TWO TYPES OF RUBIO DE FRANCIA OPERATORS ON TRIEBEL–LIZORKIN AND BESOV SPACES

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ABSTRACT. We discuss generalizations of Rubio de Francia's inequality for Triebel–Lizorkin and Besov spaces, continuing the research from [5]. Two versions of Rubio de Francia's operator are discussed: it is shown that a rotation factor is needed for the boundedness of the operator in some smooth spaces while it is not essential in other spaces. We study the operators on some "end" spaces of the Triebel–Lizorkin scale and then use usual interpolation methods.

1. INTRODUCTION

If f is a function in $L^2([0,1])$ and I is an interval in $\mathbb{Z}$, then by $(M_I f)(x)$ we denote the exponential polynomial $(\widehat{f} \mathbb{1}_I)^\vee(x) = \sum_{n \in I} \widehat{f}(n) e^{2\pi i n x}$. For any collection $\mathcal{I}$ of pairwise disjoint intervals $I \subset \mathbb{Z}$ such that $\bigcup_{I \in \mathcal{I}} I = \mathbb{Z}$, we have

$$\|f\|_{L^2} = \left\| \left(\sum_{I \in \mathcal{I}} |M_I f|^2 \right)^{1/2} \right\|_{L^2}. \tag{1}$$

This is an equivalent reformulation[1] of Parseval's identity, one of the most fundamental results in harmonic analysis. For brevity, we can write the right expression in (1) as $\left\| \{M_I f\}_{I \in \mathcal{I}} \right\|_{L^2(l^2)}$.

In form (1), Parseval's identity has an extension to the spaces $L^p([0,1])$. Namely, for $2 \le p < \infty$, we have the following two-sided inequality:

$$c_p \left\| \{M_I f\}_{I \in \mathcal{I}} \right\|_{L^p(l^2)} \le \|f\|_{L^p} \le C_p \left\| \{M_J f\}_{J \in \mathcal{J}} \right\|_{L^p(l^2)}, \tag{2}$$

where $\mathcal{I}$ is an arbitrary collection of pairwise disjoint intervals in $\mathbb{Z}$, the collection $\mathcal{J}$ is defined as

$$\mathcal{J} \stackrel{\text{def}}{=} \left\{ (-2^{k+1}, -2^k] \right\}_{k \in \mathbb{Z}^+} \cup \left\{ \{0\} \right\} \cup \left\{ [2^k, 2^{k+1}) \right\}_{k \in \mathbb{Z}^+},$$

and the constants c_p and C_p depend only on p (in particular, c_p does not depend on the choice of $\mathcal{I}$). The left inequality have been obtained by Rubio de Francia [6] in 1983, and the right inequality is the classical Littlewood–Paley theorem (see, e.g., the exposition in [8]). By duality, if we interchange the left and the right expressions in (2), we obtain correct estimates for $1 < p \le 2$, provided $\bigcup_{I \in \mathcal{I}} I = \mathbb{Z}$.

In what follows, we consider the whole line $\mathbb{R}$ instead of $[0,1]$. In such a context, the Fourier transform $\widehat{f}$ is also defined on $\mathbb{R}$ (so we consider collections of intervals

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[1]We are talking about the equivalence of two correct statements in the sense that they are direct consequences of each other.

on $\mathbb{R}$) and relation (2) remains true, provided k runs over the whole $\mathbb{Z}$ in the definition of $\mathcal{J}$. In fact, the corresponding results are usually presented precisely in this form (see [6, 8]).

Next, we note that L^p-classes do not exhaust the set of spaces studied in harmonic analysis. In addition to them, there are many normed spaces that seem, at first glance, to have no direct connection with each other: Sobolev spaces, the BMO-space, Hölder–Zygmund classes of smooth functions, etc. But it is known that the corresponding norms can be written in a uniform way: all these spaces belong to the scale of Triebel–Lizorkin and Besov spaces. In this article, we outline an overall picture: we discuss generalizations of Rubio de Francia's inequality for a substantial part of Besov–Triebel–Lizorkin scale (which includes all of the spaces listed above). In this general context, we raise and answer a subtle question concerning the presence or absence of the rotations in the operators that correspond to Rubio de Francia's inequality.

Now, let $\mathcal{I} = \{I_m\} = \big\{[a_m, b_m]\big\}$ be a finite or countable collection of pairwise disjoint intervals in $\mathbb{R}$ such that

$$(3) \qquad\qquad\qquad 0 \notin (a_m, b_m)$$

for any m. Suppose φ is a Schwartz function such that $\operatorname{supp} \widehat{\varphi} \subset (0, 1)$ (in particular, $\operatorname{supp} \widehat{\varphi}$ is separated from 0 and 1). We introduce the functions φ_m corresponding to the intervals I_m:

$$(4) \qquad\qquad\qquad \widehat{\varphi}_m(t) = \widehat{\varphi}\left(\frac{t - a_m}{b_m - a_m}\right).$$

