Anders Alexander Andersen

Simplicial Sheaves and Classification Problems

Master's thesis in Mathematical Sciences
Supervisor: Gereon Quick
June 2021

Master's thesis

NTNU
Norwegian University of Science and Technology
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Kunnskap for en bedre verden

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Abstract. This thesis aims to explore and expand on the ideas in [FH13]. To this end, we first give a brief exposition on some of the needed preliminaries, including tensors, bundles, connections, and the Chern-Weil homomorphism. Going forward, we follow the aforementioned paper and introduce the language of presheaves, simplicial sets, and simplicial sheaves, before finding a classifying space for all smooth principal bundles with connection in the category of simplicial presheaves and, using some abstract homotopy theory, show that the conjugation-invariant polynomials on the Lie algebra induce all the natural differential forms one can construct from a connection. Lastly, we depart from the paper and explore what happens in the case of holomorphic principal bundles, and find a classifying space for all holomorphic principal bundles with holomorphic connection.

Sammendrag Denne avhandlingen forsøker å utforske og å videreføre ideene fra [FH13]. For å oppnå dette, tar vi først for oss noen av forkunnskapene som trengs, inkludert tensorer, bunter, koblinger, og Chern-Weil-homomorfien. Etter det, så følger vi utredningen fra artikkelen, og introduserer språket brukt for å beskrive preknipper, simplisielle mengder, og simplisielle knipper, for så å finne et klassifiseringsrom for alle glatte hovedbunter med kobling i kategorien av simplisielle preknipper og, ved bruk av abstrakt homotopiteori, viser at de konjugat-invariante polynomene på Lie-algebraen induserer alle de naturlige differensialformene en kan konstruere fra en kobling. Til slutt viker vi fra artikkelen for å utforske hva som skjer når hovedbuntene er holomorfe, og finner et klassifiseringsrom for alle holomorfe hovedbunter med holomorf kobling.

Preface

This thesis is the accumulation of my master studies in mathematics at NTNU. It was written during the academic year of 2020-2021, from September through May. My supervisor was Gereon Quick.

The main goal of this thesis was to explore some of the interplay between geometry and homotopy theory, and "bridge the gap" between what (I think) every master student in topology or geometry should know and a small portion of modern research in mathematics. The topic, chosen by my supervisor, was to study the paper titled *Chern-Weil Forms and Abstract Homotopy Theory*, written by Freed and Hopkins in 2013 (see [FH13]), and then investigate further what would happen if we worked with holomorphic principal bundles instead of smooth principal bundles.

As the prerequisites for understanding the paper are not too many[1], I set out to write a thesis which my fellow classmates from other disciplines of mathematics could read too, and hopefully make more people as fond of algebraic topology as I am. A physical copy of this thesis can be found and read at "Deltakontoret" by anyone interested, and so I have written it for someone who only knows general topology, basic category theory, as well as basic group and ring theory. Thus, younger students of mathematics and eager physics students can probably read it too.

How to read this thesis

Depending on your background, I have different suggestions as to how to work though the thesis. No matter your knowledge, I suggest you read the Introduction, and all the chapter summaries first—they are given at the very start of each chapter—before starting at the appropriate chapter.

The expert, who knows what simplicial sheaves are, can jump straight ahead to chapter 3.

[1]If one ignores abstract homotopy theory.

Most master students of topology or geometry (including myself a year ago) should probably start with chapter 2, and only look at chapter 1 whenever a word or any notation is unclear. I, for example, did not know what connections on bundles were before starting this thesis, and so I would advice anyone in a similar position to read only section 1.4 and section 1.5 from chapter 1, either before reading chapter 2 or whenever necessary.

For other students (of mathematics or physics), I suggest judging for yourself where to start, based on the Table of Contents and how much you understood from the chapter summaries. Good general starting points are chapter 2, section 1.4, and chapter 1.

Acknowledgments

The completion of this thesis depended on the existence of several people and institutions, and sadly I cannot extend my gratitude to everyone. Still, I would like to give it a try.

First and foremost, I would like to thank Gereon Quick for making me interested in algebraic topology, suggesting a fruitful topic for my thesis, and always helping me when things have been difficult. He has always answered my questions, no matter how stupid they have been. I also want to thank my fellow classmates, especially those sitting in room 395B at Matteland, for many great discussions (about mathematics and otherwise), and making my years at university a great experience. I am very thankful for the existence of Delta, my student organization, as they have made a great community for students of mathematics and physics.

Anders Alexander Andersen
Trondheim, 2021

Contents

Introduction

In this section, we give a short introduction to the topic at hand. After this, an overview of each chapter is given. Lastly, notation and conventions used in the thesis are explained. The reader unfamiliar with the precise meaning of all the mathematical jargon used in this introduction should not be afraid or discouraged, for most of it is defined later on.

Classifications, invariants and generalized manifolds

A general goal in mathematics is to classify interesting objects. This entails obtaining a classification list, which enumerates all the objects in a non-redundant way, meaning for two essentially equal objects, we only list one of them. The precise meaning of "essentially equal" depends on how coarse or fine one wants the classification to be. But in the pursuit of such a list, one often quickly encounter the notion of an invariant. Let us see how, with a least interesting, but simple example.

Consider the task of classifying all finite non-empty sets, where the relations are bijections between sets. One might immediately notice that two finite non-empty sets X and Y are essentially the same whenever they have the same number of elements, or have the same *cardinality* as it is called. And so we have our first invariant, namely the cardinality. Any finite set is in bijection with some set $\{1, 2, \ldots, n\}$ for some natural number $n \in \mathbb{N}$. Thus we know what all finite non-empty sets "look like", with the help of our invariant, and we can say that we have classified them.

Another, possibly more exiting, type of object one would want to classify is the triangles. Call two triangles equal whenever there is a distance preserving bijection between them, or *isometry* as it is called. For two triangles X and Y, an isometry $X \longrightarrow Y$ maps any two points of X to two points of Y in such a way that the distance between the two points in Y is equal to the distance of the two points in X. Examples of isometries are rotations around points in the plane, reflections across lines in the plane, and translations in the plane. As we might notice, these three types of isometries would preserve not only the distances, but the area as well. And in general, as areas come from lengths, it might not be too surprising that all types of isometries preserve the area of the triangles (although area preserving maps are not always isometries). Thus we come to the conclusion that the area of a triangle is an invariant of triangles with respect to the isometries.

But what is an invariant in general, for other types of objects? Historically speaking, the concept of an invariant underwent several changes and generalizations. In [FH13], they mention that in the 19$^{\text{th}}$ century, invariant theory was essentially the study of certain polynomials on representation spaces of groups. Given a group G and a linear representation $G \longrightarrow \mathrm{GL}(V)$, one seeks polynomials $V \longrightarrow \mathbb{R}$ on the representation space V that are unchanged under the induced action on V from the group G. These polynomials were dubbed *invariant polynomials*. But during the late 19$^{\text{th}}$ century, Felix Klein, while working in Erlangen, came up with a new geometric perspective on invariants. Today, his work is known as the *Erlangen program*, and it approximately states that geometric invariants are not the invariant polynomials, but can be regarded as certain "numerical" values associated to geometric objects, like in the example above with areas of a triangles. In [Kle93], Klein states a view of geometry in which a geometry was associated to a group of transformations, and that this group should provide invariants. See for example Chapter 42 in [Gra05] for historic remarks on the Erlangen program.

The concept of an invariant evolved further until one arrived at a very general meaning. In broad terms, an *invariant* is a property of a mathematical object which remains unchanged after a certain type of transformation. The particular type of object and the type of transformation is decided beforehand, like when we decided that the transformations to consider were only the isometries. However, as the reader familiar with abstract nonsense probably realizes, this definition begs for a reformulation using the language of category theory. Those tools allows us to give a modern and precise definition of an invariant, in the spirit of Klein's program. If we let the collection of all mathematical objects we are interested in be all the objects of some category, and let the transformations we consider be all the morphisms in our category, then an invariant is a functor mapping out of this category, or a closely related one. Of course this definition encapsulates all the different definitions throughout history, including invariant polynomials and Klein's geometric formulation. The problem we investigate in this thesis asks for invariants of principal G-bundles with G-connection, where G is some Lie group.

In the 1970s, Chern and Weil showed that certain invariant polynomials on the Lie algebra $\mathfrak{g}$ of a Lie group G determine differential forms in such a way that two isomorphic principal G-bundles with connections give the same differential form if the isomorphism preserves the connection. The map associating each invariant polynomial to its invariant differential form is called the Chern-Weil homomorphism. We see that the principal G-bundles with connection and the connection preserving bundle isomorphisms form a category, and that the invariants are the differential forms. One could wonder if there are other invariants than those arising from these polynomials. The main result (Theorem 7.20) in [FH13] says that these differential forms are the only natural differential forms one can construct from a G-connection. This result uses an idea common in algebraic topology, namely that of a *classifying space*.

The classical example is the classifying space BG of a Lie group[2] G. The space BG is given together with a closely related space, EG, and a map $EG \longrightarrow BG$ such that EG is a principal G-bundle over BG, called the *universal bundle*. Without getting into details, we should mention that BG is called the classifying space because for any principal G-bundle $\pi\colon E \longrightarrow B$, there exists a continuous map $\varphi\colon B \longrightarrow BG$ such that the diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\pi"] & EG \arrow[d] \\ B \arrow[r, "\varphi"] & BG \end{tikzcd}$$

is a pullback diagram. We call φ the *classifying map*. In less abstract terms, any twisting in the total space E is described in the twisting in EG via the classifying map φ. For example if $G = \mathbb{R}$, the classifying space $BG = \{*\}$ is just a point, with cover $EG = \mathbb{R}$. If $G = \mathbb{Z}_2$, the classifying space is $B\mathbb{Z}_2 = \mathbb{P}^\infty$ the infinite dimensional real projective space, with cover $E\mathbb{Z}_2 = S^\infty$ the infinite dimensional sphere. One can wonder how BG is constructed and if it even exist for all G. But this is ensured by Brown's representability theorem.

[2]We can actually consider general topological groups G, but that would be too much of a digression for this introduction.

Let us for a brief moment consider the category of principal G-bundles with bundle isomorphisms as morphisms. Note that the cohomology functor $H^\bullet$ is a functor mapping out of our category. Hence any cohomological invariant on principal G-bundles, considered as a natural functor of fibre bundles to a cohomology class of the base space, can be found in the cohomology ring $H^\bullet(BG)$. Another way of saying this is that each characteristic class correspond to an element of $H^\bullet(BG)$, and so all cohomological invariants of principal G-bundles are elements of the cohomology ring. This is why the classifying spaces BG are useful to us.

If we instead now consider the category of principal G-bundles with a connection, and let the morphisms be bundle isomorphisms that in addition preserve the connection, we seek a classifying space $B_\nabla G$ with total space $E_\nabla G$ and a *universal connection* ∇^{univ} such that for any principal G-bundle $E \longrightarrow B$ with connection ∇, the diagram

$$\begin{tikzcd} E \arrow[r, "\psi"] \arrow[d, "\pi"] & E_\nabla G \arrow[d] \\ B \arrow[r, "\varphi"] & B_\nabla G \end{tikzcd}$$

is a pullback diagram, and in addition that the pullback $\psi^*\nabla^{\mathrm{univ}}$ of the universal connection is the connection ∇ on $\pi\colon E \longrightarrow B$.

Just as we saw that $B\mathbb{Z}_2$ was a infinitely dimensional manifold $\mathbb{P}^\infty$, all earlier attempt at construction a universal principal G-bundle with universal connection has resulted in infinite dimensional manifolds, see for example [NR63] or [Sch80]. The drawback is that the classifying maps are not unique. What is new in [FH13] is that they sidestep this problem and get unique classifying maps. The uniqueness actually makes the computation of $H^\bullet(B_\nabla G)$ feasible in practice. And to bypass the problem with infinite dimensions, they move out of the category of smooth manifolds, and take the reader on a journey that leads to *generalized manifolds*. Generalized manifolds are just simplicial sheaves on manifolds, and every manifold is, naturally, a simplicial sheaf. In the end, using results from abstract homotopy theory, they show that the computation of all the invariants only uses concepts from differential geometry and invariant theory. Thus, ultimately, the calculation never needs the ideas of simplicial sheaves.

The general goal of this thesis is to explore the interplay between geometry and abstract homotopy theory, and present it in such a way that master's students in mathematics (and possibly physics) can understand the main takeaways. More specifically, a primary goal we have is explaining, in more detail, the arguments used in [FH13] to show how one can calculate all the invariants of principal G-bundles with connection. A secondary goal is to see what happens in the complex world of complex manifolds. It is not obvious beforehand that Freed and Hopkins' techniques transfer over, since not all holomorphic bundles have a holomorphic connection. But still, using their ideas, we construct an analogous universal bundle $E_{\nabla,\mathbb{C}}G \longrightarrow B_{\nabla,\mathbb{C}}G$ with a universal connection ∇^{univ}, endowed with similar properties to that of $E_\nabla G \longrightarrow B_\nabla G$ from the differential world. Just as in the abovementioned paper, we take the reader on a voyage, starting geometrically with manifolds, going through the abstract world of simplicial sheaves, and ending with the geometry again. A more detailed layout is given now.

Overview of the Thesis

This thesis is divided into two parts, named The Differential Case and The Holomorphic Case respectively. At the end, there is one appendix. As the name suggests, the first part concerns itself with *differential* constructions, such as smooth manifolds, smooth bundles, differential forms, smooth connections, etc. In this part, we work with the field $\mathbb{R}$ of the real numbers. This part contains the first three chapters of the thesis, containing only known theory. The second part goes further, and considers what happens when we require all our constructions to be *holomorphic* as well, like complex manifolds, holomorphic bundles, holomorphic connections, etc. Here, the field we work with is the field $\mathbb{C}$ of complex numbers. This part has the fourth and final chapter, with some original findings. The appendix supplies some of the theory of manifolds and Lie groups.

Chapter 1. The first chapter can be considered supplementary, as it covers much of the needed preliminaries. The goal of this chapter is threefold: We want to (1) introduce the setting and language, (2) motivate the topic to the uninitiated, and (3) aid in the understanding of the topic. This chapter covers tensors, manifolds, smooth principal bundles, differential forms, connections on principal bundles and the Chern-Weil homomorphism.

Chapter 2. In the second chapter, we follow §3, §4, and §5 of [FH13]. Inspired by the way Freed and Hopkins introduces and proves their main result, we here lay out the context thoroughly to formulate the theorem. This journey goes through presheaves and sheaves on manifolds, groupoids, simplicial sets, simplicial presheaves, and weak equivalences, sprinkled with examples throughout. In particular, we walk all the steps to construct the universal bundle the classifying space $B_\nabla G$.

Chapter 3. In the last chapter of part I, we summarize the construction of the universal bundle, and present, in greater detail than [FH13], the proof that the universal bundle $E_\nabla G \longrightarrow B_\nabla G$ with universal connection ∇^{univ} is a universal bundle. We briefly mention the relevant results needed from abstract homotopy theory before defining the de Rham complex of a simplicial presheaf. The main theorems of the paper are found here.

Chapter 4. The very last chapter of this thesis contains most of the original work. In this chapter, we first explain what is meant by "holomorphic" in different contexts, e.g. holomorphic functions, holomorphic forms, and holomorphic bundles. We then look at two ways of defining holomorphic connections on principal bundles, before constructing the universal holomorphic bundle with universal connection. Lastly, we summarize all the original findings in theorem 4.4.3.

Notation, conventions and assumptions

By *abstract nonsense*, we mean category theory. By *natural*, we mean functorial in some sense. And the word *canonical* means what is usually means. When defining a term, we often emphasize it, like with the three previous concepts.

No diagram commutes unless specified. All categories will be assumed to be locally small. For any category $\mathscr{C}$, the set of morphisms $X \longrightarrow Y$ will be denoted by $\mathscr{C}(X, Y)$, or sometimes $\mathrm{Hom}_{\mathscr{C}}(X, Y)$. Specific categories will usually be denoted by two to four bold letters. In particular, **Set** is the category of sets with functions of sets as morphisms, **Man** is the category of smooth finite dimensional manifolds with smooth maps as morphisms, and $\mathbf{Vect}_{\mathbb{R}}$ is the category of finite dimensional real vector spaces, with linear maps as morphisms.

Unless otherwise stated, all manifolds will be assumed smooth and finite dimensional. The symbol G is used to denote a Lie group unless otherwise stated, and its Lie algebra is denoted $\mathfrak{g} = T_e G$.

If we have a map, say f, out of some manifold M (and into whatever), we often write f_p for the value $f(p)$ at p, unless the domain is not too interesting (e.g. the reals $\mathbb{R}$). This is to hide one layer of nested functions.

Throughout this thesis, there are generalizations, generalizations of the generalizations, and so on. An example is a manifold, which also is a sheaf on manifolds, and a simplicial sheaf on manifolds. Because of the tautological nature of these constructions, we have problem with notation and terminology. For example, for a smooth map of manifolds, we can consider the pullback. But there is a completely parallel notion of this for presheaves on manifolds, and simplicial presheaves on manifolds as well. And these parallel notions turn out to give back the original notion when working with the manifolds considered as a sheaves on manifolds, or simplicial sheaves on manifolds. Hence, we need to make a choice in what to call the new notions. A new name clears any doubt in what the word refers to, but the same name makes more sense when the concepts coincide. We choose to give corresponding notions the same names, and fully embrace this. The same goes for notation. Thus the reader should always make sure it is clear what is meant by these schizophrenic notions.

Part I

The Differential Case

CHAPTER 1

Preliminary Preliminaries

This thesis is about how to classify principal G-bundles with connections. To understand what a principal G-bundle is, and what a connection on it means, we, in this chapter, recall the definitions and give a exposition on these subjects. As mentioned earlier, the goal of this chapter is to introduce the language used, aid in understanding of the topic, and motivate the topic to the uninitiated.

We start with a brief exposition of tensors, which allows us to make multilinear maps into linear ones. As a consequence, we can turn alternating maps into linear ones as well, which will become a very useful tool.

With the algebraic prerequisites out of the way, we recall what smooth manifolds and Lie groups are, and mention what their tangent spaces look like. This allows us to define vector bundles and principal G-bundles, and the maps between them. In each case, we have two important examples, namely the trivial bundle and the pullback bundle, which we explore in detail.

Moving on, we use the insights gained about tensors and vector bundles, and define differential forms over a manifold, which are just families of certain alternating maps on the tangent spaces of the manifold. They encode vital information about the smooth structure of the manifold, and this is captured by the de Rham complex and the de Rham cohomology groups of the manifold, which we briefly mention. We also look at a slight generalization of differential forms, namely vector valued differential forms.

Armed with all these weapons, we study connections on bundles. The main reason we discuss them in depth is because we need to gain insight into how to define holomorphic G-connections in part II.

Finally, when we know what principal G-bundles with connections are, we look at how invariants of these objects have been studied in the past. This includes a brief exposition on what the set $I^k(G)$ of invariant polynomials on the Lie algebra $\mathfrak{g}$ is, as well as the Chern-Weil homomorphism.

1.1. Tensors

Later on, we work with operations on vector spaces that are not linear, like the Lie bracket, which is bilinear. And, in general, these multilinear maps are harder to understand than linear maps. So in this section, we introduce a tool used to convert multilinear maps into linear ones, namely the tensor product, and study some properties and identities of it. Almost all of the material here can be found in [Tu17] and [Lee13].

1.1.1. Multilinear algebra

In kindergarten, one learns that linear algebra is the study of linear maps between linear spaces. Multilinear algebra is a slight generalization of the concepts within that branch, and is the study of multilinear maps on vector spaces.

In this section, one might find the profusion of of indices off-putting, but there is no easy way around them, so please bear with it.

Suppose $V_1, \dots, V_k, W$ are finite dimensional real vector spaces. A map $\alpha\colon V_1 \times \cdots V_k \longrightarrow W$ is called $\mathbb{R}$-*multilinear*, or just *multilinear*, if it is linear as a function of each variable separately when the others are held fixed. If $k = 1$, this is the same as a linear map. If $k = 2$, we sometimes say α is *bilinear*. We write $L(V_1, \dots, V_k; W)$ for the collection of all these multilinear maps, which is a vector space[1].

Some examples of multilinear maps are the dot product, the cross product, the determinant, and, as mentioned in the introduction for this section, the Lie bracket.

An important special case of multilinear maps is when the target space W is the underlying field $\mathbb{R}$. Suppose $\alpha \in L(V_1, \dots, V_k; \mathbb{R})$, $\beta \in L(W_1, \dots, W_l; \mathbb{R})$. Define a function

$$\alpha \otimes \beta\colon V_1 \times \cdots \times V_k \times W_1 \times \ldots \times W_l \longrightarrow \mathbb{R}$$

by

$$\alpha \otimes \beta(v_1, \dots, v_k, w_1, \dots, w_l) = \alpha(v_1, \dots, v_k)\beta(w_1, \dots, w_l)$$

where multiplication of real numbers is written juxtaposition. As α and β are linear in each coordinate, so is $\alpha \otimes \beta$, hence $\alpha \otimes \beta \in L(V_1, \dots, V_k, W_1, \dots, W_l; \mathbb{R})$. We call this new map the *tensor product* of α and β. Since multiplication in $\mathbb{R}$ is associative, the tensor product $\alpha \otimes \beta \otimes \gamma$ of three multilinear maps α, β, γ is independent of

[1] under point-wise addition and scalar multiplication.

bracket placement. Thus, we can form the tensor product of arbitrary many multilinear maps. In particular, if ω_i is an element of the dual space $V_i^\vee = \mathrm{Hom}_{\mathbb{R}}(V_i, \mathbb{R})$, then $\omega_1 \otimes \cdots \otimes \omega_k \in L(V_1, \dots, V_k; \mathbb{R})$ is the multilinear map given by

$$\omega_1 \otimes \cdots \otimes \omega_k(v_1 \dots, v_k) = \omega_1(v_1) \cdots \omega_k(v_k).$$

These types of multilinear forms actually form a basis on the set $L(V_1, \dots, V_k; \mathbb{R})$ of multilinear maps with values in $\mathbb{R}$.

It is a shame we need to restrict ourselves to $\mathbb{R}$ as the target space, but as we can see, it is completely necessary when forming tensor products, as we do not know how to multiply elements of (generally different) vector spaces. The construction of abstract tensors remedy this.

1.1.2. Abstract tensors and tensor products

Let V, W, Z be real vector spaces. A map $\varphi\colon V \times W \longrightarrow Z$ is called $\mathbb{R}$-*balanced* if it is $\mathbb{Z}$-bilinear, and has the *tensor property*, meaning

(1) $\varphi(v, w_1 + w_2) = \varphi(v, w_1) + \varphi(v, w_2)$;
(2) $\varphi(v_1 + v_2, w) = \varphi(v_1, w) + \varphi(v_2, w)$; and
(3) $\varphi(rv, w) = \varphi(v, rw)$.

A *tensor product* is a vector space $V \otimes_{\mathbb{R}} W$ together with an $\mathbb{R}$-balanced map

$$\otimes\colon V \times W \longrightarrow V \otimes_{\mathbb{R}} W,$$

which is universal in the following sense: for any $\mathbb{R}$-bilinear map $\varphi\colon V \times W \longrightarrow Z$, there is a unique linear map $h\colon V \otimes_R W \longrightarrow Z$ such that $\varphi = h \circ \otimes$. This property can be illustrated by the following commutative diagram:

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "\text{bilinear } \varphi"'] & V \otimes_{\mathbb{R}} W \arrow[d, dashed, "\exists!\text{ linear } h"] \\ & Z \end{tikzcd}$$

We write $v \otimes w = \otimes(v, w)$, and call such an element an *elementary tensor*. Note that not all elements are elementary tensors, as $\otimes$ is not necessarily surjective.

We can see from the diagram that any multilinear map φ has a corresponding unique linear map h, where the domain is changed from $V \times W$ to $V \otimes_{\mathbb{R}} W$. But we do not know, a priori, that there are any tensor products at all. But if they do, we see from the definition that $\mathrm{Hom}_{\mathbb{R}}(V \otimes W, Z) \cong L(V, W; Z)$. And luckily:

THEOREM 1.1.1. *Tensor products exist and are unique up to isomorphism.*

We do not prove this, but briefly mention a suitable candidate for the tensor product. Proofs exist in virtually any textbook covering modules.

CONSTRUCTION 1.1.2. Consider the vector space $\mathrm{Free}(V \times W)$, whose basis is the set of all ordered pairs $(v, w) \in V \times W$, meaning any element is uniquely determined by a finite linear combination

$$\sum_i r_i(v_i, w_i).$$

In this vector space, we have the subspace S spanned by all elements of the form:

$$\begin{gathered}(v, w_1 + w_2) - (v, w_1) + (v, w_2),\\ (v_1 + v_2, w) - (v_1, w) + (v_2, w),\\ (rv, w) - r(v, w),\\ (v, rw) - r(v, w).\end{gathered}$$

(Note that these four relations are almost identical to the three properties of an $\mathbb{R}$-balanced map.) The quotient module $\mathrm{Free}(V \times W)/S$ satisfies the required properties of $V \otimes_{\mathbb{R}} W$

The tensor product satisfies some nice properties.

PROPOSITION 1.1.3 (Properties of the tensor product). *Let V, W and Z be vector spaces. Then we have the following canonical isomorphisms.*

(1) $V \otimes W \cong W \otimes V$*;*
(2) $V \otimes \mathbb{R} \cong V$*;*
(3) $V^\vee \otimes W = \mathrm{Hom}_{\mathbb{R}}(V, W)$*; and*
(4) $V^\vee \otimes W^\vee \cong (V \otimes W)^\vee$*.*

In addition, $- \otimes -\colon \mathbf{Vect}_{\mathbb{R}} \times \mathbf{Vect}_{\mathbb{R}}\colon \longrightarrow \mathbf{Vect}_{\mathbb{R}}$ *is a bifunctor.*

PROOF. One can either construct the maps explicitly, or use theorem 1.1.1. □

We can iterate these tensor products, and it is not too hard to show that the process is associative, i.e.

$$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3).$$

Thus it makes sense to talk about the tensor product $V_1 \otimes \cdots \otimes V_k$ of k vector spaces. As $\mathrm{Hom}_{\mathbb{R}}(V \otimes W, Z) \cong L(V, W; Z)$ (by theorem 1.1.1), a consequence of proposition 1.1.3 is that we have the following universal property:

$$\begin{tikzcd} V_1 \times \cdots \times V_k \arrow[r, "\otimes"] \arrow[dr, "\text{multilinear } \varphi"'] & V_1 \otimes \cdots \otimes V_k \arrow[d, dashed, "\exists! \text{ linear } h"] \\ & Z \end{tikzcd}$$

This means that we have the bijection

$$\mathrm{Hom}_{\mathbb{R}}(V_1 \otimes \cdots \otimes V_k, Z) \cong L(V_1, \dots, V_k; Z).$$

In particular, as $V_1^\vee \otimes \cdots \otimes V_k^\vee \cong (V_1 \otimes \cdots \otimes V_k)^\vee$, we see that elements of

$$\underbrace{V^\vee \otimes \cdots \otimes V^\vee}_{k \text{ times}}$$

are the same thing as multilinear maps

$$\alpha\colon \underbrace{V \times \cdots \times V}_{k \text{ times}} \longrightarrow \mathbb{R}.$$

As these special types of multilinear maps are extremely useful, they have earned themselves a name: *(covariant) k-tensors on V*, or just *tensors*, and we write $T^k(V^\vee)$ for the set of all such maps. This seems like abuse of language, as tensors are elements of tensor product spaces, but lest we forget that these multilinear maps correspond to elements of $V^\vee \otimes \cdots \otimes V^\vee$ in a one-to-one fashion. Henceforth, we use the notation $T^k(V^\vee)$ to denote either the abstract tensor product space $V^\vee \otimes \cdots \otimes V^\vee$, or the space $L(V, \dots, V; \mathbb{R})$. Thus, elements of $T^k(V^\vee)$ are either finite sums of elementary tensors $v_1 \otimes \cdots \otimes v_k$, or multilinear maps $\alpha\colon V^k \longrightarrow \mathbb{R}$.

