Endre Sørmo Rundsveen

Torsion, Cotorsion and Tilting in Abelian Categories

Master's thesis in Mathematathical Sciences
Supervisor: Professor Steffen Oppermann
June 2021

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Abstract

We prove a correspondence between Tilting subcategories and Cotorsion Torsion triples in abelian categories with enough projectives. These structures are then shown to induce an equivalence of subcategories. We also prove that certain type of cotorsion pairs in categories of quiver-representations can be described locally in the underlying abelian category. These results are applied to show that some classes of Multiparameter Persistence Modules are of finite or tame representation type.

Sammendrag

Vi beviser en korrespondanse mellom "Tilting"-underkategorier og Kotorsjon Torsjons tripler i abelske kategorier med nok projektive. Deretter vises det at disse triplene induserer en ekvivalens av underkategorier. Vi beviser også at enkelte type kotorsjonspar i representasjonskategorier kan beskrives lokalt i den underliggende abelske kategorien. Resultatene anvendes til å vise at enkelte familier av "Multiparameter Persistence" Moduler er av endelig eller tam representasjonstype.

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1 Introduction

Tilting theory has throughout the end of the last century up until today been, and most certainly will be for the foreseeable future, a great tool in the study of algebraic structures. The exposition given in [17] brings to light the usefulness of the theory. In the classical setting of tilting one works over finitely generated modules over a finite-dimensional algebra A. However, in this thesis, the tilting theory studied will, following [5], be defined in an abelian category with enough projectives. This allows us to develop the dual notion of cotilting by simply passing to the opposite category. We will see that tilting induces a torsion theory in addition to a complete cotorsion theory in the given abelian category. In fact, we will show as one of our main results that there is a correspondence between what we call cotorsion torsion triples and tilting theories in the category, i.e.

Theorem 1.1 (Theorem 3.55). *Let $\mathcal{A}$ be an abelian category with enough projectives. Then the two constructions*

$$\begin{aligned} \{\text{tilting subcategories}\} &\leftrightarrow \{\text{cotorsion torsion triples}\} \\ \mathbb{T} &\mapsto (\{X \in {}^{\perp_1}\mathbb{T} \mid \operatorname{pdim} \leq 1\}, \operatorname{Fac}\mathbb{T}, \mathbb{T}^{\perp}) \\ \mathcal{C} \cap \mathcal{T} &\leftarrow\!\shortmid (\mathcal{C}, \mathcal{T}, \mathcal{F}) \end{aligned}$$

is a bijective correspondence.

The interest given cotorsion torsion triples is justified by how they induce an equivalence between subcategories of the category. This is the second of the main results presented in this thesis. Namely

Theorem 1.2 (Theorem 3.36). *Let $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ be a cotorsion torsion triple in an abelian category $\mathcal{A}$. Then there is an equivalence of subcategories*

$$F \simeq \frac{\mathcal{C}}{\mathcal{C} \cap \mathcal{T}}$$

which was discovered by [5] and independently in [6] as noted by Bauer et al..

In categories of representations of (possibly infinite, but rooted) quivers over an abelian category, it was shown by Holm et al. and Odabaşı that certain complete cotorsion pairs can be described locally by complete cotorsion pairs of the underlying abelian category [16, 23]. These results are given here as the third and final of our main results in this thesis, in the special case of finite acyclic quivers.

Theorem 1.3 (Proposition 4.29 and Theorem 4.32). *Let $(\mathcal{C}, \mathcal{D})$ be a complete cotorsion pair in an abelian category $\mathcal{A}$ and Q a finite acyclic quiver.*

(i) If $\mathcal{A}$ has enough injectives, then $(\Gamma(\mathcal{C}), \mathrm{Rep}(Q, \mathcal{D}))$ is a complete cotorsion pair in $\mathrm{Rep}(Q, \mathcal{A})$, and

(ii) If $\mathcal{A}$ has enough projectives, then $(\mathrm{Rep}(Q, \mathcal{C}), \Lambda(\mathcal{D}))$ is a complete cotorsion pair in $\mathrm{Rep}(Q, \mathcal{A})$.

where

$$\Gamma(\mathcal{C}) = \left\{ F \in \mathrm{Rep}(Q, \mathcal{A}) \;\middle|\; \begin{array}{l} \textit{the canonical morphism } \coprod_{\alpha \in Q_1(*, x)} F(i(\alpha)) \xrightarrow{\gamma_x^F} F(x) \\ \textit{is mono and } \mathrm{Cok}(\gamma_x^F) \in \mathcal{C} \;\forall\; x \in Q_0 \end{array} \right\},$$

and

$$\Lambda(\mathcal{D}) = \left\{ F \in \mathrm{Rep}(Q, \mathcal{A}) \;\middle|\; \begin{array}{l} \textit{the canonical morphism } F(x) \xrightarrow{\lambda_x^F} \prod_{\alpha \in Q_1(x, *)} F(t(\alpha)) \\ \textit{is epi and } \mathrm{Ker}(\lambda_x^F) \in \mathcal{C} \;\forall\; x \in Q_0 \end{array} \right\}.$$

These induced cotorsion pairs of representations are then studied to see when they are a part of a cotorsion torsion triple and thus further induces a tilting theory of representations. The first cotorsion pair $(\Gamma(\mathcal{C}), \mathrm{Rep}(Q, \mathcal{D}))$ does in fact induce a cotorsion torsion triple if and only if the original cotorsion pair $(\mathcal{C}, \mathcal{D})$ is the cotorsion part of a cotorsion torsion triple in $\mathcal{A}$. The second cotorsion pair is seen to only induce a cotorsion torsion triple when the original cotorsion pair is the trivial pair $(\mathrm{Proj}\mathcal{A}, \mathcal{A})$.

The thesis assumes only knowledge up to and including that which one obtains through introductory courses of Homological Algebra and Representation Theory of Quivers. However, for the benefit of the reader, Section 2 and the start of Section 4, contain some of the key results which would have been learnt through such courses. In addition, Section 2 states a few results regarding Krull-Schmidt categories with reference to proofs, and sets the stage for the rest of the thesis with some preliminary definitions and results. In particular, the notions of approximations and orthogonality are introduced here.

In Section 3 we introduce the concepts of torsion, cotorsion and tilting, and prove our first two main results. This section mainly follows in the footsteps of [5].

Section 4 is devoted to introducing representations and developing our final main result. Towards the end of this section, we also study how this relates to tilting, and we end the section as well as the the whole thesis by applying the developed theory to grid representations that arises in Topological Data Analysis. The first part of this section follows closely the treatment given in [16, 23], and the last section follows the last part of [5].

Throughout we have tried to make the thesis as self-contained as possible. However, in Section 2 we have seen it more favourable to refer to other sources for most of the proofs which wouldn't have contributed significantly to the understanding of the main story. In the appendix we have gathered a couple miscellaneous proofs as well as a short introduction to the field of Topological Data Analysis in an effort to provide background on the application of the last section.

2 Additive Categories

In this section we will recall a few definitions and results regarding additive and abelian categories. We will also take a look at orthogonal subcategories, Krull-Schmidt categories and approximations. These concepts will help us build the theory surrounding tilting. Note that we do implicitly assume throughout the thesis that our categories are skeletally small, or in other words the isomorphism classes of objects form a set.

An ideal of an additive category $\mathcal{E}$ is a subfunctor $\mathcal{I}(-,-)$ of the additive hom bifunctor $\mathrm{Hom}_{\mathcal{E}}(-,-)$ going from $\mathcal{E}^{\mathrm{op}} \times \mathcal{E}$ into the category of abelian categories. To every ideal of $\mathcal{E}$ we have an *additive quotient* category $\mathcal{E}/\mathcal{I}$, which consists of the same objects as $\mathcal{E}$, but whose hom sets are quotient groups by the groups arising from $\mathcal{I}(-,-)$. That is, for objects $A, B \in \mathcal{E}$, the hom set $\mathrm{Hom}_{\mathcal{E}/\mathcal{I}}(A,B)$ is equal to the quotient $\mathrm{Hom}_{\mathcal{E}}(A,B)/\mathcal{I}(A,B)$. To any quotient category there is a canonical full and dense projection functor $\mathcal{E} \xrightarrow{\pi_{\mathcal{I}}} E/\mathcal{I}$, which is universal in the sense that any other additive functor $F\colon \mathcal{E} \to \mathcal{X}$ such that $F(\phi) = 0$ for every $\phi \in \mathcal{I}$, factors through it.

$$\begin{tikzcd} \mathcal{E} \arrow[r, "F"] \arrow[d, "\pi_{\mathcal{I}}"'] & \mathcal{X} \\ \mathcal{E}/\mathcal{I} \arrow[ur, dashed] & \end{tikzcd}$$

We will only be interested in the case where the ideal is generated from a full subcategory $\mathcal{X} \subseteq \mathcal{E}$. That is, the quotient of $\mathcal{E}$ by $\mathcal{X}$ is the quotient category $\mathcal{E}/[\mathcal{X}]$ where $[\mathcal{X}]$ is the ideal consisting of all morphisms factoring through an object in $\mathcal{X}$. We will usually drop the brackets from the notation whenever it do not lead to any confusion. For a more thorough treatment on ideals of additive categories, we refer to [1, Appendix A.3] or [29].

Two monomorphisms with the same codomain in an additive category are *equivalent* if there exists an isomorphism of the domains which is compatible in the natural way. That is, the monomorphisms $f\colon X' \hookrightarrow X$ and $g\colon X'' \hookrightarrow X$ are equivalent if and only if we have an isomorphism $X' \cong X''$ such that the diagram

$$\begin{tikzcd} X' \arrow[dr, hook, "f"] \arrow[dd, "\cong"'] & \\ & X \\ X'' \arrow[ur, hook, "g"'] & \end{tikzcd}$$

commute. The equivalence class of a monomorphism $f\colon X' \hookrightarrow X$ is called a *subobject* of X and is often referred to by a representative domain X' if it do not lead to ambiguity of the morphisms, we write $X' \subseteq X$. Two epimorphisms with the same domain are equivalent in the dual fashion, and the equivalence class of an epimorphism $X \to X'$ is called a *factor object* or *quotient object* of X, often only referred to by the codomain X'.

We define the *sum* of a collection subobjects $\{X_i\}$ of X to be the smallest subobject of X respective to the partial ordering given by $\subseteq$, which contains all the subobjects. We write $\sum_i X_i$ for the sum, and if the collection consists of only two objects X', X'', it is written as $X' + X''$. The *intersection* of a collection of subobjects $\{X_i\}$ of X is defined as the largest subobject contained by all the subobjects, and it is denoted by $\cap_i X_i$.

Any subcategory $\mathcal{X} \subseteq \mathcal{E}$ of an additive category induces two other subcategories related to the notion of subobjects and factor objects. Namely the subobject category $\operatorname{Sub}\mathcal{X}$ consisting of every object U admitting a monomorphism into a direct sum of objects in $\mathcal{X}$, and the factor category $\operatorname{Fac}\mathcal{X}$ consisting of objects F admitting an epimorphism from a direct sum of objects in X, i.e.

$$\operatorname{Sub}(\mathcal{X}) = \left\{ U \in \mathcal{E} \mid \exists\; U \hookrightarrow \bigoplus_{i=1}^{n} X_i,\ X_i \in \mathcal{X} \right\},$$

and

$$\operatorname{Fac}(\mathcal{X}) = \left\{ F \in \mathcal{E} \mid \exists\; \bigoplus_{i=1}^{n} X_i \twoheadrightarrow E,\ X_i \in \mathcal{X} \right\}.$$

Another subcategory related to $\mathcal{X}$ is the smallest additive subcategory of $\mathcal{E}$ containing $\mathcal{X}$, which we denote by $\operatorname{add}\mathcal{X}$. It is given by all direct summands of finite direct sums of objects in $\mathcal{X}$, that is,

$$\operatorname{add}\mathcal{X} = \left\{ E \in \mathcal{E} \mid E \oplus E' = \bigoplus_{i=1}^{n} X_i,\ X_i \in \mathcal{X} \right\}.$$

The notions of noetherian and artinian also generalizes to the categorical setting. An object $X \in \mathcal{E}$ is called *noetherian* if every ascending chain of subobjects $X_1 \subseteq X_2 \subseteq X_3 \subseteq \ldots \subseteq X$ stabilizes, i.e. there is some integer n such that $X_i = X_{i+1}$ for every $i \geq n$. Similarly, X is called *artinian* if every descending chain of subobjects $\ldots \subseteq X_3 \subseteq X_2 \subseteq X_1 \subseteq X$ stabilizes. Specifically, any subobject of a noetherian object X is contained in a maximal subobject $X' \subseteq X$. A category in which every object is noetherian, respectively artinian, is called *noetherian*, respectively *artinian*.

2.1 Abelian Categories

As the reader is assumed to have knowledge equivalent to that obtained through an introductory course for Homological Algebra, we assume that most of the results of abelian categories in the following section are known. They are, however, included as a benefit for the reader, as they will be used frequently throughout. The reader can look to the notes on abelian categories in [25] or the appendix of [1], for a more in-depth treatment.

Lemma 2.1 ([25, Thm. 14.2]). *Consider the following exact commutative diagram in an abelian category*

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}(f) \arrow[r] \arrow[d, "k"] & A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[r] \arrow[d, "h"] & \operatorname{Cok}(f) \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & \operatorname{Ker}(i) \arrow[r] & C \arrow[r, "i"] & D \arrow[r] & \operatorname{Cok}(i) \arrow[r] & 0
\end{tikzcd}$$

Then

- *the middle square is a pullback if and only if k is an isomorphism and c is a monomorphism, and*

- *the middle square is a pushout if and only if k is an epimorphism and c is an isomorphism.*

Lemma 2.2 ([25, Cor. 13.8])**.** *Monomorphisms and epimorphisms are stable under pullbacks and pushouts in an abelian category. Furher, the resulting square from the pullback of an epi is also a pushout square. Dually, the resulting square from the pushout of a mono is also a pullback square.*

Lemma 2.3 (Snake lemma [25, Thm. 14.3])**.** *Consider the following commutative diagram with exact rows and columns, in an abelian category*

$$\begin{tikzcd}
 & \mathrm{Ker}(f_1) \arrow[r, dashed] \arrow[d, hook] & \mathrm{Ker}(f_2) \arrow[r, dashed] \arrow[d, hook] & \mathrm{Ker}(f_3) \arrow[d, hook] \arrow[dddll, dashed, rounded corners] & \\
 & A \arrow[r, "a"] \arrow[d, "f_1"] & B \arrow[r, "b"] \arrow[d, "f_2"] & C \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & A' \arrow[r, "a'"] \arrow[d] & B' \arrow[r, "b'"] \arrow[d] & C' \arrow[d] & \\
 & \mathrm{Cok}(f_1) \arrow[r, dashed] & \mathrm{Cok}(f_2) \arrow[r, dashed] & \mathrm{Cok}(f_3) &
\end{tikzcd}$$

Then the dashed morphisms exists, making the whole diagram commutative, and the sequence

$$\mathrm{Ker}(f_1) \to \mathrm{Ker}(f_2) \to \mathrm{Ker}(f_3) \to \mathrm{Cok}(f_1) \to \mathrm{Cok}(f_2) \to \mathrm{Cok}(f_3)$$

is exact. Further, if a is a monomorphism and b' an epimorphism, then we have the exact sequence

$$0 \to \mathrm{Ker}(f_1) \to \mathrm{Ker}(f_2) \to \mathrm{Ker}(f_3) \to \mathrm{Cok}(f_1) \to \mathrm{Cok}(f_2) \to \mathrm{Cok}(f_3) \to 0$$

Lemma 2.4 (Horsheshoe Lemma, [25, Prop. 23.8])**.** *Let*

$$0 \to A \to B \to C \to 0$$

be a short exact sequence in an abelian category. If A admit a projective resolution $P_A^\bullet$ and C admit a projective resolution $P_C^\bullet$, then B admits a projective resolution $P_B^\bullet$ where $P_B^i = P_A^i \oplus P_C^i$, which is compatible with $P_A^\bullet$ and $P_B^\bullet$ in the natural way.

We can also note that in an abelian category we can describe sums and intersection of subobjects more explicit. It can be seen (Lemma A.1) that the intersection of two subobjects $B \hookrightarrow A$ and $C \hookrightarrow A$ of an object A coincide with the pullback

$$\begin{tikzcd} B \cap C \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, hook] \\ C \arrow[r, hook] & A \end{tikzcd},$$

and the sum of them coincide with the pushout

$$\begin{tikzcd} B \cap C \arrow[r, hook] \arrow[d, hook] & B \arrow[d, hook] \\ C \arrow[r, hook] \arrow[ur, phantom, "\ulcorner", very near end] & B + C \end{tikzcd}.$$

The sum can also be seen to be the image of the canonical map $B \oplus C \to A$.

2.1.1 Extensions

We would like to work with the derived functors $\mathrm{Ext}^n_{\mathcal{A}}(A, -)$ and $\mathrm{Ext}^n_{\mathcal{A}}(-, B)$ in cases where the abelian category do not necessarily have enough projectives. Hence, we would like to have an equivalent notion that do not require projective resolutions. This is found in *Yoneda extensions*. The reader is referred to [24] and [22, Chapter 7] for proofs of the following claims, and for further properties. In the special case of 1-extensions the reader can also see [25, chapter 27].

An *n-extension* of an object A to an object B in an abelian category $\mathcal{A}$ is an exact sequence with $n+2$ terms on the form

$$\mathbf{X}: \qquad 0 \longrightarrow B \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow A \longrightarrow 0 \ .$$

The class of all such extensions from A to B is denoted by $\mathcal{E}^n(A, B)$. Two n-extensions $\mathbf{X}$ and $\mathbf{Y}$ are called *similar*, which is denoted by $\mathbf{X} \to \mathbf{Y}$, if we can find morphisms making the following diagram commutative

$$\begin{tikzcd} \mathbf{X}: & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X_n \arrow[r] \arrow[d, dashed] & \dots \arrow[r] & X_1 \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d, equal] & 0 \\ \mathbf{Y}: & 0 \arrow[r] & B \arrow[r] & Y_n \arrow[r] & \dots \arrow[r] & Y_1 \arrow[r] & A \arrow[r] & 0 \end{tikzcd},$$

When $n = 1$, this reduces to demanding there exists some isomorphism making the following diagram commutative

$$\begin{tikzcd}
\mathbf{X}: & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, dashed, "\cong"] & A \arrow[r] \arrow[d, equal] & 0 \\
\mathbf{Y}: & 0 \arrow[r] & B \arrow[r] & Y \arrow[r] & A \arrow[r] & 0
\end{tikzcd}.$$

We say that two n-extensions $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{E}^n(A, B)$ are equivalent if there exists some n-extension $\mathbf{Z} \in \mathcal{E}^n(A, B)$ such that $\mathbf{X} \leftarrow \mathbf{Z} \rightarrow \mathbf{Y}$. This gives us an equivalence relation upon $\mathcal{E}^n(A, B)$. The collection of equivalence classes under this equivalence relation is denoted by $\mathrm{YExt}^n_{\mathcal{A}}(A, B)$ and is called an *Yoneda extension group*. Any morphism $f\colon B \to B'$ induces a well-defined morphism $f \cdot -\colon \mathrm{YExt}^n_{\mathcal{A}}(A, B) \to \mathrm{YExt}^n_{\mathcal{A}}(A, B')$, given on representatives $\mathbf{X} \in \mathcal{E}^n(A, B)$ by a pushout along $B \xrightarrow{f} B'$ and $B \to X_n$,

$$\begin{tikzcd}
\mathbf{X}: & 0 \arrow[r] & B \arrow[r] \arrow[d, "f"'] & X_n \arrow[r] \arrow[d] & X_{n-1} \arrow[r] \arrow[d, equal] & \dots \arrow[r] & A \arrow[r] \arrow[d, equal] & 0 \\
f \cdot \mathbf{X}: & 0 \arrow[r] & B' \arrow[r] & P \arrow[r] \arrow[ul, phantom, "\ulcorner"] & X_{n-1} \arrow[r] & \dots \arrow[r] & A \arrow[r] & 0
\end{tikzcd}.$$

Also any morphism $g\colon A' \to A$ induces a well-defined morphism $-\cdot g\colon \mathrm{YExt}^n_{\mathcal{A}}(A, B) \to \mathrm{YExt}^n_{\mathcal{A}}(A', B)$, given on representatives $\mathbf{X} \in \mathcal{E}^n(A, B)$ by taking the pullback along $A' \xrightarrow{g} A$ and $X_1 \to A$,

$$\begin{tikzcd}
\mathbf{X} \cdot g: & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & \dots \arrow[r] & X_2 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[r] \arrow[d, "g"] & 0 \\
\mathbf{X}: & 0 \arrow[r] & B \arrow[r] & \dots \arrow[r] & X_2 \arrow[r] & X_1 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}.$$

A Yoneda extension group is, as the name suggest, a group where addition is defined on elements $\mathbb{X}, \mathbb{Y} \in \mathrm{YExt}^n_{\mathcal{A}}(A, B)$ as the "Baer-sum" given by

$$\mathbb{X} + \mathbb{Y} = \begin{pmatrix} 1 & 1 \end{pmatrix} (X \oplus \mathbb{Y}) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The zero object in these groups are given as the split exact sequence

$$0 \longrightarrow B \longrightarrow B \oplus A \longrightarrow A \longrightarrow 0\ ,$$

in $\mathrm{YExt}^1_{\mathcal{A}}(A,B)$ and the trivial extension

$$0 \longrightarrow B \longrightarrow B \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow A \longrightarrow A \longrightarrow 0 \ ,$$

for $\mathrm{YExt}^n_{\mathcal{A}}(A,B)$.

The usefulness of these Yoneda extensions is that we have the group isomorphism

$$\mathrm{YExt}^n_{\mathcal{A}}(A,B) \cong \mathrm{Ext}^n_{\mathcal{A}}(A,B)$$

whenever the latter exists, which is natural in both variables. That is, we have an explicit description of the derived hom-functors. When working with Ext, this isomorphism will be thought of as an identification, hence there will be given no effort in distinguishing the first from the latter in the following.

2.2 Orthogonal Subcategories

Now, having a concept of extensions in the general abelian case, we can set forth defining a collection of subcategories which will be used extensively throughout the rest of the thesis, namely orthogonal complements.

Definition 2.5. For any subcategory $\mathcal{X}$ of $\mathcal{A}$, we have the full subcategories

$$\mathcal{X}^{\perp_i} \colon = \{A \in \mathcal{A} \mid \mathrm{Ext}^i_{\mathcal{A}}(X,A) = 0 \text{ for all } X \in \mathcal{X}\},$$

and

$$^{\perp_i}\mathcal{X} \colon = \{A \in \mathcal{A} \mid \mathrm{Ext}^i_{\mathcal{A}}(A,X) = 0 \text{ for all } X \in \mathcal{X}\}.$$

$\mathcal{X}^{\perp_i}$ is called the *right i-orthogonal complement* to $\mathcal{X}$ and $^{\perp_i}\mathcal{X}$ the *left i-orthogonal complement*. When $i = 0$, the i is usually dropped from the notation.

Remark 2.6. The action of taking the right or left i-orthogonal complement is inclusion reversing. That is, if we have two subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{A}$, such that $\mathcal{X} \subseteq \mathcal{Y}$ then $^{\perp_i}\mathcal{X} \supseteq {^{\perp_i}\mathcal{Y}}$, and $\mathcal{X}^{\perp_i} \supseteq \mathcal{Y}^{\perp_i}$. We also note that $\mathcal{X} \subseteq {^{\perp_1}(\mathcal{X}^{\perp_1})}$ and $\mathcal{X} \subseteq ({^{\perp_1}\mathcal{X}})^{\perp_1}$.

Further, we also observe that by that inclusion and the inclusion reversing property we have $\mathcal{X}^{\perp_1} \subseteq [^{\perp_1}(\mathcal{X}^{\perp_1})]^{\perp_1} \subseteq \mathcal{X}^{\perp_1}$, and then necessarily $[^{\perp_1}(\mathcal{X}^{\perp_1})]^{\perp_1} = \mathcal{X}^{\perp_1}$. Similarly, $^{\perp_1}[(^{\perp_1}\mathcal{X})^{\perp_1}] = {}^{\perp_1}\mathcal{X}$.

We will derive a few immediate properties of the orthogonal complements. The proofs of these will be using the inherent duality of the left and right complements to avoid needlessly repetitious arguments. Explicitly, this means that only the properties attributed to one of the complements will be proven where a mutatis mutandis argument is needed for the other. Alternatively, all proofs associated to one of the complements can at once be used for the other, by first passing to the opposite category.

Lemma 2.7. *Let $\mathcal{X} \subseteq \mathcal{A}$ be a subcategory of an abelian category $\mathcal{A}$. Both the right and left i-orthogonal complement of $\mathcal{X}$ are closed under extensions. The right i-orthogonal complement $\mathcal{X}^{\perp_i}$ is further closed under products and the left i-orthogonal complement $^{\perp_i}\mathcal{X}$ is closed under coproducts.*

Proof. Let
$$0 \to A \to B \to C \to 0$$
be a short exact sequence such that $A, C \in \mathcal{X}^{\perp_i}$. Then by applying the hom-functor $\mathrm{Hom}_{\mathcal{A}}(X, -)$ for any $X \in \mathcal{X}$, we extract from the consequent long exact sequence, the exact sequence

$$\mathrm{Ext}^i_{\mathcal{A}}(X, A) \to \mathrm{Ext}^i_{\mathcal{A}}(X, B) \to \mathrm{Ext}^i_{\mathcal{A}}(X, C),$$

whose first and last term vanishes, forcing $\mathrm{Ext}^i_{\mathcal{A}}(X, B) = 0$, and therefore $B \in \mathcal{X}^{\perp_i}$, proving that the subcategory is closed under extensions.

Recall that for any family of objects $\{A_\alpha\}_{\alpha\in\Lambda}$ which admits a product in $\mathcal{A}$, we have
$$\mathrm{Ext}^i_{\mathcal{A}}(-, \prod_{\alpha\in\Lambda} A_\alpha) = \prod_{\alpha\in\Lambda} \mathrm{Ext}^i_{\mathcal{A}}(-, A_\alpha).$$

Hence, assuming that $\{A_\alpha\}_{\alpha\in\Lambda}$ consists of objects in $\mathcal{X}^{\perp_i}$, we have that

$$\operatorname{Ext}^i_{\mathcal{A}}(-,\prod_{\alpha\in\Lambda} A_\alpha)|_{\mathcal{X}} = \prod_{\alpha\in\Lambda}\operatorname{Ext}^i_{\mathcal{A}}(-,A_\alpha)|_{\mathcal{X}} = \prod_{\alpha\in\Lambda} 0 = 0$$

so $\prod_{\alpha\in\Lambda} A_\alpha \in \mathcal{X}^{\perp_i}$. □

Lemma 2.8. *Let $\mathcal{X} \subseteq \mathcal{A}$ be a subcategory of an abelian category $\mathcal{A}$. Then ${}^\perp\mathcal{X}$ is closed under factors while $\mathcal{X}^\perp$ is closed under subobjects.*

Proof. Let $Y \twoheadrightarrow F$ be any factor-object of an object $Y \in^\perp \mathcal{X}$. Then we have a short exact sequence

$$0 \to K \to Y \to F \to 0$$

which induces a long exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}(F,X) \to \operatorname{Hom}_{\mathcal{A}}(Y,X) \to \operatorname{Hom}_{\mathcal{A}}(K,X) \to \operatorname{Ext}^1_{\mathcal{A}}(F,X) \to \cdots$$

for any $X \in^\perp \mathcal{X}$. The third term vanishes by assumption, hence $\operatorname{Hom}_{\mathcal{A}}(F,X)$ vanishes as well. We conclude that $F \in^\perp \mathcal{X}$. □

Lemma 2.9. *Let $\mathcal{X} \subseteq \mathcal{A}$ be a subcategory of an abelian category $\mathcal{A}$. Then both ${}^{\perp_1}\mathcal{X}$ and $\mathcal{X}^{\perp_1}$ are closed under direct summands. Further, ${}^{\perp_1}\mathcal{X}$ contains all the projective objects of $\mathcal{A}$ and $\mathcal{X}^{\perp_1}$ contains all injective objects.*

Proof. If $P \in \mathcal{A}$ is projective in $\mathcal{A}$, then the hom-functor $\operatorname{Hom}_{\mathcal{A}}(P,-)$ is exact and necessarily $\operatorname{Ext}^1_{\mathcal{A}}(P,-) = 0$. Thus, $\operatorname{Proj}(\mathcal{A}) \subseteq {}^{\perp_1}\mathcal{X}$.

Now, to see that ${}^{\perp_1}\mathcal{X}$ is closed under direct summands, let $Y \oplus Y'$ be an object in ${}^{\perp_1}\mathcal{X}$, and let

$$0 \to X \to E \to Y \to 0$$

be any 1-extension of Y to an object of $\mathcal{X}$. We construct the following commutative pull-back diagram

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
& & Y' \arrow[d] \arrow[r, equal] & Y' \arrow[d] & \\
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & P \arrow[r, "e'"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \oplus Y' \arrow[r] \arrow[d, "\pi_Y"'] \arrow[l, dashed, bend right, "r"'] & 0 \\
0 \arrow[r] & X \arrow[r] & E \arrow[r, "e"'] \arrow[d] & Y \arrow[r] \arrow[d] \arrow[u, dashed, bend right, "\iota_Y"'] & 0 \\
& & 0 & 0 &
\end{tikzcd},$$

where we the middle row splits, since $\mathrm{Ext}^1_{\mathcal{A}}({}^{\perp_1}\mathcal{X}, \mathcal{X}) = 0$. Observe that

$$e(gr\iota_y) = \pi_Y e' r \iota_Y = \pi_Y \mathrm{id}_{Y\oplus Y'} \iota_Y = \pi_Y \iota_Y = \mathrm{id}_Y,$$

that is, the lower row splits, or equivalently $\mathrm{Ext}^1_{\mathcal{A}}(Y, X) = 0$. $\square$

Definition 2.10. A subcategory $\mathcal{X} \subseteq \mathcal{A}$ of an abelian category $\mathcal{A}$ is called *self-orthogonal* if it satisfies any of the following equivalent conditions

(i) $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{X}, \mathcal{X}) = 0$,

(ii) $\mathcal{X} \subseteq {}^{\perp_1}\mathcal{X}$,

(iii) $\mathcal{X} \subseteq \mathcal{X}^{\perp_1}$,

(iv) $\mathcal{X} \cap \mathcal{X}^{\perp_1} = \mathcal{X} = {}^{\perp_1}\mathcal{X} \cap \mathcal{X}$.

Lemma 2.11. *Let $\mathcal{X} \subseteq \mathcal{A}$ be a self-orthogonal subcategory of an abelian category. Then the additive closure* $\mathrm{add}\,\mathcal{X}$ *is also self-orthogonal.*

Proof. Let A, A' be any two objects in $\mathrm{add}\,(\mathcal{X})$. Then we can find objects $B, B' \in \mathcal{A}$ such that $A \oplus B = \bigoplus_{i=1}^n X_i$ and $A' \oplus B' = \bigoplus_{j=1}^m$ for $X_i, X_j \in \mathcal{X}$. Hence,

$$\mathrm{Ext}^1_{\mathcal{A}}(A \oplus B, A' \oplus B') = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \mathrm{Ext}^1_{\mathcal{A}}(X_i, X_j) = 0,$$

which we claim gives that $\mathrm{Ext}^1_{\mathcal{A}}(A, A') = 0$ as well. This is shown through a similar argument as that given for complements being closed under summands in Lemma 2.9. That is, for any short exact sequnece $A' \hookrightarrow X \twoheadrightarrow A$ we construct the following commutative diagram with exact rows by taking the pushout along the inclusion $A' \hookrightarrow A' \oplus B$, and then the pullback along the projection $A \oplus B \twoheadrightarrow A$.

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A' \oplus B' \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & A \arrow[r] & 0 \\
0 \arrow[r] & A' \oplus B' \arrow[r] & \overline{P} \arrow[r] \arrow[u] \arrow[ur, phantom, "\urcorner", very near start] & A \oplus B \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The bottom row splits as $\mathrm{Ext}^1_{\mathcal{A}}(A \oplus B, A' \oplus B') = 0$, which results in the middle row splitting, and then consequently that the upper row splits.

□

2.3 Krull-Schmidt Categories

We say that an object $X \in \mathcal{E}$ is *indecomposable* whenever $X \cong X' \oplus X''$ implies either $X' = 0$ or $X'' = 0$. An additive category in which every object admits a finite decomposition into indecomposable objects with local endomorphism rings is called a *Krull-Schmidt* category. As noted in [18], this naming originates from the Krull-Schmidt Theorem, which states that such a decomposition is guaranteed for every finite length module. An in-depth and self-containing exposition on the subject of Krull-Schmidt categories have been written by Krause in [18]. Most of the following results concerning Krull-Schmidt categories are found in said article, and the reader is referred there for the proofs which we omit.

Remark 2.12. A Krull-Schmidt category is both noetherian and artinian.

Definition 2.13. An epimorphism $\phi\colon X \to Y$ is *essential* if any morphism $\psi\colon Z \to X$ is epimorphic if and only if the composition $\phi \circ \psi\colon Z \to Y$ is epimorphic.

An epimorphism from an projective object $\phi\colon P \to X$ is a *projective cover* if it is essential.

Definition 2.14. An endomorphism $f\colon X \to X$ is said to be *split idempotent* if there exist a factorization $X \xrightarrow{r} Y \xrightarrow{s} X$ of f such that $r \circ s = \mathrm{id}_Y$. It is easily verified that a split idempotent is in fact an idempotent. We say that an additive category $\mathcal{E}$ has split idempotents if all idempotents in $\mathcal{E}$ splits.

Lemma 2.15. *If $f\colon X \to Y \to X$ is a split idempotent such that the idempotent $\mathrm{id}_X - f\colon X \to X$ also splits, then Y is a direct summand of X.*

Proof. Let
$$f\colon X \xrightarrow{r} Y \xrightarrow{s} X$$
and
$$\mathrm{id}_X - f\colon X \xrightarrow{r'} Y' \xrightarrow{s'} X$$
be split idempotents in $\mathcal{E}$. Now, consider the maps
$$\left(\begin{smallmatrix} r \\ r' \end{smallmatrix}\right)\colon X \to Y \oplus Y' \text{ and } \left(\begin{smallmatrix} s & s' \end{smallmatrix}\right)\colon Y \oplus Y' \to X,$$
which, by construction, gives
$$\left(\begin{smallmatrix} r \\ r' \end{smallmatrix}\right)\left(\begin{smallmatrix} s & s' \end{smallmatrix}\right) = \begin{pmatrix} \mathrm{id}_Y & 0 \\ 0 & \mathrm{id}_{Y'} \end{pmatrix} = \mathrm{id}_{Y \oplus Y'}$$
and
$$\left(\begin{smallmatrix} s & s' \end{smallmatrix}\right)\left(\begin{smallmatrix} r \\ r' \end{smallmatrix}\right) = s \circ r + s' \circ r' = f + (\mathrm{id}_X - f) = \mathrm{id}_X$$
That is $X \cong Y \oplus Y'$. □

Proposition 2.16 ([18, Prop. 4.1])**.** *Let R be a ring. The category of R-modules is Krull-Schmidt if and only if every finitely generated R-module admits a projective cover. If R satisfies these equivalent assertions it is called semi-perfect.*

Theorem 2.17 ([18, Thm. 4.2])**.** *Let X be an object in a Krull-Schmidt category and suppose there are two decompositions*
$$X_1 \oplus \cdots \oplus X_r \cong X \cong Y_1 \oplus \cdots \oplus Y_s$$

into objects with local endomorphism rings. Then $r = s$ and there exists some permutation π such that $X_i \cong Y_{\pi(i)}$ for $1 \leq i \leq r$.

Corollary 2.18 ([18, Cor. 4.4])**.** *An additive category is a Krull-Schmidt category if and only if it has split idempotents and the endomorphism ring of any object is semi-perfect.*

Remark 2.19. In an abelian category the condition of split idempotents is trivially satisfied, since all morphisms factor through their image. Hence, the corollary above says that an abelian category is Krull-Schmidt if and only if the endomorphism ring of any object is semi-perfect.

Remark 2.20. It can be seen that an additive quotient of a Krull-Schmidt category is itself a Krull-Schmidt category. We omit the arguments here, since it requires us to define the categorical radical.

2.4 Approximations

The notion of approximations were introduced by Auslander and Smalø while studying subcategories of mod(A) for an artin ring A [3]. In certain literature one may stumble upon the concept of pre-envelopes/covers for the same morphisms which we call right-/left-approximations [14]. It has traditionally been normal to use this naming convention when working in the category of all modules over a ring R, and the convention of Auslander when working over finitely generated modules.

In module categories we can enforce a minimality condition on every morphism, which in turn gives us minimal approximations or envelopes/covers. These minimality conditions gives in particular Wakamatsu's Lemma which relates approximations to orthogonality. We will see that Krull-Schmidt categories have minimal morphisms, and, by Wakamatsu's lemma, we can therefore later characterize cotorsion pairs in Krull-Schmidt categories through the existence of approximations.

Definition 2.21. Let $\mathcal{X}$ be a full subcategory of an additive category $\mathcal{E}$. A morphism out of $\mathcal{X}$ to an object E of $\mathcal{E}$ is called a *right $\mathcal{X}$-approximation* of E if any other morphism out of $\mathcal{X}$ to E factors through it. That is $\phi\colon X \to E$, with $X \in \mathcal{X}$, is a *right $\mathcal{X}$-approximation of E* if any morphism $X' \to E$, with $X' \in \mathcal{X}$ factors through ϕ

$$\begin{tikzcd} & X' \arrow[dl, dashed, "\exists"'] \arrow[d] \\ X \arrow[r, "\phi"] & E \end{tikzcd}.$$

Equvialently $\phi\colon X \to E$ is a right $\mathcal{X}$-approximation of E if it induces an epimorphism of functors

$$\mathrm{Hom}(-, X)|_{\mathcal{X}} \xrightarrow{\phi_*} \mathrm{Hom}(-, E)|_{\mathcal{X}}.$$

If every $E \in \mathcal{E}$ admits a right $\mathcal{X}$-approximation, then $\mathcal{X} \subseteq \mathcal{E}$ is said to be *contravariantly finite.*

Dually a *left $\mathcal{X}$-approximation of E* is a morphism $\psi\colon E \to X$ which induces an epimorphism of functors

$$\mathrm{Hom}(X, -)|_{\mathcal{X}} \xrightarrow{\psi_*} \mathrm{Hom}(E, -)|_{\mathcal{X}},$$

and if every E admits such a left $\mathcal{X}$-approximation, $\mathcal{X} \subseteq \mathcal{E}$ is *covariantly finite.*

Any subcategory $\mathcal{X} \subseteq \mathcal{E}$ which is both contra- and covariantly finite is called *functorially finite.*

Remark 2.22. The naming originates in the study of modules of an additive category $\mathcal{E}$ [See e.g. 2, 15], which are contravariantly additive functors from $\mathcal{E}$ into the category of abelian groups. In the category of these modules, one has that the representable modules, $M \cong \mathrm{Hom}_{\mathcal{E}}(-, X)$, are projective. Every module admitting an epimorphism from a projective module are called finitely generated. Thus, the existence of a right $\mathcal{X}$-approximation of E from an additively closed subcategory $\mathcal{X}$ of $\mathcal{E}$ implies that $\mathrm{Hom}_{\mathcal{E}}(-, E)|_{\mathcal{X}}$ is finitely generated as an $\mathcal{X}$-module.

Example 2.23. Let $\mathcal{E}$ be an exact category with enough projectives and let E be any object of $\mathcal{E}$. Since we have enough projectives we have the following short exact sequence

$$0 \to \Omega E \to P \xrightarrow{\pi} E \to 0,$$

which gives the short exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{E}}(-, \Omega E)|_{\mathrm{Proj}\mathcal{E}} \to \mathrm{Hom}_{\mathcal{A}}(-, P)|_{\mathrm{Proj}\mathcal{E}} \xrightarrow{i^*} \mathrm{Hom}_{\mathcal{E}}(-, E)|_{\mathrm{Proj}\mathcal{E}} \to 0.$$

Thus the full subcategory of projective objects $\mathrm{Proj}\mathcal{E} \subseteq \mathcal{E}$, is contravariantly finite.

Dually, if $\mathcal{E}$ has enough injectives, then the full subcategory $\mathrm{Inj}\mathcal{E} \subseteq \mathcal{E}$ of injective objects is covariantly finite.

If $\mathcal{E}$ is Frobenius exact, we have that $\mathrm{Proj}\mathcal{E} = \mathrm{Inj}\mathcal{E}$ is functorially finite. ♣

2.4.1 Example: The Syzygy-Subcategories

A slightly less trivial example of contravariantly finite subcategories is the Syzygy-subcategories which was used in [4]. Let $\mathcal{A}$ be an abelian category with enough projectives.

