{(M)}| \leq 1$, we have

$$|\widehat{q}_M(k)| \leq \sum_{m \in \mathbb{Z}} \sum_{p \in \mathbb{P}_M} \delta_{k,mp} = \#\{(m,p) \in \mathbb{Z} \times \mathbb{P}_M : k = mp\}.$$

The number of points (m, p) which satisfy $k = mp$, is nothing more than the numbers of prime factors of $|k|$ in $\mathbb{P}_M$. By lemma 5.6 we have a bound on the number of prime factors of $|k| \in \mathbb{N}$ inside $\mathbb{P}_M$. Thus, for any $k \in \mathbb{Z} \backslash \{0\}$ we arrive at

$$|\widehat{q}_M(k)| \leq \frac{\log |k|}{\log M}.$$

Let us now consider the case $1 \leq |k| \leq (4M)^{2+\alpha}$. From (5.19), the definition of g_M, together with (5.6) since $M \geq \Lambda$, we get

$$\begin{aligned} |\widehat{g}_M(k)| = \frac{|\widehat{q}_M(k)|}{\#\mathbb{P}_M} &\leq \frac{2 \log M \log |k|}{M \log M} = 2 \frac{\log |k|}{M} \\ &\leq 2M^{-1}(2+\alpha)(\log 4 + \log M) \leq 4(2+\alpha) M^{-1} \log M, \end{aligned}$$

where the last inequality follows from the fact that $M \geq 4$.

Let us now assume $|k| > (4M)^{2+\alpha}$, and recall from lemma 5.2 that $|a_m^{(M)}| \leq m^{-2}R^{-2}$, where $R = (4M)^{-(1+\alpha)}$. We can therefore improve the bound on $\widehat{q}_M$, namely

$$|\widehat{q}_M(k)| \leq \sum_{m\in\mathbb{Z}} \sum_{p\in\mathbb{P}_M} m^{-2}R^{-2}\delta_{k,mp}.$$

Since $k = mp$, we have a lower bound on $|m|$ given by

$$|m| = \frac{|k|}{p} \geq \frac{|k|}{2M}.$$

Thus, when combining these two inequalities, we get

$$|\widehat{q}_M(k)| \leq (2M)^2|k|^{-2}R^{-2} \sum_{m\in\mathbb{Z}} \sum_{p\in\mathbb{P}_M} \delta_{k,mp} \leq \frac{1}{4}(4M)^{4+2\alpha}|k|^{-2}\frac{\log|k|}{\log M},$$

where we used that $R^{-2} = (4M)^{2+2\alpha}$. Now, since $M \geq \Lambda$, we know that (5.6) holds, and so we can bound $\widehat{g}_M(k)$ by

$$|\widehat{g}_M(k)| \leq \frac{2\log M}{M}|\widehat{q}_M(k)| \leq 2(4M)^{3+2\alpha}|k|^{-2}\log|k| \leq 2|k|^{-\frac{1}{2+\alpha}}\log|k|.$$

It remains to show that $|\widehat{g}_M(k)| \leq AM^{-1}\log M$ whenever $|k| > (4M)^{2+\alpha}$. In order to show this, we need to investigate the function

$$h(x) := x^{-\frac{1}{2+\alpha}}\log x.$$

Notice that $|\widehat{g}_M(k)| \leq 2h(|k|)$ whenever $|k| > (4M)^{2+\alpha}$. Furthermore, we can easily see that the derivative of h satisfies

$$h'(x) = x^{-\frac{1}{2+\alpha}-1}\left(1 - \frac{1}{2+\alpha}\log x\right) < 0,$$

for all $x > e^{2+\alpha}$. In particular, if $4M > e$, then $h : [(4M)^{2+\alpha}, \infty) \to [0,\infty)$ defines a monotonically decreasing function. Since $4M \geq 16 > e$, it follows that $h((4M)^{2+\alpha}) \geq h(|k|)$ whenever $|k| > (4M)^{2+\alpha}$. As such, we can conclude that

$$|\widehat{g}_M(k)| \leq 2|k|^{-\frac{1}{2+\alpha}}\log|k| \leq 2(2+\alpha)M^{-1}(\log M + \log 4) \leq 4(2+\alpha)M^{-1}\log M,$$

since $M \geq 4$. Thus, the result follows by choosing $A(\alpha) = 4(2+\alpha)$. □

Lemma 5.8. *Assume* $M \geq \max\{4, \Lambda\}$*, and* $\phi \in C_0^{1,1}(\mathbb{R})$ *is given. Then there exists a constant* $B = B(\phi, \alpha) > 0$ *such that*

$$|\widehat{\phi g_M}(\xi) - \widehat{\phi}(\xi)| \leq BM^{-1} \log M,$$

for all $\xi \in \mathbb{R}$.