Consider two operators that transform scalar-valued functions to collections of functions by the following formulas:

$$(5) \qquad S_{\mathcal{I}}^{\varphi} f(x) \stackrel{\text{def}}{=} \big\{(f * \varphi_m)(x)\big\}_m \quad \text{and} \quad \widetilde{S}_{\mathcal{I}}^{\varphi} f(x) \stackrel{\text{def}}{=} \big\{e^{-2\pi i\, a_m x}(f * \varphi_m)(x)\big\}_m.$$

Also we introduce two corresponding families of operators

$$\mathfrak{S}_{\varphi} \stackrel{\text{def}}{=} \big\{S_{\mathcal{I}}^{\varphi}\big\}_{\mathcal{I}} \quad \text{and} \quad \widetilde{\mathfrak{S}}_{\varphi} \stackrel{\text{def}}{=} \big\{\widetilde{S}_{\mathcal{I}}^{\varphi}\big\}_{\mathcal{I}},$$

where $\mathcal{I}$ runs over all possible collections of pairwise disjoint intervals in $\mathbb{R}$ satisfying (3).

The fact that for $2 \leq p < \infty$ the family $\mathfrak{S}_{\varphi}$ is uniformly bounded from L^p to $L^p(l^2)$ is a version of Rubio de Francia's theorem where we have substituted smooth multipliers φ_m instead of $\mathbb{1}_{I_m}$.[2] Its proof is contained in considerations of [6]. In fact, Rubio de Francia deals with the family $\widetilde{\mathfrak{S}}_{\varphi}$. The matter is that the factors $e^{-2\pi i\, a_m x}$ played a significant role in the proof: their presence allows to get a Calderón–Zygmund type condition for the kernels of $\widetilde{S}_{\mathcal{I}}^{\varphi}$. But since the L^p-norms are invariant under multiplications by unimodular functions and, in particular, are rotation-invariant, the exponential functions can be dropped. Now we note that the norms in all the other Triebel–Lizorkin spaces as well as in the Besov spaces are not rotation-invariant. Therefore the boundedness of the families $\mathfrak{S}_{\varphi}$ and $\widetilde{\mathfrak{S}}_{\varphi}$ should be studied separately on such spaces.

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[2]In the original form his result cannot be extended to some of the Besov and Triebel–Lizorkin spaces (see [5]). In this article, we do not want to touch on issues that arise when dealing with non-smooth multipliers.

Some studies concerning the family with rotations can be found in [5], where the author considers pointwise estimates for the operators $\widetilde{S}_{\mathcal{I}}^{\varphi}$ in terms of sharp (oscillatory) maximal functions. In particular, the results of [5] imply that $\widetilde{\mathfrak{S}}_{\varphi}$ is uniformly bounded on the Hölder–Zygmund spaces $\dot{\mathcal{C}}_s$ as well as on BMO. But it turns out that in the context of the Besov–Triebel–Lizorkin scale those pointwise estimates give much more: we are going to rely heavily on them in our considerations below.

The family $\mathfrak{S}_{\varphi}$ is also studied below. In particular, we are going to show that it is not bounded on $\dot{\mathcal{C}}_s$ or BMO. But surprisingly, it turns out that the both of our families are uniformly bounded on some other Triebel–Lizorkin and Besov spaces with the norms that are not rotation-invariant.

2. Preliminaries

2.1. **Triebel–Lizorkin and Besov spaces.** We restrict ourselves to considering only functions on the real line $\mathbb{R}$. Let $\mathcal{S}$, $\mathcal{S}'$, and $\mathcal{P}$ be Schwartz space, the space of tempered distributions, and the space of all algebraic polynomials respectively.

Consider a function $\phi \in \mathcal{S}$ such that $\operatorname{supp}\widehat{\phi} \subset [-2, 2]$ and $\widehat{\phi} \equiv 1$ on $[-1, 1]$. If we introduce functions ϕ_j by the formula

(6) $$\widehat{\phi}_j(\xi) = \widehat{\phi}\big(2^{-j}\xi\big) - \widehat{\phi}\big(2^{-j+1}\xi\big), \quad j \in \mathbb{Z},$$

then the collection $\{\phi_j\}_{j\in\mathbb{Z}}$ will be *a resolution of unity*, i.e., we will have

(7) $$\operatorname{supp}\widehat{\phi}_j \subset \big[-2^{j+1}, -2^{j-1}\big] \cup \big[2^{j-1}, 2^{j+1}\big].$$

and

$$\sum_{j\in\mathbb{Z}} \widehat{\phi}_j \equiv 1 \quad \text{on} \quad \mathbb{R} \setminus \{0\}.$$

Definition 1. Let $0 < p < \infty$, $0 < q \leq \infty$, and $s \in \mathbb{R}$. We say that an element f of the quotient space $\mathcal{S}'/\mathcal{P}$ belongs to the homogeneous Triebel–Lizorkin space $\dot{F}_{pq}^{s}$ if

$$\|f\|_{\dot{F}_{pq}^{s}} \stackrel{\text{def}}{=} \left\|\, \left|\big\{2^{js}f * \phi_j\big\}_{j\in\mathbb{Z}}\right|_{l^q} \right\|_{L^p} < \infty.$$

If we permute the L^p- and l^q-norms, we obtain a definition of the Besov spaces $\dot{B}_{pq}^{s}$.