Definition 2.24. Let $\Omega^n(\mathcal{A})$ denote the additive closure of the collection of all n-syzygies in $\mathcal{A}$, that is

$$\mathrm{add}\left\{ K \in \mathcal{A} \;\middle|\; \begin{array}{l} \exists \text{ an exact sequence} \\ 0 \to K \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0,\ P_i \in \mathrm{Proj}(\mathcal{A}) \end{array} \right\}.$$

We are going to show that $\Omega^n(\mathcal{A})$ is contravariantly finite in $\mathcal{A}$.

Definition 2.25. Let $C^b(\mathrm{Proj}(\mathcal{A}))$, respectively $C^b(\mathrm{Inj}(\mathcal{A}))$, be the full subcategories of the bounded chain complex category $C^b(\mathcal{A})$, consisting of bounded chain complexes of projectives, respectively injectives.

Proposition 2.26. *Let $A_\bullet \in C^b(\mathcal{A})$ be a bounded $\mathcal{A}$ chain complex of length $n < \infty$. Then we can find an epimorphic right $C^b(\mathrm{Proj}(\mathcal{A}))$-approximation $P_\bullet \twoheadrightarrow A_\bullet$, such that $P_\bullet$ has length n.*

Proof. The proof is constructive. We start by finding a right epimorphic $\mathrm{Proj}(\mathcal{A})$-

approximation $\rho_1 \colon P_1 \twoheadrightarrow A_1$. Then by taking the pullback of

$$\begin{tikzcd} & P_1 \arrow[d, two heads, "\rho_1"] \\ A_2 \arrow[r, "f_1"] & A_1 \end{tikzcd}$$

we get the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(f_1) \arrow[r] \arrow[d, equal] & X_2 \arrow[r] \arrow[d, two heads] & P_1 \arrow[r] \arrow[d, two heads, "\rho_1"] & C_1 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{Ker}(f_1) \arrow[r] & A_2 \arrow[r, "f_1"] & A_1 \arrow[r] & C_1 \arrow[r] & 0 \end{tikzcd}$$

where the middle square is both a pushout and a pullback square, since ρ_1 is an epimorphism. We now find an epimorphic right $\mathrm{Proj}(\mathcal{A})$-approximation $P_2 \twoheadrightarrow X_2$. Set $\rho_2 \colon P_2 \twoheadrightarrow A_2$ to be the composition $P_2 \twoheadrightarrow X_2 \twoheadrightarrow A_2$, and $g_1 \colon P_2 \to P_1$ as the composition $P_2 \twoheadrightarrow X_2 \to P_1$. Observe that since the right-hand square of the following commutative diagram is a pushout square, we have that the outer square is also a pushout square,

$$\begin{tikzcd} P_2 \arrow[r, two heads] \arrow[d, two heads] & X_2 \arrow[r] \arrow[d, two heads] & P_1 \arrow[d, two heads, "\rho_1"] \\ A_2 \arrow[r, equal] & A_2 \arrow[r, "f_1"] & A_1 \end{tikzcd} .$$

Hence, the kernel morphism $\mathrm{Ker}(g_1) \to \mathrm{Ker}(f_1)$ is necessarily an epimorphism.

Next, we take the pullback of

$$\begin{tikzcd} & \mathrm{Ker}(f_2) \arrow[d, two heads] \\ A_2 \arrow[r, "f_1"] & \mathrm{Ker}(f_1) \end{tikzcd}$$

to obtain the commutative diagram with exact rows,

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(f_2) \arrow[r] \arrow[d, equal] & X_3 \arrow[r] \arrow[d, two heads] & \mathrm{Ker}(f_2) \arrow[r] \arrow[d, two heads] & C_2 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{Ker}(f_2) \arrow[r] & A_3 \arrow[r, "f_2"] & \mathrm{Ker}(f_1) \arrow[r] & C_2 \arrow[r] & 0 \end{tikzcd} ,$$

where the middle square is a push-out and pullback square by the same reason as above. Let now $P_3 \twoheadrightarrow X_3$ be a right epimorphic $\mathrm{Proj}(\mathcal{A})$-approximation, and set $\rho_3\colon P_3 \twoheadrightarrow A_3$ as the composition $P_3 \twoheadrightarrow X_3 \twoheadrightarrow A_3$ and $g_2\colon P_2 \to P_1$ as the composition $P_3 \twoheadrightarrow X_3 \to \mathrm{Ker}(g_1) \hookrightarrow P_2$. By the same reason as above, the kernel morphism $\mathrm{Ker}(g_2) \to \mathrm{Ker}(f_2)$ is an epimorphism. We iterate this construction up until A_n.

We are left with showing that $\rho\colon P_\bullet \twoheadrightarrow A_\bullet$ is a right approximation. Thus, let $\epsilon\colon Q_\bullet \to A_\bullet$ be any other morphism from $C^b(\mathrm{Proj}(\mathcal{A}))$. $\rho_1\colon P_1 \twoheadrightarrow A_1$ is an approximation, so ϵ_1 factors through ρ_1. Observe that we get the following commutative diagram

$$\begin{tikzcd}
Q_2 \arrow[r] \arrow[d, dashed] \arrow[dd, bend right, "\epsilon_2"'] & Q_1 \arrow[d] \arrow[dd, bend left, "\epsilon_1"] \\
X_2 \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & P_1 \arrow[d, two heads, "\rho_1"'] \\
A_2 \arrow[r, "f_1"] & A_1
\end{tikzcd}$$

where the dashed morphism arises from the pullback-property. Now, using that $P_2 \twoheadrightarrow X_2$ was an approximation, we get that ϵ_2 factors through ρ_2. Assume we have shown that ϵ_i factors through ρ_i for all $i \leq k$. If we let $\mathrm{Ker}(Q_k \to Q_{k-1}) = K$, we have a commutative diagram

$$\begin{tikzcd}
Q_{k+1} \arrow[r] \arrow[d, dotted] \arrow[dd, bend right, "\epsilon_{k+1}"'] & K \arrow[r, hook] \arrow[d, dashed] & Q_k \arrow[r] \arrow[d] & Q_{k-1} \arrow[d] \\
X_{k+1} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Ker}(g_{k-1}) \arrow[r, hook] \arrow[d, two heads, "\kappa_k"] & P_k \arrow[r, "g_{k-1}"] \arrow[d, two heads, "\rho_k"] & P_{k-1} \arrow[d, two heads, "\rho_{k-1}"] \\
A_{k+1} \arrow[r] & \mathrm{Ker}(f_{k-1}) \arrow[r, hook] & A_k \arrow[r, "f_{k-1}"] & A_{k-1}
\end{tikzcd}$$

where the dashed morphism arises from the kernel property, and the dotted morphism exists by the pullback property. Once again, by the approximation property of $P_{k+1} \twoheadrightarrow X'$ we get that $\epsilon_{k+1}\colon Q_{k+1} \to A_{k+1}$ factors through ρ_k. Thus, we conclude that $\rho\colon P_\bullet \to A_\bullet$ is a right $\mathrm{Proj}(\mathcal{A})$-approximation. $\square$

Corollary 2.27. *The full subcategory $C_n^b(\mathrm{Proj}(\mathcal{A})) \subseteq C_n^b(\mathcal{A})$ of bounded complexes of length n is contravariantly finite.*

Corollary 2.28. *Let A be a bounded chain complex of length n in $C^b(\mathcal{A})$. The epimorphic right $C^b(\mathrm{Proj}(\mathcal{A}))$-approximation $\rho\colon P_\bullet \twoheadrightarrow A_\bullet$ constructed in Proposition 2.26 induces isomorphisms $H_i(P_\bullet) \cong H_i(A_\bullet)$ for $i < n$ and an epimorphism $H_n(P_\bullet) \to H_n(A_\bullet)$.*

Proof. Since the approximation $P_i \twoheadrightarrow X_i$ is an epimorphism, we have the exact diagrams from the construction

$$\begin{tikzcd}
P_{i+1} \arrow[r] \arrow[d, two heads, "\rho_{i+1}"] & \mathrm{Ker}(g_{i-1}) \arrow[r] \arrow[d, two heads, "\kappa_i"] & C_i \arrow[r] \arrow[d, equal] & 0 \\
A_{i+1} \arrow[r] & \mathrm{Ker}(f_{i-1}) \arrow[r] & C_i \arrow[r] & 0
\end{tikzcd}$$

for all $i < n$. Thus, for $i \leq n$, we have $H_i(P_\bullet) \cong C_i \cong H_i(A_\bullet)$. When $i = n$ we have $H_n(P_\bullet) \cong \mathrm{Ker}(g_{n-1}) \twoheadrightarrow \mathrm{Ker}(f_{n-1}) \cong H_n(A_\bullet)$. □

Lemma 2.29. *For every object $A \in \mathcal{A}$, there exists a right $\Omega^n(\mathcal{A})$-approximation of A.*

Proof. Let $I_\bullet \in C^-(\mathrm{Inj}(\mathcal{A}))$ be an injective resolution of A. By truncation we find the bounded complex $\sigma^{\leq n}(I_\bullet)$, and by Proposition 2.26 we find a right $\mathrm{Proj}(\mathcal{A})$-approximation $\rho\colon P_\bullet \to \sigma^{\leq n}(I_\bullet)$. We therefore obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, two heads] & P_n \arrow[r] \arrow[d, two heads, "\rho_n"] & P_{n-1} \arrow[r] \arrow[d, two heads, "\rho_{n-1}"] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, two heads, "\rho_1"] & \mho^n(A) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & I_1 \arrow[r] & I_2 \arrow[r] & \cdots \arrow[r] & I_n \arrow[r] & \mho^n(A) \arrow[r] & 0
\end{tikzcd},$$

which is exact by Corollary 2.28. Therefore $K = \Omega^n(\mho^n(A))$ and is thus contained in $\Omega^n(\mathcal{A})$. We claim that the morphism $K \twoheadrightarrow A$ is a right $\Omega^n(A)$-approximation of A. Suppose we have a morphism $X \to A$ with $X \in \Omega^n(\mathcal{A})$, then we have an exact sequence

$$0 \to X \to Q_n \to Q_{n-1} \to \cdots \to Q_1 \to Y$$

with $Q_i \in \mathrm{Proj}\mathcal{A}$. By letting $Q_\bullet$ be the bounded complex of length n

$$Q_n \to Q_{n-1} \to \cdots \to Q_1,$$

we first obtain a morphism $Q_\bullet \to I_\bullet$ since $I_\bullet$ consists of injective elements, thus also a morphism $Q_\bullet \to \sigma^{\leq n}(I_\bullet)$. Since $\rho\colon P_\bullet \to \sigma^{\leq n}(I_\bullet)$ is a right $\mathrm{C}^{\mathrm{b}}_{\mathrm{n}}(\mathrm{Proj}\mathcal{A})$-approximation, we obtain a factorization through ρ. Thus by kernel-property, we also have a factorization of $X \to A$ through $K \to A$. □

♣

2.4.2 Minimal Approximations

As already mentioned we are interested in minimal morphisms as they relate approximations to orthogonality. We can also observe that orthogonality in an abelian category relates naturally to approximations. Hence, in categories where all morphisms decomposes into a minimal and a zero part, we obtain a correspondence between orthogonality and approximation. The last relation is given in the following lemma, before we move on to explore the other direction through minimality and Wakamatsu's lemma.

Lemma 2.30. *Let $f\colon X \to E$ be any epimorphism in an abelian category $\mathcal{A}$ with $X \in \mathcal{X}$ a full subcategory of $\mathcal{A}$. If the kernel $K = \mathrm{Ker}(f)$ of the morphism has the property that $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{X}, K) = 0$, then f is a right $\mathcal{X}$-approximation of E.*

Proof. Let $f'\colon X' \to E$ be any morphism such that $X' \in \mathcal{X}$. Then, by taking the pullback of this map along $f\colon X \to E$ we get the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[r] \arrow[d, "f'"] \arrow[l, bend right, dashed] & 0 \\ 0 \arrow[r] & K \arrow[r] & X \arrow[r, "f"] & E \arrow[r] & 0 \end{tikzcd}$$

and by hypothesis the upper row splits, thus giving the wanted factorization. □

Definition 2.31. A morphism $f\colon A \to B$ is *right minimal* if every endomorphism $g\colon A \to A$ such that $f = f \circ g$ is an isomorphism.

Conversely, $f\colon A \to B$ is *left minimal* if every endomorphism $h\colon B \to B$ such that $f = h \circ f$ is an isomorphism.

Whenever we have a right, respectively left, $\mathcal{X}$-approximation of a subcategory $\mathcal{X}$, which is also right, respectively left, minimal, we will be calling it a right, respectively left, minimal $\mathcal{X}$-approximation.

Lemma 2.32. *If $\phi_1\colon X_1 \to E$ and $\phi_2\colon X_2 \to E$ are two right minimal $\mathcal{X}$-approximations of E, then $X_1 \cong X_2$.*

Proof. The approximation propery of ϕ_1 and ϕ_2 gives us the following commutative diagrams

$$\begin{tikzcd} & X_1 \arrow[dl, dashed, "f"'] \arrow[d, "\phi_1"] \\ X_2 \arrow[r, "\phi_2"] & E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & X_2 \arrow[dl, dashed, "g"'] \arrow[d, "\phi_2"] \\ X_1 \arrow[r, "\phi_1"] & E \end{tikzcd}.$$

These diagrams gives us that

$$\phi_1 = \phi_2 \circ f = \phi_1 \circ (g \circ f) \text{ and } \phi_2 = \phi_1 \circ g = \phi_2 \circ (f \circ g)$$

and by minimality both $(g \circ f)$ and $(f \circ g)$ are automorphisms, thus giving that f and g are isomorphisms. $\square$

Through Corollary 1.4 of [19] and its dual we observe that any morphism in a Krull-Schmidt category admit a decomposition into a minimal part and a zero part.

Lemma 2.33 (Dual of Corollary 1.4 in [19])**.** *Let $\phi\colon X \to Y$ be a morphism in an abelian Krull-Schmidt category $\mathcal{A}$. Then there exist a decomposition, $X \cong X' \oplus X''$,*

$$\phi = \begin{pmatrix} \phi' & \phi'' \end{pmatrix} : X' \oplus X'' \to Y,$$

such that ϕ' is right minimal and $\phi'' = 0$. $\square$

Lemma 2.34 ([19, Corollary 1.4])**.** *Let $\psi\colon X \to Y$ be a morphism in an abelian Krull-Schmidt category $\mathcal{A}$. Then there exist a decomposition, $Y \cong Y' \oplus Y''$,*

$$\psi = \begin{pmatrix} \psi' \\ \psi'' \end{pmatrix} : X \to Y' \oplus Y'',$$

such that ψ' is left minimal and $\psi'' = 0$.

Lemma 2.35 (Wakamatsu's Lemma)**.** *Let $\mathcal{A}$ be an abelian category, and $\mathcal{X} \subseteq \mathcal{A}$ a full subcategory closed under extensions. Then*

(i) Let $\phi\colon X \to A$ be a right minimal $\mathcal{X}$-approximation of an object A, then

$$\operatorname{Ext}^1_{\mathcal{A}}(-, \operatorname{Ker}(\phi))|_{\mathcal{X}} = 0.$$

(ii) Let $\psi\colon A \to X$ be a left minimal $\mathcal{X}$-approximation of an object A, then

$$\operatorname{Ext}^1_{\mathcal{A}}(\operatorname{Cok}(\psi), -)|_{\mathcal{X}} = 0.$$

Proof. (i):
First observe that ϕ gives us a short exact sequence

$$0 \to \operatorname{Ker}(\phi) \xrightarrow{k} X \xrightarrow{c} \operatorname{Im}(\phi) \to 0$$

where $c\colon X \twoheadrightarrow \operatorname{Im}(\phi)$ is a right minimal $\mathcal{X}$-approximation of $\operatorname{Im}(\phi)$.

Consider now any short exact sequence

$$0 \to \operatorname{Ker}(\phi) \xrightarrow{m} E \xrightarrow{e} X' \to 0$$

with $X' \in \mathcal{X}$. This give rise to the following commutative diagram with exact rows and columns, where the upper left square is a push-out square,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Ker}(\phi) \arrow[r, "m"] \arrow[d, "k"] \arrow[dr, phantom, "\ulcorner", very near end] & E \arrow[r, "e"] \arrow[d, "k'"] & X' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "m'"] \arrow[d, "c"] & Y \arrow[r, "e'"] \arrow[d, "c'"] & X' \arrow[r] & 0 \\
 & \mathrm{Im}(\phi) \arrow[r, equal] \arrow[d] & \mathrm{Im}(\phi) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

$\mathcal{X}$ is closed under extensions, so we get that $Y \in \mathcal{X}$, and, necessarily, $c' \colon Y \to \mathrm{Im}(\phi)$ factors through X. That is, we get a morphism $f \colon Y \to X$ such that $c' = c \circ f$.

$$\begin{tikzcd}
Y \arrow[r, "c'"] \arrow[d, dashed, "f"'] & \mathrm{Im}(\phi) \arrow[d, hook, "i"] \\
X \arrow[ur, "c"] \arrow[r, "\phi"] & A
\end{tikzcd}.$$

Thus, we get that $c = c' \circ m' = c \circ f \circ m'$, and, since c is right minimal, we deduce that $f \circ m'$ is an isomorphism. We observe now that

$$c \circ (f \circ m')^{-1} \circ f \circ k' = c \circ f \circ k' = c' \circ k' = 0$$

so $(f \circ m')^{-1} \circ f \circ k'$ factors through $\mathrm{Ker}(\phi)$, that is, we obtain a morphism $g \colon E \to \mathrm{Ker}(\phi)$ fitting into the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r, "k'"] \arrow[dd, dashed, "g"'] & Y \arrow[r, "c'"] \arrow[d, "f"] & \mathrm{Im}(\phi) \arrow[r] \arrow[dd, equal] & 0 \\
 & & X \arrow[d, "(f \circ m')^{-1}"] & & \\
0 \arrow[r] & \mathrm{Ker}(\phi) \arrow[r, "k"] & X \arrow[r, "c"] & \mathrm{Im}(\phi) \arrow[r] & 0
\end{tikzcd}.$$

We claim that g is a right inverse to $m \colon \mathrm{Ker}(\phi) \hookrightarrow E$. In order to see this, we observe that

$$k = (f \circ m')^{-1} \circ f \circ m' \circ k = (f \circ m')^{-1} \circ f \circ k' \circ m = k \circ g \circ m$$

which gives that $\mathrm{id}_{\mathrm{Ker}(\phi)} = g \circ m$, since k is mono. That is, g is a right inverse as we wanted, further the short exact sequence

$$0 \to \mathrm{Ker}(\phi) \xrightarrow{m} E \xrightarrow{e} X' \to 0$$

splits, so

$$\mathrm{Ext}^1_{\mathcal{A}}(-, \mathrm{Ker}(\phi))|_{\mathcal{X}} = 0.$$

The second assertion is proven by applying the first part of the lemma in the opposite category. $\square$

3 Torsion, Cotorsion and Tilting

We are now ready to set forth on the study of tilting. For the most part, we will follow the treatment given by Bauer et al. in [5], but some parts of the story will diverge slightly. We will see that tilting is in a bijective correspondance with cotorsion torsion triples, and that these triples induce an equivalence between certain subcategories. However, before we can prove any of these results, we have to make clear what torsion and cotorsion is. Hence, we will start by giving the axioms and some results of torsion, before moving on to cotorsion. Then cotorsion torsion triples will be defined and the induced equivalence proven, before tilting is defined in the last part of the section.

3.1 Torsion Pairs

Dickson introduced in [12] a set of axioms for decomposing a subcomplete[1] abelian category $\mathcal{A}$ into hom-orthogonal complements, based upon the concept of torsion in abelian groups. The abelian category in this thesis is not generally assumed to be subcomplete, hence our notion of torsion is a bit more restrictive than that of Dickson. However, in the more restrictive setting, our definition do coincide with the original axioms.

In an abelian group we have torsion elements, which are elements that are annihilated when multiplied by some non-zero integer. Every finitely generated abelian group decomposes into a direct sum of torsion groups on the form $\mathbb{Z}/p\mathbb{Z}$ for some prime $p \in \mathbb{N}$ and free groups $\mathbb{Z}$.

In a general abelian category these properties are approximatied through the following set of axioms describing torsion.

Definition 3.1. A *torsion pair* in an abelian category $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full

[1]Dickson used the term subcomplete about abelian categories which for any object had a set of subobjects, and in which any family of subobjects admitted a sum and intersection.

subcategories of $\mathcal{A}$ closed under isomorphisms, such that

(i) $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$, and

(ii) for any object A in $\mathcal{A}$, there is a short exact sequence

$$0 \to \mathtt{t}A \to A \to \mathtt{f}A \to 0$$

with $\mathtt{t}A \in \mathcal{T}$ and $\mathtt{f}A \in \mathcal{F}$.

If $(\mathcal{T}, \mathcal{F})$ is a torsion pair, then $\mathcal{T}$ is called a *torsion class* and $\mathcal{F}$ is called a *torsion-free class.*

Example 3.2. The trivial examples of torsion pairs in $\mathcal{A}$ are $(0, \mathcal{A})$ and $(\mathcal{A}, 0)$. ♣

Example 3.3. Let $(\mathcal{T}, \mathcal{F})$ be a pair of full categories in the category of finitely generated modules over a principal ideal domain R, given by

$$\begin{aligned} \mathcal{T} &= \mathrm{add}(\{R/rR \mid r \neq 0 \text{ unit}\}), \\ \mathcal{F} &= \mathrm{add}(R). \end{aligned}$$

We claim that these subcategories forms a torsion pair. To prove this let us look at $R/rR \in \mathcal{T}$ and $R \in \mathcal{F}$. For any homomorphism $f : R/rR \to R$ we have $0 = f(r)f(r^{-1}) = f(rr^{-1}) = f(1)$. Hence $\mathrm{Hom}_{\mathrm{mod}(R)}(R/rR, R) = 0$. For any other finitely generated module M over R, we have a structure theorem which gives us a decomposition into torsion and torsion free modules [See e.g. 7]. That is, any finitely generated module M over a PID R is of the form

$$M \cong R^s \oplus \left(\bigoplus_{i=1}^{t} R/r_iR \right).$$

This guarantees the existence of a short exact sequence

$$0 \to \left(\bigoplus_{i=1}^{t} R/r_iR \right) \to M \to R^s \to 0.$$

$(\mathcal{T}, \mathcal{F})$ is therefore a torsion pair in $\mathrm{mod}(R)$. ♣

Example 3.4. In the category of finite-dimensional representation over mod($\mathbf{k}$) of the the *zig-zag* quiver Q,

$$\underset{1}{\bullet} \longleftarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet}$$

we have the indecomposable representations

$$P_1: \quad \mathbf{k} \longleftarrow 0 \longrightarrow 0 \qquad P_2: \quad \mathbf{k} \xleftarrow{1} \mathbf{k} \xrightarrow{1} \mathbf{k} \qquad P_3: \quad 0 \longleftarrow 0 \longrightarrow \mathbf{k}$$

$$I_1: \quad \mathbf{k} \xleftarrow{1} \mathbf{k} \longrightarrow 0 \qquad I_2: \quad 0 \longleftarrow \mathbf{k} \longrightarrow 0 \qquad I_3: \quad 0 \longleftarrow \mathbf{k} \xrightarrow{1} \mathbf{k}$$

Let

$$\mathcal{T} = \operatorname{add}\{\ \mathbf{k} \xleftarrow{1} \mathbf{k} \longrightarrow 0 \ \oplus\ 0 \longleftarrow \mathbf{k} \longrightarrow 0 \ \oplus\ 0 \longleftarrow \mathbf{k} \xrightarrow{1} \mathbf{k}\ \},$$
$$\mathcal{F} = \operatorname{add}\{\ \mathbf{k} \longleftarrow 0 \longrightarrow 0 \ \oplus\ \mathbf{k} \xleftarrow{1} \mathbf{k} \xrightarrow{1} \mathbf{k} \ \oplus\ 0 \longleftarrow 0 \longrightarrow \mathbf{k}\ \}.$$

It is easy to see that $\operatorname{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$ and since every representation V over mod($\mathbf{k}$) decomposes into a direct sum of these indecomposables, we obtain a short exact sequence

$$0 \to \mathtt{t}V \to V \to \mathtt{f}V \to 0$$

for all representations $V \in \operatorname{rep}(Q, \mathbf{k})$. Thus, $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\operatorname{rep}(Q, \mathbf{k})$.

♣

Example 3.5. Over the *linear* quiver

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet}$$

we have a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\operatorname{rep}(Q, \mathbf{k})$ given by

$$\mathcal{T} = \operatorname{add}\{\ 0 \longrightarrow \mathbf{k} \longrightarrow 0 \ \oplus\ \mathbf{k} \longrightarrow \mathbf{k} \longrightarrow 0 \ \oplus\ \mathbf{k} \longrightarrow 0 \longrightarrow 0\ \},$$

and

$$\mathcal{F} = \operatorname{add}\{\ 0 \longrightarrow 0 \longrightarrow \mathbf{k} \ \oplus\ 0 \longrightarrow \mathbf{k} \longrightarrow \mathbf{k} \ \oplus\ \mathbf{k} \longrightarrow \mathbf{k} \longrightarrow \mathbf{k}\ \}.$$

We also have the torsion pair $(\mathcal{T}', \mathcal{F}')$ given by

$$\mathcal{T}' = \operatorname{add}\{\ \mathbf{k} \longrightarrow \mathbf{k} \longrightarrow \mathbf{k} \ \oplus\ \mathbf{k} \longrightarrow \mathbf{k} \longrightarrow 0 \ \oplus\ \mathbf{k} \longrightarrow 0 \longrightarrow 0\ \},$$

and

$$\mathcal{F}' = \text{add}\{\ 0 \longrightarrow 0 \longrightarrow \mathbf{k}\ \oplus\ 0 \longrightarrow \mathbf{k} \longrightarrow \mathbf{k}\ \oplus\ 0 \longrightarrow \mathbf{k} \longrightarrow 0\ \}.$$

♣

Example 3.6. All the examples of torsion pairs given thus far has been such that the union of the torsion and torsion-free class is the whole of $\mathcal{A}$, i.e. $\mathcal{T} \cup \mathcal{F} = \mathcal{A}$, this is not a general property of torsion pair. In $\text{rep}(Q, \mathbf{k})$ of the linear quiver from above, we can observe that the subcategories

$$\mathcal{T} = \text{add}\{\ \mathbf{k} \longrightarrow \mathbf{k} \longrightarrow 0\ \oplus\ \mathbf{k} \longrightarrow 0 \longrightarrow 0\ \oplus\ 0 \longrightarrow \mathbf{k} \longrightarrow 0\ \},$$
$$\mathcal{F} = \text{add}\{\ \mathbf{k} \longrightarrow \mathbf{k} \longrightarrow \mathbf{k}\ \oplus\ 0 \longrightarrow 0 \longrightarrow \mathbf{k}\ \}.$$

form a torsion pair $(\mathcal{T}, \mathcal{F})$, but the indecomposable representation $0 \longrightarrow \mathbf{k} \longrightarrow \mathbf{k}$ is not in either of them. ♣

Observe that if $(\mathcal{T}, \mathcal{F})$ is a torsion pair of $\mathcal{A}$ and we have an object $X \in^{\perp} \mathcal{F}$, then we get the exact sequence

$$0 \to \mathtt{t}X \to X \xrightarrow{0} \mathtt{f}X \to 0.$$

That is, $X \cong \mathtt{t}X \in \mathcal{T}$. Thus ${}^{\perp}\mathcal{F} \subseteq \mathcal{T}$, and since the converse inclusion is trivial we have equality. The equality $\mathcal{T}^{\perp} = \mathcal{F}$ follows similarly. From this and the results on orthogonality in Lemma 2.7 and Lemma 2.8, we conclude that that the following corollary holds.

Corollary 3.7. *If $(\mathcal{T}, \mathcal{F})$ is a torsion pair of $\mathcal{A}$, then*

$$\mathcal{F} = \mathcal{T}^{\perp} = \{A \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(T, A) = 0\ \forall\ T \in \mathcal{T}\},$$

and

$$\mathcal{T} =^{\perp} \mathcal{F} = \{A \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(A, F) = 0\ \forall\ F \in \mathcal{F}\}.$$

Further, both $\mathcal{T}$ and $\mathcal{F}$ are extension-closed, $\mathcal{T}$ is closed under factors and coproducts, and $\mathcal{F}$ is closed under subobjects and products. □

Lemma 3.8. *The short exact sequence arising from a torsion pair is functorial. In other words, given a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, the assignments*

$$\begin{aligned} \mathtt{t} : \mathcal{A} &\to \mathcal{T} \\ A &\mapsto \mathtt{t}A \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathtt{f} : \mathcal{A} &\to \mathcal{F} \\ A &\mapsto \mathtt{f}A \end{aligned}$$

are functors.

Further, they appear in adjoint pairs $(\mathrm{inc}_{\mathcal{T}}, \mathtt{t})$ *and* $(\mathtt{f}, \mathrm{inc}_{\mathcal{F}})$, *where* $\mathrm{inc}_{\mathcal{X}} \colon \mathcal{X} \hookrightarrow \mathcal{A}$ *is the canonical inclusion functor.*

Proof. Let $A, A' \in \mathcal{A}$ and $g \in \mathrm{Hom}_{\mathcal{A}}(A, A')$. Observe that the composition

$$\phi : \mathtt{t}A \hookrightarrow A \xrightarrow{g} A' \twoheadrightarrow \mathtt{f}A'$$

is zero, since $\phi \in \mathrm{Hom}_{\mathcal{A}}(\mathtt{t}A, \mathtt{f}A') = 0$. The universal property of $\mathtt{t}A'$ as kernel of $A \twoheadrightarrow \mathtt{f}A'$ gives a unique morphism $\mathtt{t}g : \mathtt{t}A \to \mathtt{t}A'$. Likewise, using the universal cokernel property of $\mathtt{f}A$, we get the unique morphism $\mathtt{f}g : \mathtt{f}A \to \mathtt{f}A'$.

$$\begin{tikzcd} \mathtt{t}A \arrow[r, hook] \arrow[d, dashed, "\mathtt{t}g"] & A \arrow[r, two heads] \arrow[d, "g"] & \mathtt{f}A \arrow[d, dashed, "\mathtt{f}g"] \\ \mathtt{t}A' \arrow[r, hook] & A' \arrow[r, two heads] & \mathtt{f}A' \end{tikzcd} .$$

For the adjoint claims, we apply the hom-functors $\mathrm{Hom}_{\mathcal{A}}(T, -)$ and $\mathrm{Hom}_{\mathcal{A}}(-, F)$ where $T \in \mathcal{T}$ and $F \in \mathcal{F}$ on

$$0 \to \mathtt{t}A \to A \to \mathtt{f}A \to 0,$$

obtaining natural isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(T, A) \cong \mathrm{Hom}_{\mathcal{A}}(T, \mathtt{t}A) \text{ and } \mathrm{Hom}_{\mathcal{A}}(\mathtt{f}A, F) \cong \mathrm{Hom}_{\mathcal{A}}(A, F).$$

□

Remark 3.9. For a torsion pair $(\mathcal{T}, \mathcal{F})$, we have that $\mathtt{t}$ is a subfunctor of the identity functor in $\mathcal{A}$. Such functors are often called a *pre-radical* on $\mathcal{A}$. Moreover, a pre-radical $\mathtt{r}$ on $\mathcal{A}$ is said to be *idempotent* if $\mathtt{r}(\mathtt{r}(-)) = \mathtt{r}(-)$. If $\mathtt{r}(X/\mathtt{r}X) = 0$[2] for every object $X \in \mathcal{A}$, then the pre-radical is *radical* on $\mathcal{A}$. Thus $\mathtt{t}$ is an idempotent radical of $\mathcal{A}$. In fact, we also have the converse direction; any idempotent radical give rise to a torsion pair, as shown in the following proposition.

Proposition 3.10 ([12, Theorem 2.8]). *Let* $\mathtt{r}$ *be a subfunctor of the identity in an abelian category* $\mathcal{A}$ *such that* $\mathtt{r}(\mathtt{r}(X)) = \mathtt{r}(X)$ *and* $\mathtt{r}(X/\mathtt{r}X) = 0$. *Then the pair of subcategories* $(\mathcal{T}, \mathcal{F})$ *given by*

$$\mathcal{T} = \{T \in \mathcal{A} \mid \mathtt{r}T = T\}$$

and

$$\mathcal{F} = \{F \in \mathcal{A} \mid \mathtt{r}F = 0\}$$

is a torsion pair of $\mathcal{A}$.

Proof. Let $f \in \mathrm{Hom}_{\mathcal{A}}(T, F)$ for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$. $\mathtt{r}$ is a subfunctor of the identity, so we obtain the following commutative diagram

$$\begin{tikzcd} \mathtt{r}T \arrow[r, equal] \arrow[d, "\mathtt{r}f"] & T \arrow[d, "f"] \\ \mathtt{r}F \arrow[r, hook] & F \end{tikzcd},$$

and since $\mathtt{r}F = 0$, we see that every morphism in $\mathrm{Hom}_{\mathcal{A}}(T, F)$ factors through zero, thus

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0.$$

From the hypothesis it follows that

$$0 \to \mathtt{r}A \to A \to A/\mathtt{r}A \to 0$$

satisfies the second axiom for being a torsion pair. □

We have seen that it is necessary for a torsion class to be closed under extensions, factors and coproduct. When the ambient abelian category is locally small and

[2] Where $X/\mathtt{r}X$ denote the cokernel $\mathrm{Cok}[\mathtt{r}X \hookrightarrow X]$.

bicomplete, we can in fact show that these properties are sufficient as well [12]. This is done by constructing the so-called *trace* of any object $A \in \mathcal{A}$ in $\mathcal{T}$, which is a functor given by

$$\mathtt{t}(-) = \sum_{\substack{f \in \mathrm{Hom}_{\mathcal{A}}(T,-) \\ [T] \in [\mathcal{T}]}} \mathrm{Im}(f) \colon \mathcal{A} \to \mathcal{T},$$

where $[\mathcal{T}]$ denotes the isomorphism classes of objects in $\mathcal{T}$.

Observe that the trace is a a right $\mathcal{T}$-approximation, hence implying a relationship between being torsion and contravariantly finite. In fact, any torsion subcategory is contravariantly finite. This is easily seen by observing that any morphism $T \to A$ for T in a torsion subcategory $\mathcal{T}$ gives the exact diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathtt{t}T \arrow[r, equal] \arrow[d] & T \arrow[r] \arrow[d] & 0 & \\ 0 \arrow[r] & \mathtt{t}A \arrow[r] & A \arrow[r] & \mathtt{f}A \arrow[r] & 0 \end{tikzcd}.$$

Conversely, we get the following sufficient criterion for being a torsion subcategory.

Lemma 3.11. *Let $\mathcal{A}$ be an abelian category, and $\mathcal{T}$ a contravariantly finite full subcategory closed under extensions and factors. Then $(\mathcal{T}, \mathcal{T}^{\perp})$ is a torsion pair.*

Proof. Let A be any object of $\mathcal{A}$, and $\phi \colon T \to A$ be a right $\mathcal{T}$-approximation of A. $\mathcal{T}$ is closed under factors, so we can assume that ϕ is in fact an monomorphism. Then we obtain the following short exact sequence

$$0 \to T \xrightarrow{\phi} A \xrightarrow{c} \mathrm{Cok}(\phi) \to 0,$$

which we want to show is of the desired form, i.e. $\mathrm{Cok}(\phi) \in \mathcal{T}^{\perp}$. Therefore, let $f \colon T' \to \mathrm{Cok}(\phi)$ be any morphism from $\mathcal{T}$ and form the exact diagram

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r, "\phi'"] \arrow[d, equal] & X \arrow[r, "c'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner"] & T' \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & T \arrow[r, "\phi"] & A \arrow[r, "c"] & \mathrm{Cok}(\phi) \arrow[r] & 0 \end{tikzcd}$$

where the right square is a pullback-square. $\mathcal{T}$ is extension-closed, so X lie in $\mathcal{T}$, thus f' factors through ϕ. Consequently, the top row splits. Now, $f\colon T' \to \mathrm{Cok}(\phi)$ factors as

$$T' \xrightarrow{g} X \xrightarrow{f'} A \xrightarrow{c} \mathrm{Cok}(\phi),$$

where g is the right inverse of c'. Using the fact that ϕ is a right $\mathcal{T}$-approximation gives us a factorization $T' \xrightarrow{h} T \xrightarrow{\phi} A$ of $f' \circ g$. Combining this, we see that $f\colon T' \to \mathrm{Cok}(\phi)$ factors through $c \circ \phi = 0$, so we conclude that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{D}, \mathrm{Cok}(\phi)) = 0$, and $(\mathcal{T}, \mathcal{T}^{\perp})$ is a torsion pair. □

In noetherian categories we can even see that any full subcategory closed under factors is contravariantly finite. This allows us to drop the condition of $\mathcal{T}$ being contravariantly finite in order to be a torsion class.

Lemma 3.12. *Let $\mathcal{A}$ be a noetherian abelian category, and $\mathcal{T} \subseteq \mathcal{A}$ a full subcategory closed under factors and extensions. Then $\mathcal{T}$ is contravariantly finite and therefore a torsion class in $\mathcal{A}$.*

Proof. We start by claiming that any object A of $\mathcal{A}$ has a unique maximal subobject that lies in $\mathcal{T}$. A is noetherian by assumption, so it suffices to show the uniqueness claim. Assume therefore that T and T' are two maximal subobjects that lie in $\mathcal{T}$. The sum of these two objects is a subobject of A by definition and trivially contains T and T' in a natural way. Further, since $T + T'$ is equal to the image of the canonical morphism $T \oplus T' \to A$, it must also lie in $\mathcal{T}$. This contradicts the maximality of T and T', thus proving the existence of a unique maximal subobject of A that lie in $\mathcal{T}$. Let us denote this subobject by $\mathtt{t}A$.

Now, we claim that the subobject $\mathtt{t}A \hookrightarrow A$ is a right $\mathcal{T}$-approximation. Let, therefore, $f\colon T \to A$ be any other morphism from $\mathcal{T}$, and observe that the image of this morphism must necessarily factor through $\mathtt{t}A$ by maximality. That is,

$\mathtt{t}A \hookrightarrow A$ is a right $\mathcal{T}$-approximation.

$$\begin{tikzcd}
T \arrow[rr, "f"] \arrow[dr, two heads] & & A \\
& \mathrm{Im}(f) \arrow[ur, hook] \arrow[dr, hook] & \\
& & \mathtt{t}A \arrow[uu, hook]
\end{tikzcd}.$$

□

3.2 Cotorsion Pairs

Where torsion pairs are hom-orthogonal, we have Ext-orthogonality in the related concept of cotorsion pairs. Cotorsion was first introduced for abelian groups in [28] by Salce, but is easily generelized to a general abelian category as we will see in the following section.

Definition 3.13. A *cotorsion pair* of $\mathcal{A}$ is a pair of full subcategories $(\mathcal{C}, \mathcal{D})$ such that $\mathcal{D}$ is left 1-orthogonal to $\mathcal{C}$, and $\mathcal{C}$ is similarly right 1-orthogonal to $\mathcal{D}$, i.e.

$$\mathcal{C} = {}^{\perp_1}\mathcal{D} = \{A \in \mathcal{A} \mid \mathrm{Ext}^1_{\mathcal{A}}(A, D) = 0 \text{ for all } D \in \mathcal{D}\}$$

and

$$\mathcal{D} = \mathcal{C}^{\perp_1} = \{A \in \mathcal{A} \mid \mathrm{Ext}^1_{\mathcal{A}}(C, A) = 0 \text{ for all } C \in \mathcal{C}\}.$$

A cotorsion pair $(\mathcal{C}, \mathcal{D})$ is said to have *enough projectives* if for any object A in $\mathcal{A}$ we have an exact sequence

$$0 \to \mathtt{d}A \to \mathtt{c}A \to A \to 0$$

with $\mathtt{d}A \in \mathcal{D}$ and $\mathtt{c}A \in \mathcal{C}$. Equally, $(\mathcal{C}, \mathcal{D})$ is said to have *enough injectives* if for any object A in $\mathcal{A}$ we have an exact sequence

$$0 \to A \to \tilde{\mathtt{d}}A \to \tilde{\mathtt{c}}A \to 0$$

with $\tilde{c}A$ in $\mathcal{C}$ and $\tilde{\mathtt{d}}A$ in $\mathcal{D}$.

A cotorsion pair $(\mathcal{C}, \mathcal{D})$ which has both enough injectives and projectives, is called *complete.*

Remark 3.14. We have chosen to not follow the convention used in [5] where the existence of enough projectives and injectives is a part of being cotorsion, and instead use the convention of calling such pairs complete cotorsion. This is done in an effort to not unnecessarily over-complicate the notation for the constructions in Section 4.2. Further, this allows us to state Salce's Lemma (Lemma 3.25) in its commonly known form.