Proof. We start by recalling one of the basic properties of the Fourier transform, namely

$$\widehat{e_y \phi}(\xi) = \widehat{\phi}(\xi + y), \quad \text{where} \quad e_y(x) := e^{-2\pi i y x}.$$

In particular, if we write g_M as a Fourier series, then

$$\widehat{\phi g_M}(\xi) = \sum_{n\in\mathbb{Z}} \widehat{g}_M(n) \widehat{e_{-n}\phi}(\xi) = \sum_{n\in\mathbb{Z}} \widehat{g}_M(n) \widehat{\phi}(\xi - n).$$

From the way g_M was defined, $\widehat{g}_M(0) = 1$, which leads to

$$\widehat{\phi g_M}(\xi) - \widehat{\phi}(\xi) = \sum_{n\in\mathbb{Z}, n\neq 0} \widehat{g}_M(n) \widehat{\phi}(\xi - n) = \sum_{|n|\in\mathbb{N}} \widehat{g}_M(n) \widehat{\phi}(\xi - n). \tag{5.23}$$

Since $\phi \in C_0^{1,1}(\mathbb{R})$, we can use integration by parts twice. Thus, for every $\xi \neq 0$,

$$\begin{aligned} |\widehat{\phi}(\xi)| = \left| \int_{\mathbb{R}} e^{-2\pi i \xi \cdot x} \phi(x) dx \right| &= \frac{1}{4\pi^2 |\xi|^2} \left| \int_{\mathbb{R}} e^{-2\pi i \xi \cdot x} \phi''(x) dx \right| \\ &\leq \|\phi''\|_\infty \frac{\lambda(\mathrm{supp}(\phi))}{4\pi^2} |\xi|^{-2} = C|\xi|^{-2}. \end{aligned}$$

If $\xi = 0$, we have the bound $|\widehat{\phi}(0)| \leq \|\phi\|_{L^1}$. Actually, the last bound holds for any $\xi \in \mathbb{R}$. As such, there is a constant $C_0 = C_0(\phi)$ such that

$$|\widehat{\phi}(\xi)| \leq C_0 (1 + |\xi|)^{-2}.$$

Using the decay of $\widehat{\phi}$, we can easily achieve a bound for (5.23), namely

$$\begin{aligned} |\widehat{\phi g_M}(\xi) - \widehat{\phi}(\xi)| \leq &\sum_{|n|\in\mathbb{N}} |\widehat{g}_M(n)| |\widehat{\phi}(\xi - n)| \\ \leq &C_0 \sum_{|n|\in\mathbb{N}} |\widehat{g}_M(n)| (1 + |\xi - n|)^{-2}. \end{aligned} \tag{5.24}$$

Since $M \geq \max\{4, \Lambda\}$, we can use lemma 5.7 to conclude that

$$|\widehat{g}_M(n)| \leq AM^{-1} \log M.$$

It therefore follows from (5.24), that

$$|\widehat{\phi g_M}(\xi) - \widehat{\phi}(\xi)| \leq C_0 AM^{-1} \log M \sum_{|n| \in \mathbb{N}} (1 + |\xi - n|)^{-2}. \tag{5.25}$$

We now want to investigate the sum which arises in (5.25). If we fix $\xi \in \mathbb{R}$, then we can note that $\xi \in [n_\xi, n_\xi + 1]$ for some $n_\xi \in \mathbb{Z}$. As such, for any $m \neq n_\xi$ we have that $|\xi - m| \geq |n_\xi - m| \in \mathbb{N}$. We can therefore bound the sum as

$$\begin{aligned} \sum_{|n| \in \mathbb{N}} (1 + |\xi - n|)^{-2} =& \sum_{n \leq \xi} (1 + |\xi - n|)^{-2} + \sum_{n > \xi} (1 + |\xi - n|)^{-2} \\ \leq& 2 \sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{3}. \end{aligned} \tag{5.26}$$

It follows from (5.25) that we have the bound

$$|\widehat{\phi g_M}(\xi) - \widehat{\phi}(\xi)| \leq \ BM^{-1} \log M,$$

where $B = C_0 A \pi^2/3$. □

We are now ready to present the proof of theorem 5.4.