Definition 2. Let $0 < p \leq \infty$, $0 < q \leq \infty$, and $s \in \mathbb{R}$. We say that $f \in \mathcal{S}'/\mathcal{P}$ belongs to the homogeneous Besov space $\dot{B}_{pq}^{s}$ if

$$\|f\|_{\dot{B}_{pq}^{s}} \stackrel{\text{def}}{=} \left|\left\{\left\|2^{js}f * \phi_j\right\|_{L^p}\right\}_{j\in\mathbb{Z}}\right|_{l^q} < \infty.$$

Note that we have not define the spaces $\dot{F}_{\infty q}^{s}$. It turns out that a direct extension of Definition 1 to $p = \infty$ is not reasonable. Such a space would depend on the choice of a dyadic resolution of unity participating in the definition (see [9]). A correct definition of $\dot{F}_{\infty q}^{s}$ follows from duality arguments and can be found, e.g., in [2, 9, 11]. There are some well-known facts about Triebel–Lizorkin and Besov spaces.

Proposition 1. *We have*
  (i) $\dot{F}_{pp}^{s} = \dot{B}_{pp}^{s}$ *if* $0 < p < \infty$;
  (ii) $\dot{B}_{\infty\infty}^{s} \cong \dot{\mathcal{C}}^{s}$ *if* $s > 0$;
  (iii) $\dot{F}_{p2}^{k} \cong \dot{W}_{k}^{p}$ *if* $1 < p < \infty$ *and* $k \in \mathbb{Z}_{+}$;

(iv) $\dot{F}^0_{\infty 2} \cong \mathrm{BMO}$.

Here by $\dot{\mathcal{C}}^s$, $s > 0$, we denote the homogeneous Hölder–Zygmund spaces. The corresponding definition can be found, e.g., in [11, 1.4.5]. In the same place the Besov norm is presented in the form that immediately implies (ii). Here we only note that if $s \notin \mathbb{Z}_+$, then the norm in $\dot{\mathcal{C}}^s$ is equivalent to the corresponding Hölder norm:

$$\|f\|_{\dot{\mathcal{C}}^s} \cong \sup_{x \neq y} \frac{\left|f^{(k)}(x) - f^{(k)}(y)\right|}{|x - y|^{s-k}}, \quad k = [s].$$

Concerning (iii) and (iv), see [10, Chapter 5]. Here $\dot{W}^p_k$ are homogeneous Sobolev spaces, and (iii) includes, in particular, the fact that $\dot{F}^0_{p2} = L^p$, $1 < p < \infty$.

2.2. **Sharp maximal functions.** Let $\mathcal{P}_i$ be the space of algebraic polynomials of degree strictly less than i. We agree that $\mathcal{P}_0 = \{0\}$.

Definition 3. Suppose[3] $1 \leq p < \infty$, $i \in \mathbb{Z}_+$, and $s \in [0, i]$. Let h be a measurable function on $\mathbb{R}$. We define the maximal function $M^p_{i,s} h$ by the formula

$$M^p_{i,s} h(x) \overset{\mathrm{def}}{=} \sup_{I \ni x} \inf_{P} \frac{1}{|I|^s} \left(\frac{1}{|I|} \int_I |h - P|^p \right)^{1/p},$$

where the supremum is taken over all the intervals containing x and the infimum is taken over all the polynomials $P \in \mathcal{P}_i$.

Definition 4. Let $1 \leq p < \infty$ and $s > 0$. Suppose $f \in L^1_{\mathrm{loc}} / \mathcal{P}_{[s]+1}$. We say that $f \in \dot{\mathcal{C}}^s_p$ if

$$\|f\|_{\dot{\mathcal{C}}^s_p} \overset{\mathrm{def}}{=} \left\| M^p_{[s]+1,s} f \right\|_{L^p} < \infty.$$

We can extend this definition to $\dot{\mathcal{C}}^s_\infty$. It is known (see [1, 4] and the exposition in [3]) that the quantities $\|M^p_{i,s} f\|_{L^\infty}$ are equivalent for various p, and so we put

(8) $$\|f\|_{\dot{\mathcal{C}}^s_\infty} \overset{\mathrm{def}}{=} \left\| M^2_{[s]+1,s} f \right\|_{L^\infty}.$$

We have (see [1, 3, 4])

$$\dot{\mathcal{C}}^s_\infty \cong \dot{\mathcal{C}}^s.$$

Following Triebel [11, 1.7.2], we put

(9) $$\dot{F}^s_{\infty\infty} \overset{\mathrm{def}}{=} \dot{B}^s_{\infty\infty} \cong \dot{\mathcal{C}}^s, \quad s > 0,$$

and state the following fact.