Example 3.15. Let $\mathcal{A}$ be an abelian category with enough projectives $\mathrm{Proj}\mathcal{A} \subseteq \mathcal{A}$. Then $(\mathrm{Proj}\mathcal{A}, \mathcal{A})$ is a cotorsion pair with enough projectives. In fact, since we can always form the short exact sequence

$$0 \to A \xrightarrow{\sim} A \to 0 \to 0$$

for any object $A \in \mathcal{A}$, we have that $(\mathrm{Proj}\mathcal{A}, \mathcal{A})$ is a complete cotorsion pair.

Dually, if $\mathcal{A}$ has enough injectives $\mathrm{Inj}\mathcal{A} \subseteq \mathcal{A}$, then $(\mathcal{A}, \mathrm{Inj}\mathcal{A})$ is a complete cotorsion pair. ♣

Example 3.16. Let us fix some abelian category $\mathcal{A}$ with enough projectives and injectives. We consider the functor category $\mathrm{Fun}(\mathbb{Z}^+, \mathcal{A})$ of all functors from the ordered category of positive integers to $\mathcal{A}$. Equivalently, we can describe this by the category $\mathrm{rep}(\vec{A}^\infty, \mathcal{A})$ of $\mathcal{A}$-valued representations of the infinite quiver

$$\vec{A}^\infty: \qquad \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet} \longrightarrow \underset{4}{\bullet} \longrightarrow \cdots .$$

The pair $(\mathcal{C}, \mathcal{D})$ of subcategories of $\mathrm{Rep}(\vec{A}^\infty, \mathcal{A})$, given by

$$\mathcal{C} = \{F \in \mathrm{Rep}(\vec{A}^\infty, A) \mid \text{All structure maps of } F \text{ are mono}\},$$
$$\mathcal{D} = \mathrm{Rep}(\vec{A}^\infty, \mathrm{Inj}\mathcal{A}) = \mathrm{Fun}(\mathbb{Z}^+, \mathrm{Inj}\mathcal{A}),$$

is a cotorsion pair [16, Theorem A]. Further, Odabaşı has shown that this cotorsion pair is in fact complete whenever $\mathcal{A}$ has exact small coproducts [23, Theorem 4.6].

Considering the ordered category of negative integers, or equivalently the quiver

$$\vec{A}_\infty: \qquad \cdots \longrightarrow \underset{4}{\bullet} \longrightarrow \underset{3}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{1}{\bullet}$$

we have the cotorsion pair $(\mathcal{C}, \mathcal{D})$ of $\mathrm{Rep}(\overrightarrow{A_\infty}, \mathcal{A})$ given by

$$\mathcal{C} = \mathrm{Rep}(\overrightarrow{A_\infty}, \mathrm{Proj}\mathcal{A}) = \mathrm{Fun}(\mathbb{Z}^-, \mathrm{Proj}\mathcal{A}),$$
$$\mathcal{D} = \{F \in \mathrm{Rep}(\overrightarrow{A_\infty}, A) \mid \text{All structure maps of } F \text{ are epi}\}.$$

which is complete if $\mathcal{A}$ has exact small products. ♣

Remark 3.17. The above example is an application of the main results presented by Holm et al. in [16] and Odabaşı in [23]. In section 4.2 we will prove those results for finite acyclic quivers. From there we obtain the next two examples.

Example 3.18. Let us fix an abelian category $\mathcal{A}$ with enough projectives and the zig-zag quiver

$$Q\colon \qquad \underset{1}{\bullet} \xleftarrow{\alpha} \underset{2}{\bullet} \xrightarrow{\beta} \underset{3}{\bullet}$$

Then $(\mathcal{C}, \mathcal{D})$ given by

$$\mathcal{C} = \mathrm{Rep}(Q, \mathrm{Proj}\mathcal{A}),$$
$$\mathcal{D} = \{F \in \mathrm{Rep}(Q, \mathcal{A}) \mid \text{The canonical morphism } F_2 \to F_1 \oplus F_3 \text{ is epic.}\}$$

is a complete cotorsion pair in $\mathrm{Rep}(Q, \mathcal{A})$.

Let us verify the cotorsion claim by showing that ${}^{\perp_1}\mathcal{D} \subseteq \mathcal{C}$ and $\mathcal{C}^{\perp_1} = \mathcal{D}$. Then we will show that the pair has enough injectives, which by the soon to be shown Salce's Lemma (Lemma 3.25) is sufficient for it to be complete.

Assume first that we have a representation $F \in \mathrm{Rep}(Q, \mathcal{A})$,

$$F_1 \xleftarrow{F\alpha} F_2 \xrightarrow{F\beta} F_3 \ ,$$

in ${}^{\perp_1}\mathcal{D}$, that is $\mathrm{Ext}^1_{\mathcal{A}}(F, -)|_{\mathcal{D}} = 0$. We want to show that $F \in \mathcal{C}$, which we will do by showing that F_1, F_2, F_3 are projective objects in $\mathcal{A}$. We start by showing that F_2 is projective. Consider the following exact diagram

$$\begin{tikzcd} & & & F_2 \arrow[d] & \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{A}$, which gives us the following exact commutative diagram where the right square is a pullback

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}.$$

From this, we construct the following commutative diagram with exact rows, where the columns are representations of Q,

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & F_1 \arrow[r] & F_1 \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u] \arrow[d] & X \arrow[r] \arrow[u] \arrow[d] & F_2 \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & F_3 \arrow[r] & F_3 \arrow[r] & 0
\end{tikzcd}.$$

This sequence splits by assumption, as the representation $0 \longleftarrow A \longrightarrow 0$ clearly lies in $\mathcal{D}$. Specifically, we see that

$$0 \to A \to X \to F_2 \to 0$$

splits, and thus, we obtain a factorization

$$\begin{tikzcd}
 & & X \arrow[d] & F_2 \arrow[l] \arrow[d] & \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

giving that F_2 is projective. By symmetry, it is enough to show that F_1 also is projective for F to be in $\mathcal{C} = \operatorname{Rep}(Q, \operatorname{Proj}\mathcal{A})$ and thus ${}^{\perp_1}\mathcal{D} \subseteq \mathcal{C}$. Therefore, let us consider the exact diagram

$$\begin{tikzcd}
 & & & F_1 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd},$$

which by constructing the pullback, gives the following commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd} .$$

From this, we obtain the following short exact sequence of representations, by taking the pullback along $F_2 \xrightarrow{F\alpha} F_1$ and $Y \to F_1$,

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] & Y \arrow[r] & F_1 \arrow[r] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[u, "1"] \arrow[d] & Z \arrow[r] \arrow[u] \arrow[d] \arrow[ur, phantom, "\urcorner", very near start] & F_2 \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & F_3 \arrow[r] & F_3 \arrow[r] & 0
\end{tikzcd} .$$

Once again, by assumption, this sequence and specifically

$$0 \to A' \to Y \to F_1 \to 0$$

splits, giving the factorization

$$\begin{tikzcd}
 & & Y \arrow[d] & F_1 \arrow[l] \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd} .$$

That is, F_1 is projective, hence F lie in $\mathcal{C}$ and ${}^{\perp_1}\mathcal{D} \subseteq \mathcal{C}$.

Now, we want to show that $\mathcal{D} = \mathcal{C}^{\perp_1}$. Let us therefore assume we have $F \in \mathcal{C}^{\perp_1}$. Explicitly, for any representation $C \in \mathcal{C} = \mathrm{Rep}(Q, \mathrm{Proj}\mathcal{A})$ with projective resolution

$$\cdots \to P^{-1} \to P^0 \to C \to 0$$

we have an epimorphism

$$\mathrm{Hom}_{\mathrm{Rep}(Q,\mathcal{A})}(P^0, F) \to \mathrm{Hom}_{\mathrm{Rep}(Q,\mathcal{A})}(P^{-1}, F). \tag{1}$$

Observe that we have the short exact sequence of representations

$$\begin{tikzcd}
0 \arrow[r] & C_2 \arrow[r, "\binom{1}{-f}"] & C_2 \oplus C_1 \arrow[r, "(f\ 1)"] & C_1 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] \arrow[d] & C_2 \arrow[r, "1"] \arrow[u, "\binom{1}{0}"] \arrow[d, "\binom{1}{0}"'] & C_2 \arrow[r] \arrow[u, "f"'] \arrow[d, "g"'] & 0 \\
0 \arrow[r] & C_2 \arrow[r, "\binom{1}{-g}"] & C_2 \oplus C_3 \arrow[r, "(g\ 1)"] & C_3 \arrow[r] & 0
\end{tikzcd},$$

which, in fact, is a projective resolution of C since $C \in \mathrm{Rep}(Q, \mathrm{Proj}\mathcal{A})$. Any morphism $P^0 \to F$ is on the form

$$\begin{tikzcd}
C_2 \oplus C_1 \arrow[r, "(F\alpha\circ y\ x)"] & F_1 \\
C_2 \arrow[r, "y"] \arrow[u, "\binom{1}{0}"] \arrow[d, "\binom{1}{0}"'] & F_2 \arrow[u, "F\alpha"'] \arrow[d, "F\beta"'] \\
C_2 \oplus C_3 \arrow[r, "(F\beta\circ y\ z)"] & F_3
\end{tikzcd}$$

and any morphism $P^{-1} \to F$ is on the form

$$\begin{tikzcd}
C_2 \arrow[r, "x'"] & F_1 \\
0 \arrow[r] \arrow[u] \arrow[d] & F_2 \arrow[u, "F\alpha"'] \arrow[d, "F\beta"'] \\
C_2 \arrow[r, "z'"] & F_3
\end{tikzcd}.$$

Hence the condition that the morphism (1) is an epimorphism is equivalent to

$$\mathcal{A}(C_1, F_1) \oplus \mathcal{A}(C_2, F_2) \oplus \mathcal{A}(C_3, F_3) \xrightarrow{\begin{pmatrix} -\circ f & F\alpha\circ - & 0 \\ 0 & F\beta\circ - & -\circ g \end{pmatrix}} \mathcal{A}(C_2, F_1) \oplus \mathcal{A}(C_2, F_3)$$

being epimorphic. Now any morphism $C \to F$ is on the form

$$\begin{tikzcd}
C_1 \arrow[r, "x"] & F_1 \\
C_2 \arrow[r, "y"] \arrow[u, "f"] \arrow[d, "g"'] & F_2 \arrow[u, "F\alpha"'] \arrow[d, "F\beta"'] \\
C_3 \arrow[r, "z"] & F_3
\end{tikzcd}$$

i.e. $x \circ f = F\alpha \circ y$ and $z \circ g = F\beta \circ y$. Thus we also get the sufficient and necessary condition that

$$\mathcal{A}(C_2, F_2) \xrightarrow{\left(\begin{smallmatrix} F\alpha\circ - \\ F\beta\circ - \end{smallmatrix}\right)} \mathcal{A}(C_2, F_1) \oplus \mathcal{A}(C_2, F_3) \cong \mathcal{A}(C_2, F_1 \oplus F_3) \tag{2}$$

is epimorphic for every projective object $C_2 \in \operatorname{Proj}\mathcal{A}$. $\mathcal{A}$ has enough projectives, so specifically, we know there is an epimorphism $C_2 \twoheadrightarrow F_1 \oplus F_3$ for some projective $C_2 \in \operatorname{Proj}\mathcal{A}$. Hence, we have that the morphism (2) is epimorphic if and only if $F_2 \to F_1 \oplus F_3$ is epimorphic. That is, F lie in $C^{\perp_1}$ if and only if it lie in $\mathcal{D}$, equivalently $\mathcal{C}^{\perp_1} = \mathcal{D}$. Now, we have

$$\mathcal{C} \subseteq {}^{\perp_1}(\mathcal{C}^{\perp_1}) = {}^{\perp_1}\mathcal{D} \subseteq \mathcal{C},$$

which implies $\mathcal{C} = {}^{\perp_1}\mathcal{D}$, and consequently $(\mathcal{C}, \mathcal{D})$ is a cotorsion pair.

To conclude, we are now left with showing that $(\mathcal{C}, \mathcal{D})$ has enough injectives, as this, in conjunction with $\operatorname{Rep}(Q, \mathcal{A})$ having enough projectives, guarantees that $(\mathcal{C}, \mathcal{D})$ is complete by Salce's Lemma. In that endevaour, let us fix some representation $F \in \operatorname{Rep}(Q, \mathcal{A})$

$$F_1 \xleftarrow{F\alpha} F_2 \xrightarrow{F\beta} F_3 \ .$$

$\mathcal{A}$ has enough projectives, so we find an epimorphism

$$\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \colon P \twoheadrightarrow F_1 \oplus F_2$$

from some projective object $P \in \operatorname{Proj}\mathcal{A}$. Observe then that since this epimorphism is equal to the composition

$$P \xhookrightarrow{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} F_2 \oplus P \xrightarrow{\left(\begin{smallmatrix} F\alpha & x \\ F\beta & y \end{smallmatrix}\right)} F_1 \oplus F_3,$$

we get that the last morphism of the composition is an epimorphism. Thus we obtain the short exact sequence of representations

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r, "1"] & F_1 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & F_2 \arrow[r, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] \arrow[u, "F\alpha"] \arrow[d, "F\beta"'] & F_2 \oplus P \arrow[r, "(0\ 1)"] \arrow[u, "(F\alpha\ x)"'] \arrow[d, "(F\beta\ y)"] & P \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & F_3 \arrow[r, "1"] & F_3 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd},$$

where the last representation is in $\mathcal{C}$ and the middle representation is in $\mathcal{D}$. We conclude that $(\mathcal{C}, \mathcal{D})$ has enough injectives, and therefore is complete. ♣

Example 3.19. Dually, let $\mathcal{A}$ be an abelian category with enough injectives. Then in the category of representaions of the zig-zag quiver

$$Q\colon \qquad \underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet} \xleftarrow{\beta} \underset{3}{\bullet}$$

we have the complete cotorsion pair $(\mathcal{C}, \mathcal{D})$ given by

$$\mathcal{C} = \{F \in \mathrm{Rep}(Q, \mathcal{A}) \mid \text{The canonical morphism } F_1 \oplus F_3 \to F_2 \text{ is mono.}\},$$
$$\mathcal{D} = \mathrm{Rep}(Q, \mathrm{Inj}\mathcal{A}).$$

♣

Example 3.20. Any class of objects $\mathcal{X} \subseteq \mathcal{A}$ in an abelian category $\mathcal{A}$ gives us the two cotorsion pairs $({}^{\perp_1}(\mathcal{X}^{\perp_1}), \mathcal{X}^{\perp_1})$ and $({}^{\perp_1}\mathcal{X}, ({}^{\perp_1}\mathcal{X})^{\perp_1})$. We say that the cotorsion pair is *generated*, respectively *cogenerated* by $\mathcal{X}$. ♣

Remark 3.21. The usefullness of generated cotorsion pairs were demonstrated by Eklof and Trlifaj, who proved that for module categories all cotorsion pairs which is generated by a set of modules are complete [13, Theorem 10].

From Lemma 2.7 and Lemma 2.9 we know that a cotorsion pair possesses the following basic properties.

Corollary 3.22. *Let $(\mathcal{C}, \mathcal{D})$ be a cotorsion pair, then both $\mathcal{C}$ and $\mathcal{D}$ are closed under direct summands and extensions. Further*

(i) $\mathcal{C}$ contains all projective objects of $\mathcal{A}$ and is closed under coproducts.

(ii) $\mathcal{D}$ contains all injective objects of $\mathcal{A}$ and is closed under products. □

In Lemma 2.30 we found that a morphism $f\colon X \to E$ from a full subcategory $\mathcal{X}$ of an abelian category is a right-approximation if $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{X}, \mathrm{Ker}(f)) = 0$, i.e. if $\mathrm{Ker}(f) \in \mathcal{X}^{\perp_1}$. This result and it's dual gives us the following result for cotorsion pairs with enough injectives or projectives respectively.

Corollary 3.23. *Let $(\mathcal{C}, \mathcal{D})$ be a cotorsion pair in $\mathcal{A}$.*

(i) If $(\mathcal{C}, \mathcal{D})$ has enough injectives, then $\mathcal{D}$ is covariantly finite in $\mathcal{A}$.

(ii) If $(\mathcal{C}, \mathcal{D})$ has enough projectives, then $\mathcal{C}$ is contravariantly finite in $\mathcal{A}$. □

The following corollary to Wakamatsu's Lemma, Lemma 2.35, gives a converse to the above when we work in a Krull-Schmidt category. In fact, the proof generalizes to abelian categories where every morphism decomposes as in Lemma 2.33 and Lemma 2.34.

Corollary 3.24. *Let $(\mathcal{C}, \mathcal{D})$ be a cotorsion pair in an abelian Krull-Schmidt category $\mathcal{A}$ with enough projectives. If $\mathcal{C}$ is contravaritantly finite, then $(\mathcal{C}, \mathcal{D})$ has enough projectives. Dually, if $\mathcal{D}$ is covariantly finite, then $(\mathcal{C}, \mathcal{D})$ has enough injectives.*

Proof. We prove the first claim, the second follows dually. Let $\phi\colon C \to E$ be any right $\mathcal{C}$-approximation of an object $E \in \mathcal{A}$. First, observe that if ϕ is not right minimal, then Lemma 2.33 and the fact that $\mathcal{C}$ is closed under direct summands gives us a right minimal $\mathcal{C}$-approximation $\phi'\colon C' \to E$. Observe also that since $\mathcal{A}$ has enough projectives, we have an epimorphism $P \twoheadrightarrow E$ with $P \in \operatorname{Proj}(\mathcal{A}) \subseteq \mathcal{C}$, which factors through ϕ', thus forcing ϕ' to be an epimorphism.

Now Lemma 2.35 gives us that

$$\operatorname{Ext}^1_{\mathcal{A}}(\mathcal{C}, \operatorname{Ker}(\phi')) = 0,$$

or, equivalently, $\operatorname{Ker}(\phi') \in \mathcal{C}^{\perp_1} = \mathcal{D}$. We therefore have the short exact sequence

$$0 \to \operatorname{Ker}(\phi') \to C' \xrightarrow{\phi'} E \to 0$$

with $C' \in \mathcal{C}$ and $\operatorname{Ker}(\phi') \in \mathcal{D}$. □

The dual statement for an abelian Krull-Schmidt category with enough injectives holds as well.

Salce showed in his article [28, Corollary 2.4] that in the category of abelian groups, every cotorsion pair has enough projectives if and only if it has enough injectives. This result generalizes to hold in any abelian category $\mathcal{A}$ with enough projectives and injectives, as is seen from the following lemma, which is commonly called Salce's lemma.

Lemma 3.25 (Salce's Lemma)**.** *Let $\mathcal{A}$ be an abelian category with enough projectives, and $(\mathcal{C}, \mathcal{D})$ a cotorsion pair in $\mathcal{A}$ with enough injectives. Then $(\mathcal{C}, \mathcal{D})$ is complete, that is it also has enough projectives. The dual assertion for $\mathcal{A}$ having enough injectives and $(\mathcal{C}, \mathcal{D})$ enough projectives holds as well.*

Proof. Let A be any object of $\mathcal{A}$. We can find an epimorphism $P \twoheadrightarrow A$ from a projective object P, since $\mathcal{A}$ has enough projectives, and by taking the kernel of this map we have a short exact sequence

$$0 \to K \to P \to A \to 0.$$

$(\mathcal{C}, \mathcal{D})$ has enough injectives, so we find a short exact sequence

$$0 \to K \to \tilde{\mathtt{d}}K \to \tilde{\mathtt{c}}K \to 0,$$

with $\tilde{\mathtt{d}}K \in \mathcal{D}$ and $\tilde{\mathtt{c}}K \in \mathcal{D}$. By taking the push-out along $K \hookrightarrow P$ and $K \hookrightarrow \tilde{\mathtt{d}}K$, we obtain the following exact commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \tilde{\mathtt{d}}K \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & A \arrow[r] & 0 \\
 & \tilde{\mathtt{c}}K \arrow[r, equal] \arrow[d] & \tilde{\mathtt{c}}K \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}.$$

Observe that $P \in \mathcal{C}$, since $\mathrm{Proj}\mathcal{A} \subseteq \mathcal{C}$ and $\tilde{\mathtt{c}}K \in \mathcal{C}$ by construction, so the pushout X lies in $\mathcal{C}$, by closure of extensions. The second row is now a short exact sequence on the wanted form. The dual statement is proven similarly. $\square$

3.2.1 Adjoint Functors

We would like for complete cotorsion pairs to have natural functors associated to them through their short exact sequences, like we had for torsion pairs. Unfortunately, the short exact sequences are not necessarily unique and therefore neither functorial. However, we can in some sense approximate the same property by going to an appropriate quotient category. This will be made clear in an instant.

Definition 3.26. The subcategory $\mathcal{K} = \mathcal{C} \cap \mathcal{D} \subseteq \mathcal{A}$ associated to a cotorsion pair $(\mathcal{C}, \mathcal{D})$ is called the *core* of the pair. For any full, additive subcategory $\mathcal{X}$ of $\mathcal{A}$ containing $\mathcal{K}$, let $\mathcal{X}/\mathcal{K}$ denote the ideal qoutient of $\mathcal{X}$ by $\mathcal{K}$. If $f \in \operatorname{Hom}_{\mathcal{X}}(A, B)$ is a morphism in $\mathcal{X}$, let $\bar{f}$ denote the image in $\mathcal{X}/\mathcal{K}$.

Lemma 3.27. *For any complete cotorsion pair $(\mathcal{C}, \mathcal{D})$ in $\mathcal{A}$ we have that any morphism from an object in $\mathcal{C}$ to an object in $\mathcal{D}$, factors through an object in $\mathcal{K}$. That is,*

$$\operatorname{Hom}_{\mathcal{A}/\mathcal{K}}(\mathcal{C}/\mathcal{K}, \mathcal{D}/\mathcal{K}) = 0.$$

Further, the constructions $\mathtt{d}, \tilde{\mathtt{d}}$ and $\mathtt{c}, \tilde{\mathtt{c}}$ induce functors from $\mathcal{A}/\mathcal{K}$ to $\mathcal{D}/\mathcal{K}$ and $\mathcal{C}/\mathcal{K}$ respectively.

Proof. Let $f \colon C \to D$ be any morphism with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. By using that $(\mathcal{C}, \mathcal{D})$ has enough projectives, we obtain the short exact sequence

$$0 \to \mathtt{d}D \xrightarrow{m} \mathtt{c}D \xrightarrow{e} D \to 0,$$

where necessarily $\mathtt{c}D \in \mathcal{C} \cap \mathcal{D}$, since $\mathcal{D}$ is closed under extensions. This sequence gives us the following short exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}(C, \mathtt{d}D) \xrightarrow{m^*} \operatorname{Hom}_{\mathcal{A}}(C, \mathtt{c}D) \xrightarrow{e^*} \operatorname{Hom}_{\mathcal{A}}(C, D) \to 0,$$

since $\operatorname{Ext}^1_{\mathcal{A}}(C, \mathtt{d}D) = 0$. Observe that since e^* is an epimorphism, we can find a morphism $h \in \operatorname{Hom}_{\mathcal{A}}(C, \mathtt{c}D)$ such that $f = eh$.

$$\begin{tikzcd}
 & C \arrow[d, "f"] \arrow[dl, dashed] & \\
\mathtt{c}D \arrow[r, two heads] & D \arrow[r] & 0
\end{tikzcd}$$

Thus any morphism from $\mathcal{C}$ to $\mathcal{D}$ factor through an object in $\mathcal{C} \cap \mathcal{D}$.

Now for the functorality claim, observe that any morphism $g\colon A \to B$ in $\mathcal{A}$, fits into the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathtt{d}A \arrow[r] \arrow[d, dashed, "\mathtt{d}g"] & \mathtt{c}A \arrow[r] \arrow[d, dashed, "\mathtt{c}g"] & A \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & \mathtt{d}B \arrow[r] & \mathtt{c}B \arrow[r] & B \arrow[r] & 0 \end{tikzcd}.$$

The existence of the lift from g to $\mathtt{c}g$ follows from $\mathtt{c}B \to B$ being a right $\mathcal{C}$-approximation of B by Lemma 2.30. The existence of this lift guarantees the existence of $\mathtt{d}g$, by the universal property of kernels.

Finally, we are left with checking uniqueness of the lifts in $\mathcal{C}/\mathcal{K}$ and $\mathcal{D}/\mathcal{K}$. We do this by showing that any lifts of the zero map $0\colon A \to B$ factors through an object of $\mathcal{K}$. If $\mathtt{c}g$ is a lift of the zero map, we observe that it factors through some morphism $h\colon \mathtt{c}A \to \mathtt{d}B$ by the kernel property of $\mathtt{d}B \hookrightarrow \mathtt{c}B$. In fact, $\mathtt{d}g$ must also factor through this morphism. Now, observe that by the first part of our lemma h factors through some object $K \in \mathcal{K}$, and thus so does $\mathtt{c}g$ and $\mathtt{d}g$. □

Lemma 3.28. *Let $(\mathcal{C}, \mathcal{D})$ be a complete cotorsion pair in $\mathcal{A}$, with core $\mathcal{K}$. Let $A \in \mathcal{A}$ be any object in $\mathcal{A}$, and*

$$0 \to \mathtt{d}A \xrightarrow{m} \mathtt{c}A \xrightarrow{e} A \to 0$$

a short exact sequence with $\mathtt{d}A \in \mathcal{D}$ and $\mathtt{c}A \in \mathcal{D}$. Then $\bar{e}^* = \mathrm{Hom}_{\mathcal{A}/\mathcal{K}}(C, \bar{e})$ is an isomorphism for every $C \in \mathcal{C}$. That is

$$\bar{e}^*\colon \mathrm{Hom}_{\mathcal{A}/\mathcal{K}}(C, \mathtt{c}A) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}/\mathcal{K}}(C, A)$$

Proof. As in Lemma 3.27, we get a short exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(C, \mathtt{d}A) \xrightarrow{m^*} \mathrm{Hom}_{\mathcal{A}}(C, \mathtt{c}A) \xrightarrow{e^*} \mathrm{Hom}_{\mathcal{A}}(C, A) \to 0,$$

for $C \in \mathcal{C}$, hence $\bar{e}^*$ is epimorphic.

In order to show that $\bar{e}^*$ is also mono, we let $f \in \mathrm{Hom}_{\mathcal{A}}(C, \mathtt{c}A)$ be any morphism such that $\bar{e}^*(f) = \bar{e} \circ \bar{f} = 0$. Thus, we can find an object $I \in \mathcal{K}$, such that we have the following commutative diagram with exact rows

$$\begin{tikzcd} & & C \arrow[r, "i"] \arrow[d, "f"] & I \arrow[d, "j"] & \\ 0 \arrow[r] & \mathtt{d}A \arrow[r, "m"] & \mathtt{c}A \arrow[r, "e"] & A \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{A}$. I lies specifically in $\mathcal{C}$, so we get a new short exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(I, \mathtt{d}A) \to \mathrm{Hom}_{\mathcal{A}}(I, \mathtt{c}A) \to \mathrm{Hom}_{\mathcal{A}}(I, A) \to 0.$$

We can therefore find a morphism $g \in \mathrm{Hom}_{\mathcal{A}}(I, \mathtt{c}A)$ such that $e \circ g = j$. Thus we have

$$0 = e \circ f - j \circ i = e \circ f - e \circ g \circ i = e^*(f - g \circ i),$$

and, by exactness above, we have $\mathrm{Ker}(e^*) = \mathrm{Im}(m^*)$, so there is a morphism $h \in \mathrm{Hom}_{\mathcal{A}}(C, \mathtt{d}A)$ such that $m \circ h = f - g \circ i$. By Lemma 3.27, h factors through some object $I' \in \mathcal{K}$

$$\begin{tikzcd} & I' \arrow[d, "j'"'] & C \arrow[l, "i'"'] \arrow[r, "i"] \arrow[d, "f"] \arrow[dl, dotted, "h"] & I \arrow[d, "j"] \arrow[dl, dashed, "g"] & \\ 0 \arrow[r] & \mathtt{d}A \arrow[r, "m"] & \mathtt{c}A \arrow[r, "e"] & A \arrow[r] & 0 \end{tikzcd},$$

so $f = m \circ h + g \circ i = m \circ j' \circ i' + g \circ i$. Thus, we have obtained the following factorization of f

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\binom{i}{i'}"'] & & \mathtt{c}A \\ & I \oplus I' \arrow[ur, "(g \;\; j' \circ m)"'] & \end{tikzcd},$$

and since both $\mathcal{C}$ and $\mathcal{D}$ are closed under extensions, we have $I \oplus I' \in \mathcal{K}$, and therefore $\bar{f} = 0$. We conclude that $\bar{e}^*$ is an isomorphism. $\square$

Corollary 3.29. *The inclusion functor $\mathcal{C}/\mathcal{K} \hookrightarrow \mathcal{A}/\mathcal{K}$ is a left adjoint to the induced functor $\mathtt{c}' \colon \mathcal{A}/\mathcal{K} \to \mathcal{C}/\mathcal{K}$.*

We also obtain the dual results, which we state without proof.

Lemma 3.30. *Let $(\mathcal{C}, \mathcal{D})$ be a complete cotorsion pair in $\mathcal{A}$, with core $\mathcal{K}$. Let $A \in \mathcal{A}$ be any object in $\mathcal{A}$, and*

$$0 \to A \xrightarrow{m} \tilde{\mathtt{d}}A \xrightarrow{e} \tilde{\mathtt{c}}A \to 0$$

a short exact sequence with $\tilde{\mathtt{d}}A \in \mathcal{D}$ and $\tilde{\mathtt{c}}A \in \mathcal{D}$. Then $\bar{m}_* = \operatorname{Hom}_{\mathcal{A}/\mathcal{K}}(\bar{m}, D)$ is an isomorphism for every $D \in \mathcal{D}$. That is

$$\bar{m}_* \colon \operatorname{Hom}_{\mathcal{A}/\mathcal{K}}(A, D) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}/\mathcal{K}}(\tilde{\mathtt{d}}A, D)$$

Corollary 3.31. *The inclusion functor $\mathcal{D}/\mathcal{K} \hookrightarrow \mathcal{A}/\mathcal{K}$ is a right adjoint to the induced functor $\tilde{\mathtt{d}}' \colon \mathcal{A}/\mathcal{K} \to \mathcal{D}/\mathcal{K}$.*

3.3 Cotorsion Torsion Triples

Now we can define what we mean by a cotorsion torsion triple.

Definition 3.32. A *cotorsion torsion triple* is a triple of subcategories $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, such that $(\mathcal{C}, \mathcal{T})$ is a complete cotorsion pair, and $(\mathcal{T}, \mathcal{F})$ is a torsion pair.

Additionally, we define a *torsion cotorsion triple* as a triple of full subcategories $(\mathcal{T}, \mathcal{F}, \mathcal{D})$, such that $(\mathcal{T}, \mathcal{F})$ is a torsion pair, and $(\mathcal{F}, \mathcal{D})$ is a complete cotorsion pair.

Remark 3.33. We can observe that torsion and cotorsion is self-dual, in the sense that if $(\mathcal{X}, \mathcal{Y})$ is either torsion or cotorsion in $\mathcal{A}$, then $(\mathcal{Y}^{\mathrm{op}}, \mathcal{X}^{\mathrm{op}})$ is either torsion or cotorsion in $\mathcal{A}^{\mathrm{op}}$. Therefore, a triple of subcategories $(\mathcal{C}, \mathcal{D}, \mathcal{F})$ in $\mathcal{A}$ is a cotorsion torsion triple if and only if the triple $(\mathcal{F}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}})$ is a torsion cotorsion triple in $\mathcal{A}^{\mathrm{op}}$.

As we saw in the previous section, the cotorsion and cotorsion-free parts of a cotorsion pair are contravariantly and covariantly finite respectively. Let us now show that in the presence of the functors of a torsion pairs, these properties extend to the core of the cotorsion pair as well.

Lemma 3.34. *Let $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ be a cotorsion torsion triple, then $\mathcal{C} \cap \mathcal{T}$ is contravariantly finite. Analogously, if we have a torsion cotorsion triple $(\mathcal{T}, \mathcal{F}, \mathcal{D})$, then $\mathcal{F} \cap \mathcal{D}$ is covariantly finite.*

Proof. We will only prove the first statement of the lemma, the proof of the second statement follows similarly. Let A be any object in $\mathcal{A}$. Let $f\colon X \to A$, be any morphism from $\mathcal{C} \cap \mathcal{T}$ to A. By post-composition with $A \twoheadrightarrow \mathtt{f}A$ we see that f factors through $\mathtt{t}A \hookrightarrow A$ by the kernel property since $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$.

$$\begin{tikzcd} & & X \arrow[dl, dashed, "\exists!"'] \arrow[d] & & \\ 0 \arrow[r] & \mathtt{t}A \arrow[r] & A \arrow[r] & \mathtt{f}A \arrow[r] & 0 \end{tikzcd} .$$

From Lemma 2.30, we know that the last morphism of

$$0 \to \mathtt{dt}A \to \mathtt{ct}A \to \mathtt{t}A \to 0$$

is a right $\mathcal{C}$-approximation of $\mathtt{t}A$, thus f factors further through $\mathtt{ct}A$. Since $\mathcal{T}$ is extension-closed we have $\mathtt{ct}A \in \mathcal{C} \cap \mathcal{T}$, so we have obtained a right $\mathcal{C} \cap \mathcal{T}$-approximation of $\mathcal{A}$.

$$\begin{tikzcd} & & X \arrow[dll, dotted] \arrow[dl, dashed, "\exists!"'] \arrow[d, "f"] & \\ \mathtt{ct}A \arrow[r, two heads] & \mathtt{t}A \arrow[r, hook] & A \arrow[r, two heads] & \mathtt{f}A \end{tikzcd}$$

$\square$

Lemma 3.35. *Let $(\mathcal{C}, \mathcal{D})$ be a complete cotorsion pair in an abelian category $\mathcal{A}$. Then $\mathcal{D}$ is closed under factors if and only if all objects in $\mathcal{C}$ have projective dimension at most one.*

In particular, if $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ is a cotorsion torsion triple, then all objects in $\mathcal{C}$ have projective dimension at most one.

Proof. We start by assuming that $\mathcal{D}$ is closed under factors. Let us consider a 2-extension of an object C in $\mathcal{C}$ to any object $A \in \mathcal{A}$,

$$0 \to A \xrightarrow{m} E \xrightarrow{f} F \xrightarrow{e} C \to 0.$$

Using that $(\mathcal{C}, \mathcal{D})$ is a cotorsion pair, we find the exact sequence

$$0 \to E \to \tilde{\mathtt{d}}E \to \tilde{\mathtt{c}}E \to 0,$$

where the monomorphism $E \hookrightarrow \tilde{\mathtt{d}}E$ is of particular interest. By constructing the push-out along this monomorphism and $f\colon E \to F$, we find the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "m"] \arrow[d, equal] & E \arrow[r, "f"] \arrow[d, hook] & F \arrow[r, "e"] \arrow[d, hook] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "m'"] & \tilde{\mathtt{d}}E \arrow[r, "f'"] & F' \arrow[r, "e'"] \arrow[ul, phantom, "\ulcorner", very near start] & C \arrow[r] & 0
\end{tikzcd}$$

From the diagram above, we can use the exactness of the bottom row to extract the short exact sequence

$$0 \to \mathrm{Im}(f') \to F' \to C \to 0$$

$\mathcal{D}$ is closed under factors, so in particular we have that the image of $f'\colon \tilde{\mathtt{d}}E \to F'$ lies in $\mathcal{D}$. Therefore we know that $\mathrm{Ext}^1_{\mathcal{A}}(C, \mathrm{Im}(f)) = 0$, and in particular the short exact sequence above splits. Therefore we can form the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, hook, "m"] \arrow[d, equal] & E \arrow[r, "f"] \arrow[d, hook] & F \arrow[r, two heads, "e"] \arrow[d, hook] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, hook, "m'"] & \tilde{\mathtt{d}}E \arrow[r, "\binom{f}{0}"] & \mathrm{Im}(f) \oplus C \arrow[r, two heads, "(0\ 1)"] & C \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r, equal] \arrow[u] & A \arrow[r, "0"] \arrow[u, "\binom{m'}{0}"] & C \arrow[r, equal] \arrow[u, "\binom{0}{1}"] & C \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

which gives us that our original 2-extension is equivalent to the trivial 2-extension from C to A, and therefore also $\mathrm{Ext}^2_{\mathcal{A}}(C, -) = 0$. That is, the projective dimension of C is at most one.

Now, assume that the projective dimension of C it at most one, or equivalently $\mathrm{Ext}^2_{\mathcal{A}}(C, -) = 0$. Let

$$0 \to A \to D \to B \to 0$$

be any short exact sequence where D is an object of $\mathcal{D}$, then by applying $\mathrm{Hom}_{\mathcal{A}}(C,-)$ we get the exact sequence

$$\mathrm{Ext}^1_{\mathcal{A}}(C,D) \to \mathrm{Ext}^1_{\mathcal{A}}(C,B) \to \mathrm{Ext}^2_{\mathcal{A}}(C,A)$$

where the first and last component is zero by assumption, and thus $\mathrm{Ext}^1_{\mathcal{A}}(C,B) = 0$, i.e. $B \in \mathcal{C}^{\perp_1} = \mathcal{D}$. That is, $\mathcal{D}$ is closed under factors.

In conclusion, if $(\mathcal{C},\mathcal{T},\mathcal{F})$ is a cotorsion torsion pair, we get that since $(\mathcal{T},\mathcal{F})$ is a torsion pair, $\mathcal{T}$ is closed under factors. Thus $\mathcal{C}$ has projective dimension at most one. ☐

Theorem 3.36. *Let $(\mathcal{C},\mathcal{T},\mathcal{F})$ be a cotorsion torsion triple in an abelian category $\mathcal{A}$. Then $\mathtt{c}\circ\pi_{\mathcal{K}}\colon \mathcal{A}\to\mathcal{C}/\mathcal{K}$ and $\mathtt{f}\colon\mathcal{A}\to\mathcal{F}$ induce mutually inverse equivalences*

$$\mathcal{F} \xhookrightarrow{\mathrm{inc}_{\mathcal{F}}} \mathcal{A} \xrightarrow{\pi_{\mathcal{K}}} \mathcal{A}/\mathcal{K} \xrightarrow{\mathtt{c}} \mathcal{C}/\mathcal{K}$$

and

$$\mathcal{C}/\mathcal{K} \xhookrightarrow{\mathrm{inc}_{\mathcal{C}/\mathcal{K}}} \mathcal{A}/\mathcal{K} \to \mathcal{F}$$

such that $F \simeq \mathcal{C}/\mathcal{K}$

Proof. Recall that $\mathtt{f}$ is left adjoint to the inclusion $\mathcal{F}\hookrightarrow\mathcal{A}$, and $\mathtt{c}\colon\mathcal{A}/\mathcal{K}$ is right adjoint to the inclusion $\mathcal{C}/\mathcal{K}\hookrightarrow\mathcal{A}/\mathcal{K}$. Observe also that the functor $\mathtt{f}\colon\mathcal{A}\to\mathcal{F}$ satisfies $\mathtt{f}(\mathcal{T})=0$. Therefore by the universal property of quotient categories we obtain a functor $\overline{\mathtt{f}}\colon\mathcal{A}/\mathcal{K}\to\mathcal{F}$ such that $\mathtt{f}=\overline{\mathtt{f}}\circ\pi_{\mathcal{K}}$.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, hook, shift right] & \mathcal{A} \arrow[l, shift right, "\mathtt{f}"'] \arrow[d, "\pi_{\mathcal{K}}"] \arrow[dr, bend left, "\mathtt{c}\circ\pi_{\mathcal{K}}"] & \\
& \mathcal{A}/\mathcal{K} \arrow[ul, bend left, "\overline{\mathtt{f}}"] \arrow[r, shift left, "\mathtt{c}"] & \mathcal{C}/\mathcal{K} \arrow[l, hook, shift left]
\end{tikzcd}$$

Denote the restriction $\mathcal{C}/\mathcal{K}\hookrightarrow\mathcal{A}/\mathcal{K}\xrightarrow{\overline{\mathtt{f}}}\mathcal{F}$, by $\overline{\mathtt{f}}|_{\mathcal{C}}$, and the restriction $\mathcal{F}\hookrightarrow\mathcal{A}\xrightarrow{\mathtt{c}\circ\pi_{\mathcal{K}}}\mathcal{C}/\mathcal{K}$ by $\mathtt{c}|_{\mathcal{F}}$. We want to show that $\overline{\mathtt{f}}|_{\mathcal{C}}$ and $\mathtt{c}|_{\mathcal{F}}$ are quasi-inverses.

