Ole Berre

The Derived Category of Exact Categories and Classification of Exact Structures

Master's thesis in Mathematical Sciences
Supervisor: Professor Steffen Oppermann
June 2021

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Abstract

In this thesis we define the derived category of an exact category through Verdier localization. In addition we classify exact structures on idempotent complete categories through module categories. Lastly we apply our classification theorems and Auslander-Reiten theory to explicitly find the derived categories of all exact structures on representations over the following quivers.

$$1 \to 2,\ 1 \to 2 \leftarrow 3 \text{ and } 1 \to 2 \to 3$$

Sammendrag

I denne avhandlingen vil vi se på Verdier-lokalisering og asykliske komplekser for å kunne definere den deriverte kategorien. I tillegg finner vi en klassifisering av eksakte strukturer på idempotent komplette kategorier gjennom modulkategorier. Til slutt vil vi andvende klassifikasjonssetningene og Auslander Reiten-teori for å finne de deriverte kategoriene av alle eksakte strukturer på representasjoner over følgende kogger.

$$1 \to 2,\ 1 \to 2 \leftarrow 3 \text{ og } 1 \to 2 \to 3$$

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Contents

Introduction

This thesis was written in 2021 under supervision of Professor Steffen Oppermann at the Norwegian University of Science and Technology. The goal of the thesis is to define the derived category of an exact category, find a classification of exact structures and look at examples where we apply these.

We will look at exact categories, localization, derived categories and classification of exact structures through module functor categories. The reader does not need any prior knowledge of these subjects. Prior knowledge of additive, abelian and triangulated categories is highly recommended although some results are included for the benefit of the reader in Appendix A and Appendix B.

The thesis starts by defining exact categories. We will use a convenient self-dual presentation of the axioms which are due to Yoneda [1]. The rest of section one is then used to get familiar with exact categories and to prove results needed later in the thesis.

The second section is used to look at idempotent complete categories. We will see a constructive proof of how to fully faithfully embed any additive category into an idempotent complete category. Furthermore we look at a weaker form of idempotent completion and look at how this affects an exact category.

We start working our way towards defining the derived category of an exact category by introducing localization in section three. We will look at multiplicative systems in order to find an explicit construction of the localization of a category. In particular we will aim for the Verdier localization of a triangulated category with respect to a multiplicative system constructed from a triangulated subcategory.

In section four we investigate chain complexes of exact categories. In particular we look at acyclic complexes and some of their properties. This will lead us to the definition of the derived category as the homotopy category with the subcategory of acyclic complexes will fit into the definition of Verdier localization. At the end of section four we develop some tools which will help us find derived categories on different exact structures in the examples at the end of the thesis.

In section five we work towards a classification of exact structures on an idempotent complete category $\mathscr{A}$ through module categories. In particular we will construct a bijection between the exact structures on $\mathscr{A}$ and certain Serre subcategories of $\mathbf{mod}\,\mathscr{A}$. At the end of section five we consider the case where the class of all kernel-cokernel pairs forms an exact category $(\mathscr{A}, \mathscr{E})$ with enough projectives. In particular this includes abelian categories.

In section six we reformulate the classification result in the case of categories of finite type and see that we can restrict our attention to finitely generated projective Γ modules

$\operatorname{proj}\Gamma$ for some noetherian ring Γ. This enables us to introduce a class of exact categories that will turn out to be controlled by simple modules satisfying the 2-regular condition. Towards the end of section six we relate our findings to Auslander-Reiten translations. We will see that given a nice noetherian R-algebra Γ we have a correspondence between admissible exact structures on $\operatorname{proj}\Gamma$ and sets of dotted arrows (AR translations) in the Auslander-Reiten quiver of Γ. No background in AR-theory is given but the relevant results and references to sources containing proofs are given in appendix C.

The last section is used to combine our classification(s) and the derived category of an exact categories in examples on representations over the following quivers.

$$1 \to 2,\ 1 \to 2 \leftarrow 3 \text{ and } 1 \to 2 \to 3$$

In section one, two and four we mainly follow the exposition article “Exact Categories” by Theo Bühler [2]. Section three is based upon results found in the article “Derived categories, resolutions, and Brown representability” by Henning Krause [3]. In section five and six our work is based upon the work of Haruhisa Enomoto in the the article “Classifications of exact structures and Cohen-Macaulay-finite algebras” [4].

1 Exact Categories

The contents of this section is based on Bühler's article [2, chapter 1-3]. Some proofs have inspiration from Hansen [5].

In this section we look at some of the basics regarding exact categories needed for the rest of the thesis. Unless otherwise stated we are working in an exact category throughout the section.

1.1 Definition

In this subsection we will give the definition of an exact category, draw some consequences of the axioms and go through a standard example.

Definition 1.1.1. Let $\mathscr{A}$ be an additive category. A *kernel-cokernel pair* (i, p) in $\mathscr{A}$ is a pair of composable morphisms

$$A \xrightarrow{i} B \xrightarrow{p} C$$

such that i is a kernel of p and p is a cokernel of i.

Definition 1.1.2. Let $\mathscr{E}$ be a fixed class of kernel-cokernel pairs on an additive category $\mathscr{A}$. An *admissible monic* is a morphism i for which there exists a morphism p such that $(i, p) \in \mathscr{E}$. *Admissible epics* are defined dually. In diagrams we will often denote admissible monics by $\rightarrowtail$ and admissible epics by $\twoheadrightarrow$.

Definition 1.1.3. An *exact structure* on $\mathscr{A}$ is a class $\mathscr{E}$ of kernel-cokernel pairs which is closed under isomorphisms and satisfies the following axioms.

(E0) For all objects $A \in \mathscr{A}$ the identity morphism Id_A is an admissible monic.

($\mathrm{E0}^{\mathrm{op}}$) For all objects $A \in \mathscr{A}$ the identity morphism Id_A is an admissible epic.

(E1) The class of admissible monics is closed under composition.

($\mathrm{E1}^{\mathrm{op}}$) The class of admissible epics is closed under composition.

(E2) The push out of an admissible monic along an arbitrary morphism exists and yields an admissible monic.

($\mathrm{E2}^{\mathrm{op}}$) The pull back of an admissible epic along an arbitrary morphism exists and yields an admissible epic.

The following diagrams illustrate (E2) and (E2$^{\mathrm{op}}$) respectively.

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d, dashed] \\ A' \arrow[r, tail, dashed] & B' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, two heads, dashed] \arrow[d, dashed] & B \arrow[d] \\ A' \arrow[r, two heads] & B' \end{tikzcd}$$

Definition 1.1.4. An *exact category* is a pair $(\mathscr{A}, \mathscr{E})$ consisting of an additive category $\mathscr{A}$ and an exact structure $\mathscr{E}$ on $\mathscr{A}$. Elements of $\mathscr{E}$ will be called *short exact sequnces.*

Remark 1.1.5. We note the following properties from the axioms.

(1) $\mathscr{E}$ is an exact structure on $\mathscr{A}$ if and only if $\mathscr{E}^{op}$ is an exact structure on $\mathscr{A}^{op}$.

(2) Isomorphisms are admissible monics and admissible epics. Let $f : A \to B$ be an isomorphism. Consider the commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{Id}_A"'] & B \arrow[r] \arrow[d, "f^{-1}"'] & 0 \arrow[d] \\ A \arrow[r, tail, "\mathrm{Id}_A"] & A \arrow[r, two heads] & 0 \end{tikzcd}$$

Since $\mathscr{E}$ is closed under isomorphisms and the axioms are self dual f is both admissible epic and admissible mono.

(3) Let $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C \in \mathscr{E}$ and let $h : A \to D$ be a morphism. Then by (E2) there exists a pushout given by the left square of the following diagram.

$$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d, "h"'] & B \arrow[r, two heads, "g"] \arrow[d, "h'"] & C \arrow[d, equal] \\ D \arrow[r, tail, "f'"] & P \arrow[r, dashed, "g'"] & C \end{tikzcd}$$

By A.1 there exists a morphism $g' : P \to C$ such that the diagram commutes and $(f', g') \in \mathscr{E}$. Dually given $i : D \to C$ we have a pullback

$$\begin{tikzcd} A \arrow[r, dashed, "f'"] \arrow[d, equal] & P \arrow[r, two heads, "g'"] \arrow[d, "i'"] & D \arrow[d, "i"] \\ A \arrow[r, tail, "f"] & B \arrow[r, two heads, "g"] & C \end{tikzcd}$$

and a morphism $f' : A \to P$ such that $(f', g') \in \mathscr{E}$.

(4) An admissible epimorphism is always an epimorphism since it is a cokernel. To see this let $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C \in \mathscr{E}$. Assume $h_1 g = h_2 g$ for $h_1, h_2 : C \to D$. Then

$h_1gf = 0$, since g is a cokernel we get a unique morphism $i : C \to D$ such that $ig = h_1g$. However both h_1 and h_2 satisfies this. Hence $h_1 = h_2$ and g is epic.

$$\begin{tikzcd} A \arrow[r, tail, "f"] & B \arrow[r, two heads, "g"] \arrow[dr, "h_1g"'] & C \arrow[d, dashed, "\exists! i"] \\ & & D \end{tikzcd}$$

Dually an admissible monomorphism is always monic as it is a kernel.

(5) The following are equivalent to (E0) and $(\text{E0})^{\text{op}}$ respectively.

(E0)' For any object $A \in \mathscr{A}$ the zero morphism $A \to 0$ is admissible epic.

(E0^{op})' For any object $A \in \mathscr{A}$ the zero morphism $0 \to A$ is admissible monic.

To see this let (E0) hold. Then Id_A is an admissible monic. $A \to 0$ is a cokernel of Id_A hence $A \to 0$ is admissible epic. Conversely if (E0)' hold we get Id_A is a kernel of $A \to 0$ and hence admissible monic. The second equivalence is dual.

Example 1.1.6. Let $\mathscr{A}$ be an abelian category. Consider the class

$$\mathscr{E} = \{X \to Y \to Z | 0 \to X \to Y \to Z \to 0 \text{ exact in } \mathscr{A}\}$$

in other words $\mathscr{E}$ is the class of short exact sequences as defined for abelian categories. Then $(\mathscr{A}, \mathscr{E})$ is an exact category. To verify this we check the axioms one by one, including the implicit one. For the implicit axiom consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"'] & B \arrow[r, two heads, "p"] \arrow[d, "g"'] & C \arrow[d, "h"'] \\ A' \arrow[r, "i'"] & B' \arrow[r, "p'"] & C' \end{tikzcd}$$

Where (i, p) is a kernel cokernel pair, and the vertical arrows f, g and h are isomorphisms. We need to show (i', p') is a kernel cokernel pair. By commutativity of the diagram we know i' is monic and p' is epic. By A.4 it suffies to show i' is a kernel of p'. express i' and p' as follows $i' = gif^{-1}$ and $p' = hpg^{-1}$. Note that if^{-1} is a kernel of hp as f and h are isomorphisms. We also easily see by the diagram

$$\begin{tikzcd} A \arrow[r, tail, "i"] & B \arrow[r, two heads, "p"] \arrow[d, bend right, "g"'] & C \arrow[d, "h"'] \\ A' \arrow[u, "f^{-1}"] \arrow[r, "i'"] & B' \arrow[r, "p'"] \arrow[u, bend right, "g^{-1}"'] & C' \end{tikzcd}$$

that if^{-1} is a kernel of $hp \iff gif^{-1} = i'$ is a kernel of $hpg^{-1} = p'$.Hence $(i', p') \in \mathscr{E}$ and the class is closed under isomorphisms. For the explicit axioms note that an admissible monic is precisely a monomorphism since an admissible monic is a kernel, and hence monic. Conversely in an abelian category, every monomorphism forms a kernel cokernel pair with any of its cokernels. The first and second pairs of axioms follows by this observation. Every identity is an isomorphism, hence a monic epimorphism, and therefore an admissible monic and admissible epic. Compositions of admissible epics are epic, hence also admissible epics. The same dually hold for monics. The remaining axioms (E2) and ($E2^{op}$) follows from A.6.

1.2 Some useful results

Proposition 1.2.1. The sequence $A \overset{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}{\rightarrowtail} A \oplus B \overset{(0\ 1)}{\twoheadrightarrow} B$ is short exact.

Proof. Consider the following pushout square

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] & B \arrow[d, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"] \\ A \arrow[r, tail, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & A \oplus B \end{tikzcd}$$

The upper and left arrow is admissible mono by (E0$^{\mathrm{op}}$)'. Furthermore the bottom arrow is admissible mono by (E2). As $A \oplus B \xrightarrow{(0\ 1)} B$ is a cokernel of $A \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} A \oplus B$ we are done. ■

Proposition 1.2.2. The direct sum of two short exact sequences is a short exact sequence.

Proof. Let $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$ and $A' \overset{f'}{\rightarrowtail} B' \overset{g'}{\twoheadrightarrow} C'$ be short exact sequences. First we claim that for every object D the sequence

$$D \oplus A \xrightarrow{\left(\begin{smallmatrix}1&0\\0&f\end{smallmatrix}\right)} D \oplus B \xrightarrow{(0\ g)} C$$

is exact. Let $(\,0\ 1\,) : D \oplus B \to B$. Then $(\,0\ g\,) = g(\,0\ 1\,)$ and we get that $(\,0\ g\,)$ is epic by (E1$^{\mathrm{op}}$) and proposition 1.2.1. The first morphism in the sequence is a kernel of $(\,0\ g\,)$ and therefore admissible monic. Hence the sequence is exact. Similarly we get that the following sequence is exact

$$A \oplus D \xrightarrow{\left(\begin{smallmatrix}f&0\\0&1\end{smallmatrix}\right)} B \oplus D \xrightarrow{(g\ 0)} C$$

Now consider the morphism.

$$A \oplus A' \xrightarrow{\left(\begin{smallmatrix}f&0\\0&f'\end{smallmatrix}\right)} B \oplus B'$$

We observe this is the composition of two admissible monics by our previous claim.

$$A \oplus A' \overset{\left(\begin{smallmatrix}f&0\\0&1\end{smallmatrix}\right)}{\rightarrowtail} B \oplus A' \overset{\left(\begin{smallmatrix}1&0\\0&f'\end{smallmatrix}\right)}{\rightarrowtail} B \oplus B'$$

Hence it is admissible monic. As $\begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix}$ is a cokernel of $\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}$ it is admissible epic and we have our desired exact sequence.

$$A \oplus A' \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} B \oplus B' \xrightarrow{\begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix}} C \oplus C'$$

■

Example 1.2.3. An exact structure $\mathscr{E}$ of an additive category $\mathscr{A}$ is an additive subcategory of the additive category $\mathscr{A}^{\to\to}$ of composable morphisms in $\mathscr{A}$ by Proposition 1.2.2.

Proposition 1.2.4. Given a commutative square

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"'] & B \arrow[d, "f'"] \\ A' \arrow[r, tail, "i'"] & B' \end{tikzcd}$$

where the horizontal arrows are admissible monics. The following are equivalent

(1) The square is a pushout.

(2) The sequence $A \xrightarrow{\begin{pmatrix} i \\ -f \end{pmatrix}} B \oplus A' \xrightarrow{(f' \ i')} B'$ is exact

(3) The square is bicartesian (Both a pushout and a pullback).

(4) The square is part of a commutative diagram

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"'] & B \arrow[r, two heads, "p"] \arrow[d, "f'"] & C \arrow[d, equal] \\ A' \arrow[r, tail, "i'"] & B' \arrow[r, two heads, "p'"] & C \end{tikzcd}$$

with exact rows.

Proof. We show (1) ⇒ (2) ⇒ (3) ⇒ (1) , (1) ⇒ (4) and (4) ⇒ (2)

(1) ⇒ (2) : The push out property is equivalent to $(f' \ i')$ being a cokernel of $\begin{pmatrix} -i \\ f \end{pmatrix}$ by Lemma A.5. So we prove the latter. It suffices to show $\begin{pmatrix} -i \\ f \end{pmatrix}$ is admissible monic. This follows from [E1] as $\begin{pmatrix} -i \\ f \end{pmatrix}$ is equal to the following composition of morphisms.

$$A \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} A \oplus A' \xrightarrow[\cong]{\begin{pmatrix} 1 & 0 \\ -f & 1 \end{pmatrix}} A \oplus A' \xrightarrow{\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}} B \oplus A'$$

These are all admissible monics by Proposition 1.2.1, Remark 1.1.5 (2.) and argument in proof of proposition 1.2.2 respectively.

(2) $\Rightarrow$ (3) : follows from A.5.

(3) $\Rightarrow$ (1) : Automatic.

(1) $\Rightarrow$ (4) : Since i is an admissible monic it is the kernel of some admissible epic $p : B \twoheadrightarrow C$. By Lemma A.1 there exist $p' : B' \twoheadrightarrow C$ such that the desired diagram commutes and p' is the cokernel of i'. Since i' is admissible monic by assumption we are done.

(4) $\Rightarrow$ (2) : As p and p' are admissible epics we have the following pullback.

$$\begin{tikzcd} P \arrow[r, two heads, "q'"] \arrow[d, two heads, "q"] & B \arrow[d, two heads, "p"] \\ B' \arrow[r, two heads, "p'"] & C \end{tikzcd}$$

By using the dual of (1) $\Rightarrow$ (4) we can find the following commutative diagram with short exact rows and columns.

$$\begin{tikzcd} & A \arrow[r, equal] \arrow[d, tail, "j"] & A \arrow[d, tail, "i"] \\ A' \arrow[r, tail, "j'"] \arrow[d, equal] & P \arrow[r, two heads, "q'"] \arrow[d, two heads, "q"] & B \arrow[d, two heads, "p"] \\ A' \arrow[r, tail, "i'"] & B' \arrow[r, two heads, "p'"] & C \end{tikzcd}$$

Our goal is to show that $A \xrightarrow{i'} P \xrightarrow{q} B'$ is isomorphic to the sequence

$$A \xrightarrow{\binom{-i}{f}} B \oplus A' \xrightarrow{(f'\ i')} B'$$

Since $p = p'f'$ we get by the following pullback property a unique morphism $k : B \to P$.

$$\begin{tikzcd} B \arrow[dr, dashed, "k"] \arrow[drr, bend left, "\mathrm{Id}_B"] \arrow[ddr, bend right, "f'"'] & & \\ & P \arrow[r, two heads, "q'"] \arrow[d, two heads, "q"] & B \arrow[d, two heads, "p"] \\ & B' \arrow[r, two heads, "p'"] & C \end{tikzcd}$$

Now $q'k = \mathrm{Id}_B \implies q'kq' = q' \implies q'(\mathrm{Id}_P - kq') = 0$. Since j' is a kernel q' there exists a unique morphism $l : P \to A'$ such that $j'l = \mathrm{Id}_P - kq'$. By Remark 1.1.5 (4) j' is monic. This yields

$$j'lk = (\mathrm{Id}_P - kq')k = 0 \implies lk = 0$$
$$j'lj' = (\mathrm{Id}_P - kq')j' = j' \implies lj' = \mathrm{Id}_{A'}$$

Simmilarly since i' is monic we get

$$i'lj = qj'lj = q(\mathrm{Id}_P - kq')j = qj - (qk)(q'j) = -f'i = -i'f \implies lj = -f$$

The morphisms

$$B \oplus A' \xrightarrow{(\,k\ j'\,)} P \qquad \text{and} \qquad P \xrightarrow{\binom{q'}{l}} B \oplus A'$$

are mutually inverses by the following equalities.

$$(\,k\ j'\,)\begin{pmatrix} q' \\ l \end{pmatrix} = kq' + j'l = kq' + \mathrm{Id}_P - kq' = \mathrm{Id}_P \text{ and } \begin{pmatrix} q' \\ l \end{pmatrix}(\,k\ j'\,) = \begin{pmatrix} q'k & q'j' \\ lk & lj' \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_B & 0 \\ 0 & \mathrm{Id}_{A'} \end{pmatrix}$$

Note that

$$(\,f'\ i'\,)\begin{pmatrix} q' \\ l \end{pmatrix} = q(\,k\ j'\,)\begin{pmatrix} q' \\ l \end{pmatrix} = q \text{ and } -\mathrm{Id}_A\begin{pmatrix} -i \\ f \end{pmatrix} = \begin{pmatrix} i \\ -f \end{pmatrix} = \begin{pmatrix} q' \\ l \end{pmatrix}j$$

Hence we have the following isomorphism between sequences

$$\begin{tikzcd}
A \arrow[r, tail, "j"] \arrow[d, "-\mathrm{Id}_A"'] & P \arrow[r, two heads, "q"] \arrow[d, "\binom{q'}{l}"] & B' \arrow[d, equal] \\
A \arrow[r, "\binom{-i}{f}"] & B \oplus A \arrow[r, "(\,f'\ i'\,)"] & B'
\end{tikzcd}$$

and the bottom sequence is exact as desired.

■

Corollary 1.2.5. The following rectangle, where the right square is a push out and the left square a pullback

$$\begin{tikzcd}
A \arrow[r, two heads, "f"] \arrow[d, "a"] & B \arrow[r, tail, "g"] \arrow[d, "b"] & C \arrow[d, "c"] \\
A' \arrow[r, two heads, "f'"] & B' \arrow[r, tail, "g'"] & C'
\end{tikzcd}$$

is bicartesian. Furthermore the sequence $A \xrightarrow{\binom{-a}{gf}} A' \oplus C \xrightarrow{(\,g'f'\ c\,)} C'$ is exact.

Proof. By Proposition 1.2.4 and its dual we get that both squares are bicartesian. By Lemma A.7 the rectangle is bicartesian. Hence we have proved the first part. Note that $(\,g'f'\ c\,) = (\,g'\ c\,)\begin{pmatrix} f' & 0 \\ 0 & \mathrm{Id}_C \end{pmatrix}$. This is a composition of two admissible epics by Proposition 1.2.4 and part of argument in proof of Proposition 1.2.2. As $\binom{-a}{gf}$ is a kernel of $(\,g'f'\ c\,)$ we are done. ■

Corollary 1.2.6. Given the push out diagram

$$\begin{tikzcd}
A' \arrow[r, tail, "i'"] \arrow[d, "a"'] & B' \arrow[d, "b"] \\
A \arrow[r, tail, "i"] & B
\end{tikzcd}$$

We have the following

(1) if $j' : B' \twoheadrightarrow C'$ is a cokernel of i' then the unique morphism $j : B \to C'$ such that $ji = 0$ and $jb = j'$ is a cokernel of i.

(2) If $j : B \to C$ is a cokernel of i then $j' = jb$ is a cokernel of i'.

Proof. The first part we see in the proof of A.1 (The dual of what we did). For the second part let $j : B \to C$ be a cokernel of i. We have $jbi' = jia = 0$ as j is a cokernel of i. Next we show the universal property. Let $t : B' \to T$ be another morphism such that $ti' = 0$. Then we get by the following universal property a unique morphism $h : B \to T$.

$$\begin{tikzcd}
A' \arrow[r, tail, "i'"] \arrow[d, "a"'] & B' \arrow[d, "b"] \arrow[ddr, bend left, "t"] & \\
A \arrow[r, tail, "i"] \arrow[drr, bend right, "0"'] & B \arrow[dr, dashed, "h"] & \\
 & & T
\end{tikzcd}$$

Now as j is a cokernel of i we get a unique morphism $f : C \to T$ such that $fj = h$. Now we have

$$\begin{tikzcd}
A' \arrow[r, tail, "i'"] & B' \arrow[r, "b"] \arrow[dr, "t"'] & B \arrow[r, two heads, "j"] \arrow[d, dashed, "h"] & C \arrow[dl, dashed, "f"] \\
 & & T &
\end{tikzcd}$$

and see $t = hb = fjb$ completing the argument. ■

Proposition 1.2.7. Let $i : A \to B$ be a morphism in $(\mathscr{A}, \mathscr{E})$ admitting a cokernel. If there exists a morphism $j : B \to C$ in $\mathscr{A}$ such that the composition $ji : A \rightarrowtail C$ is an admissible monic then i is admissible monic.

Proof. Let $c : B \to D$ be a cokernel of i. From the pushout diagram

$$\begin{tikzcd}
A \arrow[r, tail, "ji"] \arrow[d, "i"'] & C \arrow[d] \\
B \arrow[r, tail] & E
\end{tikzcd}$$

and Proposition 1.2.4 we get that $\binom{i}{ji} : A \to B \oplus C$ is admissible monic. As we have that $\begin{pmatrix} \mathrm{Id}_B & 0 \\ -j & \mathrm{Id}_C \end{pmatrix} : B \oplus C \to B \oplus C$ is an isomorphism it is in particular admissible monic. Hence we have that $\binom{i}{0} = \begin{pmatrix} \mathrm{Id}_B & 0 \\ -j & \mathrm{Id}_C \end{pmatrix} \binom{i}{ji}$ is an admissible monic as well. Since $\begin{pmatrix} c & 0 \\ 0 & \mathrm{Id}_C \end{pmatrix}$ is a cokernel of $\binom{i}{0}$ it is admissible epic. Consider the following diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, equal] & B \arrow[r, "c"] \arrow[d, "\binom{\mathrm{Id}}{0}"] & D \arrow[d, "\binom{\mathrm{Id}}{0}"] \\
A \arrow[r, tail, "\binom{i}{0}"'] & B \oplus C \arrow[r, two heads, "\begin{pmatrix} c & 0 \\ 0 & \mathrm{Id}_C \end{pmatrix}"'] & D \oplus C
\end{tikzcd}$$

c is an admissible epic as the right hand square will be a pullback. By Lemma A.1 i is a kernel of c and we are done. ■

Corollary 1.2.8. Let (i, p) and (i', p') be two pairs of composable morphisms. If the direct sum $i \oplus i', p \oplus p'$ is exact, then (i, p) and (i', p') are both exact. In other words $\mathscr{E}$ is closed under direct summands.

Proof. We easily note that (i, p) and (i', p') are kernel cokernel pairs. Furthermore we see that $\binom{1}{0} i = \begin{pmatrix} i & 0 \\ 0 & i' \end{pmatrix} \binom{1}{0}$ which is admissible monic. Hence by Proposition 1.2.7 we get that i is admissible monic and we are done. ■

1.3 Some classic diagram lemmas

In this section we show three classic results that are well known in the case of abelian categories. In particular we will show the small 5-lemma, Noether isomorphism and 3x3 lemma for exact categories.

Proposition 1.3.1. The pull-back of an admissible monic along an admissible epic yields an admissible monic.

Proof. Consider the diagram

$$\begin{tikzcd}
A' \arrow[r, "i'"] \arrow[d, two heads, "e'"] & B' \arrow[r, two heads, "pe"] \arrow[d, two heads, "e"] & C' \arrow[d, equal, "c"] \\
A \arrow[r, tail, "i"'] & B \arrow[r, two heads, "p"] & C
\end{tikzcd}$$

Where the left square is a pullback that exist by (E2$^{\mathrm{op}}$). Let p be a cokernel of i, hence admissible epic. This gives us that pe is admissible epic as it is a composition of two admissible epics. We get by Lemma A.3 that i' is monic. In order to show i' is admissible monic we show it is is a kernel of pe. pei' is clearly 0, we show the universal property. Let $g' : X \to B'$ be such that $peg' = 0$. Since i is a kernel of p there exists unique $f : X \to A$ such that $eg' = if$. Applying the universal property of the pullback we get a unique $f' : X \to A'$ such that $e'f' = f$ and $i'f' = g'$. Since i' is monic we get that f' is the unique morphism such that $i'f' = g'$ and we are done. ∎

Proposition 1.3.2. A morphism (a, b, c) from a short exact sequence $A' \rightarrowtail B' \twoheadrightarrow C'$ to another short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ factors over a short exact sequence $A \rightarrowtail D \twoheadrightarrow C'$

$$\begin{tikzcd}
A' \arrow[r, tail, "f'"] \arrow[d, "a"] \arrow[dr, phantom, "\mathrm{BC}"] & B' \arrow[r, two heads, "g'"] \arrow[d, "b'"] & C' \arrow[d, equal] \\
A \arrow[r, tail, "m"] \arrow[d, equal] & D \arrow[r, two heads, "e"] \arrow[d, "b''"] \arrow[dr, phantom, "\mathrm{BC}"] & C' \arrow[d, "c"] \\
A \arrow[r, tail, "f"] & B \arrow[r, two heads, "g"] & C
\end{tikzcd}$$

such that the two squares marked BC is bicartesian. In particular there is an isomorphism between the pushout under f' and a and the pullback over g and c.

Proof. Form the pushout under f' and a to obtain the object D and the morphisms m and b'. Let $e : D \to C'$ be the unique morphism such that $eb' = g'$ and $em = 0$. By

Corollary 1.2.6 e is a cokernel of m. Let $b'' : D \to B$ be the unique morphism such that $b''b' = b$ and $b''m = f$. The following diagrams show the construction.

$$
\begin{tikzcd}
A' \arrow[r, tail, "f'"] \arrow[d, "a"'] & B' \arrow[d, "b'"] \arrow[ddr, bend left, "g'"] & \\
A \arrow[r, tail, "m"] \arrow[drr, bend right, "0"'] & D \arrow[dr, dashed, two heads, "e"] & \\
 & & C'
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, tail, "f'"] \arrow[d, "a"'] & B' \arrow[d, "b'"] \arrow[ddr, bend left, "b"] & \\
A \arrow[r, tail, "m"] \arrow[drr, bend right, "f"'] & D \arrow[dr, dashed, "b''"] & \\
 & & B
\end{tikzcd}
$$

By Proposition 1.2.4 we get that the top left square is bicartesian. The lower right square commutes as a and $b = b''b'$ determines c uniquely by Lemma A.2. By the dual of Proposition 1.2.4 we get that the lower right square is bicartesian. ■

Corollary 1.3.3 (Five Lemma)**.** Consider a morphism between short exact sequences.

$$
\begin{tikzcd}
A \arrow[r, tail] \arrow[d, "a"] & B \arrow[r, two heads] \arrow[d, "b"] & C \arrow[d, "c"] \\
A' \arrow[r, tail] & B' \arrow[r, two heads] & C'
\end{tikzcd}
$$

if a and c are isomorphisms so is b. Furthermore if if a and c are admissible monics (epics) so is b.

Proof. First we assume a and c are isomorphisms. As isomorphisms are preserved by push-outs and pull-backs the diagram in Proposition 1.3.2 yields that b is the composition of two isomorphisms, and hence an isomorphism. Now let a and c be admissible monics. It follows from the diagram in Proposition 1.3.2 together with (E2) and Proposition 1.3.1 that b now is the composition of two admissible monics, and hence admissible monic. The case of admissible epics is dual. ■

Proposition 1.3.4 ("Noether Isomorphism")**.** Consider the diagram

$$
\begin{tikzcd}
A \arrow[r, tail, "f"] \arrow[d, equal] & B \arrow[r, two heads, "g"] \arrow[d, tail, "i"] & X \arrow[d, dashed, tail, "s"] \\
A \arrow[r, tail, "a"] & C \arrow[r, two heads, "b"] \arrow[d, two heads, "p"] & Y \arrow[d, dashed, two heads, "t"] \\
 & Z \arrow[r, equal] & Z
\end{tikzcd}
$$

where the first two horizontal rows and the middle column are short exact. Then the rightmost column exist, is short exact and is uniquely determined by the requirement that it makes the diagram commutative. Furthermore the upper right hand square is bicartesian.

Proof. By Lemma A.2 we have a unique morphism $s : X \to Y$ making the upper right square commute. By the dual of 1.2.4 we now get that the upper right square is bicartesian. Since the upper left square is bicartesian we get that s is admissible monic by (E2). By Lemma A.1 we can now find a t that is the cokernel of s and makes the lower square commute. ■

Proposition 1.3.5 (3x3 Lemma)**.** Consider a commutative diagram

$$
\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, tail, "a"] & B' \arrow[r, "g'"] \arrow[d, tail, "b"] & C' \arrow[d, tail, "c"] \\
A \arrow[r, "f"] \arrow[d, two heads, "a'"] & B \arrow[r, "g"] \arrow[d, two heads, "b'"] & C \arrow[d, two heads, "c'"] \\
A'' \arrow[r, "f''"] & B'' \arrow[r, "g''"] & C''
\end{tikzcd}
$$

in which the columns are exact. Assume in addition one of the following

(1) The two outer rows are exact and $gf = 0$.

(2) The middle and top row is exact.

(3) The middle and bottom row is exact.

Then the remaining row is exact.

Proof. First we assume (1) hold. We start by forming a pushout under g' and b. This gives us the following diagram with exact rows and columns.

$$
\begin{tikzcd}
A' \arrow[r, tail, "f'"] \arrow[d, equal] & B' \arrow[r, two heads, "g'"] \arrow[d, tail, "b"] & C' \arrow[d, tail, "k"] \\
A' \arrow[r, tail, "i"] & B \arrow[r, two heads, "j"] \arrow[d, two heads, "b'"] & D \arrow[d, two heads, "k'"] \\
 & B'' \arrow[r, equal] & B''
\end{tikzcd}
$$

By Corollary 1.2.6 the cokernel k' of k is determined by $k'j = b'$ and $k'k = 0$. j is epic by the dual of Proposition 1.3.1. Furthermore we see that $i = bf'$ is a kernel of the admissible epic j by the dual of Corollary 1.2.6. By using the pushout property of

$B'C'BD$ we get unique d' making the following commute.

$$\begin{tikzcd}
B' \arrow[r, "g'"] \arrow[d, "b"'] & C' \arrow[d, "k"] \arrow[ddr, bend left, "c"] & \\
B \arrow[r, "j"] \arrow[drr, bend right, "g"'] & D \arrow[dr, dashed, "d'"] & \\
& & C
\end{tikzcd}$$

Consider the following diagram with exact rows.

$$\begin{tikzcd}
C' \arrow[r, tail, "k"] \arrow[d, equal] & D \arrow[r, two heads, "k'"] \arrow[d, "d'"] & B'' \arrow[d, two heads, "g''"] \\
C' \arrow[r, tail, "c"] & C \arrow[r, two heads, "c'"] & C''
\end{tikzcd}$$

The left square commutes by our previous diagram. The right square commutes as $c'd'j = c'g = g''b' = g''k'j$ and j is epic. By the dual of Proposition 1.2.4 $DCB''C''$ is a pullback square. Hence d' is admissible epic. Consequently $g = d'j$ is admissible epic. The unique morphism $d : A'' \to D$ such that $k'd = f''$ and $d'd = 0$ is a kernel of d' by dual of Corollary 1.2.6. By the pullback property of $DCB''C''$ the diagram

$$\begin{tikzcd}
A' \arrow[r, equal] \arrow[d, "a"] & A' \arrow[d, "i"] & \\
A \arrow[r, "f"] \arrow[d, "a'"] & B \arrow[r, "g"] \arrow[d, "j"] & C \arrow[d, equal] \\
A'' \arrow[r, tail, "d"] & D \arrow[r, two heads, "d'"] & C
\end{tikzcd}$$

commutes as k' and d' are epic, $k'da' = f''a' = b'f = k'jf$ and $d'da' = 0 = gf = d'jf$. Note that this is where the assumption $gf = 0$ came into play. By the dual of Proposition 1.2.4 $ABA''D$ is bicartesian. Now f is a kernel of g by Proposition 1.3.1 and the middle row is exact.

Next we assume (2) hold. Apply Proposition 1.3.2 to the two upper rows to obtain the commutative diagram

$$\begin{tikzcd}
A' \arrow[r, tail, "f'"] \arrow[d, tail, "a"] \arrow[dr, phantom, "\mathrm{BC}"] & B' \arrow[r, two heads, "g'"] \arrow[d, tail, "i"] & C' \arrow[d, equal] \\
A \arrow[r, tail, "\tilde{f}"] \arrow[d, equal] & D \arrow[r, two heads, "\tilde{g}"] \arrow[d, tail, "j"] \arrow[dr, phantom, "\mathrm{BC}"] & C' \arrow[d, tail, "c"] \\
A \arrow[r, tail, "f"] & B \arrow[r, two heads, "g"] & C
\end{tikzcd}$$

where $ji = b$. Note that i is admissible monic by [E2] and j is admissible monic by Proposition 1.3.1. By Corollary 1.2.6 the morphism $i' : D \to A''$ determined by $ii' = 0$

and $i'\tilde{f} = a'$ is a cokernel of i and the morphism $j' : B \to C''$ given by $j' = c'g = g''b'$ is a cokernel of j as shown in the following diagrams.

$$
\begin{tikzcd}
A' \arrow[r, tail, "f'"] \arrow[d, tail, "a"'] & B' \arrow[d, tail, "i"] \arrow[ddr, bend left, "0"] & \\
A \arrow[r, tail, "\tilde{f}"] \arrow[drr, bend right, "a'"'] & D \arrow[dr, dashed, two heads, "i'"] & \\
 & & A''
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[r, two heads, "g'"] \arrow[d, tail, "j"] & C' \arrow[d, tail, "c"] \\
B \arrow[r, two heads, "g"] \arrow[d, dashed, two heads, "j'"] & C \arrow[d, two heads, "c'"] \\
C'' \arrow[r, equal] & C''
\end{tikzcd}
$$

Now if we show the diagram

$$
\begin{tikzcd}
B' \arrow[r, tail, "i"] \arrow[d, equal] & D \arrow[r, two heads, "i'"] \arrow[d, tail, "j"] & A'' \arrow[d, "f''"] \\
B' \arrow[r, tail, "b"] & B \arrow[r, two heads, "b'"] \arrow[d, two heads, "j'"] & B'' \arrow[d, "g''"] \\
 & C'' \arrow[r, equal] & C''
\end{tikzcd}
$$

is commutative we are done by Noether isomorphism (Proposition 1.3.4). All that remains to show is that $f''i' = b'j$ as the rest hold by construction. We will show both is a solution to the pushout problem

$$
\begin{tikzcd}
A' \arrow[r, tail, "f'"] \arrow[d, tail, "a"'] & B' \arrow[d, tail, "i"] \arrow[ddr, bend left, "0"] & \\
A \arrow[r, tail, "\tilde{f}"] \arrow[drr, bend right, "b'f"'] & D \arrow[dr, dashed, "\exists!"] & \\
 & & B''
\end{tikzcd}
$$

Then by pushout property they will be equal. We have

$$(f''i')i = 0 = b'b = (b'j)i \qquad \text{and} \qquad (b'j)\tilde{f} = b'f = f''a' = (f''i')\tilde{f}$$

which together with

$$(f''i'\tilde{f})a = (f''i'i)f' \qquad \text{and} \qquad (b'j\tilde{f})a = f''a'a = 0 = b'bf' = (b'ji)f'$$

proves the claim. Assuming (3) yields a dual proof of assuming (2). ■

1.4 Admissible morphisms

In this subsection we look at admissible morphisms and some of their properties. These will show up when we later investigate acyclic complexes in Section 4. Towards the end of the subsection we will see that an exact category consisting of only admissible morphisms is abelian.

Definition 1.4.1. A morphism $f : A \to B$ in an exact category is called *admissible* if it factors as a composition of an admissible monic with an admissible epic. This will often be denoted $\xrightarrow{\circ}$.

$$\begin{tikzcd}
A \arrow[rr, "f", "\circ" marking] \arrow[dr, two heads, "e"'] & & B \\
& I \arrow[ur, tail, "m"'] &
\end{tikzcd}$$

Proposition 1.4.2. The factorization of an admissible morphism is unique up to unique isomorphism.

Proof. Let $A \xrightarrow{\circ} B$ be a admissible morphism. Consider the commutative square

$$\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[d, two heads, "e'"'] & I \arrow[d, tail, "m"] \arrow[dl, dashed, shift right, "i'"'] \\
I' \arrow[r, tail, "m'"'] \arrow[ur, dashed, shift right, "i"'] & B
\end{tikzcd}$$

with two different factorisation of f. We need to show there is unique mutual inverses i and i' making the diagram commute. Let c be a cokernel of m'. As $cme = cm'e' = 0$ and e is epic we get $cm = 0$. By universal property of m' as a kernel we get unique morphism $i' : I \to I'$. Finding i is dual. By $m'e' = mii'e \implies ii' = \mathrm{Id}_I$ and $me = m'i'ie' \implies i'i = \mathrm{Id}_{I'}$ we get i and i' are mutual inverses. ■

Lemma 1.4.3. Let f be an admissible morphism with the following factorisation.

$$\begin{tikzcd}
A \arrow[rr, "f", "\circ" marking] \arrow[dr, two heads, "e"'] & & B \\
& I \arrow[ur, tail, "m"'] &
\end{tikzcd}$$

Then we have if k is a kernel of e then it is a kernel of f. Dually if c is a cokernel of m then it is a cokernel of f.

Proof. We only prove the first part as the second part is dual. Let k be a kernel of e. We show it is a kernel of f. We see composition is zero by $fk = mek = 0$. Now let t be another morphism such that $ft = 0$. Then we have $ft = met = 0$. Since m is monic we get that $et = 0$. Since k is a kernel of e we get a unique map h such that $kh = t$. Hence k is a kernel of f. ■

Definition 1.4.4. The *analysis* of an admissible morphism is the diagram

$$\begin{tikzcd}
& A \arrow[rr, "f", "\circ" marking] \arrow[dr, two heads, "e"'] & & B \arrow[dr, two heads, "c"] & \\
K \arrow[ur, tail, "k"] & & I \arrow[ur, tail, "m"'] & & C
\end{tikzcd}$$

where k is a kernel, c is a cokernel, e is a coimage, and m is an image of f. The isomorphism classes of K, I and C are well defined by Proposition 1.4.2 and Lemma 1.4.3

We are now able to make sense of the following definition composing multiple admissible morphisms.

Definition 1.4.5. A sequence of admissible morphisms

$$\begin{tikzcd}
A \arrow[rr, "\circ" marking] \arrow[dr, two heads] & & B \arrow[rr, "\circ" marking] \arrow[dr, two heads] & & C \\
& I \arrow[ur, tail] & & I' \arrow[ur, tail] &
\end{tikzcd}$$

is called *acyclic* if $I \to B \to I'$ is short exact. Longer sequences of admissible morphisms are acyclic if the sequence given by any two consecutive morphisms is exact.

Example 1.4.6. Given an exact category $(\mathscr{A}, \mathscr{E})$ where all morphisms are admissible we have that $\mathscr{A}$ is abelian. Given $A \xrightarrow{f} B$ in $\mathscr{A}$ it follows from 1.4.3 that f has a kernel and a cokernel. To see that that every monic is a kernel let $i : A \longrightarrow B$ be monic. As i is admissible we have

$$\begin{tikzcd}
K \arrow[r, "k"] & A \arrow[rr, "i"] \arrow[dr, two heads, "e"'] & & B \\
& & I \arrow[ur, tail, "m"'] &
\end{tikzcd}$$

where (k, e) is a kernel cokernel pair. We note that $mek = ik = 0$ Since i is monic we get $k = 0$. Hence e is an isomorphism. Since m is a kernel by assumption so is f. By a dual argument every epic is a cokernel.

2 Idempotent Completion

2.1 Idempotent completion

This section is based on bühler's article [2, Chapter 6 and 7]

A lot of the results we will see throughout the thesis only hold in idempotent complete categories. It is therefore necessary to mention this class of additive categories. We will see a constructive proof of how to fully faithfully embed any additive category into an idempotent complete category. Furthermore we look at a weaker form of idempotent completion and briefly look at some results on how this and idempotent completion affects an exact category.

Definition 2.1.1. An additive category $\mathscr{A}$ is called *idempotent complete* if for every idempotent $p : A \to A$ there is a decomposition $A \cong K \oplus I$ of $\mathscr{A}$ such that $p \cong \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Proposition 2.1.2. An additive category $\mathscr{A}$ is idempotent complete if and only if every idempotent has a kernel.

Proof. Suppose every idempotent has a kernel. Let $k : K \to A$ be a kernel of the idempotent $p : A \to A$. Let $i : I \to A$ be a kernel of the idempotent $(\mathrm{Id}_A - p)$. Then we get the following diagrams by the universal property as $p(\mathrm{Id}_A - p) = (\mathrm{Id}_A - p)p = 0$.

$$\begin{tikzcd} K \arrow[r, "k"] & A \arrow[r, "p"] & A \\ A \arrow[u, dashed, "l"] \arrow[ur, "(1-p)"'] & & \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "i"] & A \arrow[r, "(1-p)"] & A \\ A \arrow[u, dashed, "j"] \arrow[ur, "p"'] & & \end{tikzcd}$$

We have $kli = (1-p)i = 0$ hence $li = 0$ as k is monic. We also get $lk = \mathrm{Id}_K$ as $klk = (1-p)k = pk + (1-p)k = k$. Similarly we get $jk = 0$ and $ji = \mathrm{Id}_I$. Hence we get that $\left(\begin{smallmatrix} k & i \end{smallmatrix}\right) : K \oplus I \to A$ and $\left(\begin{smallmatrix} l \\ j \end{smallmatrix}\right) : A \to K \oplus I$ are mutual inverses and therefore $A \cong K \oplus I$. Furthermore $p = \left(\begin{smallmatrix} l \\ j \end{smallmatrix}\right) p \left(\begin{smallmatrix} k & i \end{smallmatrix}\right) = \left(\begin{smallmatrix} l \\ j \end{smallmatrix}\right) ij \left(\begin{smallmatrix} k & i \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Id}_I \end{smallmatrix}\right)$ as desired. Conversely let $\mathscr{A}$ be idempotent complete. Then a kernel of an idempotent $\left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Id}_B \end{smallmatrix}\right) : A \oplus B \to A \oplus B$ is $\left(\begin{smallmatrix} \mathrm{Id}_A \\ 0 \end{smallmatrix}\right) : A \to A \oplus B$. ■

Remark 2.1.3. Note that the first part of the proof gives us that every idempotent p in an idempotent complete category can be decomposed to $\mathrm{Ker}(p) \oplus \mathrm{Im}(p)$ with $p \cong \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Id}_{\mathrm{Im}(p)} \end{smallmatrix}\right)$.