Proposition 2. *If $1 \leq p \leq \infty$ and $s > 0$, then for $f \in \dot{F}^s_{p\infty}$ we have*

(10) $$\dot{F}^s_{p\infty} \cong \dot{\mathcal{C}}^s_p.$$

This proposition is a consequence of [7, Theorem 1].

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[3]A wider range of parameters p and s can be considered in this context, but those that are indicated here suffice for our goals.

2.3. **Interpolation.** The interpolation between Triebel–Lizorkin spaces is one of the main components of our subsequent considerations.

Proposition 3. *Interpolating between $\dot{F}^s_{pq}$-spaces, we can obtain another Triebel–Lizorkin space as well as a Besov space depending on the interpolation method we use.*

    (i) *Let $s_0, s_1 \in \mathbb{R}$, $1 \le q_0 < \infty$, $1 \le q_1 \le \infty$, and $1 \le p_0, p_1 < \infty$. Suppose*

$$0 < \theta < 1, \quad s = (1-\theta)s_0 + \theta s_1,$$

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad and \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

    Applying the complex interpolation method, we have

(11) $$\left[\dot{F}^{s_0}_{p_0 q_0}, \dot{F}^{s_1}_{p_1 q_1} \right]_\theta = \dot{F}^s_{pq}.$$

    (ii) *Let $s_0, s_1 \in \mathbb{R}$, $s_0 \ne s_1$, $0 < q_0, q, q_1 \le \infty$, and $0 < p < \infty$. As above, suppose*

$$0 < \theta < 1 \quad and \quad s = (1-\theta)s_0 + \theta s_1.$$

    Applying the real interpolation method, we have

(12) $$\left(\dot{F}^{s_0}_{pq_0}, \dot{F}^{s_1}_{pq_1} \right)_{\theta, q} = \dot{B}^s_{pq}.$$

Part (i) of this theorem is contained in [2, Corollary 8.3]. Here $[\cdot, \cdot]_\theta$ is the classical complex interpolation method with the interpolation property. Concerning part (ii), see [10, 2.4.2, 5.2.5]

2.4. **Vector-valued spaces.** Let X be a Triebel–Lizorkin ($p \ne \infty$) or Besov space. Then by X_* we denote the space of sequences

$$f = \{f_m\}_{m \in \mathbb{N}}, \quad f_m \in \mathcal{S}'/\mathcal{P},$$

equipped with the corresponding norm where we substitute lengths in l^2 instead of absolute values. For example, if $X = \dot{F}^s_{pq}$, then X_* has the norm

$$\|f\|_{X_*} = \left\| \left| \left\{ \left| \left\{ 2^{js} f_m * \phi_j \right\}_{m \in \mathbb{N}} \right|_{l^2} \right\}_j \right|_{l^q} \right\|_{L^p}.$$

We leave the reader to determine what will be the norm in X_* if we put $X = \dot{B}^s_{pq}$. By X_N, $N \in \mathbb{N}$, we denote the subspace in X_* consisting of sequences such that $f_m = 0$ for $m > N$. Similarly substituting l^2-norms instead of absolute values, we can also introduce the maximal functions $M^p_{i,s}$ as well as the spaces $\dot{\mathcal{C}}^s_p$ (see Definitions 3 and 4) for finite or countable collections of functions.

Since there is no difference whether we deal with absolute values or with lengths of finite-dimensional vectors, we can assert the following.

Fact 1. All aforecited facts on Triebel–Lizorkin or Besov spaces X remain true for the corresponding spaces X_N independently on N.

Next, since the l^2-norm is a limit of an increasing non-negative sequence, we have

$$\left\| \{f_m\}_{m \in \mathbb{N}} \right\|_{X_*} = \lim_{N \to \infty} \left\| \{f_1, \dots, f_N, 0, \dots\} \right\|_{X_N}$$

and, therefore, it suffice to deal only with the spaces X_N. Namely, we can state the following fact.

Fact 2. If for finite collections $\mathcal{I} = \{I_m\}_{m=1}^N$ of intervals the operators $S_{\mathcal{I}}^\varphi$ and $\widetilde{S}_{\mathcal{I}}^\varphi$ are bounded from X to X_N uniformly in N and $\mathcal{I}$, then this remains true for countable collections $\mathcal{I}$: the families $\widetilde{\mathfrak{S}}_\varphi$ and $\mathfrak{S}_\varphi$ are uniformly bounded from X to X_*.

Using considerations from [3, 5], we can prove the following proposition.

Proposition 4. *Suppose $2 \le p < \infty$, $i \in \mathbb{Z}_+$, and $s \in [0, i)$. If f is a measurable function such that $M_{i,s}^p f$ is finite at least at one point, then $f \in \mathcal{S}'$ and we have the following pointwise estimate:*

$$(13) \qquad M_{i,s}^p \left(\widetilde{S}_{\mathcal{I}}^\varphi f \right) \le C M_{i,s}^p f,$$

where the constant C does not depend on $\mathcal{I}$ or f.