This construction is actually functorial, and so we have a functor

$$T^k\colon \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}},$$

which maps a vector space V to the k-fold tensor product $V \otimes \cdots \otimes V = V^{\otimes k}$. For morphisms, we just map $f\colon V_1 \longrightarrow V_2$ to

$$f \otimes \cdots \otimes f\colon V_1 \otimes \cdots \otimes V_1 \longrightarrow V_2 \otimes \cdots \otimes V_2.$$

Lastly, to get rid of the dependence of the integer k, we sum over all $k = 0, 1, \dots,$ and we are left with a functor

$$T\colon \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}},$$

sending V to the infinete direct sum

$$T(V) = \bigoplus_{k=0}^{\infty} T^k(V) = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

called the *tensor algebra*. We now study $T^k(V^\vee)$ in more detail.

1.1.3. Symmetric and alternating tensors

In general, it is hard to say what happens when the arguments of a multilinear map are permuted. For example, the dot product, which is a map $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, is invariant under transposition of the arguments:

$$v_1 \cdot v_2 = v_2 \cdot v_1.$$

On the other hand, the cross product, which is a map $\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, will change sign when the arguments are permuted:

$$v_1 \times v_2 = -v_2 \times v_1.$$

In this subsection, we describe the language created to capture these effects. Then we will see that that they, similarly to the abstract tensor product, give rise to universal properties.

Let $\alpha\colon V \times \cdots \times V \longrightarrow Z$ be a multilinear map, and $i, j = 1, \ldots, k$ be distinct integers. We call α *symmetric* if the value is unchanged whenever any two arguments are interchanged, i.e.

$$\alpha(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = \alpha(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

On the other hand, we say α is *alternating*, or *skew-symmetric*, if the sign is changed whenever any two arguments are interchanged, i.e.

$$\alpha(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\alpha(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

The collection of all symmetric multilinear maps is denoted $\operatorname{Sym}^k(V, Z)$, and $\operatorname{Alt}^k(V, Z)$ is the set of all alternating multilinear maps. These are both vector subspaces of $L(V, \cdots, V; Z)$.

Just as we, for any module Z, can make multilinear maps $V \times \cdots \times V \longrightarrow Z$ into linear maps $V \otimes \cdots \otimes V \longrightarrow Z$, we have correspondences

$$\begin{tikzcd}
V \times \cdots \times V \arrow[r, "\wedge"] \arrow[dr, "\text{alternating } \alpha"'] & \bigwedge^k V \arrow[d, dashed, "\exists!\text{ linear } h"] \\
 & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
V \times \cdots \times V \arrow[r, "\odot"] \arrow[dr, "\text{symmetric } \alpha"'] & \Sigma^k V \arrow[d, dashed, "\exists!\text{ linear } h"] \\
 & Z
\end{tikzcd}$$

of alternating maps. To find these spaces $\bigwedge^k V$ and $\operatorname{Sym}^k V$, we will focus our attention on $\operatorname{Sym}^k(V, \mathbb{R})$ and $\operatorname{Alt}^k(V, \mathbb{R})$.

There are two functors

$$\operatorname{Sym}\colon \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}, \quad \operatorname{Alt}\colon \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}},$$

called the *symmetrization functor* and the *alternation functor*, respectively. Given a vector space V,

$$\mathrm{Sym}(V) = \Sigma V, \quad \mathrm{Alt}(V) = \bigwedge V,$$

called the *symmetric algebra* and the *exterior algebra*, respectively. Both these vector spaces are some quotients of $T(V)$. If we define the subspaces

$$I_\Sigma(V) = \mathrm{span}\{v_1 \otimes v_2 - v_2 \otimes v_1 \mid v_1, v_2 \in T(V)\},$$
$$I_\bigwedge(V) = \mathrm{span}\{v \otimes v \mid v \in T(V)\},$$

then ΣV and $\bigwedge V$ are defined as follows:

$$\Sigma V = T(V)/I_\Sigma(V), \quad \bigwedge V = T(V)/I_\bigwedge(V)$$

From these quotients, it is clear what Sym and Alt does to morphisms.

Given an elementary tensor $v_1 \otimes \cdots \otimes v_k \in T(V)$, we denote the image under the projections as

$$v_1 \odot \cdots \odot v_k \in \Sigma V, \quad v_1 \wedge \cdots \wedge v_k \in \bigwedge V,$$

and we call these elements *decomposable*. The operation $\odot$ is called the *symmetric product*, and $\wedge$ is called the *wedge product*. The *k-th symmetric power of V*, $\Sigma^k V$, and the *k-th exterior power of V*, $\bigwedge^k V$ are the images of $T^k(V)$ under their respective projections. Thus, there are canonical isomorphisms

$$\Sigma^k V \cong \frac{T^k(V)}{T^k(V) \cap I_\Sigma(V)}, \quad \bigwedge^k V = \frac{T^k(V)}{T^k(V) \cap I_\bigwedge(V)}.$$

By the first isomorphism theorem, the direct sums of these quotients must be isomorphic to the quotients of the direct sums. Hence the symmetric and the exerior algebras are graded, and we have

$$\Sigma V = \bigoplus_{k=0}^{\infty} \Sigma^k V, \quad \bigwedge V = \bigoplus_{k=0}^{\infty} \bigwedge^k V.$$

Now that we have this grading, we can define the Koszul complex.

DEFINITION 1.1.4. The *Koszul complex* is the differential graded algebra

$$\mathrm{Kos}^\bullet V = \bigwedge^\bullet V \otimes \Sigma_2^\bullet V,$$

where, for $v \in \bigwedge^1 V = V$ and $\widetilde{v} \in \Sigma^1 V = V$, we have the differential

$$dv = \widetilde{v}, \quad d\widetilde{v} = 0.$$

Here $\Sigma_2^\bullet V$ denotes $\Sigma^\bullet V$, but graded by twice the degree. Explicitly, the Koszul complex starts of as

$$\mathbb{R} \longrightarrow \bigwedge^1 V \longrightarrow \bigwedge^2 V \otimes \Sigma^1 V \longrightarrow \bigwedge^3 V \longrightarrow \bigwedge^4 V \otimes \mathrm{Sym}^2 \cdots$$

For our purposes, we do not need to know much more about the symmetric algebra, so from now on we focus on the exterior algebra.

1.1.4. Properties of the wedge product

Just as the map $\otimes\colon V^k \longrightarrow V^{\otimes k}$ sending $(v_1, \ldots, v_k)$ to $v_1 \otimes \cdots \otimes v_k$ has a certain universal property (see theorem 1.1.1), the induced map

$$\wedge\colon V^k \longrightarrow \bigwedge^k V, \quad (v_1, \ldots, v_k) \longmapsto v_1 \wedge \cdots \wedge v_k,$$

has a universal property too.

THEOREM 1.1.5. *For any real vector space Z and any alternating map $\varphi\colon V^k \longrightarrow Z$, there is a unique linear map $h\colon \bigwedge^k V \longrightarrow Z$ such that the diagram*

$$\begin{tikzcd} V \times \cdots \times V \arrow[r, "\wedge"] \arrow[dr, "\text{alternating } \alpha"'] & \bigwedge^k V \arrow[d, dashed, "\exists!\ \text{linear } h"] \\ & Z \end{tikzcd}$$

commutes.

PROOF. This follows from theorem 1.1.1. □

We now look at how to express the basis of $\bigwedge^k V$.

As $\bigwedge V$ is the quotient of $T(V)$ by elements $v \otimes v$, we immediately get that

$$0 = (u+v) \wedge (u+v) = u \wedge u + u \wedge v + v \wedge u + v \wedge v = u \wedge v + v \wedge u,$$

hence $u \wedge v = -v \wedge u$. We also, for $u \in \bigwedge^k V$ and $v \in \bigwedge^l V$, have that

$$u \wedge v = (-1)^{kl} v \wedge u.$$

This follows from the fact that $u \wedge v = -v \wedge u$. A corollary of this is that transposition of two vectors in a decomposable chain introduces a minus sign, i.e.

$$v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k = -v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_k.$$

And in general, as a permutation σ is the composition of transpositions, we get that

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = (\operatorname{sign} \sigma) v_1 \wedge \cdots \wedge v_k.$$

This result, combined with theorem 1.1.5, means that for a basis $\{e_1, \ldots, e_n\}$ of V, the wedge product of them all is not equal to 0. And thus, for $1 \leq i_1 < \cdots < i_k \leq n$, we have

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \neq 0.$$

It can further be shown that the set

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \;|\; 1 \leq i_1 < \cdots < i_k \leq n\},$$

forms a basis of $\bigwedge^k V$. The reader good at combinatorics will easily see that the dimension of $\bigwedge^k V$ is $\binom{\dim V}{k}$.

Another consequence of theorem 1.1.5 is that we have a canonical isomorphism

$$\mathrm{Alt}^k(V) \cong \left(\bigwedge^k V\right)^{\vee}.$$

To see this, note that $L(V, \ldots, V; \mathbb{R})$ is canonically isomorphic to $T^k(V)^{\vee}$ by proposition proposition 1.1.3. The above isomorphism is the one induced from the isomorphism $L(V, \ldots, V; \mathbb{R}) \cong T^k(V)^{\vee}$.

Just based on dimensions, we see that there is an isormophism

$$\bigwedge^k(V^{\vee}) \cong \left(\bigwedge^k V\right)^{\vee}.$$

Using the technology of non-degenerate pairings, it can also be shown that the isomorphism is canonical, but we skip this digression. Still, we have the following:

Proposition 1.1.6. *For any finite dimensional vector space V, there are canonical isomorphisms*

$$\bigwedge^k(V^{\vee}) \cong \left(\bigwedge^k V\right)^{\vee} \cong \mathrm{Alt}^k(V).$$

This result will be useful when working with differential forms.

1.2. Bundles

Now that we (hopefully) know all the linear and multilinear algebra necessary for this thesis, we recall what smooth manifolds are, before defining smooth fibre bundles, smooth vector bundles, and smooth principal bundles.

Again, all of the material can be found in [Tu17] and [Lee13].

1.2.1. Smooth manifolds

We recall some basic definitions of manifolds and tangent spaces. See appendix A.1 for a more in-depth exposition.

A *topological manifold* M is just a "nice enough"[2] topological space that is also *locally Euclidean*, meaning that at each point $p \in M$ there is an open neighborhood $U \subseteq M$ of p that is homeomorphic to an open set $V \subseteq \mathbb{R}^n$ of the Euclidean space $\mathbb{R}^n$. An *atlas A on M* is just a collection of *charts*

$$(U_\alpha, \varphi_\alpha \colon U_\alpha \subseteq M \longrightarrow V_\alpha \subseteq \mathbb{R}^n), \quad \varphi_\alpha \text{ homeomorphism},$$

such that $\bigcup_\alpha U_\alpha = M$. We call the atlas *smooth* if all the *transition maps*

$$\varphi_\alpha \circ (\varphi_\beta)^{-1} \colon \varphi_\beta(U_\alpha \cap U_\beta) \longrightarrow \varphi_\alpha(U_\alpha \cap U_\beta)$$

are smooth maps on $\mathbb{R}^n$ in the usual vector calculus sense, i.e. infinitely differentiable. The union $A \cup A'$ of two smooth atlases is not necessarily a smooth atlas, but when it is, we call the two atlases *smoothly equivalent*. As the name suggests, this relation is an equivalence relation. Thus we define a *smooth structure $\mathscr{A}$ on M* to be such an equivalence class, i.e. a collection $\mathscr{A} = \{A, A', \dots\}$ of smoothly equivalent atlases.

A *smooth manifold* is just a topological manifold with a smooth structure $\mathscr{A}$. Note that the union $A \cup A' \cup \cdots$ of all atlases belonging to a smooth structure $\mathscr{A}$ is, by definition, a smooth atlas, which we call the *maximal smooth atlas*. The maximal atlas is denoted by $\mathscr{A}_{\max}$, i.e.

$$\mathscr{A}_{\max} = \bigcup_{A \in \mathscr{A}} A = \{(U, f \colon U \longrightarrow V)\},$$

where U is open in M, V is open in $\mathbb{R}^n$, and f is a homeomorphism. From now on, all manifolds will be assumed smooth, unless otherwise stated. When we talk about charts of some manifold, we will always assume it comes from $\mathscr{A}_{\max}$.

There is a one-to-one correspondence between smooth structures on M and maximal smooth atlases.

One can construct new manifolds from old ones. We have the following:

Meta-theorem 1.2.1. *Many canonical construction in topology gives rise to a construction of smooth manifolds.*

In the spirit of this meta-theorem, we list two examples.

[2]Hausdorff and second countable, to be specific.

EXAMPLE 1.2.2. Let M, M_1 and M_2 be smooth manifolds.

(1) The product space $M_1 \times M_2$ can be made into a smooth manifold, and the dimension is equal to the sum of the dimensions of M and N.
(2) For an open set U of M, the subspace U can be made into a smooth manifold, and the dimension is equal that of M.

We say that a map $F\colon M_1 \longrightarrow M_2$ between manifolds M_1 and M_2 is *smooth* if for every chart (U_1, φ) of M_1 and every chart (U_2, ψ) of M_2, the dashed arrow

$$\begin{tikzcd}
U_1 \arrow[rr, "F"] & & U_2 \arrow[d, "\psi"] \\
\varphi\big(U_1 \cap f^{-1}(U_2)\big) \arrow[u, "\varphi^{-1}"] \arrow[rr, dashed, "\psi\circ F\circ\varphi^{-1}"] & & V_2
\end{tikzcd}$$

is a smooth map between Euclidean spaces. The collection of all smooth maps $F\colon M \longrightarrow N$ is denoted $C^\infty(M, N)$, which is a ring[3]. One can see that the composition of smooth maps $F\colon M_1 \longrightarrow M_2$ and $G\colon M_2 \longrightarrow M_3$ is again smooth by considering the following commutative diagram:

$$\begin{tikzcd}
U \arrow[r, "F"] & V \arrow[r, "G"] \arrow[d, shift right, "\theta"'] & W \arrow[d, "\psi"] \\
\varphi(U) \arrow[u, "\varphi^{-1}"] \arrow[r, dashed] & \theta(V) \arrow[u, shift right, "\theta^{-1}"'] \arrow[r, dashed] & \psi(W)
\end{tikzcd}$$

A *diffeomorphism* is a bijective smooth map $f\colon M_1 \longrightarrow M_2$ whose inverse $f^{-1}\colon M_2 \longrightarrow M_1$ is also smooth. Thus, (almost) by definition, the homeomorphisms of the maximal atlas $\mathscr{A}_{\max}$ are all diffeomorphisms. Note that the identity map $\mathrm{id}_M\colon M \longrightarrow M$ is not only smooth, but also a diffeomorphism. In total, this means we have a category **Man** of finite dimensional smooth manifolds, with smooth maps as morphisms.

As all manifolds are sets, we can talk about *pointed manifolds* (M, p), which are just manifolds with a distinct chosen point $p \in M$; and *pointed maps of manifolds* $f\colon (M_1, p) \longrightarrow (M_2, q)$, which are just smooth maps $f\colon M_1 \longrightarrow M_2$ such that $f(p) = q$. These objects and morphisms constitute a category, denoted $\mathbf{Man}_*$.[4]

We now move over to tangent spaces. There is a functor

$$T_*\colon \mathbf{Man}_* \longrightarrow \mathbf{Vect}_{\mathbb{R}},$$

[3]under point-wise addition and multiplication.

[4]The $*$ in the subscript signifies that the objects are pointed.

assigning any pointed manifold (M, p) its tangent space T_pM, which is a real vector space. More specifically, $T_pM \subseteq \operatorname{Hom}_{\mathbb{R}}\big(C^\infty(M, \mathbb{R}), \mathbb{R}\big)$ is the set of *point-derivations at* p on M, i.e. the collection of linear maps $v\colon C^\infty(M, \mathbb{R}) \longrightarrow \mathbb{R}$ satisfying the *Leibniz rule*:

$$v(fg) = f(p)vg + g(p)vf.$$

For a smooth map $F\colon (M_1, p) \longrightarrow (M_2, q)$, the functor T_* assigns F to the linear map $F_{*,p}\colon T_pM_1 \longrightarrow T_{F(p)}M_2$, called the *differential of* F *at* p. For any $v \in T_pM_1$, the differential of F at p of v is the map

$$F_{*,p}(v) = v(- \circ F)\colon C^\infty(M_2, \mathbb{R}) \longrightarrow \mathbb{R}, \quad (f\colon M_2 \to \mathbb{R}) \longmapsto v(f \circ F).$$

It is not immediately clear from the definition that this is a derivation at q, but quick calculation (see remark A.1.18) shows that

$$F_{*,p}(v)(fg) = f(q)F_{*,p}(v)g + g(q)F_{*,p}(v)f.$$

When p is clear from context, we often omit it from the notation, and just write F_* instead of $F_{*,p}$.

The fact that T_* is a functor means, in particular, that $(\mathrm{id}_M)_{*,p} = \mathrm{id}_{T_pM}$ and that for

$$M_1 \xrightarrow{F} M_2 \xrightarrow{G} M_3,$$

we have $(G \circ F)_* = G_* \circ F_*$, which is called the *product rule*. Also, diffeomorphisms are sent to isomorphisms:

$$(F_{*,p})^{-1} = (F^{-1})_{*,F(p)}.$$

The typical example of a tangent vector at p, or point-derivation at p, is the partial derivatives. For $M = \mathbb{R}^n$, and $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$, the partial derivatives are the usual suspects

$$\left.\frac{\partial}{\partial x_i}\right|_p f = \left.\frac{\partial f}{\partial x_i}\right|_p = \lim_{h \to 0} \frac{f(p_1, \dots, p_i + h, \dots, p_n) - f(x_1, \dots, x_n)}{h},$$

where $x_1, \dots, x_n$ are the standard coordinates on $\mathbb{R}^n$ and we write $p = (p_1, \dots, p_n)$. For the partial derivatives on general manifolds, look at example A.1.14. These partial derivatives always form a basis for our tangent spaces. Thus we see that the dimension of any tangent space T_pM is the same as the dimension of the underlying smooth manifold M.

Knowing what tangent spaces of M are, we can study the *tangent bundle* TM, which is just the smooth manifold[5]

$$TM = \bigsqcup_{p\in M} T_pM = \{(p,v) \ \mid \ p \in M, v \in T_pM\}.$$

The dimension of TM is the sum $\dim M + \dim T_pM = 2 \times \dim M$. We have a functor $T\colon \mathbf{Man} \longrightarrow \mathbf{Man}$, assigning the tangent bundle to each manifold. For any smooth function $F\colon M_1 \longrightarrow M_2$, the functor T assigns F to the map

$$F_*\colon TM_1 \longrightarrow TM_2, \quad (p,v) \longmapsto T_*(F) = F_{*,p}(v),$$

called the *global differential.* It is smooth by the canonical induced smooth structures on the tangent bundles, and because F is smooth. The functoriality $(G\circ F)_* = G_* \circ F_*$, come from the product rule for the differential at each point. Similarily, $(\mathrm{id}_M)_* = \mathrm{id}_{TM}$.

1.2.2. Fibre bundles

In the following subsection, we recall what a smooth fibre budle is. We never need them in their full generality, but as we need smooth vector bundles, smooth principal bundles, holomorphic vector bundles, and holomorphic principal bundles, all of which are examples of smooth fibre bundles, we thought it might be worth knowing about the general structure and idea governing them all. We first need to know about local trivializations, which resemble the notion of an atlas.

Let π be a smooth surjection

$$\pi\colon E \longrightarrow M,$$

between two smooth manifolds, let F be some smooth manifold, and let $\mathscr{U} = \{U_\alpha\}$ be some cover of M. If, for every set $\pi^{-1}(U_\alpha) \subseteq E$, there is a diffeomorphism $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$, such that the diagram

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi\restriction_{U_\alpha}"'] & & U_\alpha \times F \arrow[dl, "\mathrm{proj}_{U_\alpha}"] \\
& U_\alpha &
\end{tikzcd}$$

[5] It is not immedeately clear how this is a topololgical space, much less a smooth manifold. But the maximal atlas on M determine the open sets on TM, and we can assign M a canonical smooth structure from the maximal atlas.

commutes, we call the collection $\{(U_\alpha, \varphi_\alpha)\}$ a *local trivialization (of* π*, subordinate to* $\mathscr{U}$*) with fiber* F, or just *local trivialization* for brevity. For each point $p \in M$, define

$$E_p = \pi^{-1}(p).$$

We say that the maps φ_α are *fiber preserving*, as the restriction $\varphi_\alpha \restriction_{E_p}$ over a point $p \in M$ is just a map $E_p \longrightarrow \{p\} \times F \approx F$, and so each manifold E_p is diffeomorphic to F.

Definition 1.2.3. A *fibre bundle* is a collection of data (E, M, π, F), where

(1) M, E, F are smooth manifolds and $\pi\colon E \longrightarrow M$ is a smooth surjection; and
(2) $\pi\colon E \longrightarrow M$ has a local trivialization with fiber F.

We call E the *total space* and B the *base space.* We call the space $E_p = \pi^{-1}(p)$ the *fiber over* p, as it is diffeomorphic to the fiber F (by the fiber preserving diffeomorphisms). Often, $\pi\colon E \longrightarrow B$, or $E \longrightarrow B$, or even just E is used to denote the whole fibre bundle.

Given two fibre budles

$$\pi_1\colon E_1 \longrightarrow M_1, \quad \pi_2\colon E_2 \longrightarrow M_2,$$

with the same fiber F, and a commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\varphi} & E_2 \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\overline{\varphi}} & M_2 \end{array}$$

we say that $(\varphi, \overline{\varphi})$ is a *bundle map* if the diagram commutes. This is the same as requiring the fiber $(E_1)_p$ over p to be mapped to the fibre $(E_2)_{\overline{\varphi}(p)}$ over $\overline{\varphi}(p)$ for each $p \in M_1$. We often say that φ is a *bundle map covering* $\overline{\varphi}$, or even, by abuse of language, that $\varphi\colon E_1 \longrightarrow E_2$ (as opposed to $(\varphi, \overline{\varphi})$) is a *bundle map.*

If the base spaces are equal, say to some manifold M, and $\overline{\varphi} = \mathrm{id}_M$, we say (φ, id_M) is a *bundle map over* M. This map carries the fiber $(E_1)_p$ over p to the fiber $(E_2)_p$ over the same basepoint. If the bundle map φ over M is bijective, and φ^{-1} is also a bundle map over M, then φ is called a *bundle isomorphism over* M, and the bundles are said to be *isomorphic.* This can be illustrated by the following commutative diagram:

$$\begin{tikzcd} E_1 \arrow[rr, shift left, "\varphi"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[ll, shift left, "\varphi^{-1}"] \arrow[dl, "\pi_2"] \\ & M & \end{tikzcd}$$

We could list lots of explicit examples of bundles in this section, but as we shall see, both smooth vector bundles and smooth principal bundles give rise to a plethora of such examples in and of them selves, so we drop mentioning them here. But we need to look at a few key examples:

EXAMPLE 1.2.4 (The trivial bundle). Let M be an arbitrary smooth manifold, and define $E = M \times F$. Then

$$\mathrm{proj}_M\colon M \times F \longrightarrow M, \quad (p, f) \longmapsto p,$$

is a smooth surjection. As the set $\{M\}$ covers M, and $\mathrm{id}_{M\times F}$ is a diffeomorphism $\mathrm{proj}_M^{-1}(M) \longrightarrow M \times F$, the set $\{(M, \mathrm{id}_{M\times F})\}$ is a local trivialization with fiber F. As this is the most trivial example of a fibre bundle, we call it the *trivial bundle over M*.

Inspired by the last example, we call a fibre bundle $E \longrightarrow M$ *trivial* if it is isomorphic, as a fibre bundle, to the trivial bundle $M \times F$. Note that, locally, fibre bundles are always trivial, as they admit local trivializations.

EXAMPLE 1.2.5 (Pullback bundle). If $F\colon M_1 \longrightarrow M_2$ is a smooth map, and

$$\pi\colon E \longrightarrow M_2$$

is a fibre bundle, we can define a new bundle over M_1, which is dependent on F and E, hence denoted F^*E. As a set, the total space is

$$F^*E = \{(m, e) \in M_1 \times E \mid F(n) = \pi(e)\} \subseteq M_1 \times E.$$

We endow this with the subspace topology of the product topology of $M_1 \times E$. The projection maps

$$\begin{aligned} \pi'\colon F^* &\longrightarrow M_1, & F'\colon F^*E &\longrightarrow E, \\ (m, e) &\longmapsto m, & (m, e) &\longmapsto e, \end{aligned}$$

fit into the commutative diagram

$$\begin{tikzcd} F^*E \arrow[r, "F'"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ M_1 \arrow[r, "F"] & M_2 \end{tikzcd}$$

We call $\pi' \colon F^*E \longrightarrow M_1$ for the *pullback bundle.*

The next proposition will help explain the smooth structure of the pullback bundle, as bundles are locally trivial.

Proposition 1.2.6. *The pullback bundle of a trivial bundle is trivial. Meaning, for $F \colon M_1 \longrightarrow M_2$ then $F^*(M_2 \times F)$ can be given a smooth structure such that it is isomorphic to $M_1 \times F$.*

Proof. As $F^*(M_2 \times F) = \{(m_1, (m_2, e)) \in M_1 \times (M_2 \times F) \mid F(m_1) = \pi(m_2, e) = m_2\}$, the map

$$\sigma \colon F^*(M_2 \times F) \longrightarrow M_1 \times F, \quad \big(m_1, (F(m_2), e)\big) \longmapsto (m_1, e),$$

is a fibre preserving homeomorphism. It gives the pullback bundle its smooth structure, and hence the bundle is trivial. $\square$

Theorem 1.2.7. *The pullback of a fibre bundle can be given a smooth structure.*

Proof. As every fibre bundle is locally trivial, i.e. they look like $U \times F \longrightarrow U$, the pullback F^*E is, by proposition 1.2.6, locally trivial and looks like $F^{-1}(U) \times F \longrightarrow F^{-1}(U)$. And so we have described the local trivializations, making F^*E into a fibre bundle. $\square$

The pullback bundle has a certain nice property we will exploit later. In layman's terms, the total space is the "largest" total space over M_1, in the sense that any bundle map covering $F \colon M_1 \longrightarrow M_2$, must factor through F^*E.