Proposition 2.1.4. Every additive category $\mathscr{A}$ can be fully faithfully embedded into an idempotent complete category $\mathscr{A}^\wedge$.

Proof. We start by constructing the category $\mathscr{A}^\wedge$. The objects are pairs (A,p) where $A \in \mathscr{A}$ and p is an idempotent in $\mathscr{A}$. The morphisms are defined by the following $\mathrm{Hom}_{\mathscr{A}^\wedge}((A,p),(B,q)) = q \circ \mathrm{Hom}_{\mathscr{A}}(A,B) \circ p$ where the composition is induced by $\mathscr{A}$. Note that $\mathrm{Id}_{(A,p)} = p$. It is easy to verify that $\mathscr{A}^\wedge$ is additive with the biproduct $(A,p) \oplus (B,q) = (A \oplus B, p \oplus q)$. Now the functor $i_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}^\wedge$ given by $i_{\mathscr{A}}(A) = (A, \mathrm{Id}_A)$ and $i_{\mathscr{A}}(f) = \mathrm{Id}\, f\, \mathrm{Id} = f$ is fully faithful by construction. All that remains is to see that $\mathscr{A}^\wedge$ is idempotent complete. Let pfp be an idempotent of (A,p) in $\mathscr{A}^\wedge$. A fortiori pfp is also idempotent in $\mathscr{A}$. Now we can see $(A,p) \cong (A, p - pfp) \oplus (A, pfp)$ by the mutual inverses $\begin{pmatrix} p-pfp \\ pfp \end{pmatrix}$ and $\begin{pmatrix} p-pfp & pfp \end{pmatrix}$. We see that $\begin{pmatrix} p-pfp \\ pfp \end{pmatrix} pfp \begin{pmatrix} p-pfp & pfp \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & pfp \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$. Hence $\mathscr{A}^\wedge$ is idempotent complete. ■

Definition 2.1.5. In a category $\mathscr{C}$ a morphism $r : B \to C$ is called a *retraction* if there exists a *section* $s : C \to B$ such that $rs = \mathrm{Id}_C$. A *coretraction* is defined dually i.e. $c : A \to B$ is a coretraction if it admits a section $s : B \to A$ such that $sc = \mathrm{Id}_A$.

Remark 2.1.6. Retractions are epics, and coretractions are monics. Furthermore a section of a retraction is a coretraction and a section of a coretraction is a retraction.

Lemma 2.1.7. Let $r : B \to C$ be a retraction with section $s : C \to B$. Then the idempotent sr give rise to a splitting of B if r admits a kernel $k : A \to B$.

Proof. Note that $r = rsr \implies r(\mathrm{Id}_B - sr) = 0$. Hence by the kernel property of k there exists unique $t : B \to A$ such that $kt = \mathrm{Id}_B - sr$. It follows that k is a coretraction since $ktk = (\mathrm{Id}_B - sr)k = k - srk = k \implies tk = \mathrm{Id}_A$. Furthermore we have that $kts = (\mathrm{Id}_B - sr)s = s - s = 0$, hence $ts = 0$ as k is monic. Now we have that the morphism $\begin{pmatrix} k & s \end{pmatrix} : A \oplus C \to B$ is an isomorphism with inverse $\begin{pmatrix} t \\ r \end{pmatrix}$. In particular the sequence $A \to B \to C$ is isomorphic to $A \to A \oplus C \to C$. ■

Lemma 2.1.8. In an additive category the following are equivalent

(1) Every coretraction has a cokernel.

(2) Every retraction has a kernel.

Proof. By duality it suffies to show (2) implies (1). let $c : C \to B$ be a coretraction with section $s : B \to C$. Then s is a retraction and has kernel $k : A \to B$ by assumption. By the proof of Lemma 2.1.7 k is a coretraction with section $t : B \to A$. By the conclusion in Lemma 2.1.7 this is a cokernel. ■

Definition 2.1.9. If the conditions of 2.1.8 hold then $\mathscr{A}$ is said to be *weakly idempotent complete.*

Lemma 2.1.10. Let $(\mathscr{A}, \mathscr{E})$ be an exact category. The following are equivalent.

(1) The additive category $\mathscr{A}$ is weakly idempotent complete.

(2) Every coretraction is an admissible monic.

(3) Every retraction is an admissible epic.

Proof. We show (1) $\iff$ (3), (1) $\iff$ (2) is dual. For (1) $\Rightarrow$ (3) let r be a retraction. r admits a kernel by assumption. Therefore we get by Lemma 2.1.7 that the sequence $A \xrightarrow{k} B \xrightarrow{r} C$ is isomorphic to $A \to A \oplus C \to C$. This is exact by Proposition 1.2.1. Hence r is admissible epic. Conversely we know every admissible epic has a kernel. ■

Proposition 2.1.11. Let $(\mathscr{A}, \mathscr{E})$ be an exact category. Then the following are equivalent.

(1) The additive category $\mathscr{A}$ is weakly idempotent complete.

(2) Let $f : A \to B$ and $g : B \to C$ be morphisms. If gf is admissible epic then g is an admissible epic.

(3) Let $f : A \to B$ and $g : B \to C$ be morphisms. If gf is admissible monic then f is admissible monic.

Proof. (1) $\Rightarrow$ (2) : We form the pullback over g and gf and consider the following universal property.

$$\begin{tikzcd}
A \arrow[dr, dashed, "\exists! h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\mathrm{Id}_A"'] & & \\
& B' \arrow[r, two heads, "f'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\
& A \arrow[r, two heads, "gf"] & C
\end{tikzcd}$$

g' is a retraction as $g'h = \mathrm{Id}_A$. By assumption g' has a kernel $k : K \to B'$. We claim $f'k$ is a kernel of g. By the diagram

$$
\begin{tikzcd}
K \arrow[dr, dashed, "k"] \arrow[drr, bend left, "kf'"] \arrow[ddr, bend right, "0"'] & & \\
& B' \arrow[r, two heads, "f'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\
& A \arrow[r, two heads, "gf"] & C
\end{tikzcd}
$$

We see $gf'k = gfg'k = 0$. Let $t : T \to B$ be another morphism such that $gt = 0$. By the following two universal properties we find our desired unique morphism $i : T \to K'$.

$$
\begin{tikzcd}
T \arrow[dr, dashed, "s"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "0"'] & & \\
& B' \arrow[r, two heads, "f'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\
& A \arrow[r, two heads, "gf"] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[r, "k"] & B' \arrow[r, "g"] & B \\
T \arrow[u, dashed, "i"] \arrow[ur, "s"'] & &
\end{tikzcd}
$$

Hence we found a kernel of g. By the dual of Proposition 1.2.7 g is admissible epic. $(2) \Rightarrow (1)$: Let g be a retraction. Then there exist s such that $gs = \mathrm{Id}$. Since Id is admissible epic we get that g is admissible epic by assumption. By 2.1.10 we are done. Showing (1) $\iff$ (3) is dual. ■

3 Localization

This section is based on Krause's article [3, Chapter 3]

In this sections we will work our way to the definition of the Verdier localization. This will be applied later in order to define the derived category of an exact category.

3.1 Localization with a multiplicative system

In this subsection we will define localization and introduce multiplicative systems to give an explicit construction.

Definition 3.1.1. Let $\mathscr{C}$ be a category. Let S be a class of maps in $\mathscr{C}$. The *localization* of $\mathscr{C}$ with respect to S is a category $S^{-1}\mathscr{C}$ together with a functor $Q : \mathscr{C} \to S^{-1}\mathscr{C}$ such that the following hold.

(L1) Qf is an isomorphism for all $f \in S$.

(L2) For any functor $F : \mathscr{C} \to \mathscr{D}$ such that Ff is an isomorphism for all $f \in S$, there exist an unique functor $H : S^{-1}\mathscr{C} \to \mathscr{D}$ such that $HQ = F$.

In order to give an explicit construction of $S^{-1}\mathscr{C}$ we need to put some constraints on S.

Definition 3.1.2. Let S be a class of maps in $\mathscr{C}$. S is a *multiplicative system* if the following hold.

(MS1)
- If f and g are composable morphisms in S then gf is in S.
- The identity map Id_A is in S for all objects in $\mathscr{C}$.

(MS2) Let $s : B \to C \in S$. Then every pair of morphisms $f : A \to C$ and $g : B \to D$ in $\mathscr{C}$ can be completed to a pair of commutative diagrams

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, dashed, "\hat{s}_1"'] & B \arrow[d, "s"] \\ A \arrow[r, "f"] & C \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "g"] \arrow[d, "s"'] & D \arrow[d, dashed, "\hat{s}_2"] \\ C \arrow[r, dashed] & X' \end{tikzcd}$$

such that $\hat{s}_1$ and $\hat{s}_2$ are in S.

(MS3) Let $f, g : A \to B$ be morphisms in $\mathscr{C}$. Then there exist some $s_1 : A' \to A \in S$ such that $fs_1 = gs_1$ if and only if there exist some $s_2 : B \to B' \in S$ such that $s_2 f = s_2 g$.

With this definition we can define an explicit construction of the localisation of $\mathscr{C}$ with respect to S.

Construction 3.1.3. Let $\mathscr{C}$ be a category, and S multiplicative system. Then we get the following description of the category $S^{-1}\mathscr{C}$.

1. $\mathbf{Ob}S^{-1}\mathscr{C} = \mathbf{Ob}\mathscr{C}$.

2. Morphisms in $S^{-1}\mathscr{C}$ is given by pairs (f, s) $\;X \xrightarrow{f} Y' \xleftarrow{s} Y\;$ with $s \in S$, up to an equivalence (given below).

3. Two morphisms (f_1, s_1) and (f_2, s_2) are equivalent if there exists a third morphism $(f_3, s_3) \in S^{-1}\mathscr{C}$ and morphisms $u, v \in \mathscr{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
 & Y_1 \arrow[d, "v"'] & \\
X \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & Y_3 & Y \arrow[ul, "s_1"'] \arrow[l, "s_3"'] \arrow[dl, "s_2"] \\
 & Y_2 \arrow[u, "u"] &
\end{tikzcd}$$

4. Composition of two equivalence classes (f_1, s_1) and (f_2, s_2) is given by (uf_1, vs_2). u and v are obtained by the commutative diagram below, which exist by (MS2).

$$\begin{tikzcd}
 & & Z' & & \\
 & X' \arrow[ur, dashed, "u"] & & Y' \arrow[ul, dashed, "v"'] & \\
X \arrow[ur, "f_1"] & & Y \arrow[ul, "s_1"'] \arrow[ur, "f_2"] & & Z \arrow[ul, "s_2"']
\end{tikzcd}$$

5. The identity of an object X in $S^{-1}\mathscr{C}$ is $(\mathrm{Id}_X, \mathrm{Id}_X)$.

The following two results will justify our construction.

Proposition 3.1.4. Given the setup in 3.1.3 we have the following

1. The relation defined is an equivalence relation.

2. The composition rule given is well defined on equivalence classes.

3. Composition is associative.

4. The identity morphisms in $S^{-1}\mathscr{C}$ satisfy the identity axioms for a category.

Proving 1-4 shows $S^{-1}\mathscr{C}$ is a category.

Proof. 1. The relation is reflexive as (f_1, s_1) is equivalent to (f_1, s_1) by letting the third morphism be (f_1, s_1) and the morphisms u and v be identity. The relation is symmetric as the diagram will still commute if we "flip" it. To show transitivity we let $(f_1, s_1) \sim (f_2, s_2)$ and $(f_2, s_2) \sim (f_3, s_3)$. We then have the following commutative diagram

$$\begin{tikzcd}
& C_1 \arrow[d, "u'"] & \\
& C' & \\
A \arrow[uur, "f_1"] \arrow[ur, "f'"] \arrow[r, "f_2"] \arrow[dr, "f''"] \arrow[ddr, "f_3"'] & C_2 \arrow[u, "v'"'] \arrow[d, "u''"] & B \arrow[uul, "s_1"'] \arrow[ul, "s'"'] \arrow[l, "s_2"'] \arrow[dl, "s''"] \arrow[ddl, "s_3"] \\
& C'' & \\
& C_3 \arrow[u, "v''"'] &
\end{tikzcd}$$

where $s_1, s_2, s_3, s', s'' \in S$. By (MS2) we find the morphisms $\hat{s}_1, \hat{s}_2$ by the following commutative square.

$$\begin{tikzcd}
B \arrow[r, "s''"] \arrow[d, "s'"'] & C'' \arrow[d, dashed, "\hat{s}_2"] \\
C' \arrow[r, dashed, "\hat{s}_1"] & D
\end{tikzcd}$$

Where $\hat{s}_2 \in S$. By (MS 3) we get the following right diagram with $\hat{s}_3 \in S$ by the following left diagram.

$$\begin{tikzcd}
B \arrow[r, "s_2"] & C_2 \arrow[r, "u''"] \arrow[d, "v'"] & C'' \arrow[d, "\hat{s}_2"] \\
& C' \arrow[r, "\hat{s}_1"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
C_2 \arrow[r, "u''"] \arrow[d, "v'"] & C'' \arrow[d, "\hat{s}_2"] & \\
C' \arrow[r, "\hat{s}_1"] & D \arrow[r, dashed, "\hat{s}_3"] & E
\end{tikzcd}$$

Now we have the diagram

$$
\begin{tikzcd}
 & C_1 \arrow[d, "\hat{s}_3\hat{s}_1u'"] & \\
A \arrow[ur, bend left, "f_1"] \arrow[r, "\hat{s}_3\hat{s}_2f''"] \arrow[dr, bend right, "f_3"'] & E & B \arrow[ul, bend right, "s_1"'] \arrow[l, "\hat{s}_3\hat{s}_2s''"'] \arrow[dl, bend left, "s_3"] \\
 & C_3 \arrow[u, "\hat{s}_3\hat{s}_2v''"'] &
\end{tikzcd}
$$

$\hat{s}_3\hat{s}_2s''$ is in S as it is a composition of maps in S. We check that triangle 1,2,3 and 4 commutes. All the relations are in the commutative diagrams above.

1. $\hat{s}_3\hat{s}_1u'f_1 = \hat{s}_3\hat{s}_1f' = \hat{s}_3\hat{s}_1v'f_2 = \hat{s}_3\hat{s}_2u''f_2 = \hat{s}_3\hat{s}_2f''$

2. $\hat{s}_3\hat{s}_1u's_1 = \hat{s}_3\hat{s}_1s' = \hat{s}_3\hat{s}_1v's_2 = \hat{s}_3\hat{s}_2u''s_2 = \hat{s}_3\hat{s}_2s''$

3. $\hat{s}_3\hat{s}_2v''f_3 = \hat{s}_3\hat{s}_2f''$

4. $\hat{s}_3\hat{s}_2v''s_3 = \hat{s}_3\hat{s}_2s''$

2. Let (f_1, s_1) (f_2, s_2) be composable morphisms. Suppose we choose two different representations of the composition as given in commutative diagram below.

$$
\begin{tikzcd}
 & & D_1 & & \\
 & & D_2 & & \\
 & A' \arrow[uur, dashed, "u_1"] \arrow[ur, dashed, "u_2"'] & & B' \arrow[uul, dashed, "v_1"'] \arrow[ul, dashed, "v_2"] & \\
A \arrow[ur, "f_1"] & & B \arrow[ul, "s_1"'] \arrow[ur, "f_2"] & & C \arrow[ul, "s_2"']
\end{tikzcd}
$$

Note $s_1, s_2, v_1, v_2 \in S$. We need to show $(u_1f_1, v_1s_2) \sim (u_2f_1, v_2s_2)$. By (MS 2) we get the following commutative diagram with $\hat{s}_1 \in S$.

$$
\begin{tikzcd}
B' \arrow[r, "v_1"] \arrow[d, "v_2"'] & D_1 \arrow[d, dashed, "\hat{s}_1"] \\
D_2 \arrow[r, dashed, "\hat{s}_2"] & X
\end{tikzcd}
$$

Consider the compositions $\begin{tikzcd} B \arrow[r, "s_1"] & A' \arrow[r, bend left, "\hat{s}_1 u_1"] \arrow[r, bend right, "\hat{s}_2 u_2"'] & X \end{tikzcd}$. By commutativity in the two diagrams above we get $\hat{s}_1 u_1 s_1 = \hat{s}_1 v_1 f_2 = \hat{s}_2 v_2 f_2 = \hat{s}_2 u_2 s_1$. Hence by (MS3) we have $\begin{tikzcd} A' \arrow[r, bend left, "\hat{s}_1 u_1"] \arrow[r, bend right, "\hat{s}_2 u_2"'] & X \arrow[r, dashed, "\hat{s}_3"] & Y \end{tikzcd}$ where $\hat{s}_3 \in S$ and $\hat{s}_3 \hat{s}_1 u_1 = \hat{s}_3 \hat{s}_2 u_2$. Now we have the following commutative diagram.

$$\begin{tikzcd}
 & D_1 \arrow[d, "\hat{s}_3\hat{s}_1"] & \\
A \arrow[ur, bend left, "u_1 f_1"] \arrow[r, "\hat{s}_3\hat{s}_1 u_1 f_1", "=\hat{s}_3\hat{s}_2 u_2 f_1"'] \arrow[dr, bend right, "u_2 f_1"'] & Y & C \arrow[ul, bend right, "v_1 s_2"'] \arrow[l, "\hat{s}_3\hat{s}_1 v_1 s_2"', "=\hat{s}_3\hat{s}_2 v_2 s_2"] \arrow[dl, bend left, "v_2 s_2"] \\
 & D_2 \arrow[u, "\hat{s}_3\hat{s}_2"'] &
\end{tikzcd}$$

Where $\hat{s}_3\hat{s}_1 v_1 s_2 = \hat{s}_3\hat{s}_2 v_2 s_2 \in S$ by (MS1). Hence $(u_1 f_1, v_1 s_2) \sim (u_2 f_1, v_2 s_2)$.

3. Let (f_1, s_1), (f_2, s_2) and (f_3, s_3) be composable morphisms. Consider the following commutative diagram, where the dotted arrows arise from (MS2).

$$\begin{tikzcd}
 & & & D \arrow[d, "s_1"] \\
 & & C \arrow[r, "f_1"] \arrow[d, "s_2"] & C' \arrow[d, dashed, "s_5"] \\
 & B \arrow[r, "f_2"] \arrow[d, "s_3"] & B' \arrow[r, dashed, "f_4"] \arrow[d, dashed, "s_4"] & B'' \arrow[d, dashed, "s_6"] \\
A \arrow[r, "f_3"] & A' \arrow[r, dashed, "f_5"] & A'' \arrow[r, dashed, "f_6"] & C''
\end{tikzcd}$$

where $s_i \in S$. We see that it does not matter in which order we complete the diagram, and we get. $(f_1, s_1)((f_2, s_2)(f_3, s_3)) = ((f_1, s_1)(f_2, s_2))(f_3, s_3) = (f_6 f_5 f_3, s_6 s_5 s_1)$.

4. Given (f_1, s_1) and (f_2, s_2) composable with $(\mathrm{Id}_B, \mathrm{Id}_B)$ on the left and right respectively. The statement is verified by the following diagrams.

$$\begin{tikzcd}
 & & A & & \\
 & A \arrow[ur, dashed, "\mathrm{Id}"] & & B \arrow[ul, dashed, "s_1"'] & \\
A \arrow[ur, "f_1"] & & B \arrow[ul, "s_1"'] \arrow[ur, "\mathrm{Id}"] & & B \arrow[ul, "\mathrm{Id}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & C & & \\
 & B \arrow[ur, dashed, "f_2"] & & C \arrow[ul, dashed, "\mathrm{Id}"'] & \\
B \arrow[ur, "\mathrm{Id}"] & & B \arrow[ul, "\mathrm{Id}"'] \arrow[ur, "f_2"] & & C \arrow[ul, "s_2"']
\end{tikzcd}$$

∎

Proposition 3.1.5. Let $\mathscr{C}$ be a category, and S a multiplicative system. Then the following hold:

1. $Q : \mathscr{C} \to S^{-1}\mathscr{C}$, given by $A \mapsto A$ on objects, and $f \mapsto (f, \mathrm{Id})$ on morphisms is a functor.

2. For all $s \in S$ we have Qs is an isomorphism.

3. If $F : \mathscr{C} \to \mathscr{D}$ is another functor such that Fs is an isomorphism for all $s \in S$. Then there exist an unique functor $G : S^{-1}\mathscr{C} \to \mathscr{D}$ such that $GQ = F$.

Proof. 1. Let $A \in Ob\mathscr{C}$. Then $Q\,\mathrm{Id}_A = (\mathrm{Id}_A, \mathrm{Id}_A) = \mathrm{Id}_{QA}$. Let f and g be composable in $\mathscr{C}$. Then $Q(gf) = (gf, \mathrm{Id}) = (g, \mathrm{Id})(f, \mathrm{Id}) = QgQf$. Hence Q is a functor.

2. Let $s : A \to B \in S$ then $Qs = (s, \mathrm{Id}_B)$. We claim the inverse is (Id_B, s). In the left diagram we see $(s, \mathrm{Id}_B)(\mathrm{Id}_B, s)$ in the right we see it is in the same equivalence class as Id_A.

$$\begin{tikzcd}
 & & B & & \\
 & B \arrow[ur, "\mathrm{Id}"] & & B \arrow[ul, "\mathrm{Id}"'] & \\
A \arrow[ur, "s"] & & B \arrow[ul, "\mathrm{Id}"'] \arrow[ur, "\mathrm{Id}"] & & A \arrow[ul, "s"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B \arrow[d, "\mathrm{Id}"'] & \\
A \arrow[ur, "s"] \arrow[r, "s"] \arrow[dr, "\mathrm{Id}"'] & B & A \arrow[ul, "s"'] \arrow[l, "s"'] \arrow[dl, "\mathrm{Id}"] \\
 & A \arrow[u, "s"] &
\end{tikzcd}$$

By drawing diagram for $(\mathrm{Id}_B, s)(s, \mathrm{Id}_B)$ we immediately get it is equal to Id_B.

3. Let $F : \mathscr{C} \to \mathscr{D}$ we need to show there is a unique $G : S^{-1}\mathscr{C} \to D$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, "Q"] \arrow[dr, "F"'] & & S^{-1}\mathscr{C} \arrow[dl, dashed, "G"] \\
 & \mathscr{D} &
\end{tikzcd}$$

We define $G(A) = F(A)$ for objects, which is the only choice. For morphism we also only get one choice as we need composition to be well defined. $G(f, s) = G(s)^{-1}G(f)$.

■

Remark 3.1.6. 3.1.4 and 3.1.5 shows that construction 3.1.3 is a localisation.

3.2 Defining Verdier localization

In this subsection we define the Verdier localization. In order to do this we will introduce a class of morphisms in a triangulated category. Then we will show this is a multiplicative system compatible with triangulation. This then gives us the specific localization.

Definition 3.2.1. Let $\mathcal{T}$ be a triangulated category and S a multiplicative system. Then S is *compatible with triangulation* if the following hold.

(MS4) Given $s \in S$, the map $s[n]$ belongs to S for all $n \in \mathbb{Z}$.

(MS5) Given a map (f_1, f_2, f_3) between two triangles where $f_1, f_2 \in S$ there exists a map (f_1, f_2, f_3') where $f_3' \in S$.

Definition 3.2.2. Let $\mathcal{T}$ be a triangulated category. Let $\mathscr{S}$ be a triangulated subcategory as defined in B.10. We denote by $S(\mathscr{S})$ the class of morphisms in $\mathcal{T}$ such that the cone of f belongs to $\mathscr{S}$.

Proposition 3.2.3. Let $\mathcal{T}$ be a triangulated category, and $\mathscr{S}$ a triangulated subcategory. Then $S(\mathscr{S})$ is a multiplicative system which is compatible with triangulation.

Proof. We check the axioms one by one.

(MS1) Let $f, g \in S(\mathscr{S})$ Consider the following diagram which arise from (TR4).

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r] \arrow[d, "g"] & \mathrm{Cone}(f) \arrow[r] \arrow[d, dashed] & A[1] \arrow[d, equal] \\
A \arrow[r, "gf"] & C \arrow[r] \arrow[d] & \mathrm{Cone}(gf) \arrow[r] \arrow[d, dashed] & A[1] \\
& \mathrm{Cone}(g) \arrow[r, equal] \arrow[d] & \mathrm{Cone}(g) \arrow[d] & \\
& B[1] \arrow[r] & \mathrm{Cone}(f)[1] &
\end{tikzcd}$$

The second column is an exact triangle by (TR4). We see $\mathrm{Cone}(f), \mathrm{Cone}(g) \in \mathscr{S}$ as $f, g \in S(\mathscr{S})$. By (TS2) $\mathrm{Cone}(gf) \in \mathscr{S}$. Hence $gf \in S(\mathscr{S})$. For the identity we know by (TR1)+(TR2) $S \to S \to 0 \to S[1]$ is an exact triangle for $S \in \mathscr{S}$. By (TS2) we get that $0 \in \mathscr{S}$. Hence $\mathrm{Cone}(\mathrm{Id}_A) = 0 \in \mathscr{S}$ for all $A \in \mathcal{T}$. Therefore $\mathrm{Id}_A \in S(\mathscr{S})$ for all $A \in \mathcal{T}$.

(MS2) Let f be in $\mathcal{T}$ and s in $S(\mathscr{S})$. We need to find the dashed arrows f' and s' in the following diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, dashed, "s'"] \\ C \arrow[r, dashed, "f'"] & X \end{tikzcd}$$

where $s' \in S(\mathscr{S})$. We complete s to an exact triangle, and then shift it to get the following triangle.

$$\mathrm{Cone}(s)[-1] \xrightarrow{g} A \xrightarrow{s} C \longrightarrow \mathrm{Cone}(s)$$

Now we can construct the following diagram.

$$\begin{tikzcd} \mathrm{Cone}(s)[-1] \arrow[r, equal] \arrow[d, "g"'] & \mathrm{Cone}(s)[-1] \arrow[d, "fg"] \\ A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, "s'"] \\ C \arrow[r, dashed, "f'"] \arrow[d] & \mathrm{Cone}(fg) \arrow[d] \\ \mathrm{Cone}(s) \arrow[r, equal] & \mathrm{Cone}(s) \end{tikzcd}$$

By (TR3) we get the existence of f'. By (TR2) we can shift the triangle in the right column to get

$$B \xrightarrow{s'} \mathrm{Cone}(fg) \longrightarrow \mathrm{Cone}(s) \longrightarrow B[1]$$

We see that $\mathrm{Cone}(s') = \mathrm{Cone}(s) \in \mathscr{S}$. Hence $s' \in S(\mathscr{S})$. The other half of (MS2) is dual.

(MS3) Let $s \in S(\mathscr{S})$ and $f, g \in \mathcal{T}$ such that $fs = gs$. We need s' in $S(\mathscr{S})$ such that $s'f = s'g$. We can complete the following diagram with the dashed arrow t

$$\begin{tikzcd} A' \arrow[r, "s"] & A \arrow[r, "h"] \arrow[d, "(f-g)"] & \mathrm{Cone}(s) \arrow[r] \arrow[dl, dashed, bend left, "t"] & A'[1] \\ & B & & \end{tikzcd}$$

since $(f-g)s = 0$ and h is a weak cokernel of s by Corollary B.5. Now we can complete an exact triangle for t.

$$\mathrm{Cone}(s) \xrightarrow{t} B \xrightarrow{s'} B' \longrightarrow \mathrm{Cone}(s)[1]$$

Where $s'(f-g) = s'th = 0$ as s' is a weak cokernel of t by Corollary B.5. Hence $s'f = s'g$. By shifting the triangle (MS2) we get

$$B \xrightarrow{s'} B' \longrightarrow \mathrm{Cone}(s)[1] \longrightarrow B[1]$$

As $\mathrm{Cone}(s) \in \mathscr{S}$ we get $\mathrm{Cone}(s)[1] \in \mathscr{S}$ by (TS1). Hence $s' \in S(\mathscr{S})$. The other half of (MS3) is dual.

(MS4) Let $s : A \to B \in S(\mathscr{S})$ then we can complete s to an exact triangle and shift three times to get the exact triangle

$$A[1] \xrightarrow{s[1]} B[1] \longrightarrow \mathrm{Cone}(s)[1] \longrightarrow A[2]$$

By (TS1) $\mathrm{Cone}(s)[1] \in \mathscr{S}$ and so $s[1] \in S(\mathscr{S})$.

(MS5) Let (f_1, f_2, f_3) be a map between two triangles

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f_1"] & B \arrow[r] \arrow[d, "f_2"] & C \arrow[r] \arrow[d, "f_3"] & A[1] \arrow[d, "f_1[1]"] \\
A' \arrow[r] & B' \arrow[r] & B' \arrow[r] & A'[1]
\end{tikzcd}$$

where $f_1, f_2 \in S(\mathscr{S})$. Consider the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, "f_2"] & C \arrow[r] \arrow[d, dashed, "s_1"] & A[1] \arrow[d, equal] \\
A \arrow[r, "f_2 g"] \arrow[d, "f_1"] & B' \arrow[r] \arrow[d, equal] & \mathrm{Cone}(f_2 g) \arrow[r] \arrow[d, dashed, "s_2"] & A[1] \arrow[d, "f_1[1]"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd} \qquad (*)$$

We claim there exist s_1 and s_2 in $S(\mathscr{S})$ that are compatible with the diagram above. For s_1 consider the following diagram completed by (TR4).

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, equal] & B \arrow[r] \arrow[d, "f_2"] & C \arrow[r] \arrow[d, dashed, "s_1"] & A[1] \arrow[d, equal] \\
A \arrow[r, "f_2 g"] & B' \arrow[r] \arrow[d] & \mathrm{Cone}(f_2 g) \arrow[r] \arrow[d, dashed] & A[1] \\
 & \mathrm{Cone}(f_2) \arrow[r, equal] \arrow[d] & \mathrm{Cone}(s_1) \arrow[d, dashed] & \\
 & B[1] \arrow[r] & C[1] &
\end{tikzcd}$$

Since $\mathrm{Cone}(f_2) = \mathrm{Cone}(s_1) \in \mathscr{S}$ we get $s_1 \in S(\mathscr{S})$. To find s_2 consider the following completion by (TR4).

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & A' \arrow[r] \arrow[d] & \mathrm{Cone}(f) \arrow[r] \arrow[d, dashed] & A[1] \arrow[d, equal] \\
A \arrow[r, "f_2 g"] & B' \arrow[r] \arrow[d] & \mathrm{Cone}(f_2 g) \arrow[r] \arrow[d, dashed, "s_2"] & A[1] \arrow[d] \\
 & C' \arrow[r, equal] \arrow[d] & C' \arrow[r] \arrow[d] & A'[1] \\
 & A'[1] \arrow[r] & \mathrm{Cone}(f_1)[1] &
\end{tikzcd}$$

We see that s_2 fits in our diagram $(*)$. We can shift the triangle in the second column (TR2) to get.

$$\mathrm{Cone}(f_2 g) \xrightarrow{s_2} C' \longrightarrow \mathrm{Cone}(f_1)[1] \longrightarrow \mathrm{Cone}(f_2 g)[1]$$

Hence $\mathrm{Cone}(s_2) = \mathrm{Cone}(f_1)[1]$ which is in $\mathscr{S}$ by assumption and (TS1). Hence $s_2 \in S(\mathscr{S})$. Now we have the morphism $(f_1, f_2, s_2 s_1)$ between the original triangles as needed.

$\blacksquare$

Definition 3.2.4. Let $\mathcal{T}$ be a triangulated category. Let $\mathscr{S}$ be a triangulated subcategory. The *Verdier localization* of $\mathcal{T}$ with respect to $\mathscr{S}$ is defined by

$$\mathcal{T}/\mathscr{S} = (S(\mathscr{S}))^{-1}\mathcal{T}$$

Together with the functor $Q : \mathcal{T} \to \mathcal{T}/\mathscr{S}$ given in Proposition 3.1.5.

3.3 Properties of Verdier localization

In this subsection we go through results regarding localization and particularly Verdier localization. When we later define the derived category through Verdier localization we know these result will hold.

Lemma 3.3.1. Let $\mathscr{C}$ be a category and let S be a multiplicative system on $\mathscr{C}$. Let $(f,s),(g,s) : A \to B$ be morphisms in $S^{-1}\mathscr{C}$. Then $(f,s) \cong (g,s)$ if and only if there exists a morphism $a : B' \to B''$ in $\mathscr{C}$ such that $as \in S$ and $af = ag$.

Proof. Since $(f,s) \cong (g,s)$ we have the commutative diagram

$$\begin{tikzcd} & B' \arrow[d, "u"] & \\ A \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & C & B \arrow[ul, "s"'] \arrow[l, "t"'] \arrow[dl, "s"] \\ & B' \arrow[u, "v"'] & \end{tikzcd}$$

with $t \in S$. In particular $us = t = vs$. Hence by (MS3) there exists $s' : C \to B''$ in S such that $s'u = s'v$. Let $a = s'u = s'v$ then we have $as = s'us = s't$ wich is in S by (MS1). Furthermore we have $af = s'uf = s'h = s'vg = ag$ which concludes the proof. ∎

Lemma 3.3.2. Let $\mathscr{C}$ be a category, S a multiplicative system. If $f : A \to B$ and $f' : A' \to B'$ are two morphisms in $\mathscr{C}$ and the following diagram commutes in $S^{-1}\mathscr{C}$

$$\begin{tikzcd} Q(A) \arrow[r, "a"] \arrow[d, "Q(f)"'] & Q(A') \arrow[d, "Q(f')"] \\ Q(B) \arrow[r, "b"] & Q(B') \end{tikzcd}$$

where Q is the localization functor. Then there exists a morphism f'' such that the following diagram commutes in $\mathscr{C}$

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"] & A'' \arrow[d, "f''"] & A' \arrow[l, "s"'] \arrow[d, "f'"] \\ B \arrow[r, "i"'] & B'' & B' \arrow[l, "q"] \end{tikzcd}$$

where $s, q \in S$, $a = (g,s)$ and $b \cong (i,q)$.

Proof. Let $a = (g, s)$ where $g : A \to A''$ and $s : A' \to A''$. We use (MS2) to obtain the following diagram with a commutative square and $t \in S$.

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"] & A'' \arrow[d, dashed, "\hat{f}"] & A' \arrow[l, "s"'] \arrow[d, "f'"] \\
B & \widehat{B} & B' \arrow[l, dashed, "t"]
\end{tikzcd}$$

Let $b = (i, q)$ where $i : B \to C$ and $q : B' \to C$ then use (MS2) to get the commutative square

$$\begin{tikzcd}
B' \arrow[r, "q"] \arrow[d, "t"'] & C \arrow[d, dashed, "j"] \\
\widehat{B} \arrow[r, dashed, "r"'] & \widehat{B}'
\end{tikzcd}$$

with $r \in S$. By the following commutative diagram

$$\begin{tikzcd}
 & C \arrow[d, "j"] & \\
B \arrow[ur, "i"] \arrow[r, "ji"] \arrow[dr, "ji"'] & \widehat{B}' & B' \arrow[ul, "q"'] \arrow[l, "rt"'] \arrow[dl, "rt"] \\
 & \widehat{B}' \arrow[u, "\mathrm{Id}"'] &
\end{tikzcd}$$

we get $(ji, rt) \cong (i, q)$. Now we have the following diagram where the right square commutes.

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & A'' \arrow[d, "r\hat{f}"] & A' \arrow[l, "s"'] \arrow[d, "f'"] \\
B \arrow[r, "ji"'] & \widehat{B}' & B' \arrow[l, "rt"]
\end{tikzcd}$$

We calculate that $bQ(f) = (ji, rt)(f, \mathrm{Id}) = (jif, rt)$ and $Q(f')a = (f', \mathrm{Id})(g, s) = (r\hat{f}g, rt)$. By assumption we know $bQ(f) = Q(f')a$. Thus we have $(r\hat{f}g, rt) = (jif, rt)$. We apply Lemma 3.3.1 to get a d such that drt is in S and $d(jif) = d(r\hat{f}g)$. We note that $(ji, rt) \cong (dji, drt)$ by the following diagram.

$$\begin{tikzcd}
 & \widehat{B}' \arrow[d, "d"] & \\
B \arrow[ur, "ji"] \arrow[r, "dji"] \arrow[dr, "dji"'] & B'' & B' \arrow[ul, "rt"'] \arrow[l, "drt"'] \arrow[dl, "drt"] \\
 & B'' \arrow[u, "\mathrm{Id}"'] &
\end{tikzcd}$$

Now we have the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"] & A'' \arrow[d, "dr\hat{f}"] & A' \arrow[l, "s"'] \arrow[d, "f'"] \\
B \arrow[r, "dji"'] & B'' & B \arrow[l, "dr\hat{t}"]
\end{tikzcd}$$

and we are done with $(dji, drt) \cong (i, q) = b$ and $f'' = dr\hat{f}$.

■

Theorem 3.3.3. Let $\mathcal{T}$ be a triangulated category, and S a multiplicative system compatible with triangulation. Then the localization $S^{-1}\mathcal{T}$ inherits a unique triangulated structure such that the localization functor $Q : \mathcal{T} \to S^{-1}\mathcal{T}$ is exact.

Proof. The equivalence $[1] : \mathcal{T} \to \mathcal{T}$ induces a equivalence $S^{-1}[1] : S^{-1}\mathcal{T} \to S^{-1}\mathcal{T}$ given by $(f, s) \to (f[1], s[1])$. By (MS4) $s[1] \in S$. This clearly commutes with the localization functor $Q : \mathcal{T} \to S^{-1}\mathcal{T}$. Now let the exact triangles in $S^{-1}\mathcal{T}$ be all triangles isomorphic to images of exact triangles in $\mathcal{T}$. This gives us that Q is exact by construction, and all that is left is to verify (TR1)-(TR4).

(TR1) : (a) The class of exact triangles is closed under isomorphism by construction. (b) let $A \in S^{-1}\mathcal{T}$ then $0 \to A \xrightarrow{(\mathrm{Id}_A, \mathrm{Id}_A)} A \to 0$ is the image of $0 \to A \xrightarrow{\mathrm{Id}_A} A \to 0$ and hence an exact triangle. (c) Let $(f, s) \in S^{-1}\mathcal{T}$. Consider a completion of f into a triangle $A \xrightarrow{f} B' \xrightarrow{g} C \xrightarrow{h} A[1]$ in $\mathcal{T}$. This has the image $A \xrightarrow{(f, \mathrm{Id}_{B'})} B' \xrightarrow{(g, \mathrm{Id}_C)} C \xrightarrow{(f, \mathrm{Id}_{A[1]})} A[1]$ in $S^{-1}\mathcal{T}$ which gives rise to the isomorphism

$$\begin{tikzcd}
A \arrow[r, "{(f,\mathrm{Id}_B)}"] \arrow[d, equal] & B' \arrow[r, "{(g,\mathrm{Id}_C)}"] & C \arrow[r, "{(g,\mathrm{Id}_{A[1]})}"] \arrow[d, equal] & A[1] \arrow[d, equal] \\
A \arrow[r, "{(f,s)}"'] & B \arrow[u, "{(s,\mathrm{Id}_B)}"] \arrow[r, "{(gs,\mathrm{Id}_C)}"'] & C \arrow[r, "{(h,\mathrm{Id}_{A[1]})}"'] & A[1]
\end{tikzcd}$$

Hence we found an completion of (f, s) into an exact triangle.