Let us start with a morphism $\phi\colon A \to B$ in $\mathcal{F}$, then by the same argument as in the proof of 3.27, we find the commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathtt{d}A \arrow[r] \arrow[d, dashed, "\mathtt{d}\phi"] & \mathtt{c}A \arrow[r] \arrow[d, dashed, "\mathtt{c}\phi"] & A \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & \mathtt{d}B \arrow[r] & \mathtt{c}B \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

where $\mathtt{c}\phi$ is uniquely determined in $\mathcal{C}/\mathcal{K}$. Now, by applying $\mathtt{f}$ we obtain the exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \overbrace{\mathtt{fd}A}^{=0} \arrow[r] \arrow[d, "\mathtt{fd}\phi"] & \mathtt{fc}A \arrow[r] \arrow[d, "\mathtt{fc}\phi"] & \mathtt{f}A \arrow[r] \arrow[d, "\mathtt{f}\phi"] & 0 \\ 0 \arrow[r] & \underbrace{\mathtt{fd}B}_{=0} \arrow[r] & \mathtt{fc}B \arrow[r] & \mathtt{f}B \arrow[r] & 0 \end{tikzcd}$$

and by using that $\mathtt{f}$ acts as identity on all objects of $\mathcal{F}$, we conclude $\overline{\mathtt{f}|_{\mathcal{C}}} \circ \mathtt{c}|_{\mathcal{F}} \cong \mathrm{id}_{\mathcal{F}}$.

Now let, $\overline{\phi}\colon C \to C'$ be any morphism in $\mathcal{C}/\mathcal{K}$ and let $\phi\colon C \to C'$ be a preimage of $\overline{\phi}$ in $\mathcal{C}$. This preimage gives rise to the following exact commutative diagram in $\mathcal{A}$

$$\begin{tikzcd} 0 \arrow[r] & \mathtt{t}C \arrow[r] \arrow[d, "\mathtt{t}\phi"] & C \arrow[r] \arrow[d, "\phi"] & \mathtt{f}C \arrow[r] \arrow[d, "\mathtt{f}\phi"] & 0 \\ 0 \arrow[r] & \mathtt{t}C' \arrow[r] & C' \arrow[r] & \mathtt{f}C' \arrow[r] & 0 \end{tikzcd}$$

with $\mathtt{t}C, \mathtt{t}C' \in \mathcal{T}$ and $\mathtt{f}C, \mathtt{f}C' \in \mathcal{F}$. Now, observe that ϕ is a lift of $\mathtt{f}\phi$, and we know that the lift of $\mathtt{f}\phi$ is unique in $\mathcal{C}/\mathcal{K}$, thus

$$\overline{\phi} = \mathtt{c} \circ \pi_{\mathcal{K}} \circ \mathtt{f}(\phi) = \mathtt{c}|_{\mathcal{F}} \circ \overline{\mathtt{f}}|_{\mathcal{C}}(\overline{\phi})$$

giving $\mathtt{c}|_{\mathcal{F}} \circ \overline{\mathtt{f}}|_{\mathcal{C}} = \mathrm{id}_{C/\mathcal{K}}$. We conclude that

$$F \simeq \mathcal{C}/\mathcal{K}$$

□

By the duality remarked earlier, we also obtain the dual result of this theorem.

Theorem 3.37. *Let $(\mathcal{T}, \mathcal{F}, \mathcal{D})$ be a torsion cotorsion triple in an abelian category $\mathcal{A}$. Then $\mathtt{d} \circ \pi_{\mathcal{K}}\colon \mathcal{A} \to \mathcal{D}/\mathcal{K}$ and $\mathtt{t}\colon \mathcal{A} \to \mathcal{T}$ induce mutually inverse equivalences such that $\mathcal{T} \simeq \mathcal{D}/\mathcal{K}$*

3.4 Tilting

Definition 3.38. Let $\mathcal{A}$ be an abelian category with enough projectives. An additively closed full subcategory $\mathbb{T}$ of $\mathcal{A}$, i.e. $\mathbb{T} = \operatorname{add}(\mathbb{T})$, is a *weak tilting* subcategory if

(i) it is self-orthogonal, i.e. $\operatorname{Ext}^1_{\mathcal{A}}(T_1, T_2) = 0$ for all $T_1, T_2 \in \mathbb{T}$,

(ii) any object $T \in \mathbb{T}$ has projective dimension at most 1, that is, it appears in a short exact sequence

$$0 \to P_1 \to P_0 \to T \to 0$$

with P_i projective in $\mathcal{A}$, and

(iii) for any P projective in $\mathcal{A}$, there is a short exact sequence

$$0 \to P \to T_0 \to T_1 \to 0$$

with $T_i \in \mathbb{T}$.

A weak tilting subcategory $\mathbb{T} \subseteq \mathcal{A}$ is *tilting* provided $\mathbb{T}$ is contravariantly finite in $\mathcal{A}$.

Example 3.39. Let $\mathcal{A}$ be an abelian category with enough projectives, then $\operatorname{Proj}(\mathcal{A})$ is a tilting subcategory of $\mathcal{A}$. ♣

Example 3.40. Let $\mathbf{k}$ be a field. We will work in the abelian category of $\operatorname{mod}(\mathbf{k})$ valued representations of the quiver Q with commutativity relations

$$\begin{tikzcd}
\underset{1}{\bullet} \arrow[r] & \underset{4}{\bullet} \\
\underset{2}{\bullet} \arrow[u] \arrow[r] \arrow[d] \arrow[ur, dash, dotted] \arrow[dr, dash, dotted] & \underset{5}{\bullet} \arrow[u] \arrow[d] \\
\underset{3}{\bullet} \arrow[r] & \underset{6}{\bullet}
\end{tikzcd}$$

We claim that $\mathbb{T} = \mathrm{add}\{T_1, T_2, T_3, T_4, T_5, T_6\} \subseteq \mathrm{rep}(Q, \mathbf{k})$ where

$$T_1 = \begin{matrix} \mathbf{k} \to 0 \\ \uparrow \quad \uparrow \\ 0 \to 0 \\ \downarrow \quad \downarrow \\ 0 \to 0 \end{matrix},\ T_2 = \begin{matrix} \mathbf{k} \to 0 \\ \uparrow \quad \uparrow \\ \mathbf{k} \to 0 \\ \downarrow \quad \downarrow \\ \mathbf{k} \to 0 \end{matrix},\ T_3 = \begin{matrix} 0 \to 0 \\ \uparrow \quad \uparrow \\ 0 \to 0 \\ \downarrow \quad \downarrow \\ \mathbf{k} \to 0 \end{matrix},\ T_4 = \begin{matrix} \mathbf{k} \to \mathbf{k} \\ \uparrow \quad \uparrow \\ 0 \to 0 \\ \downarrow \quad \downarrow \\ 0 \to 0 \end{matrix},\ T_5 = \begin{matrix} \mathbf{k} \to \mathbf{k} \\ \uparrow \quad \uparrow \\ \mathbf{k} \to \mathbf{k} \\ \downarrow \quad \downarrow \\ \mathbf{k} \to \mathbf{k} \end{matrix},\ T_6 = \begin{matrix} 0 \to 0 \\ \uparrow \quad \uparrow \\ 0 \to 0 \\ \downarrow \quad \downarrow \\ \mathbf{k} \to \mathbf{k} \end{matrix}$$

is a weakly tilting subcategory of $\mathrm{rep}(Q, \mathbf{k})$. It is easily verified that $\mathbb{T}$ is self-orthogonal. It is in fact sufficient to check that $\mathrm{Ext}^1_{\mathcal{A}}(T_i, T_j) = 0$ for $1 \leq i, j \leq 6$. We can observe that for

$$(i, j) \in \{(1,3), (3,1), (1,6), (6,1), (3,4), (4,3), (4,6), (6,4)\},$$

T_i and T_j have non-zero components in non-adjacent positions, forcing

$$\mathrm{Ext}^1_{\mathcal{A}}(T_i, T_j) = 0.$$

Note also that T_5 is projective, thus

$$\mathrm{Ext}^1_{\mathcal{A}}(T_5, -) = 0.$$

We are left with checking $\mathrm{Ext}^1_{\mathcal{A}}(T_i, T_j) = 0$ for

$$(i, j) \in \left\{ \begin{matrix} (1,1), (1,2), (1,4), (1,5), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,2), \\ (3,3), (3,5), (3,6), (4,1), (4,2), (4,4), (4,5), (6,2), (6,3), (6,5), (6,6) \end{matrix} \right\},$$

which by symmetry reduces to checking for

$$\{(1,1), (1,2), (1,4), (1,5), (2,1), (2,2), (2,4), (2,5), (4,1), (4,2), (4,4), (4,5)\}.$$

We check explicitly for $(1,2)$, the rest are left as an exercise. Consider therefore the short exact sequence

$$0 \to T_2 \to E \to T_1 \to 0$$

which reduces to the exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{k} \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d] & \mathbf{k} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbf{k} \arrow[r, equal] & E_2 \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

which clearly splits.

Now, let us verify that all projective elements admit a coresolution of objects from $\mathbb{T}$. It will be shown in Remark 3.46 that it is enough to verify the existence of such coresolutions for a subclass of projective objects which additively generates the rest. That is, it is enough to check the existence for the indecomposable projectives.

$$P_1 = \begin{smallmatrix} \mathbf{k} & \to & \mathbf{k} \\ \uparrow & & \uparrow \\ 0 & \to & 0 \\ \downarrow & & \downarrow \\ 0 & \to & 0 \end{smallmatrix},\ P_2 = \begin{smallmatrix} \mathbf{k} & \to & \mathbf{k} \\ \uparrow & & \uparrow \\ \mathbf{k} & \to & \mathbf{k} \\ \downarrow & & \downarrow \\ \mathbf{k} & \to & \mathbf{k} \end{smallmatrix},\ P_3 = \begin{smallmatrix} 0 & \to & 0 \\ \uparrow & & \uparrow \\ 0 & \to & 0 \\ \downarrow & & \downarrow \\ \mathbf{k} & \to & \mathbf{k} \end{smallmatrix},\ P_4 = \begin{smallmatrix} 0 & \to & \mathbf{k} \\ \uparrow & & \uparrow \\ 0 & \to & 0 \\ \downarrow & & \downarrow \\ 0 & \to & 0 \end{smallmatrix},\ P_5 = \begin{smallmatrix} 0 & \to & \mathbf{k} \\ \uparrow & & \uparrow \\ 0 & \to & \mathbf{k} \\ \downarrow & & \downarrow \\ 0 & \to & \mathbf{k} \end{smallmatrix},\ P_6 = \begin{smallmatrix} 0 & \to & 0 \\ \uparrow & & \uparrow \\ 0 & \to & 0 \\ \downarrow & & \downarrow \\ 0 & \to & \mathbf{k} \end{smallmatrix}$$

We obtain the following coresolutions

$$0 \to P_1 \to T_4 \to 0$$
$$0 \to P_2 \to T_5 \to 0$$
$$0 \to P_3 \to T_6 \to 0$$
$$0 \to P_4 \to T_4 \to T_1 \to 0$$
$$0 \to P_5 \to T_5 \to T_2 \to 0$$
$$0 \to P_6 \to T_6 \to T_6 \to 0$$

where the three first doubles as projective resolutions of T_4, T_5 and T_6. To verify that $\mathrm{pd}\mathbb{T} \leq 1$ we observe that we have the following projective resolutions

$$0 \to P_4 \to P_1 \to T_1 \to 0$$
$$0 \to P_5 \to P_2 \to T_2 \to 0$$
$$0 \to P_6 \to P_3 \to T_3 \to 0$$
$$0 \to P_1 \to T_4 \to 0$$
$$0 \to P_2 \to T_5 \to 0$$
$$0 \to P_3 \to T_6 \to 0$$

which by the same proceeding result as noted above, is enough to guarantee $\mathrm{pd}T \leq 1$ for all $T \in \mathbb{T}$.

Observe that the weakly tilting subcategory $\mathbb{T} = \mathrm{add}\{T_1, T_2, T_3, T_4, T_5, T_6\}$ admits an object which additively generates the whole subcategory, i.e.

$$\mathrm{add}\left(\bigoplus_{i=1}^{6} T_i\right) = \mathbb{T}.$$

This is an example of what we shortly will be calling a tilting object. ♣

3.4.1 Tilting Objects

When we are traditionally talking of tilting we are thinking of a module with trivial extensions, projective dimension at most 1 and which additive closure admits a coresolution for every projective object. Even though our definition seems to be less limiting we will see that our notion of tilting and the traditional notion coincide in particularly well-behaved categories. Examples of such include categories of finitely generated modules over an artin ring.

Definition 3.41. Let $\mathcal{A}$ be an abelian category with enough projectives. An object $T \in \mathrm{mod}(A)$ is called a *tilting object* if

(i) $\mathrm{Ext}^1_A(T, T) = 0$

(ii) $\mathrm{pd}(T) \leq 1$

(iii) For each projective object $P \in \mathrm{Proj}\mathcal{A}$ there exists a short exact sequence

$$0 \to P \to T' \to T'' \to 0$$

where T' and T'' lie in $\mathrm{add}(T)$.

Proposition 3.42. *Let $\mathcal{A}$ be an abelian category with enough projectives, such that* $\mathrm{Proj}(\mathcal{A}) = \mathrm{add}(P)$ *for some projective object $P \in \mathcal{A}$. Then, if $\mathbb{T}$ is a tilting subcategory of $\mathcal{A}$, we can find a tilting object $T \in \mathbb{T}$ such that* $\mathbb{T} = \mathrm{add}(T)$.

Proof. We will prove that $\mathbb{T} = \text{add}(T_1 \oplus T_2)$ where T_1, T_2 are objects of $\mathbb{T}$ such that we have the short exact sequence

$$0 \to P \to T_1 \to T_2 \to 0$$

Let us first prove that any projective object $P' \in \text{Proj}(\mathcal{A}) = \text{add}(P)$ admits a short exact sequence

$$0 \to P' \to T_1' \to T_2' \to 0$$

where $T_1', T_2' \in \text{add}(T_1 \oplus T_2)$. By assumption we have for some projective object $Q \in \text{add}(P)$ and positive integer $n > 0$, that $P' \oplus Q = P^n$, and therefore also a short exact sequence

$$0 \to P' \oplus Q \xrightarrow{(f\ g)} T_1^n \to T_2^n \to 0$$

We can now form the exact commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & T_1^n \arrow[r, equal] \arrow[d, "\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)"] & T_1^n \arrow[d] & \\
0 \arrow[r] & P' \oplus Q \arrow[r, "\left(\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right)"] \arrow[d, equal] & T_1^n \oplus T_1^n \arrow[r] \arrow[d, "(1\ 1)"] & \text{Cok}(f) \oplus \text{Cok}(g) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P' \oplus Q \arrow[r, "(f\ g)"] & T_1^n \arrow[r] \arrow[d] & T_2^n \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where the right column splits since $\text{Ext}^1_{\mathcal{A}}(T_2^n, T_1^n) = 0$, thus both $\text{Cok}(f)$ and $\text{Cok}(g)$ lie in $\text{add}(T_1 \oplus T_2)$. We extract the short exact sequence

$$0 \to P' \to T_1^n \to \text{Cok}(f) \to 0$$

which is on the wanted form.

We will now show that any object $\overline{T} \in \mathbb{T}$ of the tilting subcategory lies in $\text{add}(T_1 \oplus T_2)$. We start by fixing a projective resolution of $\overline{T}$,

$$0 \to P_2 \xrightarrow{m} P_1 \xrightarrow{e} \overline{T} \to 0$$

and for $P_i \in \operatorname{Proj}(\mathcal{A})$ we find the short exact sequence

$$0 \to P_i \xrightarrow{\phi_i} T_i' \xrightarrow{\psi_i} T_i'' \to 0$$

with $T_i', T_i'' \in \operatorname{add}(T_1 \oplus T_2)$ for $i = 1, 2$. These two sequences gives us the epimorphisms

$$\operatorname{Hom}_{\mathcal{A}}(\phi_i, -)|_{\mathbb{T}} \colon \operatorname{Hom}_{\mathcal{A}}(T_i', -)|_{\mathbb{T}} \twoheadrightarrow \operatorname{Hom}_{\mathcal{A}}(P_i, -)|_{\mathbb{T}}$$

since $\operatorname{Ext}^1_{\mathcal{A}}(T_i'', -)|_{\mathbb{T}} = 0$. Equivalently, ϕ_i is a left $\mathbb{T}$-approximation of P_i. We can therefore construct the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & P_2 \arrow[r, "a"] \arrow[d, "c"] & P_1 \arrow[r, "b"] \arrow[d, "d"] & \overline{T} \arrow[r] \arrow[d, equal] & 0 \\
 & T_2' \arrow[r, dashed, "e"] \arrow[d, "g"] & T_1' \arrow[r, dashed, "f"] \arrow[d, "h"] & \overline{T} & \\
 & T_2'' \arrow[r, dotted, "i"] \arrow[urr, dotted, "j"] \arrow[d] & T_1'' \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

where the upper row and the two columns are exact. The dashed morphisms exists because of the approximation property remarked above. Note that the second row do not necessarily compose to zero, but that by commutativity we have $f \circ e \circ c = b \circ a = 0$. Hence, the cokernel property of c give rise to the dotted morphisms.

To finish our proof it is enough to show that the sequence

$$0 \to T_2' \xrightarrow{\left(\begin{smallmatrix} e \\ -g \end{smallmatrix}\right)} T_1' \oplus T_2'' \xrightarrow{\left(\begin{smallmatrix} f & j \\ h & i \end{smallmatrix}\right)} T_1'' \oplus \overline{T} \to 0$$

is exact, since it then would split and give $\overline{T} \in \operatorname{add}(T_1' \oplus T_2'') \subseteq \operatorname{add}(T_1 \oplus T_2)$. We split the proof of exactness into the three following lemmas. $\square$

Lemma 3.43. $\left(\begin{smallmatrix} e \\ -g \end{smallmatrix}\right)$ *is a monomorphism.*

Proof. Let $K \overset{k}{\hookrightarrow} T_2'$ be the kernel of $\left(\begin{smallmatrix} e \\ -g \end{smallmatrix}\right)$. Specifically, we get that $g \circ k = 0 = e \circ k$, and thus k factors through c. By commutivity of the diagram above we have now that $0 = e \circ k$ factors through $e \circ c = d \circ a$ where $d \circ a$ is mono, so k must further factor through 0. Thus $K = 0$ and the map is mono.

$$\begin{tikzcd}
0 \arrow[r] & P_2 \arrow[r, hook, "a"] \arrow[d, hook, "c"] & P_1 \arrow[d, hook, "d"] \\
K \arrow[u, dotted] \arrow[ur, dashed] \arrow[r, hook, "k"] & T_2' \arrow[r, "e"] \arrow[d, two heads, "-g"] & T_1' \arrow[d, hook] \\
& T_2'' \arrow[r, hook] & T_2'' \oplus T_1'
\end{tikzcd}$$

□

To ease the complexity of the next two lemmas we will prove them by use of elements and diagram chasing.

Lemma 3.44. $\left(\begin{smallmatrix} f & j \\ h & i \end{smallmatrix}\right)$ *is an epimorphism*

Proof. We fix some element $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in T_1'' \oplus \overline{T}$. h is an epimorphism, so we find some preimage $z \in T_1'$ of y along h. Also b is an epimorphism, so we can find a preimage $v \in P_1$ of $f(z) - x$ along b. The element $\left(\begin{smallmatrix} d(v) \\ 0 \end{smallmatrix}\right)$ lie in $T_1' \oplus T_2''$ and

$$\begin{pmatrix} f & j \\ h & i \end{pmatrix} \begin{pmatrix} z - d(v) \\ 0 \end{pmatrix} = \begin{pmatrix} f(z) - f(d(v)) \\ h(z) - h(d(v)) \end{pmatrix} = \begin{pmatrix} f(z) - f(z) + x \\ y - 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

We conclude that the morphism is in fact an epimorphism. □

Lemma 3.45. $\mathrm{Im}\left(\begin{smallmatrix} e \\ -g \end{smallmatrix}\right) = \mathrm{Ker}\left(\begin{smallmatrix} f & j \\ h & i \end{smallmatrix}\right)$.

Proof. We observe at once that by commutivity we get

$$\left(\begin{smallmatrix} f & j \\ h & i \end{smallmatrix}\right)\left(\begin{smallmatrix} e \\ -g \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$$

giving us the inclusion $\subseteq$. For the converse inclusion, let us fix some element $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathrm{Ker}\left(\begin{smallmatrix} f & j \\ h & i \end{smallmatrix}\right)$. We start by lifting $y \in T_2''$ along the epimorphism g to some element $z \in T_2'$, thus obtaining

$$0 = f(x) + j(y) = f(x) + j(g(z) = f(x + e(z))$$
$$0 = h(x) + i(y) = h(x) + i(g(z)) = h(x + e(z))$$

$x + e(z)$ lie in the kernel of h, so in turn it lie in the image of d. Let therefore $v \in P_1$ be such that $d(v) = x + e(z)$. Further, $0 = f(d(v)) = b(v)$ so v lie in the kernel of b and therefore also in the image of a. Hence we find $w \in P_2$ such that $a(w) = v$. Observe that the object $c(w) - z \in T_2''$ is such that

$$\begin{pmatrix} e \\ -g \end{pmatrix} (c(w) - z) = \begin{pmatrix} e(c(w)) - e(z) \\ -g(c(w)) + g(z) \end{pmatrix} = \begin{pmatrix} d(a(w)) - e(z) \\ -0 + g(z) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

This establishes the converse inclusion, which concludes the proof. □

Remark 3.46. In the definition of a tilting subcategory we can relax the requirements a bit. We will be using a similar argument as those presented in the start of the proof of Proposition 3.42, to show that we only need to show that (iii) holds for the objects in a subcategory $\mathbb{P} \subseteq \mathrm{Proj}\mathcal{A}$ which additively generates $\mathrm{Proj}\mathcal{A}$, i.e. $\mathrm{add}\mathbb{P} = \mathrm{Proj}\mathcal{A}$.

Observe that if we have $Q, P \in \mathbb{P}$, with short exact sequences

$$0 \to Q \to T_0 \to T_1 \to 0$$

and

$$0 \to P \to T_0' \to T_1' \to 0$$

such that $T_i, T_i' \in \mathbb{T}$, then $T_i \oplus T_i'$ lie in $\mathbb{T}$. Thus we get our desired short exact sequence

$$0 \to P \oplus Q \to T_0 \oplus T_0' \to T_1 \oplus T_1' \to 0,$$

and all direct sums of objects in $\mathbb{P}$ have a two-term $\mathbb{T}$-coresolution. The same holds true for summands, in fact, assume $P \oplus Q$ fits into the short exact sequence

$$0 \to P \oplus Q \xrightarrow{(f\ g)} T_0 \to T_1 \to 0$$

with $T_i \in \mathbb{T}$. Then we have the commutative exact diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & T_0 \arrow[r, equal] \arrow[d, "\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)"] & T_0 \arrow[d] & \\
0 \arrow[r] & P' \oplus Q \arrow[r, "\left(\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right)"] \arrow[d, equal] & T_0 \oplus T_0 \arrow[r] \arrow[d, "(1\ 1)"] & \mathrm{Cok}(f) \oplus \mathrm{Cok}(g) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P' \oplus Q \arrow[r, "(f\ g)"] & T_0 \arrow[r] \arrow[d] & T_1 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

$\mathrm{Ext}^1_{\mathcal{A}}(T_0, T_1) = 0$, so we have that the short exact sequence in the right hand column splits, and therefore we see that both $\mathrm{Cok}(f)$ and $\mathrm{Cok}(g)$ lies in $\mathbb{T}$.

We can also observe that the second requirement (ii) only needs to be verified for a subcategory $\mathbb{T}' \subseteq \mathbb{T}$ which additively generates the rest.

Proposition 3.47. *Let $\mathcal{A}$ be a noetherian abelian category with enough projectives and assume that $\mathbb{T}$ is a weak tilting subcategory of $\mathcal{A}$. Then $\mathbb{T}$ is automatically contraviariantly finite, and therefore already tilting.*

Proof. By the equivalent definition of contravariantly finite, we will show that for any object $X \in \mathcal{A}$, the functor

$$\mathrm{Hom}_{\mathcal{A}}(-, X)|_{\mathbb{T}} \colon \mathbb{T}^{\mathrm{op}} \to \mathbf{Ab}$$

is finitely generated. Let us start by showing that any object $X \in \mathcal{A}$ has a unique maximal subobject $\mathtt{t}X \hookrightarrow X$ with the property that it is an epimorphic image of an object in $\mathbb{T}$. The existence of such maximal objects are guaranteed by $\mathcal{A}$ being noetherian, thus we only need to check uniqueness. Suppose we have two maximal subobjects Y_1, Y_2 of X such that there exist epimorphisms $T_i \twoheadrightarrow Y_i$ with $T_i \in \mathbb{T}$. The image of the induced morphism $Y_1 \oplus Y_2 \to X$ contains both Y_1 and Y_2, and admits an epimorphism from an object in $\mathbb{T}$ by precomposition with

$T_1 \oplus T_2 \twoheadrightarrow Y_1 \oplus Y_2$. This contradicts the maximality of Y_1 and Y_2, and we conclude that there is a unique maximal subobject $\mathtt{t}X$ of X with the given properties.

Let $f\colon T \to M$ be any morphism from an object $X \in \mathbb{T}$. This morphism factorizes uniquely through the subobject $\mathrm{Im}(f) \subseteq M$, and by the maximality of $\mathtt{t}M$ we can observe that the inclusion $\mathrm{Im}(f) \hookrightarrow M$ also factors uniquely through $\mathtt{t}M$.

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, two heads] & & M \\
& \mathrm{Im}(f) \arrow[ur, hook] \arrow[dr, hook] & \\
& & \mathtt{t}M \arrow[uu, hook]
\end{tikzcd}$$

This factorization gives that $\mathrm{Hom}_{\mathcal{A}}(-,M)|_{\mathbb{T}} \cong \mathrm{Hom}_{\mathcal{A}}(-,\mathtt{t}M)|_{\mathbb{T}}$, since we have for every $X \in \mathbb{T}$ the natural bijections where $f\colon X \to M$ is sent to the composition $X \twoheadrightarrow \mathrm{Im}(f) \hookrightarrow \mathtt{t}M$, and $g\colon X \to \mathtt{t}M$ is sent to the composition $X \to \mathtt{t}M \hookrightarrow M$.

The given maximal epimorphism $T \overset{\phi}{\twoheadrightarrow} \mathtt{t}M$ gives us the short exact sequence

$$0 \to \mathrm{Ker}(\phi) \to T \xrightarrow{\phi} \mathtt{t}M \to 0$$

which in turn gives rise to the exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(-,T)|_{\mathbb{T}} \xrightarrow{\phi_*} \mathrm{Hom}_{\mathcal{A}}(-,\mathtt{t}M)|_{\mathbb{T}} \to \mathrm{Ext}^1_{\mathcal{A}}(-,\mathrm{Ker}(\phi))|_{\mathbb{T}} \to \mathrm{Ext}^1_{\mathcal{A}}(-,T)|_{\mathbb{T}}$$

of $\mathbb{T}$-modules. The last term is zero by assumption, so this gives us a short exact sequence

$$0 \to \mathrm{Im}(\phi_*) \to \mathrm{Hom}_{\mathcal{A}}(-,M)|_{\mathbb{T}} \to \mathrm{Ext}^1_{\mathcal{A}}(-,\mathrm{Ker}(\phi))|_{\mathbb{T}} \to 0$$

where we have used that $\mathrm{Hom}_{\mathcal{A}}(-,M) \cong \mathrm{Hom}_{\mathcal{A}}(-,\mathtt{t}M)$. Now, from the definition we have the epimorphism $\mathrm{Hom}_{\mathcal{A}}(-,T)|_{\mathbb{T}} \twoheadrightarrow \mathrm{Im}(\phi_*)$, or equivalently that $\mathrm{Im}(\phi_*)$ is finitely generated as a $\mathbb{T}$-module. Hence, by the Horseshoe lemma we only need to check that $\mathrm{Ext}^1_{\mathcal{A}}(-,\mathrm{Ker}(\phi))|_{\mathbb{T}}$ is finitely generated for $\mathrm{Hom}_{\mathcal{A}}(-,\mathtt{t}M)|_{\mathbb{T}}$ to be so as well.

$\mathcal{A}$ has enough projectives, so we can find a short exact sequence $K' \hookrightarrow P \twoheadrightarrow \mathrm{Ker}(\phi)$ with P projective. This gives rise to a long exact sequence, from which we extract the exact sequence

$$\mathrm{Ext}^1_{\mathcal{A}}(-,P) \to \mathrm{Ext}^1_{\mathcal{A}}(-,\mathrm{Ker}(\phi)) \to \mathrm{Ext}^2_{\mathcal{A}}(-,K')$$

The last term of this sequence vanishes when restricting to $\mathbb{T}$, since any weakly tilted subcategory has projective dimension at most 1, giving an epimorphism $\mathrm{Ext}^1_{\mathcal{A}}(-,P) \twoheadrightarrow \mathrm{Ext}^1_{\mathcal{A}}(-,\mathrm{Ker}(\phi))$. We now construct the exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(-,T^1)|_{\mathbb{T}} \to \mathrm{Ext}^1_{\mathcal{A}}(-,P)|_{\mathbb{T}} \to \mathrm{Ext}^1_{\mathcal{A}}(-,T^0)|_{\mathbb{T}}$$

by applying the left exact hom-functor to the coresolution $P \hookrightarrow T^0 \twoheadrightarrow T^1$, of P in $\mathbb{T}$. $\mathbb{T}$ is self-orthogonal so the last term of this sequence vanishes, giving us an epimorphism $\mathrm{Hom}_{\mathcal{A}}(-,T^1)|_{\mathbb{T}} \twoheadrightarrow \mathrm{Ext}^1_{\mathcal{A}}(-,P)|_{\mathbb{T}}$. Now, by postcomposing this with the previous epimorphism we have that $\mathrm{Ext}^1_{\mathcal{A}}(-,\mathrm{Ker}(\phi))|_{\mathbb{T}}$ is finitely generated, finishing our proof. □

We can now conclude that the weakly tilting in Example 3.40 was in fact an example of a tilting subcategory.

Lemma 3.48. *Let T be a tilting object in an abelian category $\mathcal{A}$ with enough projectives. The additively closed subcategory $\mathbb{T} = \mathrm{add}(T)$ is a weak tilting subcategory. If $\mathcal{A}$ is also noetherian, then $\mathbb{T}$ is tilting.*

Proof. $\{T\}$ is a self-orthogonal subcategory of $\mathcal{A}$, so by Lemma 2.11 we get that $\mathrm{add}(T)$ is also self-orthogonal. The last two conditions for being weakly tilting follows from Remark 3.46. If $\mathcal{A}$ is further a noetherian category, then by Proposition 3.47 we get that $\mathbb{T} = \mathrm{add}(T)$ is contravariantly finite, thus tilting. □

Remark 3.49. We can now conclude that the notion of tilting subcategory presented here coincides with the traditional notion of tilting in the category of finitely generated modules over an artinian ring.

3.4.2 Properties of Tilting

In the classical setting, a tilting module T, give rise to a torsion class $\mathcal{T}$ epimorphically generated by direct sums of T, i.e. $\mathcal{T} = \mathrm{Fac}\{T\}$ [1, Lemma VI.2.3]. We will now show that our notion of tilting subcategories recaptures this property. In fact, we will see that for a tilting subcategory $\mathbb{T}$, $\mathrm{Fac}(\mathbb{T})$ is not only a torsion class, but also a cotorsion-free class of a complete cotorsion pair. That, is any tilting subcategory give rise to a cotorsion torsion triple. Further, we establish that it is in fact a correspondence between cotorsion torsion triples of an abelian category and tilting subcategories.

Remark 3.50. The factor category $\mathrm{Fac}\,\mathbb{T}$ for a weakly tilting $\mathbb{T} \subseteq \mathcal{A}$ consists of all objects $X \in \mathcal{A}$ such that there exist an epimorphism $T \twoheadrightarrow X$ from an object $T \in \mathbb{T}$, since $\mathbb{T}$ is closed under finite direct sums and summands.

Lemma 3.51. *Let $\mathbb{T} \subseteq \mathcal{A}$ be a weakly tilting subcategory of an abelian category $\mathcal{A}$ with enough projectives. Then $\mathrm{Fac}\,\mathbb{T} = \mathbb{T}^{\perp_1}$.*

Proof. Assume $X \in \mathrm{Fac}\,\mathbb{T}$, or equivalently that we have a short exact sequence

$$0 \to K \to T \to X \to 0$$

with $T \in \mathbb{T}$. The subcategory $\mathbb{T}$ has projective dimension at most 1, so $\mathrm{Ext}^2_{\mathcal{A}}(\mathbb{T}, K) = 0$, thus we obtain the exact sequence

$$\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{T}, K) \to \mathrm{Ext}^1_{\mathcal{A}}(\mathbb{T}, T) \to \mathrm{Ext}^1_{\mathcal{A}}(\mathbb{T}, X) \to 0$$

Now, recall that every weakly tilted subcategory is self-orthogonal, so $\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{T}, T) = 0$ which forces the last term to also be zero. We therefore have

$$\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{T}, \mathrm{Fac}\,\mathbb{T}) = 0$$

or in other words $\mathrm{Fac}\,\mathbb{T} \subseteq \mathbb{T}^{\perp_1}$. To show the converse inclusion, let us assume that we have $Y \in \mathbb{T}^{\perp_1}$ that is $\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{T}, Y) = 0$. $\mathcal{A}$ has enough projectives, so we pick an epimorphism $P \twoheadrightarrow Y$ from a projective object $P \in \mathrm{Proj}\mathcal{A}$. From the third property of a weakly tilting we can now construct the following push-out diagram

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & T' \arrow[r] \arrow[d, two heads] & T'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r] & X \arrow[r] & T'' \arrow[r] & 0 \end{tikzcd}$$

with $T', T'' \in \mathbb{T}$. Observe that by construction we have $X \in \operatorname{Fac}\mathbb{T}$. Further, since $Y \in \mathbb{T}^{\perp_1}$, we have that the lower row splits, so we obtain an epimorphism

$$T' \twoheadrightarrow X \twoheadrightarrow Y$$

by composition, thus $Y \in \operatorname{Fac}\mathbb{T}$. $\square$

Proposition 3.52. *Let $\mathcal{A}$ be an abelian category with enough projectives. If $\mathbb{T} \subseteq \mathcal{A}$ is a tilting subcategory, then*

$$(\operatorname{Fac}\mathbb{T}, \mathbb{T}^{\perp})$$

is a torsion pair.

Proof. Let X be any object of $\operatorname{Fac}\mathbb{T}$, then we have an epimorphism $T \twoheadrightarrow X$ for some $T \in \mathbb{T}$. Now, for any $Y \in \mathbb{T}^{\perp} = \{A \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(\mathbb{T}, A) = 0\}$, we get a monomorphism

$$0 \to \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(T, Y) = 0$$

hence $Y \in (\operatorname{Fac}\mathbb{T})^{\perp}$, that is $\mathbb{T}^{\perp} \subseteq (\operatorname{Fac}\mathbb{T})^{\perp}$. The converse inclusion is easily seen from the fact that $\mathbb{T}$ is a subcategory of $\operatorname{Fac}\mathbb{T}$.

To finish the proof of our claim, we need to find a suitable short exact sequence for any object $A \in \mathcal{A}$. We start by finding a right $\mathbb{T}$-approximation $\phi\colon T \to A$ for A. This morphism gives us the short exact sequence

$$0 \to \mathrm{Im}(\phi) \to A \to \mathrm{Cok}(\phi) \to 0$$

with $\mathrm{Im}(\phi) \in \operatorname{Fac}\mathbb{T}$, which in turn give rise to a long exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(-, \mathrm{Im}(\phi))|_{\mathbb{T}} \longrightarrow \mathrm{Hom}_{\mathcal{A}}(-, A)|_{\mathbb{T}} \longrightarrow \cdots$$

The first morphism of this sequence acts as a part of the factorization

$$\begin{tikzcd}[column sep=small] \operatorname{Hom}_{\mathcal{A}}(-,T)|_{\mathbb{T}} \arrow[rr, two heads] \arrow[dr] & & \operatorname{Hom}_{\mathcal{A}}(-,A)|_{\mathbb{T}} \\ & \operatorname{Hom}_{\mathcal{A}}(-,\operatorname{Im}(\phi))|_{\mathbb{T}} \arrow[ur] & \end{tikzcd}$$

and must therefore be an epimorphism. Hence, the second morphism in the sequence factors through zero, and we get the monomorphism

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{A}}(-,\operatorname{Cok}(\phi))|_{\mathbb{T}} \longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(-,\operatorname{Im}(\phi))|_{\mathbb{T}}$$

Now, by Lemma 3.51 we have that $\operatorname{Im}(\phi) \in \operatorname{Fac}\mathbb{T} = \mathbb{T}^{\perp_1}$, and consequently

$$\operatorname{Hom}_{\mathcal{A}}(-,\operatorname{Cok}(\phi))|_{\mathbb{T}} = 0$$

or equivalently $\operatorname{Cok}(\phi) \in \mathbb{T}^{\perp}$. We conclude that $(\operatorname{Fac}\mathbb{T}, \mathbb{T}^{\perp})$ is in fact a torsion pair. $\square$

Proposition 3.53. *Let $\mathbb{T}$ be a weakly tilting in an abelian category $\mathcal{A}$ with enough projectives. Then*

$$({}^{\perp_1}(\operatorname{Fac}\mathbb{T}), \operatorname{Fac}\mathbb{T})$$

is a complete cotorsion pair, and

$${}^{\perp_1}(\operatorname{Fac}\mathbb{T}) = \{X \in {}^{\perp_1}\mathbb{T} \mid \operatorname{pdim} X \leq 1\}$$

Whenever $\mathbb{T}$ is also contravariantly finite, the intersection of the cotorsion and cotorsion-free part is the whole of $\mathbb{T}$, i.e.

$${}^{\perp_1}(\operatorname{Fac}\mathbb{T}) \cap \operatorname{Fac}\mathbb{T} = \mathbb{T}$$

Proof. The first condition of cotorsion pairs stipulates that the pair has to be right 1-orthogonal and left 1-orthogonal of each other. ${}^{\perp_1}(\operatorname{Fac}\mathbb{T})$ is already left 1-orthogonal of $\operatorname{Fac}\mathbb{T}$, so we are left with showing that

$$(\operatorname{Fac}\mathbb{T})^{\perp_1} = ({}^{\perp_1}(\operatorname{Fac}\mathbb{T}))^{\perp_1}$$

In the first part of the proof for Proposition 3.52 it was shown that $\operatorname{Ext}^1_{\mathcal{A}}(-,X)|_{\mathbb{T}} = 0$ for any $X \in \operatorname{Fac}\mathbb{T}$. That is $\mathbb{T} \subseteq {}^{\perp_1}(\operatorname{Fac}\mathbb{T})$, so we obtain the inclusion

$$\mathbb{T}^{\perp_1} \supseteq ({}^{\perp_1}(\operatorname{Fac}\mathbb{T}))^{\perp_1}$$

For any subcategory $\mathcal{X} \subseteq \mathcal{A}$ we have $\mathcal{X} \subseteq ({}^{\perp_1}\mathcal{X})^{\perp_1}$, so in fact

$$\operatorname{Fac}\mathbb{T} \subseteq ({}^{\perp_1}(\operatorname{Fac}\mathbb{T}))^{\perp_1} \subseteq \mathbb{T}^{\perp_1}$$

Now again by Proposition 3.52 we have the equality $\mathbb{T}^{\perp_1} = \operatorname{Fac}\mathbb{T}$, and thus $\operatorname{Fac}\mathbb{T} = ({}^{\perp_1}(\operatorname{Fac}\mathbb{T}))^{\perp_1}$.

We are now left with showing that the cotorsion pair is complete. $\mathcal{A}$ has enough projectives, so by Salce's lemma (Lemma 3.25) it suffices to show that the pair has enough injectives. Let therefore $P \twoheadrightarrow A$ be an epimorphism from a projective to any object A in $\mathcal{A}$, and $P_0 \hookrightarrow T_0 \twoheadrightarrow T_1$ be a coresulution of P in $\mathbb{T}$. We now construct the push-out along $P \twoheadrightarrow A$ and $P \hookrightarrow T_0$ and obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & T_0 \arrow[r] \arrow[d, two heads] & T_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & X \arrow[r] & T_1 \arrow[r] & 0
\end{tikzcd}$$

X clearly lies in $\operatorname{Fac}\mathbb{T}$ and T_1 lie in $\mathbb{T} \subseteq {}^{\perp_1}(\operatorname{Fac}\mathbb{T})$, so the lower row is our desired short exact sequence. Thus $({}^{\perp_1}\operatorname{Fac}\mathbb{T}, \operatorname{Fac}\mathbb{T})$ has enough injectives.