Proposition 1.2.8 (Universal property of the pullback bundle). *The pullback of a fibre bundle is a pullback in the category-theoretic sense, meaning that if we have some total space E' over M_1, with projection map $f \colon E' \longrightarrow M_1$, and a bundle map $g \colon E' \longrightarrow E$ covering F*

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d, "f"'] & E \arrow[d, "\pi"] \\ M_1 \arrow[r, "F"] & M_2 \end{tikzcd}$$

then g factors uniquely through F^*E, i.e. we have the following commutative diagram:

$$\begin{tikzcd}
E' \arrow[r, dashed, "\exists!"] \arrow[dr, "f"'] \arrow[rr, bend left, "g"] & F^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\
& M_1 \arrow[r, "F"] & M_2
\end{tikzcd}$$

PROOF. We need to show existence and uniqueness of the dashed arrow from E' to F^*E. Call the map ψ. We first show that ψ is unique.

For all $p \in E'$, commutativity of the diagram forces $\psi(p) = (f(p), g(p)) \in M_1 \times E$. So, if any such ψ exists, it must of this form, i.e. it must be unique.

Because g covers F, the fiber over $f(p)$ is mapped to the fiber over $F\big(f(p)\big)$. And so, for any $p \in E'$,

$$F\big(f(p)\big) = \pi\big(g(p)\big),$$

so $\big(f(p), g(p)\big) \in F^*E$. So the map ψ indeed exits. $\square$

Before concluding this section, we introduce some needed terminology.

DEFINITION 1.2.9. Let $\pi\colon E \longrightarrow M$ be a fibre bundle. A *smooth section* of E is a map

$$s\colon M \longrightarrow E$$

such that $\pi \circ s = \mathrm{id}_M$. That is, $s(p) \in E_p$. The set of all sections is often denoted $\Gamma(E)$.

1.2.3. Vector bundles

DEFINITION 1.2.10. An *n-dimensional smooth vector bundle* $\pi\colon V \longrightarrow M$ is a fibre bundle $(V, M, \pi, \mathbb{R}^n)$ with the follow extra condition:

- Each fibre preserving map $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n$ restricts to an isomorphism $\varphi_\alpha \restriction_{V_p}$ of vector spaces $V_p = \pi^{-1}(\{p\}) \longrightarrow \{p\} \times \mathbb{R}^n \cong \mathbb{R}^n$.

EXAMPLE 1.2.11 (The trivial bundle). If we replace a general fibre F from the trivial bundle in example 1.2.4 with $\mathbb{R}^n$, we get the *trivial vector bundle* $M \times \mathbb{R}^n \longrightarrow M$. It is a vector bundle because the only trivialization is the identity map $\mathrm{id}_{M\times\mathbb{R}^n}$, and this gives rise, not only to an isomorphism, but an equality $V_p = \{p\} \times \mathbb{R}^n$.

EXAMPLE 1.2.12 (The tangent bundle). For any manifold M, the tangent bundle

$$TM = \bigsqcup_{p\in M} T_pM,$$

is a vector bundle with $V_p = T_pM$. The set of sections of the tangent bundle is often written $\mathfrak{X}(M)$, i.e.

$$\Gamma(TM) = \mathfrak{X}(M).$$

EXAMPLE 1.2.13 (The pullback bundle). The pullback bundle of a vector bundle is, as a set, equal to the pullback bundle from example 1.2.5. But it can be turned into a vector bundle in a natural way, meaning the bundle map of total spaces is linear on fibers, and we have a similar universal property for vector bundles.

The notion of a map between vector bundles is similar to that of fibre bundles, as vector bundles are fibre bundles. A *vector bundle map* between two n-dimensional vector bundles is just a fibre bundle map that, when restricted to fibres, gives a linear map of vector spaces. More concretely, for vector bundles $V_1 \longrightarrow M_1$, $V_2 \longrightarrow M_2$, the bundle map $\varphi\colon V_1 \longrightarrow V_2$, which fits the commutative diagram

$$\begin{tikzcd} V_1 \arrow[r, "\varphi"] \arrow[d] & V_2 \arrow[d] \\ M_1 \arrow[r, "\overline{\varphi}"] & M_2 \end{tikzcd}$$

gives linear maps $\varphi_p = \varphi \restriction_{(V_1)_p}\colon (V_1)_p \longrightarrow (V_2)_{\overline{\varphi}(p)}$.

A *vector bundle isomorphism* φ is a fibre bundle isomorphism such that both φ and φ^{-1} are vector bundle maps.

META-THEOREM 1.2.14. *Any canonical construction in linear algebra gives rise to a geometric version for smooth vector bundles.*

Honouring this meta-theorem, we collect a list of useful constructions.

EXAMPLE 1.2.15. Let E, E_1 and E_2 be smooth vector bundles over a smooth manifold M. Let $p \in M$ be any point in M.

(1) The *direct sum* $E_1 \oplus E_2$ is the smooth vector bundle over M whose fiber $(E_1 \oplus E_2)_p$ over p is canonically isomorphic to $(E_1)_p \oplus (E_2)_p$.
(2) The *dual bundle* $(E^\vee)$ is the smooth vector bundle over M whose fibre $(E^\vee)_p$ over p is canonically isomorphic to the dual space $(E_p)^\vee = \mathrm{Hom}_{\mathbb{R}}(E_p, \mathbb{R})$.
(3) The *tensor product* $E_1 \otimes E_2$ is the smooth vector bundle over M whose fiber $(E_1 \otimes E_2)_p$ over p is canonically isomorphic to $(E_1)_p \otimes (E_2)_p$.

(4) The Hom-*bundle* $\mathrm{Hom}(E_1, E_2)$ is the smooth vector bundle over M whose fibre $\mathrm{Hom}(E_1, E_2)_p$ over p is canonically isomorphic to $\mathrm{Hom}_{\mathbb{R}}\big((E_1)_p, (E_2)_p\big)$.

(5) The *k-th exterior power* $\bigwedge^k E$ and the *k-th symmetric power* $\Sigma^k E$ are the smooth vector bundles over M whose fibers over p are canonically isomorphic to $\bigwedge^k E_p$ and $\Sigma^k E_p$.

(6) Let $\varphi\colon E_1 \longrightarrow E_2$ be a vector bundle morphism. Then there exists smooth vector bundles $\ker(\varphi)$ and $\mathrm{coker}(\varphi)$ over M such that the fibers are isomorphic to $\ker(\varphi_p\colon (E_1)_p \longrightarrow (E_2)_p)$ and $\mathrm{coker}(\varphi_p\colon (E_1)_p \longrightarrow (E_2)_p)$, respectively. In particular, we can create *short exact sequences* of vector bundles.

Mix and match the examples for an even greater list of examples.

There are some redundancies in the list above. For example, the dual bundle $(E^\vee)$ is equal to the Hom-bundle $\mathrm{Hom}(E, M \times \mathbb{R})$. Also, as $\mathrm{Hom}\big((E_1)_p, (E_2)_p\big)$ is canonically isomorphic to $\big((E_1)_p\big)^\vee \otimes (E_2)_p$ (see proposition 1.1.3), we have a canonical isomorphism $\mathrm{Hom}(E_1, E_2) \cong (E_1)^\vee \otimes E_2$ of vector bundles. But the Hom-bundle is useful in itself because the sections $\Gamma\big(\mathrm{Hom}(E_1, E_2)\big)$ of the Hom-bundle are in a one-to-one correspondence with vector bundle maps $E_1 \longrightarrow E_2$ over M. This is because any smooth section $p \longmapsto \mathrm{Hom}(E_1, E_2)_p$ yields a map $\varphi_p \in \mathrm{Hom}_{\mathbb{R}}\big((E_1)_p, (E_2)_p\big)$ which varies smoothly with $p \in M$, which is precisely what a vector bundle map $\varphi\colon E_1 \longrightarrow E_2$ is. Conversely, any vector bundle map $\varphi\colon E_1 \longrightarrow E_2$ determines linear maps on the fibres that vary smoothly with p, and thus yields a smooth section $M \longrightarrow \mathrm{Hom}(E_1, E_2)$.

For any smooth vector bundle $V \longrightarrow M$, the collection $\Gamma(V)$ of all smooth sections is both a real vector space and a $C^\infty(M, \mathbb{R})$-module. Let $s\colon M \longrightarrow V$ be a smooth section, meaning $s(p) \in V_p$. For any scalar $a \in \mathbb{R}$ and any smooth $f\colon M \longrightarrow \mathbb{R}$, the scalar products

$$as\colon M \longrightarrow V, \quad f \cdot s\colon M \longrightarrow V,$$

are defined point-wise, i.e.

$$(as)(p) = as(p), \quad (f \cdot s)(p) = f(p)s(p),$$

where multiplication in the reals is written juxtaposition.

For part II, we need to know what *smooth complex vector bundles* are. They are the exact same beasts as smooth (real) vector bundles, but with complex vector spaces as fibers, and the fibre preserving maps restrict to $\mathbb{C}$-linear maps.

1.2.4. Principal bundles

First we recall some terminology of group actions and G-spaces. See appendix A.2.2 for a more in-depth exposition this.

Let G be a group. A *right G-set* is a set M with a right action

$$\begin{aligned} M \times G &\longrightarrow M, \\ (p, g) &\longmapsto p \cdot g, \end{aligned}$$

Here, *right action* means that $p \cdot e = p$ and $(p \cdot g_1) \cdot g_2 = p \cdot (g_1 g_2)$. Any point $(p, g) \in M \times G$ gives rise to two maps:

$$\begin{aligned} (-) \cdot g \colon M &\longrightarrow M, & p \cdot (-) \colon G &\longrightarrow M, \\ p &\longmapsto p \cdot g, & g &\longmapsto p \cdot g \end{aligned}$$

We call a right action *free* if all the maps $(-) \cdot g$, exept $(-) \cdot e$, are fixed point-free, meaning $p \cdot g = p$ implies $g = e$. We call a right action *transitive* if all the maps $p \cdot (-)$ are surjective, meaning for any pair $p, q \in M$ there is some $g \in G$ such that $p \cdot g = q$. Note that together, free and transitive is quite strong. In fact, such an action has the property that there is a unique g sending p to q. We call such actions *regular*.

Recall that a Lie group is a group G that is a smooth manifold. The constant map $* \longmapsto e$, the inversion map $g \longmapsto g^{-1}$ and the multiplication map $(g, h) \longmapsto gh$ are also all smooth by definition. (See definition A.2.1). A *right G-space* is a G-set M that is a smooth manifold, where G is a Lie group, and the right action is a smooth map $M \times G \longrightarrow M$.

A *right G-equivariance* is a map $\varphi \colon M_1 \longrightarrow M_2$ of right G-spaces that preserve the action, meaning $\varphi(p \cdot g) = \varphi(p) \cdot g$. We also call φ *equivariant*.

Armed with this terminology, we can define principal bundles.

Definition 1.2.16. Let G be a Lie group. A *principal G-bundle* $\pi \colon E \longrightarrow M$ is a fibre bundle (E, M, π, G) with the following extra conditions:

- E is a G-space, and for each point $p \in M$ the action restricted to each space $E_p = \pi^{-1}(p)$ is regular; and
- each fibre preserving map $\varphi_\alpha \colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times G$ restricts to a G-equivariant map $\varphi_\alpha \restriction_{E_p}$, i.e. for all $g \in G$, and for all $x \in \pi^{-1}(U_\alpha)$, we have

$$\varphi_\alpha(x \cdot g) = \varphi_\alpha(x) \cdot g,$$

where the action $\varphi_\alpha(x) \cdot g$ is just group multiplication on the right coordinate in $U_\alpha \times G$.

Let E_p be a fibre of a principal G-bundle $E \longrightarrow M$. Since the action is free and transitive on E_p, we have that for each pair $p, q \in M$ there is a unique element $g \in G$ such that $p \cdot g = q$. In other words, every pair $p \cdot g$ determines a point in y, and vice versa. And so the base space M is equal to the orbit space $\{p \cdot G\}_{p \in M} = E/G$. This characterization is sometimes useful.

Example 1.2.17 (The trivial bundle). The most trivial example of a principal G-bundle is again the trivial bundle from example 1.2.4, i.e.

$$M \times G \longrightarrow M,$$

where the smooth right action is just group multiplication on the right coordinate, which is smooth. The action is free and transitive on fibers, because binary operations satisfy those criterion.

One should note the following:

Proposition 1.2.18. *Let $\pi \colon E \longrightarrow M$ be a principal bundle. Then it is trivial if and only if there is a smooth global section $M \longrightarrow E$.*

Proof. If we have a global section $s \colon M \longrightarrow E$, we can construct the bundle map

$$\varphi \colon M \times G \longrightarrow E, \quad (p, g) \longmapsto s(p) \cdot g,$$

covering the identity. This is in fact a diffeomorphism.

If E is trivial, we have some diffeomorphism $M \times G \longrightarrow E$. The composition of the trivial section $p \longmapsto (p, e)$ with φ determines a smooth global section $M \longrightarrow E$. □

Example 1.2.19 (The pullback bundle). The pullback bundle of a principal G-bundle is again, as a set, equal to the pullback bundle from example 1.2.5. But it too can be turned into a principal G-bundle in a natural way, meaning the bundle map of total spaces is equivariant on fibers, and we have a similar universal property for principal G-bundles.

Again, maps between principal bundles are defined using fibre bundle maps. A *principal bundle map* between two principal bundles is just a fibre bundle map that is equivariant. More concretely, for principal G-bundles $E_1 \longrightarrow M_1$, $E_2 \longrightarrow M_2$, the bundle map $\varphi \colon E_1 \longrightarrow E_2$ is G-equivariant and fits in the following commutative diagram:

$$\begin{tikzcd} E_1 \arrow[r, "\varphi"] \arrow[d] & E_2 \arrow[d] \\ M_1 \arrow[r] & M_2 \end{tikzcd}$$

A *principal bundle isomorphism* φ is a fibre bundle isomorphism such that both φ and φ^{-1} are principal bundle maps. When $\varphi\colon E \longrightarrow E$, we sometimes call φ a *gauge transformation*.

1.3. Differential forms on manifolds

We will now use the tools from algebraic topology developed in section 1.1, and apply them to the smooth bundles from section 1.2.

The material covered here is a blend of theory from [Tu17] and [Lee13].

1.3.1. Ordinary differential forms

Let M be a n-dimensional manifold. We construct a vector bundle over M. The fibre over any point $p \in M$ is the vector space of alternating k-tensors on the tangent space of M at p. To put it differently, the vector bundle is the k-th exterior product of the dual bundle of the tangent bundle. In symbols, the fiber at p is the space $\bigwedge^k T_p^* M$. The total space is the disjoint union

$$\bigwedge^k T^*M = \bigwedge^k (TM)^\vee = \bigsqcup_{p \in M} \bigwedge^k T_p^* M,$$

of these fibers. The smooth structure is perhaps not immediately clear, but it is ensured by meta theorem 1.2.14: just as TM is a smooth vector bundle, T^*M is a smooth vector bundle, and thus $\bigwedge^k T^*M$ has a smooth structure. The smooth surjection is the map

$$\pi\colon \bigwedge^k T^*M \longrightarrow M, \quad (p, \omega) \longmapsto p,$$

and this completes the construction.

Note that, by proposition 1.1.6. we could just as easily constructed a smooth vector bundle

$$\mathrm{Alt}^k TM = \bigsqcup_{p \in M} \mathrm{Alt}^k(T_p M),$$

over M, which would be isomorphic to $\bigwedge^k T^*M$. In practice, we use whichever is most practical.

DEFINITION 1.3.1. A *differential k-form*, or *k-form* for short, is a section of the smooth vector bundle $\bigwedge^k TM$, i.e. a smooth map

$$\omega\colon M \longrightarrow \bigwedge^k T^*M,$$

such that $\pi \circ \omega = \mathrm{id}_M$, or equivalently, that $\omega_p \in \bigwedge^k T_p^*M$. The collection of all differential k-forms is denoted by $\Omega^k(M)$, i.e.

$$\Omega^k(M) = \Gamma\Big(\bigwedge^k T^*M\Big).$$

As the bundles $\bigwedge^k T^*M$ and $\mathrm{Alt}^k\, TM$ are isomorphic, we can view any k-form ω as a smooth map on M such that the multilinear map

$$\omega_p\colon T_pM \times \cdots \times T_pM \longrightarrow \mathbb{R},$$

is alternating. In this sense, we say k-forms take values in $\mathbb{R}$. Note that $\Omega^k(M)$ is both a vector space over $\mathbb{R}$ and a module over the ring $C^\infty(M, \mathbb{R})$ where the algebraic structures are defined point-wise, as mentioned at the end of section 1.2.3. The *wedge product* $\omega \wedge \omega'$ of two differential forms $\omega \in \Omega^k(M)$ and $\omega \in \Omega^l(M)$ is also defined point-wise, meaning

$$(\omega \wedge \omega')_p = \omega_p \wedge \omega'_p.$$

This new wedge product inherits all the properties of the wedge product from section 1.1.4. For example, for any $\omega \in \Omega^k(M)$,

$$\omega \wedge \omega = 0.$$

We collect all the useful results in a proposition.

PROPOSITION 1.3.2. *Let $\omega \in \Omega^k(M)$ and $\omega \in \Omega^l(M)$. Then the following hold:*

(1) $\omega \wedge \omega' = (-1)^{kl} \omega' \wedge \omega$.
(2) $\omega_{\sigma(1)} \wedge \cdots \wedge \omega_{\sigma(n)} = \mathrm{sign}(\sigma)\omega_1 \wedge \cdots \wedge \omega_n$.

PROOF. As these properties hold point-wise, and the wedge product is defined point-wise, there is nothing more to show. □

If $F\colon M_1 \longrightarrow M_2$ is a smooth map of manifolds, and ω is a differential form on the target manifold M_2, the *pullback* $F^*\omega$ is a differential form on the domain manifold M_1. Point-wise, $(F^*\omega)_p$ is the map $T_pM \times \cdots \times T_pM \longrightarrow \mathbb{R}$ defined as

$$(F^*\omega)_p(X_1, \ldots, X_k) = \omega_{F(p)}\big(F_{*,p}(X_1), \ldots, F_{*,p}(X_k)\big).$$

Since this map is $\mathbb{R}$-linear, and the differential at p satisfies the product rule, we have a functor

$$\Omega^k\colon \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}\,.$$

This functor also preserves the wedge product, meaning

$$F^*(\omega_1 \wedge \omega_2) = (F^*\omega_1) \wedge (F^*\omega_2).$$

All that to say, the construction of differential forms is natural. We now explore what they look like.

Locally, the k-forms have a particular look. In any chart of an n-dimensional smooth manifold M, we essentially work with Euclidean space $\mathbb{R}^n$, and so it easier to get an understanding of what is going on. For $k = 0$, the differential 0-forms are just maps $f_p\colon T_pM^0 \longrightarrow \mathbb{R}$, i.e. for each $p \in M$ we have a real number in $\mathbb{R}$. So $\Omega^0(M) = C^\infty(M, \mathbb{R})$. The case $k = 1$ is a little bit more technical, but vastly important. As mentioned in section 1.2.1, the partial derivatives $\frac{\partial}{\partial x_i}$ form a basis of T_pM. We denote the dual basis of $T_p^*M = (T_pM)^\vee$ by dx_i. As we saw in section 1.1.4, they form a basis of $\bigwedge^1 T_p^*M$. And so at a point $p \in M$, any 1-form ω is a linear combination

$$\omega_p = \sum_{i=1}^{n} \omega_i(p)dx_i.$$

As ω is smooth, we must have smooth functions $\omega_1, \ldots, \omega_n\colon M \longrightarrow \mathbb{R}$ such the equation holds at any point. So any 1-form locally looks like such a sum. In general, by the nature of manifolds, we do not know how to express ω globally, but the local expressions is all we need. For higher k, a basis of T_p^*M is the set

$$\{dx_{i_1} \wedge \cdots \wedge d_{i_k} \;\mid\; 1 \leq i_1 < \cdots < i_k \leq n\}.$$

We can use the same observations as above to conclude that, locally, a k-form ω looks like

$$\omega_p = \sum_I \omega_I dx_{i_1} \wedge \cdots \wedge d_{i_k},$$

where the sum is taken over all multi-indices $I = (i_1, \ldots, i_k)$ such that $1 \leq i_1 < \cdots < i_k \leq n$.

There is a natural way to get a 1-form from a 0-form f. At a point $p \in M$, and for $X \in T_pM$, define the 1-form df as

$$df_p(X) = Xf.$$

This is called the *differential of* f, and is always a 1-form when f is smooth. Using the observations above, we can see that

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

One should note that not all 1-forms are differentials of 0-forms. It turns out that a necessary condition for ω to be equal to df is that

$$\frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j} = 0,$$

in every coordinate chart. And so we recognize that the expression on the left hand side is an important one, and we should give it a meaning of its own. For any 1-form ω, define $d\omega$ as the 2-form which locally is expressed as

$$d\omega = \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j} \right) dx_i \wedge dx_j.$$

It immediately follows that $\omega = df$ if and only $d\omega = 0$ in each chart.

It turns out that this definition lifts to a well defined global construction, independent of chart. And so we extend d to get a differential of higher k-forms as well.

Definition 1.3.3. Let M be a smooth n-dimensional manifold, and let ω be a k-form on M, which locally can be expressed as

$$\omega_p = \sum \omega_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Define $d\colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ as the map which locally sends ω to

$$d\omega = \sum \frac{\partial \omega_I}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge d_{i_k}.$$

Strictly speaking, we do not know that the local expression of $d\omega$ is independent of chart, but it is. It is also equal to the expressions we had of df and $d\omega$ for $k = 0$ and $k = 1$. Furthermore, the construction of d is compatible with pullbacks, i.e. for a smooth map $F\colon M_1 \longrightarrow M_2$, we have

$$F^* d\omega = dF^* \omega.$$

Recall that for $k = 0$, we observed that $d\omega = 0$ if and only if $\omega = df$. Thus $d(df)$ is always equal to 0. And in general, we have

$$d \circ d = 0,$$

as a map $\Omega^{k-1}(M) \longrightarrow \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$. This means we get a complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \cdots$$

This is called the *de Rham complex* of M. We measure how far this sequence is from being exact by the *de Rham cohomology groups*, which are defined as

$$H^k_{dR}(M) = \frac{\ker(d\colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M))}{\operatorname{im}(d\colon \Omega^{k-1}(M) \longrightarrow \Omega^k(M))}.$$

This is all we need to know about differential forms with values in $\mathbb{R}$, and move over to the more general case.

1.3.2. Vector-valued differential forms

In this section, we will use the general isomorphisms collected in section 1.1 to get two perspectives on vector valued differential forms which are of equal importance.

Let M be a smooth manifold, and $p \in M$ any point in M. As we have seen in section 1.3.1, the universal property of the exterior power gives a one-to-one correspondence between alternating k-linear maps on the tangent space T_pM and linear maps $\bigwedge^k T_pM \longrightarrow \mathbb{R}$. In symbols,

$$\operatorname{Alt}^k(T_pM) \cong \bigwedge^k T_p^*M.$$

It is under this isomorphism we view differential k-forms as smooth sections of the smooth vector bundle $\bigwedge^k T^*M$, and section 1.3.1 was all about these forms. In this section however, we generalize these sections which are smooth maps on M that, for each $p \in M$, are alternating maps

$$T_pM \times \cdots \times T_pM \longrightarrow \mathbb{R},$$

to smooth maps on M that, for all $p \in M$, are alternating maps

$$T_pM \times \cdots \times T_pM \longrightarrow V,$$

for an arbitrary finite dimensional vector space V. By theorem 1.1.5, the diagram

$$\begin{tikzcd} T_pM^k \arrow[r, "\wedge"] \arrow[dr, "\omega"'] & \bigwedge^k T_pM \arrow[d, dashed, "\exists!\, \widetilde{\omega}"] \\ & V \end{tikzcd}$$

commutes, and the correspondence between ω and $\widetilde{\omega}$ is one-to-one, i.e.

$$\operatorname{Alt}^k(T, V) \cong \operatorname{Hom}_{\mathbb{R}}(\bigwedge^k T, V).$$

As $V^\vee \otimes W \cong \mathrm{Hom}_{\mathbb{R}}(V, W)$ (see proposition 1.1.3), we have the following isomorphisms:

$$\mathrm{Hom}_{\mathbb{R}}(\bigwedge^k T_pM, V) \cong \left(\bigwedge^k T_pM\right)^\vee \otimes V \cong \left(\bigwedge^k T_p^*M\right) \otimes V.$$

Thus, $\mathrm{Alt}^k(T_pM, V) \cong \left(\bigwedge^k T_p^*M\right) \otimes V$, and we see that an element of $\left(\bigwedge^k T_pM\right) \otimes V$ can be thought of as an alternating map

$$\omega \colon T_pM \times \cdots \times T_pM \longrightarrow V.$$

And so we have the following definition.