(TR2) : Given an triangle $A \xrightarrow{(f,s)} B \xrightarrow{(g,t)} C \xrightarrow{(h,u)} A[1]$ in $S^{-1}\mathcal{T}$ we know it is isomorphic to the image of an exact triangle, say (a, b, c) in $\mathcal{T}$. $((g, t), (h, u), -(f[1], s[1]))$ and $((-(h[-1], u[-1]), (f, s), (g, t))$ will then be isomorphic to the image of (a, b, c) shifted.
(TR3) : First we want to reduce the problem to only looking at images of exact triangles in $\mathcal{T}$. Suppose (TR3) hold for images of exact triangles in $\mathcal{T}$. Given the solid part of

the following diagram

$$
\begin{tikzcd}
A \arrow[r, "{(a,s)}"] \arrow[d, "{(f_1,w_1)}"] & B \arrow[r, "{(b,t)}"] \arrow[d, "{(f_2,w_2)}"] & C \arrow[r, "{(c,u)}"] \arrow[d, dashed, "x"] & A[1] \arrow[d] \\
A' \arrow[r, "{(a',s')}"'] & B' \arrow[r, "{(b',t')}"'] & C' \arrow[r, "{(b',u')}"'] & A'[1]
\end{tikzcd}
$$

we need to find x. We find triangles isomorphic to these which are images of exact triangles in $\mathcal{T}$.

$$
\begin{tikzcd}
\tilde{A} \arrow[r, "{(\tilde{a},\mathrm{Id})}"] \arrow[d, "i_1", "\cong"'] & \tilde{B} \arrow[r, "{(\tilde{b},\mathrm{Id})}"] \arrow[d, "i_2", "\cong"'] & \tilde{C} \arrow[r, "{(\tilde{c},\mathrm{Id})}"] \arrow[d, "i_3", "\cong"'] \arrow[ddd, bend left, "{(\tilde{f},\tilde{s})}"] & \tilde{A}[1] \arrow[d, "\cong"] \\
A \arrow[r, "{(a,s)}"] \arrow[d, "{(f_1,w_1)}"] & B \arrow[r, "{(b,t)}"] \arrow[d, "{(f_2,w_2)}"] & C \arrow[r, "{(c,u)}"] \arrow[d, dashed, "x"] & A[1] \arrow[d] \\
A' \arrow[r, "{(a',s')}"'] \arrow[d, "j_1", "\cong"'] & B' \arrow[r, "{(b',t')}"'] \arrow[d, "j_2", "\cong"'] & C' \arrow[r, "{(b',u')}"'] \arrow[d, "j_3", "\cong"'] & A'[1] \arrow[d, "\cong"] \\
\tilde{A}' \arrow[r, "{(\tilde{a}',\mathrm{Id})}"] & \tilde{B}' \arrow[r, "{(\tilde{b}',\mathrm{Id})}"] & \tilde{C}' \arrow[r, "{(\tilde{c}',\mathrm{Id})}"] & \tilde{A}'[1]
\end{tikzcd}
$$

where we get $(\tilde{f},\tilde{s})$ by (TR3) for images. Now let $x = j_3^{-1}(\tilde{f},\tilde{s})i_3^{-1}$. Then we get

$$
\begin{aligned}
x(b,t) &= j_3^{-1}(\tilde{f},\tilde{s})i_3^{-1}(b,t) \\
&= j_3^{-1}(\tilde{f},\tilde{s})i_3^{-1}i_3(\tilde{b},\mathrm{Id})i_2^{-1} \\
&= j_3^{-1}(\tilde{f},\tilde{s})(\tilde{b},\mathrm{Id})i_2^{-1} \\
&= j_3^{-1}(\tilde{b}',\mathrm{Id})j_2(f_2,w_2)i_2i_2^{-1} \\
&= j_3^{-1}(\tilde{b}',\mathrm{Id})j_2(f_2,w_2) \\
&= (b',t')j_2^{-1}j_2(f_2,w_2) = (b',t')(f_2,w_2)
\end{aligned}
$$

similarly $(b',u')x = (f_1[1],w_1[1])(c,u)$ Hence we can reduce the problem to images of exact triangles. Let $A \xrightarrow{(a,\mathrm{Id})} B \xrightarrow{(b,\mathrm{Id})} C \xrightarrow{(c,\mathrm{Id})} A[1]$ and $A' \xrightarrow{(a',\mathrm{Id})} B' \xrightarrow{(b',\mathrm{Id})} C' \xrightarrow{(c',\mathrm{Id})} A'[1]$ be images of exact triangles in $\mathcal{T}$ such that we have the solid part of the diagram.

$$
\begin{tikzcd}
A \arrow[r, "{(a,\mathrm{Id})}"] \arrow[d, "{(f_1,s_1)}"'] & B \arrow[r, "{(b,\mathrm{Id})}"] \arrow[d, "{(f_2,s_2)}"] & C \arrow[r, "{(c,\mathrm{Id})}"] \arrow[d, dashed] & A[1] \arrow[d, "{(f_1[1],s_1[1])}"] \\
A' \arrow[r, "{(a',\mathrm{Id})}"'] & B' \arrow[r, "{(b',\mathrm{Id})}"'] & C' \arrow[r, "{(c',\mathrm{Id})}"'] & A'[1]
\end{tikzcd}
$$

By Lemma 3.3.2 we find a'' such that the following diagram commutes in $\mathcal{T}$.

$$
\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "a"'] & A'' \arrow[d, "a''"] & A' \arrow[l, "s_1"'] \arrow[d, "a'"] \\
B \arrow[r, "f_2"'] & B'' & B' \arrow[l, "s_2"]
\end{tikzcd}
$$

By (TR3) in $\mathcal{T}$ and (MS5) we get the following diagram with $s_3 \in S$.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "f_1"] & B \arrow[r, "b"] \arrow[d, "f_2"] & C \arrow[r, "c"] \arrow[d, "f_3"] & A'[1] \arrow[d, "f_1[1]"] \\
A'' \arrow[r, "a''"] & B'' \arrow[r, "b''"] & C'' \arrow[r, "c''"] & A''[1] \\
A' \arrow[r, "a'"] \arrow[u, "s_1"] & B' \arrow[r, "b'"] \arrow[u, "s_2"'] & C' \arrow[r, "c'"] \arrow[u, "s_3"'] & A'[1] \arrow[u, "s_1[1]"]
\end{tikzcd}$$

Now it is straightforward to verify that the following will be a morphism of triangles.

$$\begin{tikzcd}
A \arrow[r, "(a,\mathrm{Id})"] \arrow[d, "(f_1,s_1)"'] & B \arrow[r, "(b,\mathrm{Id})"] \arrow[d, "(f_2,s_2)"] & C \arrow[r, "(c,\mathrm{Id})"] \arrow[d, "(f_3,s_3)"] & A[1] \arrow[d, "(f_1[1],s_1[1])"] \\
A' \arrow[r, "(a',\mathrm{Id})"'] & B' \arrow[r, "(b',\mathrm{Id})"'] & C' \arrow[r, "(c',\mathrm{Id})"'] & A'[1]
\end{tikzcd}$$

(TR4) : First we want to reduce the problem to looking at images of exact triangles in $\mathcal{T}$. So suppose (TR4) Hold for images of exact triangles in $\mathcal{T}$. consider the following diagram, where we are given the solid part.

$$\begin{tikzcd}
A \arrow[r, "(a_1,s_1)"] \arrow[d, equal] & B \arrow[r, "(a_2,s_2)"] \arrow[d, "(f_1,u_1)"] & C \arrow[r, "(a_3,s_3)"] \arrow[d, dashed, "x"] & A[1] \arrow[d, equal] \\
A \arrow[r, "(b_1,t_1)"] & B' \arrow[r, "(b_2,t_2)"] \arrow[d, "(g_1,v_1)"] & C' \arrow[r, "(b_3,t_3)"] \arrow[d, dashed, "y"] & A[1] \arrow[d, "(a_1[1],s_1[1])"] \\
& A' \arrow[r, equal] \arrow[d, "(h_1,w_1)"] & A' \arrow[r, "(h_1,w_1)"] \arrow[d, "(h_2,w_2)"] & B[1] \\
& B[1] \arrow[r, "(a_2[1],s_1[1])"'] & C[1] &
\end{tikzcd}$$

We need to fill in the dotted morphisms x and y making the diagram commute. Using (L1), (TR3) and B.6 we can construct the following isomorphisms, where the bottom triangles will be images of triangles in $\mathcal{T}$.

$$\begin{tikzcd}
A \arrow[r, "(a_1,s_1)"] \arrow[d, equal] & B \arrow[r, "(a_2,s_2)"] \arrow[d, "\cong"', "(s_1,\mathrm{Id})"] & C \arrow[r, "(a_3,s_3)"] \arrow[d, "\cong"', "i_1"] & A[1] \arrow[d, equal] \\
A \arrow[r, "(a_1,\mathrm{Id})"] & \tilde{B} \arrow[r, "(c_1,\mathrm{Id})"] & \mathrm{Cone}(a_1) \arrow[r, "(c_2,\mathrm{Id})"] & A[1]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "(b_1,t_1)"] \arrow[d, equal] & B' \arrow[r, "(b_2,t_2)"] \arrow[d, "\cong"', "(\hat{u}_1u_1,\mathrm{Id})"] & C' \arrow[r, "(b_3,t_3)"] \arrow[d, "\cong"', "i_2"] & A[1] \arrow[d, equal] \\
A \arrow[r, "(\hat{a}_1a_1,\mathrm{Id})"] & \widehat{B} \arrow[r, "(d_1,\mathrm{Id})"] & \mathrm{Cone}(\hat{a}_1a_1) \arrow[r, "(d_2,\mathrm{Id})"] & A[1]
\end{tikzcd}$$

Where $\hat{a}_1$ and $\hat{u}_1$ comes from the composition of (a_1, s_1) and (f_1, u_1).

$$\begin{tikzcd}
 & & \widehat{B} & & \\
 & \tilde{B} \arrow[ur, "\hat{a}_1"] & & \tilde{C} \arrow[ul, "\hat{u}_1"'] & \\
A \arrow[ur, "a_1"] & & B \arrow[ul, "s_1"'] \arrow[ur, "f_1"] & & B' \arrow[ul, "u_1"']
\end{tikzcd}$$

Observe that $(\hat{a}_1, \mathrm{Id})(s_1, \mathrm{Id}) = (\hat{u}_1 f_1, \mathrm{Id}) = (\hat{a}_1 s_1, \mathrm{Id}) = (\hat{u}_1 u_1, \mathrm{Id})(f_1, u_1)$. We therefore get by (L1), (TR3) and Lemma B.6 the following isomorphism where the bottom is an image of a triangle in $\mathcal{T}$.

$$\begin{tikzcd}
B \arrow[r, "(f_1,u_1)"] \arrow[d, "(s_1,\mathrm{Id})", "\cong"'] & B' \arrow[r, "(g_1,v_1)"] \arrow[d, "(\hat{u}_1 u_1,\mathrm{Id})", "\cong"'] & A' \arrow[r, "(h_1,w_1)"] \arrow[d, "i_3", "\cong"'] & B[1] \arrow[d, "\cong"'] \\
\tilde{B} \arrow[r, "(\hat{a}_1,\mathrm{Id})"'] & \widehat{B} \arrow[r, "(e_1,\mathrm{Id})"'] & \mathrm{Cone}(\hat{a}_1) \arrow[r, "(e_2,\mathrm{Id})"'] & \tilde{B}[1]
\end{tikzcd}$$

Now by our assumption we can complete (TR4) in the following way for our images of triangles in $\mathcal{T}$.

$$\begin{tikzcd}
A \arrow[r, "(a_1,\mathrm{Id})"] \arrow[d, equal] & \tilde{B} \arrow[r, "(c_1,\mathrm{Id})"] \arrow[d, "(\hat{a}_1,\mathrm{Id})"] & \mathrm{Cone}(a_1) \arrow[r, "(c_2,\mathrm{Id})"] \arrow[d, "(\tilde{f},\tilde{s})"] & A[1] \arrow[d, equal] \\
A \arrow[r, "(\hat{a}_1 a_1,\mathrm{Id})"] & \widehat{B} \arrow[r, "(d_1,\mathrm{Id})"] \arrow[d, "(e_1,\mathrm{Id})"] & \mathrm{Cone}(\hat{a}_1 a_1) \arrow[r] \arrow[d, "(\tilde{g},\tilde{t})"] & A[1] \arrow[d] \\
 & \mathrm{Cone}(\hat{a}_1) \arrow[r, equal] \arrow[d, "(e_2,\mathrm{Id})"] & \mathrm{Cone}(\hat{a}_1) \arrow[r] \arrow[d] & \tilde{B}[1] \\
 & \tilde{B}[1] \arrow[r] & \mathrm{Cone}(a_1)[1] &
\end{tikzcd}$$

Now we let $x = i_2^- 1(\tilde{f}, \tilde{s})$ and $y = i_3^{-1}(\tilde{g}, \tilde{t}) i_2$. This will give us the morphisms we are looking for. We check that the top middle square in our original setup commutes. Checking the three other squares are similar.

$$\begin{aligned}
x(a_2, s_2) &= i_2^{-1}(\tilde{f}, \tilde{g}) i_1 (a_2, s_2) \\
&= i_2^{-1}(\tilde{f}, \tilde{g})(c_1, \mathrm{Id})(s_1, \mathrm{Id}) \\
&= i_2^{-1}(d_1, \mathrm{Id})(\hat{a}_1, \mathrm{Id})(s_1, \mathrm{Id}) \\
&= (b_2, t_2)(\hat{u}_1 u_1, \mathrm{Id})^{-1}(\hat{a}_1, \mathrm{Id})(s_1, \mathrm{Id}) \\
&= (b_2, t_2)(\hat{u}_1 u_1, \mathrm{Id})^{-1}(\hat{u}_1 u_1, \mathrm{Id})(f_1, u_1) \\
&= (b_2, t_2)(f_1, u_1)
\end{aligned}$$

It is clear that (TR4) hold for images of a triangles in $\mathcal{T}$. We simply complete (TR4) in $\mathcal{T}$ and then apply our localization functor on the whole diagram. ■

Lemma 3.3.4. Let $\mathcal{T}$ be a triangulated category. Let S be a multiplicative system compatible with triangulation. Then the following are equivalent for a map $f : A \to B$ in $\mathcal{T}$.

(1) The localization functor $Q : \mathcal{T} \to S^{-1}\mathcal{T}$ annihilates f.

(2) there exists a map $g : B \to C$ in S such that $gf = 0$.

(3) The map f factors through the cone of a map in S.

Proof. (1) $\iff$ (2) : Let $f : A \to B$ Then we get $Qf = 0 \iff (f, \mathrm{Id}_B) \cong (0, \mathrm{Id}_B) \iff$ the following commutes for some $g \in S$.

$$\begin{tikzcd}
 & B \arrow[d, "g"] & \\
A \arrow[ur, "f"] \arrow[r, "0"] \arrow[dr, "0"'] & C & B \arrow[l, "g"] \arrow[ul, "\mathrm{Id}"'] \arrow[dl, "\mathrm{Id}"] \\
 & B \arrow[u, "g"] &
\end{tikzcd}$$

(2) $\iff$ (3) : Let $f : A \to B$ and $g : B \to C \in S$ Consider the triangle $B \xrightarrow{g} C \to \mathrm{Cone}(g) \to B[1]$. We can shift this triangle and consider the following diagram.

$$\begin{tikzcd}
A \arrow[r, "\mathrm{Id}"] \arrow[d, dashed] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & A[1] \arrow[d, dashed] \\
\mathrm{Cone}(g)[-1] \arrow[r] & B \arrow[r, "g"] & C \arrow[r] & \mathrm{Cone}(g)
\end{tikzcd}$$

By (TR2) and (TR3) the dotted arrows exist such that the outer squares commute if and only if $gf = 0$ and the middle square commutes. Hence we are done.

■

Definition 3.3.5. Let $\mathcal{T}$ be a triangulated category. Let $F : \mathcal{T} \to \mathscr{C}$ be an additive functor. The *kernel of* F, denoted $\mathrm{Ker}(F)$ is the full subcategory of $\mathcal{T}$ that consist of all objects $X \in \mathcal{T}$ such that $FX = 0$.

Lemma 3.3.6. If $F : \mathcal{T} \to \mathcal{U}$ is an exact functor of triangulated categories, then $\mathrm{Ker}(F)$ is thick.

Proof. (TS1) and (TS2) is clear as the functor is exact. (TS3) comes from the fact that an exact functor in particular is additive. ■

Proposition 3.3.7. Let $\mathcal{T}$ be a triangulated category and $\mathscr{S}$ a triangulated subcategory. Then the Verdier localization $\mathcal{T}/\mathscr{S}$ from definition 3.2.4 has the following properties.

1. $\mathcal{T}/\mathscr{S}$ carries a unique triangulated structure such that $Q : \mathcal{T} \to \mathcal{T}/\mathscr{S}$ is exact.

2. The kernel $\mathrm{Ker}(Q)$ is the smallest thick subcategory containing $\mathscr{S}$.

3. Every exact functor $F : \mathcal{T} \to \mathcal{U}$ annihilating $\mathscr{S}$ factors through Q via an exact functor $G : \mathcal{T}/\mathscr{S} \to \mathcal{U}$.

Proof.

1. This is a special case of Theorem 3.3.3.

2. Ker(Q) is thick by Lemma 3.3.6. To see $\mathscr{S} \subseteq \mathrm{Ker}(Q)$ let $s : A \to B \in \mathscr{S}$. Then by (TS2) the triangle $A \xrightarrow{s} B \xrightarrow{\hat{s}} \mathrm{Cone}(s) \to A[1]$ is in $\mathscr{S}$, and $\hat{s}s = 0$. By Lemma 3.3.4 $Qs = 0$ and $s \in \mathrm{Ker}(Q)$. For minimality suppose $\mathcal{U}$ is a thick subcategory such that $\mathscr{S} \subseteq \mathcal{U} \subseteq \mathrm{Ker}(Q)$. Let $A \oplus A' \in \mathcal{U}$. Since $\mathcal{U} \subseteq \mathrm{Ker}(Q)$ and Q is additive we get $0 = F(A \oplus A') = F(A) \oplus F(A')$. Hence $F(A) = 0$ and $\mathrm{Ker}(Q) = \mathcal{U}$. Hence $\mathrm{Ker}(Q)$ is the smallest thick subcategory containing $\mathscr{S}$.

3. Let $F : \mathcal{T} \to \mathcal{U}$ be an exact functor that annihilates $\mathscr{S}$. Let $f \in S(\mathscr{S})$. Considering the following morphism of triangles that arise from (TR2)+(TR3)

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & F(A) \arrow[r, "F(f)"] \arrow[d, "\mathrm{Id}"] & F(B) \arrow[r] \arrow[d, dashed, "g"] & \mathrm{Cone}(F(f)) = 0 \arrow[d] \\
0 \arrow[r] & F(A) \arrow[r, "\mathrm{Id}"] & F(A) \arrow[r] & 0
\end{tikzcd}$$

We get that $F(f)$ is invertible. Hence we get that F inverts every map in $S(\mathscr{S})$. By (L2) we get a unique functor $G : \mathcal{T}/\mathscr{S} \to \mathcal{U}$ such that $F = GQ$. Let Δ be an exact triangle in $\mathcal{T}/\mathscr{S}$ Then $\Delta \cong Q\Gamma$ for an exact triangle in $\mathcal{T}$. Since F is exact $F\Gamma$ is an exact triangle in $\mathcal{U}$. Furthermore $G\Delta \cong F\Gamma$ and thus G is exact.

■

4 Chain Complexes of Exact Categories

The contents of this section is based on Bühler's article [2, Chapter 9]. $\mathscr{A}$ is an additive category unless otherwise stated.

In this section we investigate chain complexes of exact categories. In particular we look at acyclic complexes, some of their properties and how to define the derived category of an exact category through Verdier localization. In the last section of the thesis we will see the derived category in examples.

4.1 Definition and basic properties

Before defining acyclic complexes for exact categories we recall some of the definitions and properties of chain complexes of an additive category. We will also see that short exact sequences in an exact category are closed under homotopy equivalences, this will be needed later in the thesis when we classify exact structures.

Definition 4.1.1. A *(cochain) complex* of an additive category $\mathscr{A}$ is a sequence of objects and morphisms in $\mathscr{A}$

$$A^\bullet = \dots \xrightarrow{d_A^{n-2}} A^{n-1} \xrightarrow{d_A^{n-1}} A^n \xrightarrow{d_A^n} A^{n+1} \xrightarrow{d_A^{n+1}} \dots$$

such that $d_A^n d_A^{n-1} = 0$ for all n. A *chain map* $f^\bullet : A^\bullet \to B^\bullet$ between two chain complexes is morphisms f^n in each degree such that the following diagram commutes.

$$\begin{tikzcd}
\dots \arrow[r, "d_A^{n-2}"] & A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d, "f^{n-1}"] & A^n \arrow[r, "d_A^n"] \arrow[d, "f^n"] & A^{n+1} \arrow[r, "d_A^{n+1}"] \arrow[d, "f^{n+1}"] & \dots \\
\dots \arrow[r, "d_B^{n-2}"] & B^{n-1} \arrow[r, "d_B^{n-1}"] & B^n \arrow[r, "d_B^n"] & B^{n+1} \arrow[r, "d_B^{n+1}"] & \dots
\end{tikzcd}$$

We denote the category where objects are complexes and morphisms are chain maps by $\mathbf{Ch}(\mathscr{A})$.

Remark 4.1.2. $\mathbf{Ch}(\mathscr{A})$ is an additive category where $\dots \to 0 \to 0 \to 0 \to \dots$ is the zero object. The coproduct of $A^\bullet$ and $B^\bullet$ is the complex with the coproducts $A^n \oplus B^n$ as objects and $\begin{pmatrix} d_A^n & 0 \\ 0 & d_B^n \end{pmatrix}$ as morphisms.

Example 4.1.3. If $(\mathscr{A}, \mathscr{E})$ is an exact category then $(\mathbf{Ch}(\mathscr{A}), \mathbf{Ch}(\mathscr{E}))$ is an exact category where

$$\mathbf{Ch}(\mathscr{E}) = \{A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \mid A^n \xrightarrow{f^n} B^n \xrightarrow{g^n} C^n \in \mathscr{E} \text{ for all } n\}$$

it is easy to see (E0) and (E0$^{\text{op}}$) hold as $0^\bullet \in \mathbf{Ch}\,\mathscr{A}$ and $\mathrm{Id}_{A^\bullet}$ has Id_{A^n} in each degree, which is admissible epic and monic in $(\mathscr{A}, \mathscr{E})$. (E1) and (E1$^{\text{op}}$) hold as composition is done pointwise (in each degree). We justify (E2), (E2$^{\text{op}}$) is dual. We need to complete the following pushout.

$$\begin{tikzcd}
A^\bullet \arrow[r, tail, "i^\bullet"] \arrow[d, "f^\bullet"'] & B^\bullet \arrow[d, dashed, "\hat{f}^\bullet"] \\
C^\bullet \arrow[r, tail, dashed, "\hat{i}^\bullet"'] & X^\bullet
\end{tikzcd}$$

We know pushouts is taken in each degree. The following use of the universal property shows we have morphisms $d^n : X^n \to X^{n+1}$

$$\begin{tikzcd}
A^n \arrow[r, tail, "i^n"] \arrow[d, "f^n"'] & B^n \arrow[d, "\hat{f}^n"'] \arrow[ddr, bend left, "\hat{f}^{n+1} d_B^n"] & \\
C^n \arrow[r, tail, "\hat{i}^n"] \arrow[drr, bend right, "\hat{i}^{n+1} d_C^n"'] & X^n \arrow[dr, dashed, "d_X^n"] & \\
 & & X^{n+1}
\end{tikzcd}$$

Such that the following diagrams commutes for each n.

$$\begin{tikzcd}
C^n \arrow[r, "d_c^n"] \arrow[d, tail, "\hat{i}^n"'] & C^{n+1} \arrow[d, tail, "\hat{i}^{n+1}"] \\
X^n \arrow[r, "d_X^n"] & X^{n+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
B^n \arrow[r, "d_B^n"] \arrow[d, "\hat{f}^n"'] & B^{n+1} \arrow[d, "\hat{f}^{n+1}"] \\
X^n \arrow[r, "d_X^n"] & X^{n+1}
\end{tikzcd}$$

Thus $\hat{i}^\bullet$ is admissible monic as desired. One can easily check $d_X^{n+1} d_X^n = 0$ by the following universal property.

$$\begin{tikzcd}
A^n \arrow[r, tail, "i^n"] \arrow[d, "f^n"'] & B^n \arrow[d, "\hat{f}^n"'] \arrow[ddr, bend left, "\hat{f}^{n+2} d_B^{n+1} d_B^n = 0"] & \\
C^n \arrow[r, tail, "\hat{i}^n"] \arrow[drr, bend right, "\hat{i}^{n+2} d_C^{n+1} d_C^n = 0"'] & X^n \arrow[dr, dashed] & \\
 & & X^{n+2}
\end{tikzcd}$$

Thus $X^\bullet$ is a chain complex, and we have $\hat{i}^\bullet$ and $\hat{f}^\bullet$ completing the pushout.

Definition 4.1.4. The *mapping cone* of a chain map $f^\bullet : A^\bullet \to B^\bullet$ is the chain complex $\mathrm{Cone}(f)$ where

$$\mathrm{Cone}(f)^n = A^{n+1} \oplus B^n \text{ and } d^n_{\mathrm{Cone}(f)} = \begin{pmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{pmatrix}$$

Remark 4.1.5. Cone(f) is a chain complex as

$$d^{n+1}_{\mathrm{Cone}(f)} d^n_{\mathrm{Cone}(f)} = \begin{pmatrix} -d_A^{n+2} & 0 \\ f^{n+2} & d_B^{n+1} \end{pmatrix} \begin{pmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^{n} \end{pmatrix} = \begin{pmatrix} d_A^{n+2} d_A^{n+1} & 0 \\ -f^{n+2} d_A^{n+1} + d_B^{n+1} f^{n+1} & d_B^{n+1} d_B^n \end{pmatrix}$$

which is $0^\bullet$ as $A^\bullet, B^\bullet$ are chain complexes, and the fact that squares commute in the chain map $f^\bullet : A^\bullet \to B^\bullet$.

Remark 4.1.6. The mapping cone defines a canonical functor from chain maps in $\mathbf{Ch}(\mathscr{A})$ to $\mathbf{Ch}(\mathscr{A})$.

Definition 4.1.7. The *translation functor* $[n] : \mathbf{Ch}(\mathscr{A}) \to \mathbf{Ch}(\mathscr{A})$ is defined to be $A[n]^i = A^{i+n}$ on components and $d_A^n[n] = (-1)^n d_A^{n+i}$ on differentials. For chain maps $f[n]^i = f^{i+n}$. The translation functor is clearly an additive auto-equivalence. The inverse is $[-n]$.

Remark 4.1.8. This gives us a new notation for the mapping cone namely $A^\bullet[1] \oplus B^\bullet$.

Definition 4.1.9. The *strict triangle* over a chain map $f^\bullet : A^\bullet \to B^\bullet$ is the 3-periodic sequence.

$$A \xrightarrow{f} B \xrightarrow{\binom{0}{1}} \mathrm{Cone}(f) \xrightarrow{(1\ 0)} A[1] \xrightarrow{f[1]} B[1] \xrightarrow{\binom{0}{1}[1]} \mathrm{Cone}(f)[1] \xrightarrow{(1\ 0)[1]} \ldots$$

Definition 4.1.10. A chain map $f : A^\bullet \to B^\bullet$ is *chain homotopic to zero* if there exists morphisms $h^n : A^n \to B^{n-1}$ in each degree such that $f^n = d_B^{n-1} h^n + h^{n+1} d_A^n$. A chain complex $A^\bullet$ is called *null homotopic* if Id_A is chain homotopic to zero. Two morphisms $f^\bullet$ and $g^\bullet$ are called *homotopic* if $f^\bullet - g^\bullet$ is null homotopic. if $f^\bullet$ is homotopic to $g^\bullet$ we will sometimes write $f^\bullet \simeq g^\bullet$. We say two complexes $A^\bullet, B^\bullet$ are *homotopy equivalent* if there exists chain maps $f^\bullet : A^\bullet \to B^\bullet$ and $g^\bullet : B^\bullet \to A^\bullet$ such that $f^\bullet g^\bullet \simeq \mathrm{Id}_{B^\bullet}$ and $g^\bullet f^\bullet \simeq \mathrm{Id}_{A^\bullet}$.

Proposition 4.1.11. In an exact category $(\mathscr{A}, \mathscr{E})$ short exact sequences are closed under homotopy equivalences.

Proof. Let $A \xrightarrow{a} B \xrightarrow{b} C$ be a short exact sequence homotopy equivalent to the sequence $X \xrightarrow{x} Y \xrightarrow{y} Z$. In other words we have the diagram

$$\begin{tikzcd}
A \arrow[r, tail, "a"'] \arrow[d, shift right, "f_1"'] & B \arrow[r, two heads, "b"'] \arrow[d, shift right, "f_2"'] \arrow[l, dashed, bend right, "h_a"'] & C \arrow[d, shift right, "f_3"'] \arrow[l, dashed, bend right, "h_b"'] \\
X \arrow[r, "x"] \arrow[u, shift right, "g_1"'] & Y \arrow[r, "y"] \arrow[u, shift right, "g_2"'] \arrow[l, dashed, bend left, "h_x"] & Z \arrow[u, shift right, "g_3"'] \arrow[l, dashed, bend left, "h_y"]
\end{tikzcd}$$

Such that

- The solid part commutes.
- $1_A = g_1 f_1 + h_a a$
- $1_B = g_2 f_2 + h_b b + a h_a$
- $1_C = g_3 f_3 + b h_b$
- $1_X = f_1 g_1 + h_x x$
- $1_Y = f_2 g_2 + h_y y + x h_x$
- $1_Z = f_3 g_3 + y h_y$

Our plan is to construct a new homotopy, then use a dual argument to get another new homotopy which will turn out to be an isomorphism. Consider the following pushout and universal property

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "f_1"'] & B \arrow[d, "f_2'"'] \arrow[ddr, bend left, "f_2"] & \\
X \arrow[r, "a'"] \arrow[drr, bend right, "x"'] & PO \arrow[dr, dashed, "\exists! f_2''"] & \\
 & & Y
\end{tikzcd}$$

This gives us the opportunity to consider the following diagram where $g_2'' = f_2' g_2 + a' h_x$.

$$\begin{tikzcd}
A \arrow[r, tail, "a"] \arrow[d, "f_1"'] & B \arrow[r, two heads, "b"] \arrow[d, "f_2'"'] & C \arrow[d, equal] \\
X \arrow[r, tail, "a'"] \arrow[d, equal] & PO \arrow[r, two heads, "b'"] \arrow[d, shift right, "f_2''"'] & C \arrow[d, shift right, "f_3"'] \\
X \arrow[r, "x"] & Y \arrow[r, "y"] \arrow[u, shift right, "g_2''"'] & Z \arrow[u, shift right, "g_3"']
\end{tikzcd} \tag{4.1.1}$$

The top squares commute by A.1. The bottom left square starting at X commute by our previous diagram and

$$g_2'' x = (f_2 g_2 + a' h_x) x = f_2' a g_1 + a'(\mathrm{Id}_X - f_1 g_1) = a' f_1 g_1 + a' - a' f_1 g_1 = a'$$

The bottom right square commutes starting at Y as

$$b' g_2'' = b'(f_2' g_2 + a' h_x) = b' f_2' g_2 + b' a' h_x = b' f_2' g_2 = b g_2 = g_3 y$$

Now we show the bottom right square commutes starting at PO. We note that $y f_2'' f_2' =$

$yf_2 = f_3b = f_3b'f_2'$. Thus considering the universal property

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "f_1"'] & B \arrow[d, "f_2'"'] \arrow[ddr, bend left, "f_3 b"] & \\
X \arrow[r, "a'"] \arrow[drr, bend right, "yx"'] & PO \arrow[dr, dashed, "\exists!"] & \\
& & Z
\end{tikzcd}$$

We see $f_3b' = yf_2''$ by uniqueness of the dotted arrow. Thus the diagram (4.1.1) commutes. Next we claim there exists l such that $h_xf_2 - f_1h_a = lb$. As b is a cokernel of a it is sufficient to show $(h_xf_2 - f_1h_a)a = 0$. This hold as

$$f_1h_aa = f_1(\mathrm{Id}_A - g_1f_1) = f_1 - f_1g_1f_1 = (\mathrm{Id}_X - f_1g_1)f_1 = h_xxf_1 = h_xf_2a$$

Now we claim h_y and $h_b' = f_2'h_b - a'l$ gives the following homotopy equivalence.

$$\begin{tikzcd}
PO \arrow[r, two heads, "b'"'] \arrow[d, shift right, "f_2''"'] & C \arrow[l, dashed, bend right, "h_b'"'] \arrow[d, shift right, "f_3"'] \\
Y \arrow[r, "y"] \arrow[u, shift right, "g_2''"'] & Z \arrow[l, dashed, bend left, "h_y"] \arrow[u, shift right, "g_3"']
\end{tikzcd}$$

Three of the equality's are straightforward

- $f_2''g_2'' + h_yy = f_2''(f_2'g_2 + a'h_x) + h_yy = f_2g_2 + xh_x + h_yy = \mathrm{Id}_Y$
- $f_3g_3 + yh_y = \mathrm{Id}_Z$
- $g_3f_3 + b'h_b' = g_3f_3 + b'(f_2'h_b - a'l) = g_3f_3 + b'f_2'h_b - b'a'l = g_3f_3 + bh_b = \mathrm{Id}_C$

To see $\mathrm{Id}_{PO} = g_2''f_2'' + h_b'b'$ we remember PO is a pushout and Lemma A.5 to realize $\begin{pmatrix} a' & f_2' \end{pmatrix}$ is epic and therefore

$$\mathrm{Id}_{PO} = g_2''f_2'' + h_b'b' \iff a' = (g_2''f_2'' + h_b'b')a' \text{ and } f_2' = (g_2''f_2'' + h_b'b')f_2'$$

We check if the right equality's above hold. We have

$$\begin{aligned}
(g_2''f_2'' + h_b'b')a' &= g_2''f_2''a' + h_b'b'a' \\
&= g_2''f_2''a' \\
&= (f_2'g_2 + a'h_x)x \\
&= f_2'g_2x + a'h_xx \\
&= f_2'ag_1 + a'(1_X - f_1g_1) \\
&= a'f_1g_1 + a' - a'f_1g_1 = a'
\end{aligned}$$

and

$$\begin{aligned}
(g_2''f_2'' + h_b'b')f_2' &= g_2''f_2''f_2' + h_b'b'f_2' = g_2''f_2 + h_b'b \\
&= (f_2'g_2 + a'h_x)f_2 + (f_2'h_b - a'l)b \\
&= f_2'g_2f_2 + a'h_xf_2 + f_2'h_bb - a'lb \\
&= f_2'g_2f_2 + a'h_xf_2 + f_2'h_bb - a'(h_xf_2 - f_1h_a) \\
&= f_2'g_2f_2 + f_2'h_bb + a'f_1h_a \\
&= f_2'g_2f_2 + f_2'h_bb + f_2'ah_a \\
&= f_2'(g_2f_2 + h_bb + ah_a) = f_2'
\end{aligned}$$

Thus we see we have an homotopy equivalence. We can expand this to the following homotopy equivalence.

$$\begin{tikzcd}
X \arrow[r, "a'"'] \arrow[d, equal] & PO \arrow[l, dashed, bend right, "0"'] \arrow[r, "b'"'] \arrow[d, shift right, "f_2''"'] & C \arrow[l, dashed, bend right, "h_b'"'] \arrow[d, shift right, "f_3"'] \\
X \arrow[r, "x"] & Y \arrow[l, dashed, bend left, "0"] \arrow[r, "y"] \arrow[u, shift right, "g_2''"'] & Z \arrow[l, dashed, bend left, "h_y"] \arrow[u, shift right, "g_3"']
\end{tikzcd}$$

Now we can form a pullback over g_3 and b' and use a dual argument to get the following homotopy.

$$\begin{tikzcd}
X \arrow[r, tail, "a''"'] \arrow[d, equal] & PB \arrow[l, dashed, bend right, "h_a'"'] \arrow[r, two heads, "b''"'] \arrow[d, shift right, "\hat{f}_2"'] & Z \arrow[l, dashed, bend right, "0"'] \arrow[d, equal] \\
X \arrow[r, "x"] & Y \arrow[l, dashed, bend left, "0"] \arrow[r, "y"] \arrow[u, shift right, "\hat{g}_2"'] & Z \arrow[l, dashed, bend left, "0"]
\end{tikzcd}$$

Now we show $\hat{f}_2\hat{g}_2$ and $\hat{g}_2\hat{f}_2$ are isomorphisms. It is given by the homotopy that $\hat{f}_2\hat{g}_2 = \mathrm{Id}_Y$. For the second we first note that $h_a'a'' = 0$ since $\mathrm{Id}_X = \mathrm{Id}_X + h_a'a''$. This gives us that $(a''h_a')^2 = 0$ and $(\mathrm{Id}_{PB} - a''h_a')(\mathrm{Id}_{PB} - a''h_a') = \mathrm{Id}_{PB}$. From our homotopy we see $\hat{g}_2\hat{f}_2 = \mathrm{Id}_{PB} - a''h_a'$ and thus $\hat{g}_2\hat{f}_2$ is an isomorphism. As short exact sequences are closed under isomorphism we are done.

■

Definition 4.1.12. An *ideal* in an additive category $\mathscr{A}$ is a collection $\mathcal{I} = \{\mathcal{I}(A,B)\}_{A,B\in\mathrm{Ob}(\mathscr{A})}$ of abelian subgroups $\mathcal{I}(A,B) \subseteq \mathrm{Hom}_{\mathscr{A}}(A,B)$ such that the following hold.

1. If $h : B \to C \in \mathcal{I}(B,C)$. Then hf and gh are in $\{\mathcal{I}(A,B)\}_{A,B\in\mathbf{Ob}(\mathscr{A})}$ for every $f \in \mathrm{Hom}_{\mathscr{A}}(A,B)$ and $g \in \mathrm{Hom}_{\mathscr{A}}(C,D)$.

2. If $h_1, h_2 \in \mathcal{I}$ then $h_1 \oplus h_2 \in \mathcal{I}$.

Remark 4.1.13. Note that an ideal $\mathcal{I}$ in $\mathscr{A}$ gives rise to a new category $\widehat{\mathscr{A}}$ where we get $\mathbf{Ob}\,\widehat{\mathscr{A}} = \mathbf{Ob}\,\mathscr{A}$ and $\mathrm{Hom}_{\widehat{\mathscr{A}}}(A,B) = \mathrm{Hom}_{\mathscr{A}}(A,B)/\mathcal{I}(A,B)$. We can make this Hom-sets as $\mathcal{I}(A,B)$ is an abelian subgroup of $\mathrm{Hom}_{\mathscr{A}}(A,B)$. Furthermore property (1) makes composition of morphisms well defined.

Lemma 4.1.14. The collection $N = \{N(A^\bullet, B^\bullet)\}_{A^\bullet, B^\bullet \in \mathbf{Ob}(\mathbf{Ch}\,\mathscr{A})}$ of chain maps homotopic to zero is an ideal in $\mathbf{Ch}\,\mathscr{A}$.

Proof. To show $N(A^\bullet, B^\bullet)$ is a subgroup let $f^\bullet, g^\bullet \in N(A^\bullet, B^\bullet)$. Then we have

$$f^n = d_B^{n-1} h_1^n + h_1^{n+1} d_A^n \text{ and } g^n = d_B^{n-1} h_2^n + h_2^{n+1} d_A^n$$

We need to show $(f^\bullet - g^\bullet) = (f-g)^\bullet \in N(A^\bullet, B^\bullet)$. This is straightforward as

$$(f-g)^n = d_B^{n-1} h_1^n + h_1^{n+1} d_A^n - (d_B^{n-1} h_2^n + h_2^{n+1} d_A^n) = d_B^{n-1}(h_1^n - h_2^n) + (h_1^{n+1} - h_2^{n+1}) d_A^n$$

yielding the maps needed, namely $h^n = h_1^n - h_2^n$. For the absorption property let $f \in N(A^\bullet, B^\bullet)$ and $g \in \mathrm{Hom}_{\mathbf{Ch}(\mathscr{A})}(B^\bullet, C^\bullet)$. Then we have $f^n = d_B^{n-1} h_1^n + h_1^{n+1} d_A^n$. Let $\tilde{h}_n = g^{n-1} h_n$. Then we have

$$\begin{aligned} d_C^{n-1} \tilde{h}^n + \tilde{h}^{n+1} d_A^n &= d_C^{n-1} g^{n-1} h^n + g^n h^{n+1} d_A^n \\ &= g^n d_B^{n-1} h^n + g^n (f^n - d_B^{n-1} h^n) \\ &= g^n f^n \end{aligned}$$

Hence $g^\bullet f^\bullet \in N(A,B)$. Absorption from the other side is shown similarly. For the sum we note that if $f_1, f_2 \in N$ then we have $f_1^n = d_B^{n-1} h_1^n + h_1^{n+1} d_A^n$ and $f_2^n = d_B^{n-1} h_2^n + h_2^{n+1} d_A^n$ and therefore

$$(f_1 \oplus f_2)^n = (d_B^{n-1} \oplus d_B^{n-1})(h_1^n \oplus h_2^n) + (h_1^{n+1} \oplus h_2^{n+1})(d_A^n \oplus d_A^n)$$

Hence we have shown N is an ideal. ■

Definition 4.1.15. The *homotopy category* $\mathcal{K}(\mathscr{A})$ is the category where $\mathbf{Ob}(\mathcal{K}(\mathscr{A})) = \mathbf{Ob}(\mathbf{Ch}(A))$ and $\mathrm{Hom}_{\mathcal{K}(\mathscr{A})}(A,B) = \mathrm{Hom}_{\mathbf{Ch}(\mathscr{A})}(A,B)/N(A,B)$

Remark 4.1.16. In $\mathcal{K}(\mathscr{A})$ morphisms are considered the same if their difference is null-homotopic, every null homotopic complex is isomorphic to zero. Note that $\mathcal{K}(\mathscr{A})$ is well defined by Lemma 4.1.14.

Remark 4.1.17. It follows from the definition that $\mathcal{K}(\mathscr{A})$ inherits the additive structure of $\mathbf{Ch}(\mathscr{A})$. The Hom-sets are quotients of abelian groups, thus abelian groups. However $\mathcal{K}(\mathscr{A})$ will rarely be abelian or exact with respect to a non-trivial exact structure.

Example 4.1.18. $\mathcal{K}(\mathscr{A})$ is a triangulated category where the exact triangles are induced by the strict triangles in $\mathbf{Ch}(\mathscr{A})$ from Definition 4.1.9.

4.2 Acyclic complexes

In this subsection we go through the theory needed to define the derived category. We will see that acyclic complexes form a triangulated subcategory of $\mathcal{K}(\mathscr{A})$ which gives us the ability to apply Verdier localization. Furthermore we will see that the acyclic complexes form a thick subcategory if and only if we have an idempotent complete category.

Definition 4.2.1. A chain complex $A^\bullet$ over an exact category $(\mathscr{A}, \mathscr{E})$ is called acyclic if each differential d_A^n factors as $A^n \xrightarrow{e_A^n} Z^{n+1}A \xrightarrow{i_A^{n+1}} A^{n+1}$ in such a way that each sequence $Z^nA \xrightarrow{i_A^n} A^n \xrightarrow{e_A^n} Z^{n+1}A$ is exact. In other words it is a complex where each differential is admissible, and two consecutive morphisms are acyclic. See Definition 1.4.1 and 1.4.5.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{n-1} \arrow[rr, "d_A^{n-1}"] \arrow[dr, two heads, "e_A^{n-1}"'] & & A^n \arrow[rr, "d_A^n"] \arrow[dr, two heads, "e_A^n"'] & & A^{n+1} \arrow[r] & \cdots \\
& & Z^nA \arrow[ur, tail, "i_A^n"'] & & Z^{n+1}A \arrow[ur, tail, "i_A^{n+1}"'] & &
\end{tikzcd}$$

Remark 4.2.2. By Proposition 1.4.2, Lemma 1.4.3 and Definition 1.4.4 we know that Z^nA is a kernel of d_A^n, an image and coimage of d_A^{n-1} and a cokernel of d_A^{n-2}.

Lemma 4.2.3. The mapping cone of a chain map $f^\bullet : A^\bullet \to B^\bullet$ between acyclic complexes is acyclic.

Proof. First we claim there exists unique morphisms g^n making the following diagram commute.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{n-1} \arrow[rr, "d_A^{n-1}"] \arrow[dr, two heads, "e_A^{n-1}"'] \arrow[ddd, "f^{n-1}"'] & & A^n \arrow[rr, "d_A^n"] \arrow[dr, two heads, "e_A^n"'] \arrow[ddd, "f^n"'] & & A^{n+1} \arrow[r] \arrow[ddd, "f^{n+1}"] & \cdots \\
& & Z^nA \arrow[ur, tail, "i_A^n"'] \arrow[d, dashed, "g^n"] & & Z^{n+1}A \arrow[ur, tail, "i_A^{n+1}"'] \arrow[d, dashed, "g^{n+1}"] & & \\
& & Z^nB \arrow[dr, tail, "i_B^n"] & & Z^{n+1}B \arrow[dr, tail, "i_B^{n+1}"] & & \\
\cdots \arrow[r] & B^{n-1} \arrow[rr, "d_B^{n-1}"'] \arrow[ur, two heads, "e_B^{n-1}"] & & B^n \arrow[rr, "d_B^n"'] \arrow[ur, two heads, "e_B^n"] & & B^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

We see $e_B^{n-1}f^{n-1}d_A^{n-2} = e_B^{n-1}d_B^{n-2}f^{n-2} = 0$ as e_B^{n-1} is a cokernel of d_B^{n-2}. Since e_A^{n-1} is a cokernel of d_A^{n-2} we get a unique map $g^n : Z^nA \to Z^nB$ such that $e_B^{n-1}f^{n-1} = g^ne_A^{n-1}$. We also see that $d_B^nf^ni_A^n = f^{n+1}d_A^ni_A^n = 0$ as i_A^n is a kernel of d_A^n. As i_B^n is a kernel of

d_B^n we get a unique map $\tilde{g}^n : Z^nA \to Z^nB$ such that $f^n i_A^n = i_B^n \tilde{g}^n$. Now we have the following setup where $\tilde{g}^n$ makes the right triangle commute, g^n makes the left triangle commute and the square commutes.

$$\begin{tikzcd}
A^{n-2} \arrow[r, "d_A^{n-2}"] & A^{n-1} \arrow[r, "e_A^{n-1}"] \arrow[dr, "e_B^{n-1} f^{n-1}"'] & Z^nA \arrow[d, tail, dashed, shift right, "g^n"'] \arrow[d, tail, dashed, shift left, "\tilde{g}^n"] \arrow[dr, "f^n i_A^n"] & & \\
 & & Z^nB \arrow[r, "i_B^n"'] & B^n \arrow[r, "d_B^n"'] & B^{n+1}
\end{tikzcd}$$

Now $f^n i_A^n e_A^{n-1} = i_B^n e_B^{n-1} f^{n-1} \implies i_B^n \tilde{g}^n e_A^{n-1} = i_B^n g^n e_A^{n-1} \implies g^n = \tilde{g}^n$. Hence we have unique g^n making the diagram commute. Note that we can do this for every degree. By Proposition 1.3.2 we find objects Z^nC fitting into the commutative diagram

$$\begin{tikzcd}
A^{n-1} \arrow[rr, "d_A^{n-1}"] \arrow[dr, two heads, "e_A^{n-1}"] \arrow[dd, "f_1^{n-1}"'] & & A^n \arrow[rr, "d_A^n"] \arrow[dr, two heads, "e_A^n"] \arrow[dd, "f_1^n"'] & & A^{n+1} \arrow[dd, "f_1^{n+1}"] \\
 & Z^nA \arrow[ur, tail, "i_A^n"'] \arrow[dd, "g^n"] \arrow[dr, phantom, "\mathrm{BC}"] & & Z^{n+1}A \arrow[ur, tail, "i_A^{n+1}"'] \arrow[dd, "g^{n+1}"] \arrow[dr, phantom, "\mathrm{BC}"] & \\
Z^{n-1}C \arrow[ur, two heads, "h^{n-1}"] \arrow[dd, "f_2^{n-1}"'] \arrow[dr, phantom, "\mathrm{BC}"] & & Z^nC \arrow[ur, two heads, "h^n"'] \arrow[dd, "f_2^n"'] \arrow[dr, phantom, "\mathrm{BC}"] & & Z^{n+1}C \arrow[dd, "f_2^{n+1}"] \\
 & Z^nB \arrow[ur, tail, "k^n"] \arrow[dr, tail, "i_B^n"] & & Z^{n+1}B \arrow[ur, tail] \arrow[dr, tail, "i_B^{n+1}"] & \\
B^{n-1} \arrow[rr, "d_B^{n-1}"] \arrow[ur, two heads, "e_B^{n-1}"] & & B^n \arrow[rr, "d_B^n"] \arrow[ur, two heads, "e_B^n"] & & B^{n+1}
\end{tikzcd}$$

where $f_2^n f_1^n = f^n$ and the quadrilaterals marked BC are bicartesian. Recall the objects Z^nC are obtained by forming the pushouts under i_A^n and g^n (alternatively pullbacks over e_B^n and g^{n+1}) and that $Z_B^n \rightarrowtail Z^nC \twoheadrightarrow Z^{n+1}A$ is short exact. Now we have by Corollary 1.2.5 that the following sequence is exact.

$$\begin{tikzcd}
Z^{n-1}C \arrow[r, tail, "\begin{pmatrix} -i_A^n h^{n-1} \\ f_2^{n-1} \end{pmatrix}"] & A^n \oplus B^{n-1} \arrow[r, two heads, "\begin{pmatrix} f_1^n & k^n e_B^{n-1} \end{pmatrix}"] & Z^nC
\end{tikzcd}$$

We then get the commutative diagram

$$\begin{tikzcd}
A^n \oplus B^{n-1} \arrow[rr, "\begin{pmatrix} -d_A^n & 0 \\ f^n & d_B^{n-1} \end{pmatrix}"] \arrow[dr, two heads, "\begin{pmatrix} f_1^n & k^n e_B^{n-1} \end{pmatrix}"] & & A^{n+1} \oplus B^n \arrow[rr, "\begin{pmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{pmatrix}"] \arrow[dr, two heads, "\begin{pmatrix} f_1^{n+1} & k^{n+1} e_B^n \end{pmatrix}"] & & A^{n+2} \oplus B^{n+1} \\
 & Z^nC \arrow[ur, tail, "\begin{pmatrix} -i_A^{n+1} h^n \\ f_2^n \end{pmatrix}"'] & & Z^{n+1}C \arrow[ur, tail, "\begin{pmatrix} -i_A^{n+2} h^{n+1} \\ f_2^{n+1} \end{pmatrix}"'] &
\end{tikzcd}$$

in each degree. Hence Cone(f) is acyclic. ∎

Definition 4.2.4. We let $\mathbf{Ac}(\mathscr{A})$ denote the full subcategory of the homotopy category $\mathcal{K}(\mathscr{A})$ consisting of acyclic complexes.