Proof of theorem 5.4. Fix $\delta > 0$ and $\phi \in C_0^{1,1}(\mathbb{R})$. We begin by making two general observations, and then proceed by treating the cases of small and large values of ξ separately.

First of all, for each constant $B > 0$, there exist another constant $C = C(B, \delta)$, for which

$$B \leq \delta \log\log (e^e + |\xi|),$$

when $|\xi| \geq C$. This follows from the fact that $\log\log (e^e + |\xi|)$ tends to infinity as $|\xi| \to \infty$, while δ and B are independent of ξ.

Secondly, by construction we have

$$\theta(0) = 1^{-\frac{1}{2+\alpha}} \log e \log\log e^e = 1.$$

Moreover, the function $\theta(\xi)$ has no zeros on each finite interval $[0, a)$ for $a > 0$. As such, there exists a constant $c(a) > 0$ such that

$$\inf_{|\xi| \in [0,a)} \theta(\xi) \geq c > 0.$$

Whence, for each constant $B > 0$ there exists an integer $N = N(B, a, \delta)$ such that

$$BM^{-1}\log M \leq \delta c \leq \inf_{|\xi|\in[0,a)} \delta\theta(\xi),$$

for all $M \geq N$ as the function $M^{-1}\log M$ tends to zero as $M \to \infty$.

Recall that we can apply lemma 5.8 for any $M \geq \max\{4, \Lambda\}$. Let us first consider values of ξ in the range $|\xi| < 2(4M)^{2+\alpha}$ for some $M \geq \max\{4, \Lambda\}$. By rewriting the inequality, we have

$$M > 2^{-\frac{5+2\alpha}{2+\alpha}}|\xi|^{\frac{1}{2+\alpha}}, \quad M^{-1} < 2^{\frac{5+2\alpha}{2+\alpha}}|\xi|^{-\frac{1}{2+\alpha}}.$$

Applying lemma 5.8, together with the fact that $x^{-1}\log x$ is a monotonically decreasing function for $x > e$, there is a constant $B = B(\phi, \alpha)$ such that

$$\begin{aligned}
|\widehat{\phi g_M}(\xi) - \widehat{\phi}(\xi)| \leq & BM^{-1}\log(M) \\
\leq & B2^{\frac{5+2\alpha}{2+\alpha}}|\xi|^{-\frac{1}{2+\alpha}}\left(\frac{1}{2+\alpha}\log|\xi| - \frac{5+2\alpha}{2+\alpha}\log 2\right) \\
\leq & B'|\xi|^{-\frac{1}{2+\alpha}}\log|\xi| \leq B_1(1+|\xi|)^{-\frac{1}{2+\alpha}}\log(e+|\xi|),
\end{aligned}$$

for some constant $B_1 = B_1(\phi, \alpha)$ whenever $2^{-\frac{5+2\alpha}{2+\alpha}}|\xi|^{\frac{1}{2+\alpha}} > e$. In fact, this ensures that $|\xi| > 2^{5+2\alpha}$. Note that there is nothing special about the choice of M, as long as $M \geq \max\{4, \Lambda\}$ such that we can apply lemma 5.8.

By the first observation there exists a constant $C = C(B_1, \delta)$ such that for all $|\xi| \geq C$,

$$B_1 \leq \delta \log\log(e^e + |\xi|).$$

Now, define $C_1 = C_1(\phi, \alpha, \delta) = \max\{C(B_1, \delta), e^{2+\alpha}2^{5+2\alpha}\}$. Then for all $M > 2^{-\frac{5+2\alpha}{2+2\alpha}}C_1^{\frac{1}{2+\alpha}}$, it follows that

$$\begin{aligned}
|\widehat{\phi g_M}(\xi) - \widehat{\phi}(\xi)| \leq & \delta(1+|\xi|)^{-\frac{1}{2+\alpha}}\log(e+|\xi|)\log\log(e^e+|\xi|) \\
\leq & \delta\theta(\xi),
\end{aligned}$$

whenever $C_1 \leq |\xi| < 2(4M)^{2+\alpha}$.