Definition 1.3.4. Let M be a smooth manifold, V be a finite dimensional vector space, and use V to denote the trivial vector bundle $M \times V \longrightarrow M$ from example 1.2.11. A *V-valued k-form* on M is a smooth section of the tensor product bundle $\bigwedge^k T^*M \otimes V$. The collection of all these sections is denoted by

$$\Omega^k(M; V) = \Gamma\left(\left(\bigwedge^k T^*M\right) \otimes V\right).$$

We would like to extend the definition of wedge product to vector valued forms as well, but since the product is ultimately taken in $\mathbb{R}$, we have a problem, as there is no natural product on V. But, as $V \otimes V$ has a natural product, we could create a wedge product $\Omega^k(M; V) \times \Omega^l(M; V) \longrightarrow \Omega^{k+l}(M; V \otimes V)$. As the tensor product can be taken between two arbitrary vector spaces V, W, we generalize in this manner. The *wedge product* $\omega \wedge \omega'$ of V valued k-form $\omega \in \Omega^k(M; V)$ and W-valued l-form $\omega' \in \Omega^l(M; W)$ is the $V \otimes W$-valued $(k+l)$-form which point-wise looks like

$$\omega \wedge \omega'(v_1, \dots, v_{k+l}) = \sum_{\sigma \in S_{k+l}} \mathrm{sign}(\sigma)\omega(v_{\sigma(1)}, \dots) \otimes \omega'(v_{\sigma(k+1)}, \dots).$$

We have seen that $\Omega^k(M)$ is functorial in M, and as $\Omega^k(M; \mathbb{R}) = \Omega^k(M)$, we see that $\Omega^k(M, \mathbb{R})$ is functorial in M as well. But it turns out we can make $\Omega^k(M, V)$ functorial in both arguments M and V. In M, it is contravariant, while it is covariant in V. For a linear map $P \colon V \longrightarrow W$, we have the map

$$P \colon \Omega^k(M; V) \longrightarrow \Omega(M; W)$$

which is just pointwise composition. Meaning, given $p \in M$, any alternating map $\omega_p \colon T_pM \times \cdots \times T_pM \longrightarrow V$ is sent to $P\omega_p = P \circ$

$\omega_p \colon T_pM \times \cdots \times T_pM \longrightarrow W$, where

$$(P\omega_p)(v_1, \ldots, v_k) = P\big(\omega_p(v_1, \ldots, v_k)\big) \in W.$$

As we defined $\Omega^k(M;P)$ via composition, it clearly preserves composition. Thus, we have a functor

$$\Omega^k(-,-)\colon \mathbf{Man}^{\mathrm{op}} \times \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$$

All the observations we made about ordinary differential forms hold for vector valued differential forms as well. The pullback of a V-valued differential form is a V-valued differential form. We can take the exterior derivative of a vector valued form, and in turn get a *V-valued de Rham complex*

$$\Omega^0(M;V) \xrightarrow{\ d\ } \Omega^1(M;V) \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \Omega^k(M;V) \xrightarrow{\ d\ } \cdots$$

The new exterior differential $d\colon \Omega^k(M;V) \longrightarrow \Omega^{k+1}(M;V)$ commutes with pullbacks, i.e. for a smooth map $F\colon M_1 \longrightarrow M_2$, we have

$$F^* d\omega = dF^*\omega.$$

1.4. Connections on bundles

A connection, in its simplest form, is a geometric object which connects tangent vectors at different points in the manifold. In this section, we will see how the notion changes when we talk about fibre bundles, vector bundles, and finally principal bundles.

The relevant material can be found in [Tu17] and [Dup03].

1.4.1. Connections on fibre bundles

Given a fibre bundle $\pi\colon E \longrightarrow M$, we form the derivative of the projection map $\pi_*\colon TE \longrightarrow TM$. Since this is linear on fibers, we can form kernels, and we we arrive at the canonically defined smooth vector bundle

$$VE = \ker \pi_* \subseteq TE,$$

called the *vertical bundle*. Any fibre of VE is called a *vertical space*, and vectors in these spaces are *vertical vectors*. We use the adjective "vertical" because the vertical vectors are tangent to the the fibres of the base space M. Meaning, any vertical space VE_x over $x \in E$ is the tangent space $T_xE_{\pi(x)}$ of the fibre $E_{\pi(x)}$. But π_* is seldom injective, so at any point $x \in E$, we would have a non-trivial complementary vector space HE_x such that T_xE is the direct sum of VE_x and HE_x. A *horizontal space* is a choice of such a subspace of T_xE so that $T_xE = VE_x \oplus HE_x$.

A crucial difference between the vectical spaces and the horizontal spaces is that, while the vertical ones are uniquely defined, the horizontal spaces are not. At each point, there are infinitely many choices when forming the direct sum. Moreover, while the (disjoint) union of all the vertical spaces gives the vertical bundle VE, we are not guaranteed that the (disjoint) union of horizontal spaces over all of M gives a bundle HE such that $TE = VE \oplus HE$, even if HE is a smooth vector bundle. But whenever there exists a bundle HE such that $TE = VE \oplus HE$, we call HE a *horizontal distribution.* Informally, we can observe that as $x \in E$ varies the spaces VE_x and HE_x vary as well. But since $VE_x = T_x E_{\pi(x)}$, any movement of x within the fibre $E_{\pi(x)}$ only change VE_x, and not HE_x. Conversely, movement of x in E along the base space M is reflected in the HE_x component, and not VE_x. Thus, if one wants to understand how E is globally, a starting point would be to understand the horizontal distribution HE, as this tracks "twists" of E. A *connection* is a smooth bundle morphism $v\colon TE \longrightarrow TE$ such that $v^2 = v$, and $v \restriction_{VE} = VE$.

As one might notice

Proposition 1.4.1. *A horizontal distribution gives rise to a connection, and a connection determines a horizontal distribution.*

Proof. First, assume that we have a horizontal distribution of TE, i.e. $TE = VE \oplus HE$. Then the projection map $v\colon TE \longrightarrow VE$ can be considered as a map $v\colon TE \longrightarrow TE$ as any element $X \in TE$ can be written as $X = v(X) + h(X)$, where $v(X) \in VE$ and $h(X) \in HE$. It is clear that this is a bundle map.

Conversely, if we have a bundle map $v\colon TE \longrightarrow TE$ such that it restricts to the identity map on the vertical bundle VE, then the kernel

$$\ker(v) \subseteq TE,$$

is a bundle consisting of precisely the elements $X \in TE$ without any vertical component (as $v(X) = 0$ for $X \in \ker(v)$). This identifies a decomposition $TE = VE \oplus \ker(v)$, and so $\ker(v)$ is a horizontal distribution. $\square$

Thus, finding a distribution HE is the same as finding a connection v, and we can understand the twisting of E by finding a connection v. In the language of abstract nonsense, a horisontal distribution is equivalent to a splitting v of the following short exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & VE \arrow[r] & TE \arrow[l, bend right, "v"'] \arrow[r] & HE \arrow[r] & 0 \end{tikzcd}$$

We will now see what happens when our fibres have more structure.

1.4.2. Connections on vector bundles

Consider a smooth vector field Y on a manifold and let $X_p \in T_pM$ be a tangent vector at $p \in M$. If one wants to take the derivative of Y in the direction X_p, one would need to know the values of Y in a neighborhood of p and compare them, as the derivative would depend on the difference $Y_q - Y_p$. But as Y_q and Y_p lie in totally different vector spaces, there is no clear meaning of the minus sign. A linear connection tries to resolve this.

Let $\pi\colon E \longrightarrow M$ be a smooth vector bundle. Then a connection on E (regarded only as a fibre bundle) is a choice of horizontal distribution HE such that $TE = VE \oplus HE$. This means HE_x depends smoothly on $x \in E$. If, in addition, HE_x depends linearly on x, we get a linear connection. To be precise, let $S_\lambda\colon E \longrightarrow E$ denote the smooth map which is scalar multiplication by λ. We call the distribution HE a *linear horizontal distribution* if at any point $x \in E$, and for any scalar $\lambda \in \mathbb{R}$, the image of the horizontal space at x under the differential of S_λ at x is equal to the horizontal space at λx. In symbols:

$$(S_\lambda)_{*,x}(HE_x) = HE_{\lambda x}.$$

As we mentioned introductory-wise, connections try to solve the conundrum of taking the difference between vectors of different vector spaces of a vector bundle. So start with a section $s \in \Gamma(E)$. This has differential $s_*\colon TM \longrightarrow TE$, mapping any tangent vector X_p at p to the tangent vector s_*X_p at $s(p)$. This is a bundle map in $\mathrm{Hom}_{\mathbb{R}}(TM, TE)$, and so taking the global differential of a section gives a map $\Gamma(E) \longrightarrow \Gamma\big(\mathrm{Hom}_{\mathbb{R}}(TM, TE)\big)$. Since $\mathrm{Hom}_{\mathbb{R}}(TM, E) = (TM)^\vee \otimes E$ (see example 1.2.15), and $(TM)^\vee = T^*M$, we have a map

$$d\colon \Gamma(E) \longrightarrow \Gamma(T^*M \otimes TE)$$

So we have a canonical way of assigning smooth sections of $\Gamma(E)$ to smooth sections of $\Gamma(T^*M \otimes TE)$. But we want the target to be $\Gamma(E)$. And this is where the connection $v\colon TE \longrightarrow VE$ comes in.

As we noted in section 1.4.1, the vertical space VE_x at $x \in E$ is equal to the tangent space $T_xE_{\pi(x)}$. But now, $E_{\pi(x)}$ is a vector space as well, so the tangent space $T_xE_{\pi(x)}$ of the vector space $E_{\pi(x)}$ is canonically isomorphic to the vector space $E_{\pi(x)}$ itself. This means that given a tangent vector $X_p \in T_pM$, the map s_* maps X_p to a tangent vector in $T_{s(p)}E$, and the connection v can kill the horizontal component, leaving us with something in $VE_{s(p)} = T_{s(p)}E_{\pi(s(p))} =$

$T_{s(p)}E_p \cong E_p$. All this is to say that we can assign any smooth section $s \in \Gamma(E)$ to a linear map

$$T_pM \longrightarrow T_{s(p)}E \longrightarrow VE_{s(p)} = T_{s(p)}E_p \cong E_p,$$

that varies smoothly with $p \in M$, giving a vector bundle map in $\mathrm{Hom}_{\mathbb{R}}(TM, E)$. And as $\Gamma\big(\mathrm{Hom}_{\mathbb{R}}(TM, E)\big) = \Gamma(T^*M \otimes E)$, we are left with a map

$$\nabla\colon \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$$

If we specify that the distribution HE is in addition a linear horizontal distribution, meaning $(S_\lambda)_*(HE) = HE$, then it can be shown that ∇ becomes a $\mathbb{R}$-linear map which satisfies the Leibniz rule, namely that

$$\nabla(fs) = df \otimes s + f\nabla s,$$

where $d\colon C^\infty(M, \mathbb{R}) = \Omega^0(M) \longrightarrow \Omega^1(M)$ is the exterior derivative from definition 1.3.3. This result follows from a long and tedious computation (see [Lee09], p. 521), but no new ideas are needed. Thus we are lead to the following definition: a *linear connection* on a smooth vector bundle $\pi\colon E \longrightarrow M$ is a map

$$\nabla\colon \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E),$$

that is linear and satisfies the Leibniz rule.

Just as distributions on fibre bundles are equivalent to connections, linear distributions are equivalent to linear connections:

Proposition 1.4.2. *A linear horizontal distribution gives rise to a linear connection, and a linear connection determines a horizontal distribution.*

Proof. We have seen in this subsection that a linear distribution gives a linear connection. So we are done if we can show that a linear connection ∇ gives some linear horizontal distribution HE. The idea is very similar to the fibre bundle case, where the kernel $\ker(v)$ of a connection v is a horizontal distribution. The only problem here is that $\ker(\nabla)$ is not a subset of TE, and even if it was, we would not know if this distribution was linear. The solution is to canonically inject the kernel of ∇ into TE in such a way that this new vector bundle is orthogonal to the vertical bundle VE. Since the construction is canonical, we get the linearity $(S_\lambda)_*(HE) = HE$ for free.

Let $s\colon M \longrightarrow E$ be a smooth section such that $s(p) = x \in E$. Then ∇s is a smooth section of $\Gamma(T^*M \otimes E) = \Gamma\big(\mathrm{Hom}_{\mathbb{R}}(TM, E)\big)$, i.e. a vector bundle map $\nabla s\colon TM \longrightarrow E$ over M. But this determines a map

$\nabla s\colon \Gamma(TM) \longrightarrow \Gamma(E)$ by sending any smooth vector field $X\colon M \longrightarrow TM$ to the section

$$\nabla s \circ X\colon M \longrightarrow E.$$

At each $p \in M$, we have a vector $(\nabla s \circ X)_p \in E_p$. But as $E_p \cong T_{s(p)}E_p$, and each fiber embedds into the total space $i\colon E_p \hookrightarrow E$, the derivative at $s(p)$ gives a map

$$i_*\colon T_{s(p)}E_p \longrightarrow T_{s(p)}E,$$

and we can consider $(\nabla s \circ X)_p$ as an element of $T_{s(p)}E$. If we subtract away any vertical component, we are left with a completely horizontal vector. And as the horizontal component comes from M, subtracting with $s_*X_p \in T_pE$ gives the result. That is to say

$$HE_x = \{i_*(\nabla s \circ X)_{\pi(x)} - s_*X_{\pi(x)} \mid s \in \Gamma(E), X \in \Gamma(TM)\},$$

has the property $HE_x \oplus VE_x = T_xE$. It is straight forward to use the Leibniz rule to check that $(S_\lambda)_{*,x}(HE_x) = HE_{\lambda x}$, which completes the proof. □

It can be shown that any finite linear combination of connections is a connection, provided the coefficients add up to 1. So, using partition of unity, we get that every smooth vector bundle has a connection.

1.4.3. Connections on principal bundles

The perhaps most apparent difference between vector bundles and principal bundles is that principal bundles need not have vector spaces as fibers, but general Lie groups. Therefore, they do not have *vector* fields, but sections $X_p \in E_p$. So, it is not intuitively clear what it means to take a derivative in any "direction". Still, we can generalize connections from the fibre bundle case if we impose some conditions on these new connections.

Let $\pi\colon E \longrightarrow M$ be a principal G-bundle, i.e. a right G-space E over M such that the action becomes transitive and free when restricted to the fibres E_p for $p \in M$, and so that all the fibre preserving maps are G-equivariant over the fibres. If we have a distribution $HE \subseteq TE$, then we call it principal if the right action permutes the horizontal spaces. To be precise, let $R_g\colon E \longrightarrow E$ denote the smooth action which is right multiplication with g. We call the distribution HE a *principal horizontal distribution* if at any point $x \in E$, and for any element $g \in G$, the image of the horizontal space at g under the differential of R_g at x is equal to the horizontal space at xg. In symbols:

$$(R_g)_{*,x}(HE_x) = HE_{xg}.$$

Recall that there is an injection $\mathfrak{g} \longrightarrow \Gamma(TE)$, sending $X \in \mathfrak{g}$ to its fundamental vector field $X^\sharp \colon E \longrightarrow TE$, such that, for each $x \in E$, we have $X^\sharp_x \in T_xE$. Remember that each $x \in E$ gives a map,

$$f_x \colon G \longrightarrow E, \quad g \longrightarrow xg,$$

determined by the action on E. The derivative $(f_x)_{*,e}$ at the identity $e \in G$ is then a map $\mathfrak{g} \longrightarrow T_xE$. This dictates the vector $X^\sharp_x = (v_x)_*X \in T_xE$. Collecting all these vectors, we see that $X^\sharp$ is in fact a smooth section of TM. But what is more, the fundamental vector fields are actually vertical. Straightforward calculation gives

$$\pi_{*,x}(X^\sharp_x) = \pi_{*,x}\big((v_x)_{*,e}X\big) = (\pi \circ v_x)_{*,e}X,$$

and since $\pi \circ f_x \colon G \longrightarrow M$ is constant equal to $\pi(x)$, the derivative $(\pi \circ f_x)_{*,e} = 0$. So $X^\sharp \in \ker(\pi_*)$, meaning $X^\sharp$ is vertical. Thus, for principal G-bundles, we have a map

$$(f_x)_{*,e} \colon \mathfrak{g} \longrightarrow VE_x \subseteq T_xE.$$

This can be seen to be an isomorphism. Hence the vertical tangent vectors at a point are precisely the fundamental vectors. The isomorphism $(f_x)_{*,e}$ actually determines a $\mathfrak{g}$-valued 1-form ∇ on E if we have a horizontal distribution. At $x \in E$, it is defined as the composition

$$\nabla_x = \big((f_x)_{*,e}\big)^{-1} \circ v \colon T_xE \longrightarrow V_xE \longrightarrow \mathfrak{g},$$

and we immediately see that $\nabla \colon M \longrightarrow \big(\bigwedge^1 T^*E\big) \otimes \mathfrak{g}$ is a $\mathfrak{g}$-valued 1-form on the total space E. If we furthermore specify that the distribution HE is in addition a principal horizontal distribution, then it can be shown that, for all $g \in G$, the pullback

$$R_g^*\nabla = \mathrm{Ad}_{g^{-1}} \nabla.$$

We call this property of a $\mathfrak{g}$-valued 1-form for *G-equivariance*. To see that ∇ is G-equivariant, we first use the fact that the vertical tangent vectors are the fundamental ones. This implies that the projection v sends any fundamental vector $X^\sharp_x$ to itself, as it is vertical. So

$$\nabla_x(X^\sharp_x) = \big((f_x)_{*,e}\big)^{-1}\big(v(X^\sharp_x)\big) = \big((f_x)_{*,e}\big)^{-1}(X^\sharp_x) = X.$$

This allows us to easily check the G-equivariance at vertical and horizontal tangent vectors separately. For a vertical vector $Y_x \in V_xE$, it is equal to some fundamental vector $X^\sharp_x$, and so

$$(R_g^*\nabla)_x(Y_x) = \nabla_{xg}\big((R_g)_{*,x}X^\sharp_x\big).$$

We have $(R_g)_* \circ (f_x)_* = (f_{xg})_* \circ \mathrm{Ad}_{g^{-1}}$ by the chain rule. So, on the fundamental vector fields, $R_g^* X_x^\sharp = (\mathrm{Ad}_{g^{-1}} X)^\sharp$. Hence

$$(R_g^*\nabla)_x(Y_x) = \nabla_{xg}\big((\mathrm{Ad}_{g^{-1}} X)^\sharp\big) = \mathrm{Ad}_{g^{-1}} X = \mathrm{Ad}_{g^{-1}} \nabla_x(X_x^\sharp).$$

So we see ∇ is G-equivariant on vertical vectors. On a horizontal vector $Y_x \in H_xE$, we instantly get

$$(R_g^*\nabla)_x(Y_x) = \nabla_{xg}\big((R_g)_{*,x} Y_x\big) = 0 = \mathrm{Ad}_{g^{-1}} \nabla_x(Y_x),$$

by right-invariance on the horizontal distribution.

The upshot is that G-equivariance holds on any vector $Y_x \in T_xE$ whenever $TE = VE \oplus HE$. So a principal horizontal distribution induces a G-equivariant splitting ∇. Thus we are lead to the following definition: a *G-connection* on a principal G-bundle $\pi\colon E \longrightarrow M$ is $\mathfrak{g}$-valued 1-form $\nabla \in \Omega^1(E;\mathfrak{g})$ such that

$$R_g^*\nabla = \mathrm{Ad}_{g^-1}\nabla.$$

and it splits, i.e. $\nabla_x \circ (f_x)_* = \mathrm{id}_\mathfrak{g}$, where $f_x\colon G \longrightarrow E$ is the map $f_x(g) = x \cdot g$.

For the third time, we have the following:

Proposition 1.4.3. *A principal connection gives rise to a principal horizontal distribution, and a principal horizontal distribution gives rise to a principal connection.*

Proof. We have seen in this subsection that a principal horizontal distribution induces a principal connection. What is left to show is that a G-connection ∇ gives a horizontal distribution HE such that

$$(R_g)_{*,x}(HE_x) = HE_{xg}.$$

The idea again is to let $HE_x = \ker(\nabla)$. If $X_x \in H_xE$, then we obtain

$$\nabla_{xg}\big((R_g)_{*,x}X\big) = (R_g^*\nabla)_x(X) = \mathrm{Ad}_{g^{-1}}\big(\nabla_x(X)\big) = 0,$$

hence $(R_g)_{*,x}X_x \in H_{xg}$, and we are done. $\square$

We now look at a most important example of a connection.

Example 1.4.4 (The Maurer-Cartan form). Let $E = M \times G$ be the trivial G-bundle from example 1.2.17. We will now define the *Maurer-Cartan form*, $\nabla_{MC} \in \Omega^1(M \times G;\mathfrak{g})$, which will serve as the most trivial example. Recall that for $g \in G$, the left translation map $L_{g^{-1}}\colon G \longrightarrow G$ is a diffeomorphism of G. So we have a map

$$L_{g^{-1}} \circ \mathrm{proj}_G\colon M \times G \longrightarrow G \longrightarrow G, \quad (p, g) \longmapsto e$$

Thus, $(L_{g^{-1}} \circ \mathrm{proj}_G) \circ f_{(p,g)}$ sends h to h again, i.e.

$$(L_{g^{-1}} \circ \mathrm{proj}_G) \circ f_{(p,g)} = \mathrm{id}_G .$$

With this in mind, define ∇_{MC} to be the differential of $L_{g^{-1}} \circ \mathrm{proj}_G$, i.e. for any point $(p,g) \in E \times G$, we have

$$(\nabla_{MC})_{(p,g)} = (L_{g^{-1}} \circ \mathrm{proj}_G)_{*,(p,g)} \colon T_{(p,g)}(M \times G) \longrightarrow \mathfrak{g}.$$

It is a splitting of $0 \to \mathfrak{g} \xrightarrow{v_x} T_x E \xrightarrow{\pi_{*,x}} T_{\pi(x)} M \to 0$ because $(L_{g^{-1}} \circ \mathrm{proj}_G) \circ f_{(p,g)} = \mathrm{id}_G$. To see that it satisfies the equivariance property, note that $(\nabla_{MC})_{(p,g)} = \mathrm{proj}_G^*(L_{g^{-1}})_*$. Thus

$$(R_h^* \nabla_{MC})_{(p,h)} = (R_h^*)(\mathrm{proj}_G^*(L_{g^{-1}})_*) = \mathrm{proj}_G^*(R_h)^*(L_{g^{-1}})_*.$$

Simple calculation shows $R_h^*(L_{g^{-1}})_* = \mathrm{Ad}_{h^{-1}} \circ (L_{g^{-1}})_*$, and so by applying proj_G^* to both sides, we are done.

We should note that the pullback of a connection determines a connection. It follows from the fact that the pullback is natural, as discussed in section 1.3.2. This implies that all principal bundles have a connection. The last fact follows from the local triviality of principal bundles and partition of unity. A principal bundle locally looks like $U \times G \longrightarrow U$, and we can assign each of these their Maurer-Cartan form ∇_{MC}. Thus a partition of unity gives us a globally defined 1-form, which is a connection.

1.5. The Chern-Weil homomorphism

We now know what a principal G-bundle with a connection is. When trying to classify them, Chern and Weil found lots of invariants, one for each invariant polynomial. In this section, we explore these invariants, and the Chern-Weil homomorphism.

The theory is taken from [Dup03].

1.5.1. Invariant polynomials

Recall, from section 1.1.3, that $\Sigma^k V$ is (canonically isomorphic to) the quotient of $V \otimes \cdots \otimes V$ (k times) by all elements $v_1 \otimes v_2 - v_2 \otimes v_1$, and that we write $v_1 \odot \cdots \odot v_k$ for the projected images of elementary tensors $v_1 \otimes \cdots \otimes v_k$. The space $\Sigma^k(V^\vee)$ is isomorphic to the space $\mathrm{Sym}^k(V, \mathbb{R})$ of symmetric multilinear maps

$$\alpha \colon V \times \cdots \times V \longrightarrow \mathbb{R}.$$

We now let $V = \mathfrak{g}$ be the Lie algebra of the Lie group G. Then the adjoint representation Ad of G on the Lie algebra $\mathfrak{g}$ induces an action of G on $\Sigma^k(\mathfrak{g}^\vee)$. So, for a given symmetric map $\alpha\colon \mathfrak{g}\times\cdots\times\mathfrak{g}\longrightarrow\mathbb{R}$, and a point $g\in G$, we define $g\alpha$ to be the map

$$g\alpha(X_1,\dots,X_n)=\alpha(\mathrm{Ad}_{g^{-1}}(X_1),\dots,\mathrm{Ad}_{g^{-1}}(X_n)).$$

We call $\alpha\in\Sigma^k(\mathfrak{g}^\vee)$ an *invariant polynomial* (even though it is a k-linear map[6]) if for all $g\in G$, we have $g\alpha=\alpha$, and the ring of all invariant polynomials of $\Sigma^k(\mathfrak{g}^\vee)$ is denoted $I^k(G)$. Just as we have a grading $\Sigma^\bullet(\mathfrak{g}^\vee)=\bigoplus_{k=0}^\infty\Sigma^k(\mathfrak{g}^\vee)$, we define the subring

$$I^\bullet(G)=\bigoplus_{k=0}^\infty I^k(G)=\Sigma^\bullet(\mathfrak{g}^\vee)^G\subseteq\Sigma^\bullet(\mathfrak{g}^\vee).$$

We now combine the terminology from this subsection to the vector valued forms of section 1.3.2.

1.5.2. The curvature form

Let M be any manifold, and V and W be finite dimensional vector spaces. Recall, from section 1.3.2, that we have a wedge product

$$\Omega^k(M;V)\times\Omega^l(M;W)\longrightarrow\Omega^{k+l}(M;V\otimes W).$$

If $V=W=\mathfrak{g}$, the wedge product of two $\mathfrak{g}$-valued forms is a $\mathfrak{g}\otimes\mathfrak{g}$-valued form. Since the Lie bracket is a linear map $\mathfrak{g}\otimes\mathfrak{g}\longrightarrow\mathfrak{g}$, and $\Omega^{2k}(M;\mathfrak{g}\otimes\mathfrak{g})$ is functorial in the last argument, we have a canonical way of mapping $\mathfrak{g}\otimes\mathfrak{g}$-forms to $\mathfrak{g}$-forms. Specifically, for two $\mathfrak{g}$ valued forms $\omega\in\Omega^k(M;\mathfrak{g})$ and $\omega'\in\Omega^l(M;\mathfrak{g})$, we define $[\omega,\omega']\in\Omega^{k+l}(M;\mathfrak{g})$ to be the $\mathfrak{g}$-valued $(k+l)$-form defined point-wise as the composition

$$[\omega,\omega']_p=[-,-]\circ(\omega_p\wedge\omega'_p)\colon T_pM\times\cdots\times T_pM\longrightarrow\mathfrak{g}\otimes\mathfrak{g}\longrightarrow\mathfrak{g}.$$

DEFINITION 1.5.1. Let $\nabla\in\Omega^1(E;\mathfrak{g})$ be a connection on a principal bundle $E\longrightarrow M$. The *curvature form* $F_\nabla\in\Omega^2(E)$ of ∇ is the $\mathfrak{g}$-valued 2-form defined by the *structural equation* of ∇:

$$d\nabla=-\frac{1}{2}[\nabla,\nabla]+F_\nabla,$$

where $d\colon\Omega^1(E;\mathfrak{g})\longrightarrow\Omega^2(E;\mathfrak{g})$ is the exterior derivative from section 1.3.2.

[6]The reason we use the word "polynomial" is because $\Sigma^k(V^\vee)$ is isomorphic, as a vector space, to the space $\mathbb{R}[x_1,\dots,x_{\dim V}]^k$ of all homogeneous polynomials of degree k. Hence we can think of α as a polynomial. It can also be shown that the rings $\Sigma^\bullet(V^\vee)$ and $\mathbb{R}[x_1,\dots,x_{\dim V}]$ are isomorphic. The correspondence takes use of something called *polynomial functions*, and the process of turning a polynomial function to a multilinear one is often called *polarization*.