In order to show the equality

$${}^{\perp_1}(\operatorname{Fac}\mathbb{T}) = \{X \in {}^{\perp_1}\mathbb{T} \mid \operatorname{pdim} X \leq 1\}$$

we start by observing that since $\mathbb{T} \subseteq \operatorname{Fac}\mathbb{T}$ we have the inclusion ${}^{\perp_1}\operatorname{Fac}\mathbb{T} \subseteq {}^{\perp_1}\mathbb{T}$. In addition to being a cotorsion class, we have by Proposition 3.52 that ${}^{\perp_1}(\operatorname{Fac}\mathbb{T})$ is a torsion class and therefore closed under factors. Now by Lemma 3.35, we know that a cotorsion class which is closed under factors has projective dimension less than or equal to 1, hence

$${}^{\perp_1}(\operatorname{Fac}\mathbb{T}) \subseteq \{X \in {}^{\perp_1}\mathbb{T} \mid \operatorname{pdim} X \leq 1\}.$$

For the other inclusion, let $X \in {}^{\perp_1}\mathbb{T}$ have projective dimension at most one. Let F be an object in $\operatorname{Fac}\mathbb{T}$ and choose an epimorphism $\phi\colon T \twoheadrightarrow F$ with $T \in \mathbb{T}$. From this epimorphism we get the exact sequence

$$\operatorname{Ext}^1_{\mathcal{A}}(X,T) \to \operatorname{Ext}^1_{\mathcal{A}}(X,F) \to \operatorname{Ext}^2_{\mathcal{A}}(X,\operatorname{Ker}(\phi))$$

where the outer two terms vanish, thus forcing $\operatorname{Ext}^1_{\mathcal{A}}(X,F) = 0$. Hence X also lies in ${}^{\perp_1}(\operatorname{Fac}\mathbb{T})$ and therefore the other inclusion is established.

Assume now that $\mathbb{T}$ is contravariantly finite and let $X \in {}^{\perp_1}(\operatorname{Fac}\mathbb{T}) \cap \operatorname{Fac}\mathbb{T}$. We choose an epimorphism $T' \twoheadrightarrow X$ for $T \in \mathbb{T}$ and use that $\mathbb{T}$ is contravariantly finite to obtain a right $\mathbb{T}$-approximation $\phi\colon T \twoheadrightarrow X$

$$\begin{tikzcd} & T \arrow[d, two heads, "\phi"] \\ T' \arrow[ur, dashed] \arrow[r, two heads] & X \end{tikzcd}$$

which is necessarily also an epimorphism. From the short exact sequence $\operatorname{Ker}(\phi) \hookrightarrow T \xrightarrow{\phi} X$ we obtain the exact sequence

$$\operatorname{Hom}_{\mathcal{A}}(-,T) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(-,X) \longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(-,\operatorname{Ker}(\phi)) \longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(-,T)$$

where the first map must be an epimorphism since ϕ is a right $\mathbb{T}$-approximation, hence the second morphism factors through zero. In addition when restricting to $\mathbb{T}$, the last term vanish since $\mathbb{T}$ is self-orthogonal. Hence, by exactness we must have $\operatorname{Ext}^1_{\mathcal{A}}(-,\operatorname{Ker}(\phi))|_{\mathbb{T}} = 0$, or equivalently $\operatorname{Ker}(\phi) \in \mathbb{T}^{\perp_1} = \operatorname{Fac}\mathbb{T}$. Now, since $X \in {}^{\perp_1}(\operatorname{Fac}\mathbb{T})$, we observe that the short exact sequence

$$0 \to \operatorname{Ker}(\phi) \to T \to X \to 0$$

splits, and thus $\operatorname{Ker}(\phi) \oplus X \cong T$. Further, since $\mathbb{T}$ is closed under summands, X must lie in $\mathbb{T}$, and we have ${}^{\perp_1}(\operatorname{Fac}\mathbb{T}) \cap \operatorname{Fac}\mathbb{T} \subseteq \mathbb{T}$. As already noted $\mathbb{T}$ is a subcategory of both $\operatorname{Fac}\mathbb{T}$ and ${}^{\perp_1}(\operatorname{Fac}\mathbb{T})$, so we have in fact the equality we wanted. $\square$

Proposition 3.54. *Let $\mathcal{A}$ be an abelian category with enough projectives, and $(\mathcal{C},\mathcal{T},\mathcal{F})$ be a cotorsion torsion triple in $\mathcal{A}$. Then $\mathcal{C} \cap \mathcal{T}$ is a tilting subcategory of $\mathcal{A}$.*

Proof. We start by observing that we have the natural inclusion

$$\operatorname{Ext}^1_{\mathcal{A}}(\mathcal{C} \cap \mathcal{T}, \mathcal{C} \cap \mathcal{T}) \subseteq \operatorname{Ext}^1_{\mathcal{A}}(\mathcal{C}, \mathcal{T})$$

where the last term is zero by $(\mathcal{C}, \mathcal{T})$ being a cotorsion pair. Next, we have by Lemma 3.35 that every object in $\mathcal{C}$ has projective dimension at most one. Thus all subcategories of $\mathcal{C}$ will also exhibit the same property. In particular the subcategory $\mathcal{C} \cap \mathcal{T} \subseteq \mathcal{C}$ satisfies the second property of being weakly tilted in $\mathcal{A}$.

Now, for the third condition for $\mathcal{C} \cap \mathcal{T}$ being weakly tilting, we let $P \in \mathcal{A}$ be a projective element, and

$$0 \to P \to \tilde{\mathtt{d}}P \to \tilde{\mathtt{c}}P \to 0$$

be a short exact sequence as in the definition of $(\mathcal{C}, \mathcal{T})$ being a cotorsion pair, i.e. $\tilde{\mathtt{d}}P \in \mathcal{T}$ and $\tilde{\mathtt{c}}P \in \mathcal{C}$. $\mathcal{T}$ is closed under quotients, so $\tilde{\mathtt{c}}P$ also lies in $\mathcal{T}$. $\mathcal{C}$ contains all the projective objects in $\mathcal{A}$, and is closed under extensions, thus $P \in \mathcal{C}$ and therefore $\tilde{\mathtt{d}}P \in \mathcal{C}$. We conclude that both $\tilde{\mathtt{d}}P$ and $\tilde{\mathtt{c}}P$ lie in $\mathcal{C} \cap \mathcal{T}$ and $\mathcal{C} \cap \mathcal{T}$ is weakly tilting.

Recall that for a cotorsion torsion triple $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ it was shown in Lemma 3.34 that $\mathcal{C} \cap \mathcal{T}$ is contravariantly finite. We conclude that $\mathcal{C} \cap \mathcal{T}$ is a tilting subcategory.

□

We have seen that a tilting subcategory induces a cotorsion torsion triple and conversely that a cotorsion torsion triple induces a tilting subcategory. These constructions do in fact induce a correspondence as we will see shortly.

Theorem 3.55. *Let $\mathcal{A}$ be an abelian category with enough projectives. Then the two constructions*

$$\begin{aligned} \{\text{tilting subcategories}\} &\leftrightarrow \{\text{cotorsion torsion triples}\} \\ \mathbb{T} &\mapsto (\{X \in {}^{\perp_1}\mathbb{T} \mid \operatorname{pdim} \leq 1\}, \operatorname{Fac}\mathbb{T}, \mathbb{T}^{\perp}) \\ \mathcal{C} \cap \mathcal{T} &\leftarrow\!\shortmid (\mathcal{C}, \mathcal{T}, \mathcal{F}) \end{aligned}$$

is a bijective correspondence.

Proof. The map taking a cotorsion torsion triple to a tilting subcategory is given in Proposition 3.54 and the converse map is given through Proposition 3.52 and Proposition 3.53. What is left to do is to show that these maps are inverses of each other. If we start with a tilting subcategory, we have by Proposition 3.53 that $\mathbb{T} = {}^{\perp_1}\mathrm{Fac}\,\mathbb{T} \cap \mathrm{Fac}\,\mathbb{T}$, i.e. the maps gives back the original tilting subcategory.

Conversely, if we start with a cotorsion torsion triple $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ and construct the tilting subcategory $\mathbb{T} = \mathcal{C} \cap \mathcal{T}$, we want to recover $(\mathcal{C}, \mathcal{D}, \mathcal{F})$. A cotorsion torsion triple is uniquely given from one of it's component, so it is enough to show that $\mathcal{T} = \mathrm{Fac}(\mathcal{C} \cap \mathcal{T})$. $\mathcal{T}$ is closed under factors, since it is a torsion class, thus $\mathrm{Fac}(\mathcal{C} \cap \mathcal{T}) \subseteq \mathrm{Fac}(\mathcal{T}) = \mathcal{T}$. For the inclusion $\mathcal{T} \subseteq \mathrm{Fac}(\mathcal{C} \cap \mathcal{T})$, construct the short exact sequence

$$0 \to \mathtt{d}T \to \mathtt{c}T \to T \to 0$$

with $\mathtt{d}T \in \mathcal{T}$ and $\mathtt{c}T \in \mathcal{C}$ for any $T \in \mathcal{T}$. $\mathcal{T}$ is closed under extensions, so $\mathtt{c}T \in \mathcal{C} \cap \mathcal{T}$. That is, we have an epimorphism $\mathtt{c}T \twoheadrightarrow T$ from an object in $\mathcal{C} \cap \mathcal{T}$, thus $T \in \mathrm{Fac}(\mathcal{C} \cap \mathcal{T})$. Consequently $\mathcal{T} \subseteq \mathrm{Fac}(\mathcal{C} \cap \mathcal{D})$, concluding our proof. $\square$

From this theorem and Theorem 3.36 we get the following immediate corollary.

Corollary 3.56. *Let $\mathbb{T}$ be a tilting subcategory in an abelian category $\mathcal{A}$ with enough projectives. There is an equivalence*

$$\frac{\{X \in {}^{\perp_1}\mathbb{T} \mid \mathrm{pdim} \leq 1\}}{\mathbb{T}} \cong \mathbb{T}^{\perp}$$

3.5 Cotilting

We can observe that all the result concerning tilting has corresponding dual results in abelian categories with enough injectives. This observation relies partly on the fact that the concepts of torsion and cotorsion are self-dual. With this in mind we define the dual of tilting, namely *cotilting.*

Definition 3.57. Let $\mathcal{A}$ be an abelian category with enough injectives. An additivel closed full subcategory $\mathbb{C}$ of $\mathcal{A}$ is a *weak cotilting* subcategory if

(i) it is self-orthogonal,

(ii) any object $C \in \mathbb{C}$ has injective dimension at most 1, and

(iii) for any I injective in $\mathcal{A}$ there is a short exact sequence

$$0 \to C_1 \to C_0 \to I \to 0$$

with $C_0, C_1 \in \mathbb{C}$.

A weak cotilting subcategory $\mathbb{C} \subseteq \mathcal{A}$ is *cotilting* provided it is also covariantly finite in $\mathcal{A}$.

Proposition 3.58 (Dual of Proposition 3.52 and Proposition 3.53). *Let $\mathcal{A}$ be an abelian category with enough injectives. If $\mathbb{C} \subseteq \mathcal{A}$ is weakly cotilting, then* $\operatorname{Sub}\mathbb{C} = {}^{\perp_1}\mathbb{C}$, $(\operatorname{Sub}\mathbb{C})^{\perp_1} = \{X \in \mathcal{C}^{\perp_1} \mid \operatorname{idim} X \leq 1\}$, *and*

$$(\operatorname{Sub}\mathbb{C}, (\operatorname{Sub}\mathbb{C})^{\perp_1})$$

is a cotorsion pair. If $\mathbb{C}$ is in addition covariantly finite, then

$$({}^{\perp}\mathbb{C}, \operatorname{Sub}\mathbb{C})$$

is a torsion pair and

$$(\operatorname{Sub}\mathbb{C})^{\perp_1} \cap \operatorname{Sub}\mathbb{C} = \mathbb{C}$$

Theorem 3.59 (Dual of Theorem 3.55). *Let $\mathcal{A}$ be an abelian category with enough injectives. Then there is a bijective correspondence between cotilting subcategories $\mathbb{C}$ of $\mathcal{A}$ and torsion cotorsion triples $(\mathcal{T}, \mathcal{F}, \mathcal{D})$ in $\mathcal{A}$, given by*

$$\mathbb{C} \mapsto ({}^{\perp}\mathbb{C}, \operatorname{Sub}\mathbb{C}, \{X \in \mathbb{C}^{\perp_1} \mid \operatorname{idim} X \leq 1\}$$
$$\mathcal{F} \cap \mathcal{D} \leftarrow\!\shortmid (\mathcal{T}, \mathcal{F}, \mathcal{D}),$$

Corollary 3.60 (Dual of Corollary 3.56). *In an abelian category $\mathcal{A}$ with enough injectives, any cotilting subcategory $\mathbb{C}$ gives an equivalence*

$$\frac{\{X \in \mathbb{C}^{\perp_1} \mid \operatorname{idim} X \leq 1\}}{\mathbb{C}} \simeq {}^{\perp}\mathbb{C}$$

4 Representations

We are now ready to study cotorsion and tilting in the abelian categories of quiver representations valued over an abelian category $\mathcal{A}$. We start by recalling a few fundamental facts about quivers and fix some notation conventions. In Section 4.1 we describe classical results of projective and injective representations through the use of adjoints pair between the category of representations and the underlying abelian category. Then in Section 4.2 we give a description of certain complete cotorsion pairs of representation through their local behaviour in $\mathcal{A}$, before we explore how these pairs correspond to tilting and cotilting in Section 4.3 and Section 4.4. At the end of Section 4.4 we also give an application of the discovered results to Multiparameter Persistence Modules.

Recall that a quiver is an oriented graph Q consisting of a set of vertices Q_0 and a set of arrows Q_1. We will be denoting the initial vertex of an arrow α by $i(\alpha)$ and the terminal vertex with $t(\alpha)$, that is, for the quiver

$$\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "\alpha"'] \arrow[r, "\beta"] & \underset{3}{\bullet} \end{tikzcd}$$

we have $i(\alpha) = 2$ and $t(\alpha) = 1$. A *path* of a quiver, will be a concatenation of arrows such that their initial and terminal vertices correspond nicely, that is $p = \alpha_n \alpha_{n-1} \cdots \alpha_1$ is a path if $i(\alpha_i) = t(\alpha_{i-1})$. The length of a path is the number of arrows it consists of. We will be denoting the set of paths starting in vertex x and ending in vertex y by $Q(x, y)$, the set of all non-trivial paths by $Q_{\geq 1}(x, y)$, and all arrows starting at x and ending in y by $Q_1(x, y)$. If we want to consider all paths ending in x, we will denote that set by $Q(*, x)$. We similarly define the sets $Q(x, *), Q_{\geq 1}(*, x), Q_{\geq 1}(x, *), Q_1(*, x)$ and $Q_1(x, *)$.

We can observe that any quiver Q generate a small category where the objects are the vertices and the morphisms are the paths. Hence, a classical quiver representation can be thought of as a functor from the quiver into the module category of an algebra A. In this text we further generalize this and think of representations as functors from the quiver-category into some abelian category $\mathcal{A}$. This is done so that we can utilize the inherent duality of $\mathcal{A}$. One can observe that the opposite category of a module category is not necessarily a module category, but the opposite of an abelian category is always an abelian category. Hence,

Definition 4.1. A *representation* of a small category $\mathcal{X}$ (over $\mathcal{A}$) is a covariant functor

$$F\colon \mathcal{X} \to \mathcal{A}$$

into an abelian category $\mathcal{A}$.

We denote the category of all representations of a quiver (valued over $\mathcal{A}$) by $\mathrm{Rep}(Q, \mathcal{A})$, and whenever $\mathcal{A}$ is a module category over an algebra $\mathcal{A}$ we denote it by $\mathrm{Rep}(Q, A)$. Note that any quiver in the following will be assumed to be finite and acyclic.

Remark 4.2. Recall that the opposite of a functor $F\colon \mathcal{C} \to \mathcal{D}$ is the functor $F^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ such that $F^{\mathrm{op}}(C) = F(C)$ for any object $C \in \mathcal{C}$, and $F^{\mathrm{op}}(\phi) = F(\phi)$for any morphism $\phi \in \mathcal{C}$. Thus, $\mathrm{Rep}(Q, \mathcal{A})^{\mathrm{op}} = \mathrm{Rep}(Q^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}})$.

Lemma 4.3. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor, such that $F^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ has a left adjoint $G\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$, then the functor $G^{\mathrm{op}}\colon \mathcal{D} \to \mathcal{C}$ is a right adjoint of F.*

Proof. The assumption gives us a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(G-, -) \xrightarrow{\eta} \mathrm{Hom}_{\mathcal{D}^{\mathrm{op}}}(-, F^{\mathrm{op}}-)$$

Which is the same as having commutative squares with horizontal isomorphisms

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(GD, C) \arrow[r, "\eta_{C,D}"] \arrow[d, "\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Gf^{\mathrm{op}}, g^{\mathrm{op}})"] & \mathrm{Hom}_{\mathcal{D}^{\mathrm{op}}}(D, F^{\mathrm{op}}C) \arrow[d, "\mathrm{Hom}_{\mathcal{D}^{\mathrm{op}}}(f^{\mathrm{op}}, F^{\mathrm{op}}g^{\mathrm{op}})"] \\
\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(GD', C') \arrow[r, "\eta_{C',D'}"] & \mathrm{Hom}_{\mathcal{D}^{\mathrm{op}}}(D', F^{\mathrm{op}}C')
\end{tikzcd}$$

for every morphism $f^{\mathrm{op}}\colon D' \to D$ in $\mathcal{D}^{\mathrm{op}}$ and $g^{\mathrm{op}}\colon C \to C'$ in $\mathcal{C}^{\mathrm{op}}$. Now, passing to the opposite setting, this induces the commutative squares,

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(C, G^{\mathrm{op}}D) & \mathrm{Hom}_{\mathcal{D}}(FC, D) \arrow[l, "{\eta_{C,D}}^{\mathrm{op}}"'] \\
\mathrm{Hom}_{\mathcal{C}}(C', G^{\mathrm{op}}D') \arrow[u, "\mathrm{Hom}_{\mathcal{C}}(g, G^{\mathrm{op}}f)"] & \mathrm{Hom}_{\mathcal{D}}(FC', D') \arrow[l, "{\eta_{C',D'}}^{\mathrm{op}}"'] \arrow[u, "\mathrm{Hom}_{\mathcal{D}}(Fg, f)"]
\end{tikzcd}$$

with isomorphic horizontal morphisms, for each morphism $f\colon D \to D'$ in $\mathcal{D}$ and each morphism $g\colon C' \to C$ in $\mathcal{C}$. Thus we have a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(-, G^{\mathrm{op}}-) \xleftarrow{\eta^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}^{\mathrm{op}}}(F-, -)$$

and we see that G^{op} is an right adjoint to F. □

Remark 4.4. To save space, we will find it useful to sometimes adopt the convention of denoting the morphism sets of a category $\mathcal{C}$ by $\mathcal{C}(-,-)$ in the following, that is for pairs of objects $C, D \in \mathcal{C}$ we have $\mathcal{C}(C, D) = \mathrm{Hom}_{\mathcal{C}}(C, D)$

4.1 Projective and Injective Representations

We would now like to describe how the projective and injective representations look like in the category $\mathrm{Rep}(Q, \mathcal{A})$. In the special case where $\mathcal{A} = \mathrm{mod}(A)$ for an algebra A, we will rediscover the descriptions we are used to. The description relies on constructing adjoint functors between $\mathcal{A}$ and $\mathrm{Rep}(Q, \mathcal{A})$ and is a specialization of the corresponding construction in [16].

We start by observing that for every vertex $x \in Q$ there is an exact *evaluation functor*

$$-x\colon \mathrm{Rep}(X, \mathcal{A}) \to \mathcal{A}$$

taking any representation $F \in \mathrm{Rep}(Q, \mathcal{A})$ to it's evalutaion at x, and any morphism $\phi \in \mathrm{Rep}(Q, \mathcal{A})(F, G)$ to the morphism $\phi_x\colon Fx \to Fy$. In the other direction we can define the following functor.

Definition 4.5. For every vertex $x \in Q$ we have an exact functor

$$P_x\colon \mathcal{A} \to \mathrm{Rep}(Q, \mathcal{A})$$

sending an object A in $\mathcal{A}$ to the representation $P_x(A)$ given on each vertex $y \in Q$ by the zero object in $\mathcal{A}$ if $Q(x, y)$ is empty and else by

$$P_x(A)(y) = \bigoplus_{p \in Q(x,y)} A^{(p)}.$$

Here the superscript only acts as an identifier for which path the summand arises from, that is $A^{(p)} = A$.

The internal morphisms of the representation are given as follows. Let $\iota_p \colon A^{(p)} \hookrightarrow P_x(A)(y)$ be the canonical inclusion morphism into the direct sum in $\mathcal{A}$, and note that for every path $q \in Q(y,z)$ and $p \in Q(x,y)$, there is a path $qp \in Q(x,z)$. Now

$$P_x(A)(q) \colon P_x(A)(y) \to P_x(A)(z)$$

is the unique morphism in $\mathcal{A}$ making the following square commutative

$$\begin{tikzcd}
A^{(p)} \arrow[r, equal] \arrow[d, hook, "\iota_p"] & A \arrow[r, equal] & A^{(qp)} \arrow[d, hook, "\iota_{qp}"] \\
P_x(A)(y) \arrow[rr, "P_x(A)(q)"] & & P_x(A)(z)
\end{tikzcd} \tag{3}$$

Explicitly, $P_x(A)(y)$ maps the summand $A^{(p)}$ identically to $A^{(qp)}$.

Example 4.6. For the quiver

$$\begin{tikzcd}
\underset{(1,3)}{\bullet} \arrow[r] & \underset{(2,3)}{\bullet} \arrow[r] & \underset{(3,3)}{\bullet} \arrow[r] & \underset{(4,3)}{\bullet} \\
\underset{(1,2)}{\bullet} \arrow[r] \arrow[u] & \underset{(2,2)}{\bullet} \arrow[r] \arrow[u] & \underset{(3,2)}{\bullet} \arrow[r] \arrow[u] & \underset{(4,2)}{\bullet} \arrow[u] \\
\underset{(1,1)}{\bullet} \arrow[r] \arrow[u] & \underset{(2,1)}{\bullet} \arrow[r] \arrow[u] & \underset{(3,1)}{\bullet} \arrow[r] \arrow[u] & \underset{(4,1)}{\bullet} \arrow[u]
\end{tikzcd}$$

we have that $P_{(2,2)}$ is given by

$$\begin{tikzcd}
\underset{(1,3)}{0} \arrow[r] & \underset{(2,3)}{A} \arrow[r] & \underset{(3,3)}{A^2} \arrow[r] & \underset{(4,3)}{A^3} \\
\underset{(1,2)}{0} \arrow[r] \arrow[u] & \underset{(2,2)}{A} \arrow[r] \arrow[u] & \underset{(3,2)}{A} \arrow[r] \arrow[u] & \underset{(4,2)}{A} \arrow[u] \\
\underset{(1,1)}{0} \arrow[r] \arrow[u] & \underset{(2,1)}{0} \arrow[r] \arrow[u] & \underset{(3,1)}{0} \arrow[r] \arrow[u] & \underset{(4,1)}{0} \arrow[u]
\end{tikzcd}$$

♣

Proposition 4.7. *P_x is left adjoint to the evaluation functor $-x$.*

Proof. To prove the statement we construct natural isomorphisms

$$\mathrm{Rep}(Q,\mathcal{A})(P_x(-),-) \underset{\xi}{\overset{\eta}{\rightleftarrows}} \mathcal{A}(-,-x)$$

in the following manner. Let $A \in \mathcal{A}$ and $F \in \mathrm{Rep}(Q,\mathcal{A})$. Define

$$\eta_{A,F} \colon \mathrm{Rep}(Q,\mathcal{A})(P_x(A),F) \to \mathcal{A}(A,Fx)$$

as

$$\phi \mapsto \phi_x \circ \iota_{\mathrm{id}_x}$$

The other transformation

$$\xi_{A,F} \colon \mathcal{A}(A,Fx) \to \mathrm{Rep}(Q,\mathcal{A})(P_x(A),F)$$

is a bit more delicate. Let us fix a morphism $\psi \in \mathcal{A}(A,Fx)$. If there are no paths from x to y in Q, $\xi(\psi)_x$ is forced to be the zero map since $P_x(A)(y) = 0$. When there is at least one path $p \in Q(x,y)$, we set $\xi(\psi)_y$ as the unique morphism which fits into the commutative square

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, hook, "\iota_p"] & Fx \arrow[d, "Fp"] \\ P_x(A)(y) \arrow[r, "\xi(\psi)_y"] & Fy \end{tikzcd} \tag{4}$$

for every path $p \in Q(x,y)$, whose existence and uniqueness follows from the universal property of coproducts. In order to see that $\xi(\psi)$ is in fact a morphism of representations, we now claim that for any path $q \in Q(y,z)$, the following square commutes

$$\begin{tikzcd} P_x(A)(y) \arrow[r, "P_x(A)(q)"] \arrow[d, "\xi(\psi)_y"] & P_x(A)(z) \arrow[d, "\xi(\psi)_z"] \\ F(y) \arrow[r, "Fq"] & F(z) \end{tikzcd}$$

If there are no paths from x to y in Q, then $P_x(A)(y) = 0$, so in that case our claim holds. Therefore we assume that there is at least one path $p \in Q(x,y)$. Now, for each of these morphisms, we have by using the defining Diagram (4) and (3) that

$$F(q) \circ \xi(\psi)_y \circ \iota_p = F(q) \circ F(p) \circ \psi = F(qp)\psi = \xi(\psi)_z \circ \iota_{qp} = \xi(\psi)_z \circ P_x(A)(q) \circ \iota_p$$

By the universal property of the coproduct $P_x(A)(y)$ we summize that

$$F(q) \circ \xi(\psi)_y = \xi(\psi)_z \circ P_x(A)(q),$$

which proves our claim. Now we claim that the two given natural transformations are in fact mutual inverse.

Let $\psi \in \mathcal{A}(A, Fx)$ be a morphism in $\mathcal{A}$. We have $\eta(\xi(\psi)) = \xi(\psi)_x \circ \iota_{\mathrm{id}_x}$ and by the defining Diagram (4), this is equal to $F(\mathrm{id}_x) \circ \psi = \mathrm{id}_{Fx} \circ \psi = \psi$. Thus, $\eta \circ \xi$ is the identity on $\mathcal{A}(A, Fx)$.

Now, let $\phi \in \mathrm{Rep}(Q, \mathcal{A})(P_x(A), F)$ be a morphism in $\mathrm{Rep}(Q, \mathcal{A})$. If there are no paths from x to y in Q, then $P_x(A)(y) = 0$ and we have necessarily that $\phi_y = 0$ and $\xi(\eta(\phi)) = 0$, so we can assume there is at least one path from x to y. We will now utilize the universal property of coproducts once again to establish that $\phi_y = \xi(\eta(\phi))_y$. Let $p \in Q(x, y)$ be any path. We have that

$$\xi(\eta(\phi))_y \circ \iota_p = F(p) \circ \eta(\phi)$$

by Diagram (4), and

$$F(p) \circ \eta(\phi) = F(p) \circ \phi_x \circ \iota_{\mathrm{id}_x}$$

from the definition of η. Now, using that ϕ is a morphism in $\mathrm{Rep}(Q, \mathcal{A})$ we get

$$F(p) \circ \phi_x \circ \iota_{\mathrm{id}_x} = \phi_y \circ P_x(A)(p) \circ \iota_{\mathrm{id}_x}$$

Finally, using the defining property of $P_x(A)(p)$ in Diagram (3), we get

$$\phi_y \circ P_x(A)(p) \circ \iota_{\mathrm{id}_x} = \phi_y \circ \iota_p$$

That is,

$$\xi(\eta(\phi))_y \circ \iota_p = \phi_y \circ \iota_p$$

and by the universal property of coproducts we conclude that $\xi(\eta(\phi))_y = \phi_y$, so $\xi \circ \eta$ is the identity on $\mathrm{Rep}(Q, \mathcal{A})(P_x(A), F)$. Thus, ξ and η is mutual inverses and we have established the needed isomorphism. $\square$

We can now make use of the fact that $\mathrm{Rep}(Q, \mathcal{A})^{\mathrm{op}} = \mathrm{Rep}(Q^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}})$ and Lemma 4.3 to find a right adjoint to $-x$ as well. Let us therefore define the dual functor of P_x.

Definition 4.8. For every vertex x in Q we have an exact functor

$$I_x \colon \mathcal{A} \to \mathrm{Rep}(Q, \mathcal{A})$$

given by sending any object A in $\mathcal{A}$ to the representation $I_x(A)$ given on each vertex $y \in Q$ by the zero object if $Q(y, x)$ is empty and else by

$$I_x(A)(y) = \bigoplus_{p \in Q(y,x)} A^{(p)}$$

where the superscript is an identifier for the otherwise identical summands. Let $\pi_p \colon I_x(A)(y) \twoheadrightarrow A^{(p)}$ be the canonical projection morphism from the direct sum in $\mathcal{A}$.

The internal morphisms of the representation are dual to those of P_x. That is, for each path $q \in Q(y, z)$ and $p \in Q(z, x)$, the related morphism $I_x(A)(q) \colon I_x(A)(y) \to I_x(A)(z)$ takes the summand $A^{(qp)}$ identically to $A^{(p)}$. Equivalently, $I_x(A)(q)$ is the unique morphism making the following square commutative;

$$\begin{tikzcd}
I_x(A)(y) \arrow[rr, "I_x(A)(q)"] \arrow[d, "\pi_{qp}"] & & I_x(A)(z) \arrow[d, "\pi_p"] \\
A^{(qp)} \arrow[r, equal] & A \arrow[r, equal] & A^{(p)}
\end{tikzcd}$$

Example 4.9. For the quiver of Example 4.6 we have that $I_{(2,2)}(A)$ is given by

$$\begin{tikzcd}
\underset{(1,3)}{0} \arrow[r] & \underset{(2,3)}{0} \arrow[r] & \underset{(3,3)}{0} \arrow[r] & \underset{(4,3)}{0} \\
\underset{(1,2)}{A} \arrow[r] \arrow[u] & \underset{(2,2)}{A} \arrow[r] \arrow[u] & \underset{(3,2)}{0} \arrow[r] \arrow[u] & \underset{(4,2)}{0} \arrow[u] \\
\underset{(1,1)}{A^2} \arrow[r] \arrow[u] & \underset{(2,1)}{A} \arrow[r] \arrow[u] & \underset{(3,1)}{0} \arrow[r] \arrow[u] & \underset{(4,1)}{0} \arrow[u]
\end{tikzcd}$$

♣

Proposition 4.10. *I_x is a right adjoint to the evaluation functor $-x$ for every vertex $x \in Q$.*

Proof. The opposite functor $-x^{\mathrm{op}}$ is the evaluation functor from $\mathrm{Rep}(Q^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}}) = \mathrm{Rep}(Q, \mathcal{A})^{\mathrm{op}}$ to $\mathcal{A}^{\mathrm{op}}$, which we know has a left adjoint $P_x \colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Rep}(Q, \mathcal{A})^{\mathrm{op}}$. By Lemma 4.3 we therefore know that ${P_x}^{\mathrm{op}} \colon \mathcal{A} \to \mathrm{Rep}(Q, \mathcal{A})$ is a right adjoint to $-x \colon \mathrm{Rep}(Q, \mathcal{A}) \to \mathcal{A}$, which by the previous construction is equal to I_x. □

We will now see that with the adjoint functors of the evaluation map, we can send projective or injective objects in $\mathcal{A}$ to projective or injective objects in $\mathrm{Rep}(Q, \mathcal{A})$ respectively. Under some further constraints we can in fact guarantee that if $\mathcal{A}$ has enough projectives or injectives, then $\mathrm{Rep}(Q, \mathcal{A})$ inherits the same property.

Lemma 4.11. *If $P \in \mathrm{Proj}\mathcal{A}$ is a projective object in $\mathcal{A}$, then $P_x(P)$ is projective in $\mathrm{Rep}(Q, \mathcal{A})$ for every $x \in Q_0$. Dually, if $I \in \mathrm{Inj}\mathcal{A}$ is injective in $\mathcal{A}$, then $I_x(I)$ is injective in $\mathrm{Rep}(Q, \mathcal{A})$ for every $x \in Q_0$*

Proof. We prove the first claim, the second follows dually.

Each evaluation functor $-x$ is exact, so by the left-adjunction P_x we obtain that for each projective object P in $\mathcal{A}$, the hom-functor $\mathrm{Hom}_{\mathrm{Rep}(Q,\mathcal{A})}(P_x(P), -) \cong \mathrm{Hom}_{\mathcal{A}}(P, -x)$ is exact. Thus $P_x(P)$ is projective. □

Lemma 4.12. *Let $\mathcal{A}$ have enough projectives. Then $\mathrm{Rep}(Q, \mathcal{A})$ also has enough projectives. Dually, let $\mathcal{A}$ have enough injectives. Then $\mathrm{Rep}(Q, \mathcal{A})$ also has enough injectives.*

Proof. Assume that $\mathcal{A}$ has enough projectives $\mathrm{Proj}\mathcal{A} \subseteq \mathcal{A}$. The adjunction $(P_x, -x)$ give rise to the counit transformation, $\epsilon^x \colon P_x \circ -x \to \mathrm{id}_{\mathrm{Rep}(Q,\mathcal{A})}$, which in turn gives us a morphism $\epsilon^x_F \colon P_x(Fx) \to F$ for every $x \in Q$. We obtain by the universal property a morphism from the coproduct $\bigoplus_{x \in Q_0} P_x(Fx)$ to F, which we claim is an epimorphism.

$$\begin{tikzcd}
P_x(Fx) \arrow[r, "\epsilon_x^F"] \arrow[d, hook] & F \\
\bigoplus_{x\in Q_0} P_x(Fx) \arrow[ur, dashed] &
\end{tikzcd}$$

It is enough to show that ρ_x is epimorphic for every $x \in Q_0$, since cokernels are computed component-wise. Thus, apply $-x$ to the commutative triangle above. Observe that ρ_x is epimorphic if and only if $-x(\epsilon_F^x)$ is epimorphic. Recall that since $(P_x, -x)$ are adjoints with unit η^x and counit ϵ^x, we have that $\mathrm{id}_{Fx} = -x(\epsilon_F^x) \circ \eta_{Fx}^x$. In other words, $-x(\epsilon_F^x)$ is a split epimorphism with right inverse η_{Fx}^x. We conclude that our claim holds, i.e. ρ is epimorphic.

Now, for every $x \in Q_0$, we obtain an epimorphism $p^x \colon P^x \twoheadrightarrow Fx$ with P^x projective in $\mathcal{A}$. P_x is exact, so we have in turn an epimorphism $P_x(P^x) \twoheadrightarrow P_x(Fx)$ in $\mathrm{Rep}(Q, \mathcal{A})$. The coproduct $\bigoplus_{x\in Q_0} P_x(P^x)$ is also projective, since it is a small coproduct of projectives, and coproducts preserves epimorphisms, so we have in fact obtained an epimorphic composition from a projective object to F.

$$\bigoplus_{x\in Q_0} P_x(P^x) \twoheadrightarrow \bigoplus_{x\in Q_0} P_x(Fx) \overset{\rho}{\twoheadrightarrow} F$$

The dual assertions regarding enough injectives $\mathrm{Inj}\mathcal{A} \subseteq \mathcal{A}$, is proven by applying the first part in the opposite category. $\square$

Lemma 4.13. *Let $\mathcal{A}$ have enough projectives. Then*

$$\mathrm{ProjRep}(Q, \mathcal{A}) = \mathrm{add}\{P_x(P) \mid x \in Q_0 \textit{ and } P \in \mathrm{Proj}\mathcal{A}\}$$

Proof. As we have seen in Lemma 4.11 $P_x(P)$ is projective, so we must necessarily have the inclusion " $\supseteq$ ". For the other inclusion, we take some projective object $P' \in \mathrm{Rep}(Q, \mathcal{A})$, and observe that the epimorphism from Lemma 4.12 splits,

$$\begin{tikzcd}
 & P' \arrow[dl, dashed] \arrow[d, equal] & \\
\bigoplus_{x\in Q_0} P_x(P^x) \arrow[r, two heads] & P' \arrow[r] & 0
\end{tikzcd}$$

Thus, P' is a summand of $\bigoplus_{x \in Q_0} P_x(P^x)$ and therefore lies in $\operatorname{add}\{P_x(P) \mid x \in Q_0 \text{ and } P \in \operatorname{Proj}\mathcal{A}\}$. We have both inclusions, so the sets must be equal. $\square$

4.2 Cotorsion

A natural question to ask when considering cotorsion in categories of representations is what, if any, relation cotorsion pair of the underlying abelian category has to those in the representation category. This has recently been studied by Holm et al. in [16]. They found that for left (right) rooted quivers Q[3] and an abelian category $\mathcal{A}$ satisfying certain conditions, the representations inherit cotorsion pairs from $\mathcal{A}$. It was also shown that hereditary cotorsion pairs[4] of the underlying category gave hereditary cotorsion pairs in the category of representations.

Holm et al. also asked whether complete cotorsion pairs in the abelian category induces complete cotorsion pairs in the category of representations over a (left) right rooted category. Odabaşı answered this in the affirmative when the pair in addition were hereditary [23]. Recently, in a preprint by Di et al. [11], the hereditary condition has been removed.

We will in this section reproduce this relation between cotorsion in representations and the underlying category, in the special case when the quiver is finite and acyclic. This allows us to greatly simplify the arguments, and relax the conditions put on the underlying abelian category a bit. The proofs are constructive in nature and not necessarily intuitive nor informative. The reader can therefore skip past them without losing too much understanding of the story.

As in the previous subsection we let Q denote a finite acyclic quiver.

[3] A left rooted quiver, is a quiver which do not admit the left infinite linear quiver $\vec{A}_\infty : \cdots \underset{3}{\bullet} \to \underset{2}{\bullet} \to \underset{1}{\bullet}$ as a subquiver.

[4] A cotorsion pair $(\mathcal{C}, \mathcal{D})$ is hereditary if $\operatorname{Ext}^i_{\mathcal{A}}(\mathcal{C}, \mathcal{D}) = 0$ for all $i \geq 1$.

Definition 4.14. For each vertex $x \in Q$, we have the exact *stalk functor*

$$s_x \colon \mathcal{A} \to \operatorname{Rep}(Q, \mathcal{A})$$

given by sending an object $A \in \mathcal{A}$ to the representation $s_x(A)$ where

$$s_x(A)(y) = \begin{cases} A & \text{if } y = x \\ 0 & \text{else.} \end{cases}$$

To each representation $F \in \operatorname{Rep}(Q, \mathcal{A})$ we have for each evaluation $F(x)$ two morphisms in $\mathcal{A}$ that we will be using quite frequently in the following, namely

$$\gamma_x^F \colon \coprod_{\alpha \in Q_1(*,x)} F(i(\alpha)) \to F(x)$$

and

$$\lambda_x^F \colon F(x) \to \prod_{\beta \in Q_1(x,*)} F(t(\beta))$$

given as the unique morphisms fitting into the following diagrams

$$\begin{tikzcd} F(i(\alpha')) \arrow[d, "\iota_{\alpha'}"] \arrow[dr, "F(\alpha')"] & \\ \coprod\limits_{\alpha \colon \in Q_1(*,x)} F(i(\alpha)) \arrow[r, dashed, "\gamma_x^F"] & F(x) \end{tikzcd} \qquad \forall \alpha' \in Q_1(*, x)$$

$$\begin{tikzcd} F(x) \arrow[r, dashed, "\lambda_x^F"] \arrow[dr, "F(\beta')"'] & \prod\limits_{\beta \in Q_1(x,*)} F(t(\beta)) \arrow[d, "\pi_{\beta'}"] \\ & F(t(\beta')) \end{tikzcd} \qquad \forall \beta' \in Q_1(x, *)$$

where $\iota_{\alpha'}$ are the canonical inclusion morphisms and $\pi_{\beta'}$ the canonical projection morphisms. Note that since we are working over a finite acylic quiver, both the product and coproduct in the preceding is nothing but direct sums.

Remark 4.15. A necessary, but not sufficient condition for γ_x^F to be a monomorphism, is that all internal morphism of the representation $F \in \operatorname{Rep}(Q, \mathcal{A})$ are monomorphisms. That it is not sufficient can be seen by representations of the zig-zag quiver

$$Q\colon \qquad \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longleftarrow \underset{3}{\bullet}$$

valued over the category of finite dimensional $\mathbf{k}$-vector spaces for a field $\mathbf{k}$. The representation

$$\mathbf{k} \xrightarrow{1} \mathbf{k} \xleftarrow{1} \mathbf{k}$$

clearly has monomorphic structure morphisms, but

$$\gamma_2^F = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\colon \mathbf{k} \oplus \mathbf{k} \to \mathbf{k}$$

is not a monomorphism.