For the case $|\xi| < C_1$, it follows from the second observation that there exists an $N = N(B, C_1, \delta)$ such that

$$|\widehat{\phi g_M}(\xi) - \widehat{\phi}(\xi)| \leq BM^{-1}\log(M) \leq \delta\theta(\xi),$$

whenever

$$M \geq M_1 = M_1(\phi, \alpha, \delta) := \max\left\{4, \Lambda, N, \left[2^{-\frac{5+2\alpha}{2+2\alpha}}C_1^{\frac{1}{2+\alpha}}\right] + 1\right\},$$

where $[\cdot]$ denotes the integer part. This shows that for all $M \geq M_1$,

$$|\widehat{\phi g_M}(\xi) - \widehat{\phi}(\xi)| \leq \delta\theta(\xi), \quad |\xi| < 2(4M)^{2+\alpha}. \tag{5.27}$$

Now assume that $|\xi| \geq 2(4M)^{2+\alpha}$ for some M, and recall the expression we found in (5.24),

$$|\widehat{\phi g_M}(\xi) - \widehat{\phi}(\xi)| \leq C_0 \sum_{|n| \in \mathbb{N}} |\widehat{g}_M(n)|(1 + |\xi - n|)^{-2}.$$

The sum can be divided into two parts, namely $|\xi - n| < |\xi|/2$ and $|\xi - n| \geq |\xi|/2$. In the first sum, where $|\xi - n| < |\xi|/2$, we must have $|\xi|/2 < |n| < 3|\xi|/2$. Whence it follows that

$$\begin{aligned}\sum_{|\xi - n| < \frac{|\xi|}{2}} |\widehat{g}_M(n)|(1 + |\xi - n|)^{-2} \leq & \sup_{\frac{|\xi|}{2} < |n| < \frac{3}{2}|\xi|} |\widehat{g}_M(n)| \sum_{|n| \in \mathbb{N}} (1 + |\xi - n|)^{-2} \\ \leq & 2 \sup_{|n| > \frac{|\xi|}{2}} |\widehat{g}_M(n)| \sum_{n \in \mathbb{N}} \frac{1}{n^2},\end{aligned}$$

where the last inequality follows from (5.26), as well as taking the supremum over a larger set. Since $|\xi|/2 \geq (4M)^{2+\alpha}$, it follows from the second result in lemma 5.7 that we can bound the supremum, and so

$$\begin{aligned}2 \sup_{|n| > \frac{|\xi|}{2} \geq (4M)^{2+\alpha}} |\widehat{g}_M(n)| \sum_{n \in \mathbb{N}} \frac{1}{n^2} \leq & \frac{\pi^2}{3} A \left(\frac{|\xi|}{2}\right)^{-\frac{1}{2+\alpha}} \log\left(\frac{|\xi|}{2}\right) \\ \leq & A_1 (1 + |\xi|)^{-\frac{1}{2+\alpha}} \log(e + |\xi|),\end{aligned}$$

for some constant A_1. Here we have again used the fact that $|\widehat{g}_M(n)|$ is bounded by a monotonically decreasing function by lemma 5.7.

Now, by the first observation there exists a constant $M_2 = M_2(\phi, \alpha, \delta)$, such that $C_0 A_1 \leq \delta/2 \log\log(e^e + |\xi|)$, whenever $|\xi| \geq 2(4M_2)^{2+\alpha}$. Thus, we have

$$C_0 \sum_{|\xi - n| < \frac{|\xi|}{2}} |\widehat{g}_M(n)|(1 + |\xi - n|)^{-2} \leq \frac{\delta}{2}\theta(\xi), \tag{5.28}$$

whenever $|\xi| \geq 2(4M)^{2+\alpha}$ for all $M \geq \max\{4, \Lambda, M_2\}$.