If we now wedge the curvature form F_∇ with itself k times, we have

$$F_\nabla^k = F_\nabla \wedge \cdots \wedge F_\nabla \in \Omega^{2k}(E; \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}).$$

As any invariant polynomial $P \in I^k(G) \subseteq \Sigma^k(\mathfrak{g}^\vee)$ defines a map $P\colon \mathfrak{g} \otimes \cdots \otimes \mathfrak{g} \longrightarrow \mathbb{R}$, and as $\Omega^{2k}(E; \mathfrak{g} \otimes \cdots \otimes \mathfrak{g})$ is functorial in the last argument, we get a $2k$-form $P(F_\nabla^k) \in \Omega^{2k}(E)$, defined point-wise as the composition

$$P(F_\nabla^k)_p = P \circ (F_\nabla^k)_p \colon T_pM \times \cdots \times T_pM \longrightarrow \mathfrak{g} \otimes \cdots \otimes \mathfrak{g} \longrightarrow \mathbb{R}.$$

One could wonder if there are any forms on the base space M such that they map to $P(F_\nabla^k)$ under the pullback $\pi^*\colon \Omega^{2k}(M) \longrightarrow \Omega^{2k}(E)$. We do not show it here, and we never explicitly need the construction, but it turns out there exists a unique $2k$-form on M which pulls back to $P(F_\nabla^k) \in \Omega^{2k}(E)$ by π^* (see [Dup03], p. 76). This new form gets a name.

Definition 1.5.2. Let $\pi\colon E \longrightarrow M$ be a principal G-bundle, and let $P \in I^k(G)$ be an invariant polynomial. The unique $2k$-form on M which pulls back to $P(F_\nabla^k) \in \Omega^{2k}(E)$ is called the *characteristic form* corresponding to P, and is, by abuse of notation, also denoted $P(F_\nabla^k) \in \Omega^{2k}(M)$.

We would not care about the characteristic form, let alone give it such a pompous name, if it had no nice properties. And indeed it has several, but we need only one of them, namely

Proposition 1.5.3. *Let $(\varphi, \overline{\varphi})$ be a bundle map between two principal G-bundles $\pi_1\colon E_1 \longrightarrow M_1$, and $\pi_2\colon E_2 \longrightarrow M_2$, i.e. the diagram*

$$\begin{tikzcd} E_1 \arrow[r, "\varphi"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "\overline{\varphi}"] & M_2 \end{tikzcd}$$

commutes, and let $P \in I^k(G)$ be an invariant polynomial. Then for any connection $\nabla \in \Omega^1(E_2; \mathfrak{g})$ on E_2, the pullback $\varphi^*\nabla \in \Omega^1(E_1; \mathfrak{g})$ is a connection on E_1 and we have

$$P(F_{\varphi^*\nabla}^k) = \overline{\varphi}^*\big(P(F_\nabla^k)\big).$$

Proof. In section 1.4.3, we saw that the pullback of a connection is a connection. Hence $\varphi^*\nabla \in \Omega^1(E_1; \mathfrak{g})$ is a connection on E_1. Since $F_\nabla = d\nabla + \frac{1}{2}[\nabla, \nabla]$, we get

$$\varphi^* F_\nabla = \varphi^*(d\nabla) + \varphi^*\left(\frac{1}{2}[\nabla, \nabla]\right) = d\varphi^*\nabla + \frac{1}{2}\varphi^*\big([\nabla, \nabla]\big),$$

using the fact that d commutes with pullbacks to get $\varphi^*(d\nabla) = d\varphi^*\nabla$. It is straightforward to check that $\varphi^*\big([\nabla, \nabla]\big) = [\varphi^*\nabla, \varphi^*\nabla]$, and hence $F_{\varphi^*\nabla}$ satisfies the structural equation, making it the curvature form. Thus, in the total spaces, we have that $\varphi^*\big(P(F_\nabla^k)\big) = P(F_{\varphi^*\nabla}^k) \in \Omega^{2k}(E_1)$. As $\pi_1^*\colon \Omega^{2k}(M_1) \longrightarrow \Omega^{2k}(E_1)$ is injective, we are done. $\square$

The proposition above is saying that given a principal bundle with a connection, the pullback of the characteristic form corresponding to any invariant polynomial defines a characteristic form corresponding to the particular invariant polynomial in a canonical way. In particular, when the bundle map is an isomorphism, we get that gauge equivalent connections have the same characteristic form.

Definition 1.5.4. Let $\pi\colon E \longrightarrow M$ be a principal G-bundle with connection $\nabla \in \Omega^1(E; \mathfrak{g})$. For any invariant polynomial $P \in I^k(G)$, define $w(E; P)$ to be the cohomology class of the characteristic form corresponding to P, meaning $w(E; P) = [P(F_\nabla^k)] \in H_{dR}^{2k}(M)$. This defines a mapping

$$w(E; -)\colon I^k(G) \longrightarrow H_{dR}^{2k}(M),$$

called the *Chern-Weil homomorphism.*

We call each $w(-; P)$ an invariant because of proposition 1.5.3. But we need to know what other invariants for these objects are. A *characteristic class* is a rule c such that for each principal G-bundle $\pi\colon E \longrightarrow M$, we assign

$$(E, \pi, M, G) \longmapsto c(E) \in H_{dR}^\bullet(M),$$

and for each principal G-bundle map

$$\begin{array}{ccc} E_1 & \xrightarrow{\varphi} & E_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{\overline{\varphi}} & M_2 \end{array}$$

we have $c(E_2) = \overline{\varphi}^*\big(c(E_1)\big)$. We can formalize the definition, and call a characteristic class a natural transformation from a certain functor into the cohomology functor $H^\bullet$, but it introduces unnecessary complications, hence we do not.

It follows immediately that for each invariant polynomial P, the rule $w(-; P)$ is a charateristic class. Moreover, these characteristic classes also take the connection into account. When a characteristic class does this, we sometimes call it an *invariant associated to connections.*

The main result of [FH13] is that there are no more invariants associated to connections than the characteristic classes $w(-,P)$. We discuss this in section 3.3.2.

CHAPTER 2

From Manifolds to Simplicial Sheaves

In this chapter, we start exploring the ideas explained in [FH13]. In particular, we show how any the category of manifolds embeds into the category of presheaves. We then go on to construct a universal space of differential forms equipped with a universal differential form. This construction has the nice property that each differential form ω on a manifold determines a unique map from the manifold into this universal space with the property that the pullback of the universal form is ω. There maps are called the classifying maps, and will be useful in chapter 3.

Moving on, we mention what sheaves and stalks are, and mention the sheafification process of turning any presheaf into a sheaf.

Crucial to the construction of the universal bundle is the concept of a groupoid. We mention how a set determines a groupoid and how an action determines a groupoid. We look at how the category of principal G-bundles with connection is a groupoid. We then define simplicial sets, and see how a groupoid naturally determines a simplicial set. Thus a set determines a simplicial set, an action determines a simplicial set, and the category of principal G-bundles with a connection determines a simplicial set.

To introduce the notion of a weak equivalence of simplicial sheaves, we need to know what a weak equivalence of simplicial set is. This is defined using the geometric realization functor, hence we introduce it. With all of this in mind, we finally define simplicial presheaves and simplicial sheaves, and give some examples. Specifically, we mention how a sheaf induces a simplicial sheaf, using our example of how to turn a set into a simplicial set. We also show how an action on a sheaf naturally determines a simplicial sheaf, analogue to how an action on a set determines a simplicial set. Finally, we mention what differential forms on simplicial presheaves are (something glossed over in [FH13]), and define weak equivalences of simplicial manifolds.

2.1. Presheaves on manifolds

Recall that a presheaf on $\mathscr{C}$ is a contravariant functor from some category $\mathscr{C}$ to the category of sets **Set**. With this terminology in mind, we define a presheaf on manifolds to be a contravariant functor from **Man** to **Set**, or in other words:

DEFINITION 2.1.1. A *presheaf on manifolds*, or often just *presheaf* for brevity, is a functor

$$\mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set}\,.$$

2.1.1. Examples

Any such presheaf on manifolds should be thought of as a geometric object—a generalization of manifolds. To warrant this claim, we explain why a manifold can be viewed as a presheaf. The prototypical example is

EXAMPLE 2.1.2 (The associated presheaf). Let X be any smooth finite dimensional manifold. The *associated presheaf of* X, noted as $\mathcal{F}_X$, is the functor

$$\mathcal{F}_X\colon \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set}, \quad M \longmapsto \mathbf{Man}(M, X).$$

The functor $\mathcal{F}_X$ is just the contravariant Hom-functor, $\mathrm{Hom}_{\mathbf{Man}}(-, X)$. The idea behind $\mathcal{F}_X$ is analogous to what one does in simplicial homology. In that setting, one tries to understand X by looking at the sets $\mathrm{Sing}_n(X) = \mathbf{Top}(\Delta^n, X)$ of continuous maps from the "test spaces" Δ^n, called the *standard n-simplices*[1], into X. The difference is that now the test spaces we consider are not the standard n-simplices Δ^n, but finite dimensional smooth manifolds M, and the maps are not only continuous, but also smooth. Succinctly, $\mathbf{Top}(\Delta^n, X)$ is replaced with $\mathbf{Man}(M, X)$.

Another example that is important for this thesis is

EXAMPLE 2.1.3 (The universal space of differential forms). Recall, from section 1.3.1, that Ω^k is a functor $\mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$, assigning to each manifold M the space of differential k-forms $\Omega^k(M) = \Gamma(M, \bigwedge^k T^*M)$, and sending each smooth map $f\colon M_1 \longrightarrow M_2$ to its pullback $f^*\colon \Omega^k(M_2) \longrightarrow \Omega^k(M_1)$. It is a simple matter to consider the spaces $\Omega^k(M)$ as sets, and thus we have a presheaf

$$\Omega^k\colon \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set},$$

[1] We will return to them again in section 2.3, and in particular example 2.3.9.

on manifolds. More generally, we can fix a finite dimensional vector space V, and let $\Omega^k(M;V)$ be the set of V-valued differential k-forms, i.e.

$$\Omega^k(M;V) = \Gamma\bigg(\Big(\bigwedge^k T^*M\Big) \otimes V\bigg),$$

as described in section 1.3.2. This defines a presheaf of manifolds as well. We sometimes write $\Omega^k \otimes V$ for the functor $\Omega^k(-;V)$, reflecting the equation above. In particular, if we let $V = \mathfrak{g}$ be the Lie algebra of some Lie group G, we have the presheaf

$$\Omega^k \otimes \mathfrak{g}\colon\ \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set},$$

on manifolds.

2.1.2. Maps of presheaves

If we want to be serious about presheaves on manifolds, we should seek a notion of maps between them, which would allow us to talk about their geometry. This will yield a category **Pre** of presheaves. For the reader familiar with abstract nonsense, this is (of course) just the presheaf category.

Let $\mathcal{F}, \mathcal{G}\colon\ \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be two presheaves on manifolds. Then a *natural transformation* η from $\mathcal{F}$ to $\mathcal{G}$, written either as $\eta\colon \mathcal{F} \Longrightarrow \mathcal{G}$ or $\eta\colon \mathcal{F} \longrightarrow \mathcal{G}$ depending on perspective, is an operation associating with each manifold M a morphism $\eta_M\colon \mathcal{F}(M) \longrightarrow \mathcal{G}(M)$ in such a way that for any smooth map $f\colon M_1 \longrightarrow M_2$, the diagram

$$\begin{tikzcd}
\mathcal{F}(M_1) \arrow[d, "\eta_M"'] & \mathcal{F}(M_2) \arrow[l, "\mathcal{F}(f)"'] \arrow[d, "\eta_N"] \\
\mathcal{G}(M_1) & \mathcal{G}(M_2) \arrow[l, "\mathcal{G}(f)"']
\end{tikzcd}$$

commutes. The map η_M is called the *component of* η *at* M. The collection of all natural transformations $\mathcal{F} \longrightarrow \mathcal{G}$ is denoted $\mathbf{Pre}(\mathcal{F}, \mathcal{G})$. This choice of definition has an enjoyable property:

LEMMA 2.1.4 (Yoneda). *For any presheaf $\mathcal{F}$, evaluation on X determines a bijection $\mathbf{Pre}(\mathcal{F}_X, \mathcal{F}) \cong \mathcal{F}(X)$ of sets.*

IDEA OF PROOF. We need to show that every natural transformation determines an element in $\mathcal{F}(X)$, and that every element in $\mathcal{F}(X)$ is determined by such a transformation. Note that id_X is a point in $\mathcal{F}_X(X)$. So for a natural transformation $\eta\colon \mathcal{F}_X \Longrightarrow \mathcal{F}$, evaluation at this point gives an element $\eta_X(\mathrm{id}_X) \in \mathcal{F}(X)$. This identification is a one-to-one correspondence, and the inverse identifies $x \in \mathcal{F}(X)$ with the natural transformation ξ^x which associates to each manifold M the morphism $\xi^x_M \colon \mathcal{F}_X(M) \longrightarrow \mathcal{F}$ sending g to $\mathcal{F}(g)(x)$. See for example [MM94] for a proper proof. $\square$

Because of this lemma, every element of $\mathcal{F}(X)$ is associated to a map $\mathcal{F}_X \longrightarrow \mathcal{F}$. And since we view $\mathcal{F}_X$ as a generalized version of X, we sometimes write an element of $\mathcal{F}(X)$ as a map

$$X \longrightarrow \mathcal{F}.$$

This notation seems to suggest that for smooth manifolds X and Y, the maps $\mathcal{F}_X \longrightarrow \mathcal{F}_Y$ of associated presheaves are the same as smooth maps $X \longrightarrow Y$. But is this right? The following corollary answers this in the in the affirmative.

COROLLARY 2.1.5 (Yoneda embedding). *The set of maps $\mathcal{F}_X \longrightarrow \mathcal{F}_Y$ is in one-to-one correspondence with the set of smooth maps $X \longrightarrow Y$, i.e.*

$$\mathbf{Pre}(\mathcal{F}_X, \mathcal{F}_Y) \cong \mathcal{F}_Y(X) = \mathbf{Man}(X, Y).$$

We can interpret this corollary as saying the maps $\mathcal{F}_X \longrightarrow \mathcal{F}_Y$ are smooth, as each such map actually corresponds to a smooth map $X \longrightarrow Y$. It seems the associated presheaves actually remember the smooth structure of their underlying smooth manifolds. And so we could wonder if $\mathcal{F}_X$ inherits other smooth properties of the underlying manifold. For example, one could ponder if it makes sense to define differential forms on $\mathcal{F}_X$. This is the subject of the following subsection.

2.1.3. The universal space of differential forms

The Yoneda lemma, lemma 2.1.4, is saying that for $\mathcal{F} = \Omega^\bullet$, we have

$$\mathbf{Pre}(\mathcal{F}_X, \Omega^\bullet) \cong \Omega^\bullet(X).$$

And so we see that each differential form on X corresponds to a natural transformation $\mathcal{F}_X \Longrightarrow \Omega^\bullet$. Thus, if we define the set of differential forms on the associated presheaf $\mathcal{F}_X$ to be set of natural transformations from $\mathcal{F}_X$ to $\Omega^\bullet$, i.e. $\Omega^\bullet(\mathcal{F}_X) = \mathbf{Pre}(\mathcal{F}_X, \Omega^\bullet)$, we automatically get that $\Omega^\bullet(\mathcal{F}_X) \cong \Omega^\bullet(X)$. But this is too narrow of a definition, because we want to consider differential forms on all types of presheaves $\mathcal{F}$. This is what is done in

DEFINITION 2.1.6. Let $\mathcal{F} \colon \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be any presheaf. Define the collection of differential forms on $\mathcal{F}$ as the set $\Omega^\bullet(\mathcal{F}) =$

$\mathbf{Pre}(\mathcal{F}, \Omega^\bullet)$. In other words, we consider the functor

$$\Omega^\bullet \colon \mathbf{Pre}^{\mathrm{op}} \longrightarrow \mathbf{Set}, \quad \mathcal{F} \longmapsto \mathbf{Pre}(\mathcal{F}, \Omega^\bullet).$$

For a map $\varphi \colon \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ of presheaves, we call the map

$$\varphi^* = \Omega^\bullet(\varphi) \colon \Omega^\bullet(\mathcal{F}_2) \longrightarrow \Omega^\bullet(\mathcal{F}_1)$$

the *pullback* of φ. This sends any differential form $\omega \in \Omega^\bullet(\mathcal{F}_2)$ to the composition

$$\varphi^*\omega = \omega \circ \varphi \colon \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \Omega^\bullet,$$

and thus $\varphi^*\omega \in \Omega^\bullet(\mathcal{F}_1)$.

Note that in the above definition we abuse both notation and terminology. The symbol $\Omega^\bullet$ means both the "old" functor $\mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ from example 2.1.3, and the "newer" functor $\Omega^\bullet \colon \mathbf{Pre} \longrightarrow \mathbf{Set}$ (from definition 2.1.6 of course). The word "pullback" similarly means the value of $\Omega^\bullet$ when applied to a morphism, no matter which of the categories $\mathbf{Man}^{\mathrm{op}}$ or $\mathbf{Pre}^{\mathrm{op}}$ it comes from. It is important to distinguish these differences, but luckily it is always clear from context.

Also note that a consequence of lemma 2.1.4 is that the collection of all differential forms on the associated presheaf $\mathcal{F}_X$ is actually, tautologically, the same as the collection of differential forms on X, since

$$\Omega^\bullet(\mathcal{F}_X) = \mathbf{Pre}(\mathcal{F}_X, \Omega^\bullet) \cong \Omega^\bullet(X).$$

This tautological property is hinting at something more, and we explore it now, in

Construction 2.1.7 (The classifying maps). Let X be a smooth manifold and M a test manifolds, and let $\omega \in \Omega^k(X)$. Define a map

$$\varphi_M \colon \mathbf{Man}(M, X) \longrightarrow \Omega^k(M), \quad f \longmapsto \varphi(f) = f^*\omega \in \Omega^k(M),$$

sending any smooth map $f \colon M \longrightarrow X$ from the test manifold M, to its pullback $f^*\omega$ of the form ω by f. This, as the subscript might suggest, gives rise to a natural transformation $\varphi \colon \mathbf{Man}(-, X) \Longrightarrow \Omega^k$, or, more to the point, a map

$$\varphi \colon \mathcal{F}_X \longrightarrow \Omega^k.$$

Observe that the pullback of this map φ between presheaves is a map

$$\varphi^* \colon \mathbf{Pre}(\Omega^k, \Omega^k) \longrightarrow \mathbf{Pre}(\mathcal{F}_X, \Omega^k) \cong \Omega^k(X).$$

Since $\varphi^*(\mathrm{id}_{\Omega^k}) = \mathrm{id}_{\Omega^k} \circ \varphi = \varphi$, the general Yoneda isomorphism tells us that φ can be regarded as the element $\varphi_X(\mathrm{id}_X) = \mathrm{id}_X^* \omega = \omega \in \Omega^k(X)$.

If we take a step back, we see that we have actually showed that there is a (canonically defined) map $\varphi\colon \mathcal{F}_X \longrightarrow \Omega^k$ with the property that all differential forms $\omega \in \Omega^k(X)$ pulls back to one universal differential form, namely $\mathrm{id}_{\Omega^k} \in \Omega^k(\Omega^k)$. There are no other maps $\mathcal{F}_X \longrightarrow \Omega^k$ with this property. (This is shown easily by assuming we have two such maps that have equal pullbacks.) Meaning φ is unique. Thus we call the unique maps φ the *classifying maps*.

All that to say, we have now constructed a *universal space of differential forms* $\Omega^\bullet$, and a *universal de Rham complex*, the latter being the complex

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^k \xrightarrow{d} \cdots$$

which on a test manifold M is the de Rham complex on M, from section 1.3.1. We again stress that the classifying maps from any manifold X into this universal objects are unique, and that there exists some unique form such that any differential form on X is pulled back from the unique one.

Before moving on, we expand on some of the ideas of presheaves on manifolds, glossed over in [FH13]. We have a natural extension of differential forms on manifolds, as seen in definition 2.1.6. The same is true for the vector valued differential forms, which we discussed in section 1.3.2. For some finite dimensional vector space V, the set of V-valued differential forms on a presheaf $\mathcal{F}$ is the set $\mathbf{Pre}(\mathcal{F}, \Omega^\bullet \otimes V)$ of natural transformations from $\mathcal{F}$ to $\Omega^\bullet \otimes V$. In particular, a G-connection $\nabla \in \Omega^1(E; \mathfrak{g})$ on a principal G-bundle $E \longrightarrow X$ induces a $\mathfrak{g}$-valued differential 1-form on $\mathcal{F}_E$, i.e. we have some unique natural transformation $\nabla\colon \mathcal{F}_E \longrightarrow \Omega^1 \otimes \mathfrak{g}$.

2.2. Sheaves on manifolds and their stalks

Let $\mathcal{F}\colon \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a presheaf, and $\{U_i\}_{i \in I}$ be any cover of an arbitrary test manifold M. For any two indices $i, j \in I$, the inclusions

$$\begin{tikzcd} U_i \cap U_j \arrow[r] \arrow[d] & U_i \arrow[d] \\ U_j \arrow[r] & M \end{tikzcd}$$

induces maps

$$\begin{tikzcd} \mathcal{F}(U_i \cap U_j) & \mathcal{F}(U_i) \arrow[l] \\ \mathcal{F}(U_j) \arrow[u] & \mathcal{F}(M) \arrow[l] \arrow[u] \end{tikzcd}$$

This allows us to define three maps:

$$\mathcal{F}(M) \longrightarrow \prod_i \mathcal{F}(U_i), \quad \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j).$$

The first map sends an element to the product of all the values of the induced maps $\mathcal{F}(M) \longrightarrow \mathcal{F}(U_i)$. The first of the remaining two maps sends an element coming from coordinate $k \in I$ to the product of the values induced from the inclusion functions $\mathcal{F}(U_i) \longrightarrow \mathcal{F}(U_i \cap U_k)$, while the other map, analogously, coordinate-wise comes from the inclusion functions $\mathcal{F}(U_j) \longrightarrow \mathcal{F}(U_k \cap U_j)$.

2.2.1. Sheaves

DEFINITION 2.2.1. A presheaf $\mathcal{F}\colon \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a *sheaf* if, given any test manifold M in $\mathbf{Man}$ and any open cover $\{U_i\}_{i\in I}$ of M

$$\mathcal{F}(M) \longrightarrow \textstyle\prod_i \mathcal{F}(U_i) \rightrightarrows \textstyle\prod_{i,j} \mathcal{F}(U_i \cap U_j),$$

is an equalizer diagram.

Recall that an element in $\mathcal{F}(X)$ is sometimes written as a map $X \longrightarrow \mathcal{F}$ because of the Yoneda lemma. The diagram in definition 2.2.1 being an equalizer amounts to two things:

(1) If $s, t\colon M \longrightarrow \mathcal{F}$ are such that the diagram

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & M \arrow[d, "s"] \\ M \arrow[r, "t"] & \mathcal{F} \end{tikzcd}$$

commutes for each $i \in I$, then $s = t$.

(2) If, for each $i \in I$, there is a map $s_i\colon U_i \longrightarrow \mathcal{F}$ such that

$$\begin{tikzcd} U_i \cap U_j \arrow[r] \arrow[d] & U_i \arrow[d, "s_i"] \\ U_j \arrow[r, "s_j"] & \mathcal{F} \end{tikzcd}$$

commutes, then there is some $s\colon M \longrightarrow \mathcal{F}$ such that each s_i factors through s.

With words, the first property is saying that maps which agree locally, must agree globally, while the second property is saying that given a collection of local sections which agree on overlaps must glue to a global section. These properties are called *locality* and *gluing*, respectively.

PROPOSITION 2.2.2. *Let $\mathcal{F}$ be a presheaf. Then the following are equivalent:*

(1) The presheaf $\mathcal{F}$ is a sheaf.
(2) For each manifold M and each open cover of M, the presheaf satisfies the locality and gluing property.

Using this proposition, we identify some sheaves.

EXAMPLE 2.2.3 (The associated presheaf). Let X be a smooth manifold, and $\mathcal{F}_X$ be the associated presheaf from example 2.1.2. The Gluing Lemma for Smooth Maps ([p. 35 in Lee]) in topology says that given manifolds M and X, an open cover $\{U_i\}_{i\in I}$ of M, and smooth maps $s_i\colon U_i \longrightarrow X$ such that

$$\begin{tikzcd} U_i \cap U_j \arrow[r] \arrow[d] & U_i \arrow[d, "s_i"] \\ U_j \arrow[r, "s_j"] & X \end{tikzcd}$$

commutes, there exists a unique smooth map $s\colon M \longrightarrow X$ such that all the maps s_i factor through it. Replacing X with $\mathcal{F}_X$ shows that the associated presheaf has the locality and gluing property, and by proposition 2.2.2 it must be a sheaf.

EXAMPLE 2.2.4 (The universal space of differential forms). Let M be a smooth manifold with open cover $\{U_i\}_{i\in I}$. Although Ω^k satisfies locality by definition, the gluing property is not straight forward to check. But note that we have inclusions

$$\bigsqcup_{p\in U_i\cap U_j} \bigwedge^k \left(T_p^*(U_i\cap U_j)\right) \subseteq \bigsqcup_{p\in U_i\cap U_j} \bigwedge^k (T_p^* M) \subseteq \bigsqcup_{p\in X} \bigwedge^k (T_p^* M),$$

so $\bigwedge^k T^*(U_i \cap U_j) \subseteq \bigwedge T^*X$. This means a k-form $\omega\colon U_i \cap U_j \longrightarrow \bigwedge^k T^*(U_i \cap U_j)$ can be considered as a map $\omega\colon U_i \cap U_j \longrightarrow \bigwedge^k T^*M$, and we see that the gluing property is satisfied.

2.2.2. Sheafification and stalks

There is a functor from the category of presheaves **Pre** to the category of sheaves **Sh**, (where the morphisms defined identically to those in **Pre**) denoted by

$$\mathbf{a}\colon \mathbf{Pre} \longrightarrow \mathbf{Sh}\,.$$

Although we do not prove it, this functor has the universal property that if $\mathcal{F}$ is a presheaf, and $\mathcal{F}'$ is any sheaf, and there is a map $\mathcal{F} \longrightarrow \mathcal{F}'$, then there is a unique sheaf map $\mathbf{a}\mathcal{F} \longrightarrow \mathcal{F}'$ making the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr] & \mathbf{a}\mathcal{F} \arrow[d, dashed, "\exists!"] \\ & \mathcal{F}' \end{tikzcd}$$

commute. We call $\mathbf{a}$ the *associated sheafification functor*. A good general reference is [MM94], see Chapter III, Section 3 for a complete description of the sheaf $\mathbf{a}\mathcal{F}$.

By proposition 2.2.2, it is clear that a sheaf is a tool for systematically tracking locally defined data attached to the open sets of a space. It is therefore reasonable to attempt to isolate the behavior of a sheaf at an arbitrary single fixed point of the space. This could be done by looking at smaller and smaller neighborhoods of the point, essentially taking a direct limit. But since all manifolds are locally diffeomorphic to an open ball centered at the origin, we could instead just calculate the direct limit of such balls.