Remark 4.2.5. By Proposition 1.2.2 we get that $\mathbf{Ac}(\mathscr{A})$ it is a full additive subcategory of $\mathcal{K}(\mathscr{A})$.

Corollary 4.2.6. The subcategory of acyclic complexes $\mathbf{Ac}(\mathscr{A})$ is a triangulated subcategory of $\mathcal{K}(\mathscr{A})$.

Proof. If $A^\bullet$ is acyclic then clearly $A^\bullet[1]$ is acyclic. If $A^\bullet \to B^\bullet \to \text{Cone} \to A^\bullet[1]$ is an exact triangle with two of $\{A^\bullet, B^\bullet, C^\bullet\}$ acyclic we get by Lemma 4.2.3 and (TR2) that the third also is. ∎

Lemma 4.2.7. Let $(\mathscr{A}, \mathscr{E})$ be idempotent complete. Let $A^\bullet$ an acyclic complex in $\mathbf{Ch}(\mathscr{A})$. Then the following hold.

(1) If $f^\bullet : X^\bullet \to A^\bullet$ is a coretraction in $\mathcal{K}(\mathscr{A})$ then $X^\bullet$ is acyclic.

(2) If $f^\bullet : A^\bullet \to X^\bullet$ is a retraction in $\mathcal{K}(\mathscr{A})$ then $X^\bullet$ is acyclic.

Proof. We prove (1), (2) is dual. Let $f^\bullet : X^\bullet \to A^\bullet$ be a coretraction in $\mathcal{K}(\mathscr{A})$. Then there exists a section $s^\bullet : A^\bullet \to X^\bullet$ such that $s^\bullet f^\bullet \simeq \mathrm{Id}_{X^\bullet}$. Hence there exist morphisms $h^n : X^n \to X^{n+1}$ such that $s^n f^n - \mathrm{Id}_X = d_X^{n-1} h^n + h^{n+1} d_X^n$. We claim that we without loss of generality can assume f has a left inverse in $\mathbf{Ch}(\mathscr{A})$. Consider the chain complex $(IX)^\bullet$ with objects $(IX)^n = X^n \oplus X^{n+1}$ and differentials $d_{IX}^n = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. This is acyclic as we have

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-1} \oplus X^n \arrow[rr, "\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)"] \arrow[dr, two heads, "(0\ 1)"'] & & X^n \oplus X^{n+1} \arrow[rr, "\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)"] \arrow[dr, two heads, "(0\ 1)"'] & & X^{n+1} \oplus X^{n+2} \arrow[r] & \cdots \\
& & X^n \arrow[ur, tail, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] & & X^{n+1} \arrow[ur, tail, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] & &
\end{tikzcd}$$

and by Proposition 1.2.1 $X^n \overset{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}{\rightarrowtail} X^n \oplus X^{n+1} \overset{(0\ 1)}{\twoheadrightarrow} X^{n+1}$ is exact. We have a canonical chain map $i_X^\bullet : X^\bullet \to (IX)^\bullet$ given by $\begin{pmatrix} \mathrm{Id}_{X^n} \\ d_X^n \end{pmatrix} : X^n \to X^n \oplus X^{n+1}$. Furthermore we have a chain map $\begin{pmatrix} f^\bullet \\ i_X^\bullet \end{pmatrix} : X^\bullet \to A^\bullet \oplus (IX)^\bullet$. Since $A^\bullet \oplus (IX)^\bullet$ is a direct sum of two acyclic complexes, we can use Proposition 1.2.2 to see $A^\bullet \oplus (IX)^\bullet$ is acyclic. $\begin{pmatrix} f^\bullet \\ i_X^\bullet \end{pmatrix}$ has a left inverse in $\mathbf{Ch}(\mathscr{A})$ namely $\left(s^n \ \ -d_X^{n-1} h^n \ \ -h^{n+1} \right) : A^n \oplus X^n \oplus X^{n-1}$ in each degree.

$$\begin{tikzcd}[column sep=large]
A^{n-1}\oplus X^{n-1}\oplus X^{n} \arrow[r, "\begin{pmatrix} d_A^{n-1} & 0 & 0\\ 0 & 0 & \mathrm{Id}_{X^n}\\ 0 & 0 & 0\end{pmatrix}"] \arrow[d, "\begin{pmatrix} s^{n-1} & -d_X^{n-2}h^{n-1} & -h^{n}\end{pmatrix}"] & A^{n}\oplus X^{n}\oplus X^{n+1} \arrow[r, "\begin{pmatrix} d_A^{n} & 0 & 0\\ 0 & 0 & \mathrm{Id}_{X^{n+1}}\\ 0 & 0 & 0\end{pmatrix}"] \arrow[d, "\begin{pmatrix} s^{n} & -d_X^{n-1}h^{n} & -h^{n+1}\end{pmatrix}"] & A^{n+1}\oplus X^{n+1}\oplus X^{n+2} \arrow[d, "\begin{pmatrix} s^{n+1} & -d_X^{n}h^{n+1} & -h^{n+2}\end{pmatrix}"] \\
X^{n-1} \arrow[r, "d_X^{n-1}"] & X^{n} \arrow[r, "d_X^{n}"] & X^{n+1}
\end{tikzcd}$$

We see this is a chain map as

$$\begin{aligned}
\begin{pmatrix} s^{n+1} & -d_X^{n}h^{n+1} & -h^{n+2}\end{pmatrix}\begin{pmatrix} d_A^{n} & 0 & 0\\ 0 & 0 & \mathrm{Id}_{X^{n+1}}\\ 0 & 0 & 0\end{pmatrix} &= s^{n+1}d_A^{n} - d_x^{n}h^{n} + 1\\
&= d_X^{n+1}s^{n} - d_X^{n}h^{n+1}\\
&= d_X^{n+1}s^{n} - d_X^{n}d_X^{n-1}h^{n} - d_X^{n} + 1h^{n+1}\\
&= d_X^{n}\begin{pmatrix} s^{n} & -d_X^{n-1}h^{n} & -h^{n+1}\end{pmatrix}
\end{aligned}$$

Then we note it is a left inverse as

$$\begin{pmatrix} s^{n} & -d_X^{n-1}h^{n} & -h^{n+1}\end{pmatrix}\begin{pmatrix} f^{n}\\ \mathrm{Id}_{X^n}\\ d_X^{n}\end{pmatrix} = s^{n}f^{n} - d_X^{n-1}h^{n} - h^{n+1}d_X^{n} = \mathrm{Id}_{X^n}$$

Now we can replace $A^\bullet$ with our acyclic complex $A^\bullet \oplus (IX)^\bullet$. Hence we simply assume our coretraction $f^\bullet : X^\bullet \to A^\bullet$ has a left inverse $s^\bullet$ in $\mathbf{Ch}(\mathscr{A})$ as section. Now we note that $f^\bullet s^\bullet$ is idempotent as $(f^\bullet s^\bullet)^2 = f^\bullet s^\bullet f^\bullet s^\bullet = f^\bullet s^\bullet$. Hence we get $A^\bullet \cong B^\bullet \oplus C^\bullet$ and $f^\bullet s^\bullet = \left(\begin{smallmatrix} 0 & 0\\ 0 & \mathrm{Id}_C\end{smallmatrix}\right)$ since $\mathscr{A}$ is idempotent complete. This gives us that the sequences $Z^nA \to A \to Z^{n+1}A$ decompose into two exact sequences. Thus we apply Corollary 1.2.8 to get that $B^\bullet$ and $C^\bullet$ are acyclic. Now we have the following commutative diagrams where $s^\bullet = \begin{pmatrix} s_1^\bullet & s_2^\bullet\end{pmatrix}$ and $f^\bullet = \begin{pmatrix} f_1^\bullet\\ f_2^\bullet\end{pmatrix}$

$$\begin{tikzcd}
X^\bullet \arrow[r, swap, "\begin{pmatrix} f_1^\bullet\\ f_2^\bullet\end{pmatrix}"] \arrow[rr, bend left, "\mathrm{Id}_{X^\bullet}"] & B^\bullet\oplus C^\bullet \arrow[r, swap, "\begin{pmatrix} s_1^\bullet & s_2^\bullet\end{pmatrix}"] & X^\bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
B^\bullet\oplus C^\bullet \arrow[r, swap, "\begin{pmatrix} s_1^\bullet & s_2^\bullet\end{pmatrix}"] \arrow[rr, bend left, "\left(\begin{smallmatrix} 0 & 0\\ 0 & \mathrm{Id}_{C^\bullet}\end{smallmatrix}\right)"] & X^\bullet \arrow[r, swap, "\begin{pmatrix} f_1^\bullet\\ f_2^\bullet\end{pmatrix}"] & B^\bullet\oplus C^\bullet
\end{tikzcd}$$

We want to show $s_2^\bullet$ and $f_2^\bullet$ are mutual inverses. $\begin{pmatrix} f_1^\bullet\\ f_2^\bullet\end{pmatrix}\begin{pmatrix} s_1^\bullet & s_2^\bullet\end{pmatrix} = \left(\begin{smallmatrix} 0 & 0\\ 0 & \mathrm{Id}_{C^\bullet}\end{smallmatrix}\right)$ which gives us $f_1^\bullet s_1^\bullet = f_1^\bullet s_2^\bullet = f_2^\bullet s_1^\bullet = 0$ and $f_2^\bullet s_2^\bullet = \mathrm{Id}_{C^\bullet}$. For the other direction we see

$$\mathrm{Id}_{X^\bullet} = (\mathrm{Id}_{X^\bullet})^2 = s^\bullet f^\bullet s^\bullet f^\bullet = s^\bullet\, \mathrm{Id}_{C^\bullet}\, f^\bullet = \begin{pmatrix} s_1^\bullet & s_2^\bullet\end{pmatrix}\left(\begin{smallmatrix} 0 & 0\\ 0 & \mathrm{Id}_{C^\bullet}\end{smallmatrix}\right)\begin{pmatrix} f_1^\bullet\\ f_2^\bullet\end{pmatrix} = s_2^\bullet f_2^\bullet$$

Hence $f_2^\bullet$ and $s_2^\bullet$ are mutual inverses, $X^\bullet \cong C^\bullet$ and we are done as $C^\bullet$ is acyclic.

■

Proposition 4.2.8. The following are equivalent for an exact category $\mathscr{A}$.

(1) Every null-homotopic complex in $\mathbf{Ch}(\mathscr{A})$ is acyclic.

(2) The category $\mathscr{A}$ is idempotent complete.

(3) The class of acyclic complexes is closed under isomorphisms in $\mathcal{K}(\mathscr{A})$.

Proof. (1) $\Rightarrow$ (2) : Let $e : A \to A$ be an idempotent of $\mathscr{A}$. Consider the complex

$$... \xrightarrow{1-e} A \xrightarrow{e} A \xrightarrow{1-e} A \xrightarrow{e} ...$$

This is null homotopic, to see this set $h^n = \mathrm{Id}_A$. By assumption the complex is also acyclic. Hence e has a kernel and by Proposition 2.1.2 we are done.

(2) $\Rightarrow$ (3) : Let $A^\bullet$ be acyclic. Let $X^\bullet \cong A^\bullet$ in $\mathcal{K}(\mathscr{A})$ then the induced isomorphism $f : X^\bullet \to A^\bullet$ is in particular a coretraction hence by Lemma 4.2.7 $X^\bullet$ is acyclic.

(3) $\Rightarrow$ (1) : A null homotopic complex $X^\bullet$ is isomorphic to $0^\bullet$ in $\mathcal{K}(\mathscr{A})$. Hence it is acyclic by assumption as $0^\bullet$ is acyclic. ∎

Corollary 4.2.9. The triangulated subcategory $\mathbf{Ac}(\mathscr{A})$ of $\mathcal{K}(\mathscr{A})$ is thick if and only if $\mathscr{A}$ is idempotent complete.

Proof. Let $\mathscr{A}$ be idempotent complete. Let $A^\bullet = B^\bullet \oplus C^\bullet$ be in $\mathbf{Ac}(\mathscr{A})$. Then we get by similar argument as in Proposition 4.2.7 that $B^\bullet$ and $C^\bullet$ are acyclic. Proposition 4.2.8 ((2) $\Rightarrow$ (3)) gives that $\mathbf{Ac}(\mathscr{A})$ is strictly full. Conversely if $\mathbf{Ac}(\mathscr{A})$ is thick we get by Proposition 4.2.8((3) $\Rightarrow$ (2)) that $\mathscr{A}$ is idempotent complete. ∎

4.3 Bounded complexes

In this subsection we consider bounded complexes. We will see that for bounded complexes it suffice that $\mathscr{A}$ is weakly idempotent complete in order for $\mathbf{Ac}^{\mathrm{b}}(\mathscr{A})$ to be thick.

Definition 4.3.1. A complex $A^\bullet$ is called *left bounded* if $A^n = 0$ for all $n < k$ for some $k \in \mathbb{Z}$. It is called *right bounded* if $A^n = 0$ for all $n > k$ for some $k \in \mathbb{Z}$. If $A^\bullet$ is right bounded and left bounded it is called *bounded*.

Definition 4.3.2. We denote the full subcategories of $\mathcal{K}(\mathscr{A})$ generated by the left bounded, right bounded and bounded complexes by $\mathcal{K}^+(\mathscr{A})$, $\mathcal{K}^-(\mathscr{A})$ and $\mathcal{K}^{\mathrm{b}}(\mathscr{A})$ respectively.

Remark 4.3.3. Note that $\mathcal{K}^{\mathrm{b}}(\mathscr{A}) = \mathcal{K}^+(\mathscr{A}) \cap \mathcal{K}^-(\mathscr{A})$. Also note K^* is *not* closed under isomorphisms in $\mathcal{K}(\mathscr{A})$ for $* \in \{+,-,b\}$ unless $\mathscr{A} = 0$.

Definition 4.3.4. For $* \in \{+,-,b\}$ we define $\mathbf{Ac}^*(\mathscr{A}) = \mathcal{K}^*(\mathscr{A}) \cap \mathbf{Ac}(\mathscr{A})$.

Remark 4.3.5. Let $* \in \{+,-,\mathrm{b}\}$. We see that $\mathcal{K}^*(\mathscr{A})$ is a full triangulated subcategory of $\mathcal{K}(\mathscr{A})$. Furthermore by Proposition 4.2.3 $\mathbf{Ac}^*(\mathscr{A})$ is a full triangulated subcategory of $\mathcal{K}^*(\mathscr{A})$.

Proposition 4.3.6. The following are equivalent.

(1) The subcategories $\mathbf{Ac}^+(\mathscr{A})$ and $\mathbf{Ac}^-(\mathscr{A})$ of $\mathcal{K}^+(\mathscr{A})$ and $\mathcal{K}^-(\mathscr{A})$ respectively are thick.

(2) The subcategory $\mathbf{Ac}^{\mathrm{b}}(\mathscr{A})$ of $\mathcal{K}^{\mathrm{b}}(\mathscr{A})$ is thick.

(3) The category $\mathscr{A}$ is weakly idempotent complete.

Proof. (1) $\Rightarrow$ (2): Hold as $\mathbf{Ac}^{\mathrm{b}}(\mathscr{A}) = \mathbf{Ac}^+(\mathscr{A}) \cap \mathbf{Ac}^-(\mathscr{A})$.

(2) $\Rightarrow$ (3) : Let $s : B \to A$ and $t : A \to B$ be such that $ts = \mathrm{Id}_B$. In other words t is a retraction with section s. We need to show t has a kernel. Consider the complex $A^\bullet$ given by

$$\dots \longrightarrow 0 \longrightarrow B \xrightarrow{s} A \xrightarrow{1-st} A \xrightarrow{t} B \longrightarrow 0 \longrightarrow \dots$$

We want to show this is an acyclic complex. Note that $A^\bullet$ is a direct summand of

$A^\bullet \oplus A^\bullet[1]$. Consider the acyclic complex $X^\bullet$ given by

$$\begin{tikzcd}
B \arrow[rr, "\binom{1}{0}"] \arrow[dr, two heads, "\mathrm{Id}"'] & & B\oplus A \arrow[rr, "\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[dr, two heads, "(0\ 1)"'] & & A\oplus A \arrow[rr, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}"] \arrow[dr, two heads, "(1\ 0)"'] & & A\oplus B \arrow[rr, "(0\ 1)"] \arrow[dr, two heads, "(0\ 1)"'] & & B \\
& B \arrow[ur, tail, "\binom{1}{0}"'] & & A \arrow[ur, tail, "\binom{0}{1}"'] & & A \arrow[ur, tail, "\binom{1}{0}"'] & & B \arrow[ur, tail, "\mathrm{Id}"'] &
\end{tikzcd}$$

We claim that there is an isomorphism $f^\bullet : X^\bullet \to A^\bullet \oplus A^\bullet[1]$ given by

$$\begin{tikzcd}
B \arrow[r, "\binom{1}{0}"] \arrow[d, "\mathrm{Id}_B"'] & B\oplus A \arrow[r, "\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & -t \\ s & 1-st \end{pmatrix}"'] & A\oplus A \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} -1+st & st \\ st & 1-st \end{pmatrix}"'] & A\oplus B \arrow[r, "(0\ 1)"] \arrow[d, "\begin{pmatrix} 1-st & -s \\ t & 0 \end{pmatrix}"'] & B \arrow[d, "\mathrm{Id}_B"'] \\
B \arrow[r, "\binom{0}{s}"'] & B\oplus A \arrow[r, "\begin{pmatrix} -s & 0 \\ 0 & 1-st \end{pmatrix}"'] & A\oplus A \arrow[r, "\begin{pmatrix} -1+st & 0 \\ 0 & t \end{pmatrix}"'] & A\oplus B \arrow[r, "(-t\ 0)"'] & B
\end{tikzcd}$$

It is clear that this is a chain map by multiplying matrices. One can verify by multiplying matrices that the following is a chain map $g^\bullet : A^\bullet \oplus A^\bullet[1] \to X^\bullet$ that is an inverse of $f^\bullet$ Keeping in mind that $(st)^2 = st$.

$$\begin{tikzcd}
B \arrow[r, "\binom{0}{s}"] \arrow[d, "\mathrm{Id}"] & B\oplus A \arrow[r, "\begin{pmatrix} -s & 0 \\ 0 & 1-st \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & t \\ -s & 1-st \end{pmatrix}"'] & A\oplus A \arrow[r, "\begin{pmatrix} -1+st & 0 \\ 0 & t \end{pmatrix}"] \arrow[d, "\begin{pmatrix} st-1 & st \\ st & 1-st \end{pmatrix}"'] & A\oplus B \arrow[r, "(-t\ 0)"] \arrow[d, "\begin{pmatrix} 1-st & s \\ -t & 0 \end{pmatrix}"'] & B \arrow[d, "\mathrm{Id}"] \\
B \arrow[r, "\binom{1}{0}"'] & B\oplus A \arrow[r, "\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}"'] & A\oplus A \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}"'] & A\oplus B \arrow[r, "(0\ 1)"'] & B
\end{tikzcd}$$

Thus we see that $A^\bullet \oplus A^\bullet[1]$ is acyclic. As $\mathbf{Ac}^{\mathrm{b}}(\mathscr{A})$ is thick it follows that $A^\bullet$ is acyclic, furthermore t has a kernel by Remark 4.2.2. Thus we get are done by Lemma 2.1.10.

(3) $\Rightarrow$ (1): Let $A^\bullet \oplus B^\bullet$ be in $\mathbf{Ac}^+(\mathscr{A})$. We want to show $A^\bullet \in \mathbf{Ac}^+(\mathscr{A})$. Since $\mathcal{K}(\mathscr{A})$ is additive we get $f^\bullet : A^\bullet \to A^\bullet \oplus B^\bullet$, $s^\bullet : A^\bullet \oplus B^\bullet \to A^\bullet$ and morphisms $h^n : A^n \to A^{n-1}$ such that $s^n f^n - \mathrm{Id}_A^n = d_{A\oplus B}^{n-1} h^n + h^{n+1} d_A^n$. By replacing $A^\bullet \oplus B^\bullet$ by $A^\bullet \oplus B^\bullet \oplus (IA)^\bullet$ as in the proof of 4.2.7 we can assume $s^\bullet$ is a left inverse of $f^\bullet$ in $\mathbf{Ch}^+(\mathscr{A})$. Since $\mathscr{A}$ is weakly idempotent complete we get by Proposition 2.1.11 that each s^n is admissible epic and each f^n is admissible monic. Since $A^\bullet \oplus B^\bullet$ and $A^\bullet$ are both left bounded we can assume that $A^n = A^n \oplus B^n = 0$ for all $n < 0$. It follows that $d_{A\oplus B}^0 : A^0 \oplus B^0 \rightarrowtail A^1 \oplus B^1$ is admissible monic as $A^\bullet \oplus B^\bullet$ is acyclic. This gives us that $d_{A\oplus B}^0 f^0 = f^1 d_A^0$ is an admissible monic. By Proposition 2.1.11 we get that d_A^0 is an admissible monic. Let $e_A^1 : A^1 \to Z^2 A^\bullet$ be a cokernel of d_A^0 and let $e_{A\oplus B}^1 : A^1 \oplus B^1 \to Z^2 A^\bullet \oplus B^\bullet$ be a cokernel of $d_{A\oplus B}^1$. By Lemma A.2 we get the unique dotted morphisms making the following

diagram commute.

$$\begin{tikzcd}
A^0 \arrow[r, tail, "d^0_A"] \arrow[d, tail, "f^0"] & A^1 \arrow[r, two heads, "e^1_A"] \arrow[d, tail, "f^1"] & Z^2A \arrow[d, tail, dashed, "g^2"] \\
A^0 \oplus B^0 \arrow[r, tail, "d^0_{A\oplus B}"] \arrow[d, two heads, "s^0"] & A^1 \oplus B^1 \arrow[r, two heads, "e^1_{A\oplus B}"] \arrow[d, two heads, "s^1"] & Z^2A \oplus B \arrow[d, two heads, dashed, "t^2"] \\
A^0 \arrow[r, tail, "d^0_A"'] & A^1 \arrow[r, two heads, "e^1_A"'] & Z^2A
\end{tikzcd}$$

We see that $t^2g^2e^1_A = e^1_A s^1 f^1 = e^1_A \operatorname{Id}_{A_1} = \operatorname{Id}_{Z^2A} e^1_A$. Hence $t^2g^2 = \operatorname{Id}_{Z^2A}$ as e^1_A is epic. By Proposition 2.1.11 we get that t^2 is admissible epic and g^2 is admissible monic. Since $A^\bullet \oplus B^\bullet$ is acyclic we have a unique admissible monic $m^2_{A\oplus B} : Z^2A \oplus B \rightarrowtail A^2 \oplus B^2$ such that $m^2_{A\oplus B}e^1_{A\oplus B} = d^1_{A\oplus B}$. Furthermore as $A^\bullet$ is a chain complex we get by the cokernel property a unique morphism $m^2_A : Z^2A \to A^2$ such that $m^2_A e^1_A = d^1_A$. Now we have $f^2m^2_Ae^1_A = f^2d^1_A = d^1_{A\oplus B}f^1 = m^2_{A\oplus B}e^1_{A\oplus B}f^1 = m^2_{A\oplus B}g^2e^1_A$. Hence we get $f^2m^2_A = m^2_{A\oplus B}g^2$ as e^1_A is epic. Similarly $m^2_A t^2 = s^2m^2_{A\oplus B}$ as $e^1_{A\oplus B}$ is epic. Now we have the commutative diagram

$$\begin{tikzcd}
Z^2A \arrow[r, "m^2_A"] \arrow[d, tail, "g^2"] & A^2 \arrow[d, tail, "f^2"] \\
Z^2A \arrow[r, tail, "m^2_{A\oplus B}"] \arrow[d, two heads, "t^2"] & A^2 \oplus B^2 \arrow[d, two heads, "s^2"] \\
Z^2A \arrow[r, "m^2_A"'] & A^2
\end{tikzcd}$$

By the commutative in the top square and Proposition 2.1.11 we get that m^2_A is admissible monic. By induction we now see that $A^\bullet$ is acyclic. Showing that $\mathbf{Ac}^-(\mathscr{A})$ is thick in $\mathcal{K}^-(\mathscr{A})$ is dual. ∎

4.4 The derived category

In this section we define quasi isomorphism for exact categories and the derived category. We will see that the definitions makes the most sense when $\mathscr{A}$ is idempotent complete. We will also see that the derived category of $(\mathscr{A}, \mathscr{E})$ where $\mathscr{E}$ is split exact sequences gives back the homotopy category. Lastly we go through a result needed at the end of the thesis where we look at examples.

Definition 4.4.1. A chain map is called a *quasi isomorphism* if its mapping cone is homotopy equivalent to an acyclic complex.

Example 4.4.2. Let $p : A \to A$ be an idempotent in an exact category $(\mathscr{A}, \mathscr{E})$ which does not split. Then the complex $A^\bullet$ given by

$$... \xrightarrow{1-p} A \xrightarrow{p} A \xrightarrow{1-p} A \xrightarrow{p} ...$$

is null homotopic but *not* acyclic. We see that $f^\bullet : 0^\bullet \to A^\bullet$ is a chain homotopy equivalence, thus a quasi isomorphism but $\mathrm{Cone}(f) = A^\bullet$ fails to be acyclic. Working in an idempotent complete category "fixes" issues like this as all null-homotopic complexes will be acyclic.

Example 4.4.3. If $\mathscr{A}$ is idempotent complete we get by Proposition 4.2.8 that a chain map f is a quasi isomorphism if and only if $\mathrm{Cone}(f)$ is acyclic.

As we have seen $\mathbf{Ac}(\mathscr{A})$ is a triangulated subcategory of $\mathcal{K}(\mathscr{A})$, where given $f \in \mathbf{Ac}(\mathscr{A})$ we also have $\mathrm{Cone}(f) \in \mathbf{Ac}(\mathscr{A})$ by Lemma 4.2.3. We see that $\mathbf{Ac}(\mathscr{A})$ fits into Definition 3.2.2, and thus $S(\mathbf{Ac}(\mathscr{A}))$ is a multiplicative system by Proposition 3.2.3. Recalling the Verdier localization in Definition 3.2.4 we have the ability to make sense of the following definition.

Definition 4.4.4. The *derived category* of the exact category $\mathscr{A}$ is defined to be the Verdier quotient

$$\mathcal{D}(\mathscr{A}) = \mathcal{K}(\mathscr{A}) / \mathbf{Ac}(\mathscr{A})$$

Remark 4.4.5. We note the following

1. Construction 3.1.3 hold in $\mathcal{D}(\mathscr{A})$.

2. Considering Example 4.4.2 we see the definition can be hard to unravel if $\mathscr{A}$ fails to be idempotent complete.

3. If $\mathscr{A}$ is idempotent complete then a chain map becomes and isomorphism in $\mathcal{D}(\mathscr{A})$ if and only if its cone is acyclic by Proposition 4.2.8.

Example 4.4.6. Let $(\mathscr{A}, \mathscr{E})$ be an exact category where $\mathscr{E}$ is all split exact sequences. Let $A^\bullet$ be the acyclic complex

$$\begin{tikzcd}
\dots \arrow[r] & A^{n-1} \arrow[rr, "d_A^{n-1}"] \arrow[dr, two heads, "e_A^{n-1}"'] & & A^n \arrow[rr, "d_A^n"] \arrow[dr, two heads, "e_A^n"'] & & A^{n+1} \arrow[r] & \dots \\
& & Z^nA \arrow[ur, tail, "i_A^n"'] & & Z^{n+1}A \arrow[ur, tail, "i_A^{n+1}"'] & &
\end{tikzcd}$$

We know $Z^jA \rightarrowtail A^j \twoheadrightarrow Z^{j+1}A$ is in $\mathscr{E}$ for all j. Thus it is an split exact sequence. This gives us that $A^j = Z^jA \oplus Z^{j+1}A$, $i_A^j = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, $e_A^j = \left(\begin{smallmatrix}0&1\end{smallmatrix}\right)$ for all j. This gives us $d_A^j = i_A^{j+1}e_A^j = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ for all j. Now consider the homotopy

$$\begin{tikzcd}
\dots \arrow[r] & Z^{n-1}A \oplus Z^nA \arrow[r, "\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)"] \arrow[d, "\mathrm{Id}"'] & Z^nA \oplus Z^{n+1}A \arrow[r, "\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)"] \arrow[d, "\mathrm{Id}"'] \arrow[dl, "\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)"'] & Z^{n+1}A \oplus Z^{n+2}A \arrow[r] \arrow[d, "\mathrm{Id}"'] \arrow[dl, "\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)"'] & \dots \\
\dots \arrow[r] & Z^{n-1}A \oplus Z^nA \arrow[r, "\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)"] & Z^nA \oplus Z^{n+1}A \arrow[r, "\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)"] & Z^{n+1}A \oplus Z^{n+2}A \arrow[r] & \dots
\end{tikzcd}$$

We see that $\mathrm{Id}_{A^\bullet}$ is null homotopic, thus $A^\bullet \cong 0^\bullet$ in $\mathcal{K}(\mathscr{A})$. This yields the result

$$\mathcal{D}(\mathscr{A}, \mathscr{E}) \cong \mathcal{K}(\mathscr{A})$$

Lemma 4.4.7. Let $A^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathscr{A})$ be such that $\mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, A^\bullet[i]) = 0$ for all acyclic complexes $E^\bullet$. Then every quasi-isomorphism $s^\bullet : A^\bullet \to B^\bullet$ is monic.

Proof. As $s : A^\bullet \to B^\bullet$ is a quasi isomorphism the cone is acyclic. We have the following sequence from triangles.

$$\mathrm{Cone}(s^\bullet)[-1] \xrightarrow{0^\bullet} A^\bullet \xrightarrow{s^\bullet} B^\bullet \longrightarrow \mathrm{Cone}(s^\bullet) \xrightarrow{0^\bullet} A^\bullet[1]$$

As morphisms in triangles are weak kernels and cokernels we are done. ■

Definition 4.4.8. Let $\mathcal{T}$ be a triangulated category. Let $\mathscr{S}$ be a full triangulated subcategory. An $\mathscr{S}$ *approximation* of an object $T \in \mathcal{T}$ is an object $S \in \mathscr{S}$ together with a morphism $f : S \to T$ with the following property. For all $S' \in \mathscr{S}$ with a morphism $S' \to T$ we can find a morphism $S' \to S$ such that the following commutes.

$$\begin{tikzcd}
S \arrow[rr] & & T \\
& S' \arrow[ul, dashed] \arrow[ur] &
\end{tikzcd}$$

Example 4.4.9. Let $(\mathscr{A}, \mathscr{E})$ be an idempotent complete category where $\mathscr{E}$ has finitely many indecomposable sequences and all Hom-sets have finite dimension. Then there exist an $\mathbf{Ac}(\mathscr{A})$-approximation for all $T \in \mathcal{K}^{\mathrm{b}}(\mathscr{A})$. One can simply take

$$\bigoplus_{E \text{ acyclic indecomposable}} \; \bigoplus_{\text{Basis of } \mathrm{Hom}(E,X)} E \to T$$

.

Proposition 4.4.10. Let $(\mathscr{A}, \mathscr{E})$ be idempotent complete where $\mathscr{E}$ has finitely many indecomposable sequences and all Hom-sets have finite dimension. Let

$$\mathscr{H} = \{A^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathscr{A}) \,|\, \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, A^\bullet[i]) = 0 \text{ for all } E^\bullet \in \mathbf{Ac}^{\mathrm{b}}(\mathscr{A})\}$$

Then the canonical functor $Q : \mathscr{H} \to \mathscr{D}^{\mathrm{b}}(\mathscr{A})$ is an equivalence.

Proof. We show that functor is full faithfull and dense. Consider

$$\phi : \mathrm{Hom}_{\mathscr{H}}(A^\bullet, B^\bullet) \to \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\mathscr{A})}(A^\bullet, B^\bullet)$$

Let $(f^\bullet, s^\bullet) \in \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\mathscr{A})}(A^\bullet, B^\bullet)$. We know $s^\bullet$ has left inverse s' by Lemma 3.3.4. Using the diagram

$$\begin{tikzcd}
 & X^\bullet \arrow[d, "s'^\bullet"] & \\
A^\bullet \arrow[ur, "f^\bullet"] \arrow[r, "s'^\bullet f^\bullet"] \arrow[dr, "s'^\bullet f^\bullet"'] & B^\bullet & B^\bullet \arrow[ul, "s^\bullet"'] \arrow[l, "\mathrm{Id}"'] \arrow[dl, "\mathrm{Id}"] \\
 & B^\bullet \arrow[u, "\mathrm{Id}"] &
\end{tikzcd}$$

we see $(f^\bullet, s^\bullet) = (s'^\bullet f^\bullet, \mathrm{Id}_{B^\bullet})$ in $\mathscr{D}^{\mathrm{b}}(\mathscr{A})$. Thus $\phi(s'^\bullet f^\bullet) = (f^\bullet, s^\bullet)$ and our functor is full. To see it is faithfull let $\phi(f^\bullet) = 0^\bullet$. Then we have the diagram

$$\begin{tikzcd}
 & B^\bullet \arrow[d, "s^\bullet"] & \\
A^\bullet \arrow[ur, "f^\bullet"] \arrow[r, "0"] \arrow[dr, "0"'] & X^\bullet & B^\bullet \arrow[ul, "\mathrm{Id}"'] \arrow[l, "s^\bullet"'] \arrow[dl, "\mathrm{Id}"] \\
 & B^\bullet \arrow[u, "s^\bullet"] &
\end{tikzcd}$$

and see $s^\bullet f^\bullet = 0$. By Lemma 4.4.7 we have that $s^\bullet$ monic in this case. Thus $f^\bullet = 0$ and the functor is faithfull. Lastly we show it is dense. For convenience we denote $(f^\bullet \circ -)$ by f^*. Let $A^\bullet \in \mathscr{D}^{\mathrm{b}}(\mathscr{A})$. Consider an $\mathbf{Ac}(\mathscr{A})$-approximation $f^\bullet : T^\bullet \to A^\bullet$ in $\mathcal{K}^{\mathrm{b}}(\mathscr{A})$. Then we have a triangle

$$T^\bullet \xrightarrow{f^\bullet} A^\bullet \xrightarrow{g^\bullet} \mathrm{Cone}(f) \xrightarrow{h^\bullet} T^\bullet[1]$$

Let $E^\bullet \in \mathbf{Ac}^{\mathrm{b}}(\mathscr{A})$. We claim the following sequence is exact.

$$\mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, \mathrm{Cone}(f^\bullet)[-1]) \xrightarrow{-h[-1]^*} \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, T^\bullet) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, A^\bullet) \to 0 \tag{4.4.1}$$

f^* is epic as $f^\bullet$ is an approximation. The sequence is exact in the middle as Hom is homological (Proposition B.9). Thus our claim holds. By Lemma 4.2.3 Cone(f) is acyclic. This also gives that Cone$(f)[-1]$ is acyclic. Thus we have an $\mathbf{Ac}^{\mathrm{b}}(\mathscr{A})$-approximation $r^\bullet : S^\bullet \to \mathrm{Cone}(f^\bullet)[-1]$. This gives us the exact sequence

$$\mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, S^\bullet) \xrightarrow{r^*} \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, \mathrm{Cone}(f^\bullet)[-1]) \to 0 \tag{4.4.2}$$

By combining the two sequences (4.4.1) and (4.4.2) we obtain the following exact sequence.

$$\mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, S^\bullet) \xrightarrow{(-h[-1]r)^*} \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, T^\bullet) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, A^\bullet) \to 0 \tag{4.4.3}$$

Now consider the triangle

$$S^\bullet \xrightarrow{-h^\bullet[-1]r^\bullet} T^\bullet \xrightarrow{s^\bullet} \mathrm{Cone}(-h^\bullet[-1]r^\bullet) \to S^\bullet[1]$$

which yields the following exact sequence as Hom is homological.

$$\mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, S^\bullet) \to \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, T^\bullet) \to \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, \mathrm{Cone}(-h^\bullet[-1]r^\bullet))$$

By recalling the sequence (4.4.3) we observe that $\mathrm{Im}(s^*) \cong \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, A^\bullet)$ and get the monomorphism $u^* : \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, A^\bullet) \to \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, \mathrm{Cone}(-h^\bullet[-1]r^\bullet)$. Next we note $s^\bullet$ is a weak cokernel of $-h^\bullet[-1]r^\bullet$ thus there exists a morphism $a^\bullet$ such that $a^\bullet s^\bullet = f^\bullet$. Furthermore we note that $\mathrm{Cone}(-h^\bullet[-1]r^\bullet)$ is acyclic as $S^\bullet$ and $T^\bullet$ are acyclic. As $f^\bullet$ is an $\mathbf{Ac}(\mathscr{A})$-approximation this gives us that there exists $t^\bullet : \mathrm{Cone}(-h^\bullet[-1]r^\bullet) \to T^\bullet$ such that $a^\bullet = t^\bullet f^\bullet$. Now we have the following diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, T^\bullet) \arrow[rr, "s^*"] \arrow[dr, two heads, "f^*"'] & & \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, \mathrm{Cone}(-h^\bullet[-1]r^\bullet)) \arrow[ll, bend right, "t^*"'] \arrow[dl, bend left, "a^*"] \\
& \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(E^\bullet, A^\bullet) \arrow[ur, tail, "u^*"] &
\end{tikzcd}$$

where $s^* = u^* f^*$, $a^* = f^* t^*$ and $f^* = a^* s^*$. This gives us that $f^* t^* u^* f^* = a^* s^* = f^*$. As f^* is epic this yields $f^* t^* u^* = \mathrm{Id}$. As everything is natural for acyclic complexes this implies that $f^* : \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}}(-, T) \to \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}}(-, A)$ is a split epic of functors $\mathbf{Ac}^{\mathrm{b}}(\mathscr{A})^{\mathrm{op}} \to \mathbf{Ab}$, where the splitting is given by $t^* u^*$. As $(\mathscr{A}, \mathscr{E})$ is idempotent complete and $t^* u^* f^*$ is idempotent we get $\mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}}(-, T^\bullet) \cong \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}}(-, T_1^\bullet) \oplus \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}}(-, T_2^\bullet)$. Note that $T_2^\bullet$ is acyclic as $\mathbf{Ac}^{\mathrm{b}}(\mathscr{A})$ is thick by Corollary 4.2.9. That our category is idempotent complete also gives us the commutative diagram.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}}(-, T_1^\bullet) \oplus \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}}(-, T_2^\bullet) \arrow[r, two heads, "f^*"'] \arrow[rr, bend left, "\left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{smallmatrix}\right)"] & \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}}(-, A^\bullet) \arrow[r, tail, "t^* u^*"'] & \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}}(-, T_1^\bullet) \oplus \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}}(-, T_2^\bullet)
\end{tikzcd}$$

This gives us a natural isomorphism $\mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(-, A^\bullet) \cong \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathscr{A})}(-, T_2^\bullet)$ on functors $\mathbf{Ac}^{\mathrm{b}}(\mathscr{A})^{\mathrm{op}} \to \mathbf{Ab}$ and we are done by Yoneda lemma. ∎

5 Classification of Exact Structures

In this section we will find a classification of exact structures. Our main theorem will be a bijection between exact structures on an idempotent complete category $\mathscr{A}$ and certain subcategories of the module category of $\mathscr{A}$. Furthermore we will look at the theorem in a less general case.

For the rest of the thesis we assume that all categories are skeletally small, that is the isomorphism classes of objects form a set. Furthermore all subcategories are assumed to be full and closed under isomorphisms.

5.1 Module functor categories

This subsection is based on Enomotos article [4, Chapter 2.1] and Prest's article [6, Chapter 1.2].

Before we are able to work our way to classifying exact structures we need some knowledge about module functor categories. In this subsection we go through the basics and see how it relates to kernel-cokernel pairs.

Definition 5.1.1. Let $\mathscr{A}$ be an additive category. A *right $\mathscr{A}$-module M* is a contravariant additive functor $M : \mathscr{A}^{op} \to \mathbf{Ab}$. Where $\mathbf{Ab}$ denotes abelian groups.

Proposition 5.1.2. The class of right $\mathscr{A}$ modules with natural transformations as morphisms forms an abelian category.

Proof. Easily follows from the fact that $\mathbf{Ab}$ is abelian. ■

Definition 5.1.3. The abelian category of right $\mathscr{A}$ modules will be denoted $\mathbf{Mod}\,\mathscr{A}$.

Lemma 5.1.4. $\operatorname{Hom}_{\mathscr{A}}(-, A)$ is a projective object of $\mathbf{Mod}\,\mathscr{A}$ for all $A \in \mathscr{A}$.

Proof. Let $\pi : M \twoheadrightarrow N$ be an epic, and $f : \operatorname{Hom}_{\mathscr{A}}(-, A) \to N$ a morphism in $\mathbf{Mod}\,\mathscr{A}$. By Yoneda lemma f corresponds to some $a \in N(A)$. As π is epic there exists $b \in M(A)$ mapping to a. Let $g \in \operatorname{Hom}(\operatorname{Hom}(-, A), M)$ Yoneda correspond to b. Now we have found g such that $\pi g = f$ and we are done. ■

Proposition 5.1.5. $\mathbf{Mod}\,\mathscr{A}$ has enough projectives.

Proof. Let $M : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab} \in \mathbf{Mod}\,\mathscr{A}$. Let $[A]$ be the isomorphism class represented by $A \in \mathscr{A}$. Define the morphism $\pi : \bigoplus_{[A]\in\mathscr{A}} \bigoplus_{a\in MA} \mathrm{Hom}_{\mathscr{A}}(-, A) \to M$ to have component at a the morphism $f_a : \mathrm{Hom}_{\mathscr{A}}(-, A) \to MA$ which Yoneda corresponds to a. As $\mathrm{Hom}_{\mathscr{A}}(-, A)$ is projective, so is $\bigoplus_{[A]\in\mathscr{A}} \bigoplus_{a\in MA} \mathrm{Hom}_{\mathscr{A}}(-, A)$ and π is epic by construction. ■

Proposition 5.1.6. In $\mathbf{Mod}\,\mathscr{A}$ The projective objects are precisely direct summands of direct sums of representable functors.

Proof. Let $P : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ be a projective object in $\mathbf{Mod}\,\mathscr{A}$. By the construction in last proposition we have epic morphism $\pi : \bigoplus_{[A]\in\mathscr{A}} \bigoplus_{a\in PA} \mathrm{Hom}_{\mathscr{A}}(-, A) \twoheadrightarrow P$. As P is projective we have that there exist f_1 such that the following commute.

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "f_1"'] \arrow[d, equal] \\
\bigoplus_{[A]\in\mathscr{A}} \bigoplus_{a\in PA} \mathrm{Hom}_{\mathscr{A}}(-, A) \arrow[r, two heads, "\pi"] & P
\end{tikzcd}$$

If we show f_1 is split monic, we are done as P then will be a direct summand of $\bigoplus_{[A]\in\mathscr{A}} \bigoplus_{a\in PA} \mathrm{Hom}_{\mathscr{A}}(-, A)$. We easily see this as $\mathrm{Id}_P = \pi f_1$. ■

Remark 5.1.7. We will use the notation $P_X := \mathrm{Hom}_{\mathscr{A}}(-, X)$ and $P^X := \mathrm{Hom}_{\mathscr{A}}(X, -)$.

Definition 5.1.8. An $\mathscr{A}$ module M is called *finitely generated* if there exists an epic $P_X \twoheadrightarrow M$ for some $X \in \mathscr{A}$. We denote the category of finitely generated $\mathscr{A}$-modules $\mathbf{mod}\,\mathscr{A}$.

Proposition 5.1.9. $\mathscr{A}$ is idempotent complete if and only if the essential image of the Yoneda embedding $P_{(-)} : \mathscr{A} \to \mathbf{Mod}\,\mathscr{A}$ consist of all finitely generated projective $\mathscr{A}$-modules.