Lemma 4.16. *γ_x^F induces a functor $c_x\colon \mathrm{Rep}(Q, \mathcal{A}) \to \mathcal{A}$ given by $F \mapsto \mathrm{Cok}\gamma_x^F$. Similarly, λ_x^F induces a functor $k_x\colon \mathrm{Rep}(Q, \mathcal{A}) \to \mathcal{A}$ given by $\mathrm{Ker}\lambda_x^F$.*

Proof. Let $\phi \in \mathrm{Rep}(Q, \mathcal{A})(F, G)$ be a morphism of representations, this gives a unique morphism $\tilde{\phi}$ fitting into the commutative diagrams below

$$\begin{tikzcd} F(i(\alpha')) \arrow[r, "\iota^F_{\alpha'}"] \arrow[d, "\phi_{i(\alpha')}"] & \coprod\limits_{\alpha \in Q_1(*,x)} F(i(\alpha)) \arrow[d, dashed, "\tilde{\phi}"] \\ G(i(\alpha')) \arrow[r, "\iota^G_{\alpha'}"] & \coprod\limits_{\alpha \in Q_1(*,x)} G(i(\alpha)) \end{tikzcd} \qquad \forall \alpha' \in Q_1(*, x)$$

Now, we claim that $\phi_x\colon F(x) \to G(x)$ fits into the following commutative diagram

$$\begin{tikzcd} \coprod\limits_{\alpha \in Q_1(*,x)} F(i(\alpha)) \arrow[r, "\gamma_x^F"] \arrow[d, "\tilde{\phi}"] & F(x) \arrow[r, two heads] \arrow[d, "\phi_x"] & \mathrm{Cok}\gamma_x^F \arrow[r] & 0 \\ \coprod\limits_{\alpha \in Q_1(*,x)} G(i(\alpha)) \arrow[r, "\gamma_x^G"] & G(x) \arrow[r, two heads] & \mathrm{Cok}\gamma_x^F \arrow[r] & 0 \end{tikzcd}$$

This can be seen by observing that for every arrow $\alpha' \in Q_1(*, x)$ we have from $\phi \in \mathrm{Rep}(Q, \mathcal{A})(F, G)$ being a morphism of representations that $\phi_x \circ F(\alpha') =$

$G(\alpha') \circ \phi_{i(\alpha')}$. In addition, from the construction of γ_x^F we know that $F(\alpha')$ factors as $\gamma_x^F \circ \iota_{\alpha'}^F$ for the inclusion $\iota_{\alpha'}^F\colon F(i(\alpha')) \to \coprod_{\alpha \in Q_1(*,x)} F(i(\alpha))$, and $G(\alpha')$ as $\gamma_x^G \circ \iota_{\alpha'}^G$ for the inclusion $\iota_{\alpha'}^G\colon G(i(\alpha')) \to \coprod_{\alpha \in Q_1(*,x)} G(i(\alpha))$. Thus, we have

$$\phi_x \circ \gamma_x^F \circ \iota_{\alpha'}^F = \phi_x \circ F(\alpha') = G(\alpha') \circ \phi_{i(\alpha')} = \gamma_x^G \circ \iota_{\alpha'}^G \circ \phi_{i(\alpha')} = \gamma_x^G \circ \tilde{\phi} \circ \iota_{\alpha'}^F$$

and by the universal propery of coproducts, we conclude that $\phi_x \circ \gamma_x^F = \gamma_x^G \circ \tilde{\phi}$ as we claimed.

$$\begin{tikzcd}
F(i(\alpha')) \arrow[dr, "\iota_{\alpha'}^F"] \arrow[drr, bend left, "F(\alpha')"] \arrow[ddd, "\eta_{i(\alpha')}"] & & \\
& \coprod\limits_{\alpha \in Q_1(*,x)} F(i(\alpha)) \arrow[r, "\gamma_x^F"] \arrow[d, "\eta'"] & F(x) \arrow[d, "\eta_x"] \\
& \coprod\limits_{\alpha \in Q_1(*,x)} G(i(\alpha)) \arrow[r, "\gamma_x^G"] & G(x) \\
G(i(\alpha')) \arrow[ur, "\iota_{\alpha'}^G"] \arrow[urr, bend right, "G(\alpha')"'] & &
\end{tikzcd}
\qquad \forall \alpha' \in Q_1(*,x)$$

Now, we can see that by the cokernel property, there is a unique induced morphism $\mathrm{Cok}\gamma_x^F \to \mathrm{Cok}\gamma_x^G$ making the diagram below commutative

$$\begin{tikzcd}
\coprod\limits_{\alpha \in Q_1(*,x)} F(i(\alpha)) \arrow[r, "\gamma_x^F"] \arrow[d, "\tilde{\phi}"] & F(x) \arrow[r, two heads] \arrow[d, "\phi_x"] & \mathrm{Cok}\gamma_x^F \arrow[r] \arrow[d, dashed] & 0 \\
\coprod\limits_{\alpha \in Q_1(*,x)} G(i(\alpha)) \arrow[r, "\gamma_x^G"] & G(x) \arrow[r, two heads] & \mathrm{Cok}\gamma_x^F \arrow[r] & 0
\end{tikzcd}$$

Observe that $\mathrm{id_F} \in \mathrm{Rep}(Q, \mathcal{A})(F, F)$ gets sent to $\mathrm{id}_{\mathrm{Cok}\gamma_x^F}$. Associativity is straightforward, albeit a bit tedious to verify, and is therefore omitted here. We conclude that $c_x\colon \mathrm{Rep}(Q, \mathcal{A}) \to \mathcal{A}$ is a functor. The claim of k_x being a functor is proven similarly. $\square$

Remark 4.17. Observe that if we let $c_x\colon \mathrm{Rep}(Q,\mathcal{A}) \to \mathcal{A}$ and $k_x\colon \mathrm{Rep}(Q,\mathcal{A})^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$, we have that ${c_x}^{\mathrm{op}} = k_x$. Thus, k_x being a functor follows from c_x being a functor.

Proposition 4.18. *Let $x \in Q$ be any vertex in a finite, acyclic quiver.*

(i) c_x is left adjoint to the stalk functor s_x.

(ii) k_x is right adjoint to the stalk functor s_x.

Proof. Here we will only be proving (i), as (ii) follows directly from (i) in light of Lemma 4.3 and Remark 4.17.

We start by constructing a natural map

$$\eta_{F,A}\colon \mathcal{A}(c_x(F), A) \to \mathrm{Rep}(Q,\mathcal{A})(F, s_x(A))$$

We have that $s_x(A)(y) = 0$ whenever $y \neq x$, so for every $\phi \in \mathcal{A}(c_x(F), A)$ we must necessarily have $\eta_{F,A}(\phi)_y = 0$ in these cases. When $y = x$, we set $\eta_{F,A}(\phi)_x = \phi \circ e$ where $e\colon F(x) \to \mathrm{Cok}\gamma_x^F$ is the cokernel morphism

$$\coprod_{\alpha \in Q_1(*,x)} F(i(\alpha)) \xrightarrow{\gamma_x^F} F(x) \overset{e}{\twoheadrightarrow} \mathrm{Cok}\gamma_x^F$$

We have $s_x(A)(y) \neq 0$ and $\eta_{F,A}(\phi)_y \neq 0$ only when $y = x$, so we are left with verifying that $\eta_{F,A}(\phi) \circ F(\alpha) = 0$ for every arrow $\alpha \in Q_1(*,x)$, in order to see that $\eta_{F,A}(\phi)$ is a morphism of representations. Observe therefore that $F(\alpha)$ factors through γ_x^F by construction, thus $\eta_{F,A}(\phi) \circ F(\alpha) = \phi \circ e \circ F(\alpha) = 0$.

Now, let us construct a natural map in the opposite direction

$$\xi_{F,A}\colon \mathrm{Rep}(Q,\mathcal{A})(F, s_x(A)) \to \mathcal{A}(c_x(F), A)$$

Let $\psi \in \mathrm{Rep}(Q,\mathcal{A})(F, s_x(A))$ be any morphism. We do have that $\psi_{t(\alpha)} \circ F(\alpha) = 0$ for any arrow $\alpha \in Q_1$. Specifically we have $0 = \psi_x \circ F(\alpha') = \psi_x \circ \gamma_x^F \circ \iota_{\alpha'}$ for every $\alpha' \in Q_0(*, x)$. Thus, we get that $\psi_x \circ \gamma_x^F = 0$ and therefore by the cokernel property, there exists a unique morphism $\xi_{F,A}(\psi)$ such that $\phi_x = \xi_{F,A}(\psi) \circ e$. Let ξ be defined to send any $\psi \in \mathrm{Rep}(Q, \mathcal{A})$ to these unique morphisms.

$$\begin{tikzcd}
F(i(\alpha')) \arrow[d, hook, "\iota_{\alpha'}"] \arrow[dr, "F(\alpha')"] & & \forall \alpha' \in Q_1(*, x) & \\
\coprod_{\alpha \in Q_1(*,x)} F(i(\alpha)) \arrow[r, "\gamma_x^F"] & F(x) \arrow[r, two heads, "e"] \arrow[dr, "\psi_x"'] & \mathrm{Cok}\gamma_x^F \arrow[r] \arrow[d, dashed, "\xi_{F,A}(\psi)"] & 0 \\
& & s_x(A)(x) &
\end{tikzcd}$$

It is clear that ξ and η are mutual inverses, so we have established the natural isomorphism

$$\mathrm{Rep}(Q, \mathcal{A})(F, s_x(A)) \cong \mathcal{A}(c_x(F), A)$$

which means that (c_x, s_x) is an adjoint pair. $\square$

We will now see that the adjoint pairs obtained for the representation category extends to Ext^1, which will be essential in our proceeding proofs of cotorsion. The fact that $(P_x, -x)$ and $(-x, I_x)$, extends to Ext^1 follows easily from the following technical proof, which is proven in the appendix. The recently seen adjoint pairs (c_x, s_x) and (s_x, k_x), extends to Ext^1 for a subclass of representations $F \in \mathrm{Rep}(Q, \mathcal{A})$ and needs a bit more careful argumentation than the two first pairs.

Lemma 4.19. *Let $F \colon \mathcal{A} \to \mathcal{B}$ and $G \colon \mathcal{B} \to \mathcal{A}$ be functors between abelian categories such that F is a left adjoint of G. If for objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have that*

(i) the functor F sends every short exact sequence

$$0 \to GB \to Y \to A \to 0$$

to an exact sequence

$$0 \to FGB \to FY \to FA \to 0$$

(ii) and, the functor G sends every exact sequence

$$0 \to B \to X \to FA \to 0$$

to an exact sequence

$$0 \to GB \to GX \to GFA \to 0$$

then there is an isomorphism of abelian groups $\mathrm{YExt}^1_{\mathcal{B}}(GA, B) \cong \mathrm{YExt}^1_{\mathcal{A}}(A, FB)$.

Proof. Proof in appendix (Lemma A.2) □

Proposition 4.20. *Let x be any vertex in Q_0. For all objects $A \in \mathcal{A}$ and representations $F \in \mathrm{Rep}(Q, \mathcal{A})$ we have that*

$$\mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(P_x(A), F) \cong \mathrm{Ext}^1_{\mathcal{A}}(A, Fx)$$

and

$$\mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(F, I_x(A)) \cong \mathrm{Ext}^1_{\mathcal{A}}(Fx, A)$$

That is, the adjunctions $(P_x, -x)$ and $(-x, I_x)$ extends to Ext^1.

Proof. $-x$, P_x and I_x are exact functors, so the result follows from Lemma 4.19. □

Proposition 4.21 (Proposition 3.10 in [23], Proposition 5.4 in [16])**.** *Let x be any vertex in Q_0, $F \in \mathrm{Rep}(Q, \mathcal{A})$ any representation and $A \in \mathcal{A}$ any object.*

(i) There is an injective homomorphism of abelian groups

$$\mathrm{Ext}^1_{\mathcal{A}}(c_x(F), A) \hookrightarrow \mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(F, s_x(A))$$

If γ_x^F is a monomorphism, then the homomorphism is an isomorphism.

(ii) There is an injective homomorphism of abelian groups

$$\mathrm{Ext}^1_{\mathcal{A}}(A, k_x(F)) \hookrightarrow \mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(s_x(A), F)$$

If ψ_x^F is an epimorphism, then the homomorphism is an isomorphism.

Proof. We show (i), (ii) is dual. Let $\mathbf{E}$ be a representative of an element in $\mathrm{YExt}^1_{\mathcal{A}}(c_x(F), A)$,

$$\mathbf{E}\colon 0 \to A \xrightarrow{f} E \xrightarrow{g} c_x(F) \to 0$$

We form the following commutative diagram with exact rows and columns by taking the pullback of g and the cokernel of γ_x^F

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & A \arrow[r, equal] \arrow[d] & A \arrow[d, "f"] & \\
\coprod_{\alpha\in Q_1(*,x)} F(i(\alpha)) \arrow[r, "h"] \arrow[d, equal] & E' \arrow[r, "e'"] \arrow[d, "g'"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[r] \arrow[d, "g"] & 0 \\
\coprod_{\alpha\in Q_1(*,x)} F(i(\alpha)) \arrow[r, "\gamma_x^F"] & F(x) \arrow[r, "e"] \arrow[d] & c_x(F) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

Now, let $G \in \mathrm{Rep}(Q, \mathcal{A})$ be a representation given on each vertex $y \in Q_0$ by

$$G(y) = \begin{cases} E' & \text{if } y = x \\ F(y) & \text{else.} \end{cases}$$

And for an arrow $\alpha \in Q_1(z, y)$, we let $G(\alpha)\colon G(z) \to G(y)$ be given by

Case 1: If $z \neq x$ and $y \neq x$, then $G(\alpha) = F(\alpha)$.

Case 2: If $z \neq x$ and $y = x$, then $G(\alpha) = h \circ \iota_\alpha$ where ι_α is the inclusion corresponding to the arrow α, into the coproduct.

Case 3: If $z = x$ and $y \neq x$, then $G(\alpha) = F(\alpha) \circ g'$.

Now, we observe that the representation $G \in \mathrm{Rep}(Q, \mathcal{A})$ fits into a short exact sequence of representations,

$$\mathbf{G}\colon 0 \to s_x(A) \xrightarrow{\tilde{f}} G \xrightarrow{\tilde{g}} F \to 0$$

where the morphisms are given by

$$\tilde{f}_y = \begin{cases} f' & \text{if } y = x \\ 0 & \text{else.} \end{cases} \qquad \tilde{g}_y = \begin{cases} g' & \text{if } y = x \\ \mathrm{id}_{F(x)} & \text{else.} \end{cases}$$

Clearly, the assignment of $\mathbb{E} \in \mathrm{YExt}^1_{\mathcal{A}}(c_x(F), A)$ to $\mathbb{G} \in \mathrm{YExt}^1_{\mathrm{Rep}(Q,\mathcal{A})}(F, s_x(A))$ is well-defined. For $\mathbb{X}, \mathbb{Y} \in \mathrm{Ext}^1_{\mathcal{A}}(A, B)$ and $f \in \mathrm{Hom}_{\mathcal{A}}(B, B')$, we have

$$(\mathbb{X} + \mathbb{Y}) \cdot f = (\,1\ 1\,) \cdot \mathbb{X} \oplus \mathbb{Y} \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) f = (\,1\ 1\,) \cdot (\mathbb{X} \cdot f) \oplus (\mathbb{Y} \cdot f) \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$$

so our construction gives in fact a group homomorphism. Now for the injectivity, let us assume that $\mathbf{G}$ splits, so we have a morphism $\phi\colon G \to s_x(A)$ such that $\phi \circ \tilde{f} = \mathrm{id}_{s_x(A)}$

$$\begin{tikzcd} \mathbf{G}\colon & 0 \arrow[r] & s_x(A) \arrow[r, shift left, "\tilde{f}"] & G \arrow[l, shift left, "\phi"] \arrow[r, "\tilde{g}"] & F \arrow[r] & 0 \end{tikzcd}$$

If there are no arrows $\alpha \in Q_1$ which terminates in $x \in Q_0$, then $e'\colon E' \to E$ is the identity, and thus $f = f' = \tilde{f}_x$ is a split monomorphism, and $\mathbf{E}$ is split. If there is at least one arrow which terminates in $x \in Q_0$, we have for every such arrow $\alpha \in Q_1(*, x)$, that $\phi_x \circ h \circ \iota_\alpha = \phi_x \circ G(\alpha) = 0$, and consequently that $\phi_x \circ h = 0$. Now, $e'\colon E' \to E$ is the cokernel of h, so ϕ_x must factor through e', i.e. there exist some morphism $\psi\colon E \to A$, such that $\phi_x = \psi \circ e'$. Now,

$$\mathrm{id}_A = \phi_x \circ f' = \psi \circ e' \circ f' = \psi \circ f$$

so $\mathbf{E}$ is split. Thus, the constructed homomorphism is injective.

We now assume that $F \in \mathrm{Rep}(Q, \mathcal{A})$ is such that γ^F_x is a monomorphism, and claim that this makes the homomorphism surjective. Let

$$\mathbf{H}\colon 0 \to s_x(A) \xrightarrow{\phi} H \xrightarrow{\psi} F \to 0$$

be a short exact sequence in $\mathrm{Rep}(Q, \mathcal{A})$. This sequence induces the following commutative diagram with exact rows in $\mathcal{A}$,

$$\begin{tikzcd} 0 \arrow[r] & \coprod_{\alpha \in Q_1(*,x)} s_x(A)(i(\alpha)) \arrow[r] \arrow[d, "\gamma_x^{s_x(A)}"] & \coprod_{\alpha \in Q_1(*,x)} H(i(\alpha)) \arrow[r] \arrow[d, "\gamma_x^H"] & \coprod_{\alpha \in Q_1(*,x)} F(i(\alpha)) \arrow[r] \arrow[d, hook, "\gamma_x^F"] & 0 \\ 0 \arrow[r] & s_x(A)(x) \arrow[r] & H(x) \arrow[r] & F(x) \arrow[r] & 0 \end{tikzcd}$$

We have by assumption that $\mathrm{Ker}\gamma_x^F = 0$, so by the snake lemma we obtain the short exact sequence

$$0 \to \mathrm{Cok}\gamma_x^{s_x(A)} \xrightarrow{f} \mathrm{Cok}\gamma_x^H \xrightarrow{g} \mathrm{Cok}\gamma_x^F \to 0$$

Q is acyclic, so necessarily $\coprod_{\alpha\in Q_1(*,x)} s_x(A)(i(\alpha)) = \coprod_{\alpha\in Q_1(*,x)} 0 = 0$. Thus $\mathrm{Cok}\gamma_X^{s_x(A)} \cong A$, and we have gotten an element in $\mathrm{YExt}^1_{\mathcal{A}}(c_x(F), A)$. Further, $\coprod_{\alpha\in Q_1(*,x)} H(i(\alpha)) \cong \coprod_{\alpha\in Q_1(*,x)} F(i(\alpha))$, so we get the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & A \arrow[r, equal] \arrow[d] & A \arrow[d, "f"] & \\
0 \arrow[r] & \coprod\limits_{\alpha\in Q_1(*,x)} H(i(\alpha)) \arrow[r, "\gamma_x^H"] \arrow[d, "\cong"] & H(x) \arrow[r, "e'"] \arrow[d] & \mathrm{Cok}\gamma_x^H \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & \coprod\limits_{\alpha\in Q_1(*,x)} F(i(\alpha)) \arrow[r, "\gamma_x^F"] & F(x) \arrow[r, "e"] \arrow[d] & \mathrm{Cok}\gamma_x^F \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Observe that since $\mathrm{Ker}(e') \to \mathrm{Ker}(e)$ is an isomorphism and $\mathrm{Cok}(e') \to \mathrm{Cok}(e)$ is an isomorphism, that $H(x)$ is the pullback of f and e. We have therefore found a preimage for $\mathbb{H} \in \mathrm{YExt}^1_{\mathrm{Rep}(Q,\mathcal{A})}(F, s_x(A))$ in $\mathrm{YExt}^1_{\mathcal{A}}(c_x(F), A)$, and conclude therefore that the homomorphism is in fact an isomorphism. □

As we observed, the adjuncts of the stalk functor extends to Ext^1 for representations $F \in \mathrm{Rep}(Q, \mathcal{A})$ such that γ_x^F is monomorphic or λ_x^F is epimorphic. Hence, we would like to have some kind of condition on F for these induced morphism to be exactly that. The following constructive remark and the subsequent Proposition gives us such a condition.

Remark 4.22. Let $F \in \mathrm{Rep}(Q, \mathcal{A})$ be a representation and $\{f_\alpha\}_{\alpha\subseteq Q_1(*,x)}$ be any set of morphisms $f_\alpha \colon F(i(\alpha)) \to A$ in $\mathcal{A}$. There is a representation $G \in \mathrm{Rep}(Q, \mathcal{A})$

fitting into a short exact sequence

$$0 \to s_x(A) \xrightarrow{m} G \xrightarrow{e} F \to 0$$

given on objects as

$$G(y) = \begin{cases} F(y) & \text{if } y \neq x \\ F(x) \oplus A & \text{if } y = x \end{cases}$$

Proof. Let $G \in \mathrm{Rep}(Q, \mathcal{A})$ be given on objects as described. If $\beta \in Q_1(z, y)$ is an arrow in Q, we define the corresponding morphism $G(\beta)\colon G(z) \to G(y)$ as

Case 1: If $z \neq x$ and $y \neq x$,

$$G(\beta) = F(\beta)\colon F(z) \to F(y).$$

Case 2: If $z \neq x$ and $y = x$,

$$G(\beta) = \begin{pmatrix} F(\beta) \\ f_\beta \end{pmatrix}\colon F(z) \to F(x) \oplus A$$

Case 3: If $z = x$ and $y \neq x$,

$$G(\beta) = \begin{pmatrix} F(\beta) & 0 \end{pmatrix}\colon F(x) \oplus A \to F(y)$$

Now, G fits into the short exact sequence

$$0 \to s_x(A) \xrightarrow{m} G \xrightarrow{e} F \to 0$$

where

$$m_z = \begin{cases} \begin{pmatrix} 0 \\ \mathrm{id}_A \end{pmatrix} & \text{if } z = x \\ 0 & \text{else.} \end{cases} \qquad e_z = \begin{cases} \begin{pmatrix} \mathrm{id}_{Fx} & 0 \end{pmatrix} & \text{if } z = x \\ \mathrm{id}_{Fz} & \text{else.} \end{cases}$$

In order to verify that m and e is in fact morphisms of representations, we check whether the diagram

$$\begin{tikzcd} S_x(A)(z) \arrow[r, "m_z"] \arrow[d, "s_x(A)(\beta)=0"'] & G(z) \arrow[r, "e_z"] \arrow[d, "G(\beta)"] & F(z) \arrow[d, "F(\beta)"] \\ S_x(A)(y) \arrow[r, "m_y"] & G(y) \arrow[r, "e_y"] & F(y) \end{tikzcd}$$

commutes for every arrow $\beta\colon z \to y$ in Q. If $z = x$ we get $G(\beta) \circ m_x = \left(\begin{smallmatrix} F(\beta) & 0 \end{smallmatrix} \right) \left(\begin{smallmatrix} 0 \\ \mathrm{id}_A \end{smallmatrix} \right) = 0 = m_y \circ 0$, so the left-hand square is commutative for every arrow $\beta \in Q_1(z, y)$. Let us now go through the right-hand side for each of the cases given above.

Case 1:

$$e_y \circ G(\beta) = \mathrm{id}_{Fy} \circ F(\beta) = F(\beta) \circ \mathrm{id}_{Fz} = F(\beta) \circ e_z$$

Case 2:

$$e_x \circ G(\beta) = \left(\begin{smallmatrix} \mathrm{id}_{Fx} & 0 \end{smallmatrix} \right) \left(\begin{smallmatrix} F(\beta) \\ f_\beta \end{smallmatrix} \right) = \mathrm{id}_{Fx} \circ F(\beta) = F(\beta) \circ \mathrm{id}_{Fz} = F(z) \circ e_z$$

Case 3:

$$e_y \circ G(\beta) = \mathrm{id}_{Fy} \circ \left(\begin{smallmatrix} F(\beta) & 0 \end{smallmatrix} \right) = F(\beta) \circ \left(\begin{smallmatrix} \mathrm{id}_{Fz} & 0 \end{smallmatrix} \right) = F(\beta) \circ e_z$$

□

Proposition 4.23 ([16, Prop. 5.6]). *Let $x \in Q_0$ be any vertex, and $F \in \mathrm{Rep}(Q, \mathcal{A})$ and $A \in \mathcal{A}$ any objects. Then*

(i) If $\mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(F, s_x(A)) = 0$, then $\mathrm{Hom}_{\mathcal{A}}(\gamma^F_x, A)$ is an epimorphism. Further, if $\mathcal{A}$ has enough injectives and $\mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(F, s_x(I)) = 0$ for every injective object $I \in \mathrm{Inj}\mathcal{A}$, then γ^F_x is a monomorphism.

(ii) If $\mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(s_x(A), F) = 0$, then $\mathrm{Hom}_{\mathcal{A}}(A, \lambda^F_x)$ is an epimorphism. Further, if $\mathcal{A}$ has enough projectives and $\mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(s_x(P), F) = 0$ for every projective object $P \in \mathrm{Proj}\mathcal{A}$, then λ^F_x is an epimorphism.

Proof. (i): $\mathrm{Hom}_{\mathcal{A}}(\gamma^F_x, A)$ is an epimorphism if and only if for every morphism $f\colon \coprod_{\alpha \in Q_1(*,x)} F(i(\alpha)) \to A$ in $\mathcal{A}$, there exists a morphism $g\colon F(x) \to A$ such that the following triangle commutes

$$\begin{tikzcd} \coprod_{\alpha \in Q_1(*,x)} F(i(\alpha)) \arrow[r, "\gamma^F_x"] \arrow[d, "f"] & F(x) \arrow[dl, dashed, "g"] \\ A & \end{tikzcd}$$

By precomoposing f with the canonical injections $\iota_\alpha\colon F(i(\alpha)) \hookrightarrow \coprod_{\alpha' \in Q_1(*,x)} F(i(\alpha'))$, we obtain a collection of morphisms $\{f \circ \iota_\alpha\}_{\alpha \in Q(*,x)}$ as in the construction of Remark 4.22. Thus we get the short exact sequence

$$0 \to s_x(A) \xrightarrow{m} G \xrightarrow{e} F \to 0$$

Now, by the hypothesis, $\mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(F, s_x(A)) = 0$, so the sequence splits, and we obtain a right-inverse $r\colon F \to G$ to e. For any vertex $y \in Q_0$ not equal to x, we have $e_y = \mathrm{id}_{Fy}$, so necessarily $r_y = \mathrm{id}_{Fy}$. For the vertex x, r must be on the form

$$r_x = \begin{pmatrix} r_x^1 \\ r_x^2 \end{pmatrix} \colon F(x) \to G(x) = F(x) \oplus A$$

r_x is a right-inverse of e_x so $\mathrm{id}_{Fx} = e_x \circ r_x = \begin{pmatrix} \mathrm{id}_{Fx} & 0 \end{pmatrix} \begin{pmatrix} r_x^{(1)} \\ r_x^{(2)} \end{pmatrix} = r_x^{(1)}$. For every arrow $\alpha \in Q_1(*,x)$ terminating in x, we have a commutative diagram, from r being a morphism of representations,

$$\begin{tikzcd} F(i(\alpha)) \arrow[r, equal, "r_z = \mathrm{id}_{Fz}"] \arrow[d, "F(\alpha)"] & G(i(\alpha)) \arrow[d, "G(\alpha) = \begin{pmatrix} F(\alpha) \\ f\iota_\alpha \end{pmatrix}"] & \\ F(x) \arrow[r, "r_x = \begin{pmatrix} \mathrm{id}_{Fx} \\ r_x^2 \end{pmatrix}"] & G(x) \arrow[r, equal] & F(x) \oplus A \end{tikzcd}$$

which gives us that $r_x^{(2)} \circ F(\alpha) = f \circ \iota_\alpha$. By definition we have that $\gamma_x^F \circ \iota_\alpha = F(\alpha)$ for every arrow α terminating in x, thus $r_x^{(2)} \circ \gamma_x^F \circ \iota_\alpha = f \circ \iota_\alpha$ for every such arrow. This forces $r_x^{(2)} \circ \gamma_x^F = f$ by the universal property of the coproduct, and by setting $g = r_x^{(2)}$ we have obtained the wanted morphism, and we can conclude that $\mathrm{Hom}_{\mathcal{A}}(\gamma_x^F, A)$ is an epimorphism.

If $\mathcal{A}$ has enough injectives $\mathrm{Inj}\mathcal{A} \subseteq \mathcal{A}$, then we can find a monomorphism

$$f\colon \coprod_{\alpha \in Q_1(*,x)} F(i(\alpha)) \hookrightarrow I$$

into an injective object $I \in \mathrm{Inj}\mathcal{A}$. If also $\mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(\gamma_x^F, -)|_{\mathrm{Inj}\mathcal{A}} = 0$, then the discussion above gives that we have a morphism $g\colon F(x) \to I$, such that $f = g \circ \gamma_x^F$, thus forcing γ_x^F to also be a monomorphism.

(ii) follows similarly. $\square$

Definition 4.24. Let $\mathcal{C} \subseteq \mathcal{A}$ be a full subcategory of $\mathcal{A}$. Let

$$\begin{aligned}
\mathrm{Rep}(Q, \mathcal{C}) &= \{F \in \mathrm{Rep}(Q, \mathcal{A}) \mid F(x) \in \mathcal{C} \ \forall\ x \in Q_0\} \\
P_*(\mathcal{C}) &= \{P_x(C) \mid x \in Q_0,\ C \in \mathcal{C}\} \\
I_*(\mathcal{C}) &= \{I_x(C) \mid x \in Q_0,\ C \in \mathcal{C}\} \\
\Gamma(\mathcal{C}) &= \{F \in \mathrm{Rep}(Q, \mathcal{A}) \mid \gamma_x^F \text{ is monic and } \mathrm{Cok}\gamma_x^F \in \mathcal{C} \ \forall\ x \in Q_0\} \\
\Lambda(\mathcal{C}) &= \{F \in \mathrm{Rep}(Q, \mathcal{A}) \mid \lambda_x^F \text{ is epic and } \mathrm{Ker}\lambda_x^F \in \mathcal{C} \ \forall\ x \in Q_0\}
\end{aligned}$$

Proposition 4.25 ([16, Prop. 7.3])**.** *Let $\mathcal{B} \subseteq \mathcal{A}$ be any class of objects in $\mathcal{A}$, then*

(i) $P_*(\mathcal{B})^{\perp_1} = \mathrm{Rep}(Q, \mathcal{B}^{\perp_1})$.

(ii) ${}^{\perp_1}I_*(\mathcal{B}) = \mathrm{Rep}(Q, {}^{\perp_1}\mathcal{B})$.

(iii) if $\mathcal{A}$ has enough injectives such that $\mathrm{Inj}\mathcal{A} \subseteq \mathcal{B}$ *we have* ${}^{\perp_1}s_*(\mathcal{B}) = \Gamma({}^{\perp_1}\mathcal{B})$.

(iv) if $\mathcal{A}$ has enough projectives such that $\mathrm{Proj}\mathcal{A} \subseteq \mathcal{B}$ *we have* $s_*(\mathcal{B})^{\perp_1} = \Lambda(\mathcal{B}^{\perp_1})$.

Proof. Similarly to the preceding proofs, we will only prove (i) and (iii). (ii) and (iv) follows by analogous arguments, or by passing to the opposite setting. To prove (i) we recall from Proposition 4.20 that for any vertex $x \in Q_0$, object $A \in \mathcal{A}$ and representation $F \in \mathrm{Rep}(Q, \mathcal{A})$ we have

$$\mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(P_x(A), F) \cong \mathrm{Ext}^1_{\mathcal{A}}(A, Fx)$$

That is, if $B \in \mathcal{B}$ and $Fx \in \mathrm{Rep}(Q, \mathcal{B}^{\perp_1})$ we have

$$\mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(P_x(B), F) \cong \mathrm{Ext}^1_{\mathcal{A}}(B, Fx) = 0$$

Thus $P_*(\mathcal{B})^{\perp_1} \supseteq \mathrm{Rep}(Q, \mathcal{B}^{\perp_1})$. The converse inclusion is shown similarly.

(iii) In Proposition 4.21 we saw that for any vertex $x \in Q_0$, any representation $F \in \mathrm{Rep}(Q, \mathcal{A})$ and $A \in \mathcal{A}$ we have the inclusion

$$\mathrm{Ext}^1_{\mathcal{A}}(c_x(F), A) \hookrightarrow \mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(F, s_x(A))$$

which is an isomorphism if γ_x^F is a monomorphism. If $F \in \Gamma({}^{\perp_1}\mathcal{B})$ and $B \in \mathcal{B}$ we have for each $x \in Q_0$

$$0 = \mathrm{Ext}^1_{\mathcal{A}}(c_x(F), B) \cong \mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(F, s_x(B))$$

since $c_x(F) = \mathrm{Cok}\gamma_x^F \in^{\perp_1} \mathcal{B}$, consequently ${}^{\perp_1}s_*(\mathcal{B}) \supseteq \Gamma({}^{\perp_1}\mathcal{B})$. If $F \in^{\perp_1} s_*(\mathcal{B})$, then for each $x \in Q_0$ and $B \in \mathcal{B}$ we have

$$\mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(F, s_x(B)) = 0$$

We also have that $\mathrm{Inj}\mathcal{A} \subseteq \mathcal{B}$, so γ_x^F is a monomorphism by Proposition 4.23. Therefore we have

$$\mathrm{Ext}^1_{\mathcal{A}}(c_x(F), B) \cong \mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(F, s_x(B)) = 0$$

so $c_x(F) \in^{\perp_1} \mathcal{B}$, thus we conclude that $F \in \Gamma({}^{\perp_1}\mathcal{B})$ and $s_*(\mathcal{B}) \subseteq \Gamma({}^{\perp_1}\mathcal{B})$.

□

In the next proof we will be using a filtration of the set of vertices in our finite acyclic quiver. This filtration coincide with what Holm et al. call *transfinite sequence* in Section 2.5 of [16], for a general (possibly infinite) quiver.

Remark 4.26. The filtration of our quivers will be the set of subsets $\{Q_0(i) \subseteq Q_0\}$ given by

$$\begin{aligned}
&Q_0(0) = \varnothing \\
&Q_0(1) = \{x \in Q_0 \mid Q_1(*, x) = \varnothing, \text{ i.e. } x \text{ is initial}\} \\
&\qquad\vdots \\
&Q_0(i) = \{x \in Q_0 \mid \text{There is no arrow } \alpha \in Q_1(z,x) \text{ s.t. } z \in Q_0 \setminus Q_0(i-1)\} \\
&\qquad\vdots
\end{aligned}$$

We will denote the filtration by $\{Q_0(i)\}_{0 \leq i \leq k}$, where $k \in \mathbb{N}$ is the smallest positive integer such that $Q_0(k) = Q_0$. This integer exist since Q is assumed finite acyclic. To familiarize us with the concept, let us look at a few examples.

Example 4.27. Let Q be the quiver

$$\begin{tikzcd} \underset{5}{\bullet} \arrow[r] & \underset{6}{\bullet} \\ \underset{3}{\bullet} \arrow[u] \arrow[r] & \underset{4}{\bullet} \arrow[u] \\ \underset{1}{\bullet} \arrow[u] \arrow[r] & \underset{2}{\bullet} \arrow[u] \end{tikzcd}$$

The filtration $\{Q_0(i)\}_{0\leq i\leq 4}$ of Q is given by

$$\begin{tikzcd} \underset{5}{\circ} \arrow[r, dotted] & \underset{6}{\circ} \\ \underset{3}{\circ} \arrow[u, dotted] \arrow[r, dotted] & \underset{4}{\circ} \arrow[u, dotted] \\ \underset{1}{\circ} \arrow[u, dotted] \arrow[r, dotted] & \underset{2}{\circ} \arrow[u, dotted] \end{tikzcd} \qquad \begin{tikzcd} \underset{5}{\circ} \arrow[r, dotted] & \underset{6}{\circ} \\ \underset{3}{\circ} \arrow[u, dotted] \arrow[r, dotted] & \underset{4}{\circ} \arrow[u, dotted] \\ \underset{1}{\bullet} \arrow[u, dotted] \arrow[r, dotted] & \underset{2}{\circ} \arrow[u, dotted] \end{tikzcd} \qquad \begin{tikzcd} \underset{5}{\circ} \arrow[r, dotted] & \underset{6}{\circ} \\ \underset{3}{\bullet} \arrow[u, dotted] \arrow[r, dotted] & \underset{4}{\circ} \arrow[u, dotted] \\ \underset{1}{\bullet} \arrow[u, dotted] \arrow[r, dotted] & \underset{2}{\bullet} \arrow[u, dotted] \end{tikzcd} \qquad \begin{tikzcd} \underset{5}{\bullet} \arrow[r, dotted] & \underset{6}{\circ} \\ \underset{3}{\bullet} \arrow[u, dotted] \arrow[r, dotted] & \underset{4}{\bullet} \arrow[u, dotted] \\ \underset{1}{\bullet} \arrow[u, dotted] \arrow[r, dotted] & \underset{2}{\bullet} \arrow[u, dotted] \end{tikzcd} \qquad \begin{tikzcd} \underset{5}{\bullet} \arrow[r, dotted] & \underset{6}{\bullet} \\ \underset{3}{\bullet} \arrow[u, dotted] \arrow[r, dotted] & \underset{4}{\bullet} \arrow[u, dotted] \\ \underset{1}{\bullet} \arrow[u, dotted] \arrow[r, dotted] & \underset{2}{\bullet} \arrow[u, dotted] \end{tikzcd}$$

$Q_0(0)$ $Q_0(1)$ $Q_0(2)$ $Q_0(3)$ $Q_0(4)$

♣

Example 4.28. Let Q be the quiver

$$\begin{tikzcd} \underset{4}{\bullet} & \underset{5}{\bullet} \\ \underset{2}{\bullet} \arrow[u] \arrow[ur] & \underset{3}{\bullet} \arrow[u] \arrow[l] \\ \underset{1}{\bullet} \arrow[u] \arrow[ur] & \end{tikzcd}$$

The filtration $\{Q_0(i)\}_{0\leq i\leq 4}$ of Q is given by

$$\begin{tikzcd} \underset{4}{\circ} & \underset{5}{\circ} \\ \underset{2}{\circ} \arrow[u, dotted] \arrow[ur, dotted] & \underset{3}{\circ} \arrow[u, dotted] \arrow[l, dotted] \\ \underset{1}{\circ} \arrow[u, dotted] \arrow[ur, dotted] & \end{tikzcd} \qquad \begin{tikzcd} \underset{4}{\circ} & \underset{5}{\circ} \\ \underset{2}{\circ} \arrow[u, dotted] \arrow[ur, dotted] & \underset{3}{\circ} \arrow[u, dotted] \arrow[l, dotted] \\ \underset{1}{\bullet} \arrow[u, dotted] \arrow[ur, dotted] & \end{tikzcd} \qquad \begin{tikzcd} \underset{6}{\circ} & \underset{5}{\circ} \\ \underset{2}{\circ} \arrow[u, dotted] \arrow[ur, dotted] & \underset{3}{\bullet} \arrow[u, dotted] \arrow[l, dotted] \\ \underset{1}{\bullet} \arrow[u, dotted] \arrow[ur, dotted] & \end{tikzcd} \qquad \begin{tikzcd} \underset{4}{\circ} & \underset{5}{\circ} \\ \underset{2}{\bullet} \arrow[u, dotted] \arrow[ur, dotted] & \underset{3}{\bullet} \arrow[u, dotted] \arrow[l, dotted] \\ \underset{1}{\bullet} \arrow[u, dotted] \arrow[ur, dotted] & \end{tikzcd} \qquad \begin{tikzcd} \underset{4}{\bullet} & \underset{5}{\bullet} \\ \underset{2}{\bullet} \arrow[u, dotted] \arrow[ur, dotted] & \underset{3}{\bullet} \arrow[u, dotted] \arrow[l, dotted] \\ \underset{1}{\bullet} \arrow[u, dotted] \arrow[ur, dotted] & \end{tikzcd}$$

$Q_0(0)$ $Q_0(1)$ $Q_0(2)$ $Q_0(3)$ $Q_0(4)$

♣

Proposition 4.29 ([16, Thm. 7.4 & 7.9])**.** *Let $(\mathcal{C}, \mathcal{D})$ be a cotorsion pair in an abelian category $\mathcal{A}$.*

(i) If $\mathcal{A}$ has enough injectives, then

$$(\Gamma(\mathcal{C}), \mathrm{Rep}(Q, \mathcal{D}))$$

is a cotorsion pair generated by $P_*(\mathcal{C})$ and cogenerated by $s_*(\mathcal{D})$.

(ii) If $\mathcal{A}$ has enough projectives, then

$$(\mathrm{Rep}(Q, \mathcal{C}), \Lambda(\mathcal{D}))$$

is a cotorsion pair cogenerated by $I_*(\mathcal{D})$ and generated by $s_*(\mathcal{C})$.