In [5], a similar estimate is proved for $p = 2$ and for non-smooth multipliers $\widehat{\varphi}_m$. But Rubio de Francia's [6] theorem allows to prove that the same method can be employed for all $p \ge 2$; and the smoothness of $\widehat{\varphi}_m$ simplifies the arguments. Also we note that this is the very place where we need the set supp $\widehat{\varphi}$ to be separated from 0 and 1.

Relations (10) and (13) together with Facts 1 and 2 imply the following consequence.

Proposition 5. *Let $2 \le p \le \infty$ and $s > 0$. If we put $X = \dot{F}_{p\infty}^s$, then the family $\widetilde{\mathfrak{S}}_\varphi$ will be uniformly bounded from X to X_*.*

We also have (see [5] again) the following proposition.

Proposition 6. *Let $X = \dot{F}_{\infty 2}^0 \cong \mathrm{BMO}$. Then the family $\widetilde{\mathfrak{S}}_\varphi$ will be uniformly bounded from X to $X_* \overset{\mathrm{def}}{=} \mathrm{BMO}(l^2)$.*

3. Formulation of the results

Definition 5. We say that $\gamma \in L^1$ is non-degenerate if $\int_{-\infty}^0 e^{2\pi i t} \gamma(t)\, dt \ne 0$.

The following fact justifies the term "non-degenerate".

Fact 3. If $\gamma \in L^1$ is a non-zero function such that $\widehat{\gamma}$ is non-negative and supported in $[0, 1]$, then γ is non-degenerate.

Proof. Let $\Phi_\gamma(x) \overset{\mathrm{def}}{=} \int_{-\infty}^x e^{2\pi i t} \gamma(t)\, dt$. We have $\widehat{\Phi}_\gamma(\xi) = (2\pi i \xi)^{-1} \widehat{\gamma}(\xi - 1)$ and

$$\int_{-\infty}^0 e^{2\pi i t} \gamma(t)\, dt = \Phi_\gamma(0) = \int_{\mathbb{R}} \widehat{\Phi}_\gamma(\xi)\, d\xi = \frac{1}{2\pi i} \int_1^2 \frac{\widehat{\gamma}(\xi - 1)}{\xi}\, d\xi.$$

Since $\widehat{\gamma} \ge 0$ and does not vanish on $[0, 1]$, we get $\Phi_\gamma(0) \ne 0$ and γ is non-degenerate. $\square$

Now we are ready to present our results.

Theorem 1. *Let $X = \dot{F}_{pq}^s$. We determine various ranges for p, q, and s for each case considered below.*

  (i) *Let $2 \le p \le \infty$, $2 \le q \le \infty$, and $s > 0$. We modify this domain as follows (see also Figure 1):*
    • *if $q = 2$, then for $p \ne \infty$ we consider all $s \ge 0$;*

- *if $q = 2$ and $p = \infty$, then we consider only $s = 0$;*
- *if $p = \infty$, then we exclude $q \in (2, \infty)$ from consideration.*

If p, q, and s belong to the domain just described, then the family $\widetilde{\mathfrak{S}}_\varphi$ is uniformly bounded from X to X_*.

(ii) *For $2 \le q \le p < \infty$ and $s \ge 0$ (see Figure 3), the family $\mathfrak{S}_\varphi$ is uniformly bounded from X to X_*.*

(iii) *If $X = \dot{F}^0_{\infty 2}$ or $X = \dot{F}^s_{\infty\infty}$, $s > 0$, then there exists a collection $\mathcal{I}$ of pairwise disjoint intervals such that the operator $S^\varphi_\mathcal{I}$ is not bounded from X to X_* provided φ is non-degenerate.*

So there are Triebel–Lizorkin spaces where only the family $\widetilde{\mathfrak{S}}_\varphi$ is uniformly bounded as well as spaces where both families $\mathfrak{S}_\varphi$ and $\widetilde{\mathfrak{S}}_\varphi$ are uniformly bounded (in spite of the fact that the corresponding norms are not rotation-invariant).

Similar result holds for the Besov spaces. Namely, we have the following theorem.

Theorem 2. *Let $X = \dot{B}^s_{pq}$.*

(i) *Let $2 \le p \le \infty$, $0 < q \le \infty$, and $s > 0$. If $p = \infty$, then we exclude all $q \ne \infty$ from consideration (see Figure 2). For such p, q, and s, the family $\widetilde{\mathfrak{S}}_\varphi$ is uniformly bounded from X to X_*.*

(ii) *For $2 \le p < \infty$, $0 < q \le \infty$, and $s \ge 0$, the family $\mathfrak{S}_\varphi$ is uniformly bounded from X to X_*.*

(iii) *Let $X = \dot{B}^s_{\infty q}$ for $0 < q \le \infty$ and $s \ge 0$. Then there exists a collection $\mathcal{I}$ of pairwise disjoint intervals such that the operator $S^\varphi_\mathcal{I}$ is not bounded from X to X_* provided φ is non-degenerate.*

As we will see, there is a deep connection between Theorems 1 and 2. The point is that in order to prove their first parts, we will, in fact, interpolate between the same spaces, but applying two different methods of interpolation.