Proof. Let $\mathscr{A}$ be idempotent complete. We show that finitely generated projective modules is representable up to isomorphism, hence in the essential image of $P_{(-)}$. Let $M \in \mathbf{Mod}\,\mathscr{A}$ be finitely generated and projective. By Proposition 5.1.6 we get that M is a direct summand of some $P_A \in \mathbf{Mod}\,\mathscr{A}$. Hence we have a projection $\pi : P_A \twoheadrightarrow M$ and an inclusion $i : M \hookrightarrow P_A$. Let $f = \pi i$, then $f^2 = f$. Since the Yoneda embedding $P_{(-)}$ is fully faithfull we have a bijection between $\mathrm{End}(A)$ and $\mathrm{End}(P_A)$. This gives us that there exists some $e \in \mathrm{End}(A)$ such that $P_e = f$ and $e = e^2$. Since $\mathscr{A}$ is idempotent complete we get by Remark 2.1.3 that e is isomorphic to $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) : \mathrm{Ker}(e) \oplus \mathrm{Im}(e) \to \mathrm{Ker}(e) \oplus \mathrm{Im}(e)$. Hence we have the following diagram.

$$\begin{tikzcd} P_{\mathrm{Ker}(e)} \oplus P_{\mathrm{Im}(e)} \arrow[r, two heads, "\pi"] \arrow[rr, bend right, "\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] & M \arrow[r, tail, "i"] & P_{\mathrm{Ker}(e)} \oplus P_{\mathrm{Im}(e)} \end{tikzcd}$$

Which means $\pi = \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)$, $i = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and $M \cong \mathrm{Hom}_{\mathscr{A}}(-, \mathrm{Im}(e))$. Conversely let $e : A \to A$ be an idempotent in $\mathscr{A}$. Then we have the idempotent $P_e : P_A \to P_A$ in $\mathbf{Mod}\,\mathscr{A}$. We can factor P_e in the following way.

$$\begin{tikzcd} P_A \arrow[rr, "P_e"] \arrow[dr, two heads, "\pi"'] & & P_A \\ & \mathrm{Im}(P_A) \arrow[ur, tail, "i"'] & \end{tikzcd}$$

Since P_e is idempotent we have $i\pi i\pi = i\pi$, consequently $\pi i = \mathrm{Id}_{\mathrm{Im}(P_A)}$ as i is monic and π is epic. Hence $\mathrm{Im}(P_A)$ is a direct summand of P_A. By Proposition 5.1.6 $\mathrm{Im}(P_A)$ is projective and it is clearly finitely generated. Hence $\mathrm{Im}(P_A) = P_B$ for some $B \in \mathscr{A}$. This gives us that $P_A \cong P_{X \oplus B}$, $i = P_f$ and $\pi = P_g$ for some $X, f, g \in \mathscr{A}$. Now it follows by Yoneda lemma that $A \cong X \oplus B$, $gf = \mathrm{Id}_A$ and $i\pi = e$. This also gives $e \cong \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Id}_B \end{smallmatrix}\right)$. ■

Definition 5.1.10. A finitely presented $\mathscr{A}$ module M is an $\mathscr{A}$ module such that there exists an exact sequence $P_X \to P_Y \to M \to 0$. We denote the category of finitely presented $\mathscr{A}$ modules $\mathbf{mod}_1\,\mathscr{A}$.

Definition 5.1.11. We define the contravariant functor $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{A}) : \mathbf{Mod}\,\mathscr{A} \to \mathbf{Mod}\,\mathscr{A}^{\mathrm{op}}$ by sending an $\mathscr{A}$-module M to the following composition.

$$\begin{aligned} \mathrm{Hom}_{\mathscr{A}}(M, \mathscr{A}) : \quad & \mathscr{A} \longrightarrow \mathbf{Mod}\,\mathscr{A} \longrightarrow \mathbf{Ab} \\ & X \longmapsto P_X \longmapsto \mathrm{Hom}_{\mathbf{Mod}\,\mathscr{A}}(M, P_X) \end{aligned}$$

Where $\mathscr{A} \to \mathbf{Mod}\,\mathscr{A}$ is the Yoneda embedding. This is an left exact functor by the fact that the Hom functor is left exact.

Proposition 5.1.12. $\mathrm{Hom}_{\mathscr{A}}(P_{(-)}, \mathscr{A}) \cong P^{(-)}$ and $\mathrm{Hom}_{\mathscr{A}}(P^{(-)}, \mathscr{A}) \cong P_{(-)}$

Proof. By Yoneda lemma we have $\mathrm{Hom}_{\mathscr{A}}(P_X, \mathscr{A}) = \mathrm{Hom}(P_X, P_{(-)}) \cong P^X$ for the first and $\mathrm{Hom}_{\mathscr{A}}(P^X, \mathscr{A}) = \mathrm{Hom}(P^X, P_{(-)}) \cong P_X$ for the second. ■

Definition 5.1.13. We denote by $\mathrm{Ext}^i_{\mathscr{A}}(-, \mathscr{A}) : \mathbf{Mod}\,\mathscr{A} \to \mathbf{Mod}\,\mathscr{A}^{\mathrm{op}}$ the i-th right derived functor of $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{A})$.

With these concepts we are ready to interpret kernel-cokernel pairs in terms of modules over $\mathscr{A}$.

Proposition 5.1.14. Consider a complex $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathscr{A}$. Put $M := \mathrm{Coker}(P_g)$ in $\mathbf{Mod}(\mathscr{A})$. Then the following hold.

(1) g is epic in $\mathscr{A}$ if and only if $\mathrm{Hom}_{\mathscr{A}}(M, \mathscr{A}) = 0$.

(2) $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a kernel-cokernel pair if and only if the following are satisfied.

   (a) $0 \to P_X \xrightarrow{P_f} P_Y \xrightarrow{P_g} P_Z \to M \to 0$ is exact.

   (b) $\mathrm{Ext}^i_{\mathscr{A}}(M, \mathscr{A}) = 0$ for $i = 0, 1$.

Proof. (1) : The sequence $P_Y \xrightarrow{P_g} P_Z \twoheadrightarrow M \to 0$ is exact by assumption. Applying $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{A})$ and recalling Proposition 5.1.12 we get the following exact sequence.

$$0 \to \mathrm{Hom}_{\mathscr{A}}(M, \mathscr{A}) \to P^Z \xrightarrow{P^g} P^Y$$

Thus we see $\mathrm{Hom}_{\mathscr{A}}(M, \mathscr{A}) = 0$ if and only if P^g is monic. As Hom is left exact we see P^g is monic if and only if g is epic and we are done.

(2) : The condition (a) is equivalent with f being a kernel of g as Hom is left exact. Let (a) hold. If we consider the sequence

$$0 \to M \to P^Z \xrightarrow{P^g} P^Y \xrightarrow{P^f} P^X \to \mathrm{Ext}^1_{\mathscr{A}}(M, \mathscr{A}) \to ...$$

We see that (b) is equivalent to $0 \to P^Z \xrightarrow{P^g} P^Y \xrightarrow{P^f} P^X \to 0$ being exact. Similarly to argumentation in (1) this holds exactly when g is a cokernel of f.

■

The rest of this subsection will be used to go through Schanuel's lemma and its generalised version in order to prove a technical lemma. This lemma will be needed later in the proof of the main theorem in this section.

Lemma 5.1.15 (Schanuel's)**.** Consider the exact sequences $0 \to A \to P \to B \to 0$ and $0 \to A' \to P' \to B' \to 0$ in an Abelian (Exact) category. If P and P' are projective and $B = B'$ then $A \oplus P' \cong A' \oplus P$

Proof. We construct the following diagram where PB is a pullback.

$$\begin{tikzcd} & A' \arrow[r, equal] \arrow[d, tail] & A' \arrow[d, tail] \\ A \arrow[r, tail] \arrow[d, equal] & PB \arrow[r, two heads, "f"] \arrow[d, two heads, "g"'] & P' \arrow[d, two heads] \\ A \arrow[r, tail] & P \arrow[r, two heads] & B = B' \end{tikzcd}$$

Where both vertical and horizontal sequences are exact by assumption and Lemma A.1. Since P' is projective there exist h making the following commute.

$$\begin{tikzcd} & P' \arrow[dl, dashed, "h"'] \arrow[d, "\mathrm{Id}_{P'}"] \\ PB \arrow[r, "f"'] & P' \end{tikzcd}$$

Hence f is split epic, we have a split exact sequence and $PB \cong A \oplus P'$. Similarly g is split epic and $PB \cong A' \oplus P$ ∎

Lemma 5.1.16 (Generalised Schanuel's lemma)**.** Let $\mathscr{A}$ be an Exact category. Given two exact sequences

$$0 \longrightarrow K \xrightarrow{f_{-1}} P_0 \xrightarrow{f_0} P_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} P_n \xrightarrow{f_n} M \longrightarrow 0$$

$$0 \longrightarrow K' \xrightarrow{f'_{-1}} P'_0 \xrightarrow{f'_0} P'_1 \xrightarrow{f'_1} \dots \xrightarrow{f'_{n-1}} P'_n \xrightarrow{f'_n} M \longrightarrow 0$$

where P_i and P'_i are projective for all i then we have

$$K' \oplus P_0 \oplus P'_1 \oplus P_2 \oplus \dots \cong K \oplus P'_0 \oplus P_1 \oplus P'_2 \oplus \dots$$

Proof. We do induction on n. The case $n = 1$ is Schanuel's lemma. Suppose the lemma hold for all $k < n$. Consider the following diagram where PB is a pullback of f_{n-1} and $(\,1\ 0\,)$. We want the long sequence below to be exact.

$$\begin{tikzcd} & & & & P_{n-1} \arrow[r, "f_{n-1}"] & P_n \arrow[r, two heads, "f_n"] & M \arrow[r] \arrow[d, equal] & 0 \\ K \arrow[r, "f_{-1}"] & \dots \arrow[r] & P_{n-3} \arrow[r, "f_{n-3}"] & P_{n-2} \arrow[r, "h"] & PB \arrow[r, "g"] \arrow[u, two heads, "p"] & P_n \oplus P'_n \arrow[r, two heads, "(f_n\ f'_n)"] \arrow[u, two heads, "(1\ 0)"'] & M \arrow[r] & 0 \\ & & & & P'_n \arrow[r] \arrow[u, "i"] & P'_n \arrow[u, "\binom{0}{1}"'] & & \end{tikzcd}$$

Since $\begin{pmatrix} 1 & 0 \end{pmatrix}$ is split epic so is p using a similar argument as in our previous proof. Furthermore by A.1 we obtain the lower square of the diagram where i is a kernel of p. Hence the leftmost vertical sequence is split and $PB \cong P_{n-1} \oplus P'_n$. By commutative of the diagrams we now get $g = \begin{pmatrix} f_{n-1} & 0 \\ 0 & 1 \end{pmatrix}$ and we choose $h = \begin{pmatrix} f_{n-2} \\ 0 \end{pmatrix}$ to get an exact sequence. Similarly we get the exact sequence

$$K' \xrightarrow{f_{-1}} \dots \longrightarrow P'_{n-3} \xrightarrow{f_{n-3}} P'_{n-2} \xrightarrow{\begin{pmatrix} f'_{n-2} \\ 0 \end{pmatrix}} P'_{n-1} \oplus P_n \xrightarrow{\begin{pmatrix} 0 & 1 \\ f'_{n-1} & 0 \end{pmatrix}} P_n \oplus P'_n \xrightarrow{\begin{pmatrix} f_n & f'_n \end{pmatrix}} M \longrightarrow 0$$

Now we have the diagram

$$\begin{tikzcd}
K \arrow[r] & \dots \arrow[r] & P_{n-2} \arrow[r] & P_{n-1} \oplus P'_n \arrow[rr] \arrow[dr, two heads] & & P_n \oplus P'_n \arrow[r, two heads, "\begin{pmatrix} f_n & f'_n \end{pmatrix}"] \arrow[dd, equal] & M \arrow[r, two heads] \arrow[dd, equal] & 0 \\
& & & & \mathrm{Ker}\begin{pmatrix} f_n & f'_n \end{pmatrix} \arrow[ur, tail] \arrow[dd, equal] & & & \\
K' \arrow[r] & \dots \arrow[r] & P'_{n-2} \arrow[r] & P'_{n-1} \oplus P_n \arrow[rr] \arrow[dr, two heads] & & P_n \oplus P'_n \arrow[r, two heads, "\begin{pmatrix} f_n & f'_n \end{pmatrix}"] & M \arrow[r, two heads] & 0 \\
& & & & \mathrm{Ker}\begin{pmatrix} f_n & f'_n \end{pmatrix} \arrow[ur, tail] & & &
\end{tikzcd}$$

and the claim follows from the induction hypothesis. ■

Lemma 5.1.17. Let $\mathscr{A}$ be idempotent complete. Suppose there exists an exact sequence

$$0 \to P_X \to P_Y \to P_Z \to M \to 0$$

in $\mathbf{Mod}\,\mathscr{A}$. Then for any morphism $h : B \to C$ with $\mathrm{Coker}(P_h) = M$ there exists $A \in \mathscr{A}$ such that $\mathrm{Ker}(P_h) \cong P_A$.

Proof. By Generalised Schanuel's lemma we have $\mathrm{Ker}(P_h) \oplus P_Y \oplus P_C \cong P_X \oplus P_B \oplus P_Z$. Hence we get $\mathrm{Ker}(P_h) \oplus P_{Y \oplus C} \cong P_{X \oplus B \oplus Z}$ and $\ker(P_h)$ is projective by Proposition 5.1.6. It is also finitely generated as there is an canonical epic $P_{X \oplus B \oplus Z} \twoheadrightarrow \mathrm{Ker}(P_h)$. Since $\mathscr{A}$ is idempotent complete the claim now follows by Proposition 5.1.9. ■

5.2 Construction of maps

This subsection is based on Enomoto [4, Chapter 2.2] and Kimura [7, Chapter 2].

In order to classify exact structures through the module category we will construct maps between a class of subcategories of $\mathbf{Mod}\,\mathscr{A}$ and kernel-cokernel pairs of $\mathscr{A}$. Then we will find a mutually inverse bijection between a class of subcategories in $\mathbf{mod}\,\mathscr{A}$ and a class of kernel-cokernel pairs in $\mathscr{A}$. We start with a reformulation of Proposition 5.1.14.

Lemma 5.2.1. For an object $M \in \mathbf{Mod}\,\mathscr{A}$ the following are equivalent.

(1) There exists a kernel cokernel pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathscr{A}$ such that M is isomorphic to $\mathrm{Coker}(P_g)$.

(2) There exists an exact sequence $0 \to P_X \to P_Y \to P_Z \to M \to 0$ in $\mathbf{Mod}\,\mathscr{A}$ and $\mathrm{Ext}^i_{\mathscr{A}}(M, \mathscr{A}) = 0$ for $i = 0, 1$.

Definition 5.2.2. We denote the subcategory of $\mathbf{Mod}\,\mathscr{A}$ consisting of $\mathscr{A}$ modules satisfying the equivalent conditions above by $\mathcal{C}_2(\mathscr{A})$.

Before we start working our way to constructing maps we note that there is a duality between $\mathcal{C}_2(\mathscr{A})$ and $\mathcal{C}_2(\mathscr{A}^{\mathrm{op}})$.

Lemma 5.2.3. The functor $\mathrm{Ext}^2_{\mathscr{A}}(-, \mathscr{A})$ induces an duality $\mathcal{C}_2(\mathscr{A}) \to \mathcal{C}_2(\mathscr{A}^{\mathrm{op}})$.

Proof. Let $f : M \to N$ be a morphism in $\mathcal{C}_2(\mathscr{A})$. This gives us the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & P_C \arrow[r] \arrow[d] & P_B \arrow[r] \arrow[d] & P_A \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & P_Z \arrow[r] & P_Y \arrow[r] & P_X \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Recalling 5.1.12 we get the following diagram for $\mathrm{Ext}^2_{\mathscr{A}}(f, \mathscr{A})$.

$$\begin{tikzcd}
0 \arrow[r] & P^A \arrow[r] & P^B \arrow[r] & P^C \arrow[r] & \mathrm{Ext}^2_{\mathscr{A}}(M, \mathscr{A}) \arrow[r] & 0 \\
0 \arrow[r] & P^X \arrow[r] \arrow[u] & P^Y \arrow[r] \arrow[u] & P^Z \arrow[r] \arrow[u] & \mathrm{Ext}^2_{\mathscr{A}}(N, \mathscr{A}) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

We note $\mathrm{Ext}^2_{\mathscr{A}}(M, \mathscr{A}), \mathrm{Ext}^2_{\mathscr{A}}(N, \mathscr{A}) \in \mathcal{C}_2(\mathscr{A}^{\mathrm{op}})$ as $P^{(-)}$ are projectives is $\mathbf{Mod}(\mathscr{A}^{\mathrm{op}})$. Computing $\mathrm{Ext}^2_{\mathscr{A}^{\mathrm{op}}}(\mathrm{Ext}^2_{\mathscr{A}}(f, \mathscr{A}), \mathscr{A}))$ yields the initial diagram. Now we can conclude that $\mathrm{Ext}^2_{\mathscr{A}^{op}}(\mathrm{Ext}^2_{\mathscr{A}}(-, \mathscr{A}), \mathscr{A})$ is isomorphic to the identity functor. ■

Remark 5.2.4. For an additive category $\mathscr{A}$ with a class $\mathscr{E}$ of kernel cokernel pairs we will use the following terms. An exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathscr{E}$ will be called $\mathscr{E}$-exact, f and g will be referred to as $\mathscr{E}$-monic and $\mathscr{E}$-epic respectively.

The following definition and lemma gives us way of translating subcategories of $\mathcal{C}_2(\mathscr{A})$ into kernel-cokernel pairs of $\mathscr{A}$ and vice versa.

Definition 5.2.5. (1) Let $\mathscr{D}$ be a subcategory of $\mathcal{C}_2(\mathscr{E})$. We denote by $\mathscr{E}(\mathscr{D})$ the class of all complexes $X \xrightarrow{f} Y \xrightarrow{g} Z$ for which there exists an exact sequence

$$0 \to P_X \xrightarrow{P_f} P_Y \xrightarrow{P_g} P_Z \to M \to 0$$

in $\mathbf{Mod}(\mathscr{A})$ with $M \in \mathscr{D}$. This gives us a map $\mathscr{E}(-)$ from subcategories of $\mathcal{C}_2(\mathscr{E})$ to classes of kernel-cokernel pairs (See lemma 5.2.6).

(2) Let $\mathscr{E}$ be a class of kernel cokernel pairs in $\mathscr{A}$. We denote by $\mathscr{D}(\mathscr{E})$ the subcategory of $\mathbf{Mod}\,\mathscr{A}$ Consisting of all objects M satisfying the following condition. There exists an $\mathscr{E}$-exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ satisfying $M \cong \mathrm{Coker}(P_g)$. This also gives us a map $\mathscr{D}(-)$ from the classes of kernel cokernel pairs into subcategories of $\mathscr{C}_2(\mathscr{E})$ (See Lemma 5.2.6).

Lemma 5.2.6. The following hold:

1. Every complex $X \to Y \to Z$ in $\mathscr{E}(\mathscr{D})$ is a kernel-cokernel pair.
2. Every object $M \in \mathscr{D}(\mathscr{E})$ is in $\mathcal{C}_2(\mathscr{A})$.

Proof. This is immediate from Lemma 5.2.1. ■

Our next goal is to show $\mathcal{C}_2(\mathscr{A})$ is closed under direct summands. This will be needed at the end of the section when we find a mutually inverse bijection. In order to show this we will first introduce subcategories $\mathbf{mod}_n(\mathscr{A})$ of $\mathbf{Mod}\,\mathscr{A}$ then show these are closed under direct summands. This will help us as $\mathcal{C}_2(\mathscr{A})$ will turn out to be the intersection of two subcategories of $\mathbf{mod}_2(\mathscr{A})$.

Definition 5.2.7. We denote by $\mathbf{mod}_n(\mathscr{A})$ the subcategory of $\mathbf{Mod}\,\mathscr{A}$ consisting of all objects M such that there exists an exact sequence

$$P_n \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

in $\mathbf{Mod}\,\mathscr{A}$ where each P_i is a finitely generated projective module.

Lemma 5.2.8. For an exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ in $\mathbf{Mod}\,\mathscr{A}$ with $L \in \mathbf{mod}_{n-1}$ and $M \in \mathbf{mod}_n\,\mathscr{A}$ we have $N \in \mathbf{mod}_n\,\mathscr{A}$.

Proof. Let

$$Q^\bullet = Q_{n-1} \xrightarrow{q_{n-1}} Q_{n-2} \xrightarrow{q_{n-2}} \cdots \xrightarrow{q_1} Q_0 \xrightarrow{q_0} L \to 0$$

and

$$P^\bullet = P_n \xrightarrow{p_n} P_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

be our exact sequence where all Q_i and P_i are projectives. As all Q_i are projective we are able to construct $s_i : Q_i \to P_i$ for $0 \le i \le n-1$ iteratively by the following diagrams.

$$\begin{tikzcd}
 & Q_0 \arrow[dl, dashed, "s_0"'] \arrow[d, "fq_0"] \\
P_0 \arrow[r, two heads, "p_0"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Q_i \arrow[dl, "s_i"'] \arrow[d, "s_{i-1}q_i"] \\
P_i \arrow[r, two heads, "p_i"'] & \mathrm{Im}(p_i) \subseteq P_{i-1}
\end{tikzcd}$$

Now we have the commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, tail, "f"] & M \arrow[r, two heads, "g"] & N \arrow[r] & 0 \\
 & Q_0 \arrow[u, two heads, "q_0"] \arrow[r, "s_0"] & P_0 \arrow[u, two heads, "p_0"'] & & \\
 & Q_1 \arrow[u, "q_1"] \arrow[r, "s_1"] & P_1 \arrow[u, "p_1"'] & & \\
 & \vdots \arrow[u] & \vdots \arrow[u] & & \\
 & Q_{n-1} \arrow[u, "q_{n-1}"] \arrow[r, "s_{n-1}"] & P_{n-1} \arrow[u, "p_{n-1}"'] & & \\
 & & P_n \arrow[u, "p_n"'] & &
\end{tikzcd}$$

Consider the complexes with Q_0 and P_0 in degree 0 respectively

$$Q'^\bullet = 0 \to Q_{n-1} \xrightarrow{q_{n-1}} Q_{n-2} \xrightarrow{q_{n-2}} \cdots \xrightarrow{q_1} Q_0$$

and

$$P'^\bullet = P_n \xrightarrow{p_n} P_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_1} P_0$$

Then we have a chain map $s^\bullet : Q'^\bullet \to P'^\bullet$ and can consider the exact triangle

$$Q'^\bullet \xrightarrow{s^\bullet} P'^\bullet \to \mathrm{Cone}(s^\bullet) \to Q'^\bullet[1]$$

This gives us the following long exact sequence in homology.

$$\begin{tikzcd}
 & H_{n-1}(P'^\bullet) \arrow[r] & H_{n-1}(\mathrm{Cone}(s^\bullet)) \arrow[dll] & \\
H_{n-2}(Q'^\bullet) \arrow[r] & H_{n-2}(P'^\bullet) \arrow[r] & H_{n-2}(\mathrm{Cone}(s^\bullet)) \arrow[dll] & \\
H_{n-3}(Q'^\bullet) & \vdots & H_1(\mathrm{Cone}(s^\bullet)) \arrow[dll] & \\
H_0(Q'^\bullet) \arrow[r] & H_0(P'^\bullet) \arrow[r] & H_0(\mathrm{Cone}(s^\bullet)) \arrow[r] & 0
\end{tikzcd}$$

Since $H_i(P'^\bullet) = \begin{cases} 0 & 1 \le i \le n-1 \\ M & i = 0 \end{cases}$ and $H_i(Q'^\bullet) = \begin{cases} 0 & 1 \le i \le n-2 \\ L & i = 0 \end{cases}$

we get $H_i(\mathrm{Cone}(s^\bullet)) = \begin{cases} 0 & 1 \le i \le n-1 \\ N & i = 0 \end{cases}$ and realize we are done. ∎

Lemma 5.2.9. The subcategory $\mathbf{mod}_n\,\mathscr{A}$ is closed under direct summands in $\mathbf{Mod}\,\mathscr{A}$.

Proof. Let $A \oplus B$ be in $\mathbf{mod}_n\,\mathscr{A}$. We show $A, B \in \mathbf{mod}_n\,\mathscr{A}$ by induction. For $n = 0$ the claim is obvious. Let $n \geq 1$. Since $\mathbf{mod}_n\,\mathscr{A} \subseteq \mathbf{mod}_{n-1}\,\mathscr{A}$ we have by induction hypothesis $A, B \in \mathbf{mod}_{n-1}\,\mathscr{A}$. Now we have the short exact sequences

$$A \longrightarrow A \oplus B \longrightarrow B \qquad \text{and} \qquad B \longrightarrow A \oplus B \longrightarrow A$$

Hence by Lemma 5.2.8 we get $A, B \in \mathbf{mod}_n\,\mathscr{A}$. ∎

Lemma 5.2.10. The subcategory $\mathcal{C}_2(\mathscr{A})$ is closed under direct summands in in $\mathbf{Mod}\,\mathscr{A}$.

Proof. Note that $\mathcal{C}_2(\mathscr{A})$ is equal to the intersection of the following two subcategories of $\mathbf{mod}_2\,\mathscr{A}$.

(1) The subcategory consisting of all objects whose projective dimension are at most 2.

(2) The subcategory consisting of all objects M such that $\mathrm{Ext}^i_{\mathscr{A}}(M, \mathscr{A}) = 0$ for $i = 1, 2$.

Since these subcategories clearly are closed under direct summands in $\mathbf{Mod}\,\mathscr{A}$, so is $\mathcal{C}_2(\mathscr{A})$. ∎

Remark 5.2.11. We would have gotten away with only defining $\mathbf{mod}_2 \mathscr{A}$ but as the proofs would essentially be the same we chose to generalize. We note that Lemma 5.2.10 can be generalised to $\mathcal{C}_n(\mathscr{A})$.

Now we are ready to define two classes and use the maps in Definition 5.2.5 to get a bijection. We will see in the next subsection that this leads to an equivalence between certain Serre subcategories of $\mathbf{mod}\, \mathscr{A}$ and exact structures on $\mathscr{A}$, which will be the main result of the section.

Proposition 5.2.12. Let $\mathscr{A}$ be an additive category. Then the maps in Definition 5.2.5 induce mutually inverse bijections between the following two classes.

(1) Classes of kernel-cokernel pairs $\mathscr{E}$ in $\mathscr{A}$ satisfying the following conditions.

- (a) $\mathscr{E}$ is closed under homotopy equivalences of complexes.
- (b) $\mathscr{E}$ is closed under direct sums of complexes.
- (c) $\mathscr{E}$ is closed under direct summands.
- (d) $\mathscr{E}$ is not empty.

(2) Subcategories $\mathscr{D}$ of $\mathcal{C}_2(\mathscr{A})$ satisfying the following conditions.

- (a) $\mathscr{D}$ is closed under direct sums.
- (b) $\mathscr{D}$ is closed under direct summands.
- (c) $\mathscr{D}$ is not empty.

Proof. First we show that the maps in Definition 5.2.5 induces well-defined maps between (1) and (2).
(1) $\to$ (2) : Let $\mathscr{E}$ be a class of kernel-cokernel pairs in $\mathscr{A}$ satisfying the conditions in (1). We need to show $\mathscr{D}(\mathscr{E})$ satisfy the conditions in (2).

(a) Let M and N be in $\mathscr{D}(\mathscr{E})$. This gives $\mathscr{E}$-exact sequences $A \to B \xrightarrow{g} C$ and $X \to Y \xrightarrow{p} Z$ with $\operatorname{Coker}(P_g) = M$ and $\operatorname{Coker}(P_p) = N$. By (1)($b$) we have that $A \oplus X \to B \oplus Y \xrightarrow{g \oplus p} C \oplus Z$ is an $\mathscr{E}$-exact sequence. As Hom commutes with direct sums we have $\operatorname{Coker}(P_{g\oplus p}) \cong M \oplus N$. Hence we get that $M \oplus N$ is in $\mathscr{D}(\mathscr{E})$.

(b) Let $M_1 \oplus M_2 \in \mathscr{D}(\mathscr{E})$. We have in particular that $M_1 \oplus M_2$ is in $\mathcal{C}_2(\mathscr{A})$ which is closed under summands by Lemma 5.2.10. Hence both M_1 and M_2 are in $\mathcal{C}_2(\mathscr{A})$. This gives exact sequences
$$0 \to P_{A_i} \xrightarrow{f_i} P_{B_i} \xrightarrow{g_i} P_{C_i} \to M_i \to 0$$

For some kernel-cokernel pairs $A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$ in $\mathscr{A}$ with $i = 1, 2$. Taking direct sum of these complexes we obtain a kernel-cokernel pair.

$$A_1 \oplus A_2 \xrightarrow{f_1 \oplus f_2} B_1 \oplus B_2 \xrightarrow{g_1 \oplus g_2} C_1 \oplus C_2 \tag{5.2.1}$$

This gives us the following exact sequence.

$$0 \to P_{A_1 \oplus A_2} \xrightarrow{P_{f_1} \oplus P_{f_2}} P_{B_1 \oplus B_2} \xrightarrow{P_{g_1} \oplus P_{g_2}} P_{C_1 \oplus C_2} \to M_1 \oplus M_2 \to 0 \tag{5.2.2}$$

By the fact that $M_1 \oplus M_2$ is in $\mathscr{D}(\mathscr{E})$ there exist an $\mathscr{E}$-exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{5.2.3}$$

in $\mathscr{A}$ such that the following sequence is exact and $\operatorname{Coker}(P_g) = M_1 \oplus M_2$.

$$0 \to P_A \xrightarrow{P_f} P_B \xrightarrow{P_g} P_C \to M_1 \oplus M_2 \to 0 \tag{5.2.4}$$

As projective resolutions are homotopy equivalent we get in particular that the projective resolutions acquired from (5.2.2) and (5.2.4) are homotopy equivalent. Thus it is clear by Yoneda Lemma that (5.2.1) and (5.2.3) are homotopy equivalent in $\mathscr{A}$. By (1)(a) we get that (5.2.1) is an $\mathscr{E}$-exact sequence. This gives us by condition (1)(c) that (f_i, g_i) are $\mathscr{E}$-exact sequences. Hence we get that M_1 and M_2 are in $\mathscr{D}(\mathscr{E})$.

(c) Hold as $\mathscr{E}$ is nonempty.

(2) $\to$ (1) : Let $\mathscr{D}$ be a subcategory of $\mathcal{C}_2(\mathscr{A})$ satisfying the conditions in (2). We show $\mathscr{E}(\mathscr{D})$ satisfies the conditions in (1).

(a) Let $A \to B \to C$ be in $\mathscr{E}(\mathscr{D})$. Then we have that $0 \to P_A \to P_B \to P_C$ is projective resolution for some $M \in \mathscr{D}$. Let $X \to Y \to Z$ be homotopy equivalent to $A \to B \to C$ in $\mathscr{A}$. Then we have by the Yoneda lemma that $0 \to P_A \to P_B \to P_C$ is homotopic to $0 \to P_X \to P_Y \to P_Z$. This gives us that $0 \to P_X \to P_Y \to P_Z$ is a projective resolution for M. Hence we get that $0 \to P_X \to P_Y \to P_Z \to M \to 0$ is exact.

(b) Follows from the horseshoe lemma and (2)(a).

(c) Let $A_1 \oplus A_2 \xrightarrow{f_1 \oplus f_2} B_1 \oplus B_2 \xrightarrow{g_1 \oplus g_2} C_1 \oplus C_2$ be in $\mathscr{E}(\mathscr{D})$. Then we have the sequence

$$0 \to P_{A_1} \oplus P_{A_2} \xrightarrow{P_{f_1} \oplus P_{f_2}} P_{B_1} \oplus P_{B_2} \xrightarrow{P_{g_1} \oplus P_{g_2}} P_{C_1} \oplus P_{C_2} \to M \to 0$$

where $M \cong \operatorname{Coker}(P_{g_1}) \oplus \operatorname{Coker}(P_{g_2})$. Now the claim follows from (2)(b).

(d) Hold as $\mathscr{D}$ is nonempty.

Now we will see that the maps in Definition 5.2.5 are inverses. First we see that $\mathscr{D} = \mathscr{D}(\mathscr{E}(\mathscr{D}))$ by the following implications.

$$\begin{aligned} M \in \mathscr{D} &\iff \text{There exists a kernel-cokernel pair } A \to B \xrightarrow{g} C \text{ with } \operatorname{Coker}(P_g) \cong M \in \mathscr{D} \\ &\iff A \to B \to C \text{ is } \mathscr{E}(\mathscr{D})\text{-exact with} \operatorname{Coker}(P_g) \cong M \in \mathscr{D} \\ &\iff M \in \mathscr{D}(\mathscr{E}(\mathscr{D})) \end{aligned}$$

To see $\mathscr{E} \subseteq \mathscr{E}(\mathscr{D}(\mathscr{E}))$ let $A \to B \xrightarrow{g} C$ be an $\mathscr{E}$-exact sequence. Then we have that

$$0 \to P_A \to P_B \xrightarrow{P_g} P_C \to \operatorname{Coker}(P_g) \to 0$$

is exact. Hence $A \to B \xrightarrow{g} C \in \mathscr{E}(\mathscr{D}(\mathscr{E}))$. Now all that is left to show is $\mathscr{E}(\mathscr{D}(\mathscr{E})) \subseteq \mathscr{E}$. Let

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{5.2.5}$$

Be an $\mathscr{E}(\mathscr{D}(\mathscr{E}))$-exact sequence. Let $M := \operatorname{Coker}(P_g)$, then we have $M \in \mathscr{D}(\mathscr{E})$. This yields an $\mathscr{E}$-exact sequence

$$A' \xrightarrow{f'} B' \xrightarrow{g'} C' \tag{5.2.6}$$

such that

$$0 \to P_{A'} \to P_{B'} \to P_{C'} \to M \to 0$$

is exact. We note that the Yoneda embedding of (5.2.5) also gives a projective resolution of M. As projective resolutions are homotopy equivalent it follows from Yoneda lemma that (5.2.5) is homotopy equivalent to (5.2.6). Hence (5.2.5) is an $\mathscr{E}$-exact sequence as $\mathscr{E}$ is closed under homotopy equivalences. ■

5.3 The classification result

This subsection is based on Enomotos article [4, Chapter 2.3]

In this subsection we arrive at our main theorem. Before we arrive at Theorem 5.3.4 we need to connect our exact structures from Proposition 5.2.12(1) to Serre subcategories of $\mathbf{mod}\,\mathscr{A}$ and $\mathbf{mod}\,\mathscr{A}^{\mathrm{op}}$. But first we remind the reader that in an exact category $(\mathscr{A},\mathscr{E})$ all the conditions in Proposition 5.2.12(1) are satisfied.

Lemma 5.3.1. Let $(\mathscr{A},\mathscr{E})$ be an exact category. Then $\mathscr{E}$ satisfy all the conditions of Proposition 5.2.12(1).

Proof. 1(a) follows from Proposition 4.1.11, 1(b) follows from Proposition 1.2.2, 1(c) follows from Corollary 1.2.8 and 1(d) is by definition. ■

Definition 5.3.2. A *Serre subcategory* of an exact category $(\mathscr{A},\mathscr{E})$ is an additive subcategory $\mathscr{D}$ of $\mathscr{A}$ such that for every exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $(\mathscr{A},\mathscr{E})$ we have that B belongs to $\mathscr{D}$ if and only if A and C belong to $\mathscr{D}$.

Proposition 5.3.3. Let $\mathscr{A}$ be idempotent complete. Let $\mathscr{E}$ be a class that satisfies all the conditions of Proposition 5.2.12(1). Let $\mathscr{D} := \mathscr{D}(\mathscr{E})$. Then the following are equivalent.

(1) $\mathscr{E}$ is an exact structure on $\mathscr{A}$.

(2) $\mathscr{D}$ is a Serre subcategory of of $\mathbf{mod}\,\mathscr{A}$ and $\mathrm{Ext}^2_{\mathscr{A}}(\mathscr{D},\mathscr{A})$ is a Serre subcategory of $\mathbf{mod}\,\mathscr{A}^{\mathrm{op}}$.

Proof. (1) $\Rightarrow$ (2) : We show $\mathscr{D}$ is a Serre subcategory of $\mathbf{mod}\,\mathscr{A}$. Showing $\mathrm{Ext}^2_{\mathscr{A}}(\mathscr{D},\mathscr{A})$ is a Serre subcategory of $\mathbf{mod}\,\mathscr{A}^{\mathrm{op}}$ is dual. Let

$$0 \to M_1 \xrightarrow{i} M \xrightarrow{p} M_2 \to 0$$

be a short exact sequence in $\mathbf{mod}\,\mathscr{A}$. First suppose M_1 and M_2 are in $\mathscr{D}$. We will show M is in $\mathscr{D}$. By definition we have $\mathscr{E}$-exact sequences $A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$ such that the sequences $0 \to P_{A_i} \xrightarrow{P_{f_i}} P_{B_i} \xrightarrow{P_{g_i}} P_{C_i} \to M_i \to 0$ is exact for $i = 1, 2$. By the horseshoe lemma we have the following commutative diagram in $\mathbf{Mod}\,\mathscr{A}$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & P_{A_1} \arrow[r, "P_{f_1}"] \arrow[d] & P_{B_1} \arrow[r, "P_{g_1}"] \arrow[d] & P_{C_1} \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_{A_1 \oplus A_2} \arrow[r, "P_f"] \arrow[d] & P_{B_1 \oplus B_2} \arrow[r, "P_g"] \arrow[d] & P_{C_1 \oplus C_2} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_{A_1} \arrow[r, "P_{f_2}"] \arrow[d] & P_{B_2} \arrow[r, "P_{g_2}"] \arrow[d] & P_{C_2} \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

where the rows are exact and all but the rightmost column are split exact. Since the Yoneda embedding $P_{(-)} : \mathscr{A} \to \mathbf{Mod}\,\mathscr{A}$ is fully faithfull we obtain the following diagram in $\mathscr{A}$.

$$\begin{tikzcd}
A_1 \arrow[r, tail, "f_1"] \arrow[d, tail] & B_1 \arrow[r, two heads, "g_1"] \arrow[d, tail] & C_1 \arrow[d, tail] \\
A_1 \oplus A_2 \arrow[r, "f"] \arrow[d, two heads] & B_1 \oplus B_2 \arrow[r, "g"] \arrow[d, two heads] & C_1 \oplus C_2 \arrow[d, two heads] \\
A_2 \arrow[r, tail, "f_2"] & B_2 \arrow[r, two heads, "g_2"] & C_2
\end{tikzcd}$$

We note that the top and bottom rows are $\mathscr{E}$-exact, each column is split exact and $gf = 0$. Recall split exact sequences are $\mathscr{E}$-exact by Proposition 1.2.1. Now we can apply the 3x3 lemma (Proposition 1.3.5) to see that middle row is $\mathscr{E}$-exact. This implies $M \in \mathscr{D}$.

Now suppose that M is in $\mathscr{D}$. We start by showing M_1 is in $\mathscr{D}$. We have an $\mathscr{E}$-exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ such that $0 \to P_A \xrightarrow{P_f} P_B \xrightarrow{P_g} P_C \to M \to 0$ is exact. Since M_1 is in $\mathbf{mod}\,\mathscr{A}$ we have an epic $c : P_X \twoheadrightarrow M_1$ for some $X \in \mathscr{A}$. As P_X is projective this gives us the following diagram with a commutative square.

$$\begin{tikzcd}
 & & & P_X \arrow[r, two heads, "c"] \arrow[d, "P_d"] & M_1 \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & P_A \arrow[r, "P_f"] & P_B \arrow[r, "P_g"] & P_C \arrow[r, two heads, "h"] & M \arrow[r] & 0
\end{tikzcd}$$

As $\mathscr{E}$ is an exact structure we get by [E2] and the dual of Corollary 1.2.6 the following diagram in $\mathscr{A}$, where the right square is a pullback and both rows are $\mathscr{E}$-exact.

$$\begin{tikzcd}
A \arrow[r, tail, "a"] \arrow[d, equal] & E \arrow[r, two heads, "b"] \arrow[d, "e"] & X \arrow[d, "d"] \\
A \arrow[r, tail, "f"] & B \arrow[r, two heads, "g"] & C
\end{tikzcd} \tag{5.3.1}$$

Now we show that $0 \to P_A \xrightarrow{P_a} P_E \xrightarrow{P_b} P_X \xrightarrow{c} M_1 \to 0$ is exact. As Hom is left exact and c is epic we only need to show exactness in P_X. Consider the diagram

$$\begin{tikzcd}
P_E \arrow[r, "P_b"] \arrow[d, "P_e"'] & P_X \arrow[r, two heads, "c"] \arrow[d, "P_d"] & M_1 \arrow[d, tail, "i"] \\
P_B \arrow[r, "P_g"] & P_C \arrow[r, two heads, "h"] & M
\end{tikzcd}$$

First we show $\mathrm{Im}(P_b) \subseteq \mathrm{Ker}(c)$. Let $\psi \in \mathrm{Im}(P_b)$. Then $\psi = b\gamma$ for some $\gamma \in P_E$. By commutative of the left square we get $d\psi = ge\gamma$. As the bottom sequence is exact in P_C we know $h(ge\gamma) = 0$. By commutativity of the right square and the fact that i is monic we now get $c(\psi) = 0$. To see $\mathrm{Ker}(c) \subseteq \mathrm{Im}(P_b)$ Let $\alpha \in \mathrm{Ker}(c)$. As i is monic and the right square commutes we get $0 = hP_d(\alpha) = h(d\alpha)$. As the bottom sequence is exact in P_C we know that we can lift $d\alpha$ to some $\beta \in P_B$. This gives us $g\beta = d\alpha$. By the following universal property of our pullback square we find unique $\sigma \in P_E$ such that $b\sigma = \alpha$.

$$\begin{tikzcd}
(-) \arrow[drr, bend left, "\alpha"] \arrow[dr, dashed, "\exists!\sigma"] \arrow[ddr, bend right, "\beta"'] & & \\
& E \arrow[r, "b"] \arrow[d, "c"] & X \arrow[d, "d"] \\
& B \arrow[r, "g"] & C
\end{tikzcd}$$

Hence $\alpha \in \mathrm{Im}(P_b)$ and we have shown the sequence is exact. Now it follows that M_1 is in $\mathscr{D}$. Next we show M_2 is in $\mathscr{D}$. Using the pullback in (5.3.1) and the dual of Proposition 1.2.4 we know that the sequence $E \xrightarrow{\begin{pmatrix} b \\ -e \end{pmatrix}} X \oplus B \xrightarrow{(d\ g)} C$ is $\mathscr{E}$-exact. To see M_2 is in $\mathscr{D}$ we show the sequence $0 \to P_E \xrightarrow{\begin{pmatrix} P_b \\ P_{(-e)} \end{pmatrix}} P_X \oplus P_B \xrightarrow{(P_d\ P_g)} P_C \xrightarrow{hp} M_2 \to 0$ is exact. Where $p : M \twoheadrightarrow M_2$ and $h : P_C \twoheadrightarrow M$. As Hom is left exact and hp is a composition of two epics all we need to show is exactness at P_C. Consider the diagram.

$$\begin{tikzcd}
P_E \arrow[r, "P_b"] \arrow[d, "P_e"] & P_X \arrow[r, two heads, "c"] \arrow[d, "P_d"] & M_1 \arrow[r] \arrow[d, tail, "i"] & 0 \\
P_B \arrow[r, "P_g"] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & P_C \arrow[r, two heads, "h"] \arrow[d, equal] & M \arrow[r] \arrow[d, two heads, "p"] & 0 \\
P_X \oplus P_B \arrow[r, "(P_d\ P_g)"] & P_C \arrow[r, two heads, "ph"] & M_2 \arrow[r] & 0
\end{tikzcd}$$

We have $ph(\, P_d\ P_g\,) = phP_d + phP_g = pic + phP_g = 0$ by commutativity and exactness. This implies $\mathrm{Im}((\, P_d\ P_g\,)) \subseteq \mathrm{Ker}(ph)$. Too see $\mathrm{Ker}(ph) \subseteq \mathrm{Im}((\, P_d\ P_g\,))$ let $\alpha \in \mathrm{Ker}(ph)$. Then we have $h(\alpha) \in \mathrm{Ker}(p)$. The rightmost vertical sequence is exact, hence we know there exists $\beta \in M_1$ such that $i(\beta) = h(\alpha)$. As c is epic we have some $\gamma \in P_X$ so that $c(\gamma) = \beta$. By commutativity we now have $hP_d(\gamma) = h(\alpha) \Rightarrow h(\alpha - P_d(\gamma)) = 0$. Thus by exactness there exists $\sigma \in P_B$ such that $P_g(\sigma) = \alpha - P_d(\gamma)$. This implies $\alpha = P_g(\sigma) + P_d(\gamma)$ and we get $\alpha \in \operatorname{Im}(\begin{pmatrix} P_d & P_g \end{pmatrix})$. Now we can see that our sequence is exact and consequently M_2 is in $\mathscr{D}$. Now we can conclude that $\mathscr{D}$ is a Serre subcategory.

(2) $\Rightarrow$ (1) : $\mathscr{E}$ is clearly closed under isomorphisms, as it is closed under homotopy equivalences. By duality it suffice to show that [E0$^{\text{op}}$], [E1$^{\text{op}}$] and [E2$^{\text{op}}$] hold.

[E0$^{\text{op}}$]. Let $X \in \mathscr{A}$. By 1(d) in Proposition 5.2.12 we know there exist some $\mathscr{E}$-exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$. By 1(c) in Proposition 5.2.12 we know $0 \to 0 \to 0$ is in $\mathscr{E}$. As $0 \to 0 \to 0$ is homotopy equivalent to $0 \to X \xrightarrow{\text{Id}} X$ we are done by 1(a) in Proposition 5.2.12.