Proof. (i): Proposition 4.25 tells ut that

$$P_*(\mathcal{C})^{\perp_1} = \mathrm{Rep}(Q, \mathcal{C}^{\perp_1}) = \mathrm{Rep}(Q, \mathcal{D})$$

and

$$^{\perp_1}s_*(\mathcal{D}) = \Gamma(^{\perp_1}\mathcal{D}) = \Gamma(\mathcal{C})$$

so we must show that $\mathrm{Rep}(Q, \mathcal{D}) = \Gamma(\mathcal{C})^{\perp_1}$. Let $C \in \mathcal{C}$ and $D \in \mathcal{D}$ be any objects, then for every pair of vertices $x, y \in Q_0$, we have

$$\mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(P_x(C), s_y(D)) \cong \mathrm{Ext}^1_{\mathcal{A}}(C, s_y(D)(x))$$

since $(P_x, -x)$ is an adjoint pair. Further

$$\mathrm{Ext}^1_{\mathcal{A}}(C, s_y(D)(x)) = \begin{cases} \mathrm{Ext}^1_{\mathcal{A}}(C, D) = 0 & \text{if } y = x \\ \mathrm{Ext}^1_{\mathcal{A}}(C, 0) = 0 & \text{if } y \neq x \end{cases}$$

Thus $P_*(\mathcal{C}) \subseteq^{\perp_1} s_*(\mathcal{D})$ and

$$\mathrm{Rep}(Q, \mathcal{D}) = P_*(\mathcal{C})^{\perp_1} \supseteq (^{\perp_1}s_*(\mathcal{D}))^{\perp_1} = \Gamma(\mathcal{C})^{\perp_1}$$

In order to prove the converse inclusion, let $\{Q_0(i)\}_{0 \leq i \leq k}$ be a filtration of the vertices in Q. Now, let $F \in \mathrm{Rep}(Q, \mathcal{D})$. We construct a chain of subrepresentations

$$F_0 \overset{m_1}{\hookrightarrow} F_1 \overset{m_2}{\hookrightarrow} \cdots \overset{m_k}{\hookrightarrow} F_k = F$$

in the following way. F_i is given by

$$F_i(x) = \begin{cases} F(x) & \text{if } x \in Q_i(x) \\ 0 & \text{else.} \end{cases}$$

on objects and for any arrow $\alpha\colon x \to y$ by

$$F_i(\alpha) = \begin{cases} F(\alpha) & \text{if both } x, y \in Q_0(i) \\ 0 & \text{else.} \end{cases}$$

We clearly have monomorphisms $m_i\colon F_{i-1} \to F_i$, so we have obtained our chain of subrepresentations. Observe that

$$\operatorname{Cok}(m_i)(x) = \begin{cases} F(x) & \text{if } x \in Q_0(i) \setminus Q_0(i-1) \\ 0 & \text{else.} \end{cases}$$

Let $\alpha\colon x \to y$ be any arrow in Q_1. If x is in $Q_0(i) \setminus Q_0(i-1)$ we necessarily have that $y \in Q_0 \setminus Q_0(i)$, so $F_i(y) = 0$ and thus $\operatorname{Cok}(m_i)(\alpha) = 0$. If y is in $Q_0(i) \setminus Q_0(i-1)$ then $x \in Q_0(i-1)$, so $\operatorname{Cok}(m_i)(x) = 0$ and thus also $\operatorname{Cok}(m_i)(\alpha) = 0$. Therefore we have

$$\operatorname{Cok}(m_i) = \bigoplus_{x \in Q_0(i) \setminus Q_0(i-1)} s_x(F(x)),$$

and since F lies in $\operatorname{Rep}(Q, \mathcal{D})$, we have $s_x(F(x)) \in s_*(\mathcal{D}) \subseteq ({}^{\perp_1} s_*(\mathcal{D}))^{\perp_1} = \Gamma(\mathcal{C})^{\perp_1}$ further since $\Gamma(\mathcal{D})^{\perp_1}$ is closed under products we also have that $\operatorname{Cok}(m_i) \in \Gamma(\mathcal{D})^{\perp_1}$. We claim now that every representation F_i for $0 \le i \le k$ lies in $\Gamma(\mathcal{D})^{\perp_1}$. Obviously $F_0 = 0 \in \Gamma(\mathcal{D})^{\perp_1}$, so we assume that $F_{i-1} \in \Gamma(\mathcal{D})^{\perp_1}$, so we obtain the short exact sequence

$$0 \to F_{i-1} \xrightarrow{m_i} F_i \to \operatorname{Cok}(m_i) \to 0$$

and since $\Gamma(\mathcal{D})^{\perp_1}$ is closed under extensions, and $\operatorname{Cok}(m_i), F_{i-1} \in \Gamma(\mathcal{D})^{\perp_1}$ we conclude that $F_i \in \Gamma(\mathcal{D})^{\perp_1}$ for every $0 \le i \le k$. Thus

$$\operatorname{Rep}(Q, \mathcal{D}) \subseteq \Gamma(\mathcal{C})^{\perp_1}$$

(ii): Dual proof. □

We can describe the subcategories $\Gamma(\mathcal{C})$ and $\Lambda(\mathcal{D})$ even more explicit, as shown in the following lemma.

Lemma 4.30 ([16, Proposition 7.2]). *Let $(\mathcal{C}, \mathcal{D})$ be a cotorsion pair in an abelian category $\mathcal{A}$, and Q be a finite acyclic quiver. Then*

- *if F is a representation in $\Gamma(\mathcal{C})$, then $F_x \in \mathcal{C}$ for every vertex $x \in Q_0$, and*
- *if F is a representation in $\Lambda(\mathcal{D})$, then $F_x \in \mathcal{D}$ for every vertex $x \in Q_0$.*

Proof. The statements are proven in a dually manner, so it is sufficient to prove the first statement. Let $\{Q_0(i)\}_{0 \leq i \leq k}$ be the filtration given in Remark 4.26. We observe that for every vertex $x \in Q_0(1)$ the claim is trivially true, thus let us assume that it has been shown for every filtration step $i < n$. Let $x \in Q_0(n)$, then we obtain the short exact sequence

$$0 \to \bigoplus_{\alpha \in Q_1(y,x)} F_y \xrightarrow{\gamma_x^F} F_x \to \operatorname{Cok}(\gamma_x^F) \to 0$$

where the first term is a direct sum of objects in $\mathcal{C}$, and therefore also an object of $\mathcal{C}$. Fx is therefore trapped between two objects of $\mathcal{C}$, and since $\mathcal{C}$ is closed under extension it is also in $\mathcal{C}$. □

As an easy corollary to the preceding theorem we get the following characterization of projective and injective representation when working over an abelian category having enough injectives and projectives.

Corollary 4.31. *Let $\mathcal{A}$ be an abelian category with enough projectives and injectives, and Q a finite acyclic quiver. Then the set of projective objects in* $\operatorname{Rep}(Q, \mathcal{A})$ *is given as*

$$\operatorname{ProjRep}(Q, \mathcal{A}) = \{F \in \operatorname{Rep}(Q, \mathcal{A}) \mid \gamma_x^F \textit{ is split mono and } F_x \in \operatorname{Proj}\mathcal{A} \ \forall\ x \in Q_0\}$$

and the set of injective objects in $\operatorname{Rep}(Q, \mathcal{A})$ *is given as*

$$\operatorname{InjRep}(Q, \mathcal{A}) = \{F \in \operatorname{Rep}(Q, \mathcal{A}) \mid \lambda_x^F \textit{ is split epi and } F_x \in \operatorname{Inj}\mathcal{A} \ \forall\ x \in Q_0\}$$

Proof. We prove the claim for the projectives, the claim regarding injectives is completely dual. In $\mathcal{A}$ we have the trivial cotorsion pair $(\mathcal{A}, \mathrm{Proj}\mathcal{A})$ which induces the cotorsion pair $(\Gamma(\mathrm{Proj}\mathcal{A}), \mathrm{Rep}(Q, \mathcal{A}))$ since $\mathcal{A}$ has enough injectives. Thus, since a cotorsion pair is completely described from either part of it, we see that the cotorsion pair $(\Gamma(\mathrm{Proj}\mathcal{A}), \mathrm{Rep}(Q, \mathcal{A}))$ is nothing but the trivial cotorsion pair $(\mathrm{Rep}(Q, \mathcal{A}), \mathrm{Proj}\mathcal{A})$ of $\mathrm{Rep}(Q, \mathcal{A})$. Observing further that the short exact sequence

$$0 \to \bigoplus_{\alpha \in Q_1(y,x)} F_y \xrightarrow{\gamma_x^F} F_x \to \mathrm{Cok}\gamma_x^F \to 0 \to F_x$$

splits for every $F \in \Gamma(\mathrm{Proj}\mathcal{A})$ gives that

$$\mathrm{ProjRep}(Q, \mathcal{A}) = \Gamma(\mathrm{Proj}\mathcal{A}) = \left\{ F \in \mathrm{Rep}(Q, \mathcal{A}) \;\middle|\; \begin{array}{l} \gamma_x^F \text{ is split mono and} \\ F_x \in \mathrm{Proj}\mathcal{A} \;\forall\; x \in Q_0 \end{array} \right\}$$

□

4.2.1 Completeness

In order to show that the induced cotorsion pairs are complete, we recall that by Salce's Lemma (Lemma 3.25) we only need to show that $((\Gamma(\mathcal{C}), \mathrm{Rep}(Q, \mathcal{D}))$ has enough projectives. The proof is a specialization of the constructions used in [23] and [11]. By passing to the finite case, we are able to omit some of the more lengthy and formal arguments.

Theorem 4.32 ([23, Thm. 4.6])**.** *Adopting the setup of Proposition 4.29, we have that the induced cotorsion pairs are complete when $(\mathcal{C}, \mathcal{D})$ is complete.*

Proof. We will show the claim for the cotorsion pair $(\Gamma(\mathcal{C}), \mathrm{Rep}(Q, \mathcal{D}))$ in an abelian category with enough injectives. The other claim is shown dually.

As already noted, it suffices to show that the cotorsion pair has enough projectives. Let $F \in \mathrm{Rep}(Q, \mathcal{A})$ be any representation. We will be constructing a short exact sequence

$$0 \to D \to C \to F \to 0$$

with $C \in \Gamma(\mathcal{C})$ and $D \in \operatorname{Rep}(Q, \mathcal{D})$. First, observe that since $(\mathcal{C}, \mathcal{D})$ is a complete cotorsion pair in $\mathcal{A}$, we can construct short exact sequences

$$0 \to D'(x) \to C'(x) \to F(x) \to 0$$

for every vertex $x \in Q_0$ such that $D'(x) \in \mathcal{D}$ and $C'(x) \in \mathcal{C}$. For every arrow $\alpha\colon x \to y$ in Q_0 we can finish the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & D'(x) \arrow[r] \arrow[d, dashed, "D'(\alpha)"] & C'(x) \arrow[r] \arrow[d, dashed, "C'(\alpha)"] & F(x) \arrow[r] \arrow[d, "F(\alpha)"] & 0 \\
0 \arrow[r] & D'(y) \arrow[r] & C'(y) \arrow[r] & F(y) \arrow[r] & 0
\end{tikzcd}$$

as we observed in the proof of Lemma 3.27. Thus we have a short exact sequence

$$0 \to D' \to C' \to F \to 0$$

of representations, such that $D' \in \operatorname{Rep}(Q, \mathcal{D})$ and $C' \in \operatorname{Rep}(Q, \mathcal{C})$. Now, C' is not necessarily in $\Gamma(\mathcal{C})$, so we have to somehow remedy this. We will be doing this iteratively by means of the filtration, $\{Q_0(i)\}_{0<i\leq k}$, of Q_0 given in Remark 4.26. For each $0 \leq i \leq k$ we will be constructing a short exact sequence

$$\mathbf{C}(i)\colon 0 \to D_i \xrightarrow{m(i)} C_i \xrightarrow{e(i)} F \to 0$$

in $\operatorname{Rep}(Q, \mathcal{A})$ such that

(i) for every vertex $x \in Q_0 \setminus Q_0(i)$, $C_i(x) = C'(x)$ and $D_i(x) = D'(x)$.

(ii) for every vertex $x \in Q_0(i)$, $\gamma_x^{C_i}$ is mono, $C_i(x), \operatorname{Cok}(\gamma_x^{C_i}) \in \mathcal{C}$ and $D_i(x) \in \mathcal{D}$.

We start by setting $C_1 = C'$ and $D_1 = D'$. Assume that we have constructed $\mathbf{C}(i)$ for every $i < n$, then we construct $\mathbf{C}(n)$ as follows.

For any vertex $x \in Q_0(n) \setminus Q_0(n-1)$ we have that if there is an arrow $\alpha\colon z \to x \in Q_1$, then $z \in Q_0(n-1)$ and thus $C_{n-1}(z) \in \mathcal{C}$, so specifically $\coprod_{\alpha \in Q_1(*,x)} C_{n-1}(i(\alpha)) \in \mathcal{C}$. By using the completeness of $(\mathcal{C}, \mathcal{D})$, we now obtain a short exact sequence

$$0 \to \coprod_{\alpha' \in Q_1(*,x)} C_{n-1}(i(\alpha')) \xrightarrow{f_n^x} X_n^x \to Y_n^x \to 0$$

with $Y_n^x \in \mathcal{C}$ and $X_n^x \in \mathcal{C} \cap \mathcal{D}$. A precomposition with the canonical inclusion $\iota_\alpha \colon C_{n-1}(i(\alpha)) \hookrightarrow \coprod_{\alpha' \in Q_1(*,x)} C_{n-1}(i(\alpha'))$, gives us a monomorphism

$$f_n^x \circ \iota_\alpha \colon C_{n-1}(i(\alpha)) \hookrightarrow \coprod_{\alpha' \in Q_1(*,x)} C_{n-1}(i(\alpha')) \hookrightarrow X_n^x$$

for each arrow $\alpha \in Q_1(*,x)$. Since $x \in Q_0(n) \setminus Q_0(n-1) \subseteq Q_0 \setminus Q_0(n-1)$, we also get

$$C_{n-1}(\alpha) \colon C_{n-1}(i(\alpha)) \to C_{n-1}(x) = C'(x)$$

for these arrows. Hence, we obtain the morphism

$$\begin{pmatrix} \gamma_x^{C_{n-1}} \\ f_n^x \end{pmatrix} \colon \coprod_{\alpha' \in Q_1(*,x)} C_{n-1}(i(\alpha')) \to C'(x) \oplus X_n^x$$

which in fact is a monomorphism since f_n^x is a monomorphism.

Now, we define C_n and D_n in the following way. For vertices $x \in Q_0$ let

$$C_n(x) = \begin{cases} C_{n-1}(x) & \text{if } x \notin Q_0(n) \setminus Q_0(n-1) \\ C'(x) \oplus X_n^x & \text{else.} \end{cases}$$

and

$$D_n(x) = \begin{cases} D_n(x) & \text{if } x \notin Q_0(n) \setminus Q_0(n-1) \\ D'(x) \oplus X_n^x & \text{else.} \end{cases}$$

For arrows $\alpha \colon x \to y \in Q_1$ we define $C_n(\alpha)$ and $D_n(\alpha)$ as follows

- If $y \in Q_0(n) \setminus Q_0(n-1)$, then $x \in Q_0(n-1)$. In that case we set $C_n(\alpha)$ and $D_n(\alpha)$ as

$$C_n(\alpha) \colon C_n(x) = C_{n-1}(x) \xrightarrow{\begin{pmatrix} C_{n-1}(\alpha) \\ f_n^y \circ \iota_\alpha \end{pmatrix}} C'(y) \oplus X_n^y = C_n(y)$$

and

$$D_n(\alpha) \colon D_n(x) = D_{n-1}(x) \xrightarrow{\begin{pmatrix} D_{n-1}(\alpha) \\ f_n^y \circ \iota_\alpha \circ m(n-1)_y \end{pmatrix}} D'(y) \oplus X_n^y = D_n(y)$$

- If $x \in Q_0(n) \setminus Q_0(n-1)$, then $y \in Q_0 \setminus Q_0(n)$. We set $C_n(\alpha)$ and $D_n(\alpha)$ as the compositions

$$C_n(x) = C'(x) \oplus X_n^x \xrightarrow{\rho_C} C'(x) \xrightarrow{C'(\alpha)} C'(y) = C_n(y)$$

$$D_n(x) = D'(x) \oplus X_n^x \xrightarrow{\rho_D} D'(x) \xrightarrow{D'(\alpha)} D'(y) = D_n(y)$$

where ρ_C and ρ_D are the canonical projections.

- If neither x nor y lies in $Q_0(n) \setminus Q_0(n-1)$, then we set $C_n(\alpha) = C_{n-1}(\alpha)$ and $D_n(\alpha) = D_{n-1}(\alpha)$.

Observe that since $X_n^x \in \mathcal{C} \cap \mathcal{D}$ for every vertex $x \in Q_0(n) \setminus Q_0(n-1)$, we get that $D'(x) \oplus X_n^x \in \mathcal{D}$ and $C'(x) \oplus X_n^x \in \mathcal{C}$. Thus $D_n(x) \in \mathcal{D}$ and $C_n(x) \in \mathcal{C}$ for all $x \in Q_0$. We can also observe that for $x \in Q_0(n) \setminus Q_0(n-1)$, $\gamma_x^{C_n}$ is equal to

$$\begin{pmatrix} \gamma_x^{C_{n-1}} \\ f_n^x \end{pmatrix} \colon \coprod_{\alpha' \in Q_1(*,x)} C_{n-1}(i(\alpha')) \hookrightarrow C'(x) \oplus X_n^x$$

thus we are only left with showing $\mathrm{Cok}(\gamma_x^{C_n}) \in \mathcal{C}$ to see that C_n and D_n are of the wanted form. Therefore, consider the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & C(x) \arrow[d, hook] & & \\
0 \arrow[r] & \coprod\limits_{\alpha \in Q_1(*,x)} C_{n-1}(i(\alpha)) \arrow[r, hook, "\gamma_x^{C_n}"] \arrow[d, equal] & C'(x) \oplus X_n^x \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Cok}(\gamma_x^{C_n}) \arrow[r] \arrow[d, dashed, two heads] & 0 \\
0 \arrow[r] & \coprod\limits_{\alpha \in Q_1(*,x)} C_{n-1}(i(\alpha)) \arrow[r, hook] & X_n^x \arrow[r, two heads] & Y_n^x \arrow[r] & 0
\end{tikzcd}$$

where the dashed morphism exist because of the cokernel property. By the snake lemma, we also get that the cokernel of this morphism coincide with $C'(x)$, thus we have the short exact sequence

$$0 \to C'(x) \to \mathrm{Cok}(\gamma_x^{C_n}) \to Y_n^x \to 0$$

where the end-terms are objects of $\mathcal{C}$. We conclude that $\mathrm{Cok}(\gamma_x^{C_n}) \in \mathcal{C}$, since $\mathcal{C}$ is closed under extensions. We are thus left with showing that we do in fact have a short exact sequence

$$0 \to D_n \xrightarrow{m(n)} C_n \xrightarrow{e(n)} F \to 0$$

Thus, define

$$m(n)_x = \begin{cases} m(n-1)_x & \text{if } x \notin Q_0(n) \setminus Q_0(n-1) \\ \begin{pmatrix} m(n-1)_x & 0 \\ 0 & 1 \end{pmatrix} & \text{else.} \end{cases}$$

and

$$e(n)_x = \begin{cases} e(n-1)_x & \text{if } x \notin Q_0(n) \setminus Q_0(n-1) \\ \begin{pmatrix} e(n-1)_x & 0 \end{pmatrix} & \text{else.} \end{cases}$$

Verification that these maps are in fact morphisms of representations, are left to the reader.

Observe that for $k \in \mathbb{N}$ such that $Q_0(k) = Q_0$, we have

$$0 \to D_k \to C_k \to F \to 0$$

such that $D_k \in \mathrm{Rep}(Q, \mathcal{D})$ and $C_k \in \Gamma(\mathcal{C})$, finishing our proof.

□

4.3 Tilting

We have observed in the previous chapters that cotorsion pairs appearing together with a torsion pair is of great interest since they are in correspondence with tilting subcategories. A natural question is therefore when the cotorsion pairs obtained in Proposition 4.29 acts as a part of a cotorsion torsion triple, or equivalently, when the cotorsion-free class is also a torsion class. In this section we will observe that the cotorsion pair $(\Gamma(\mathcal{C}), \mathrm{Rep}(Q, \mathcal{D}))$ is a part of a triple exactly whenever the underlying cotorsion pair $(\mathcal{C}, \mathcal{D})$ is a part of such a triple. The other induced cotorsion pair $(\mathrm{Rep}(Q, \mathcal{C}), \Lambda(\mathcal{D})$ will be observed to only induce such a triple when $(\mathcal{C}, \mathcal{D})$ is the trivial pair $(\mathrm{Proj}\mathcal{A}, \mathcal{A})$. We start by proving the stated property of the first induced cotorsion pair.

Lemma 4.33. *Let $(\mathcal{C}, \mathcal{D})$ be a complete cotorsion pair in an be an abelian category $\mathcal{A}$ with enough injectives, and Q a finite acyclic quiver. The induced cotorsion-free subcategory $\mathrm{Rep}(Q, \mathcal{D})$ is a torsion subcategory if and only if $\mathcal{D}$ is a torsion subcategory.*

Proof. We start by assuming that $\mathrm{Rep}(Q, \mathcal{D})$ is in fact a torsion class. Any epimorphism $D \twoheadrightarrow X$ out of an object D in $\mathcal{D}$, gives an epimorphism $s_x(D) \twoheadrightarrow s_x(X)$ for some vertex $x \in Q_0$. $\mathrm{Rep}(Q, \mathcal{D})$ is closed under factors since it is assumed to be a torsion class. Hence, as $s_x(X)$ admits such an epimorphism, it must also lie in $\mathrm{Rep}(Q, \mathcal{D})$ and by definition X lies in $\mathcal{D}$. $\mathcal{D}$ is therefore closed under factors. If we also can find a right $\mathcal{D}$-approximation for any object $A \in \mathcal{A}$, we have by Lemma 3.11 that $\mathcal{D}$ is in fact also a torsion class. The obvious candidate morphism for being an approximation is that which arises from the short exact sequence

$$0 \to \mathtt{t}s_x(A) \xrightarrow{m} s_x(A) \to \mathtt{f}s_x(A) \to 0$$

in $\mathrm{Rep}(Q, \mathcal{A})$, i.e. $m_x \colon \mathtt{t}s_x(A)(x) \hookrightarrow A$. We observe easily that for every morphism $T \to A$ out of $\mathcal{D}$ in $\mathcal{A}$, we get our wanted factorization.

$$\begin{tikzcd}
 & T = s_x(T)(x) \arrow[d, dashed] \arrow[dr] & \\
0 \arrow[r] & \mathtt{t}s_x(A)(x) \arrow[r, "m_x"] & s_x(A)(x) = A
\end{tikzcd}$$

We conclude that $\mathcal{D}$ is a torsion class.

For the converse direction, assume that $\mathcal{D}$ is a torsion class. By the functoriality of $\mathtt{t} \colon \mathcal{A} \to \mathcal{T}$ and $\mathtt{f} \colon \mathcal{A} \to \mathcal{F}$, we have for every representation $F \in \mathrm{Rep}(Q, \mathcal{A})$ and path $p \in Q(x,y)$ the commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathtt{t}F(x) \arrow[r] \arrow[d, "\mathtt{t}Fp"] & F(x) \arrow[r] \arrow[d, "Fp"] & \mathtt{f}F(x) \arrow[r] \arrow[d, "\mathtt{f}Fp"] & 0 \\ 0 \arrow[r] & \mathtt{t}F(y) \arrow[r] & F(y) \arrow[r] & \mathtt{f}F(y) \arrow[r] & 0 \end{tikzcd}$$

which give rise to the representations $\mathtt{t}F(x) \in \operatorname{Rep}(Q,\mathcal{D})$ and $\mathtt{f}F \in \operatorname{Rep}(Q,\mathcal{D}^{\perp})$, as well as the short exact sequence

$$0 \to \mathtt{t}F \to F \to \mathtt{f}F \to 0$$

We conclude by the easy observation $\operatorname{Rep}(Q,\mathcal{D}^{\perp}) \subseteq \operatorname{Rep}(Q,\mathcal{D})^{\perp}$, that $\operatorname{Rep}(Q,\mathcal{D})$ is a torsion class. $\square$

This gives us that in any category with enough injectives, the induced cotorsion pair $(\Gamma(\mathcal{C}), \operatorname{Rep}(Q,\mathcal{D}))$ is a part of a cotorsion torsion triple, if and only if the original cotorsion pair was the cotorsion part of a cotorsion torsion triple. Let us therefore shift our interest to the other induced cotorsion pair.

Lemma 4.34. *Let Q be a finite acylic non-trivial quiver, and $(\mathcal{C},\mathcal{D})$ a complete cotorsion pair in an abelian category $\mathcal{A}$ with enough projectives. If*

$$(\operatorname{Rep}(Q,\mathcal{C}), \Lambda(\mathcal{D}), \Lambda(\mathcal{D})^{\perp})$$

is a cotorsion torsion triple in $\operatorname{Rep}(Q,\mathcal{A})$, then $(\mathcal{C},\mathcal{D})$ is necessarily the trivial cotorsion pair $(\operatorname{Proj}(\mathcal{A}),\mathcal{A})$.

Proof. We first claim that $\mathcal{D}$ is necessarily closed under factors. Let therefore $f\colon D \twoheadrightarrow E$ be any epimorphism such that $D \in \mathcal{D}$. If x is an initial vertex of Q, then $s_x(D)$ lie in $\Lambda(\mathcal{D})$, and we have an epimorphism $s_x(f)\colon s_x(D) \twoheadrightarrow s_x(E)$ from an object of $\Lambda(\mathcal{D})$. Now, since $\Lambda(\mathcal{D})$ is a torsion subcategory it is necessarily closed under factors, therefore $s_x(C) \in \Lambda(\mathcal{D})$ and $\operatorname{Ker}(\lambda_x^{s_x(C)}) = C \in \mathcal{D}$.

Next, we claim that $\mathcal{D}$ is also closed under subobjects. Suppose we have a short exact sequence in $\mathcal{A}$,

$$0 \to U \xrightarrow{m} D \xrightarrow{e} E \to 0$$

with $D \in \mathcal{D}$. I_x is an exact functor, hence $I_x(e)\colon I_x(D) \twoheadrightarrow I_x(E)$ is an epimorphism out of an object in $\Lambda(\mathcal{D})$. Let us now fix a terminal vertex $x \in Q_0$. We construct a new representation $F \in \operatorname{Rep}(Q, \mathcal{A})$ in the following manner. If $y \in Q_0$ is an initial vertex such that there exists at least one path $p \in Q(y, x)$, then we set $Gy = I_x(D)(y) = \bigoplus_{p \in Q(y,x)} D^{(p)}$ and otherwise $Gy = I_x(E)(y)$.

For an arrow $\beta\colon y \to z \in Q_1$ we let

$$G\beta = I_x(E)(\beta) \circ I_x(e)(y)\colon Gy = I_x(D)(y) \to I_x(E)(y) \to I_x(E)(z)$$

if $y \in Q_0$ is initial with some path $p \in Q(y, z)$ and $G\beta = I_x(E)(\beta)$ otherwise. From the epimorphism $I_x(e)$ we construct an epimorphism

$$\phi\colon I_x(D) \twoheadrightarrow G$$

given by $\phi_y = \mathrm{id}_{I_x(D)(y)}$ if y is initial with a path $p \in Q(y, x)$ and $\phi_y = I_x(E)(y)$ else. Thus $G \in \Lambda(\mathcal{D})$ as $\Lambda(\mathcal{D})$ is closed under factors and $I_x(D) \in \Lambda(\mathcal{D})$. Now, let us fix an initial vertex y which admits at least one path $p \in Q(y, x)$. We have that $\operatorname{Ker}(\lambda_y^G) \in \mathcal{D}$ fits into the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}(\lambda_y^G) \arrow[r] \arrow[d, equal] & Gy \arrow[r] \arrow[d, equal] & \bigoplus\limits_{\alpha \in Q_1(y,*)} G(t(\alpha)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Ker}(\lambda_y^G) \arrow[r] & \bigoplus\limits_{\substack{\alpha \in Q_1(y,*) \\ q \in Q(t(\alpha),x)}} D^{(q\alpha)} \arrow[r] & \bigoplus\limits_{\alpha \in Q_1(y,*)} \left(\bigoplus\limits_{p \in Q(t(\alpha),x)} E^{(q,\alpha)} \right) \arrow[r] & 0 \\
0 \arrow[r] & U^{(q'\alpha')} \arrow[r] \arrow[u, hook] & D^{(q'\alpha')} \arrow[r] \arrow[u, hook] & E^{(q',\alpha')} \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

for all paths $q'\alpha' \in Q(y, z)$, where the superscripts only acts as identifiers for otherwise identical summands. We summize that since finite direct sums preserves kernel, we have

$$\operatorname{Ker}(\lambda_y^G) = \bigoplus_{\substack{\alpha\colon y \to z \in Q_1 \\ p \in Q(z,x)}} U^{q'\alpha'} = \bigoplus_{\substack{\alpha\colon y \to z \in Q_1 \\ p \in Q(z,x)}} U,$$

and since $\mathcal{D}$ is closed under summands, we get that $U \in \mathcal{D}$. That is, $\mathcal{D}$ is closed under subobjects.

We are now ready to prove that $\mathcal{D} = \mathcal{A}$. Let A be any object of $\mathcal{A}$. $(\mathcal{C}, \mathcal{D})$ is complete, so we obtain the short exact sequence

$$0 \to A \to \tilde{\mathtt{d}}A \to \tilde{\mathtt{c}}A \to 0$$

with $\tilde{\mathtt{d}}A \in \mathcal{D}$ and $\tilde{\mathtt{c}}A \in \mathcal{C}$. That is, A is a subobject of some object in $\mathcal{D}$, and since $\mathcal{D}$ is closed under subobjects we get that $A \in \mathcal{D}$. Thus, $\mathcal{A} \subseteq \mathcal{D}$, and trivially $\mathcal{D} \subseteq \mathcal{A}$, so $\mathcal{D} = \mathcal{A}$. Then, necessarily $\mathcal{C} = \mathrm{Proj}(\mathcal{A})$. $\square$

The question which then arises is when the trivial cotorsion pair $(\mathrm{Proj}\mathcal{A}, \mathcal{A})$ of an abelian category $\mathcal{A}$ with enough projectives, do induce a cotorsion torsion triple $(\mathrm{Rep}(Q, \mathrm{Proj}\mathcal{A}), \Lambda(\mathcal{A}), \Lambda(\mathcal{A})^{\perp})$. In an effort to answer this question we observe in the following lemma that $\Lambda(\mathcal{A})$ is closed under factors, which in an noetherian category suffices for it to be a torsion class as we saw in Lemma 3.12.

Lemma 4.35. $\Lambda(\mathcal{A})$ *is closed under factors.*

Proof. Let $\phi\colon F \twoheadrightarrow G$ be any epimorphism from an representation in $\Lambda(\mathcal{A})$. Then for each vertex $x \in Q_0$ we obtain the commutative diagram

$$\begin{tikzcd}
Fx \arrow[r, two heads, "\phi_x"] \arrow[d, two heads, "\lambda_x^F"] & Gx \arrow[r] \arrow[d, "\lambda_x^G"] & 0 \\
\bigoplus\limits_{\alpha \in Q_1(x,*)} F(t(\alpha)) \arrow[r, two heads, "\oplus\phi_y"] & \bigoplus\limits_{\alpha \in Q_1(x,*)} G(t(\alpha)) \arrow[r] & 0
\end{tikzcd}$$

where we know by assumption that all but λ_x^G is epimorphic, thus forcing λ_x^G to also be epimorphic. Then, necessarily $G \in \Lambda(\mathcal{A})$, concluding our proof. $\square$

The cotorsion pair $(\mathrm{Rep}(Q, \mathrm{Proj}\mathcal{A}), \Lambda(\mathcal{A}))$ does in fact always appear as the cotorsion part of a cotorsion torsion triple $(\mathrm{Rep}(Q, \mathrm{Proj}\mathcal{A}), \Lambda(\mathcal{A}), \Lambda(\mathcal{A})^{\perp})$. This follows from the fact that $I_*(\mathrm{Proj}\mathcal{A})$ is a tilting subcategory in $\mathrm{Rep}(Q, \mathcal{A})$ as was shown by Bauer et al. in [5]. This tilting subcategory induces a cotorsion torsion triple by the correspondende given in Theorem 3.55, which in short notice will be shown to coincide with our wanted cotorsion torsion triple.

Proposition 4.36 ([5, Prop. 3.9]). *Let $\mathcal{A}$ be an abelian category with enough projectives. The subcategory $\mathbb{T} = I_*(\mathrm{Proj}\mathcal{A})$ of $\mathrm{Rep}(Q, \mathcal{A})$ is a tilting subcategory.*

Corollary 4.37. *The tilting category $I_*(\mathrm{Proj}\mathcal{A})$ induces the cotorsion torsion triple* $(\mathrm{Rep}(Q, \mathrm{Proj}\mathcal{A}), \Lambda(\mathcal{A}), \Lambda(\mathcal{A})^{\perp})$

Proof. The correspondence in Theorem 3.55 tells us that the tilting subcategory $\mathbb{T} = I_*(\mathrm{Proj}\mathcal{A})$ give rise to the cotorsion torsion triple

$$({}^{\perp_1}\mathrm{Fac}\,\mathbb{T}, \mathrm{Fac}\,\mathbb{T}, \mathbb{T}^{\perp})$$

where ${}^{\perp_1}\mathrm{Fac}\,\mathbb{T} = \{F \in {}^{\perp_1}\mathbb{T} \mid \mathrm{pdim}F \leq 1\}$. Thus, we have to show that $\mathrm{Rep}(Q, \mathrm{Proj}\mathcal{A}) = {}^{\perp_1}\mathrm{Fac}\,\mathbb{T}$ and $\mathrm{Fac}\,\mathbb{T} = \Lambda(\mathcal{A})$. In fact, if we show the first of these equalities, we get the last one at once, since one part of a cotorsion pair uniquely determines the whole, i.e.

$$\Lambda(\mathcal{A}) = \mathrm{Rep}(Q, \mathrm{Proj}\mathcal{A})^{\perp_1} = \{F \in {}^{\perp_1}\mathbb{T} \mid \mathrm{pdim}F \leq 1\}^{\perp_1} = \mathrm{Fac}\,\mathbb{T}$$

First, since $\mathcal{A}$ has enough projectives, we can find an epimorphism $P \twoheadrightarrow A$ from a projective object to any object A in $\mathcal{A}$, which after applying the exact functor I_x gives that $I_x(A) \in \mathrm{Fac}\,\mathbb{T}$ for any vertex $x \in Q_0$. Now, for any representation $F \in {}^{\perp_1}\mathrm{Fac}\,\mathbb{T}$ we see that

$$\mathrm{Ext}^1_{\mathcal{A}}(F_x, A) \cong \mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(F, I_x(A)) = 0$$

hence, F_x is projective for every vertex $x \in Q_0$, i.e. ${}^{\perp_1}\mathrm{Fac}\,\mathbb{T} \subseteq \mathrm{Rep}(Q, \mathrm{Proj}\mathcal{A})$.

For the converse inclusion, observe first that for any $F \in \mathrm{Rep}(Q, \mathrm{Proj}\mathcal{A})$ and $P \in \mathrm{Proj}\mathcal{A}$, we have

$$\mathrm{Ext}^1_{\mathrm{Rep}(Q,\mathcal{A})}(F, I_x(P)) \cong \mathrm{Ext}^1_{\mathcal{A}}(F_x, P) = 0$$

hence $F \in {}^{\perp_1}\mathbb{T}$. Therefore, we are left with showing that $\mathrm{pdim}F \leq 1$. This will be shown by an inductive argument on the number of vertices in the support of F, i.e.

$$\mathrm{Supp}F = \{x \in Q_0 \mid F_x \neq 0\}$$

We start by observing that for every vertex $x \in Q_0$ we have the following projective resolution

$$0 \to \bigoplus_{\alpha \in Q_1(x,y)} P_y(F_x) \to P_x(F_x) \to s_x(F_x) \to 0$$

so $\mathrm{pdim} s_x(F_x) \leq 1$, and consequently $\mathrm{pdim} F \leq 1$ when $|\mathrm{Supp} F| = 1$. Let us assume that the claim holds whenever $|\mathrm{Supp} F| < n$. Now, for $|\mathrm{Supp} F| = n$, we find a vertex $x \in Q_0$ such that $Fy = 0$ for every $y \in Q_0$ such that $Q_1(y,x) \neq \varnothing$. From this we have the canonical epimorphism $F \twoheadrightarrow s_x(F_x)$ which give rise to the short exact sequence

$$0 \to K \to F \to s_x(F_x) \to 0$$

where $|\mathrm{Supp} K| < n$ and $K \in \mathrm{Rep}(Q, \mathrm{Proj}\mathcal{A})$, giving by assumption that $\mathrm{pdim} K \leq 1$. Therefore by the Horseshoe lemma, we conclude that $\mathrm{pdim} F \leq 1$. $\square$

4.3.1 Representations of $\vec{A}_n$

When we are working with representations over the linear quiver

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n-1}{\bullet} \longrightarrow \underset{n}{\bullet}$$

valued over an abelian category $\mathcal{A}$ with enough projectives, we can observe that the cotorsion-free $\Lambda(\mathcal{A})$ is the class of representation where all the internal morphisms are epimorphic. From this observation we deduce that $\Lambda(\mathcal{A})^{\perp}$ must necessarily be all representations where the first vertex evaluates to the zero object. Thus, we have the following explicit description of $(\mathrm{Rep}(\vec{A}_n, \mathrm{Proj}\mathcal{A}), \Lambda(\mathcal{A}), \Lambda(\mathcal{A})^{\perp})$.

Lemma 4.38. *Let $\mathcal{A}$ be an abelian category with enough projective. There is a cotorsion torsion triple $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ given by*

$$\mathcal{C} = \mathrm{Rep}(\vec{A}_n, \mathrm{Proj}(\mathcal{A}))$$
$$\mathcal{T} = \{F \in \mathrm{Rep}(\vec{A}_n, \mathcal{A}) \mid \textit{All internal morphisms are epimorphic.}\}$$
$$\mathcal{F} = \{F \in \mathrm{Rep}(\vec{A}_n, \mathcal{A}) \mid F_1 = 0\}$$

$\square$

Combining this with the obvious isomorphism

$$\{F \in \mathrm{Rep}(\vec{A}_n, \mathcal{A}) \mid F_1 = 0\} \cong \mathrm{Rep}(\vec{A}_{n-1}, \mathcal{A})$$

and Theorem 3.36, we obtain the following corollary which was also obtained in [5, Corollary 3.14].