We also mention the following non-linear quadratic operator that transform scalar-valued functions to scalar-valued functions:

$$G^\varphi_\mathcal{I} f = \left| S^\varphi_\mathcal{I} f \right|_{l^2} = \left| \widetilde{S}^\varphi_\mathcal{I} f \right|_{l^2}$$

It is a more "rough" operator: treating it, we deal with expressions of the form $\left| |a|_{l^2} - |b|_{l^2} \right|$ instead of $|a - b|_{l^2}$ (it becomes clear what we mean if we put, e.g., $X = \dot{\mathcal{C}}^s$). We also note that in order to study the operator $G^\varphi_\mathcal{I}$, we should answer the question: can the sequences $S^\varphi_\mathcal{I} f$ or $\widetilde{S}^\varphi_\mathcal{I} f$ be presented as l^2-valued functions? We assume this neither in the definition of vector-spaced spaces X_* nor elsewhere above. Concerning this question, see, e.g., [5, Fact 2.1]. Here we do not investigate problems related to the operator $G^\varphi_\mathcal{I}$ anymore.

4. The proofs

In order to prove parts (i) and (ii) of Theorems 1 and 2, it suffices (due to Fact 2) to consider finite collections $\mathcal{I}$ of intervals that determine the operators. In this case, Fact 1 allows to employ the whole theory of Triebel–Lizorkin and Besov spaces.

4.1. **Proof of Theorem 1, part (i).** First, we consider the space $\dot{F}^k_{22}$, $k \in \mathbb{Z}_+$, which coincides with $\dot{W}^2_k$ (see Proposition 1).

Lemma 1. *If $X = \dot{F}_{22}^k$, $k \in \mathbb{Z}_+$, then the family $\widetilde{\mathfrak{S}}_\varphi$ is uniformly bounded from X to X_*.*

Proof. Let $f \in X$. We put

$$g_m(x) \overset{\text{def}}{=} e^{-2\pi i\, a_m x}(f * \varphi_m)(x)$$

(i.e., we have $\{g_m\} = \widetilde{S}_{\mathcal{I}}^\varphi f$), and by the Plancherel theorem, we can write

$$(14) \qquad \left\|\{g_m^{(k)}\}\right\|_{L^2(l^2)}^2 = \sum_m \int_0^{l_m} \left|\hat{f}(\xi + a_m)\right|^2 \left|\widehat{\varphi}_m(\xi + a_m)\right|^2 |\xi|^{2k}\, d\xi,$$

where $l_m = b_m - a_m$. Rewrite each term with $a_m \geq 0$ as

$$\int_{I_m} \left|\hat{f}(\xi)\right|^2 \left|\widehat{\varphi}_m(\xi)\right|^2 |\xi - a_m|^{2k}\, d\xi.$$

In this case we have $|\xi - a_m| \leq |\xi|$, and, therefore, it can be estimated by

$$C\int_{I_m} \left|\hat{f}(\xi)\right|^2 |\xi|^{2k}\, d\xi.$$

For all the remaining terms in (14), we have $b_m \leq 0$, because $0 \notin (a_m, b_m)$ for all m. In this case, we rewrite the discussed terms as

$$(15) \qquad \int_{-l_m}^0 \left|\hat{f}(\xi + b_m)\right|^2 \left|\widehat{\varphi}_m(\xi + b_m)\right|^2 |\xi + l_m|^{2k}\, d\xi$$

and get rid of l_m in the last factor.[4] For this we verify that for $\xi \in (-l_m, 0)$, we have

$$(16) \qquad \left|\widehat{\varphi}_m(\xi + b_m)\right|^2 |\xi + l_m|^{2k} \leq C_k |\xi|^{2k},$$

where C_k does not depend on m. We have $\widehat{\varphi}_m(\xi + b_m) = \widehat{\varphi}(1 + \xi/l_m)$, and in order to prove (16), we only need to verify that

$$\widehat{\varphi}(1 - t) \leq C_k \left(\frac{t}{1 - t}\right)^k \quad \text{for} \quad t \in (0, 1).$$

But this is true because $\widehat{\varphi}$ equals zero at 1 with all its derivatives.[5] Thus, we have that (15) can be estimated by

$$C\int_{-l_m}^0 \left|\hat{f}(\xi + b_m)\right|^2 |\xi|^{2k}\, d\xi = C\int_{I_m} \left|\hat{f}(\xi)\right|^2 |\xi - b_m|^{2k}\, d\xi.$$

But since we consider the terms where $b_m \leq 0$, the last expressions are lesser than

$$C\int_{I_m} \left|\hat{f}(\xi)\right|^2 |\xi|^{2k}\, d\xi.$$

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[4]Thus, we show that it is not significant whether we shift a_m or b_m to the origin.