To be more precise: let $\mathcal{F}$ be a presheaf, and define $B^m(r) \subseteq \mathbb{R}^m$ to be the open ball of radius r about the origin in $\mathbb{R}^m$. If $r' \leq r$, then we have a natural inclusion $B^m(r') \hookrightarrow B^m(r)$, and so we get a map $f\colon \mathcal{F}\big(B^m(r)\big) \longrightarrow \mathcal{F}\big(B^m(r')\big)$. If we form the disjoint union of all these sets, and say that for $x \in \mathcal{F}\big(B^m(r)\big)$, $x' \in \mathcal{F}\big(B^m(r')\big)$, we have $x \sim x'$ if and only if $f(x) = x'$, then the quotient set

$$\bigsqcup_{r\in\mathbb{R}_{>0}} \mathcal{F}\big(B^m(r)\big)/\sim,$$

of all equivalence classes is the direct limit $\operatorname{colim}_{r\to 0}\mathcal{F}\big(B^m(r)\big)$.

Definition 2.2.5. Let $\mathcal{F}\colon \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a presheaf. For $m \in \mathbb{N}_0$, the *m-dimensional stalk* of $\mathcal{F}$ is the direct limit

$$\operatorname*{colim}_{r\to 0}\mathcal{F}\big(B^m(r)\big).$$

2.3. Simplicial sets

Let G be a Lie group. If we want to have any shot at finding a classifying space $B_\nabla G$ of the universal principal G-bundle $E_\nabla G \longrightarrow B_\nabla G$, we at least need to know what mathematical structure $\mathscr{C}$ (or "category") sufficiently well describes the collection of G-connections over a smooth manifold M. Why? Because if we pick a structure $\mathscr{C}$ that is too general, the gauge equivalent connections over M will correspond to different, non-equivalent instances (or "objects") of $\mathscr{C}$. Similarly, if the structure is not general enough, then we are not guaranteed that gauge equivalent connections give equivalent instances. Any of these wrong choices will not yield unique classifying maps $X \longrightarrow B_\nabla G$. One possibility that history has shown works is the structure $\mathscr{C} = \mathbf{Grpd}$, the category of groupoids. We explicitly construct the needed groupoid in example 2.3.4.

For what is accomplished in [FH13], the structure of groupoids is sufficient to describe the collection of G-connections over a smooth manifold M. But a more general solution which applies more broadly is that of simplicial sets. Without getting too ahead of ourselves, we can motivate this change, from the category **Grpd** to the category $\mathbf{Set}_\Delta$ of simplicial sets, by mentioning that the latter category is more "geometric". In fact, $\mathbf{Set}_\Delta$ sits between the category **Grpd** and the category **Top** in such a way that the functor

$$\mathbf{Grpd} \longrightarrow \mathbf{Top},$$

which assigns a classifying space to each groupoid, factors through $\mathbf{Set}_\Delta$. Moreover, equivalent groupoids are sent to equivalent spaces, and this relation is also preserved under the factorization. By this, we mean that there is a commutative diagram of functors

$$\begin{tikzcd} & \mathbf{Set}_\Delta \arrow[dr] & \\ \mathbf{Grpd} \arrow[ur] \arrow[rr] & & \mathbf{Top} \end{tikzcd}$$

such that equivalent groupoids map to equivalent simplicial sets, and such that equivalent simplicial sets are mapped to equivalent spaces. We see there that indeed $\mathbf{Set}_\Delta$ is between **Grpd** and **Top**.

Note that we have not specified what is meant by "equivalent" in the paragraphs above. By this word we actually mean "weakly equivalent" and will explain the weak equivalences in section 2.3.5. First we explain what groupoids are.

2.3.1. Groupoids

Definition 2.3.1. A *groupoid* is a category $\mathcal{G}$ where all morphisms are isomorphisms, that is, all arrows are invertible. We often write

$\mathcal{G} = \{\mathcal{G}_0, \mathcal{G}_1\}$ where $\mathcal{G}_0$ is the set of objects, and $\mathcal{G}_1$ is the set of all morphisms. A groupoid is called *discrete* if for any two objects x, y, the cardinality of the set of morphisms $\mathcal{G}(x, y)$ from x to y is either 0 or 1. We call two groupoids *equivalent* if they are equivalent as categories.

If we define $\mathbf{Grpd}(\mathcal{G}, \mathcal{G}')$ to be the collection of functors $f\colon \mathcal{G} \longrightarrow \mathcal{G}'$ of groupoids, we automatically get the category of all groupoids. We call it **Grpd**.

The reason we call them groupoids is because there exists a fully faithful functor

$$\mathbf{Grp} \longrightarrow \mathbf{Grpd},$$

from the category **Grp** of groups, and so groupoids are a generalization of groups. The functor sends any group G to the groupoid often denoted BG, consisting of just one object, say p, where the set of maps is

$$\mathrm{Hom}_{BG}(p, p) = G.$$

The composition map in this (only) Hom-set of the category BG is just the group operation in G, and the identity morphism id_p is the identity element $e \in G$.

We now look at a very useful example, namely

Example 2.3.2 (Sets as discrete groupoids). Let S be any set. If we consider all the elements of S as the objects of some category, call it S, and the only arrows in this category S are the identity arrows on these objects, then S is a groupoid. Note that we essentially have $S = S_0 = S_1$; the objects and morphisms are the same. It is in fact a discrete groupoid, as there are no arrows between two different objects of S. It is common, as we have done, to denote this groupoid by the same symbol as the set.

Example 2.3.3 (Groupoid of a group action). Let G be a group, and S be a G-set, i.e. let G act on S. Define the groupoid $\mathcal{G} = \{\mathcal{G}_0, \mathcal{G}_1\}$, where $\mathcal{G}_0 = S$, and $\mathcal{G}_1 = G \times S$. It is perhaps not immediately clear how any element $(g, s) \in G \times S$ corresponds to an arrow between two elements in S, nor which two. But since G acts on S, (g, s) uniquely determines the two elements s and $g \cdot s$. So the element (g, s) is the arrow $s \longrightarrow g \cdot s$.

As mentioned in the introduction to this section, **Grpd** is a suitable choice of mathematical structure to describe the collection of G-connections on a smooth manifold M. Let us see why.

EXAMPLE 2.3.4 (The groupoid of principal bundles with connection). Let G be a Lie group and M be any smooth manifold. Use the symbol $G\,\mathbf{Bund}_\nabla(M)$ for the category where the objects are principal G-bundles over M with a connection[2]. The morphisms are principal bundle isomorphisms such that the connections are gauge equivalent. This means that for principal G-bundles $\pi_1\colon E_1 \longrightarrow M$ and $\pi_2\colon E_2 \longrightarrow M$ over M with G-connections ∇_1 and ∇_2 respectively, the morphisms between them are the G-equivariant diffeomorphisms $\varphi\colon E_1 \longrightarrow E_2$ with G-equivariant inverse φ^{-1} such that the diagram

$$\begin{tikzcd} E_1 \arrow[rr, "\varphi"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & M & \end{tikzcd}$$

commutes, and $\varphi^*\nabla_2 = \nabla_1$. As all the morphisms of this category are isomorphisms, $G\,\mathbf{Bund}_\nabla(M)$ is a groupoid.

Later, we will encounter discrete groupoids, so the following lemma will be a handy way to determine if they are equivalent or not.

LEMMA 2.3.5. *Let $f\colon \mathcal{G} \longrightarrow \mathcal{G}'$ be a functor between two discrete groupoids. If f is essentially surjective, then $\mathcal{G}$ and $\mathcal{G}'$ are equivalent as categories. That is, f is essentially surjective and fully faithful.*

PROOF. Observe that by surjectivity, the cardinality of $\mathcal{G}'(f(x), f(y))$, which is 0, or 1, must match that of $\mathcal{G}(x, y)$, so the sets are isomorphic to eachother. $\square$

As we mentioned, the functor from **Grpd** to **Top**, factorizes nicely through the category $\mathbf{Set}_\Delta$ of simplicial sets. In the following subsection, we try to motivate and understand simplicial sets.

2.3.2. Motivating simplicial sets

Recall that an *n-simplex* is a “generalized tetrahedra”[3], that is, it can be thought of as the span of $n+1$ vectors $v_0, v_1, v_2, \dots, v_n$ where the collection $\{v_i - v_0\}_{i=1}^n$ is linearly independent in Euclidean space. Each point v_i is a *vertex*, and the span of a subset of the collection is a *face* of the simplex, and is also a simplex.

[2]Hence the subscript ∇ in $G\,\mathbf{Bund}_\nabla(M)$

[3]The terminology is due to Henri Poincare in the first supplement [Poi99] of the famous *Analysis Situs*. See [Poi10] for a translated version with the term.

The standard example is the *standard n-simplex*

$$\Delta^n = \{(x_0, x_1, x_2 \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \sum_{i=0}^{n} x_i = 1\}.$$

The vertices of Δ^n are the standard basis vectors $e_0 = (1, 0, 0, \ldots, 0), e_1 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, 0, \ldots, 1)$ of $\mathbb{R}^n$. Thus Δ^0 is the point 1 in $\mathbb{R}$, Δ^1 is the line between $(1, 0)$ and $(0, 1)$ in $\mathbb{R}^2$, Δ^2 is the filled equilateral triangle with vertices $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ in $\mathbb{R}^3$, and so on.

A *(geometric) simplicial complex* X is a collection of simplices with two conditions: (1) that every face of X is in X and (2) the intersection of any two simplices of X is a face of each of them. Simply put, X consists of simplices of various dimensions glued along shared faces. A map $f \colon X \to Y$ of complexes is called a *simplicial map* if the image of any vertex in X is a vertex in Y, and the image of a simplex (with not necessarily unique vertices). If we introduce the notation $[v_{i_1}, \ldots, v_{i_k}]$ to mean a subset $\{v_{i_1}, \ldots, v_{i_k}\} \subseteq \{v_i\}_i$ of the simplices of X such that they actually form a simplex, we see that a map $f \colon X \longrightarrow Y$ is a simplicial map if for any simplex $[v_{i_1}, \ldots, v_{i_k}]$ of X, we have a simplex $[f(v_{i_1}), \ldots, f(v_{i_k})]$ of Y (with not necessarily unique vertices).

Again, Δ^n is not only a simplex, but also simplicial complex, as it is the collection of one simplex. But a more interesting example would be, say, the triangle Δ^2 with a tail (i.e. a straight line, looking like Δ^1), glued to one of the vertices of Δ^2. And this is the general principle. Simplicial complexes are built by attaching, or gluing, different dimensional simplices, as we will see in a bit.

Notice that any complex X contains copious amounts of information. To describe X, one needs to know how each simplex is embedded in some Euclidean space, how they all intersect, and more. But the only information actually needed to understand X is the collection of all vertices $\{v_i\}_i$ of X, and to know when a subset $\{v_{i_1}, \ldots, v_{i_k}\} \subseteq \{v_i\}_i$ of the vertices form a simplex $[v_{i_1}, \ldots, v_{i_k}]$. Hence we should seek a new way to define a simplicial complex, which is more abstract, but also simpler. If we denote the collection of all the vertices by X_0, the collection of all the 1-simplices by X_1, and so on, we have described the *skeleta* $X_0.X_1, X_2, \ldots$ of X. And so, the new definition could be: A *(abstract) simplicial complex* X is a collection, consisting of a set $X_0 = \{v_i\}$ (which need not be vectors) of *vertices* and an operation associating with each natural number k a set X_k, consisting of subsets of X_0 with cardinality $k+1$, such that any subset with cardinality $j+1$

of an element of X_k is an element of X_j, where $j \leq n$. This all seems very abstract, but notice that for a fixed k, the set X_k corresponds to the collection of all k-simplices. For any simplex in X_k, the last requirement in the definition says that any subset of the vertices in this simplex is a new simplex in X. This definition loses the information of the embedding into Euclidean space, but retains the combinatorics, which is the essential part.

Note that the assignment $[v_{i_1}, \ldots, v_{i_k}]$ adds simplices redundantly. Permuting the order of the verticies $v_{i_1}, \ldots, v_{i_k}$ gives a new name for the same span. For this reason, we introduce *ordered simplicial complexes*, which are the same beasts as simplicial complexes, but with the restriction that the symbol $[v_{i_1}, \ldots, v_{i_k}]$ is a simplex if and only if $v_{i_j} < v_{i_l}$ whenever $j < l$.

The standard n-simplex Δ^n, which is a simplicial complex, also can be considered an ordered simplicial complex, as it has a natural order. The order on $\mathbb{N}_0$ induces an order of the vertices of Δ^n, and thus we know when $e_i < e_j$. To distinguish between the unordered and the ordered standard n-simplex, we denote the latter by $\widehat{\Delta^n}$. If we relabel the verticies $e_0, \ldots, e_n$ of Δ^n to $0, \ldots, n$, we can think of $\widehat{\Delta^n}$ as the vertex $[0, \ldots, n]$, since $\widehat{\Delta^n} = [e_0, \ldots, e_n]$. The relabeling is actually a useful shift of perspective. Because now it is easy to see that any n-simplex $[v_{i_1}, \ldots, v_{i_n}]$ in an arbitrary (not necessarily ordered) complex X is the image of of $[0, \ldots, n] = \widehat{\Delta^n}$ under a order preserving simplicial map. Since $[0, \ldots, n]$ is a totally ordered set of n elements, we shorten the notation to just $[n]$. Meaning $[n] = \{0, \ldots, n\}$, regarded as a totally ordered set of cardinality $n + 1$. And thus $\widehat{\Delta^n}$ is equal to $[n]$ if we forget everything about Δ^n except its combinatorics.

The upshot is that any complex is made up of images of the standard ordered simplices $\widehat{\Delta^n}$ under order preserving maps. But we do not know how the faces are glued together from this point of view. This issue is resolved later. Since we have noted that $\widehat{\Delta^n}$ can describe all simplicial complexes, we should probably give the colleciton of them a name. Let $\widehat{\Delta}$ be the category consisting of the finite totally ordered sets $[n] = \{0, 1, \ldots, n\}$ as objects, and whose maps are strictly ordered functions $[m] \longrightarrow [n]$. Thus we think of the objects of $\widehat{\Delta}$ as the ordered standard simplices $\widehat{\Delta^n}$. The condition of strictly order preserving maps, as opposed to just order preserving maps, is imposed for a reason. If we imagine the objects $[n]$ of $\widehat{\Delta}$

to be the standard ordered n-simplices Δ^n, then strictly order preserving maps $[m] \longrightarrow [n]$, which only exists for $m \leq n$ (because of the pigeonhole principle) correspond to different embeddings of Δ^m to faces of Δ^n. There are exactly as many such embeddings as there are maps $[m] \longrightarrow [n]$. So $\mathrm{Hom}_{\widehat{\Delta}}([m],[n])$ correspond to the set of all inclusions of m-dimensional faces into Δ^n. But since each embedding $\Delta^m \longrightarrow \Delta^n$ can be considered as just the composite of certain embeddings $\Delta^m \longrightarrow \Delta^{m+1} \longrightarrow \cdots \longrightarrow \Delta^n$, we need only the information of the embeddings $\Delta^{n-1} \longrightarrow \Delta^n$. For $n = 0$, there are only two maps $\Delta^0 \longrightarrow \Delta^1$, corresponding to sending the vertex 1 to the verticies $(1, 0)$ or $(0, 1)$. Similarily, the line Δ^1 can be mapped to three different faces of the triangle Δ^2. If we continue in this faishion, we create a generating set (under composition) of all possible embeddings $\Delta^m \longrightarrow \Delta^n$. This is algebraically reflected by the fact that there are two strict order preserving maps $[0] \longrightarrow [1]$, three maps $[1] \longrightarrow [2]$, and so on, which generate all strict order preserving maps $[m] \longrightarrow [n]$.

In total, we have justified the fact that we can illustrate $\widehat{\Delta}$ as the diagram

$$
\begin{tikzcd}
{[0]} \arrow[r, shift left] \arrow[r, shift right] & {[1]} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & {[2]} \ \cdots
\end{tikzcd}
$$

The category $\widehat{\Delta}$ gives us back (i.e. is a generalization of) our almost rudimentary definition of abstract simplicial complexes, because for any functor $F : \widehat{\Delta} \longrightarrow \mathbf{Set}$, we could set $X_k = F([n])$.

When talking about the standard n-simplices Δ^n, one usually mentions the *face maps* $d^i_n : \Delta^{n-1} \longrightarrow \Delta^n$ which are the generating maps described above. But, equally important, are the *degeneracy maps* $s^j_n : \Delta^{n+1} \longrightarrow \Delta^n$ which, loosely speaking, maps Δ^n to the $(n+1)$-simplex with the j-th vertex duplicated. For example, there is only one map $\Delta^1 \longrightarrow \Delta^0$, namely the one sending the whole line segment of Δ^1 to the point Δ^0. The collection of all face and degeneracy maps generate any map (not necessarily strictly) order preserving map $[m] \longrightarrow [n]$. This forms a category, and we will now explore it.

2.3.3. Simplicial sets and examples

As we saw in the subsection above, we can abstract the concept of a n-simplex quite a bit. We now define them in full generality. Let Δ be the category where objects are finite non-empty totally ordered sets, and morphisms are order preserving maps between these sets. Thus the objects look like the totally ordered sets $[n] = \{0, 1, \ldots, n\}$ for $n \in \mathbb{N}_0$. We call Δ the *simplex category*. This category mimics $\widehat{\Delta}$.

As we see, the objects are the same, but it has more morphisms. The category $\widehat{\Delta}$ has only strictly order preserving maps, but Δ has any order preserving map as a morphism. And while $\widehat{\Delta}$ can be illustrated as

$$\begin{tikzcd} {[0]} \arrow[r, shift left] \arrow[r, shift right] & {[1]} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & {[2]} \; \cdots \end{tikzcd}$$

the category Δ needs more arrows, and can be illustrated as

$$\begin{tikzcd} {[0]} \arrow[r, shift left=2] \arrow[r, shift right=2] & {[1]} \arrow[l, dashed] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & {[2]} \; \cdots \arrow[l, dashed, shift right=2] \arrow[l, dashed, shift left=2] \end{tikzcd}$$

As all embeddings and face collapsing maps $\Delta^n \longrightarrow \Delta^m$ are generated (under composition) by the face maps $d_n^i \colon \Delta^{n-1} \longrightarrow \Delta^n$ and degeneracy maps $s_n^j \colon \Delta^{n+1} \longrightarrow \Delta^n$ we see that we can think of the objects of Δ not only as $\widehat{\Delta^n}$, but Δ^n with all its redundant faces. The solid arrows, going up, correspond to the face maps, while the dashed arrows correspond to the degeneracy maps. With this understanding of the simplex category, we can now define simplicial sets.

Definition 2.3.6. A *simplicial set* is a functor $F \colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$. The collection of all these contravariant functors, together with natural transformations between them, form the presheaf category $\mathbf{Set}_\Delta$. If $F_\bullet$ is a simplicial set, we define the sequence of sets $F_0, F_1, F_2, \ldots$ by $F_n = F([n])$ (whence $\bullet$ as the subscript). For $\varphi \colon [m] \longrightarrow [n]$, we sometimes denote the image under F as $F(\varphi) = \varphi^*$.

Since maps $[m] \longrightarrow [n]$ can be generated by the face and degeneracy maps, the only arrows we usually write down between the sets $F_0, F_1, F_2, \ldots$ are those coming from these maps. Thus any simplicial set can illustrated by the following diagram:

$$\begin{tikzcd} F_0 \arrow[r, dashed] & F_1 \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, dashed, shift left=2] \arrow[r, dashed, shift right=2] & F_2 \; \cdots \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] \end{tikzcd}$$

We call the solid arrows the *face maps*, and the dashed arrows are called *degeneracy maps*. This notation is quite handy, and often we denote a simplicial set $F_\bullet \colon \Delta \longrightarrow \mathbf{Set}$ by just the diagram alone.

We now move on to some examples.

Example 2.3.7 (Groupoids as simplicial sets). Let $\mathcal{G}$ be a groupoid. Recall, from definition 2.3.1, that we often write $\mathcal{G} = \{\mathcal{G}_0, \mathcal{G}_1\}$, where $\mathcal{G}_0$ is the collection of all objects of $\mathcal{G}$, while $\mathcal{G}_1$ is all the morphisms of $\mathcal{G}$. We will define a simplicial set $F(\mathcal{G})_\bullet$ associated to the groupoid $\mathcal{G}$. The idea is to let $F(\mathcal{G})_0 = \mathcal{G}_0$, and $F(\mathcal{G})_1 = \mathcal{G}_1$. In other words, the 0-simplices are the objects of the gropoid, and the 1-simplices are the arrows. For $n > 1$, define $F(\mathcal{G})_n$ to be the collection of compositions of n arrows. The first face and degeneracy maps are

$$F(\mathcal{G})_0 \overset{\mathrm{id}_{(-)}}{\dashrightarrow} F(\mathcal{G})_1, \qquad\qquad F(\mathcal{G})_0 \underset{\text{target}}{\overset{\text{source}}{\leftleftarrows}} F(\mathcal{G})_1$$

where $\mathrm{id}_{(-)}$ assigns the identity arrow to any object in $\mathcal{G}_0$, and the source and target maps maps any morphism to its domain and codomain respectively. The maps between the higher simplices are similarly defined. The degeneracy maps $F(\mathcal{G})_n \longrightarrow F(\mathcal{G})_{n+1}$ correspond to adding the identity arrow to some series of compositions, making it one arrow longer. The face maps $F\mathcal{G}_{n+1} \longrightarrow F\mathcal{G}_n$ just consider a series of compositions of length $n+1$ as a series of composition of length n by considering two arrows f and g as one arrow $(f \circ g)$.

Example 2.3.8 (Discrete simplicial sets). Let S be any set. As described in example 2.3.2, we can make S into a groupoid, also denoted S, with $S_0 = S_1 = S$. By the previous example, example 2.3.7, the groupoid S determines a simplicial set $F(S)_\bullet = S_\bullet$. By construction, $S_n = S$ for all n. Since this is the case, we usually omit $\bullet$ and write S for the simplicial set. We often call the simplicial set a *discrete simplicial set*, or a *constant simplicial set*.

Another example, which is one that remembers more of the topology of the underlying space than the discrete simplicial, is

Example 2.3.9 (Spaces as simplicial sets). Let X be a topological space. We wish to define a simplicial set $\mathrm{Sing}_\bullet(X)$. For any $n \in \mathbb{N}_0$, define $\mathrm{Sing}_n(X) = \mathbf{Top}(\Delta^n, X)$, the collection of all continuous maps from the standard n-simplex into X. This defines a functor $\mathrm{Sing}_\bullet(X)\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$, and hence we have a simplicial set. The reader familiar with singular homology will know that $\mathrm{Sing}_\bullet(X)$ contains the information of the topology of X in all dimensions. This is in contrast to the discrete simplicial set, which only contains the 0-dimensional information of X.

Example 2.3.10 (Simplicial set from a group action). Let G be a group, and S be a G-set. Recall, from example 2.3.3, that the group action determines a groupoid $\{S, G \times S\}$. By example 2.3.7, this groupoid determines a simplicial set. Explicitly, the the composition of arrows is just the product of group elements. And so we represent the simplicial set as

$$\begin{tikzcd} S \arrow[r, shift left=2] \arrow[r, shift right=2] & G \times S \arrow[l, dashed] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & G \times G \times S \arrow[l, dashed, shift left=1.5] \arrow[l, dashed, shift right=1.5] & \cdots \end{tikzcd}$$

Example 2.3.11 (The simplicial set of connections). Let G be a Lie group and M be any smooth manifold. Recall, from example 2.3.4, that the groupoid of G-connections over M was the category denoted $G\,\mathbf{Bund}_\nabla(M)$. This naturally determines a simplicial set by example 2.3.7. We denote this simplicial set as

$$B_\nabla G(M)\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}\,.$$

Example 2.3.12. Let X be a smooth manifold and $\mathscr{U} = \{U_i\}_{i\in I}$ be an open cover of X. Then we can define the groupoid $F(\mathscr{U}) = \{F(\mathscr{U})_0, F(\mathscr{U})_1\}$, where

$$F(\mathscr{U})_0 = \coprod_{i_0\in I} U_{i_0}, \quad F(\mathscr{U})_1 = \coprod_{i_0,i_1\in I} U_{i_0}\cap U_{i_1}$$

As example 2.3.7 dictates, this defines a simplicial set, which looks like the following:

$$\coprod_{i_0\in I} U_{i_0} \;\substack{\longleftarrow\\ \dashrightarrow\\ \longleftarrow}\; \coprod_{i_0,i_1\in I} U_{i_0}\cap U_{i_1} \;\substack{\longleftarrow\\ \dashrightarrow\\ \longleftarrow\\ \dashrightarrow\\ \longleftarrow}\; \coprod_{i_0,i_1,i_2\in I} U_{i_0}\cap U_{i_1}\cap U_{i_2}\cdots$$

We denote this simplicial set by $F(\mathscr{U})_\bullet$.

This last example is important. It turns out that there is a map

$$f\colon F(\mathscr{U})_\bullet \longrightarrow X,$$

where the domain X is regarded as the discrete simplicial set coming from the smooth manifold X. For each number $n \in N_0$, the induced map $f_n\colon F(\mathscr{U})_n \longrightarrow X$ is just inclusion. As this is a surjection of groupoids, we get an equivalence of groupoids by lemma 2.3.5.

2.3.4. Realization of simplicial sets

In the above subsection, we have studied objects that a priori are purely combinatorial. Now we wish to turn these combinatorical constructions to topological things. The standard process is called geometric realization. In broad terms, it glues a collection of simplices into a simplicial complex. The specific gluing depends on the information encoded in the face and degeneracy maps.

But before we get ahead of ourselves, we create a functor

$$\Delta\colon \Delta \longrightarrow \mathbf{Top}$$

from the simplex category Δ to the category of topological spaces. The functor assigns each object I in Δ to the space $\Delta^{|I|}$, that is, assigns the finite ordered set $I \cong [n]$ of cardinality $n+1$ the standard n-simplex Δ^n. For a order preserving morphism $\varphi\colon I \longrightarrow I'$ in the simplex category Δ, we define the image of φ under the functor Δ to be the corresponding embedding or collapsing map $\varphi_*\colon \Delta^{|I|} \longrightarrow \Delta^{|I'|}$ of standard n-simplices.

Definition 2.3.13. *Geometric realization* is a functor $|-|\colon \mathbf{Set}_\Delta \longrightarrow \mathbf{Top}$ from the category of simplicial sets to the category of topological spaces. Let $F\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a simplicial set. The geometric realization $|F|$ is the quotient space

$$\bigsqcup_I \Delta(I) \times F(I)/\sim,$$

where for $(x, y) \in \Delta^{|I|} \times F(I)$, and $(x', y') \in \Delta^{|I'|} \times F(I')$, we say $(x, y) \sim (x', y')$ whenever there exist a order preserving morphism $\varphi\colon I \longrightarrow I'$ such that

$$(\varphi_* x, y) = (x', \phi^* y').$$

Here $\varphi_* = \Delta(\varphi)$, and $\varphi^* = F(I)$. The topology of $|F|$ is the final topology, where all $F(I)$ have the discrete topology.