Corollary 4.39. *Let $\mathcal{A}$ be an abelian category with enough projectives. Then*

$$\frac{\mathrm{Rep}(\vec{A}_n, \mathrm{Proj}\mathcal{A})}{\mathcal{C} \cap \mathcal{T}} \simeq \mathrm{Rep}(\vec{A}_{n-1}, \mathcal{A})$$

where

$$\mathcal{C} \cap \mathcal{T} = \{F \in \mathrm{Rep}(\vec{A}_n, \mathrm{Proj}\mathcal{A}) \mid \textit{All internal morphisms are epimorphic.}\}$$

□

Remark 4.40 (Construction)**.** When working over the linear quiver $\vec{A}_n$ we can without a lot of effort describe the cotorsion torsion triple

$$(\mathrm{Rep}(\vec{A}_n, \mathrm{Proj}\mathcal{A}), \Lambda(\mathcal{A}), \Lambda(\mathcal{A})^{\perp})$$

explicit. In fact, let us start by observing that the short exact sequence $\mathtt{t}F \hookrightarrow F \twoheadrightarrow \mathtt{f}F$, of the torsion pair is given through the following exact commutative diagram, where the upper row is the representation $\mathtt{t}F \in \mathcal{T} = \Lambda(\mathcal{A})$ and the lower row is the representation $\mathtt{f}F \in \mathcal{F} = \Lambda(\mathcal{A})^{\perp}$.

$$\begin{tikzcd}
F_1 \arrow[r, two heads] \arrow[d, hook] & \mathrm{Im}(f_1) \arrow[r, two heads] \arrow[d, hook] & \mathrm{Im}(f_2 f_1) \arrow[r, two heads] \arrow[d, hook] & \cdots \arrow[r, two heads] & \mathrm{Im}(f_{n-1} \cdots f_2 f_1) \arrow[d, hook] \\
F_1 \arrow[r, "f_1"] \arrow[d, two heads] & F_2 \arrow[r, "f_2"] \arrow[d, two heads] & F_3 \arrow[r, "f_3"] \arrow[d, two heads] & \cdots \arrow[r, "f_{n-1}"] & F_n \arrow[d, two heads] \\
0 \arrow[r] & \mathrm{Cok}(f_1) \arrow[r] & \mathrm{Cok}(f_2 f_1) \arrow[r] & \cdots \arrow[r] & \mathrm{Cok}(f_{n-1} \cdots f_2 f_1)
\end{tikzcd}$$

We can also see that the short exact sequence $F \hookrightarrow \tilde{\mathtt{d}}F \twoheadrightarrow \tilde{\mathtt{c}}F$ of the cotorsion pair can be described iteratively in the following manner. We start by setting

$\tilde{\mathtt{d}}F_n = F_n$. Then after finding an epimorphism from a projective object P'_{n-1} to $\tilde{\mathtt{d}}F_n$, setting $\tilde{\mathtt{d}}F_{n-1} = F_{n-1} \oplus P'_{n-1}$. Now, iterating this process, provides us with the short exact sequence of representations on the wanted form

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d, hook] & F_2 \arrow[r] \arrow[d, hook] & F_3 \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & F_{n-1} \arrow[r] \arrow[d, hook] & F_n \arrow[d, hook] \\
F_1 \oplus P'_1 \arrow[r, two heads] \arrow[d, two heads] & F_2 \oplus P'_2 \arrow[r, two heads] \arrow[d, two heads] & F_3 \oplus P'_3 \arrow[r, two heads] \arrow[d, two heads] & \cdots \arrow[r, two heads] & F_{n-1} \oplus P'_{n-1} \arrow[r, two heads] \arrow[d, two heads] & F_n \arrow[d, two heads] \\
P'_1 \arrow[r] & P'_2 \arrow[r] & P'_3 \arrow[r] & \cdots \arrow[r] & P'_{n-1} \arrow[r] & 0
\end{tikzcd}$$

The last sequence $\mathtt{d}F \hookrightarrow \mathtt{c}F \twoheadrightarrow F$ of the cotorsion pair is a bit more involved to describe than the two preceding. We start by finding an epimorphism from a projective object P_n to F_n, and taking a pullback along this and the internal map $F_{n-1} \xrightarrow{f_{n-1}} F_n$ of F.

$$\begin{tikzcd}
\Omega F_n \arrow[r, equal] \arrow[d, hook] & \Omega F_n \arrow[d, hook] \\
X_n \arrow[r] \arrow[d, two heads] & P_n \arrow[d, two heads] \\
F_{n-1} \arrow[r] & F_n
\end{tikzcd}$$

Then we find an epimorphism from a projective object P_{n-1} to X_n which by composition gives an epimorphism down onto F_{n-1}. Denote the kernel of this epimorphism by ΩF_{n-1} and observe that the upper left square in the subsequent commutative diagram is exact

$$\begin{tikzcd}
\Omega F_{n-1} \arrow[r] \arrow[d, hook] & \Omega F_n \arrow[r, equal] \arrow[d, hook] & \Omega F_n \arrow[d, hook] \\
P_{n-1} \arrow[r, two heads] \arrow[d, two heads] & X_n \arrow[r] \arrow[d, two heads] & P_n \arrow[d, two heads] \\
F_{n-1} \arrow[r, equal] & F_{n-1} \arrow[r] & F_n
\end{tikzcd}$$

The morphism $\Omega F_{n-1} \to \Omega F_n$ is by the exactness of that square necessarily an epimorphism. Now, take the pullback along $P_{n-1} \twoheadrightarrow F_{n-1}$ and the structure morphism $F_{n-2} \xrightarrow{f_{n-2}} F_{n-1}$, and repeat until reaching F_1.

$$\begin{tikzcd}
 & \Omega F_{n-1} \arrow[r, equal] \arrow[d, hook] & \Omega F_{n-1} \arrow[r, two heads] \arrow[d, hook] & \Omega F_n \arrow[r, equal] \arrow[d, hook] & \Omega F_n \arrow[d, hook] \\
\cdots \arrow[r] & X_{n-1} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & P_{n-1} \arrow[r, two heads] \arrow[d, two heads] & X_n \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & P_n \arrow[d, two heads] \\
\cdots \arrow[r, equal] & F_{n-2} \arrow[r] & F_{n-1} \arrow[r, equal] & F_{n-1} \arrow[r] & F_n
\end{tikzcd}$$

This results in the following short exact sequence of representations

$$\begin{tikzcd}
\Omega F_1 \arrow[r, two heads] \arrow[d, hook] & \Omega F_2 \arrow[r, two heads] \arrow[d, hook] & \cdots \arrow[r, two heads] & \Omega F_{n-2} \arrow[r, two heads] \arrow[d] & \Omega F_{n-1} \arrow[r, two heads] \arrow[d, hook] & \Omega F_n \arrow[d, hook] \\
P_1 \arrow[r] \arrow[d, hook] & P_2 \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & P_{n-2} \arrow[r] \arrow[d] & P_{n-1} \arrow[r] \arrow[d, two heads] & P_n \arrow[d, two heads] \\
F_1 \arrow[r] & F_2 \arrow[r] & \cdots \arrow[r] & F_{n-2} \arrow[r] & F_{n-1} \arrow[r] & F_n
\end{tikzcd}$$

where the upper representation lie in $\mathcal{T}$ and the middle representation lie in $\mathcal{C}$.

Example 4.41. Let Q be the zig-zag quiver $\underset{1}{\bullet} \leftarrow \underset{2}{\bullet} \rightarrow \underset{3}{\bullet}$ and set $\mathcal{A}$ as the abelian category of representations of Q valued in the category of finite dimensional vector spaces over a field $\mathbf{k}$, $\operatorname{rep}(Q, \mathbf{k})$. Then we can observe that $\operatorname{Rep}(\vec{A}_2, \mathcal{A})$ is equivalent to $\operatorname{Rep}(Q', \mathbf{k})$ where Q' is the quiver with commutativity relations from Example 3.40, that is

$$\begin{tikzcd}
\underset{1}{\bullet} \arrow[r] & \underset{4}{\bullet} \\
\underset{2}{\bullet} \arrow[u] \arrow[r] \arrow[d] \arrow[ur, dotted, no head] \arrow[dr, dotted, no head] & \underset{5}{\bullet} \arrow[u] \arrow[d] \\
\underset{3}{\bullet} \arrow[r] & \underset{6}{\bullet}
\end{tikzcd}$$

Hence we observe that the tilting subcategory $\Lambda(\operatorname{Proj}[\operatorname{rep}(Q, \mathbf{k})])$ is exactly the tilting subcategory $\mathbb{T}$ of Example 3.40. ♣

4.3.2 Representations of $\underset{1}{\bullet} \leftarrow \underset{2}{\bullet} \rightarrow \underset{3}{\bullet}$

Example 4.42. Let us go back to cotorsion pair of the zig-zag quiver

$$Q\colon \qquad \underset{1}{\bullet} \xleftarrow{\ \alpha\ } \underset{2}{\bullet} \xrightarrow{\ \beta\ } \underset{3}{\bullet}$$

presented in Example 3.18. That is, in the category of representation of Q valued over an abelian category $\mathcal{A}$ with enough projectives, the pair $(\mathcal{C}, \mathcal{D})$ given by

$$\mathcal{C} = \operatorname{Rep}(Q, \operatorname{Proj}\mathcal{A})$$
$$\mathcal{D} = \{F \in \operatorname{Rep}(Q, \mathcal{A}) \mid \text{The canonical morphism } F_2 \to F_1 \oplus F_3 \text{ is epic.}\}$$

is a complete cotorsion class. This is nothing but the cotorsion pair

$$(\operatorname{Rep}(Q, \operatorname{Proj}\mathcal{A}), \Lambda(\mathcal{A}))$$

induced by the trivial cotorsion pair $(\operatorname{Proj}\mathcal{A}, \mathcal{A})$ of $\mathcal{A}$. Therefore we also know that this gives the cotorsion torsion triple $(\mathcal{C}, \mathcal{D}, \mathcal{D}^{\perp})$ and the tilting subcategory

$$\mathcal{C} \cap \mathcal{D} = \{F \in \operatorname{Rep}(Q, \operatorname{Proj}\mathcal{A}) \mid F_2 \to F_1 \oplus F_3 \text{ is epic.}\}$$

If $\mathcal{A} = \operatorname{rep}(\vec{A}_2, \mathbf{k})$ we see that in $\operatorname{Rep}(Q', \mathbf{k})$ where Q' is the quiver with commutativity relations from Example 3.40, that is

$$\begin{array}{ccc} \underset{1}{\bullet} & \longrightarrow & \underset{4}{\bullet} \\ \uparrow & & \uparrow \\ \underset{2}{\bullet} & \longrightarrow & \underset{5}{\bullet} \\ \downarrow & & \downarrow \\ \underset{3}{\bullet} & \longrightarrow & \underset{6}{\bullet} \end{array}$$

we also have the tilting subcategory

$$\mathbb{T} = \operatorname{add}\{T_1, T_2, T_3, T_4, T_5, T_6\}$$

where

$$T_1 = \begin{array}{ccc} \mathbf{k} & \to & \mathbf{k} \\ \uparrow & & \uparrow \\ \mathbf{k} & \to & \mathbf{k} \\ \downarrow & & \downarrow \\ 0 & \to & 0 \end{array},\ T_2 = \begin{array}{ccc} 0 & \to & 0 \\ \uparrow & & \uparrow \\ \mathbf{k} & \to & \mathbf{k} \\ \downarrow & & \downarrow \\ 0 & \to & 0 \end{array},\ T_3 = \begin{array}{ccc} 0 & \to & 0 \\ \uparrow & & \uparrow \\ \mathbf{k} & \to & \mathbf{k} \\ \downarrow & & \downarrow \\ \mathbf{k} & \to & \mathbf{k} \end{array},\ T_4 = \begin{array}{ccc} 0 & \to & \mathbf{k} \\ \uparrow & & \uparrow \\ 0 & \to & \mathbf{k} \\ \downarrow & & \downarrow \\ 0 & \to & 0 \end{array},\ T_5 = \begin{array}{ccc} 0 & \to & 0 \\ \uparrow & & \uparrow \\ 0 & \to & \mathbf{k} \\ \downarrow & & \downarrow \\ 0 & \to & 0 \end{array},\ T_6 = \begin{array}{ccc} 0 & \to & 0 \\ \uparrow & & \uparrow \\ 0 & \to & \mathbf{k} \\ \downarrow & & \downarrow \\ 0 & \to & \mathbf{k} \end{array}$$

♣

4.4 Cotilting

Just as cotorsion torsion triples are interesting by their relation to tilting, we have also seen that torsion cotorsion bear the same relation to cotilting. Thus an equally natural question to that asked in Section 4.3 is when the cotorsion part of the induced cotorsion pairs from Proposition 4.29 acts as a part of a torsion cotorsion triple. The self-duality of torsion and cotorsion let us answer this quickly through the dual results of the tilting-case. Let us therefore summarize the dual results for cotilting. After stating these results, we will at the end apply them to representations of about the commutative grids $\vec{A}_n \otimes \vec{A}_m$ which were mentioned in the introduction of the thesis. This application is found among a couple more in [5].

Lemma 4.43 (Dual of Lemma 4.33)**.** *Let $(\mathcal{C}, \mathcal{D})$ be a complete cotorsion pair in an abelian category $\mathcal{A}$ with enough projectives, and Q a finite acyclic quiver. The induced cotorsion subcategory* $\mathrm{Rep}(\mathcal{C}, \mathcal{A})$ *is a torsion-free class if and only if $\mathcal{C}$ is a torsion-free class.*

Lemma 4.44 (Dual of Lemma 4.34)**.** *Let Q be a non-trivial finite acyclic quiver, and $(\mathcal{C}, \mathcal{D})$ a complete cotorsion pair in an abelian category $\mathcal{A}$ with enough injectives. If*

$$({}^{\perp}\Gamma(\mathcal{C}), \Gamma(\mathcal{C}), \mathrm{Rep}(\mathcal{D}, \mathcal{A}))$$

is a torsion cotorsion triple in $\mathrm{Rep}(Q, \mathcal{A})$*, then $(\mathcal{C}, \mathcal{D})$ is necessarily the trivial cotorsion pair* $(\mathcal{A}, \mathrm{Inj}\mathcal{A})$*.*

Lemma 4.45 (Dual of Lemma 4.35)**.** *Let $\mathcal{A}$ be an abelian category with enough injectives. Then $\Gamma(\mathcal{A})$ is closed under subobjects.*

Lemma 4.46 (Dual of Corollary 4.37)**.** *Let $\mathcal{A}$ be abelian with enough injectives, then* $({}^{\perp}\Gamma(\mathcal{C}), \Gamma(\mathcal{C}), \mathrm{Rep}(\mathcal{D}, \mathcal{A}))$ *is a torsion cotorsion triple in* $\mathrm{Rep}(Q, \mathcal{A})$.

4.4.1 Representations of $\vec{A}_n$

Analogous to the discussion in Section 4.3.1 we observe that for representations over the linear quiver $\vec{A}_n$ valued over an abelian category with enough injectives, the cotorsion class $\Gamma(\mathcal{A})$ is the class of representations with monomorphic structure morphisms. Hence, we have the following dual characterization.

Lemma 4.47 (Dual of Lemma 4.38)**.** *Let $\mathcal{A}$ be an abelian category with enough injectives. There is a torsion cotorsion triple $(\mathcal{T}, \mathcal{F}, \mathcal{D})$ given by*

$$\mathcal{T} = \{F \in \mathrm{Rep}(\vec{A}_n, \mathcal{A}) \mid F_n = 0\}$$
$$\mathcal{F} = \{F \in \mathrm{Rep}(\vec{A}_n, \mathcal{A}) \mid \textit{All internal morphisms are monomorphic.}\}$$
$$\mathcal{D} = \mathrm{Rep}(\vec{A}_n, \mathrm{Inj}(\mathcal{A}))$$

□

Thus giving us the equivalence

Lemma 4.48 (Dual of Corollary 4.39)**.** *Let $\mathcal{A}$ be an abelian category with enough injectives. Then*

$$\frac{\mathrm{Rep}(\vec{A}_n, \mathrm{Inj}\mathcal{A})}{\mathcal{F} \cap \mathcal{D}} \simeq \mathrm{Rep}(\vec{A}_{n-1}, \mathcal{A})$$

where

$$\mathcal{F} \cap \mathcal{D} = \{F \in \mathrm{Rep}(\vec{A}_n, \mathrm{Inj}\mathcal{A}) \mid \textit{All internal morphisms are monomorphic.}\}$$

□

4.4.2 Application to TDA

Multiparameter persistence modules in topological data analysis (See Appendix B for a brief introduction to TDA) are a special case of quiver representation. That is, as these modules arises as the nth-homology of a filtered topological space

$(X_i)_{i\in I}$, we have for each filtration step $i \in I$ a vector space V_i over some field $\mathbf{k}$. The set I which the filtration is taken over, is a partially ordered set, such that for each $i \leq j \in I$ we have $X_i \subseteq X_j$. These inclusions induces maps $V_i \to V_j$ of the homology of them, and hence the multiparameter persistence module is a representation of I seen as a quiver with relations valued over vector spaces of $\mathbf{k}$.

The filtration on the topological space can explicitly be described as a multi-dimensional grid, which in the 2-parameter case gives us that I is the partially ordered set given by the Hasse-diagram

$$
\begin{tikzcd}
\underset{(1,n)}{\bullet} \arrow[r] & \underset{(2,n)}{\bullet} \arrow[r] & \underset{(3,n)}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{(m,n)}{\bullet} \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
\underset{(1,2)}{\bullet} \arrow[r] \arrow[u] & \underset{(2,2)}{\bullet} \arrow[r] \arrow[u] & \underset{(3,2)}{\bullet} \arrow[r] \arrow[u] & \cdots \arrow[r] & \underset{(m,2)}{\bullet} \arrow[u] \\
\underset{(1,1)}{\bullet} \arrow[r] \arrow[u] & \underset{(2,1)}{\bullet} \arrow[r] \arrow[u] & \underset{(3,1)}{\bullet} \arrow[r] \arrow[u] & \cdots \arrow[r] & \underset{(m,1)}{\bullet} \arrow[u]
\end{tikzcd}
$$

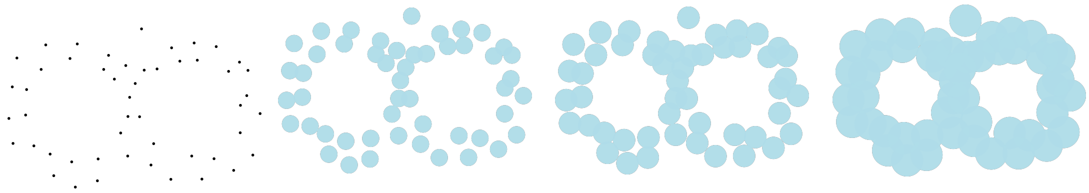
or equivalently the fully commutative grid quiver given by the same diagram. We denote such quivers by $\vec{A}_m \otimes \vec{A}_n$, and observe that we have in fact a natural equivalence

$$\mathrm{Rep}(\vec{A}_n, \mathcal{A}) \simeq \mathrm{Rep}(\vec{A}_m \otimes \vec{A}_n, \mathbf{k})$$

for $\mathcal{A} = \mathrm{Rep}(\vec{A}_m, \mathbf{k})$. To summarize, we have that the category of multiparameter persistence modules is in fact equal to $\mathrm{Rep}(\vec{A}_m \otimes \vec{A}_n, \mathbf{k})$.

Observe that persistence modules arising from the zeroth homology measures the connectivity of the space at different filtration steps. If we fix all filtration parameters except the radius, we observe that each subsequent filtration step leads to a greater overlap of the pieces. Hence, each subsequent space is either equally connected or more connected as the preceding one. This translates to the induced maps of the persistence module being surjections. Hence, we conclude that there is a non-trivial amount of cases where we can assume that the multiparameter persistence module in question has epimorphic structure morphism in at least one direction of the grid.

Example 4.49. If we have the following 4-step filtration with radius as the only parameter



we see that for each step in the filtration, more and more pieces merge. Hence each subsequent filtration step is equally or more connected than the preceding step. ♣

Motivated by the preceding observation, we denote the full subcategory of all representations of $\vec{A}_m \otimes \vec{A}_n$ with epimorphic structure morphism in the horizontal direction by $\mathrm{rep}^{e,*}(\vec{A}_m \otimes \vec{A}_n, \mathbf{k})$. Similarly the full subcategory of representations with epimorphic horizontal structure morphisms and monomorphic vertical morphisms is denoted by $\mathrm{rep}^{e,m}(\vec{A}_m \otimes \vec{A}_n, \mathbf{k})$. Applying our acquired knowledge of cotilting in the category of representations of the linear quiver $\vec{A}_n$ we observe the following.

Lemma 4.50 ([5, Corollary 3.17])**.** *There is an equivalence*

$$\frac{\mathrm{rep}^{e,*}(\vec{A}_m \otimes \vec{A}_n, \mathbf{k})}{\mathrm{rep}^{e,m}(\vec{A}_m \otimes \vec{A}_n, \mathbf{k})} \simeq \mathrm{rep}(\vec{A}_m \otimes \vec{A}_{n-1}, \mathbf{k})$$

Proof. First, observe that the injective representations of $\mathrm{Rep}(\vec{A}_n, \mathbf{k})$ are those which have epimorphic structure morphisms. Hence for $\mathcal{A} = \mathrm{Rep}(\vec{A}_m, \mathbf{k})$ we see that

$$\mathrm{Rep}(\vec{A}_n, \mathrm{Inj}\mathcal{A}) \simeq \mathrm{Rep}^{e,*}(\mathbf{k}, \vec{A}_m \otimes \vec{A}_n)$$

Further, we observe that

$$\mathcal{F} \cap \mathcal{D} = \{F \in \mathrm{Rep}(\vec{A}_n, \mathrm{Inj}\mathcal{A}) \mid \text{All internal morphisms are monomorphic.}\}$$

is necessarily equivalent to $\mathrm{Rep}^{e,m}(\mathbf{k}, \vec{A}_m \otimes \vec{A}_n)$. Thus, by Lemma 4.48 we see that

$$\frac{\mathrm{rep}^{e,*}(\vec{A}_m \otimes \vec{A}_n, \mathbf{k})}{\mathrm{rep}^{e,m}(\vec{A}_m \otimes \vec{A}_n, \mathbf{k})} \simeq \frac{\mathrm{Rep}(\vec{A}_n, \mathrm{Inj}\mathcal{A})}{\mathcal{F} \cap \mathcal{D}} \simeq \mathrm{Rep}(\vec{A}_{n-1}, \mathcal{A}) \simeq \mathrm{rep}(\vec{A}_m \otimes \vec{A}_{n-1}, \mathbf{k})$$

□

Now, by observing that the indecomposables of $\mathrm{Rep}^{e,m}(\vec{A}_m \otimes \vec{A}_n, \mathbf{k})$ are those which is equal to $\mathbf{k}$ on each vertex $(x, y) \in [1, i] \times [j, n]$ for $1 \le i \le m$ and $1 \le j \le n$[5] and zero else, we realise that there is a finite difference in the amount of indecomposables in $\mathrm{rep}^{e,*}(\vec{A}_m \otimes \vec{A}_n, \mathbf{k})$ and $\mathrm{rep}(\vec{A}_m \otimes \vec{A}_{n-1}, \mathbf{k})$. Hence, the equivalence above gives us the following corollary to Theorem B.1.

Corollary 4.51. *Let* $\mathbf{k}$ *be an algebraically closed field. The category* $\mathrm{rep}^{e,*}(\vec{A}_m \otimes \vec{A}_n, \mathbf{k})$*, is of finite representation type when*

- $m = 1$ *or* $n \le 2$,
- *or* $(m, n) \in \{(2, 3), (2, 4), (2, 5), (3, 3), (4, 3)\}$.

It is of tame representation type when

- $(m, n) \in \{(2, 6), (3, 4), (5, 3)\}$. □

Hence, we have a small amount of additional cases where we can parametrize 2-parameter persistence modules under the additional epimorphic condition. If we have a case where we can safely assume that all the internal morphisms of a 2-parameter persistence module is epimorphic, we would hope that the subcategory of such representations, which we denote by $\mathrm{rep}^{e,e}(\vec{A}_m \otimes \vec{A}_n, \mathbf{k})$, has even more additional cases where we can parametrize the resulting modules. This was observed to be the case in [5], which through the dual of Corollary 3.36 established

[5]These representations are often called *constant representations* and denoted by $\mathbf{k}_{[1,i]\times[j,n]}$.

the equivalence

$$\frac{\operatorname{rep}^{e,e}(\vec{A}_m \otimes \vec{A}_n, \mathbf{k})}{\mathbb{C}} \simeq \operatorname{rep}(\vec{A}_{m-1} \otimes \vec{A}_{n-1}, \mathbf{k})$$

where $\mathbb{C}$ is a suitable cotilting subcategory with a finite amount of indecomposable objects, and therefore also the following classification.

Lemma 4.52 (Dual of [5, Corollary 3.36]). *Let* $\mathbf{k}$ *be an algebraically closed field. The category* $\operatorname{rep}^{e,e}(\vec{A}_m \otimes \vec{A}_n, \mathbf{k})$*, is of finite representation type when*

- $m \leq 2$ *or* $n \leq 2$,
- *or* $(m, n) \in \{(3,3), (3,4), (3,5), (4,3), (5,3)\}$.

It is of tame representation type when

- $(m, n) \in \{(3,6), (4,4), (6,3)\}$. □

A Miscellaneous Results

A.1 Sums and Intersections

Lemma A.1. *Let $\mathcal{E}$ be an additive category. The sum and intersection of two subobjects $B \hookrightarrow A$ and $C \hookrightarrow A$ are described through the following pullback and pushout diagram.*

$$
\begin{tikzcd}
B \cap C \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, hook] \\
C \arrow[r, hook] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \cap C \arrow[r, hook] \arrow[d, hook] & B \arrow[d, hook] \\
C \arrow[r, hook] \arrow[ur, phantom, "\ulcorner", very near end] & B + C
\end{tikzcd}
$$

Further, the image of $B \oplus C \to A$ coincide with the sum $B + C$.

Proof. Let us start with the intersection. Let D be a subobject of A contained in both B and C, then we have by the pullback-property that it factors through $B \cap C$. The morphism $D \to B \cap C$ is necessarily a monomorphism, thus D is also contained in $B \cap C$, proving that this is in fact the intersection. Now, for the sum, we first would like to show that $B + C$ is in fact a subobject of A. The pushout diagram above give rise to the exact sequence

$$
0 \longrightarrow B \cap C \longrightarrow B \oplus C \longrightarrow A
$$

which then in connection with the pullback diagram above gives us the following commutative diagram with exact rows,

$$
\begin{tikzcd}
0 \arrow[r] & B \cap C \arrow[r] \arrow[d, equal] & B \oplus C \arrow[r] \arrow[d, equal] & B + C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \cap C \arrow[r] & B \oplus C \arrow[r] & X \arrow[r] & 0
\end{tikzcd}
$$

where $X = \operatorname{Im}(B \oplus C \to A)$. Now the five lemma tells us that the last morphism is necessarily an isomorphism, giving that $B + C$ is in fact a subobject of A. Assume now that there is a subobject Y of A which contains both B and C, then by the pushout-property, we have that $B + C \hookrightarrow A$ factors through it, and therefore $B + C$ is necessarily also a subobject of Y.

□

A.2 Adjoints in Extensions

We have that for certain pairs of adjunct functors (F, G) between abelian categories, the adjunction extends to the first extension group. In fact similar results is true for higher extension groups [16, Lemma 5.1].

Lemma A.2. *Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ be functors between abelian categories such that F is a left adjoint of G. If for objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have that*

(i) the functor F sends every exact sequence

$$0 \to GB \to Y \to A \to 0$$

to an exact sequence

$$0 \to FGB \to FY \to FA \to 0$$

(ii) and, the functor G sends every exact sequence

$$0 \to B \to X \to FA \to 0$$

to an exact sequence

$$0 \to GB \to GX \to GFA \to 0$$

then there is an isomorphism of abelian groups $\mathrm{YExt}^n_{\mathcal{B}}(GA, B) \cong \mathrm{YExt}^n_{\mathcal{A}}(A, FB).S$

Proof. By the given assumptions, we have that F and G induces well-defined group-homomorphisms

$$F(-)\colon \mathrm{YExt}^1_{\mathcal{A}}(A, GB) \to \mathrm{YExt}^1_{\mathcal{B}}(FA, FGB)$$

and

$$G(-)\colon \mathrm{YExt}^1_{\mathcal{B}}(FA, B) \to \mathrm{YExt}^1_{\mathcal{A}}(GFA, GB)$$

Let $\eta\colon \mathrm{id}_{\mathcal{A}} \to G \circ F$ be the unit and $\epsilon\colon F \circ G \to \mathrm{id}_{\mathcal{B}}$ be the counit of the adjunction. We now construct group homomorphisms $\phi\colon \mathrm{YExt}^1_{\mathcal{B}}(FA, B) \to \mathrm{YExt}^1_{\mathcal{A}}(A, GB)$ and $\psi\colon \mathrm{YExt}^1_{\mathcal{A}}(A, GB) \to \mathrm{YExt}^1_{\mathcal{B}}(FA, B)$ by the compositions,

$$\begin{tikzcd}
\mathrm{YExt}^1_{\mathcal{B}}(FA, B) \arrow[rr, "\phi"] \arrow[dr, "G(-)"'] & & \mathrm{YExt}^1_{\mathcal{A}}(A, GB) \\
& \mathrm{YExt}^1_{\mathcal{A}}(GFA, GB) \arrow[ur, "-\cdot\eta_A"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{YExt}^1_{\mathcal{A}}(A, GB) \arrow[rr, "\psi"] \arrow[dr, "F(-)"'] & & \mathrm{YExt}^1_{\mathcal{B}}(FA, B) \\
& \mathrm{YExt}^1_{\mathcal{B}}(FA, FGB) \arrow[ur, "\epsilon_B\cdot -"'] &
\end{tikzcd}$$

As we will show, these homomorphisms is inverses of each other and thus gives our desired isomorphism. Let $\mathbb{X} \in \mathrm{YExt}^1_{\mathcal{B}}(FA, B)$ be an isomorphism class represented by

$$\mathbf{X}\colon \qquad B \hookrightarrow X \twoheadrightarrow FA$$

Under ϕ this class is sent to $\phi(\mathbb{X}) \in \mathrm{YExt}^1_{\mathcal{A}}(A, GB)$ represented by the lower row in the following commutative diagram

$$\begin{tikzcd}
G(\mathbf{X})\colon & GB \arrow[r, hook] & GX \arrow[r, two heads] & GFA \\
\phi(\mathbf{X})\colon & GB \arrow[r, hook] \arrow[u, equal] & PB \arrow[r, two heads] \arrow[u] & A \arrow[u, "\eta_A"]
\end{tikzcd}$$

We now apply $F(-)$ on the whole diagram, obtaining the following commutative diagram

$$\begin{tikzcd}
F(G(\mathbf{X}))\colon & FGB \arrow[r, hook] & FGX \arrow[r, two heads] & FGFA \\
F(\phi(\mathbf{X}))\colon & FGB \arrow[r, hook] \arrow[u, equal] & F(PB) \arrow[r, two heads] \arrow[u] & FA \arrow[u, "F(\eta_A)"]
\end{tikzcd}$$

By using $\epsilon_B \cdot -$ on the lower row obtaining a representative of $\psi(\phi(\mathbb{X}))$, and using the counit ϵ on the whole upper row, we get a new commutative diagram,

$$
\begin{tikzcd}
\mathbf{X}\colon & B \arrow[r, hook] & X \arrow[r, two heads] & FA \\
F(G(\mathbf{X}))\colon & FGB \arrow[r, hook] \arrow[u, "\epsilon_B"] & FGX \arrow[r, two heads] \arrow[u, "\epsilon_X"] & FGFA \arrow[u, "\epsilon_{FA}"] \\
F(\phi(\mathbf{X}))\colon & FGB \arrow[u, equal] \arrow[d, "\epsilon_B"'] \arrow[r, hook, no head] & F(PB) \arrow[u] \arrow[r, two heads] \arrow[d] & FA \arrow[u, "F(\eta_A)"] \arrow[d, equal] \\
\psi(\phi(\mathbf{X}))\colon & B \arrow[r, hook] & PO \arrow[r, two heads] \arrow[uuu, dashed, bend right] & FA
\end{tikzcd}
$$

By the properties of units and counits, we have that $\mathrm{id}_{FA} = \epsilon_{FA} \circ F(\eta_A)$. Further, by the universal property of a push-out, we conclude that the upper and lower row in this diagram is a representative of the same isomorphism class, that is $\psi(\phi(\mathbb{X})) = \mathbb{X}$, or equivalently, $\psi \circ \phi = \mathrm{id}_{\mathrm{YExt}^1_{\mathcal{B}}(FA,B)}$.

Now, let $\mathbb{Y} \in \mathrm{YExt}^1_{\mathcal{A}}(A, GB)$ be an isomorphism class represented by

$$
\mathbf{Y}\colon \qquad GB \hookrightarrow Y \twoheadrightarrow A
$$

We follow in the same manner as above. $\mathbf{Y}$ is sent to the lower row in the commutative diagram below

$$
\begin{tikzcd}
F(\mathbf{Y})\colon & FGB \arrow[r, hook] \arrow[d, "\epsilon_B"] & FY \arrow[r, two heads] \arrow[d] & FA \arrow[d, equal] \\
\psi(\mathbf{X})\colon & B \arrow[r, hook] & PO \arrow[r, two heads] & FA
\end{tikzcd}
$$

After applying $G(-)$ on this, we get the commutative diagram,

$$
\begin{tikzcd}
G(F(\mathbf{Y}))\colon & GFGB \arrow[r, hook] \arrow[d] & GFY \arrow[r, two heads] \arrow[d] & GFA \arrow[d, equal] \\
G(\psi(\mathbf{X}))\colon & GB \arrow[r, hook] & G(PO) \arrow[r, two heads] & GFA
\end{tikzcd}
$$

Finally, by applying $- \cdot \eta_A$ on the lower row and the natural transformation

$\eta\colon \mathrm{id}_{\mathcal{A}} \to G \circ F$ on the upper row, we obtain the commutative diagram

$$\begin{tikzcd}
\mathbf{Y}\colon & GB \arrow[r, hook] \arrow[d, "\eta_{GB}"] & Y \arrow[r, two heads] \arrow[d, "\eta_Y"] \arrow[ddd, dashed, bend right] & A \arrow[d, "\eta_A"] \\
G(F(\mathbf{Y}))\colon & GFGB \arrow[r, hook] \arrow[d, "G(\epsilon_B)"] & GFY \arrow[r, two heads] \arrow[d] & GFA \arrow[d, equal] \\
G(\psi(\mathbf{X}))\colon & GB \arrow[r, hook] \arrow[d, equal] & G(PO) \arrow[r, two heads] & GFA \\
\phi(\psi(\mathbf{Y}))\colon & GB \arrow[r, hook] & PB \arrow[r, two heads] \arrow[u] & A \arrow[u, "\eta_A"']
\end{tikzcd}$$

with our original representative on the top row, and a representative of $\phi(\psi(\mathbf{Y}))$ on the lower row. By η and ϵ being unit and counit, respectively, we have that $\mathrm{id}_{GB} = G(\epsilon_B) \circ \eta_{GB}$. Furthermore, the pullback property gives us a map $\mathbf{Y} \to \phi(\psi(\mathbf{Y}))$, so we conclude that $\mathbf{Y} = \phi(\psi(\mathbf{Y}))$.

$\square$

B Topological Data Analysis

In the last twenty years topological data analysis, TDA for short, has risen as a promising addition to the data analysts' toolbox. With roots in cluster analysis it incorporates results and classification tools from algebraic topology, in an effort to make sense of seemingly non-structured data. This is the age of information, and each day there is a new surge of vast data introduced throughout all branches of science. TDA aims to bring the inherent structure of these data sets into light, in an effort to make it less overwhelming. This is done by considering the data points as sampled from some underlying geometric object, and then trying to approximate this object's topological features through (co)homological invariants. We will now give a short introduction to the field of topological data analysis. Those interested can check out [8, 26, 27] for a more in-depth introduction to TDA.

In the general process of TDA one stumbles upon representations of partially ordered sets, or persistence modules as they are called within the realm of TDA. In order to extract useful information from these modules, they are decomposed into their smallest components. The traditional concept of persistence modules are representations over totally ordered sets, e.g. $\mathbb{R}$, $\mathbb{Z}$ or a subset of these, which decomposes into interval modules whenever the ordered set is finite or each associated vector space is of finite dimension. Longer interval modules imply a more likely homological invariant associated to the underlying space. Persistence modules over totally ordered sets are often called single-parameter persistence modules.

However, in most application the resulting persistence module is over a more complex partially ordered set, e.g. $\mathbb{R}^n$, $\mathbb{Z}^n$, or subsets of these. These persistence modules are often denoted as being multi-parameter, and there are no equivalent decomposition theory for them, see e.g. [9]. In fact, we can observe that for the 2-paramater case, we have that the persistence modules in question are quiver-representation of the commutative grid $\vec{A}_m \otimes \vec{A}_n$,

$$
\begin{tikzcd}
\underset{(1,n)}{\bullet} \arrow[r] & \underset{(2,n)}{\bullet} \arrow[r] & \underset{(3,n)}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{(m,n)}{\bullet} \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
\underset{(1,2)}{\bullet} \arrow[r] \arrow[u] & \underset{(2,2)}{\bullet} \arrow[r] \arrow[u] & \underset{(3,2)}{\bullet} \arrow[r] \arrow[u] & \cdots \arrow[r] & \underset{(m,2)}{\bullet} \arrow[u] \\
\underset{(1,1)}{\bullet} \arrow[r] \arrow[u] & \underset{(2,1)}{\bullet} \arrow[r] \arrow[u] & \underset{(3,1)}{\bullet} \arrow[r] \arrow[u] & \cdots \arrow[r] & \underset{(m,1)}{\bullet} \arrow[u]
\end{tikzcd}
$$

which has a complete classification into finite or tame representation types. As would be expected by the results of [9], there is a limited amount of such cases:

Theorem B.1 ([20, Theorem 2.5],[21, Theorem 5]). *Let* $\mathbf{k}$ *be an algebraically closed field. The category* $\operatorname{rep}(\vec{A}_m \otimes \vec{A}_n, \mathbf{k})$ *of representations of the quiver with relations* $\vec{A}_m \otimes \vec{A}_n$ *valued in the category of finite dimensional* $\mathbf{k}$*-vector spcaces, is of finite representation type when*

- $m = 1$ *or* $n = 1$,
- *or* $(m, n) \in \{(2, 2), (2, 3), (2, 4), (3, 2), (4, 2)\}$.

It is of tame representation type when

- $(m, n) \in \{(2,5), (3,3), (5,2)\}$.

And in all other cases it is of wild representation type.

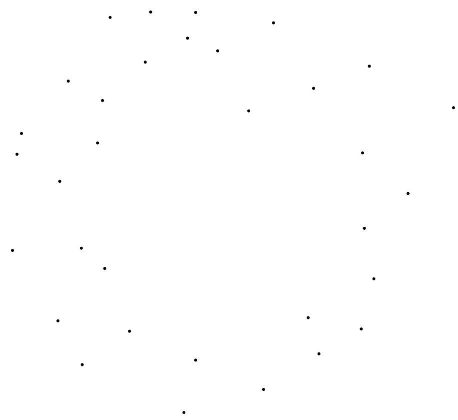
To make the intuition behind TDA clearer, we will describe in broad strokes the pipeline of a traditional application of TDA. The first step is to build a one-parameter filtrated simplicial complex from the set of data points, apply a (co)homological functor to obtain a vector space representation of the index set for the filtration and decompose this representation into what is commonly called barcodes. The quintessential example of how the filtrated simplicial complex is built is a Čech-complex.

When building the Čech complex, we first embed the set of data points, X, in a metric space, (M, d). For each value $\epsilon \in \mathbb{R}^+ = [0, \infty)$, we construct a simplicial complex, C_ϵ consisting of a k-simplex $[x_0, x_1, \ldots, x_k]$ for each subset $\{x_0, x_1, \ldots, x_k\} \subset X$ of points, such that the intersection of balls of radius ϵ and center x_i, $i = 0, 1, \ldots, k$ is non-empty. That is

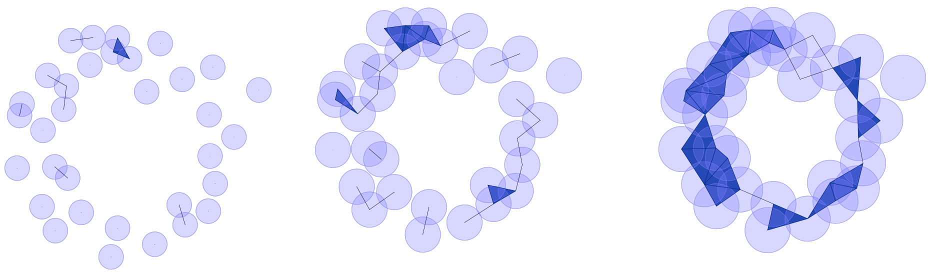
$$\bigcap_{0 \leq i \leq k} B_\epsilon(x_i) \neq \varnothing.$$

Thus, we have a filtered simplicial complex $\{C_\epsilon\}_{\epsilon \in \mathbb{R}^+}$. The sampled data point set is finite, so there will be a finite amount of *critical values* ϵ such that, $C_{\epsilon'} \neq C_\epsilon$ for $\epsilon' \leq \epsilon$. The filtration can therefore be taken over the set of critical values instead, leading to a persistence module over a finite subset of $\mathbb{R}^+$.

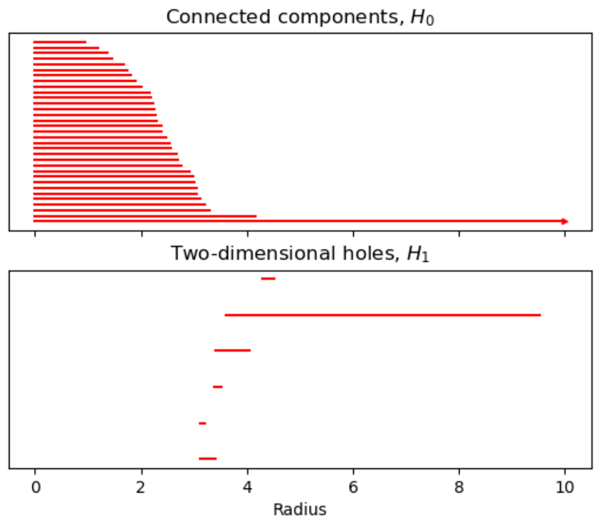
Example B.2. Say we have the data cloud given by



sampled from a circle in the plane. Through taking the union of balls with ever-increasing radius, we obtain a filtered simplicial complex.



which gives the following barcode-diagram of the first two homologies,



where each horizontal bar represents an interval module in the decomposition of the persistence module. Thus, we can deduce that the underlying structure of the data cloud most likely consists of one connected component and has a 2-dimensional hole. This correspond nicely with the fact that the data cloud were sampled from a circle in the plane. ♣

All real data sets contain a certain degree of noise. Therefore one often wants to have some way of removing noise from the data before building the filtered complex. This can be done for example by only allowing data points of a certain density to be considered, see e.g. [10], but by choosing a density treshold, the analyst has done an a priori analysis of the data, which goes against the principle of TDA. Thus one wants to consider all density treshold at the same time, which will give another natural parameter to the filtered simplicial complex and thus a multi-parameter persistence module. This is one example of a situation where more than one parameter is wanted, another situation could be that the analysts needs to consider variations of curvature.

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