[5]We could also use the fact that supp $\widehat{\varphi}$ is separated from 0 and 1, but we do not need this restriction in order to prove Lemma 1.
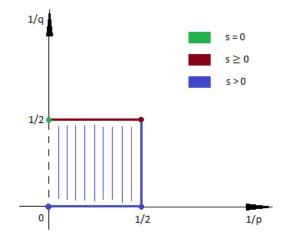
FIGURE 1. Part (i) of Theorem 1.

Combining it all together, at least for finite collections $\mathcal{I}$ we obtain

$$\left\|\widetilde{S}_{\mathcal{I}}^{\varphi}f\right\|_{X_*}^2 \cong \left\|\{g_m^{(k)}\}\right\|_{L^2(l^2)}^2 \leq C\sum_m \int_{I_m} |\hat{f}(\xi)|^2 |\xi|^{2k}\,d\xi \leq C\|f\|_X^2.$$

Due to Fact 2, the lemma is proved. □

We know that $\widetilde{\mathfrak{S}}_{\varphi}$ is bounded on $\dot{F}_{\infty 2}^0 \cong$ BMO (see Proposition 6) and on $\dot{F}_{\infty\infty}^s \cong \dot{\mathcal{C}}^s$ as well (see Proposition 5).

Finally, let $2 \leq p < \infty$. We have the boundedness on $\dot{F}_{p2}^0 \cong L^p$ (see Rubio de Francia's [6] theorem), on $\dot{F}_{22}^k$ for $k \in \mathbb{Z}_+$ (see Lemma 1 just proved), and on $\dot{F}_{p\infty}^s$ for $s > 0$ (see Proposition 5). Using the complex interpolation method (11) for the couples $\{\dot{F}_{p2}^0, \dot{F}_{22}^k\}$ and $\{\dot{F}_{p2}^0, \dot{F}_{p\infty}^s\}$, we come to the desired result (see Figure 1).

4.2. **Proof of Theorem 2, part (i).** We already know that $\widetilde{\mathfrak{S}}_{\varphi}$ is uniformly bounded on $\dot{B}_{\infty\infty}^s = \dot{\mathcal{C}}^s$. For the remaining spaces we can use the real interpolation method (12). Indeed, suppose $2 \leq p < \infty$ and $s_0, s_1 > 0$. Then part (i) of Theorem 1 implies that if we take $\{\dot{F}_{p\infty}^{s_0}, \dot{F}_{p\infty}^{s_1}\}$ or $\{\dot{F}_{p2}^{s_0}, \dot{F}_{p2}^{s_1}\}$ as an interpolation couple, we come to the desired result (see Figure 2).

4.3. **Proof of Theorem 2, part (ii).** Denote $g_m \overset{\text{def}}{=} f * \varphi_m$. By Rubio de Francia's [6] theorem, we have

$$\left\|\{g_m * \phi_j\}_m\right\|_{L^p(l^2)} = \left\|\{f * \phi_j * \varphi_m\}_m\right\|_{L^p(l^2)} \leq C\|f * \phi_j\|_{L^p}$$

and multiplying by 2^{sj} and taking l^q norms we obtain the required boundedness.

4.4. **Proof of Theorem 1, part (ii).** First, consider the spaces

$$X = \dot{F}_{p2}^k \cong \dot{W}_k^p, \quad k \in \mathbb{Z}^+, \quad 2 \leq p < \infty.$$

Suppose $f \in X$. By Rubio de Francia's [6] theorem, we have

$$\left\|\{(f * \varphi_m)^{(k)}\}\right\|_{L^p(l^2)} = \left\|\{f^{(k)} * \varphi_m\}\right\|_{L^p(l^2)} = \left\|S_{\mathcal{I}}^{\varphi}[f^{(k)}]\right\|_{L^p(l^2)} \leq C\|f\|_X.$$

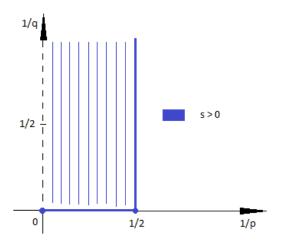Therefore, in the case being considered, we have the desired result.
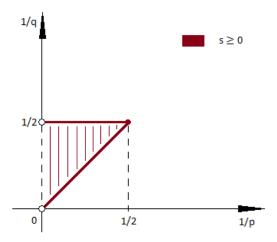
FIGURE 2. Part (i) of Theorem 2.

FIGURE 3. Part (ii) of Theorem 1.

But due to part (ii) of Theorem 2, we also know that $\mathfrak{S}_\varphi$ is uniformly bounded on the spaces

$$X = \dot{F}_{pp}^s = \dot{B}_{pp}^s, \quad 2 \leq p < \infty, \quad s \geq 0.$$

Using the complex interpolation method (11) (also see Figure 3), we conclude the proof.