[E1$^{\text{op}}$] Let $A \xrightarrow{f} B \xrightarrow{g} C$ and $X \xrightarrow{h} C \xrightarrow{k} D$ be $\mathscr{E}$-exact sequences. We show kg is an $\mathscr{E}$-epic. Let $M = \operatorname{Coker}(P_{kg})$. Consider the following commutative diagram.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
 & & & P_X \arrow[r, equal] \arrow[d, "P_h"] & P_X \arrow[d, "aP_h"] & \\
0 \arrow[r] & P_A \arrow[r, "P_f"] & P_B \arrow[r, "P_g"] \arrow[d, equal] & P_C \arrow[r, "a"] \arrow[d, "P_k"] & L \arrow[r] \arrow[d, "c"] & 0 \\
 & & P_B \arrow[r, "P_{kg}"] & P_D \arrow[r, "b"] \arrow[d, "e"] & M \arrow[r] \arrow[d, "d"] & 0 \\
 & & & N \arrow[r, equal] \arrow[d] & N \arrow[d] & \\
 & & & 0 & 0 &
\end{tikzcd}$$

Where $L = \operatorname{Coker}(P_g)$ and $N = \operatorname{Coker}(P_k)$ are in $\mathscr{D}$ by definition. Furthermore all the rows and all but the rightmost column are trivially exact. We show the rightmost column is exact. We start with exactness in L. We have $\operatorname{Im}(aP_h) \subseteq \operatorname{Ker}(c)$ as $caP_h = bP_kP_h = 0$. Conversely let $\alpha \in \operatorname{Ker}(c)$. As a is epic there exists $\beta \in P_C$ such that $a(\beta) = \alpha$. By commutativity we have $bP_k(\beta) = 0$. Hence by exactness there exists $\gamma \in P_B$ such that $P_{kg}(\gamma) = P_k(\beta)$. Now $P_k(\beta - P_g(\gamma)) = 0$ and by exactness there exists $\sigma \in P_X$ such that $P_h(\sigma) = \beta - P_g(\gamma)$. Now we have $aP_h(\sigma) = a(\beta) - aP_g(\gamma) = a(\beta) = \alpha$ which implies $\alpha \in \operatorname{Im}(aP_h)$. Next we show exactness in M. Let $\alpha \in \operatorname{Im}(c)$ then there exists $\beta \in L$ such that $c(\beta) = \alpha$. Since a is epic there exists $\gamma \in P_C$ such that $a(\gamma) = \beta$. By exactness $eP_k(\gamma) = 0$. By commutativity we get $b(P_k(\gamma)) = \alpha$ and $d(\alpha) = db(P_k(\gamma)) = eP_k(\gamma) = 0$. Hence $\operatorname{Im}(c) \subseteq \operatorname{Ker}(d)$. Conversely let $\alpha \in \operatorname{Ker}(d)$. As b is epic there exists $\beta \in P_D$ such that $b(\beta) = \alpha$. By commutativity $e(\beta) = 0$. By exactness there exists some $\gamma \in P_C$

such that $P_k(\gamma) = \beta$. By commutativity we now have $c(a(\gamma)) = \alpha$ and $\alpha \in \mathrm{Im}(c)$. One easily sees that d is epic as e is epic which gives exactness in N. Now we have that all the rows and columns are exact and are ready to show that kg is $\mathscr{E}$-epic. First we look at the exact sequence $0 \to \mathrm{Im}(aP_h) \to L \to \mathrm{Im}(c) \to 0$. We recall that L is in $\mathscr{D}$ and that $\mathscr{D}$ is a Serre subcategory to get that $\mathrm{Im}(c) \cong \mathrm{Ker}(d)$ is in $\mathscr{D}$. Next we consider the exact sequence $0 \to \ker(d) \to M \to N \to 0$. As $\ker(d)$ and N is in $\mathscr{D}$ we get that M is in $\mathscr{D}$ since $\mathscr{D}$ is a Serre subcategory. Particularly M is contained in $\mathcal{C}_2(\mathscr{A})$ and by Lemma 5.1.17 there exists a kernel-cokernel pair $Y \xrightarrow{l} B \xrightarrow{kg} D$ such that $0 \to P_Y \xrightarrow{P_l} P_B \xrightarrow{P_{kg}} P_D \to M \to 0$ is exact. This gives us that kg is an $\mathscr{E}(\mathscr{D})$-epic. We recall that $\mathscr{D} := \mathscr{D}(\mathscr{E})$ which gives us $\mathscr{E}(\mathscr{D}) = \mathscr{E}$ and we can conclude that kg is an $\mathscr{E}$-epic.

[E2$^{\mathrm{op}}$] Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an $\mathscr{E}$-exact sequence. Let $h : X \to C$ be an arbitrary morphism in $\mathscr{A}$. Then we have the following commutative diagram where $L = \mathrm{Im}(aP_h)$ and $N = \mathrm{Coker}(d)$.

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
 & & & P_X \arrow[r, "c"] \arrow[d, "P_h"] & L \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & P_A \arrow[r, "P_f"] & P_B \arrow[r, "P_g"] & P_C \arrow[r, "a"] & M \arrow[r] \arrow[d, "b"] & 0 \\
 & & & & N \arrow[d] & \\
 & & & & 0 &
\end{tikzcd}$$

Note that the rows and columns are exact. Since M is in $\mathscr{D}$ we get by assumption L and N are also in $\mathscr{D}$. In particular L and N are in $\mathcal{C}_2(\mathscr{A})$. We now want to show that the sequence $P_X \oplus P_B \xrightarrow{(P_h\ P_g)} P_C \xrightarrow{ba} N \to 0$ is exact. We immediately see that ba is epic and $ba(P_h\ P_g) = 0$. Thus it remains to show $\mathrm{Ker}(ba) \subseteq \mathrm{Im}((P_h\ P_g))$. Let $\alpha \in \mathrm{Ker}(ba)$. As the vertical sequence in our diagram is exact we see there exists $\beta \in L$ such that $d(\beta) = a(\alpha)$. We know c is epic thus there exists $\gamma \in P_X$ such that $c(\gamma) = \beta$. Now we have $a(\alpha) = a(P_h(\gamma)) \Rightarrow a(\alpha - P_h(\gamma)) = 0$. Hence there exists some $\sigma \in P_B$ such that $P_g(\sigma) = \alpha - P_h(\gamma)$. Now we have $\alpha = P_g(\sigma) + P_h(\gamma)$ consequently α is in $\mathrm{Im}((P_h\ P_g))$. Hence the sequence is exact. This gives us by Lemma 5.1.17. That we have the following exact sequence in $\mathbf{mod}(\mathscr{A})$.

$$0 \to P_E \xrightarrow{\begin{pmatrix} P_k \\ -P_l \end{pmatrix}} P_X \oplus P_B \xrightarrow{(P_h\ P_g)} P_C \xrightarrow{ba} N \to 0$$

Similar to earlier in the proof this corresponds to a pullback square of the form

$$\begin{tikzcd} E \arrow[r, "k"] \arrow[d, "l"] & X \arrow[d, "h"] \\ B \arrow[r, "g"] & C \end{tikzcd} \tag{5.3.2}$$

in $\mathscr{A}$. Hence we have showed the existence of the pullback. What remains is that k is $\mathscr{E}$-epic. Also similar to earlier we get by the pullback property of (5.3.2) that there exists a sequence $A \xrightarrow{i} E \xrightarrow{k} X$ in $\mathscr{A}$ such that $0 \to P_A \xrightarrow{P_i} P_E \xrightarrow{P_k} P_X \to L \to 0$ is exact. Thus the complex $A \xrightarrow{i} E \xrightarrow{k} X$ belongs to $\mathscr{E}(\mathscr{D})$ which is equal to $\mathscr{E}$. Hence k is an $\mathscr{E}$-epic. ■

Now we are finally able to state our main theorem. We see that it simply falls out of the preceding results.

Theorem 5.3.4. Let $\mathscr{A}$ be an idempotent complete category. Then there exists mutually inverse bijections between the following two classes.

(1) Exact structures $\mathscr{E}$ on $\mathscr{A}$.

(2) Subcategories $\mathscr{D}$ of $\mathcal{C}_2(\mathscr{A})$ satisfying the following conditions.

  (a) $\mathscr{D}$ is a serre subcategory of $\mathbf{mod}\,\mathscr{A}$.

  (b) $\mathrm{Ext}^2_{\mathscr{A}}(\mathscr{D}, \mathscr{A})$ is a serre subcategory of $\mathbf{mod}\,\mathscr{A}^{\mathrm{op}}$.

Proof. This now follows immediately from Lemma 5.3.1 and Proposition 5.3.3. ■

Next we will investigate more explicit classifications by adding structure to $\mathscr{A}$. Firstly in the following subsection we see what happens when the class of all kernel-cokernel pairs forms an exact category $(\mathscr{A}, \mathscr{E})$ with enough projectives. Then in the next section we consider categories of finite type and see modules satisfying the 2-regular condition play an important role. Towards the end we will see that when applying the theorem to quivers we get a nice correspondence between exact structures on $\operatorname{rep} Q$ and Auslander Reiten translations.

5.4 A more particular classification

As mentioned we will in this section see what happens when the class of all kernel-cokernel pairs forms an exact category $(\mathscr{A}, \mathscr{E})$ with enough projectives. Before we start working our way towards this result we go through an immediate corollary of our main result and a proposition that will be useful for the final result of this subsection.

Corollary 5.4.1. Let $\mathscr{A}$ be an idempotent complete category. Then the following are equivalent.

(1) $\mathcal{C}_2(\mathscr{A})$ and $\mathcal{C}_2(\mathscr{A}^{\mathrm{op}})$ are Serre subcategories of $\mathbf{mod}\,\mathscr{A}$ and $\mathbf{mod}\,\mathscr{A}^{\mathrm{op}}$ respectively.

(2) The class of kernel-cokernel pairs in $\mathscr{A}$ defines an exact structure of $\mathscr{A}$.

In this case there exists a bijection between exact structures on $\mathscr{A}$ and Serre subcategories of $\mathcal{C}_2(\mathscr{A})$.

Proof. Let $\mathscr{E}$ be the class of all kernel-cokernel pairs. Then we have $\mathscr{D}(\mathscr{E}) = \mathcal{C}_2(\mathscr{A})$, thus the claim follows from Proposition 5.3.3. The second part is clear from Theorem 5.3.4. ■

Proposition 5.4.2. Let $(\mathscr{A}, \mathscr{E})$ be an idempotent complete exact category. Let $\mathscr{D}$ be the corresponding Serre subcategory of $\mathbf{mod}\,\mathscr{A}$ given in Theorem 5.3.4. Then for an object $A \in \mathscr{A}$ the following are equivalent.

(1) A is projective in $\mathscr{A}$.

(2) $M(A) = 0$ for all objects $M \in \mathscr{D}$.

Proof. The category $\mathscr{D}$ consists of all functors M such that $P_B \xrightarrow{g} P_C \to M \to 0$ is exact for some admissible epic $g : B \to C$. Hence (2) is equivalent to $P_B(A) \xrightarrow{P_g(A)} P_C(A)$ being epic for every admissible epic $g : B \to C$. This is equivalent to A being an projective object of $\mathscr{A}$. ■

Next we work our way to defining the projectively stable category $\underline{\mathscr{A}}$ of a category $(\mathscr{A}, \mathscr{E})$ with enough projectives. Afterwords we will see that when the class of all kernel-cokernel pairs forms an exact category $(\mathscr{A}, \mathscr{E})$ with enough projectives we will get a correspondence between Serre subcategories of $\mathbf{mod}_1(\underline{\mathscr{A}})$ and exact structures on $\mathscr{A}$.

Proposition 5.4.3. Let $(\mathscr{A}, \mathscr{E})$ be an exact category with enough projectives. Then the class $[\mathscr{P}]$ of all morphisms factoring through an projective object forms an two sided ideal in $(\mathscr{A}, \mathscr{E})$.

Proof. We show the requirements from Definition 4.1.12 First we show the subset $[\mathscr{P}](A,B)$ consisting of morphisms $A \to B$ factoring through an projective object is a subgroup of $\mathrm{Hom}_{\mathscr{A}}(A,B)$. Let $f \in [\mathscr{P}](A,B)$. Then we have the factorization where P is projective.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \\ & P \arrow[ur, "i"'] & \end{tikzcd}$$

As $f = ip$ we have $-f = -ip$, so $-f \in [\mathscr{P}](A,B)$. We know the zero object factors through any object, thus $0 \in [\mathscr{P}](A,B)$. Now let $f, f' \in [\mathscr{P}](A,B)$ with the two following factorizations where P and P' are projectives.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \\ & P \arrow[ur, "i"'] & \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[rr, "f'"] \arrow[dr, "p'"'] & & B' \\ & P' \arrow[ur, "i'"'] & \end{tikzcd}$$

Let $\iota : P \to P \oplus P'$ and $\iota' : P' \to P \oplus P'$ be the canonical inclusions. Let $\pi : P \to P \oplus P'$ and $\pi' : P' \to P \oplus P'$ be the canonical projections. We know $P \oplus P'$ is projective as P and P' are projective. We define $g = \iota p \oplus \iota' p'$ and $h = i\pi \oplus i'\pi'$. Now we have $gf = f + f'$ and can conclude that $[\mathscr{P}](A,B)$ is a subgroup of $\mathrm{Hom}_{\mathscr{A}}(A,B)$. For the absorption property let $f \in [\mathscr{P}](A,B)$ with factorization $f = ip$ and $g : B \to C$ be a morphism. then $gf = gip$ hence it factors through an projective object. similarly one shows absorbtion from the other side. $[\mathscr{P}](A,B)$ is closed under direct sums as the direct sum of two projective objects is projective. ∎

Definition 5.4.4. Let $(\mathscr{A}, \mathscr{E})$ be an exact category with enough projectives. Then the category $\underline{\mathscr{A}} = (\mathscr{A}, \mathscr{E})/[\mathscr{P}]$ where $[\mathscr{P}]$ is the ideal from our previous proposition is called *the projectively stable category of* $\mathscr{A}$.

Lemma 5.4.5. Let $(\mathscr{A}, \mathscr{E})$ be an exact category with enough projectives. Let $\mathscr{D} := \mathscr{D}(\mathscr{E})$ be the subcategory of $\mathbf{mod}\,\mathscr{A}$ corresponding of all short exact sequences (See definition 5.2.5). Then $\mathscr{D} \cong \mathbf{mod}_1(\underline{\mathscr{A}})$

Proof. We note there is an embedding $E : \mathbf{mod}_1\underline{\mathscr{A}} \to \mathbf{mod}\,\mathscr{A}$. Denote the essential image of E by $\mathscr{D}'$. We show that following are equivalent, which will prove the Lemma.

(1) $M \in \mathscr{D}$

(2) There exists an admissible epic $g : B \to C$ in $\mathscr{A}$ and an exact sequence

$$\mathrm{Hom}_{\underline{\mathscr{A}}}(-,B) \xrightarrow{-\circ g} \mathrm{Hom}_{\underline{\mathscr{A}}}(-,C) \to M \to 0$$

(3) $M \in \mathscr{D}'$

(1) $\Rightarrow$ (2) : Let $M \in \mathscr{D}$ Then there exist an exact sequence

$$0 \to P_A \to P_B \to P_C \to M \to 0$$

corresponding to a short exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$. We show g is the admissible epic in (2). Consider the commutative diagram

$$\begin{tikzcd}
P_B \arrow[r, "P_g"] \arrow[d, two heads, "b"] & P_C \arrow[r, "p"] \arrow[d, two heads, "c"] & M \arrow[r] \arrow[d, equal] & 0 \\
\mathrm{Hom}_{\underline{\mathscr{A}}}(-,B) \arrow[r, "-\circ g"] & \mathrm{Hom}_{\underline{\mathscr{A}}}(-,C) \arrow[r, "\tilde{p}"] & M \arrow[r] & 0
\end{tikzcd}$$

We know there exists $\tilde{p}$ which is well defined and makes the diagram commute by Proposition 5.4.2. It is now straightforward to see that the bottom sequence is exact by diagram chasing.

(2) $\Rightarrow$ (1) : Similar to (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) : Automatic.

(3) $\Rightarrow$ (2) : Let $M \in \mathscr{D}'$. Then M is finitely presented. i.e. there exists a sequence $\mathrm{Hom}_{\underline{\mathscr{A}}}(-,B) \xrightarrow{-\circ g} \mathrm{Hom}_{\underline{\mathscr{A}}}(-,C) \to M \to 0$. corresponding to some morphism $g : B \to C$ By assumption there exists an admissible epic $g : P \to C$. Thus we can replace g by the admissible epic $(\, f \; g\,) : B \oplus P \to C$ and we are done. ■

Corollary 5.4.6. Let $\mathscr{A}$ be an idempotent complete category such that the class of all kernel-cokernel pairs forms an exact category $(\mathscr{A}, \mathscr{E})$ with enough projectives. Denote $\underline{\mathscr{A}}$ the protectively stable category category of $(\mathscr{A}, \mathscr{E})$. Then there exists a bijection between the following two classes.

(1) Exact structures on $\mathscr{A}$.

(2) Serre subcategories of $\mathbf{mod}_1(\underline{\mathscr{A}})$.

Proof. This follows immediately from Corollary 5.4.1 and Lemma 5.4.5. ■

Example 5.4.7. Corollary 5.4.6 in particular hold in Abelian categories with enough projectives.

6 Classifying Exact Categories of Finite Type

This section is based on Enomoto [4, Chapter 3].

In this section we start by reformulating Theorem 5.3.4 in the case of categories of finite type. Afterwords we introduce a class of exact categories that will turn out to be controlled by simple modules satisfying the 2-regular condition. In the final subsection of this section we relate our results to Auslander-Reiten theory. We will see that given a 'nice' noetherian R-algebra Γ we have a correspondence between admissible exact structures on $\operatorname{proj}\Gamma$ and sets of dotted arrows (AR translations) in the Auslander Reiten quiver of Γ.

6.1 Reformulations and the 2-regular condition

In this subsection we start by defining categories of finite type in order to reformulate Theorem 5.3.4 in terms of a noetherian ring. We then define a special class of simple Γ modules and relate it to our classification theorem. At the end of the section we will see that the additional structure has given us a way to tell when there exists non-trivial exact structures on $\mathscr{A}$.

Definition 6.1.1. Let $\mathscr{A}$ be an additive category. For $A \in \mathscr{A}$ we denote by $\operatorname{add} A$ the full subcategory consisting off all finite direct sums of copies of A and their direct summands. We say an object $M \in \mathscr{A}$ is an *additive generator* of $\mathscr{A}$ if $\operatorname{add} M = \mathscr{A}$. We say that an additive category $\mathscr{A}$ is of *finite type* if it has an additive generator.

Lemma 6.1.2. Let M be an additive generator of $\mathscr{A}$ and $\Gamma := \operatorname{End}_{\mathscr{A}}(M)$. Then we have a fully faithfull functor $\mathscr{A}(M,-) : \mathscr{A} \to \mathbf{Mod}\Gamma$. Furthermore its essential image coincides with the category $\operatorname{proj}\Gamma$ of finitely generated projective Γ modules precisely when $\mathscr{A}$ is idempotent complete.

Proof. The first part follows by Yoneda lemma. The second part is a special case of 5.1.9. ∎

Remark 6.1.3. The preceding Lemma tells us that when we deal with an idempotent complete additive category of finite type we may assume $\mathscr{A} := \operatorname{proj}\Gamma$.

We reformulate our main result from Section 5 in terms of a noetherian ring Γ. We use the same notation for $\mathcal{C}_2$.

$$\mathcal{C}_2(\Gamma) := \{M \in \mathbf{mod}\Gamma \,|\, \operatorname{pd} M_\Gamma \leq 2 \text{ and } \operatorname{Ext}^i_\Gamma(M,\Gamma) = 0 \text{ for } i = 0,1\}$$

Where pd denotes the projective dimension. Note that our restriction on Ext yields that every nonzero module in $\mathcal{C}_2(\Gamma)$ has projective dimension 2.

Theorem 6.1.4. Let Γ be a noetherian ring. Let $\mathscr{A} := \operatorname{proj}\Gamma$. Then there exists a bijection between the following two classes.

(1) Exact structures $\mathscr{E}$ on $\mathscr{A}$.

(2) Subcategories $\mathscr{D}$ of $\mathcal{C}_2(\Gamma)$ satisfying the following.

  a) $\mathscr{D}$ is a Serre subcategory of $\mathbf{mod}\Gamma$.

  b) $\operatorname{Ext}^2_\Gamma(\mathscr{D}, \Gamma)$ is a Serre subcategory of $\mathbf{mod}\Gamma^{\mathrm{op}}$.

The correspondence is given as follows.

- For a given $\mathscr{D}$ a complex $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathscr{A}$ is in $\mathscr{E}$ if and only if the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to M \to 0$ is an exact sequence in $\mathbf{mod}\Gamma$ for some $M \in \mathscr{D}$.
- For a given $\mathscr{E}$ a Γ module $M \in \mathbf{mod}\Gamma$ is in $\mathscr{D}$ if and only if there exists an admissible epic g in $\mathscr{E}$ with $M \cong \operatorname{Coker}(g)$.

Proof. This is a reformulation of Theorem 5.3.4 given an idempotent complete additive category of finite type. See Lemma 6.1.2 and Remark 6.1.3. ∎

Definition 6.1.5. Let Γ be a two sided noetherian ring. Let S be a simple right Γ module. We say S satisfies the *2-regular condition* if the following hold.

(1) $\operatorname{pd} S_\Gamma = 2$.

(2) $\operatorname{Ext}^i_\Gamma(S, \Gamma) = 0$ for $i = 0, 1$.

(3) $\operatorname{Ext}^2_\Gamma(S, \Gamma)$ is a simple left Γ module.

Definition 6.1.6. Let $\mathscr{S}$ be a set of simple modules. We denote by $\operatorname{Filt}\mathscr{S}$ the subcategory of $\mathbf{mod}\Gamma$ consisting of all modules M that satisfy the following: M has finite length and all composition factors are contained in $\mathscr{S}$.

Lemma 6.1.7. Let Γ be a Noetherian ring. Let $\mathscr{S}$ be a set of simple Γ modules. Then the following are equivalent.

(1) Every module in $\mathscr{S}$ satisfies the 2-regular condition.

(2) $\mathscr{D} := \operatorname{Filt}\mathscr{S}$ is contained in $\mathcal{C}_2(\Gamma)$ and satisfies the conditions of Theorem 6.1.4(2).

(3) There exists a subcategory $\mathscr{D}$ of $\mathcal{C}_2(\Gamma)$ containing $\mathscr{S}$ such that $\mathscr{D}$ satisfies the conditions of Theorem 6.1.4(2).

Proof. (1) $\Rightarrow$ (2): Clearly $S \in \mathcal{C}_2(\Gamma)$ for all $S \in \mathscr{S}$ as they satisfy the 2-regular condition. We know $\mathcal{C}_2(\Gamma)$ is closed under extensions by the horseshoe lemma. This implies that $\operatorname{Filt}\mathscr{S}$ is contained in $\mathcal{C}_2(\Gamma)$. Next we show $\mathscr{D}$ satisfy the conditions of Theorem 6.1.4(2). It is easy to see $\operatorname{Filt}\mathscr{S}$ is a Serre subcategory of $\mathbf{mod}\Gamma$ by the behaviour of finite length modules in short exact sequences. For the second part we recall by Lemma 5.2.3 that $\operatorname{Ext}^2_\Gamma(-,\Gamma)$ induce a duality $\mathcal{C}_2(\Gamma) \cong \mathcal{C}_2(\Gamma^{\mathrm{op}})$. Furthermore $\operatorname{Ext}^2_\Gamma(S,\Gamma)$ is a simple module for all $S \in \mathscr{S}$ by 2-regularity. Hence we get that that $\operatorname{Ext}^2_\Gamma(\mathscr{D},\Gamma) = \operatorname{Filt}\operatorname{Ext}^2_\Gamma(\mathscr{S},\Gamma)$ which yields that it is a Serre subcategory of $\mathbf{mod}\Gamma^{\mathrm{op}}$.

(2) $\Rightarrow$ (3): Automatic.

(3) $\Rightarrow$ (1): Let S be a simple module in $\mathscr{S}$. Then $S \in \mathcal{C}_2(\Gamma)$ by assumption. This gives us $\operatorname{pd} S_\Gamma = 2$ and $\operatorname{Ext}^i_\Gamma(S,\Gamma) = 0$ for $i = 0, 1$. For the last requirement recall $\operatorname{Ext}^2_\Gamma(-,\Gamma)$ gives a duality between $\mathscr{D}$ and $\operatorname{Ext}^2_\Gamma(\mathscr{D},\Gamma)$. Both of these are Abelian categories. Hence we immediately get that $\operatorname{Ext}^2_\Gamma(S,\Gamma)$ is a simple object in $\operatorname{Ext}^2_\Gamma(\mathscr{D},\Gamma)$ which implies $\operatorname{Ext}^2_\Gamma(S,\Gamma)$ is a simple left Γ module and we are done. ∎

Recall from Proposition 1.2.1 that every additive category admits a trivial exact structure where short exact sequences are split short exact sequences. The following result gives a criterion for the existence of non-trivial exact structures.

Proposition 6.1.8. Let Γ be a noetherian ring. Then $\operatorname{proj}\Gamma$ admits a non-trivial exact structure if and only if there exists a simple Γ module satisfying the 2-regular condition.

Proof. Suppose there exists a simple Γ module S satisfying the 2-regular condition. Then we have that there exists a set $\mathscr{S}$ of simple modules such that $\operatorname{Filt}\mathscr{S} \neq \emptyset$. By Lemma 6.1.7 $\operatorname{Filt}\mathscr{S}$ is a non-zero Serre subcategory satisfying the conditions of Theorem 6.1.4(2). By Theorem 6.1.4 we now have that there exists at least one non-trivial exact structure on $\operatorname{proj}\Gamma$.

Conversely suppose $\operatorname{proj}\Gamma$ has a non-trivial exact structure. By Theorem 6.1.4 we have a non-zero serre subcategory $\mathscr{D}$ of $\mathbf{mod}\Gamma$. As any non-zero Γ module has a surjection onto a simple Γ module and $\mathscr{D}$ is a serre subcategory we get that $\mathscr{D}$ contains at least one simple Γ module S. By Lemma 6.1.7 S satisfy the 2-regular condition. ∎

6.2 Admissible exact structures

In this section we introduce a class of exact categories that will turn out to be controlled by simple modules satisfying the 2-regular condition. This will lead us to bijection between exact structures in this class and sets of isomorphism classes of simple modules satisfying the 2-regular condition. In the next subsection we will see that this makes us able to find a correspondence between Auslander-Reiten translations and admissible exact structures. This will make us able to give very explicit examples.

Definition 6.2.1. For a ring Γ we denote by $\mathrm{f.l.}\,\Gamma$ the subcategory of $\mathbf{mod}\Gamma$ consisting of Γ modules of finite length. Similarly we denote by $\mathrm{f.l.}\,\mathscr{A}$ the category consisting of $\mathscr{A}$ modules of finite length.

Definition 6.2.2. Let $(\mathscr{A},\mathscr{E})$ be an exact category. Let $\mathscr{D}$ be the subcategory of $\mathbf{mod}\,\mathscr{A}$ corresponding to $\mathscr{E}$ under Theorem 5.3.4. We call $\mathscr{E}$ *admissible* if $\mathscr{D} \subseteq \mathrm{f.l.}\,\mathscr{A}$ hold.

Example 6.2.3. Let $(\operatorname{proj}\Gamma,\mathscr{E})$ be an exact category, where Γ is a noetherian ring. Let $\mathscr{D}$ be the subcategory corresponding to $\mathscr{E}$ under Theorem 6.1.4. Then it is clear that $\mathscr{E}$ is admissible if and only if $\mathscr{D} \subseteq \mathrm{f.l.}\,\Gamma$.

If Γ is an artinian ring we have that every exact structure on $\operatorname{proj}\Gamma$ is admissible. This is due to the fact that an artinian finitely generated module has finite length.

Proposition 6.2.4. Let $(\operatorname{proj}\Gamma,\mathscr{E})$ be an exact category for a noetherian ring Γ. Then it is admissible if and only if $(\operatorname{proj}\Gamma^{\mathrm{op}},\mathscr{E}^{\mathrm{op}})$ is admissible. i.e. Admissibility is left-right symmetric when Γ is noetherian.

Proof. Let $\mathscr{D}$ be the subcategory of $\mathbf{mod}\Gamma$ corresponding to $\mathscr{E}$ under Theorem 6.1.4. Then $\operatorname{Ext}^2_\Gamma(\mathscr{D},\Gamma)$ is the subcategory corresponding to $\mathscr{E}^{\mathrm{op}}$. As $\mathscr{E}$ is admissible every object in $\mathscr{D}$ has finite length as a Γ module. Hence $\mathscr{D}$ is an abelian category where every object has finite length as a Γ module. By Lemma 5.2.3 $\operatorname{Ext}^2_\Gamma(\mathscr{D},\Gamma)$ is dual to $\mathscr{D}$. Hence every object in $\operatorname{Ext}^2_{\operatorname{proj}\Gamma}(\mathscr{D},\operatorname{proj}\Gamma)$ has finite length in the abelian category $\operatorname{Ext}^2_\Gamma(\mathscr{D},\Gamma)$. As $\operatorname{Ext}^2_\Gamma(\mathscr{D},\Gamma)$ is a serre subcategory of $\mathbf{mod}\Gamma^{\mathrm{op}}$ we have $\operatorname{Ext}^2_\Gamma(\mathscr{D},\Gamma) \subseteq \mathrm{f.l.}\,\Gamma^{\mathrm{op}}$. ■

Theorem 6.2.5. Let Γ be a noetherian ring. For $\mathscr{A} := \operatorname{proj}\Gamma$ there exists a bijection between the following two classes.

(1) Admissible exact structures $\mathscr{E}$ on $\mathscr{A}$.

(2) Sets $\mathscr{S}$ of isomorphism classes of simple Γ modules satisfying the 2-regular condition.

The correspondence is given by the following.

- For a given $\mathscr{S}$ a complex $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathscr{A}$ is a short exact sequence if and only if there exists an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to M \to 0$ in $\mathbf{mod}\Gamma$ where $M \in \operatorname{Filt}\mathscr{S}$.

- For a given $\mathscr{E}$, a simple Γ module S is in $\mathscr{S}$ if and only if there exists an admissible monic $g : B \to C$ in $\mathscr{A}$ such that $\operatorname{Coker}(g) \cong S$ in $\mathbf{mod}\Gamma$.

Proof. By Theorem 6.1.4 and Proposition 6.2.4 there is a bijection between (1) and

($*$) Subcategories of $\mathcal{C}_2(\Gamma)$ satisfying the following conditions.

a) $\mathscr{D}$ is a serre subcategory of $\mathbf{mod}\Gamma$ satisfying $\mathscr{D} \subseteq \mathrm{f.l.}\,\Gamma$.

b) $\operatorname{Ext}^2_\Gamma(\mathscr{D}, \Gamma)$ is a serre subcategory of $\mathbf{mod}\Gamma^{\mathrm{op}}$.

We have mutually inverse bijections between (2) and ($*$) as follows. (1) $\to$ ($*$): We send $\mathscr{S}$ in (2) to $\mathscr{D} = \operatorname{Filt}\mathscr{S}$. ($*$) $\to$ (2) We send a $\mathscr{D}$ in ($*$) to the set $\mathscr{S}$ of simple modules contained in $\mathscr{D}$. These are both well defined by Lemma 6.1.7. These are mutually inverse to each other by definition. ∎

We note by Example 6.2.3 that in the case where Γ is artinian we can drop the assumption that $\mathscr{E}$ is admissible as all exact structures will be admissible.

6.3 Classifying exact structures by quivers

In this section we relate our findings to Auslander-Reiten theory. We will see given a 'nice' noetherian R-algebra Γ we have a correspondence between admissible exact structures on $\operatorname{proj}\Gamma$ and sets of dotted arrows (AR translations) in the Auslander-Reiten quiver of Γ. Particularly we get when Γ is artinian a classification between all exact structures on $\operatorname{proj}\Gamma$ through sets of dotted arrows. In the last section we use the result in examples to find all exact structures. There we will also find the derived categories of the different exact structures.

Definition 6.3.1. The radical of a noetherian R algebra Γ and its powers is defined as follows.

- The radical $\operatorname{rad}_{\mathscr{A}}(A,B)$ of two objects A and B in a noetherian R algebra Γ is the two sided ideal formed by all all $f \in \operatorname{Hom}_\Gamma(A,B)$ such that $\operatorname{Id}_A - gf$ is invertible for all $g \in \operatorname{Hom}_\Gamma(B,A)$.
- Given $m \geq 1$, the m^{th} power $\operatorname{rad}^m_{\mathscr{A}}(A,B) \subseteq \operatorname{rad}_{\mathscr{A}}(A,B)$ of $\operatorname{rad}_{\mathscr{A}}(A,B)$ is obtained by taking the subspace of $\operatorname{rad}_{\mathscr{A}}(A,B)$ containing all finite sums of morphism of the form
$$A = X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{f_{m-1}} X_{m-1} \xrightarrow{f_m} X_m = B$$
where $f_i : X_{i-1} \to X_i$ is in $\operatorname{rad}_{\mathscr{A}}(X_{i-1}, X_i)$ for all $i = 1, 2, \dots, m$.

Definition 6.3.2. For a noetherian R algebra Γ we define the valued quiver $Q(\Gamma)$ as follows.

- We draw an arrow from A to B if $\operatorname{rad}_{\mathscr{A}}(A,B)/\operatorname{rad}^2_{\mathscr{A}}(A,B) \neq 0$. The arrow has valuation $(d_{A,B}, d'_{A,B})$ where $d_{A,B}$(resp.$d'_{A,B}$) is the dimension of $\operatorname{rad}_\Gamma(A,B)/\operatorname{rad}^2_\Gamma(A,B)$ as a k_A vector space (resp. k_B vector space). Here $k_A := \operatorname{End}_{\mathscr{A}}(A)/\operatorname{rad}\operatorname{End}_{\mathscr{A}}(A)$ and $k_B := \operatorname{End}_{\mathscr{A}}(B)/\operatorname{rad}\operatorname{End}_{\mathscr{A}}(B)$.
- The set of vertices is $\operatorname{ind}(\operatorname{proj}\Gamma)$. That is the isomorphism classes of all indecomposable projective right Γ modules.

Definition 6.3.3. Let R be a commutative noetherian complete local ring. Let Γ be a noetherian R algebra. The *translation* τ of $P \in Q(\Gamma)$ is defined when $P/\operatorname{rad}P$ satisfy the 2-regular condition. In this case τP is the projective module Q such that $\operatorname{Hom}_\Gamma(Q,\Gamma)$ is a projective cover of the simple left Γ module $\operatorname{Ext}^2_\Gamma(P/\operatorname{rad}P,\Gamma)$. We draw a dotted arrow from P to τP whenever τP is defined.

Lemma 6.3.4. Let Λ be a noetherian R algebra over a noetherian complete local ring R. Let $\mathscr{A} = \mathbf{mod}\Lambda$. Let G be the additive generator equal to the direct sum of all indecomposables. Consider the functor $\mathrm{Hom}_{\mathscr{A}}(G,-) : \mathscr{A} \to \mathrm{proj}\,\Gamma$ (See Lemma 6.1.2). Then the AR translation (Definition C.5) corresponds to Definition 6.3.3 via $\mathrm{Hom}_{\mathscr{A}}(G,-)$.

Proof. Let τ denote the translation in Definition 6.3.3 and let τ' denote the AR translation. As $\mathrm{Hom}_{\mathscr{A}}(G,-)$ is an equivalence it is sufficient to show the following.

(1) When τ' is non-zero for an indecomposable Λ module M then $\mathrm{Hom}_{\mathscr{A}}(G,M)$ satisfy the 2-regular condition and $\tau\,\mathrm{Hom}_{\mathscr{A}}(G,M) = \mathrm{Hom}_{\mathscr{A}}(G,\tau' M)$.

(2) When τ' is zero for a indecomposable Λ module M then $\mathrm{Hom}_{\mathscr{A}}(G,M)$ do not satisfy the 2-regular condition and thus τ' is not defined.

(1) : Let M be a indecomposable non-projective Λ module, i.e. $\tau' \neq 0$. Then we have by Theorem C.6 that there exists an almost split exact sequence.

$$0 \to \tau' M \xrightarrow{f} E \xrightarrow{g} M \to 0$$

We apply $\mathrm{Hom}_{\mathscr{A}}(G,-)$ to get the exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{A}}(G,\tau' P) \to \mathrm{Hom}_{\mathscr{A}}(G,E) \to \mathrm{Hom}_{\mathscr{A}}(G,M) \xrightarrow{(G,g)} \mathrm{Coker}((G,g)) \to 0 \quad (6.3.1)$$

in $\mathrm{proj}\,\Gamma$. We claim

$$\mathrm{Coker}((G,g)) = \mathrm{Hom}_{\Lambda}(G,M)/\operatorname{rad}_{\Gamma}\mathrm{Hom}_{\Lambda}(G,M)$$

First we show $\operatorname{rad}_{\Gamma}\mathrm{Hom}_{\Lambda}(G,M)$ is equal to all non-split epimorphisms. It is standard to show that all non-split epimorphisms form a submodule. Consider a submodule $U \subseteq \mathrm{Hom}_{\Lambda}(G,M)$ containing at least one split epic. Let $e \in U$ be split-epic. Then there exists a morphism m such that $em = \mathrm{Id}_M$. Consider $\phi \in \mathrm{Hom}_{\mathscr{A}}(G,M)$ then $\phi = em\phi$ which is in U as $m\phi$ is in $\mathrm{End}_{\Gamma}(M) \subseteq \Gamma = \mathrm{End}_{\Gamma}(G)$. Thus $U = \mathrm{Hom}_{\mathscr{A}}(G,M)$ and the submodule of non-split epimorphisms is our the a maximal ideal thus the radical as we are working over a local local ring. Next we show $\mathrm{Im}(G,g) = \operatorname{rad}_{\Gamma}\mathrm{Hom}_{\Lambda}(G,M)$. We know $\mathrm{Im}(G,g)$ is all morphisms $G \to M$ that factors through a morphism $g : E \to M$. As g is right almost split this yields $\mathrm{Im}(G,g) = \operatorname{rad}_{\Gamma}\mathrm{Hom}_{\Lambda}(G,M)$. Our claim now follows. Next we show $\mathrm{Hom}_{\Lambda}(G,M)/\operatorname{rad}_{\Gamma}\mathrm{Hom}_{\Lambda}(G,M)$ satisfy the 2-regular condition. It is clear that $\mathrm{Hom}_{\Lambda}(G,M)/\operatorname{rad}_{\Gamma}\mathrm{Hom}_{\Lambda}(G,M)$ has projective dimension 2. For the two other requirements we apply $\mathrm{Hom}_{\Gamma}(\mathrm{Hom}_{\Lambda}(G,M)/\operatorname{rad}_{\Gamma}\mathrm{Hom}_{\Lambda}(G,M),\Gamma)$ to the projective part of 6.3.1 to get the sequence

$$0 \to \mathrm{Hom}_{\Gamma}(\mathrm{Hom}_{\Lambda}(G,M),\Gamma) \to \mathrm{Hom}_{\Gamma}(\mathrm{Hom}_{\Lambda}(G,E),\Gamma) \to \mathrm{Hom}_{\Gamma}(\mathrm{Hom}_{\Lambda}(G,\tau' M),\Gamma) \to 0$$

which is equal to the sequence.

$$0 \to \mathrm{Hom}_{\Lambda}(M,G) \to \mathrm{Hom}_{\Lambda}(E,G) \to \mathrm{Hom}_{\Lambda}(\tau' M,G) \to 0$$

As Hom is left exact we see the second requirement for 2-regularity is satisfied. By a dual argument of the first part of the proof we also get the third requirement. Furthermore we note that

$$\operatorname{Ext}^2_\Gamma(\operatorname{Hom}_\Lambda(G,M)/\operatorname{rad}_\Gamma \operatorname{Hom}_\Lambda(G,M),\Gamma) = \operatorname{Hom}_\Lambda(G,\tau M)/\operatorname{rad}_\Gamma \operatorname{Hom}_\Lambda(G,\tau M)$$

Which have $\operatorname{Hom}_\Lambda(G,\tau M)$ as a projective cover and yields $\tau \operatorname{Hom}_{\mathscr{A}}(G,M) = \operatorname{Hom}_{\mathscr{A}}(G,\tau' M)$.

(2) : Let M be an indecomposable projective Λ module. Then $\tau' M = 0$ by Proposition C.3. We get by Proposition C.7 a right almost split morphism $\operatorname{rad} M \to M$. Applying $\operatorname{Hom}_{\mathscr{A}}(G,-)$ then yields the exact sequence

$$0 \to \operatorname{Hom}_\Lambda(G,M) \to \operatorname{Hom}_\Lambda(G,M) \to \operatorname{Hom}_\Lambda(G,M)/\operatorname{rad}_\Gamma \operatorname{Hom}_\Lambda(G,M) \to 0$$

Thus if M is a projective indecomposable Λ module we get that $\operatorname{Hom}_\Lambda(G,M)/\operatorname{rad}_\Gamma \operatorname{Hom}_\Lambda(G,M)$ do not satisfy the 2-regular condition as

$$\operatorname{pd} \operatorname{Hom}_\Lambda(G,M)/\operatorname{rad}_\Gamma \operatorname{Hom}_\Lambda(G,M) = 1$$

■

Theorem 6.3.5. Let Γ be a noetherian R algebra over a noetherian complete local ring R. Then there exists a bijection between the following two classes.

1. Admissible exact structures on $\operatorname{proj}\Gamma$.
2. Sets of dotted arrows in $Q(\Gamma)$.

Moreover the Ausander-Reiten quiver of the exact category $(\mathscr{A},\mathscr{E})$ is given by $Q(\Gamma)$ with the dotted arrows in (2).

Proof. By definition each dotted arrow $\tau P \dashleftarrow P$ bijectively corresponds to a simple Γ module $P/\operatorname{rad} P$ satisfying the 2-regular condition. As Γ is local all the simple modules are of the form $P/\operatorname{rad} P$. Thus the correspondence follow from Theorem 6.2.5. It follows from Lemma 6.3.4 that $Q(\Gamma)$ is the Auslander-Reiten quiver of of $(\mathscr{A},\mathscr{E})$.

■

Example 6.3.6. In the situation of Theorem 6.3.5 let X be a set of dotted arrows. Let $\mathscr{E}$ be the corresponding exact structure on $\mathscr{A} = \operatorname{proj}\Gamma$. Then we have the following.

- X is empty if and only if $\mathscr{E}$ is the smallest exact structure, i.e. all split exact sequences.
- X is the set of all dotted arrows if and only if $\mathscr{E}$ is the unique maximal exact structure among the admissible ones on $\mathscr{A}$. If R is artinian this holds precisely when $\mathscr{E}$ is the unique maximal structure as all exact structures are admissible in this case.

7 Examples on Representations over Quivers

7.1 Finding the exact structures

Example 7.1.1. Consider the category $\mathscr{A} = \operatorname{rep} Q$ of representations of the quiver

$$Q: \qquad 1 \longrightarrow 2$$

Then we get the Auslander-Reiten quiver of $\mathscr{A}$ as follows

$$\begin{tikzcd} & P_2 \arrow[dr] & \\ P_1 \arrow[ur] & & I_1 \arrow[ll, dashed] \end{tikzcd}$$

As there is only one dotted arrow we know by Theorem 6.3.5 there is only two exact structures. We have the AR-sequence $(\mathrm{AR}\,1) = P_1 \to P_2 \to I_1$ as our only indecomposable non-split exact sequence in $\mathscr{A}$.

Now we consider the Auslander algebra of $\mathscr{A}$ to classify exact structures on $\operatorname{proj}\Gamma \cong \mathscr{A}$ (See Theorem 6.3.5 and Lemma 6.3.4). The Auslander algebra Γ of $\mathscr{A}$ is given by the following quiver with relations.

$$Q: \quad \begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] & & 3 \arrow[ll, dashed] \end{tikzcd}$$

In $\operatorname{proj}\Gamma$ our indecomposable non-split exact sequence will correspond to $P_3 \to P_2 \xrightarrow{g} P_1$. We have $\operatorname{Coker}(g) = S_1$ and we can now classify the exact structures using Theorem 6.2.5.

- $\{\emptyset\} \leftrightarrow \mathscr{E}_{\min} =$ All split exact sequences
- $\{S_1\} \leftrightarrow \mathscr{E}_{\max} = \{X \oplus Y | X \in \mathscr{E}_{\min}, Y \in \operatorname{add}(AR1)\} =$ All short exact sequences in $\mathscr{A}$

and note $\mathscr{E}_{\min} \subseteq \mathscr{E}_{\max}$

Example 7.1.2. Consider the category $\mathscr{A} = \operatorname{rep} Q$ of representations of the quiver

$$Q: \qquad 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$$

Then we get the Auslander-Reiten quiver of $\mathscr{A}$ as follows.

$$\begin{tikzcd}
& P_1 \arrow[dr] & & S_3 \arrow[ll, dashed] \\
S_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ll, dashed] \arrow[ur] \arrow[dr] & \\
& P_3 \arrow[ur] & & S_1 \arrow[ll, dashed]
\end{tikzcd}$$

As there are three dotted arrows we already know there are $2^3 = 8$ different exact structures on $\mathscr{A}$. We have the following indecomposable non split exact sequences in $\mathscr{A}$.