This definition is perhaps unnecessarily abstract. When we think of the category Δ as the collection of all standard n-simplices Δ^n, any simplicial set $F_\bullet$ is just a collection F_n of n-simplices with certain face and degeneracy maps. In this perspective of the simplex category, the functor Δ just assigns Δ^n to Δ^n. And so the disjoint union from the above definition just becomes a union of n-simplices $\Delta^n \times F_n$ for all n. And the quotient, which is the geometric realization of $F_\bullet$ just glues the various n-simplices of distinct n together. Thus we are left with a simplicial complex.

For example, the geometric realization of the discrete simplicial set associated to a set S, from example 2.3.8, is canonically isomorphic to S equipped with the discrete topology. The geometric realization of $\mathrm{Sing}_\bullet(X)$, from example 2.3.9, is homotopy equivalent to the space X. We postpone the description of the geometric realization of $B_\nabla G(M)$, from example 2.3.11, until later. But what we should take away from these examples is that the geometric realization of simplicial set which is in some way related to a topological space is a space almost equal to the space. We now make sense of this "almost equal".

2.3.5. Weak equivalences

As promised, we have a commutative diagram

$$\begin{tikzcd} & \mathbf{Set}_\Delta \arrow[dr, "|-|"] & \\ \mathbf{Grpd} \arrow[ur, "\text{example 2.3.7}"] \arrow[rr] & & \mathbf{Top} \end{tikzcd}$$

of functors. We also promised that weakly equivalent objects are sent to weakly equivalent objects. The purpose of this section is to come to grips with this terminology.

Definition 2.3.14. As we have three categories, **Grpd**, $\mathbf{Set}_\Delta$, and **Top**, we need to know what a weak equivalence is in each category.

(1) Two groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ are *weakly equivalent* if there is a map $f\colon \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ making them equivalent as groupoids, that is, $\mathcal{G}_1$ and $\mathcal{G}_2$ are equivalent as categories.
(2) Two topological spaces X_1 and X_2 are *weakly equivalent* if there is a continuous map $f\colon X_1 \longrightarrow X_2$ which is a weak homotopy equivalence, that is, all the induced maps

$$f_*\colon \pi_n(X_1, x_1) \longrightarrow \pi_n(X_2, x_2)$$

of homotopy groups are isomorphisms (for $n = 0$, this is an isomorphism of sets, while for $n \geq 1$, it is an isomorphism of groups).
(3) Two simplicial sets $\mathcal{F}_\bullet$ and $\mathcal{F}'_\bullet$ are *weakly equivalent* if the induced map $|F_\bullet| \longrightarrow |F'_\bullet|$ of geometric realizations is a weak homotopy equivalence.

We can immediately notice that we weak equivalence is preserved under geometric realization. But to prove that $\mathbf{Grpd} \longrightarrow \mathbf{Top}$ preserves weak equivalence is harder, so we refer the curious reader to [Seg68] (the relevant result is Proposition 2.1).

2.4. Simplicial sheaves on manifolds

In this section, we combine the ideas from section 2.1, section 2.2, and section 2.3. This fusion of structures gives rise to simplicial presheaves (and simplicial sheaves) on manifolds. It is in this world we can construct the universal principal G-bundle $E_\nabla G \longrightarrow B_\nabla G$ with universal connection ∇^{univ}.

Definition 2.4.1. A *simplicial presheaf on manifolds*, or sometimes *simplicial presheaf* for brevity, is a functor

$$\mathcal{F}_\bullet\colon \mathbf{Man}^{\text{op}} \longrightarrow \mathbf{Set}_\Delta\,.$$

A map $\mathcal{F}_\bullet \longrightarrow \mathcal{F}'_\bullet$ of simplicial presheaves is a natural transformation of functors. Thus we have a category **sPre** of simplicial presheaves.

We call a simplicial presheaf $\mathcal{F}_\bullet$ a *simplicial sheaf* if for each totally ordered finite set I in the simplex category Δ the presheaf of sets

$$\mathcal{F}_\bullet(I)\colon\ \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set},$$

is a sheaf.

Just as we sometimes view simplicial sets $F_\bullet\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ as sequences $F_0, F_1, F_2, \ldots$ by thinking the objects of the simplex category Δ as the standard simplices, we can view any simplicial presheaf $\mathcal{F}_\bullet$ as a sequence $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$ of "ordinary" presheaves.

2.4.1. Some examples of simplicial presheaves

We have already seen several examples of simplicial presheaves and simplicial sheaves, even without knowing so. We look at some examples.

Example 2.4.2 (Discrete simplicial sheaves). Let $\mathcal{F}\colon\ \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a sheaf on manifolds. This means the value $\mathcal{F}(M)$ of each test manifold M is a set. By example 2.3.8, we can make this set into a groupoid, which can in turn be turned into the discrete simplicial set $\mathcal{F}(M)_\bullet\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ with constant value $\mathcal{F}(M)_n = \mathcal{F}(M)$ for all n. Thus we get an induced simplicial sheaf $\mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set}_\Delta$ with value equal to the discrete simplicial set $\mathcal{F}(M)$ for any test manifold M. We often denote this induced simplicial sheaf by $\mathcal{F}$.

Example 2.4.3 (Representable simplicial sheaves). Just as we can embed manifolds into the category of presheaves using the associated presheaves $\mathcal{F}_X$ from example 2.1.2, we can embed simplicial manifolds into the category of simplicial presheaves. A *simplicial manifold* $X_\bullet$ is a simplicial set

$$X_0 \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; X_1 \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; X_2 \cdots$$

where each X_n is a smooth manifold and all the face and degeneracy maps are smooth. We define $\mathcal{F}_{X_\bullet}$ to be the simplicial sheaf whose value on a test manifold M in **Man** is the simplicial set

$$\mathbf{Man}(M, X_0) \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; \mathbf{Man}(M, X_1) \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; \mathbf{Man}(M, X_2) \cdots$$

The face and degeneracy maps are induced from the functor $\mathbf{Man}(M, -)$.

A special case is the *constant simplicial manifold*

$$X \leftrightarrows X \mathrel{\substack{\longleftarrow\\ \dashrightarrow\\ \longleftarrow\\ \dashrightarrow\\ \longleftarrow}} X \cdots$$

where all the maps are identity arrows. We denote this by $\mathcal{F}_X$, which is no mistake. A worry might be that this is easily confused with the associated presheaf $\mathcal{F}_X$. But it turns out that the set of natural transformations between two constant simplicial manifolds $\mathcal{F}_X$ and $\mathcal{F}_Y$ is in bijection with the set of all natural transformations between the associated presheaves $\mathcal{F}_X$ and $\mathcal{F}_Y$. We study this further in section 3.3.1.

EXAMPLE 2.4.4. Let X be a smooth manifold and $\mathscr{U} = \{U_i\}_{i \in I}$ be an open cover of X. The simplicial manifold $F(\mathscr{U})_\bullet$

$$\bigsqcup_{i_0 \in I} U_{i_0} \leftrightarrows \bigsqcup_{i_0, i_1 \in I} U_{i_0} \cap U_{i_1} \mathrel{\substack{\longleftarrow\\ \dashrightarrow\\ \longleftarrow\\ \dashrightarrow\\ \longleftarrow}} \bigsqcup_{i_0, i_1, i_2 \in I} U_{i_0} \cap U_{i_1} \cap U_{i_2} \cdots$$

from example 2.3.12 describes a representable simplicial sheaf, which we denote as $(\mathcal{F}_{\mathscr{U}})_\bullet$.

Similarly to how $F(\mathscr{U})_\bullet$ is related to the discrete discrete simplicial set X, the representable simplicial sheaf $(\mathcal{F}_{\mathscr{U}})_\bullet$ is related to the discrete simplicial sheaf $\mathcal{F}_X$. We will investigate this closer in the next section when our language is more developed. But first we look at one last example, which requires a definition first.

DEFINITION 2.4.5. Let $\mathcal{F} \colon \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a sheaf and G a Lie group. A *G-action* on $\mathcal{F}$ is a map

$$a \colon \mathcal{F}_G \times \mathcal{F} \longrightarrow \mathcal{F},$$

of sheaves (where the product $\times$ is "manfiold"-wise) such that for any test manifold M, the set $\mathcal{F}(M)$ is a $\mathcal{F}_G(M)$-set. This means that the coordinate a_M of a at M is an action

$$a_M \colon \mathcal{F}_G(M) \times \mathcal{F}(M) \longrightarrow \mathcal{F}(M)$$

from the group $\mathcal{F}_G(M) = \mathbf{Man}(M, G)$ on the set $\mathcal{F}(M)$.

Such an action determines a simplicial sheaf, which we look at in

EXAMPLE 2.4.6. Recall, from example 2.3.10, that for any Lie group G, we can turn a G-set S into a groupoid $\{S, G \times S\}$, which determined its simplicial set. We now construct the simplicial analog. Let $\mathcal{F}$ be a sheaf and $a \colon \mathcal{F}_G \times \mathcal{F} \longrightarrow \mathcal{F}$ be an action. Then the diagram

$$\mathcal{F} \leftrightarrows \mathcal{F}_G \times \mathcal{F} \mathrel{\substack{\longleftarrow\\ \dashrightarrow\\ \longleftarrow\\ \dashrightarrow\\ \longleftarrow}} \mathcal{F}_G \times \mathcal{F}_G \times \mathcal{F} \ \cdots$$

is a simplicial sheaf. The two first face maps are the projection map $\mathcal{F}_G \times \mathcal{F} \longrightarrow \mathcal{F}$ and action map a, and all higher maps are repeated group operations.

2.4.2. Differential forms on simplicial presheaves

Just as we, in section 2.1.3, expanded on the idea of differential forms on manifolds, to include differential forms on presheaves on manifolds, we again fill out some details glossed over in [FH13] and define differential forms on simplicial presheaves.

Recall, from definition 2.1.6, that we defined the set $\Omega^\bullet(\mathcal{F})$ of differential forms on a presheaf $\mathcal{F}$ on manifolds as the set of natural transformations $\mathcal{F} \longrightarrow \Omega^\bullet$. Completely analogously, for a simplicial presheaf $\mathcal{F}_\bullet$ on manifolds, we define the set $\Omega^\bullet(\mathcal{F}_\bullet)$ of *differential forms* on $\mathcal{F}_\bullet$ as the collection of all natural transformations $\mathcal{F}_\bullet \longrightarrow \Omega^\bullet$, where the target simplicial sheaf $\Omega^\bullet$ is the discrete simplicial sheaf coming from the sheaf $\Omega^\bullet$ of differential forms (from example 2.1.3). And yet again, the *pullback* of a map between simplicial sheaves is defined, analogously, as the precomposition.

It should be noted that we now have a total of three functors: $\Omega^\bullet\colon \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, assigning manifolds their differential forms; $\Omega^\bullet\colon \mathbf{Pre} \longrightarrow \mathbf{Set}$, assigning presheaves their differential forms; and $\Omega^\bullet\colon \mathbf{sPre} \longrightarrow \mathbf{Set}$, assigning simplicial sheaves their differential forms. For a morphism f in one of the domain categories of $\Omega^\bullet$, we call $f^* = \Omega^\bullet(f)$ the pullback. It should always be clear from context which functor is being used.

We can also form vector valued differential forms on simplicial presheaves. For a finite dimensional vector space V, we just define the set of *V-valued differential forms* on $\mathcal{F}_\bullet$ as the set $\Omega^\bullet(\mathcal{F}_\bullet; V) = \mathbf{sPre}(\mathcal{F}_\bullet, \Omega^\bullet \otimes V)$. Pullback are defined analogously.

2.4.3. Weak equivalences again

Recall, from section 2.2.2, that $B^m(r) \subseteq \mathbb{R}^m$ is the m-dimensional ball of radius r centered at the origin, and that the stalk of a presheaf $\mathcal{F}$ of sets on manifolds simply is defined as the direct limit

$$\operatorname*{colim}_{r\to 0} \mathcal{F}\big(B^m(r)\big).$$

Let $\mathcal{F}_\bullet$ be a simplicial presheaf. Analogous to the usual definition of stalk, we define the *m-dimensional stalk* of the simplicial presheaf $\mathcal{F}_\bullet$ to be the simplicial set

$$\operatorname*{colim}_{r\to 0} \mathcal{F}_\bullet\big(B^m(r)\big)\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}\,.$$

DEFINITION 2.4.7. A map $\mathcal{F}_\bullet \longrightarrow \mathcal{F}'_\bullet$ of simplicial presheaves is a *weak equivalence* if for each m, the induced map

$$\operatorname*{colim}_{r\to 0} \mathcal{F}_\bullet\big(B^m(r)\big)$$

on m-dimensional stalks is a weak equivalence of simplicial sets.

Before ending the chapter, we look at a weak equivalence.

PROPOSITION 2.4.8. *The inclusion map* $(\mathcal{F}_{\mathscr{U}})_\bullet \longrightarrow \mathcal{F}_X$ *is a weak equivalence of simplicial sheaves.*

CHAPTER 3

Classification of Principal Bundles with Connection

This chapter combines the material covered in chapter 1 and chapter 2. First, we reconstruct the simplicial sheaves $B_\nabla G$ and $E_\nabla G$ in full detail, before showing that $E_\nabla G$ is weakly equivalent to the discrete simplicial sheaf $\Omega^1 \otimes \mathfrak{g}$ (induced from the sheaf $\Omega^1 \otimes \mathfrak{g}$). As there is no obvious choice of projection map $E_\nabla G \longrightarrow B_\nabla G$ in our category, we construct an action on the sheaf $\Omega^1 \otimes \mathfrak{g}$, and show that the simplicial sheaf $B_\nabla^{\text{triv}} G$ induced from this action is weakly equivalent to $B_\nabla G$.

With these weak equivalences in mind, we can sensibly talk about $E_\nabla G \longrightarrow B_\nabla G$ as a "bundle". We then show, in detail, that $B_\nabla G$ is a classifying space for principal G-bundles with connection. A consequence of the unique classifying maps from chapter 2 gives unique classifying maps into the classifying space $B_\nabla G$.

As alluded to, we actually need to move out of the category of simplicial sheaves and into its homotopy category for our constructions to make sense, as the former category does not account for weak equivalences (while the latter category does). In this new category, we, using a result from abstract homotopy theory, show that the de Rham complex of a simplicial sheaf is easily computable as a certain equalizer of a diagram. Using this result, it is possible to explicitly calculate the de Rham complex of both $B_\nabla G$ and $E_\nabla G$. Because all of our constructions are tautological, we, with the help of the Chern-Weil homomorphism from chapter 1, find that the invariant polynomials in $I^k(G)$ determine all the invariants defined from connections on principal G-bundles.

3.1. The universal bundle with connection

In this section, we summarize the construction of the classifying space $B_\nabla G$, and construct the universal bundle $E_\nabla G \longrightarrow B_\nabla G$ in its entirety, and construct the universal connection ∇^{univ}.

3.1.1. Constructing the universal bundle

Before constructing the universal connection ∇^{univ}, we need to know what the simplicial sheaves $E_\nabla G$ and $B_\nabla G$ look like. This subsection is devoted to constructing them. As they stem from groupoids, we will sometimes write $E_\nabla G$ and $B_\nabla G$ for the groupoids they originate from as well.

Construction 3.1.1 ($B_\nabla G$). Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let M be any test manifold in **Man**. We define the category $G\,\mathbf{Bund}_\nabla(M)$ as the collection of principal G-bundles over M with a connection. Thus an element is a quintuple (E, π, M, G, ∇), where $\pi\colon E \longrightarrow M$ is a principal G-bundle over M, and $\nabla \in \Omega^1(E; \mathfrak{g})$ is a G-connection on the bundle. As both M and G are fixed, we can indicate any object by (E, π, ∇) for notation's sake. Morphisms in this category are certain principal bundle isomorphisms. Given two objects (E_1, π_1, ∇_1) and (E_2, π_2, ∇_2), a principal bundle isomorphism

$$\begin{tikzcd} E_1 \arrow[rr, "\varphi"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & M & \end{tikzcd}$$

covering the identity is a morphism in $G\,\mathbf{Bund}_\nabla(M)$ if the pullback $\varphi^*\nabla_2$ of the connection ∇_2 is equal to the connection ∇_1.

By definition, as all arrows in $G\,\mathbf{Bund}_\nabla(M)$ are invertible, this category is a groupoid. Like explained in example 2.3.7 and example 2.3.11, we can turn this groupoid into a simplicial set

$$B_\nabla G(M)\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}\,.$$

The 0-simplices $B_\nabla G(M)_0 = G\,\mathbf{Bund}_\nabla(M)_0$, or vertices, are the principal bundles (E, π, ∇) themselves. The 1-simplices $B_\nabla G(M)_1 = G\,\mathbf{Bund}_\nabla(M)_1$ are the principal bundle isomorphisms φ in the category. For $n > 1$, the n-simplices in $B_\nabla G(M)_n$ are just series of compositions $\varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_2 \circ \varphi_1$ with n morphisms from the category:

$$\begin{tikzcd} E_1 \arrow[r, "\varphi_1"] \arrow[drr, "\pi_1"'] & E_2 \arrow[r, "\varphi_2"] \arrow[dr, "\pi_2"] & \cdots \arrow[r, "\varphi_{n-1}"] & E_n \arrow[r, "\varphi_n"] \arrow[dl, "\pi_n"'] & E_{n+1} \arrow[dll, "\pi_{n+1}"] \\ & & M & & \end{tikzcd}$$

The first degeneracy map $B_\nabla G(M)_0 \dashrightarrow B_\nabla G(M)_1$ assigns the identity arrow to each object. The two face maps $B_\nabla G(M)_1 \longrightarrow B_\nabla G(M)_0$ assign any bundle isomorphism to its source and target, regarded as objects in the category. For $n > 1$, the face maps maps an n-simplex $\varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_2 \circ \varphi_1$ to itself, regarded as a $(n-1)$-simplex by placement of a bracket

$$\cdots \circ \varphi_{i+1} \circ \varphi_i \circ \cdots = \cdots \circ (\varphi_{i+1} \circ \varphi_i) \circ \cdots$$

The degeneracy maps maps $\varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_2 \circ \varphi_1$ to itself, regarded as a $(n+1)$-simplex by adding id_{E_i} to the composition series

$$\cdots \circ \varphi_{i+1} \circ \varphi_i \circ \cdots = \cdots \circ \varphi_{i+1} \circ \mathrm{id}_{E_i} \circ \varphi_i \circ \cdots$$

Hence have described the simplicial set

$$\begin{tikzcd} B_\nabla G(M)_0 \arrow[r, shift left=2, leftarrow] \arrow[r, dashed] \arrow[r, shift right=2, leftarrow] & B_\nabla G(M)_1 \arrow[r, shift left=4, leftarrow] \arrow[r, shift left=2, dashed] \arrow[r, leftarrow] \arrow[r, shift right=2, dashed] \arrow[r, shift right=4, leftarrow] & \cdots \end{tikzcd}$$

and denote this simplicial set as $B_\nabla G(M)$.

As this construction was done for any test manifold M, we actually get a simplicial presheaf

$$B_\nabla G\colon \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set}_\Delta,$$

assigning M the simplicial set $B_\nabla G(M)$. For a morphism $f\colon M_1 \longrightarrow M_2$, the induced map $f_0\colon B_\nabla G(M_2)_0 \longrightarrow B_\nabla G(M_1)_0$ on vertices sends any object (E, π, ∇) to its "pullback" object $(f^*E, \pi', f^*\nabla)$. On 1-simplices, the induced map $f_1\colon B_\nabla G(M_2)_1 \longrightarrow B_\nabla G(M_1)$ maps any bundle map $\varphi\colon E_1 \longrightarrow E_2$ to the induced map between pullbacks $f^*E_1 \longrightarrow f^*E_2$. As the higher simplices are described using the 1-simplices, we know how $B_\nabla G(f)$ looks like. And since the pullback is associative, the functor $B_\nabla G$ is well-defined[1].

From the construction, it is not immediately clear that $B_\nabla G$ is a simplicial sheaf. But since both connections and connection preserving isomorphisms can be glued together along open sets, the simplicial presheaf $B_\nabla G$ is a simplicial sheaf. Indeed, because connections and bundles are defined locally, if we are given two open sets U_i and U_j in an open cover $\{U_i\}$ of a test manifold M, and principal G-bundles with connections over them such that the connections agree on the intersection $U_i \cap U_j$, then they can be glued to uniquely together on $U_i \cup U_j$. Thus $B_\nabla G(M)_0$ is a sheaf. By the same reasons as before, connection preserving isomorphisms on U_i and U_j which agree on the

[1] This is actually not true. The pullback is not strictly associative in **Set**, and this is a problem. The pullback of $E \longrightarrow M_3$ by a composition $M_1 \longrightarrow M_2 \longrightarrow M_3$ is canonically isomorphic to the pullback of the pullback of E, but not *equal*. This issue is resolvable using higher category theory, but we will not look into it in this thesis.

intersection $U_i \cap U_j$ can be glued together to define a connection preserving isomorphism on $U_i \cup U_j$. So $B_\nabla G(M)_1$ is a sheaf as well. As $B_\nabla G(M)$ is built entirely from $B_\nabla G(M)_0$ and $B_\nabla G(M)_1$, we know $B_\nabla G(M)$ is a sheaf.

Construction 3.1.2 ($E_\nabla G$). Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let M be any test manifold in **Man**. We define the category $G\,\mathbf{Bund}_\nabla^{\mathrm{triv}}(M)$ as the collection of trivial principal G-bundles over M with a connection. Thus an object is a sextuplet $(E, \pi, M, G, \nabla, s)$, where (E, π, M, G, ∇) is in $G\,\mathbf{Bund}_\nabla(M)$ and $s\colon M \longrightarrow E$ is a global section (which, by proposition 1.2.18, determines the trivial structure of E). Similarly to the case above, we often write just (E, π, ∇, s) to shorten notation as M and G are fixed. For φ to be a morphism in this category, it must be a morphism in $G\,\mathbf{Bund}_\nabla(M)$, and in addition it must preserve trivialization. This means we have a diagram

$$\begin{tikzcd} E_1 \arrow[rr, "\varphi"] \arrow[dr, shift right, "\pi_1"'] & & E_2 \arrow[dl, shift left, "\pi_2"] \\ & M \arrow[ul, shift right, "s_1"'] \arrow[ur, shift left, "s_2"] & \end{tikzcd}$$

such that the inner triangle commutes, i.e. $\varphi \circ s_1 = s_2$. (The outer triangle commutes by assumption.)

By the same argument as in construction 3.1.1, this category is a groupoid. And again, completely analogous to the above construction, the groupoid determines a simplicial set $E_\nabla G(M)$ where $E_\nabla G(M)_0$ is the collection of objects (E, π, ∇, s), the 1-simplices $E_\nabla G(M)_1$ are the morphisms which respect the trivialization, and so on. And since M was an arbitrary test manifold, we get a simplicial presheaf

$$E_\nabla G\colon\ \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set}_\Delta\,.$$

Again, it may perhaps not be clear that $E_\nabla G$ is a simplicial sheaf. But as connections, connection preserving isomorphisms, and trivializations of trivial bundles can be glued together along open sets, the simplicial presheaf $E_\nabla G$ is a simplicial sheaf. The argument is identical to the case with $B_\nabla G$.

One might notice that any the set of morphisms in $E_\nabla G(M)$ from E_1 to E_2 can have maximum have cardinality 1. Indeed, if we have a morphism such that $\varphi \circ s_1 = s_2$, then we know that for each $p \in M$, we have $\varphi\big(s_1(p)\big) = s_2(p) \in (E_2)_p$. In other words, we know where one element in $(E_1)_p$, namely $s_1(p)$, is mapped to under φ. The isomorphism φ preserves fibers and is G-equivariant by assumption, and since the action on E_2 is free and transitive, we know where the rest of the elements in $(E_1)_p$ are mapped to in $(E_2)_p$. As we can do this for every $p \in M$, we see that any φ is predetermined by the sections s_1 and s_2.

Thus $E_\nabla G$, considered as a groupoid, is discrete. So we should expect $E_\nabla G$ to be weakly equivalent to some discrete simplicial sheaf.

3.1.2. Two important weak equivalences

Recall, from example 2.1.3, that we have a presheaf

$$\Omega^1 \otimes \mathfrak{g} \colon \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set}\,.$$

on manifolds of sets. On any test manifold, it produces the set $\Omega^1 \otimes \mathfrak{g}(M) = \Omega^1(M;\mathfrak{g})$ of $\mathfrak{g}$-valued differential 1-forms. We saw, in example 2.2.4, that it indeed is a sheaf. This can be turned into the constant simplicial sheaf $\Omega^1 \otimes \mathfrak{g}$, as described in example 2.3.8. For each test manifold M, we have the constant simplicial set

$$\begin{tikzcd} \Omega^1 \otimes \mathfrak{g}(M) \arrow[r, shift left=2, leftarrow] \arrow[r, dashed] \arrow[r, shift right=2, leftarrow] & \Omega^1 \otimes \mathfrak{g}(M) \arrow[r, shift left=3, leftarrow] \arrow[r, shift left=1, dashed] \arrow[r, shift right=1, leftarrow] \arrow[r, shift right=3, dashed] \arrow[r, shift right=5, leftarrow] & \cdots \end{tikzcd}$$

where all the face and degeneracy maps are identity arrows. As this comes from the gropoid $\{\Omega^1 \otimes \mathfrak{g}(M), \Omega^1 \otimes \mathfrak{g}(M)\}$, we sometimes denote this gropoid by $\Omega^1 \otimes \mathfrak{g}(M)$ as well.

PROPOSITION 3.1.3. *The discrete simplicial sheaf $\Omega^1 \otimes \mathfrak{g}$ is weakly equivalent to the simplicial sheaf $E_\nabla G$.*

PROOF. To show this, we first need to define a natural transformation

$$\psi \colon E_\nabla G \longrightarrow \Omega^1 \otimes \mathfrak{g},$$

or, equivalently, for each M, a map $\psi_M \colon E_\nabla G(M) \longrightarrow \Omega^1 \otimes \mathfrak{g}(M)$ of simplicial sets. Since both of these objects are completely determined by their groupoid structures (also denoted) $E_\nabla G(M)$ and $\Omega^1 \otimes \mathfrak{g}(M)$, we only need to show that ψ_M is an equivalence of groupoids, meaning we have maps

$$\begin{tikzcd} E_\nabla G(M) \arrow[r, shift left, "\psi_M"] & \Omega^1(M;\mathfrak{g}), \arrow[l, shift left, "\theta_M"] \end{tikzcd}$$

which determine an equivalence of categories, We first define ψ_M.