Let us now treat the case $|\xi - n| > |\xi|/2$. By applying the first result of lemma 5.7, we obtain

$$\begin{aligned}\sum_{|\xi - n| \geq \frac{|\xi|}{2}} |\widehat{g}_M(n)|(1 + |\xi - n|)^{-2} \leq & AM^{-1} \log M \sum_{|\xi - n| \geq \frac{|\xi|}{2}} |(1 + |\xi - n|)^{-2} \\ \leq & 2Ae^{-1} \sum_{k > \frac{|\xi|}{2}} k^{-2},\end{aligned}$$

where we have used the fact that $x^{-1}\log x$ has a maximum at $x = e$ for $x \geq 1$. It then follows from the integral test that

$$\sum_{k>\frac{|\xi|}{2}} k^{-2} \leq \left(4|\xi|^{-2} + \int_{\frac{|\xi|}{2}}^{\infty} x^{-2}dx\right) \leq 6|\xi|^{-1},$$

which again implies

$$\sum_{|\xi-n|\geq\frac{|\xi|}{2}} |\widehat{g}_M(n)|(1+|\xi-n|)^{-2} \leq 12Ae^{-1}|\xi|^{-1} \leq 12Ae^{-1}(1+|\xi|)^{-\frac{1}{2+\alpha}}\log{(e+|\xi|)},$$

since $|\xi| > 2(4M)^{2+\alpha}$. We can therefore find an integer $M_3 = M_3(\phi, \alpha, \delta)$, such that

$$12Ae^{-1}C_0 < \delta/2\log\log{(e^e + |\xi|)},$$

whenever $|\xi| \geq 2(4M_3)^{2+\alpha}$. It then follows that for $|\xi - n| > |\xi|/2$, that

$$\sum_{|\xi-n|\geq\frac{|\xi|}{2}} |\widehat{g}_M(n)|(1+|\xi-n|)^{-2} \leq \frac{\delta}{2}\theta(\xi) \tag{5.29}$$

whenever $|\xi| \geq 2(4M)^{2+\alpha}$ for all $M \geq \max\{4, \Lambda, M_3\}$. Thus, by combining (5.28) and (5.29), we have for any $M \geq \max\{4, \Lambda, M_2, M_3\}$ the bound,

$$|\widehat{\phi g_M}(\xi) - \widehat{\phi}(\xi)| \leq \delta\theta(\xi), \quad |\xi| \geq 2(4M)^{2+\alpha}. \tag{5.30}$$

To finish the proof, we define $M_0 = M_0(\phi, \alpha, \delta) := \max\{M_1, M_2, M_3\}$. Then for any $M \geq M_0$, we have

$$|\widehat{\phi g_M}(\xi) - \widehat{\phi}(\xi)| \leq \delta\theta(\xi),$$

for all $\xi \in \mathbb{R}$ by (5.27) and (5.30). Since $\phi \in C_0^{1,1}(\mathbb{R})$ and $\delta > 0$ was arbitrary, while α is assumed to be fixed, we can repeat the proof for any choice of ϕ and δ, and conclude that there exists an $M_0 = M_0(\phi, \delta)$ such that for any $M \geq M_0$,

$$|\widehat{\phi g_M}(\xi) - \widehat{\phi}(\xi)| \leq \delta\theta(\xi),$$

for all $\xi \in \mathbb{R}$. $\square$

References

[1] C. Bluhm, Random recursive construction of Salem sets, Ark. Mat. 34 (1996), 51-63

[2] C. Bluhm, On a theorem of Kaufman: Cantor-type construction of linear fractal Salem sets, Ark. Mat. 36 (1998), no. 2, 307-316

[3] H. Davenport, Multiplicative Number Theory, Springer-Verlag, 1980

[4] G. B. Folland, Real Analysis: Modern Techniques and Their Applications, Wiley Inter-Science, 1999

[5] K. Hambrook, Explicit Salem sets and applications to metrical Diophantine approximation, Trans. Amer. Math. Soc., 371 (2019), no. 6, 4353–4376.

[6] R. Kaufman, On the theorem of Jarník and Besicovitch, Acta Arith. 39 (1981), 265–267.

[7] P. Mattila, Fourier Analysis and Hausdorff dimension, Cambridge Univ. Press, 2015.

[8] C. Muscalu and W. Schlag, Classical and Multilinear Harmonic Analysis, vol 1, Cambridge University Press, 2013.

[9] R. Salem, On singular monotonic functions whose spectrum has a given Hausdorff dimension, Ark. Mat. 1 (1950), 353-365

[10] T. Wolff, Lectures on harmonic analysis. University Lecture Series, 29. American Mathematical Society, Providence, RI, 2003