4.5. **Proof of Theorem 2, part (iii).** Suppose that φ is non-degenerate and set

$$\mathcal{I} = \left\{ [1 + 2^m, \, 1 + 2^{m+1}] \right\}_{m \in \mathbb{Z}_-}.$$

Now consider our functions ϕ_j that are generated by the function ϕ and form a resolution of unity (see (6)). Due to (7) we have

$$\operatorname{supp} \widehat{\phi_0} \subset [-2, \, -1/2] \cup [1/2, \, 2].$$

Without loss of generality we can additionally assume that $\widehat{\phi} \equiv 1$ on $[-3/2, \, 3/2]$. Then we also have

(17) $$\widehat{\phi}_0 \equiv 1 \quad \text{on} \quad [-3/2, \, -1] \cup [1, \, 3/2].$$

We define
$$f_0 \stackrel{\text{def}}{=} \big[\exp(2\pi i \cdot) \, \mathbb{1}_{[0, +\infty)}(\cdot) \big] * \phi_0.$$
By (7) the function $f_0 * \phi_j$ does not vanish only if $j = -1,\, 0,\, 1$. This fact, together with Definition 2 and the obvious estimate $|f_0 * \phi_j| \leq C \|f_0\|_{L^\infty}$, implies
$$\|f_0\|_X \leq C \, \|f_0\|_{L^\infty} < \infty.$$

Next, we set
$$g_m \stackrel{\text{def}}{=} f_0 * \varphi_m \quad \text{and} \quad g_m^j \stackrel{\text{def}}{=} g_m * \phi_j,$$
where φ_m is the function that corresponds (see (4)) to the interval $[1 + 2^m, 1 + 2^{m+1}]$:
$$\varphi_m(x) = \exp\big(2\pi i \, (1 + 2^m) x\big) \, 2^m \varphi(2^m x).$$
If $2^{m+1} \leq 1/2$, then due to (17) we have
$$g_m^0 = \big[\exp(2\pi i \cdot) \, \mathbb{1}_{[0, +\infty)}(\cdot) \big] * \phi_0 * \phi_0 * \varphi_m$$
$$= \big[\exp(2\pi i \cdot) \, \mathbb{1}_{[0, +\infty)}(\cdot) \big] * \varphi_m = g_m.$$
Therefore, in this case we can write

(18)
$$g_m^0(x) = g_m(x) = e^{2\pi i x} \int\limits_{-\infty}^{2^m x} e^{2\pi i t} \varphi(t) \, dt$$
$$= e^{2\pi i x} \Phi_\varphi(0) + e^{2\pi i x} \big(\Phi_\varphi(2^m x) - \Phi_\varphi(0) \big),$$

where $\Phi_\varphi(x) = \int_{-\infty}^x e^{2\pi i t} \varphi(t) \, dt$. Since φ is non-degenerate (see Definition 5), we have
$$g_m^0(0) = \Phi_\varphi(0) \neq 0$$
provided $2^{m+1} \leq 1/2$. We also note that g_m^0 are continuous functions. We have
$$\big\| S_{\mathcal{I}}^\varphi f_0 \big\|_{X_*} = \left| \left\{ 2^{js} \big\| \{g_m^j\}_m \big\|_{L^\infty(l^2)} \right\}_j \right|_{l^q} \geq \big\| \{g_m^0\}_m \big\|_{L^\infty(l^2)} = +\infty.$$

4.6. **Proof of Theorem 1, part (iii).** By definition (9) we have $\dot{F}_{\infty\infty}^s = \dot{B}_{\infty\infty}^s$, $s > 0$, and it remains to prove the statement for $X = \dot{F}_{\infty 2}^0 \cong \text{BMO}$. We show that the same example as above gives an unbounded operator from X to X_*.

Consider $\{g_m\}_m$ as an element of the quotient space X_*. It is clear that $g_m(x)$ are bounded by $\|\varphi\|_1$ uniformly in m and x. Therefore, for a sequence $\{P_m\}_m$ of polynomials, the expression
$$\big\| \{g_m - P_m\}_m \big\|_{\text{BMO}(l^2)}$$
could be finite only if $\deg P_m \leq 1$ for all m.

Next, suppose $2^{m+1} \leq 1/2$ and $x \in [0, \, 1/2]$. Then by (18), we obtain
$$|g_m(x) - P_m(x) - c_m| \geq |\Phi_\varphi(0)| |\exp(2\pi i x) - a_m x - b_m| - |\Phi_\varphi(2^m x) - \Phi_\varphi(0)|.$$
Then there exist subsets $E_m \subset [0, \, 1/2]$ and a number $\gamma > 0$ such that $|E_m| > \gamma$ and
$$|g_m - P_m - c_m| > c_0 |\Phi_\varphi(0)| \quad \text{on} \quad E_m$$

provided m is small enough. Then

$$\big\|\{g_m - P_m\}_m\big\|_{\mathrm{BMO}(l^2)} = \infty.$$

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