(AR1) $0 \to P_1 \to I_2 \to S_3 \to 0$

(AR2) $0 \to P_3 \to I_2 \to S_1 \to 0$

(AR3) $0 \to S_2 \to P_1 \oplus P_3 \to I_2 \to 0$

(4) $0 \to S_2 \to P_1 \to S_1 \to 0$

(5) $0 \to S_2 \to P_3 \to S_3 \to 0$

where (AR1), (AR2) and (AR3) are the Auslander-Reiten sequences.

Now we consider the Auslander algebra of $\mathscr{A}$ to classify exact structures on $\operatorname{proj}\Gamma$ as this corresponds to $\mathscr{A}$ (See Theorem 6.3.5 and Lemma 6.3.4). The Auslander algebra Γ of $\mathscr{A}$ is given by the quiver with relations.

$$Q : \quad \begin{tikzcd}
& 2 \arrow[dr] & & 5 \arrow[ll, dashed] \\
1 \arrow[ur] \arrow[dr] & & 4 \arrow[ll, dashed] \arrow[ur] \arrow[dr] & \\
& 3 \arrow[ur] & & 6 \arrow[ll, dashed]
\end{tikzcd}$$

In $\operatorname{proj}\Gamma$ the sequences above will correspond to

(AR1) $0 \to P_5 \to P_4 \xrightarrow{g_1} P_2 \to 0$

(AR2) $0 \to P_6 \to P_4 \xrightarrow{g_2} P_3 \to 0$

(AR3) $0 \to P_4 \to P_2 \oplus P_3 \xrightarrow{g_3} P_1 \to 0$

(4) $0 \to P_6 \to P_2 \xrightarrow{g_4} P_1 \to 0$

(5) $0 \to P_5 \to P_3 \xrightarrow{g_5} P_1 \to 0$

Using Lemma 6.3.4 we get the following simple modules in $\operatorname{proj}\Gamma$ satisfying the 2-regular condition: $\operatorname{Coker}(g_1) = S_2, \operatorname{Coker}(g_2) = S_3$ and $\operatorname{Coker}(g_3) = S_1$. We find $\operatorname{Coker}(g_4)$ to be the representation

$$\begin{tikzcd}
& 0 \arrow[dr] & & 0 \\
k \arrow[ur] \arrow[dr] & & 0 \arrow[ur] \arrow[dr] & \\
& k \arrow[ur] & & 0
\end{tikzcd}$$

which has composition factors S_1 and S_3. Next we see $\operatorname{Coker}(g_5)$ is the representation

$$\begin{tikzcd}
& k \arrow[dr] & & 0 \\
k \arrow[ur] \arrow[dr] & & 0 \arrow[ur] \arrow[dr] & \\
& 0 \arrow[ur] & & 0
\end{tikzcd}$$

which has composition factors S_1 and S_2. Now we are able to apply Theorem 6.2.5 to classify all exact structures on $\operatorname{proj}\Gamma \cong \mathscr{A}$.

- $\{\emptyset\} \leftrightarrow \mathscr{E}_{\min} =$ All split exact sequences
- $\{S_2\} \leftrightarrow \mathscr{E}_1 = \{X \oplus Y | X \in \mathscr{E}_{\min}, Y \in \operatorname{add}(\mathrm{AR}\,1)\}$
- $\{S_3\} \leftrightarrow \mathscr{E}_2 = \{X \oplus Y | X \in \mathscr{E}_{\min}, Y \in \operatorname{add}(\mathrm{AR}\,2)\}$
- $\{S_1\} \leftrightarrow \mathscr{E}_3 = \{X \oplus Y | X \in \mathscr{E}_{\min}, Y \in \operatorname{add}(\mathrm{AR}\,3)\}$
- $\{S_2, S_3\} \leftrightarrow \mathscr{E}_{1,2} = \{X \oplus Y | X \in \mathscr{E}_1, Y \in \mathscr{E}_2\}$
- $\{S_1, S_2\} \leftrightarrow \mathscr{E}_{1,3,5} = \{X \oplus Y \oplus Z | X \in \mathscr{E}_1, Y \in \mathscr{E}_3\ Z \in \operatorname{add}(5)\}$
- $\{S_1, S_3\} \leftrightarrow \mathscr{E}_{2,3,4} = \{X \oplus Y \oplus Z | X \in \mathscr{E}_2, Y \in \mathscr{E}_3\ Z \in \operatorname{add}(4)\}$
- $\{S_1, S_2, S_3\} \leftrightarrow \mathscr{E}_{\max} =$ All short exact sequences in $\mathscr{A}$

Thus we have classified all the exact structures. These can be represented in the following diagram where the arrows represent inclusions.

$$\begin{tikzcd}
 & \mathscr{E}_3 \arrow[dl] \arrow[rr] & & \mathscr{E}_{1,3,5} \arrow[dl] \\
\mathscr{E}_{2,3,4} \arrow[rr] & & \mathscr{E}_{\max} & \\
 & \mathscr{E}_{\min} \arrow[uu] \arrow[dl] \arrow[rr] & & \mathscr{E}_1 \arrow[uu] \arrow[dl] \\
\mathscr{E}_2 \arrow[uu] \arrow[rr] & & \mathscr{E}_{1,2} \arrow[uu] &
\end{tikzcd}$$

Example 7.1.3. Consider the category $\mathscr{A} = \operatorname{rep} Q$ of representations of the quiver

$$Q: \qquad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

Then we get the Auslander-Reiten quiver of $\mathscr{A}$ as follows.

$$\begin{tikzcd}
 & & P_1 \arrow[dr] & & \\
 & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ll, dashed] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ll, dashed] \arrow[ur] & & S_1 \arrow[ll, dashed]
\end{tikzcd}$$

As there are three dotted arrows we already know there are $2^3 = 8$ different exact structures on $\mathscr{A}$. We have the following indecomposable non split exact sequences in $\mathscr{A}$.

(AR1) $0 \to P_3 \to P_2 \to S_2 \to 0$

(AR2) $0 \to S_2 \to I_2 \to S_1 \to 0$

(AR3) $0 \to P_2 \to P_1 \oplus S_2 \to I_2 \to 0$

(4) $0 \to P_3 \to P_1 \to I_2 \to 0$

(5) $0 \to P_2 \to P_1 \to S_1 \to 0$

where (AR1),(AR2) and (AR3) are the Auslander-Reiten sequences.

Now we consider the Auslander algebra of $\mathscr{A}$ to classify exact structures on $\operatorname{proj} \Gamma \cong$

$\mathscr{A}$. The Auslander algebra Γ of $\mathscr{A}$ is given by the following quiver with relations.

$$Q: \quad \begin{tikzcd}
 & & 3 \arrow[dr] & & \\
 & 2 \arrow[ur] \arrow[dr] & & 5 \arrow[ll, dashed] \arrow[dr] & \\
1 \arrow[ur] & & 4 \arrow[ll, dashed] \arrow[ur] & & 6 \arrow[ll, dashed]
\end{tikzcd}$$

In $\operatorname{proj}\Gamma$ the sequences above will correspond to

(AR1) $0 \to P_4 \to P_2 \xrightarrow{g_1} P_1 \to 0$

(AR2) $0 \to P_6 \to P_5 \xrightarrow{g_2} P_4 \to 0$

(AR3) $0 \to P_5 \to P_3 \oplus P_4 \xrightarrow{g_3} P_2 \to 0$

(4) $0 \to P_5 \to P_3 \xrightarrow{g_4} P_1 \to 0$

(5) $0 \to P_6 \to P_3 \xrightarrow{g_5} P_2 \to 0$

Using Lemma 6.3.4 we get the following simple modules in $\operatorname{proj}\Gamma$ satisfying the 2-regular condition: $\operatorname{Coker}(g_1) = S_1, \operatorname{Coker}(g_2) = S_4$ and $\operatorname{Coker}(g_3) = S_2$.

We find $\operatorname{Coker}(g_4)$ to be the representation

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & \\
 & k \arrow[ur] \arrow[dr] & & 0 \arrow[dr] & \\
k \arrow[ur] & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

which has composition factors S_1 and S_2. Next we see $\operatorname{Coker}(g_5)$ is the representation

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & \\
 & k \arrow[ur] \arrow[dr] & & 0 \arrow[dr] & \\
0 \arrow[ur] & & k \arrow[ur] & & 0
\end{tikzcd}$$

which has composition factors S_2 and S_4. Now we are able to apply Theorem 6.2.5 to classify all exact structures on $\operatorname{proj}\Gamma \cong \mathscr{A}$.

- $\{\emptyset\} \leftrightarrow \mathscr{E}_{\min} =$ All split exact sequences

- $\{S_1\} \leftrightarrow \mathscr{E}_1 = \{X \oplus Y | X \in \mathscr{E}_{\min}, Y \in \operatorname{add}(\mathrm{AR}\,1)\}$
- $\{S_4\} \leftrightarrow \mathscr{E}_2 = \{X \oplus Y | X \in \mathscr{E}_{\min}, Y \in \operatorname{add}(\mathrm{AR}\,2)\}$
- $\{S_2\} \leftrightarrow \mathscr{E}_3 = \{X \oplus Y | X \in \mathscr{E}_{\min}, Y \in \operatorname{add}(\mathrm{AR}\,3)\}$
- $\{S_1, S_4\} \leftrightarrow \mathscr{E}_{1,2} = \{X \oplus Y | X \in \mathscr{E}_1, Y \in \mathscr{E}_2\}$
- $\{S_1, S_2\} \leftrightarrow \mathscr{E}_{1,3,4} = \{X \oplus Y \oplus Z | X \in \mathscr{E}_1, Y \in \mathscr{E}_3, Z \in \operatorname{add}(4)\}$
- $\{S_2, S_4\} \leftrightarrow \mathscr{E}_{2,3,5} = \{X \oplus Y \oplus Z | X \in \mathscr{E}_2, Y \in \mathscr{E}_3, Z \in \operatorname{add}(5)\}$
- $\{S_1, S_2, S_4\} \leftrightarrow \mathscr{E}_{\max} =$ All short exact sequences in $\mathscr{A}$

Thus we have classified all the exact structures. These can be represented in the following diagram where the arrows represent inclusions.

$$\begin{tikzcd}
& \mathscr{E}_3 \arrow[rr] \arrow[dl] & & \mathscr{E}_{1,3,4} \arrow[dl] \\
\mathscr{E}_{2,3,5} \arrow[rr] & & \mathscr{E}_{\max} & \\
& \mathscr{E}_{\min} \arrow[uu] \arrow[rr] \arrow[dl] & & \mathscr{E}_1 \arrow[uu] \arrow[dl] \\
\mathscr{E}_2 \arrow[uu] \arrow[rr] & & \mathscr{E}_{1,2} \arrow[uu] &
\end{tikzcd}$$

7.2 Finding the derived categories of the exact categories

In this section we find the derived categories of the different exact structures found in the previous subsection.

We denote degree i of complexes by an i over degree i and let brackets denote zeroes further left and right. Which will mean that the complex

$$[A \to B \to \overset{i}{Y} \to M]$$

is concentrated in degree $i-2, i-1, i, i+1$ and have Y in degree i. We will denote by $A[i]$ the complex with A concentrated in degree zero shifted by i. We will denote triangulated AR- translations by dotted arrows in diagrams. We will refer to the repeating patterns that appear by taking triangulated AR-translations as τ-orbits or simply orbits.

Example 7.2.1. In this example we will find the derived categories of the exact structures in $\operatorname{proj}\Gamma$ from Example 7.1.1. We know that

$$\mathcal{K}^{\mathrm{b}}(\mathscr{A}) \cong \mathcal{K}^{\mathrm{b}}(\operatorname{proj}\Gamma) \cong \mathscr{D}^{\mathrm{b}}(\mathbf{mod}\Gamma)$$

where $\mathcal{K}^{\mathrm{b}}(\operatorname{proj}\Gamma)$ correspond the Auslander Reiten algebra of $\mathscr{A}$. In the proof of C.13 found in [8, Section 3.6] we have a way of obtaining all of $\mathscr{D}^{\mathrm{b}}(\mathbf{mod}\Gamma)$ through triangulated AR- translations.
We start with $P_1[0]$ which have $I_2[0]$ as an injective resolution thus $\tau(P_1[0]) = P_2[1]$. Next an injective resolution of P_2 is I_3 which yield $\tau(P_2[1]) = P_3[2]$. An injective resolution of P_3 is $I_3 \to I_2 \to I_3$. This gives us $\tau(P_3[2]) = [P_3 \to \overset{0}{P_2} \to P_1]$. We note this is quasi isomorphic to $I_1[1]$ and get $\tau([P_3 \to \overset{0}{P_2} \to P_1]) = P_1[2]$. Thus we will have the following repeating pattern.

$$\ldots \dashleftarrow P_1[0] \dashleftarrow P_2[1] \dashleftarrow P_3[2] \dashleftarrow [P_3 \to P_2 \to \overset{-1}{P_1}] \dashleftarrow P_1[2] \dashleftarrow \ldots$$

Now we need to fill in the triangles to complete the quiver of $\mathcal{K}^{\mathrm{b}}(\operatorname{proj}\Gamma)$. We have that $\mathrm{Cone}(P_2[0] \to P_1[0]) = [P_2 \to \overset{0}{P_1}]$. Next we know $\tau(P_1[0])$, $\tau(P_2[0])$ and $\tau(P_3[0])$ so we easily find the translations $\tau([P_2 \to \overset{0}{P_1}]) = [P_3 \to \overset{-1}{P_2}]$ and $\tau([P_3 \to \overset{-1}{P_2}]) = [P_2 \to \overset{-1}{P_1}]$. Now we calculate $\mathrm{Cone}([P_3 \to \overset{0}{P_2}] \to [P_2 \to \overset{0}{P_1}]) = P_2[1] \oplus [P_3 \to P_2 \to \overset{0}{P_1}]$ and note that we can complete a diagram for $\mathcal{K}^{\mathrm{b}}(\operatorname{proj}\Gamma)$.

$$\begin{tikzcd}
\cdots & P_1[0] \arrow[l, dashed] \arrow[dr] & & P_2[1] \arrow[ll, dashed] \arrow[dr] & & P_3[2] \arrow[ll, dashed] \arrow[dr] & & {[P_3 \to P_2 \to \overset{-1}{P_1}]} \arrow[ll, dashed] & \cdots \arrow[l, dashed] \\
\cdots & & {[P_2 \to \overset{0}{P_1}]} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & {[P_3 \to \overset{-1}{P_2}]} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & {[P_2 \to \overset{-1}{P_1}]} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & \cdots \arrow[ll, dashed] \\
\cdots & P_3[1] \arrow[l, dashed] \arrow[ur] & & {[P_3 \to P_2 \to \overset{0}{P_1}]} \arrow[ll, dashed] \arrow[ur] & & P_1[1] \arrow[ll, dashed] \arrow[ur] & & P_2[2] \arrow[ll, dashed] & \cdots \arrow[l, dashed]
\end{tikzcd}$$

As seen in Example 4.4.6 this also gives $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_{\min})$. Now we want to find $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_{\max})$. We recall that $P_3 \to P_2 \to P_1$ is acyclic over $(\mathscr{A}, \mathscr{E}_{\max})$. Thus we see using Proposition 4.4.10 in our diagram of $\mathcal{K}^{\mathrm{b}}(\operatorname{proj}\Gamma)$ that $[P_3 \to P_2 \to \overset{i}{P_1}], [P_3 \to \overset{i}{P_2}]$ and $P_3[i]$ are superfluous in $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_{\max})$. Now we have the following diagram for $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_{\max})$.

$$\begin{tikzcd}
\cdots & P_1[0] \arrow[l, dashed] \arrow[dr] & & P_2[1] \arrow[ll, dashed] \arrow[dr] & & {[P_2 \to \overset{-1}{P_1}]} \arrow[ll, dashed] \arrow[dr] & & P_1[2] \arrow[ll, dashed] \arrow[dr] & \cdots \arrow[l, dashed] \\
\cdots & & {[P_2 \to \overset{0}{P_1}]} \arrow[ll, dashed] \arrow[ur] & & P_1[1] \arrow[ll, dashed] \arrow[ur] & & P_2[2] \arrow[ll, dashed] \arrow[ur] & & {[P_2 \to \overset{-2}{P_1}]} \arrow[ll, dashed] & \cdots \arrow[l, dashed]
\end{tikzcd}$$

Example 7.2.2. In this example we will find the derived categories of the exact structures in $\operatorname{proj}\Gamma$ from Example 7.1.2. Tedious calculations are excluded and the reader is encouraged to check that the τ-orbits and cones are correct themselves. We start by obtaining $\mathcal{K}^{\mathrm{b}}(\operatorname{proj}\Gamma)$, starting with the τ-orbit of $P_1[0]$.

$$P_1[0] \dashleftarrow P_4[1] \dashleftarrow [P_5 \oplus P_6 \to P_4 \to \overset{0}{P_1}] \dashleftarrow [P_2 \oplus P_3 \to \overset{0}{P_1}] \dashleftarrow$$

$$\dashleftarrow [P_5 \oplus P_6 \to \overset{-1}{P_4}] \dashleftarrow [P_4 \to P_2 \oplus P_3 \to \overset{0}{P_1}] \dashleftarrow P_1[1]$$

We calculate $\mathrm{Cone}(P_4[0] \to P_1[0]) = [P_4 \to \overset{0}{P_1}]$, then we find the τ- orbit of $[P_4 \to \overset{0}{P_1}]$.

$$[P_4 \to \overset{0}{P_1}] \dashleftarrow [P_5 \oplus P_6 \to P_4^2 \to \overset{0}{P_1}] \dashleftarrow [P_5 \oplus P_6 \to P_2 \oplus P_3 \oplus P_4 \to \overset{0}{P_1^2}] \dashleftarrow [P_5 \oplus P_6 \to P_2 \oplus P_3 \oplus P_4 \to \overset{0}{P_1}] \dashleftarrow$$

$$\dashleftarrow [P_5 \oplus P_6 \to P_2 \oplus P_3 \to \overset{0}{P_1}] \dashleftarrow [P_4 \to P_2 \overset{-1}{\oplus} P_3] \dashleftarrow [P_4 \to \overset{-1}{P_1}]$$

Next we see

$$\begin{aligned}
&\mathrm{Cone}([P_5 \oplus P_6 \to P_4^2 \to \overset{1}{P_1}] \to [P_4 \to \overset{0}{P_1}]) \\
&= [P_5 \oplus P_6 \to P_4^3 \to \overset{0}{P_1^2}] \\
&\cong P_4[1] \oplus [P_5 \to P_4 \to \overset{0}{P_1}] \oplus [P_6 \to P_4 \to \overset{0}{P_1}]
\end{aligned}$$

So we calculate the orbits

$$[P_5 \to P_4 \to \overset{0}{P_1}] \dashleftarrow [P_6 \to P_2 \oplus P_4 \to \overset{0}{P_1}] \dashleftarrow [P_5 \oplus P_6 \to P_3 \oplus P_4 \to \overset{0}{P_1}] \dashleftarrow [P_5 \to P_2 \oplus P_3 \to \overset{0}{P_1}] \dashleftarrow$$

$$\dashleftarrow [P_6 \to \overset{-1}{P_2}] \dashleftarrow [P_4 \to \overset{-1}{P_3}] \dashleftarrow [P_6 \to P_4 \to \overset{-1}{P_1}]$$

And

$$[P_6 \to P_4 \to \overset{0}{P_1}] \dashleftarrow [P_5 \to P_3 \oplus P_4 \to \overset{0}{P_1}] \dashleftarrow [P_5 \oplus P_6 \to P_2 \oplus P_3 \to \overset{0}{P_1}] \dashleftarrow [P_6 \to P_2 \oplus P_3 \to \overset{0}{P_1}] \dashleftarrow$$

$$\dashleftarrow [P_5 \to \overset{-1}{P_3}] \dashleftarrow [P_4 \to \overset{-1}{P_2}] \dashleftarrow [P_5 \to P_4 \to \overset{-1}{P_1}]$$

then find the cones

$$\begin{aligned}
&\mathrm{Cone}([P_6 \to P_2 \oplus P_4 \to \overset{1}{P_1}] \to [P_5 \to P_4 \to \overset{0}{P_1}]) \\
&= [P_5 \oplus P_6 \to P_2 \oplus P_4^2 \to \overset{0}{P_1^2}] \\
&\cong [P_5 \oplus P_6 \to P_4^2 \to \overset{0}{P_1}] \oplus [P_2 \to \overset{0}{P_1}]
\end{aligned}$$

and

$$
\begin{aligned}
&\mathrm{Cone}([P_5 \to P_3 \oplus P_4 \to \overset{1}{P_1}] \to [P_6 \to P_4 \to \overset{0}{P_1}]) \\
&= [P_5 \oplus P_6 \to P_3 \oplus P_4^2 \to \overset{0}{P_1^2}] \\
&\cong [P_5 \oplus P_6 \to P_4^2 \to \overset{0}{P_1}] \oplus [P_3 \to \overset{0}{P_1}]
\end{aligned}
$$

We see that there are even more orbits to calculate. We get the two orbits

$$[P_2 \to \overset{0}{P_1}] \dashleftarrow [P_6 \to \overset{-1}{P_4}] \dashleftarrow [P_5 \to P_3 \to \overset{0}{P_1}] \dashleftarrow P_2[1] \dashleftarrow P_6[2] \dashleftarrow [P_6 \to P_4 \to \overset{-1}{P_3}] \dashleftarrow [P_3 \to \overset{-1}{P_1}]$$

$$[P_3 \to \overset{0}{P_1}] \dashleftarrow [P_5 \to \overset{-1}{P_4}] \dashleftarrow [P_6 \to P_2 \to \overset{0}{P_1}] \dashleftarrow P_3[1] \dashleftarrow P_5[2] \dashleftarrow [P_5 \to P_4 \to \overset{-1}{P_2}] \dashleftarrow [P_2 \to \overset{-1}{P_1}]$$

Now we see that we have $\mathrm{Cone}([P_6 \to \overset{0}{P_4}] \to [P_2 \to \overset{0}{P_1}]) = [P_6 \to P_2 \oplus P_4 \to \overset{0}{P_1}]$ and $\mathrm{Cone}([P_5 \to \overset{0}{P_4}] \to [P_3 \to \overset{0}{P_1}]) = [P_5 \to P_3 \oplus P_4 \to \overset{0}{P_1}]$. Thus we have found all the orbits and are able to draw the diagram for $\mathcal{K}^{\mathrm{b}}(\operatorname{proj}\Gamma)$ which is found at the end of the example together with all the derived categories. Luckily $\mathcal{K}^{\mathrm{b}}(\operatorname{proj}\Gamma) \cong \mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_{\min})$ so we only need to find the derived category for the other seven structures. Now is a good time to recall the exact structures from Example 7.1.2. Using Proposition 4.4.10 we can cross out the following in $\mathcal{K}^{\mathrm{b}}(\operatorname{proj}\Gamma)$ to obtain $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_1)$.

$$
\begin{aligned}
&[P_5 \to P_4 \to \overset{i}{P_2}], [P_5 \to P_4 \to \overset{i}{P_1}], [P_5 \oplus P_6 \to P_4 \oplus P_4 \to \overset{i}{P_1}], \\
&[P_5 \oplus P_6 \to P_4 \to \overset{i}{P_1}], [P_5 \to P_3 \oplus P_4 \to \overset{i}{P_1}], [P_5 \to \overset{i}{P_4}] \\
&[P_5 \oplus P_6 \to P_2 \oplus P_3 \oplus P_4 \to P_1 \overset{i}{\oplus} P_1], [P_5 \oplus P_6 \to P_3 \oplus P_4 \to \overset{i}{P_1}], [P_5 \to P_3 \to \overset{i}{P_1}] \\
&[P_5 \oplus P_6 \to P_3 \oplus P_4 \to \overset{i}{P_1}], [P_5 \oplus P_6 \to P_2 \oplus P_3 \oplus P_4 \to \overset{i}{P_1}], [P_5 \oplus P_6 \to \overset{i}{P_4}] \\
&[P_5 \to P_2 \oplus P_3 \to \overset{i}{P_1}], [P_5 \oplus P_6 \to P_2 \oplus P_3 \to \overset{i}{P_1}], [P_5 \to \overset{i}{P_3}], P_5[i]
\end{aligned}
$$

and we can draw the diagram for $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_1)$ with the remaining triangles. Simmilarly for $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_2)$ we cross out

$$
\begin{aligned}
&[P_6 \to P_4 \to \overset{i}{P_3}], [P_6 \to P_4 \to \overset{i}{P_1}], [P_5 \oplus P_6 \to P_4 \oplus P_4 \to \overset{i}{P_1}], \\
&[P_5 \oplus P_6 \to P_4 \to \overset{i}{P_1}], [P_6 \to P_2 \oplus P_4 \to \overset{i}{P_1}], [P_6 \to \overset{i}{P_4}] \\
&[P_5 \oplus P_6 \to P_2 \oplus P_3 \oplus P_4 \to P_1 \overset{i}{\oplus} P_1], [P_5 \oplus P_6 \to P_2 \oplus P_3 \to \overset{i}{P_1}], [P_6 \to P_2 \to \overset{i}{P_1}] \\
&[P_5 \oplus P_6 \to P_3 \oplus P_4 \to \overset{i}{P_1}], [P_5 \oplus P_6 \to P_2 \oplus P_3 \oplus P_4 \to \overset{i}{P_1}], [P_5 \oplus P_6 \to \overset{i}{P_4}] \\
&[P_5 \to P_2 \oplus P_3 \to \overset{i}{P_1}], [P_5 \oplus P_6 \to P_2 \oplus P_3 \to \overset{i}{P_1}], [P_6 \to \overset{i}{P_2}], P_6[i]
\end{aligned}
$$

and in $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_3)$ the following.

$$[P_4 \to P_2 \oplus P_3 \to \overset{i}{P_1}], [P_4 \to P_2 \overset{i}{\oplus} P_3], [P_4 \to \overset{i}{P_2}]$$

$$[P_5 \to P_4 \to \overset{i}{P_2}], [P_4 \to \overset{i}{P_3}], [P_6 \to P_4 \to \overset{i}{P_3}]$$

$$[P_4 \to \overset{i}{P_1}], P_4[i], [P_5 \to P_4 \to \overset{i}{P_1}], [P_6 \to P_4 \to \overset{i}{P_1}]$$

$$[P_5 \oplus P_6 \to P_4 \oplus P_4 \to \overset{i}{P_1}], [P_5 \oplus P_6 \to P_4 \to \overset{i}{P_1}], [P_6 \to P_2 \oplus P_4 \to \overset{i}{P_1}]$$

$$[P_6 \to \overset{i}{P_4}], [P_5 \to P_3 \oplus P_4 \to \overset{i}{P_1}], [P_5 \to \overset{i}{P_4}]$$

$$[P_5 \oplus P_6 \to P_2 \oplus P_3 \oplus P_4 \to P_1 \overset{i}{\oplus} P_1], [P_5 \oplus P_6 \to P_3 \oplus P_4 \to \overset{i}{P_1}]$$

$$[P_5 \oplus P_6 \to P_2 \oplus P_3 \to \overset{i}{P_1}], [P_5 \oplus P_6 \to P_2 \oplus P_3 \oplus P_4 \to \overset{i}{P_1}], [P_5 \oplus P_6 \to \overset{i}{P_4}]$$

Thus we can draw these too. To find $\mathscr{D}^b(\operatorname{proj}\Gamma, \mathscr{E}_{1,2})$ we can consider the diagram for $\mathscr{D}^b(\operatorname{proj}\Gamma, \mathscr{E}_1)$ and cross out the complexes that became redundant in $\mathscr{D}^b(\operatorname{proj}\Gamma, \mathscr{E}_2)$ (or vice versa). To find $\mathscr{D}^b(\operatorname{proj}\Gamma, \mathscr{E}_{2,3,4})$ we can choose to look at $\mathscr{D}^b(\operatorname{proj}\Gamma, \mathscr{E}_3)$ and note that the following "vanish".

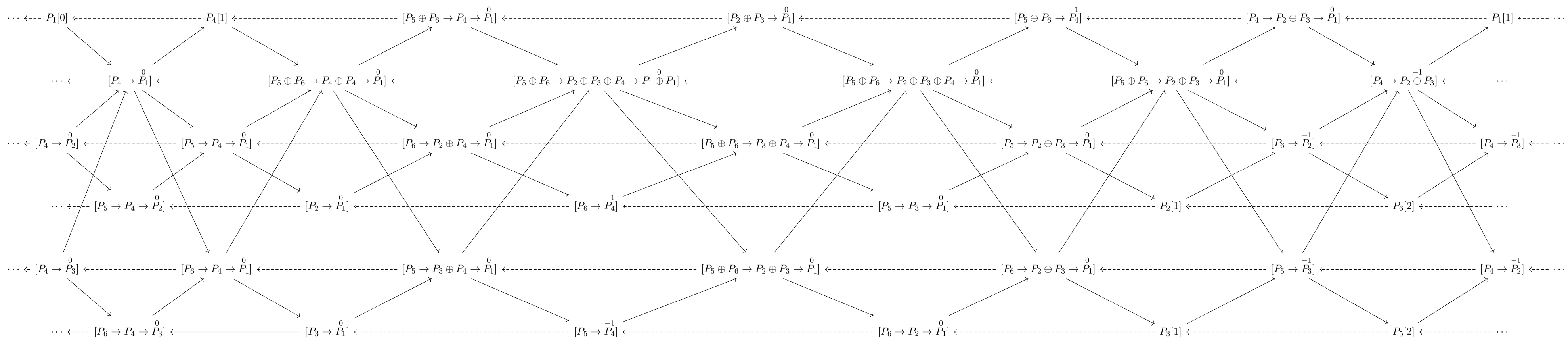
$$[P_6 \to P_2 \to \overset{i}{P_1}], [P_6 \to P_2 \oplus P_3 \to \overset{i}{P_1}], [P_5 \oplus P_6 \to P_2 \oplus P_3 \to \overset{i}{P_1}], [P_6 \to \overset{i}{P_2}], P_6[i]$$

Similarly to find $\mathscr{D}^b(\operatorname{proj}\Gamma, \mathscr{E}_{1,3,5})$ we look at $\mathscr{D}^b(\operatorname{proj}\Gamma, \mathscr{E}_3)$ and note that the following become superfluous.

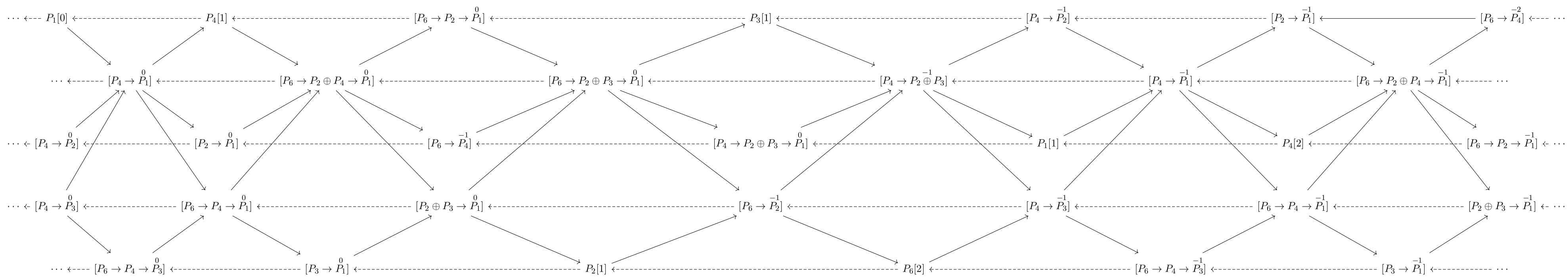
$$[P_5 \to P_3 \to \overset{i}{P_1}], [P_5 \to P_2 \oplus P_3 \to \overset{i}{P_1}], [P_5 \oplus P_6 \to P_2 \oplus P_3 \to \overset{i}{P_1}], [P_5 \to \overset{i}{P_3}], P_5[i]$$

Thus we have found the derived categories of the different exact structures which will be presented in the following four pages.

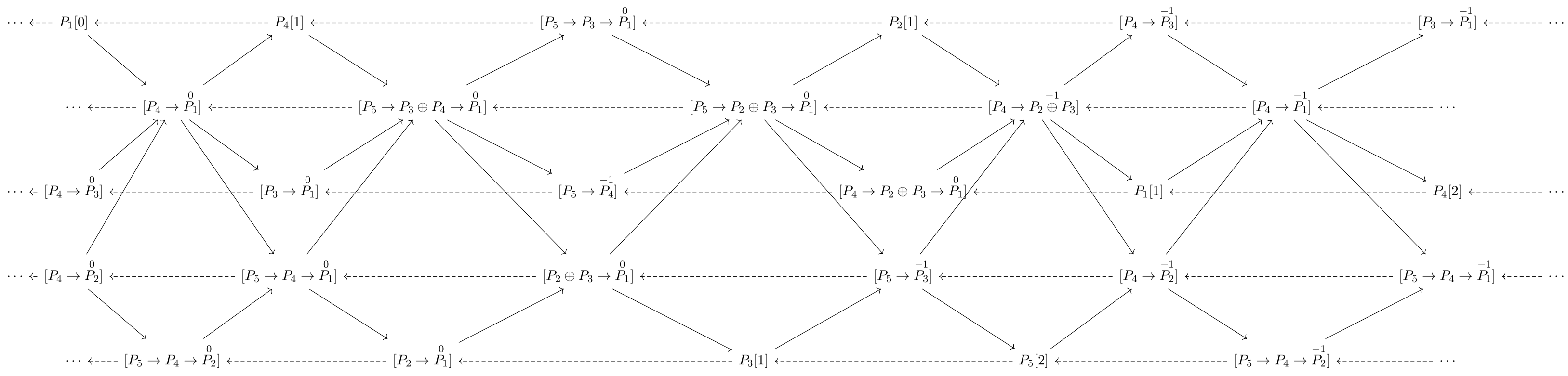
$\mathcal{K}^{\mathrm{b}}(\operatorname{proj}\Gamma) \cong \mathscr{D}(\operatorname{proj}\Gamma, \mathscr{E}_{\min})$



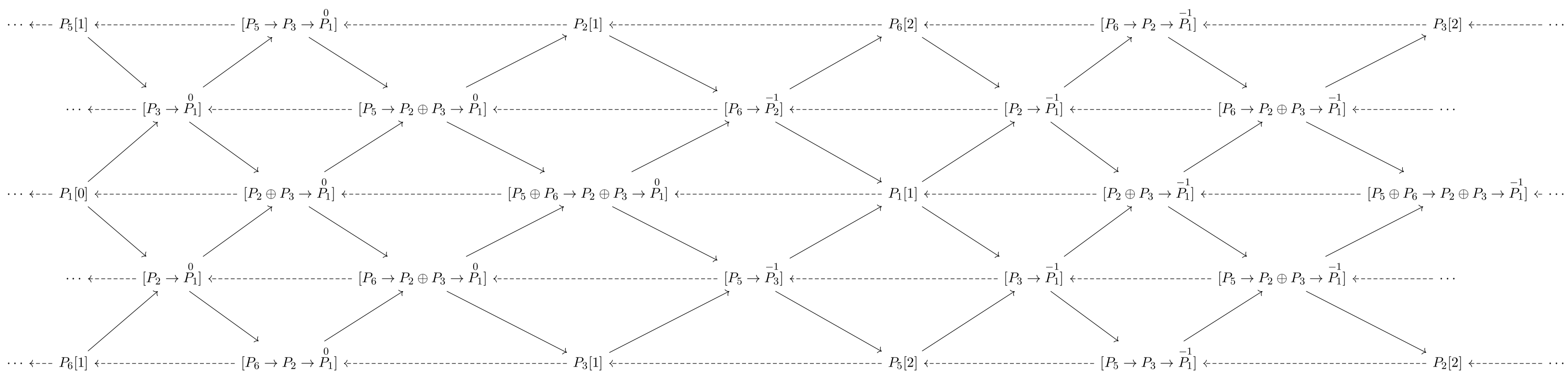
$\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_1)$



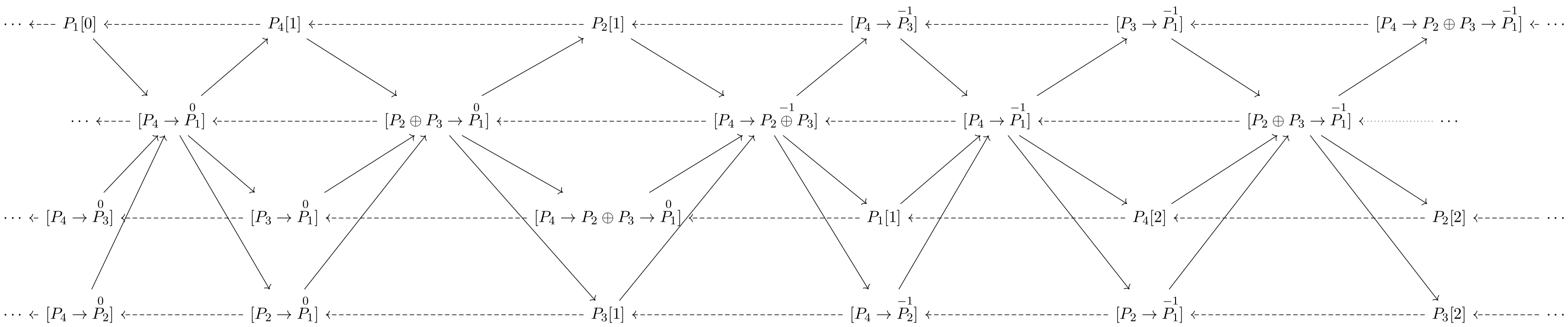
$\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_2)$



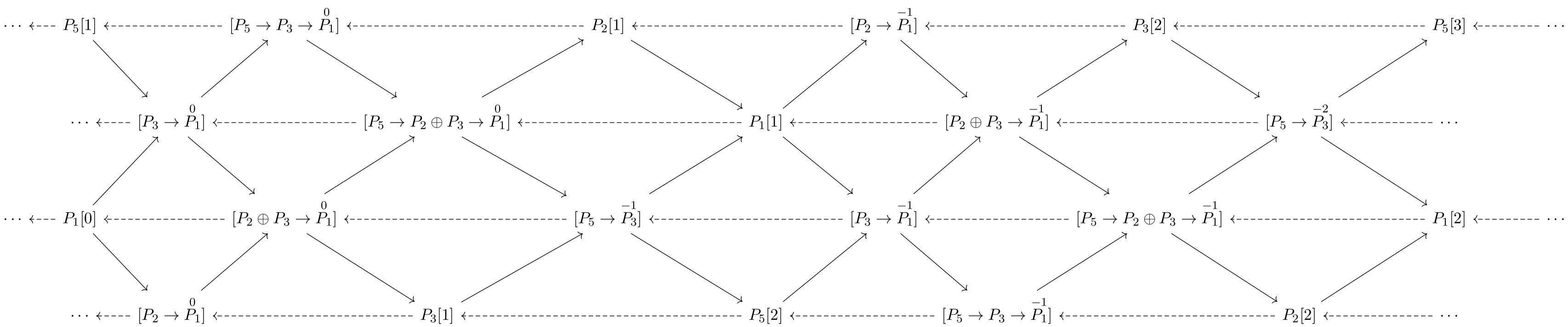
$\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_3)$



$\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_{1,2})$



$\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_{2,3,4})$



$\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_{1,3,5})$

$$
\begin{tikzcd}[column sep=small]
\cdots & P_6[1] \arrow[l] \arrow[dr] & & {[P_6 \to P_2 \to \overset{0}{P_1}]} \arrow[ll, dashed] \arrow[dr] & & P_3[1] \arrow[ll, dashed] \arrow[dr] & & {[P_3 \to \overset{-1}{P_1}]} \arrow[ll, dashed] \arrow[dr] & & P_2[2] \arrow[ll, dashed] \arrow[dr] & & P_6[3] \arrow[ll, dashed] & {} \arrow[l, dashed] \\
& \cdots & {[P_2 \to \overset{0}{P_1}]} \arrow[l, dashed] \arrow[ur] \arrow[dr] & & {[P_6 \to P_2 \oplus P_3 \to \overset{0}{P_1}]} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & P_1[1] \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & {[P_2 \oplus P_3 \to \overset{-1}{P_1}]} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & {[P_6 \to \overset{-2}{P_2}]} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & \cdots \arrow[l, dashed] & \\
\cdots & P_1[0] \arrow[l] \arrow[ur] \arrow[dr] & & {[P_2 \oplus P_3 \to \overset{0}{P_1}]} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & {[P_6 \to \overset{-1}{P_2}]} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & {[P_2 \to \overset{-1}{P_1}]} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & {[P_6 \to P_2 \oplus P_3 \to \overset{-1}{P_1}]} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & P_1[2] \arrow[ll, dashed] & \cdots \arrow[l, dashed] \\
& \cdots & {[P_3 \to \overset{0}{P_1}]} \arrow[l, dashed] \arrow[ur] & & P_2[1] \arrow[ll, dashed] \arrow[ur] & & P_6[2] \arrow[ll, dashed] \arrow[ur] & & {[P_6 \to P_2 \to \overset{-1}{P_1}]} \arrow[ll, dashed] \arrow[ur] & & P_3[2] \arrow[ll, dashed] \arrow[ur] & \cdots \arrow[l, dashed] &
\end{tikzcd}
$$

$\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_{\max})$

$$
\begin{tikzcd}[column sep=small]
& \cdots & {[P_2 \to \overset{0}{P_1}]} \arrow[l, dashed] \arrow[dr] & & P_3[1] \arrow[ll, dashed] \arrow[dr] & & {[P_3 \to \overset{-1}{P_1}]} \arrow[ll, dashed] \arrow[dr] & & P_2[2] \arrow[ll, dashed] \arrow[dr] & & {[P_2 \to \overset{-2}{P_1}]} \arrow[ll, dashed] \arrow[dr] & \cdots \arrow[l, dashed] & \\
\cdots & P_1[0] \arrow[l] \arrow[ur] \arrow[dr] & & {[P_2 \oplus P_3 \to \overset{0}{P_1}]} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & P_1[1] \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & {[P_2 \oplus P_3 \to \overset{-1}{P_1}]} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & P_1[2] \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & {[P_2 \oplus P_3 \to \overset{-2}{P_1}]} \arrow[ll, dashed] & \cdots \arrow[l] \\
& \cdots & {[P_3 \to \overset{0}{P_1}]} \arrow[l, dashed] \arrow[ur] & & P_2[1] \arrow[ll, dashed] \arrow[ur] & & {[P_2 \to \overset{-1}{P_1}]} \arrow[ll, dashed] \arrow[ur] & & P_3[2] \arrow[ll, dashed] \arrow[ur] & & {[P_3 \to \overset{-2}{P_1}]} \arrow[ll, dashed] \arrow[ur] & \cdots \arrow[l, dashed] &
\end{tikzcd}
$$

Example 7.2.3. In this example we will find the derived categories of the exact structures in $\operatorname{proj}\Gamma$ from Example 7.1.3. Tedious calculations are excluded and the reader is encouraged to check that the τ-orbits and cones are correct themselves. We start by obtaining $\mathcal{K}^{\mathrm{b}}(\operatorname{proj}\Gamma)$, starting with the $\tau-$orbit of $P_1[0]$.

$$P_1[0] \dashleftarrow P_3[1] \dashleftarrow P_6[2] \dashleftarrow [P_6 \to P_5 \to \overset{-1}{P_4}] \dashleftarrow [P_4 \to P_2 \to \overset{0}{P_1}] \dashleftarrow P_1[1]$$

We calculate $\operatorname{Cone}(P_3[0] \to P_1[0]) = [P_3 \to \overset{0}{P_1}]$, then we find the $\tau-$ orbit of $[P_3 \to \overset{0}{P_1}]$.

$$[P_3 \to \overset{0}{P_1}] \dashleftarrow [P_6 \to \overset{-1}{P_3}] \dashleftarrow [P_5 \to \overset{-1}{P_4}] \dashleftarrow$$

$$\dashleftarrow [P_6 \to P_5 \to P_2 \to \overset{0}{P_1}] \dashleftarrow [P_4 \to \overset{-1}{P_2}] \dashleftarrow [P_3 \to \overset{-1}{P_1}]$$

Next we see $\operatorname{Cone}([P_6 \to \overset{0}{P_3}] \to [P_3 \to \overset{0}{P_1}]) = [P_6 \to P_3 \to \overset{0}{P_1}] \oplus P_3[1]$ and find a new orbit.

$$[P_6 \to P_3 \to \overset{0}{P_1}] \dashleftarrow [P_5 \to P_3 \overset{-1}{\oplus} P_4] \dashleftarrow [P_5 \to P_2 \to \overset{0}{P_1}] \dashleftarrow$$

$$\dashleftarrow [P_6 \to P_5 \to \overset{-1}{P_2}] \dashleftarrow [P_3 \oplus P_4 \to \overset{-1}{P_2}] \dashleftarrow [P_6 \to P_3 \to \overset{-1}{P_1}]$$

We calculate cone again

$$\operatorname{Cone}([P_5 \to P_3 \overset{0}{\oplus} P_4] \to [P_6 \to P_3 \to \overset{0}{P_1}]) = [P_6 \to \overset{-1}{P_3}] \oplus [P_5 \to P_3 \oplus P_4 \to \overset{0}{P_1}]$$

then find a τ-orbit.

$$[P_5 \to P_3 \oplus P_4 \to \overset{0}{P_1}] \dashleftarrow [P_5 \to P_2 \oplus P_3 \to \overset{0}{P_1}] \dashleftarrow [P_5 \to \overset{-1}{P_2}] \dashleftarrow$$

$$\dashleftarrow [P_6 \to P_3 \oplus P_5 \to \overset{-1}{P_2}] \dashleftarrow [P_6 \to P_3 \oplus P_4 \to \overset{0}{P_2}] \dashleftarrow [P_5 \to P_3 \oplus P_4 \to \overset{0}{P_1}]$$

Next we get

$$\operatorname{Cone}([P_5 \to P_2 \oplus P_3 \to \overset{1}{P_1}] \to [P_5 \to P_3 \oplus P_4 \to \overset{0}{P_1}]) = [P_5 \to P_3 \overset{-1}{\oplus} P_4] \oplus [P_2 \to \overset{0}{P_1}] \oplus [P_5 \to P_3 \to \overset{0}{P_1}]$$

and calculate the two orbits

$$[P_5 \to P_3 \to \overset{0}{P_1}] \dashleftarrow P_2[1] \dashleftarrow P_5[2] \dashleftarrow [P_6 \to P_3 \to \overset{-1}{P_2}] \dashleftarrow P_4[2] \dashleftarrow [P_5 \to P_3 \to \overset{-1}{P_1}]$$

$$[P_2 \to \overset{0}{P_1}] \dashleftarrow [P_5 \to \overset{-1}{P_3}] \dashleftarrow [P_3 \to \overset{-1}{P_2}] \dashleftarrow [P_6 \to \overset{-2}{P_5}] \dashleftarrow [P_5 \to P_3 \oplus P_4 \to \overset{-1}{P_2}] \dashleftarrow [P_2 \to \overset{-1}{P_1}]$$

Now we see that we have $\mathrm{Cone}(P_2[0] \to [P_5 \to P_3 \to \overset{0}{P_1}]) = [P_5 \to P_2 \oplus P_3 \to \overset{0}{P_1}]$ and $\mathrm{Cone}([P_5 \to \overset{0}{P_3}] \to [P_2 \to \overset{0}{P_1}]) = [P_5 \to P_2 \oplus P_3 \to \overset{0}{P_1}]$. Thus we have found all the orbits and are able to draw the diagram for $\mathcal{K}^{\mathrm{b}}(\operatorname{proj}\Gamma)$ which is found at the end of the example together with all the derived categories. Now is a good time to recall the exact structures from Example 7.1.3. Using Proposition 4.4.10 we see can cross out the following in $\mathcal{K}^{\mathrm{b}}(\operatorname{proj}\Gamma)$ to obtain $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_1)$.

$[P_4 \to P_2 \to \overset{i}{P_1}], [P_4 \to \overset{i}{P_2}], [P_3 \oplus P_4 \to \overset{i}{P_2}], [P_6 \to P_3 \oplus P_4 \to \overset{i}{P_2}], [P_5 \to P_3 \oplus P_4 \to \overset{i}{P_2}],$

$P_4[i], [P_5 \to P_3 \oplus P_4 \to \overset{i}{P_1}], [P_5 \to P_3 \overset{i}{\oplus} P_4], [P_5 \to \overset{i}{P_4}], [P_6 \to P_5 \to \overset{i}{P_4}]$

and we can draw the diagram for $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_1)$ with the remaining triangles. Similarly for $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_2)$ we cross out

$[P_6 \to P_5 \to \overset{i}{P_4}], [P_6 \to P_5 \to P_2 \to \overset{i}{P_1}], [P_6 \to P_5 \to \overset{i}{P_2}], [P_6 \to P_3 \oplus P_5 \to \overset{i}{P_2}], [P_6 \to \overset{i}{P_5}],$

$[P_6 \to P_3 \to \overset{i}{P_2}], [P_6 \to P_3 \oplus P_4 \to \overset{i}{P_2}], [P_6 \to P_3 \to \overset{i}{P_1}], [P_6 \to \overset{i}{P_3}], P_6[i].$

and in $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_3)$ the following.

$[P_5 \to P_3 \oplus P_4 \to \overset{i}{P_2}], [P_5 \to P_3 \oplus P_4 \to \overset{i}{P_1}], [P_5 \to P_3 \to \overset{i}{P_1}], [P_5 \to P_3 \overset{i}{\oplus} P_4], [P_5 \to \overset{i}{P_4}],$

$[P_6 \to P_5 \to \overset{i}{P_4}], [P_5 \to P_2 \oplus P_3 \to \overset{i}{P_1}], [P_5 \to \overset{i}{P_3}], [P_5 \to P_2 \to \overset{i}{P_1}], [P_6 \to P_5 \to P_2 \to \overset{i}{P_1}],$

$[P_5 \to \overset{i}{P_2}], P_5[i], [P_6 \to P_5 \to \overset{i}{P_2}], [P_6 \to P_3 \oplus P_5 \to \overset{i}{P_2}], [P_6 \to \overset{i}{P_5}]$

Thus we can draw these too. To find $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_{1,2})$ we consider the diagram for $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_1)$ and cross out the complexes that became redundant in $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_2)$ (or vice versa). To find $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_{1,3,4})$ we can choose to look at $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_3)$ and note that the following "vanish".

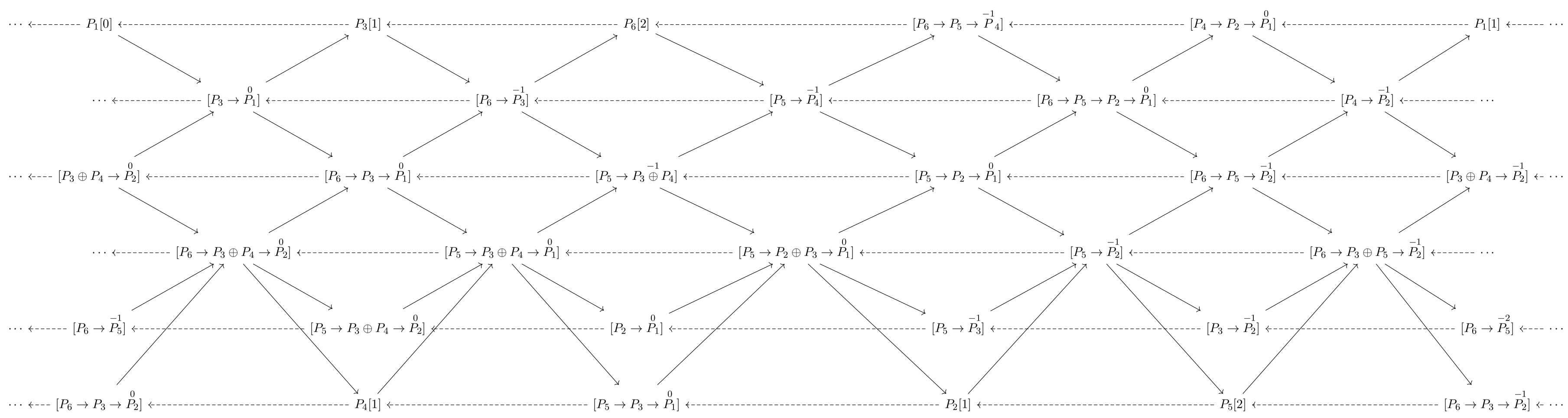
$$[P_4 \to P_2 \to \overset{i}{P_1}], [P_4 \to \overset{i}{P_2}], [P_3 \oplus P_4 \to \overset{i}{P_2}], [P_6 \to P_3 \oplus P_4 \to \overset{i}{P_2}], P_4[i]$$

Similarly to find $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_{2,3,5})$ we look at $\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_3)$ and note that the following become superfluous.

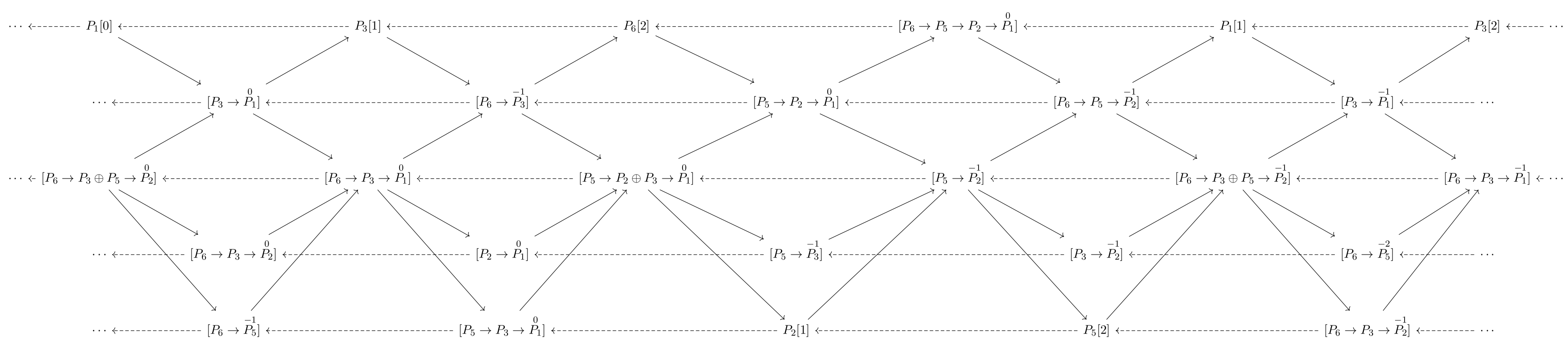
$$[P_6 \to P_3 \to \overset{i}{P_2}], [P_6 \to P_3 \oplus P_4 \to \overset{i}{P_2}], [P_6 \to P_3 \to \overset{i}{P_1}], [P_6 \to \overset{i}{P_3}], P_6[i]$$

Thus we have found the derived categories of the different exact structures which are presented in the following four pages.

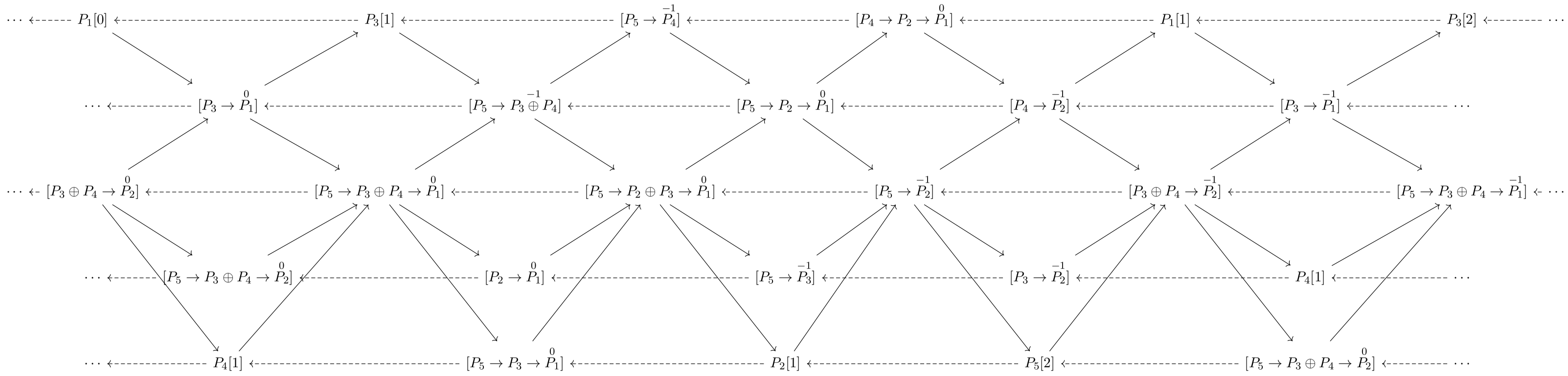
$\mathscr{K}^{\mathrm{b}}(\operatorname{proj}\Gamma) \cong \mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_{\min})$



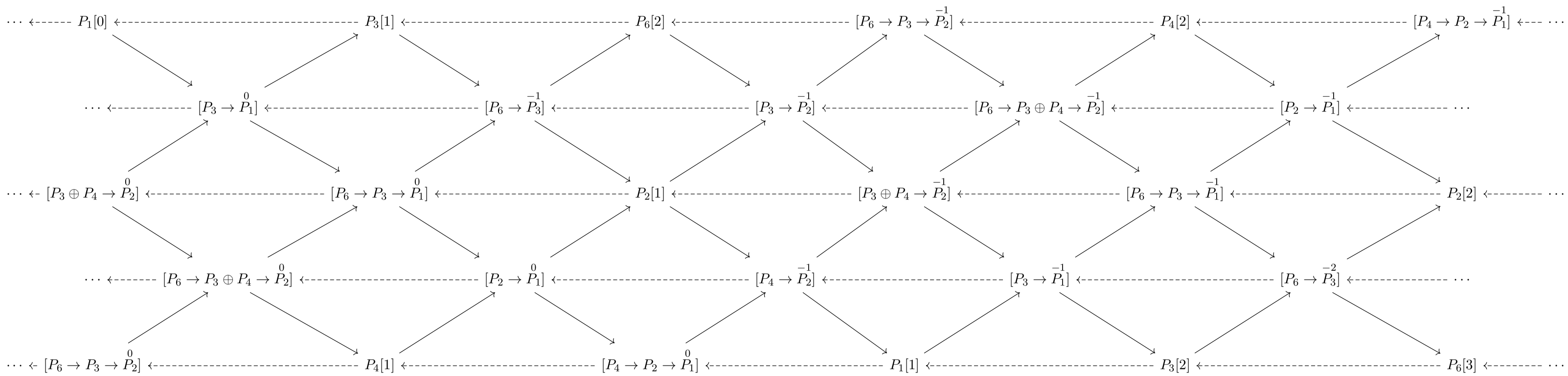
$\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_1)$



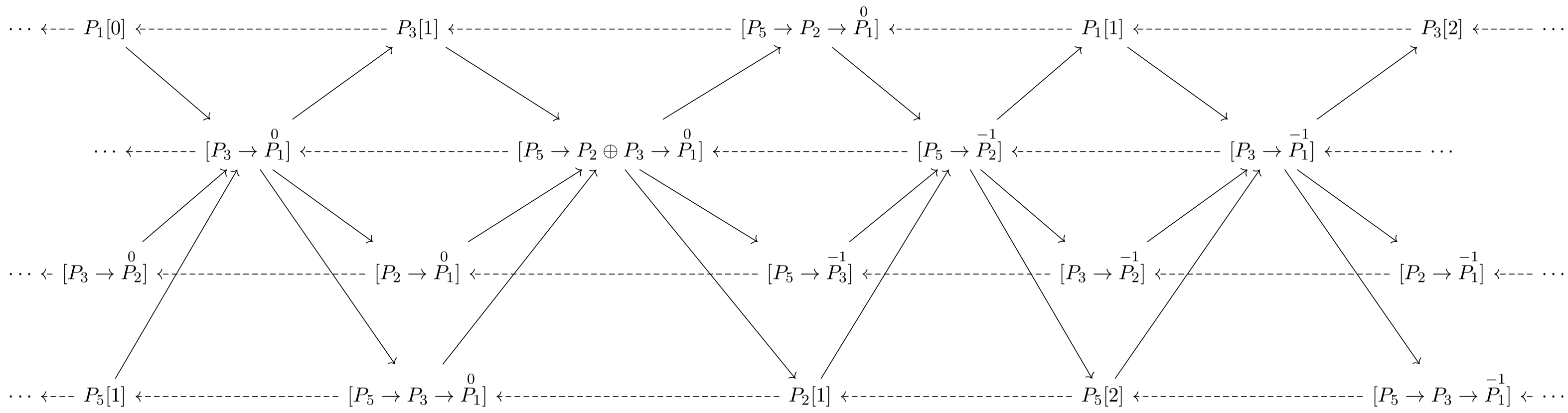
$\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_2)$



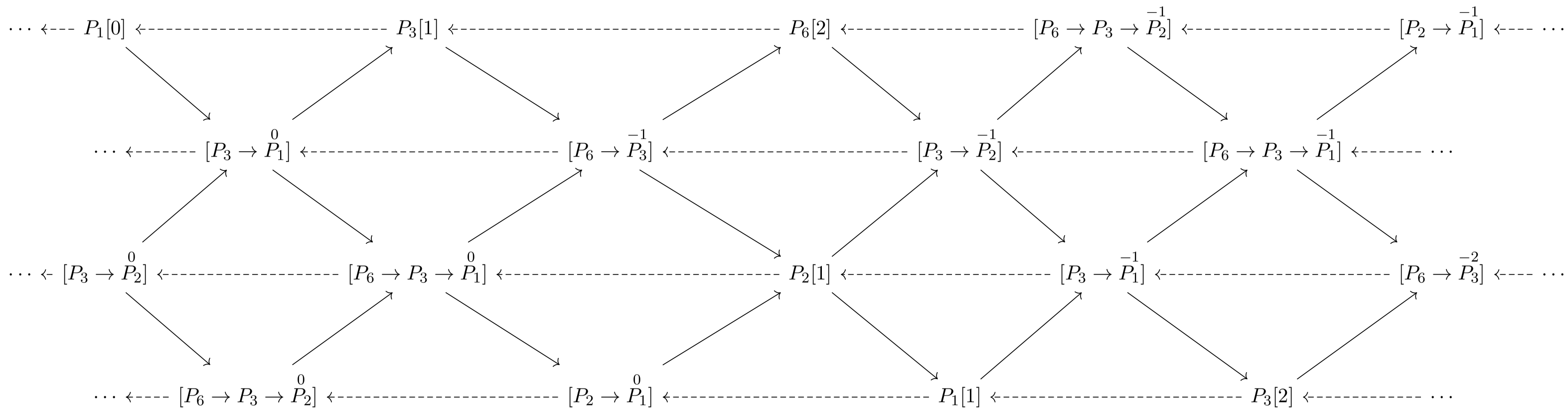
$\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_3)$



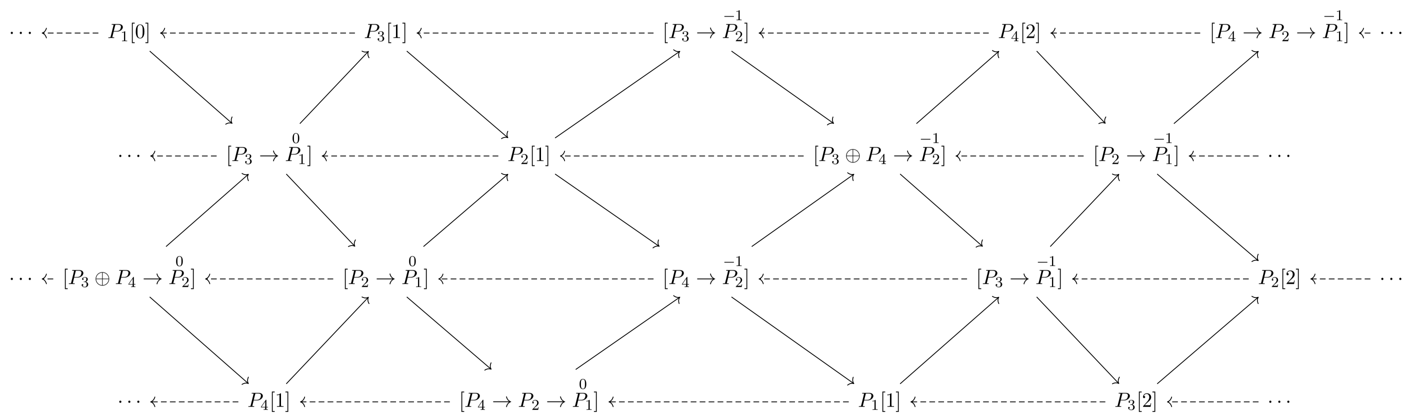
$\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_{1,2})$



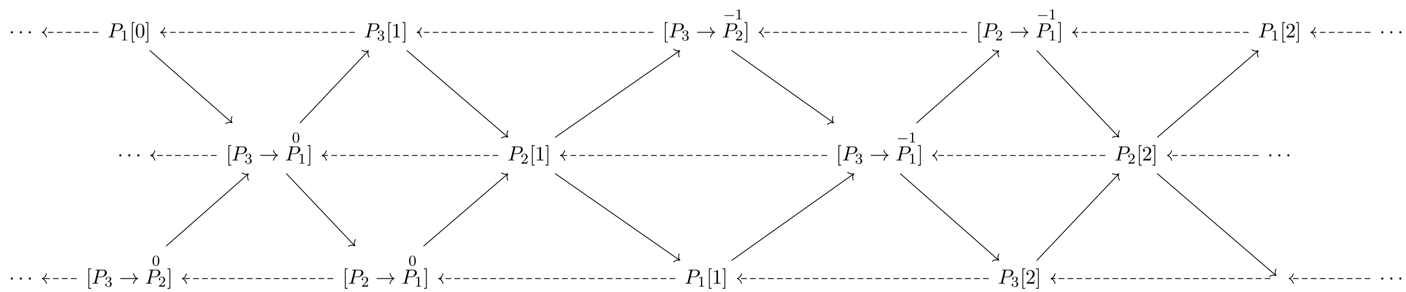
$\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_{1,3,4})$



$\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_{2,3,5})$



$\mathscr{D}^{\mathrm{b}}(\operatorname{proj}\Gamma, \mathscr{E}_{\max})$



Appendices

A Basic category results

In this section we go through some basic results. We work in a category $\mathscr{C}$ that contains zero morphisms unless otherwise specified.

Lemma A.1. Let f be a kernel of g. Let the right hand square of the following diagram be a pullback along g and h.

$$\begin{tikzcd}
A \arrow[r, dashed, "f'"] \arrow[d, equal] & P \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}$$

Then there exists a morphism $f' : A \to P$ such that the diagram above commutes, and f' is a kernel of g'. Dually if g is a cokernel of f and the left hand square is a push-out along g and f

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"] & B \arrow[r, "g"] \arrow[d, "h'"] & C \arrow[d, equal] \\
D \arrow[r, "f'"] & P \arrow[r, dashed, "g'"] & C
\end{tikzcd}$$

there exists $g' : P \to C$ such that the diagram commutes and g' is a cokernel of f'

Proof. We prove the first part, the second part is dual. Let f' be the unique morphism such that $h'f' = f$ and $g'f' = 0$. f' exist as P is a pullback, and $gf = 0 = h \circ 0$. Let $\alpha : T \to P$ be a morphism such that $g'\alpha = 0$. Then f' is the kernel of g' if there exists a unique morphism $\beta : T \to A$ such that $f'\beta = \alpha$.

$$\begin{tikzcd}
 & T \arrow[dl, dashed, "\beta"'] \arrow[d, "\alpha"] \arrow[dr, "0"] & \\
A \arrow[r, "f'"] \arrow[d, equal] & P \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[dr, dashed, "\gamma"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "f\beta"'] & & \\
 & P \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] \\
 & B \arrow[r, "g"] & C
\end{tikzcd}$$

Since f is a kernel of g, and $gh'\alpha = hg'\alpha = 0$ there exist a unique morphism $\beta : T \to A$ such that $h'\alpha = f\beta = h'f'\beta$. Furthermore the pullback property gives unique $\gamma : T \to P$ satisfying $g'\gamma = 0$ and $h'\gamma = f\beta$. We see that both α and $f'\beta$ satisfy this. Hence $\alpha = f'\beta$. ■

Lemma A.2. Given the following commutative square, where a and b have cokernels.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "b"] & B' \end{tikzcd}$$

There exists unique morphism h such that the following commutes.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"] & B \arrow[r, "c"] \arrow[d, "g"] & \mathrm{Coker}(a) \arrow[d, dashed, "h"] \\ A' \arrow[r, "b"] & B' \arrow[r, "c'"] & \mathrm{Coker}(b) \end{tikzcd}$$

Proof. By universal property of cokernels we get the following diagram.

$$\begin{tikzcd} A \arrow[r, "a"] & B \arrow[r, "c"] \arrow[dr, "c'g"'] & \mathrm{Coker}(a) \arrow[d, dashed, "\exists! h"] \\ & & \mathrm{Coker}(b) \end{tikzcd}$$

Hence we are done. ■

Lemma A.3. Given the pullback

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\ C \arrow[r, "i"] & D \end{tikzcd}$$

where i is monic we have that f is monic.

Proof. Let $\alpha, \beta : T \to A$ be such that $f\alpha = f\beta$. Then we have $hf\alpha = hf\beta$ which implies $ig\alpha = ig\beta$. Since i is monic we get $g\beta = g\alpha$. Now as both α and β fit as the dotted arrow in the following diagram

$$\begin{tikzcd} T \arrow[dr, dashed] \arrow[drr, bend left, "f\alpha=f\beta"] \arrow[ddr, bend right, "g\beta=g\alpha"'] & & \\ & A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\ & C \arrow[r, "i"] & D \end{tikzcd}$$

we get $\alpha = \beta$. ■

Proposition A.4. Let f be a monomorphism and g be an epimorphism in an abelian category. Then f is a kernel of g if and only if g is a cokernel of f.

Proof. Let $f : K \to A$ be a kernel of $g : A \to B$. We note $gf = 0$ as f is a kernel of g. Hence we only need to show that for any morphism $h : A \to T$ such that $hf = 0$ there exists a unique morphism $\sigma : B \to T$ such that $\sigma g = h$.

$$\begin{tikzcd}
K \arrow[r, "f"] & A \arrow[r, "g"] \arrow[dr, "h"'] & B \arrow[d, dashed, "\sigma"] \\
& & T
\end{tikzcd}$$

If there exists such σ we get for free that it is unique as g is epic. As we are in an abelian category g is the cokernel of some morphism $\mu : X \to A$. Since f is a kernel of g and $\mu g = 0$ there exists a unique map $\gamma : X \to K$ such that $f\gamma = \mu$.

$$\begin{tikzcd}
& & T & \\
& A \arrow[ur, "h"] \arrow[rr, "g"] & & B \arrow[ul, dashed, "\sigma"'] \\
X \arrow[ur, "\mu"] \arrow[rr, "\gamma"] & & K \arrow[ul, "f"'] &
\end{tikzcd}$$

By assumption $hf = 0$. we see that $h\mu = hf\gamma = 0 \circ h = 0$. Hence h give rise via the cokernel property of g a unique morphism σ as required. The opposite direction is dual. ■

Lemma A.5. Let $\mathscr{A}$ be an additive category. Consider the commutative square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\
C \arrow[r, "i"] & D
\end{tikzcd}$$

(1) The square is a pushout if and only if $\begin{pmatrix} h & i \end{pmatrix}$ is a cokernel of $\begin{pmatrix} -f \\ g \end{pmatrix}$.

(2) The square is a pullback if and only in $\begin{pmatrix} -f \\ g \end{pmatrix}$ is a kernel of $\begin{pmatrix} h & i \end{pmatrix}$.

Proof. We prove (1), (2) is dual. Suppose the square is a pushout.

$\begin{pmatrix} h & i \end{pmatrix}\begin{pmatrix} -f \\ g \end{pmatrix} = 0$ by commutativity of the square. Let $\begin{pmatrix} t_1 & t_2 \end{pmatrix}$ be such that $\begin{pmatrix} t_1 & t_2 \end{pmatrix}\begin{pmatrix} -f \\ g \end{pmatrix} = 0$.

We need to find unique ϕ making the following diagram commute.

$$\begin{tikzcd} A \arrow[r, "\begin{pmatrix} -f \\ g \end{pmatrix}"] & B \oplus C \arrow[r, "(h\ i)"] \arrow[dr, "(t_1\ t_2)"'] & D \arrow[d, dashed, "\exists!\phi"] \\ & & T \end{tikzcd}$$

By universal property of the pushout we find the unique ϕ by the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"'] \arrow[ddr, bend left, "t_1"] & \\ C \arrow[r, "i"] \arrow[drr, bend right, "t_2"'] & D \arrow[dr, dashed, "\exists!\phi"] & \\ & & T \end{tikzcd}$$

Conversely if $(h\ i)$ is a cokernel of $\begin{pmatrix} -f \\ g \end{pmatrix}$ we find the unique ϕ in the pushout diagram by the universal property of the cokernel. ■

Proposition A.6. Consider a square in an abelian category.

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "f_2"] & B \arrow[d, "g_1"] \\ C \arrow[r, "g_2"] & D \end{tikzcd}$$

if the square is a pushout and f_1 is a monomorphism, then g_2 is a monomorphism. Dually, if the square is a pullback and g_2 is an epimorphism then f_1 is an epimorphism.

Proof. Suppose the square is a pushout and f_1 is monic. Then $(g_1\ g_2)$ is the cokernel of $\begin{pmatrix} f_1 \\ -f_2 \end{pmatrix}$ by our previous lemma. Let $h : T \to A$ be any morphism such that $g_2 h = 0$. Then we have $0 = g_2 h = (g_1\ g_2)\begin{pmatrix} 0 \\ 1 \end{pmatrix} h = (g_1\ g_2)\begin{pmatrix} 0 \\ h \end{pmatrix}$.

As we are in an abelian category $\begin{pmatrix} f_1 \\ -f_2 \end{pmatrix}$ is a kernel of $(g_1\ g_2)$. This gives us that $\begin{pmatrix} 0 \\ h \end{pmatrix}$ factors through $\begin{pmatrix} f_1 \\ -f_2 \end{pmatrix}$ via some $t : T \to A$. As f_1 is monic we now get

$$\begin{aligned} \begin{pmatrix} 0 \\ h \end{pmatrix} = \begin{pmatrix} f_1 \circ t \\ (-f_2) \circ t \end{pmatrix} &\Rightarrow f_1 t = 0 \\ &\Rightarrow t = 0 \\ &\Rightarrow 0 = \begin{pmatrix} f_1 \\ -f_2 \end{pmatrix} t = \begin{pmatrix} 0 \\ h \end{pmatrix} \\ &\Rightarrow h = 0 \end{aligned}$$

Hence only the zero morphism precompose with g_2 to zero. Therefore g_2 is mono as the category is additive. The other part of the statement is dual. ■

Lemma A.7. given the diagram.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[d, "c"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \end{tikzcd}$$

The following hold.

(1) If both the squares are pushouts, the rectangle is a pushout

(2) If both the squares are pullbacks then the rectangle is a pullback.

Proof. We proove (2), (1) is dual.The rectangle commutes as $g'f'a = g'bf = cgf$. Given the following diagram, we need to fill in the dotted morphism $t_4 : T \to A$ uniquely.

$$\begin{tikzcd} T \arrow[drrr, bend left, "t_1"] \arrow[drr, dashed, bend left, "t_3"] \arrow[dr, dashed, "t_4"] \arrow[ddr, bend right, "t_2"'] & & & \\ & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[d, "c"] \\ & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \end{tikzcd}$$

By the universal property of the right pullback square we find $t_3 : T \to B$ then by the universal property of the left pullback square we find our desired unique t_4. ■

B Triangulated categories

This appendix is based on Oppermann's lecture notes [9, Chapter VI] and Krause's article [3, Chapter 2].

In this appendix we will go through the definitions and results necessary regarding triangulated categories.

Definition B.1. Let $\mathcal{T}$ be an additive category with an auto-equivalence $[1] : \mathcal{T} \to \mathcal{T}$. A *triangle* in $\mathcal{T}$ is a sequence (f, g, h) of maps

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

A morphism between two triangles (f, g, h) and (f', g', h') is a triple (ϕ_1, ϕ_2, ϕ_3) of maps in $\mathcal{T}$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\phi_1"] & B \arrow[r, "g"] \arrow[d, "\phi_2"] & C \arrow[r, "h"] \arrow[d, "\phi_3"] & A[1] \arrow[d, "\phi_1[1]"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & A'[1]
\end{tikzcd}$$

The category $\mathcal{T}$ is called *triangulated* if it is equipped with a class of distinguished triangles called *exact triangles* satisfying the following axioms.

(TR1) (a) The class of distinguished triangles is closed under isomorphisms.

(b) For each $A \in \mathcal{T}$ the triangle $0 \longrightarrow A \xrightarrow{\mathrm{Id}_A} A \longrightarrow 0$ is exact.

(c) Each map f fits into a triangle (f, g, h).

(TR2) Given an exact triangle (f, g, h) the triangles $(g, h, -f[1])$ and $(-h[-1], f, g)$ are also exact.

(TR3) Given exact triangles (f, g, h) and (f', g', h') each pair of maps ϕ_1 and ϕ_2 satisfying $\phi_2 f = f'\phi_1$ can be completed to a morphism of triangles.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\phi_1"] & B \arrow[r, "g"] \arrow[d, "\phi_2"] & C \arrow[r, "h"] \arrow[d, "\phi_3"] & A[1] \arrow[d, "\phi_1[1]"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & A'[1]
\end{tikzcd}$$

In other words the morphism ϕ_3 exist.

(TR4) Given the solid part of the following diagram , where the two rows and the left column are exact triangles

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d] & C' \arrow[r] \arrow[d, dashed] & A[1] \arrow[d, equal] \\
A \arrow[r] & C \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d, dashed] & A[1] \arrow[d, "f[1]"] \\
& A' \arrow[r, equal] \arrow[d, "h"'] & A' \arrow[r, "h"] \arrow[d] & B[1] \\
& B[1] \arrow[r, "g[1]"] & C'[1] &
\end{tikzcd}$$

There exist morphisms as indicated by the dashed arrows such that the second column is an exact triangle, and the whole diagram commutes.

Remark B.2. The third object C in an exact triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1]$$

is called *the cone of* f and will sometimes be denoted $\mathrm{Cone}(f)$.

Remark B.3. As $0 \to A \to A \to 0$ always is an exact triangle by (TR1), we get by (TR2) that $A \to A \to 0 \to A[1]$ also is an exact triangle.

Proposition B.4. Let $\mathcal{T}$ be a triangulated category. Let $A \to B \to C \to A[1]$ be an exact triangle, and $T \in \mathcal{T}$ then the sequences

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{T}}(T, A[n]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(T, B[n]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(T, C[n]) \longrightarrow$$

$$\mathrm{Hom}_{\mathcal{T}}(T, A[n+1]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(T, B[n+1]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(T, C[n+1]) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{T}}(C[n], T) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(B[n], T[n]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(A[n], T) \longrightarrow$$

$$\mathrm{Hom}_{\mathcal{T}}(C[n-1], T) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(B[n-1], T) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(A[n-1], T) \longrightarrow \cdots$$

are exact

Proof. We prove the second sequence is exact, showing the first one is exact is dual. By (TR2) it suffice to show that the sequence

$$\mathrm{Hom}_{\mathcal{T}}(C,T) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(B,T) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(A,T)$$

is exact. We compare our given triangle with the triangle from (TR1).

$$\begin{tikzcd}
A \arrow[r] \arrow[d, dashed, "g"] & B \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, dashed, "h"] & A[1] \arrow[d, dashed, "g[1]"] \\
0 \arrow[r] & T \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

By (TR3) the morphism g exist if and only if the morphism h exist. That is given $f \in \mathrm{Hom}_{\mathcal{T}}(B,T)$

$$f[A \to B] = 0 \iff \exists h \in \mathrm{Hom}_{\mathcal{T}}(C,T) : h[B \to C] = f$$

■

Corollary B.5. Any morphism in a triangle is a weak kernel of the next morphism, and a weak cokernel the previous morphism. ■

Lemma B.6. Let $\mathcal{T}$ be a triangulated category. Consider the following morphism of triangles

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, "f[1]"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}$$

If two of the morphisms f, g and h are isomorphisms so is the third.

Proof. For convenience we let $\mathcal{T}(-,-)$ denote $\mathrm{Hom}_{\mathcal{T}}(-,-)$. By (TR2) it is sufficient to check h is an isomorphism given f and g are isomorphisms. We apply $\mathcal{T}(-,C)$ to the initial diagram.

$$\begin{tikzcd}
\mathcal{T}(A,C) & \mathcal{T}(B,C) \arrow[l] & \mathcal{T}(C,C) \arrow[l] & \mathcal{T}(A_1[1],C) \arrow[l] & \mathcal{T}(B[1],C) \arrow[l] \\
\mathcal{T}(A',C) \arrow[u] & \mathcal{T}(B',C) \arrow[l] \arrow[u] & \mathcal{T}(C',C) \arrow[l] \arrow[u, "-\circ h"] & \mathcal{T}(A'[1],C) \arrow[l] \arrow[u] & \mathcal{T}(B'[1],C) \arrow[l] \arrow[u]
\end{tikzcd}$$

As f and g are isomorphisms we get that the left two and right two morphisms in the diagram are isomorphisms. By 5-lemma for abelian groups we get that the morphism $-\circ h : \mathcal{T}(C',C) \to \mathcal{T}(C,C)$ is an isomorphism. In particular there is $\hat{h} \in \mathcal{T}(C',C)$ such that $\hat{h}h = \mathrm{Id}_C$. Hence h is split monomorphism. Similarly applying $\mathrm{Hom}_{\mathcal{T}}(C',-)$ we get h is split epimorphism. Hence h is an isomorphism. ■

Definition B.7. An *exact functor between triangulated categories* $\mathcal{T}$ and $\mathcal{T}'$ is an additive functor $F : \mathcal{T} \to \mathcal{T}'$ with natural isomorphisms $\mu_A : F(A[1]) \to F(A)[1]$ such that for any exact triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

in $\mathcal{T}$, the triangle

$$FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \xrightarrow{Fh} \mu_A(F(A[1]))$$

is an exact triangle in $\mathcal{T}'$.

Definition B.8. An additive functor $F : \mathcal{T} \to \mathscr{A}$ from a triangulated category to an abelian category is called an *homological* if for every exact triangle

$$A \to B \to C \to A[1]$$

in $\mathcal{T}$, the following is exact in $\mathscr{A}$

$$F(A) \to F(B) \to F(C)$$

If F is contravariant we call F *cohomological* if the corresponding functor $F^{\mathrm{op}} : \mathcal{T}^{op} \to \mathscr{A}$ is homological.

Proposition B.9. Let $\mathcal{T}$ be a triangulated category. For any object $T \in \mathcal{T}$ the functor $\mathrm{Hom}(T, -)$ is homological and the functor $\mathrm{Hom}_{\mathcal{T}}(-, T)$ is cohomological.

Proof. This follows directly from B.4. ∎

Definition B.10. Let $\mathcal{T}$ be a triangulated category. A non-empty full subcategory $\mathscr{S}$ is a *triangulated subcategory* if the following conditions hold

(TS1) $A[n] \in \mathscr{S}$ for all $A \in \mathscr{S}$ and $n \in \mathbb{Z}$.

(TS2) Let $A \to B \to C \to A[1]$ be an exact triangle in $\mathcal{T}$. If two objects from $\{A, B, C\}$ belongs to $\mathscr{S}$ so does the third.

Definition B.11. Let $\mathscr{S}$ be a triangulated subcategory of $\mathcal{T}$. Then $\mathscr{S}$ is called *thick* if it is strictly full and the following hold.

(TS3) Every direct factor of an object in $\mathscr{S}$ belong to $\mathscr{S}$. That is given a decomposition $A = A' \oplus A''$ with $A \in \mathscr{S}$, we get $A', A'' \in \mathscr{S}$.

C Auslander-Reiten theory

In this appendix we give the definitions and results regarding Auslander-Reiten theory used in the thesis. The results will not be proven. We refer the interested reader to the book by Skowronski, Simson and Assem [10] and the work of Happel [8] where proofs and background for the results can be found.

Definition C.1. Let L, M and N be modules in $\mathbf{mod}A$. Then we have the following definitions.

(1) A morphism $f : L \to M$ is called *left almost split* if the following holds

   a) f is not a section.

   b) For every morphism $u : L \to U$ that is not a section there exists $u' : M \to U$ such that $u'f = u$.

(2) A morphism $g : M \to N$ is called *right almost split* if the following hold.

   a) g is not a retraction.

   b) For every morphism $v : V \to N$ that is not a retraction there exists $v' : V \to M$ such that $gv' = v$.

(3) A morphism$f : L \to M$ is called *left minimal* if every $h \in \mathrm{End}(M)$ such that $hf = f$ is an automorphism.

(4) A morphism $g : M \to N$ is called *right minimal* if every $k \in \mathrm{End}\, M$ such that $gk = g$ is an automorphism.

(5) A morphism $f : L \to M$ is called *left minimal almost split* if it is left minimal and left almost split.

(6) A morphism $g : M \to N$ is called *right minimal almost split* if it is right minimal and right almost split.

(7) A short exact sequence in $\mathbf{mod}A$

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

   is called an *almost split sequence* provided

   a) f is left minimal almost split.

   b) g is right minimal almost split.

Definition C.2. Let M be a right A module. A *minimal projective presentation* is an exact sequence $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ where p_0 and $p_1 : P_1 \to \ker(p_0)$ are projective covers. If we apply $\mathrm{Hom}_A(-, A)$ to the sequence we obtain a sequence of left A modules

$$0 \to \mathrm{Hom}_A(M, A) \to \mathrm{Hom}_A(P_0, A) \xrightarrow{p_1 \circ -} \mathrm{Hom}_A(P_1, A) \to \mathrm{Coker}(p_1 \circ -) \to 0$$

We denote $\mathrm{Coker}(p_1 \circ -)$ by $\mathrm{Tr}\, M$ and call it the *transpose of* M.

Proposition C.3. [10, Proposition 2.1(c)] M is projective if and only if $\mathrm{Tr}\, M = 0$. If M is not projective then $\mathrm{Tr}\, M$ is indecomposable and $\mathrm{Tr}(\mathrm{Tr}\, M) = M$.

Proposition C.4. [10, Proposition 2.2] The correspondance $M \mapsto \mathrm{Tr}\, M$ induces a K-linear duality functor $\mathrm{Tr} : \underline{\mathbf{mod}}A \to \underline{\mathbf{mod}}A^{\mathrm{op}}$ where $\underline{\mathbf{mod}}A$ and $\underline{\mathbf{mod}}A^{\mathrm{op}}$ denote the projectively stable categories of $\mathbf{mod}A$ and $\mathbf{mod}A^{\mathrm{op}}$ resp.

Definition C.5. The *Auslander Reiten translation* is defined to be the composition of D with Tr where D is the standard duality $D = \mathrm{Hom}_K(-, K)$. So we have the auslander reiten translation $\tau = D\,\mathrm{Tr}$.

Theorem C.6. [10, Theorem 3.1(a)] For any indecomposable non-projective right A module M there exists an almost split exact sequence $0 \to \tau M \to E \to M \to 0$ in $\mathbf{mod}A$ where τ denotes the Auslander-Reiten translation.

Proposition C.7. [10, Proposition 3.5(a)] Let P be an indecomposable projective module in $\mathbf{mod}A$. An A module homomorphism $g : M \to P$ is right minimal almost split if and only if g is a monomorphism with image equal to $\mathrm{rad}\, P$.

Definition C.8. Let A be a basic and connected finite dimensional K algebra. The quiver $\Gamma(\mathbf{mod}A)$ of $\mathbf{mod}A$ is defined as follows.

(1) The points of $\Gamma(\mathbf{mod}A)$ are the isomorphism classes $[X]$ of indecomposable modules X in $\mathbf{mod}A$.

(2) Let $[M], [N]$ be the points in $\Gamma(\mathbf{mod}A)$ corresponding to the indecomposable modules M and N in $\mathbf{mod}A$. The arrows $[M] \to [N]$ are in bijective correspondence with the vectors of a basis of the K vector space $\mathrm{Irr}(M, N) = \mathrm{rad}_K(M, N)/\,\mathrm{rad}^2_K(M, N)$.

Remark C.9. We can define in the exactly same way the quiver $\Gamma(\mathscr{C})$ of an arbitrary additive subcategory of $\mathbf{mod}A$ that is closed under direct sums and summands.

Proposition C.10. [10, Lemma 4.7] The Auslander Reiten quiver $\Gamma(\mathbf{mod}A)$ of an algebra A is a translation quiver, the translation being defined for all points $[M]$ such that M is not a projective module by $\tau[M] = [\tau M]$.

For the rest of the appendix we let $\mathcal{T}$ be a triangulated category such that $\mathrm{Hom}_{\mathcal{T}}(A, B)$ is a finite dimensional k vector space for all $A, B \in \mathcal{T}$ and assume that the endomorphism ring of an indecomposable object is local. I.e. we have that $\mathcal{T}$ is a Krull-Schmidt category.

Definition C.11. A triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is called an *Auslander-Reiten triangle* if the following hold.

1. A and C are indecomposable.

2. $h \neq 0$.

3. If $u : X \to C$ is not a retraction, then there exists $u' : X \to B$ such that $f'g = u$.

We say $\mathcal{T}$ *has Auslander-Reiten triangles* if for all indecomposable objects $C \in \mathcal{T}$ there is a triangle satisfying the conditions above. We call $A[1]$ the translation of A and sometimes denote it τA.

Lemma C.12. [8, Section 3.4] The definition above is self dual.

Theorem C.13. [8, Section 3.6] Let A be a finite dimensional k-algebra of finite global dimension. Then the derived category $\mathscr{D}^{\mathrm{b}}(A)$ has Auslander-Reiten triangles.

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