For an object (E,π,∇,s) in $E_\nabla G(M)_0$, define

$$\psi_M\big(E,\pi,\nabla,s\big) = s^*\nabla.$$

This makes sense, as G-connections are certain elements in $\Omega^1(E;\mathfrak{g})$, so the pullback $s^*\nabla$ is an element in $\Omega^1(M;\mathfrak{g})$. For an arrow $\varphi \colon E_1 \longrightarrow$

E_2 between objects $(E_1, \pi_1, \nabla_1, s_1)$, and $(E_2, \pi_2, \nabla_2, s_2)$ in $E_\nabla G_1$, define

$$\psi_M(\varphi) = s_1^* \nabla_1.$$

It may seem arbitrary to pick the pullback of the first connection ∇_1. But because $\varphi \circ s_1 = s_2$ and $\varphi^* \nabla_2 = \nabla_1$, we see that

$$s_1^* \nabla_1 = s_1^*(\varphi^* \nabla_2) = (\varphi \circ s_1)^* \nabla_2 = s_2^* \nabla_2.$$

So $\psi(\varphi) = s_1^* \nabla_1 = s_2^* \nabla_2$, thus there is not really a choice to be made.

To define $\theta_M \colon \Omega^1(M; \mathfrak{g}) \longrightarrow E_\nabla G(M)$, we need to find some trivial bundle over M. The natural choice is the trivial bundle $M \times G \longrightarrow M$. But we still need to forge some connection on it which is determined by a 1-form in $\Omega^1(M; \mathfrak{g})$. We will use the fact that we have the following bijection between sets:

$$\{\text{connections on } M \times G \longrightarrow M\} \longleftrightarrow \Omega^1(M; \mathfrak{g})$$

To see this, note that the trivial inclusion map $i \colon M \longrightarrow M \times G$ sending any point $p \in M$ to (p, e) determines a map $i^* \colon \Omega^1(M \times G; \mathfrak{g}) \longrightarrow \Omega^1(M; \mathfrak{g})$. The restriction of i^* to the set of connections on the trivial bundle $M \times G$ is the bijection we want. Observe that for $(p, g) \in M \times G$, the smooth map $\widetilde{\omega}$ defined by

$$\widetilde{\omega}_{(p,g)} = \mathrm{Ad}_{g^{-1}} \circ \omega_p \circ (\mathrm{proj}_M)_{*,(p,g)} \colon T_{(p,g)}(M \times G) \longrightarrow T_p M \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g},$$

is a 1-form on $M \times G$. If ∇_{MC} is the Maurer-Cartan form, from example 1.4.4, then for $(p, g) \in M \times G$, we define ∇_ω as

$$\nabla_\omega = \nabla_{MC} + \widetilde{\omega},$$

which can easily be checked to be a connection, and it is also the corresponding connection coming from ω. (See p. 47 of [Dup03] for a complete proof.) This determines the bijection from above. Hence we define

$$\theta_M(\omega) = (M \times G, \mathrm{proj}_M, \nabla_\omega, i).$$

To see that these maps determines equivalences of categories, we need calculate $\psi_M \circ \theta_M$ and $\theta_M \circ \psi_M$. As $i^* \nabla_\omega = \omega$, we get for free that $\psi_M \circ \theta_M(\omega) = \omega$. For the latter composition, we actually do not get the same bundle back. But we only need a connection preserving and section preserving isomorphism of bundles. As we start with a trivial bundle and end with a trivial bundle, all that is left to show is that for (E, π, ∇, s), there is some G-eqivariant diffeomorphism $\varphi \colon E \longrightarrow M \times G$ such that the inner and outer triangles of the diagram

$$\begin{tikzcd}
M \times G \arrow[rr, "\varphi"] \arrow[dr, shift right, "\mathrm{proj}_M"'] & & E \arrow[dl, shift left, "\pi"] \\
& M \arrow[ul, shift right, "i"'] \arrow[ur, shift left, "s"] &
\end{tikzcd}$$

commute, and $\varphi^*\nabla = \nabla_\omega$. We will explicitly construct φ.

As we can see from the inner triangle, any such φ must preserve sections, and so $\varphi(p, e) = s(p)$. Because φ needs to be equivariant, we must have

$$\varphi(p, g) = \varphi(p, e \cdot g) = \varphi(p, e) \cdot g = s(p) \cdot g.$$

And so we define $\varphi(p, g) = s(p) \cdot g$. Note that φ is (equal to) the composition $M \times G \longrightarrow E \times G \longrightarrow E$, where the first map is $s \times \mathrm{id}_G$ and the second is the action on the total space. So we get smoothness and equivariance for free. As $\pi \circ s = \mathrm{id}_M$, the map φ is a bundle map as well. By the equivariance, the map φ is an isomorphism because the action is free and transitive on each fibre. What is left is to show that φ preserves connection, i.e. that $\varphi^*\nabla = \nabla_\omega$. By definition

$$(\varphi^*\nabla)_{(p,g)}(X) = \nabla_{\varphi(p,g)}\big(\varphi_{*,(p,g)}(X)\big),$$

so we need to find the derivative $\varphi_{*,(p,g)}$. By the factorization of φ, we see that

$$\varphi(-, g) = R_g \circ s \colon M \longrightarrow E, \quad \varphi(p, -) = f_{s(p)} \circ \mathrm{id}_G \colon G \longrightarrow E.$$

where R_g is right multiplication by g and $f_{s(p)}(g) = s(p) \cdot g$. We denote the derivative of f_x by $(f_x)_* = v_x$. Since we can decompose $X \in T_{(p,g)}(M \times G)$ into $X = X_1 + X_2 \in T_pM \oplus T_gG$, it follows that $\varphi_{*,(p,g)}(X) = \varphi(-, g)_{*,p}X_1 + \varphi(p, -)_{*,g}X_2$. And so we see that

$$\nabla_{\varphi(p,g)}\big(\varphi_{*,(p,g)}(X)\big) = \nabla_{s(p)\cdot g}\big((R_g \circ s)_{*,p}X_1\big) + \nabla_{s(p)\cdot g}\big((f_{s(p)})_{*,g}X_2\big).$$

Recall that since ∇ is a connection, we have that $\nabla_{s(p)\cdot p} \circ v_{s(p)\cdot g} = \mathrm{id}_{\mathfrak{g}}$. But in the above expression, the subscript of $f_{s(p)}$ is lacking a factor of $\cdot g$. Hence we introduce the map $L_h \colon G \longrightarrow G$ which is left multiplication by h, and it follows immediately that $f_{s(p)} = f_{s(p)\cdot g} \circ L_{g^{-1}}$. Thus it is not too hard to see that

$$\nabla_{\varphi(p,g)}\big(\varphi_{*,(p,g)}(X)\big) = (R_g^*\nabla)_{s(p)}(s_{*,p}X_1) + \nabla_{s(p)\cdot g}\big((f_{s(p)\cdot g} \circ L_{g^{-1}})_{*,g}X_2\big),$$

where we have used the chain rule to get $(R_g \circ s)_{*,p} = (R_g)_{*,s(p)} \circ s_{*,p}$. Since ∇ is a connection, we get that $R_g^*\nabla = \mathrm{Ad}_{g^{-1}} \circ \nabla$, and it follows that

$$\nabla_{\varphi(p,g)}\big(\varphi_{*,(p,g)}(X)\big) = \mathrm{Ad}_{g^{-1}} \circ (s^*\nabla)_p(X_1) + \mathrm{id}_{\mathfrak{g}} \circ (L_{g^{-1}})_{*,g}(X_2).$$

Since $X_1 = (\mathrm{proj}_M)_*(X)$ and $X_2 = (\mathrm{proj}_G)_*(X)$, we in total have that

$$(\varphi^*\nabla)_{(p,g)}(X) = \widetilde{\omega}_{(p,g)}(X) + (\nabla_{MC})_{(p,g)}(X) = (\nabla_\omega)_{(p,g)}(X).$$

And so $\varphi^*\nabla = \nabla_\omega$, and φ is a connection preserving morphism. Thus the object $(M \times G, \mathrm{proj}_M, \nabla_{s^*\nabla}, i)$ is isomorphic to (E, π, ∇, s) in $E_\nabla G(M)$, and so the groupoids $E_\nabla G(M)$ and $\Omega^1 \otimes \mathfrak{g}(M)$ are equivalent. It follows that $E_\nabla G$ and $\Omega^1 \otimes \mathfrak{g}$ are weakly equivalent. $\square$

We have now found one weak equivalence. But the subsection title promises two, and so we continue the search.

Let us take a step back. We want to make $E_\nabla G$ into a total space over $B_\nabla G$. But the objects in $E_\nabla G$ are trivial principal G-bundles, while the objects in $B_\nabla G$ can be non-trivial principal G-bundles, so there is perhaps no "obvious" projection map $E_\nabla G \longrightarrow B_\nabla G$. Thus one might wonder if there is some base space $B_\nabla^{\mathrm{triv}} G$ of trivial bundles, say, with no trivialization chosen, such that the projection map could "forget" the choice of trivialization. If the objects of $B_\nabla G(M)$ indeed would be principal G-bundles (E, π, ∇), such that E is trivializable, but no global section is chosen, then it turns out, according to [FH13], that this construction does not yield a simplicial sheaf, but a simplicial presheaf only, because "'trivializable'" is not a local property. Luckily, we can replace this presheaf with a simplicial sheaf which is more trivial as well. Note that if $E \longrightarrow M$ is a principal G-bundle with two global sections $s, s' \colon M \longrightarrow E$, then, on each fibre E_p, we have $s(p) \cdot g_p = s'(p)$ for some unique $g_p \in G$ because the action is free and transitive on E_p. Thus, there is some unique smooth map $g \colon M \longrightarrow G$ such that $s \cdot g = s'$ globally. And so we can think of $\mathbf{Man}(M, G) = \mathcal{F}_G(M)$ as acting on the vertices of $E_\nabla G(M)$. Using the weak equivalence from proposition 3.1.3, we thus have an induced action on the set $\Omega^1(M; \mathfrak{g})$. This action can be described explicitly, as is done in

Proposition 3.1.4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and M a test manifold. For $g \in \mathbf{Man}(M, G)$, let $g \cdot i \colon M \longrightarrow M \times G$ be the map $p \longmapsto \big(p, s(p)\big)$. Then the map*

$$a_M \colon \mathcal{F}_G(M) \times \Omega^1(M; \mathfrak{g}) \longrightarrow \Omega^1(M; \mathfrak{g}),$$

sending (g, ω) to the pullback $(g \cdot i)^*\nabla_\omega$ induces a G-action

$$a \colon \mathcal{F}_G \times (\Omega^1 \otimes \mathfrak{g}) \longrightarrow \Omega^1 \otimes \mathfrak{g},$$

on the sheaf $\Omega^1 \otimes \mathfrak{g}$. Here $\nabla_\omega \in \Omega^1(M \times G; \mathfrak{g})$ is the connection from the proof of proposition 3.1.3.

PROOF. Note that the map a_M is (equal to) the composition

$$\mathbf{Man}(M,G) \times \Omega^1(M;\mathfrak{g}) \xrightarrow{\psi_M \circ a'_M \circ (\mathrm{id}_{\mathbf{Man}(M,G)} \times \theta_M)} \Omega^1(M;\mathfrak{g})$$

where we have used θ_M and ψ_M from the proof of proposition 3.1.3, and a'_M is the map

$$\big(g, (M \times G, \mathrm{proj}_M, \nabla_\omega, i)\big) \longmapsto (M \times G, \mathrm{proj}_M, \nabla_\omega, g \cdot i).$$

We need to prove that this composition determines a natural transformation, and that it makes $\Omega^1(M;\mathfrak{g})$ a $\mathbf{Man}(M,G)$-set. Let us see why it is a natural transformation first.

As θ_M and ψ_M come from natural transformations θ and ψ, we only need to show that the middle map a'_M determines a natural transformation. But as there is maximally one morphism between objects in $E_\nabla G(M)$, the commutativity from the natural tranformation diagram is trivial to check. And so a is a natural transformation.

What is left is to show that a_M makes $\Omega^1(M;\mathfrak{g})$ a $\mathbf{Man}(M,G)$-set. But as the action is induced from a'_M, and this is an action, we are done. $\square$

Now that we see that we have an action on the sheaf $\Omega^1 \otimes \mathfrak{g}$, we can produce the simplicial sheaf

$$\Omega^1 \otimes \mathfrak{g} \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; \mathcal{F}_G \times (\Omega^1 \otimes \mathfrak{g}) \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; \mathcal{F}_G \times \mathcal{F}_G \times (\Omega^1 \otimes \mathfrak{g}) \;\cdots$$

just as in example 2.4.6. Explicitly, the two first face maps are the projection map $\mathcal{F}_G \times \mathcal{F} \longrightarrow \mathcal{F}$ and action map a. This simplicicial sheaf is denoted $B^{\mathrm{triv}}_\nabla G$.

The reason we constructed $B^{\mathrm{triv}}_\nabla G$ was to determine the projection map $E_\nabla G \longrightarrow B_\nabla G$. And so we better be sure that $B^{\mathrm{triv}}_\nabla G$ is weakly equivalent to $B_\nabla G$.

PROPOSITION 3.1.5. *The discrete simplicial sheaf $B^{triv}_\nabla G$ is weakly equivalent to the simplicial sheaf $B_\nabla G$.*

PROOF. To show this, we first need to define a natural transformation

$$\Psi \colon B^{\mathrm{triv}}_\nabla G \longrightarrow B_\nabla G,$$

or, equivalently, for each M, a map $\Psi_M \colon B^{\mathrm{triv}}_\nabla G(M) \longrightarrow B_\nabla G(M)$ of simplicial sets. Since both of these objects are completely determined by their groupoid structures $\{\Omega^1(M;\mathfrak{g}), \mathbf{Man}(M,G) \times \Omega^1(M;\mathfrak{g})\}$ and $B_\nabla G(M)$, we only need to show that Ψ_M is an equivalence of groupoids, meaning we have maps

$$B_\nabla^{\mathrm{triv}}G(M) \underset{\Theta_M}{\overset{\Psi_M}{\rightleftarrows}} B_\nabla G(M)$$

which determine an equivalence of categories, We first define Ψ_M.

For an object ω in $\Omega^1(M;\mathfrak{g})$, define

$$\Psi_M(\omega) = (M \times G, \mathrm{proj}_M, \nabla_\omega),$$

which is an object in $B_\nabla G(M)_0$. As a 1-simplex in $\mathbf{Man}(M,G) \times \Omega^1(M;\mathfrak{g})$ is just an element (g,ω), which we think of as the arrow $\omega \longrightarrow (g\cdot i)^*\nabla_\omega$, we need to determine how this induces an arrow $\varphi\colon M \times G \longrightarrow M \times G$ in $B_\nabla G(M)_1$, i.e. a bundle map such that $\varphi^*\nabla_\omega = \nabla_{(g\cdot i)^*\nabla_\omega}$. Hence we start by determining what $(g\cdot i)^*\nabla_\omega$ looks like.

For any point $p \in M$ and any tangent vector $X \in T_pM$,

$$\big((g\cdot i)^*\nabla_\omega\big)_p(X) = (\nabla_\omega)_{(g\cdot i)(p)}\big((g\cdot i)_{*,p}(X)\big) = (\nabla_\omega)_{(p,g_p)}(X + g_{*,p}(X)).$$

As $\nabla_\omega = \nabla_{MC} + \widetilde{\omega}$, we see that

$$\big((g\cdot i)^*\nabla_\omega\big)_p(X) = (\nabla_{MC})_{(p,g_p)}(X + g_*X) + \widetilde{\omega}_{(p,g_p)}(X + g_*X).$$

With this in mind, we proceed by finding an isomorphism φ. Note that for $g\colon M \longrightarrow G$, we get an isomorphism

$$\varphi_g\colon M \times G \longrightarrow M \times G, \quad (p,h) \longmapsto (p, g_p h)$$

This isomorphism is (equal to) the composition

$$M \times G \xrightarrow{\Delta\times\mathrm{id}_G} M \times M \times G \xrightarrow{\mathrm{id}_M \times g \times \mathrm{id}_G} M \times G \times G \longrightarrow M \times G,$$

where $\Delta(p) = (p,p)$ is the diagonal map, and the last map is the action on the total space $M \times G$. Just as in the proof of proposition 3.1.3, we determine the pullback $\varphi_g^*\nabla_\omega)$ by finding the derivative of φ_g, using the factorization of φ_g. In the end, when calculating the derivatives, we see that $\varphi^*\nabla_\omega = \nabla_{(g\cdot i)^*\nabla_\omega}$, and so we are done. □

3.2. The classification theorem

We can summarize the previous section by saying we have weak equivalences

$$E_\nabla G \underset{\theta}{\overset{\psi}{\rightleftarrows}} \Omega^1 \otimes \mathfrak{g} \qquad\qquad B_\nabla^{\mathrm{triv}}G \underset{\Theta}{\overset{\Psi}{\rightleftarrows}} B_\nabla G$$

From proposition 3.1.4, we know that the Lie group G (or, actually $\mathcal{F}_G$) acts freely and transitively on $\Omega^1 \otimes \mathfrak{g}$ with quotient $B_\nabla^{\mathrm{triv}}G$. Thus, up to weak equivalence, we have a "principal G-bundle" $E_\nabla G \longrightarrow B_\nabla G$. We want to equip this universal bundle with a *universal connection*

$\nabla^{\text{univ}} \in \Omega^1(E_\nabla G; \mathfrak{g})$, i.e. a natural transformation $\nabla^{\text{univ}}: E_\nabla G \longrightarrow \Omega^1 \otimes \mathfrak{g}$. The element that makes the construction universal is

$$\nabla^{\text{univ}} = \psi,$$

which is the weak equivalence from above. We have the following theorem.

THEOREM 3.2.1. *Let $\pi: E \longrightarrow M$ be a principal G-bundle with a connection $\nabla \in \Omega^1(E; \mathfrak{g})$, and denote the induced discrete simplicial sheaves (as explained in example 2.4.2) by $\mathcal{F}_E$ and $\mathcal{F}_M$ respectively. Then there is a unique classifying map*

$$\begin{tikzcd} \mathcal{F}_E \arrow[r, "f"] \arrow[d] & E_\nabla G \arrow[d] \\ \mathcal{F}_X \arrow[r, "\bar{f}"] & B_\nabla G \end{tikzcd}$$

such that $f^*(\nabla^{univ}) = \nabla$.

PROOF. As an appetizer, observe that all the objects $\mathcal{F}_E, \mathcal{F}_M, E_\nabla G$, and $B_\nabla G$ stem from their defining groupoids, it suffices to work within that setting. For the rest of this proof, let M be any test manifold.

For the main course, we want to define f_M. This map should send any $g: M \longrightarrow E$ in $\mathcal{F}_E(M) = \mathbf{Man}(M, E)$ to some object (E', π', ∇', s') in $E_\nabla G(M)$. We construct this trivializable principal bundle and its connection ∇' as follows: take the pullback of E by π (i.e. itself). This determines a bundle $\widetilde{E} = \pi^* E$ with projection map $\widetilde{\pi}: \widetilde{E} \longrightarrow E$ and connection $\widetilde{\nabla} = \pi^*\nabla$. By the universal property of pullbacks (proposition 1.2.8), there is a map $\widetilde{s}: E \longrightarrow \widetilde{E}$. And since $\widetilde{s}$ commutes with everything, $\widetilde{\pi} \circ \widetilde{s} = \mathrm{id}_E$, thus it is a global section. The object $(\widetilde{E}, \widetilde{\pi}, \widetilde{\nabla}, \widetilde{s})$ is in $E_\nabla G(E)$, but not $E_\nabla G(M)$, so we take another pullback. The pullback of $\widetilde{E}$ by our chosen map g yields some bundle $\pi': E' = \varphi^*(\widetilde{E}) \longrightarrow M$. Again, we have a connection $\nabla' = g^*\widetilde{\nabla}$ on the bundle E'. And as the pullback of a trivial total space is trivial, we have an induced trivialization $s = g^*\widetilde{s}: M \longrightarrow g^*\widetilde{E}$. The situation can be illustrated by the following diagram:

$$\begin{tikzcd} g^*\widetilde{E} \arrow[r] \arrow[d, shift left, "\pi'"] & \widetilde{E} \arrow[r] \arrow[d, shift left, "\widetilde{\pi}"] & E \arrow[d, "\pi"] \\ M \arrow[r, "g"] \arrow[u, shift left, "g^*\widetilde{s}"] & E \arrow[r, "\pi"] \arrow[u, shift left, "\widetilde{s}"] & X \end{tikzcd}$$

The left-most bundle is our desired bundle, and we hence define

$$f_M(g) = (E', \pi', \nabla', s) = (g^*\widetilde{E}, \pi', g^*\widetilde{\nabla}, g^*\widetilde{s}).$$

We now check that $f^*\nabla^{\text{univ}} = \nabla$.

By construction 2.1.7, the connection ∇, which is a 1-form, determines a unique classifying map $\varphi\colon \mathcal{F}_E \longrightarrow \Omega^1 \otimes \mathfrak{g}$ such that the component φ_M at M sends any map $g\colon M \longrightarrow E$ to the pullback $g^*\nabla$. By definition, φ is an element in $\Omega^1(\mathcal{F}_E; \mathfrak{g})$, and it turns out φ is the image of ∇^{univ} under f^*. To see this, note that the pullback of f is the map

$$f^*\colon \Omega^1(E_\nabla G; \mathfrak{g}) \longrightarrow \Omega^1(\mathcal{F}_E; \mathfrak{g}), \quad \omega \longmapsto \omega \circ f.$$

For any M, the component $(f^*\omega)_M$ is a map $\mathbf{Man}(M, E) \longrightarrow \Omega^1(M; \mathfrak{g})$. And in particular, for $\omega = \nabla^{\text{univ}}$, we see that

$$(f^*\nabla^{\text{univ}})_M\colon\ \mathbf{Man}(M; E) \longrightarrow \Omega^1(M; \mathfrak{g}), \quad g \longmapsto s^*\nabla',$$

where $s = g^*s$ and $\nabla' = g^*\widetilde{\nabla}$. As f just sends a map to the pullback of some pullback, we can walk the diagram above in reverse (taking pullbacks of every map), and we see that

$$s^*\nabla' = g^* \circ \underbrace{\widetilde{s}^* \circ \widetilde{\pi}^*}_{=\text{id}_E^*} \nabla = g^*\nabla.$$

As the classifying map is the unique map with this pullback property, we have show $f^*\nabla^{\text{univ}} = \varphi$, and it follows that the pullback of ∇^{univ} is ∇. As the classifying maps is unique, f is unique.

For dessert, we define the bottom map $\overline{f}$. The constrution of f and the bundle map criterion (i.e. commutativity) forces $\overline{f}$ to be a particular map. The component $\overline{f}_M$ at M must send $g\colon M \longrightarrow X$ in $\mathbf{Man}(M, X)$ to some object (E'', π'', ∇'') in $B_\nabla G(M)$. We know that whatever this bundle is, commutativity forces

$$\overline{f}_M(\pi \circ g) = (g^*\widetilde{E}, \pi', g^*\widetilde{\nabla}).$$

And hence we are forced at defining $\overline{f}_M$ to be the map

$$\overline{f}_M(g) = (E'', \pi'', \nabla'') = (g^*E, \pi'', g^*\nabla),$$

where π'' is the projection of the pullback bundle g^*E.

In total, we have defined a map f covering $\overline{f}$, and showed that it is the only possible bundle map such that $f^*\nabla^{\text{univ}} = \nabla$. Hence we are done. $\square$

We have seen that many constructions used in this theorem have been tautological. Proceeding, we now want to calculate the cohomology of these spaces. But to do so, we need some advanced magic called abstract homotopy theory.

3.3. The de Rham complexes of the universal bundle

In this section, we first go from the category **sPre** of simplicial presheaves on manifolds to its homotopy category, summarizing the important results needed from that setting, before mentioning the main results of [FH13]. This section contains no proofs, as it is not the main focus of this thesis.

3.3.1. Abstract homotopy theory

We saw, in section 2.4.2, that we already know how to define the de Rham complex of simplicial presheaves. But in this category, it becomes increasingly difficult to actually compute most de Rham complexes. Another problem is that the category of simplicial presheaves does not care much about weak equivalences. Hence we move to new category, where our weak equivalences are cared for, and computations become feasible in practice. The idea, originally due to Quillen, is to add an inverse to every weak equivalence. Hence whenever we have a weak equivalence in our old category, we get an isomorphism in the new category. A good and more in-depth introduction is given in [FH13]. That paper is offered as a tribute to Quillen, and his monographs [Qui67] and [Qui69] are the original sources for many of the ideas.[2]

Glossing over *many* details, the main takeaway is that we have a category $\operatorname{ho}\mathbf{sPre}$ and a functor $L\colon \mathbf{sPre} \longrightarrow \operatorname{ho}\mathbf{sPre}$ such that for any two simplicial presheaves $\mathcal{F}_\bullet, \mathcal{F}'_\bullet$,

$$\{\text{weak equivalences } \mathcal{F}_\bullet \longrightarrow \mathcal{F}'_\bullet\} \longmapsto \{\text{isomorphisms } L(\mathcal{F}_\bullet) \longrightarrow L(\mathcal{F}'_\bullet)\}.$$

In addition, $\operatorname{ho}\mathbf{sPre}$ is the "smallest" such category. Meaning the functor L has the universal property that any other "homotopy category"[3] $\mathscr{C}$ and any functor K mapping weak equivalences to isomorphisms, can be lifted uniquely, in the sense that we have the following commutative diagram:

[2] This thesis, inspired by [FH13], does not study the model category of underlying structures at play, to hopefully give a more clear exposition.

[3] Again, concealing the technicalities.

$$\begin{tikzcd} \mathbf{sPre} \arrow[r, "L"] \arrow[dr, "K"'] & \operatorname{ho}\mathbf{sPre} \arrow[d, dashed, "\exists!"] \\ & \mathscr{C} \end{tikzcd}$$

Since the new category does not add additional objects, we denote the image $L\mathcal{F}_\bullet$ under L of a simplicial presheaf $\mathcal{F}_\bullet$ by $\mathcal{F}_\bullet$ as well. As we have added new morphisms, it would be interesting to know how the Hom-sets $\mathbf{sPre}(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)$ and $\operatorname{ho}\mathbf{sPre}(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)$ of two simplicial presheaves $\mathcal{F}_\bullet, \mathcal{F}'_\bullet$ are related.

PROPOSITION 3.3.1. *Let $\mathcal{P}$ in $\mathbf{Pre}$ be any presheaf, regarded as a constant simplicial presheaf in $\mathbf{sPre}$ (by example 2.4.2). Then for any simplicial presheaf $\mathcal{F}_\bullet$ in $\mathbf{sPre}$, we have bijections*

$$\operatorname{ho}\mathbf{sPre}(\mathcal{F}_\bullet, \mathcal{P}) \cong \operatorname{ho}\mathbf{sPre}(\mathcal{F}_\bullet, \mathbf{a}\mathcal{P}) \cong \mathbf{sPre}(\mathcal{F}_\bullet, \mathcal{P}),$$

where $\mathbf{a}$ is the sheafifcation functor from section 2.2.2. This means, in particular, that if $\mathcal{P}$ is a sheaf $\mathcal{F}'$, then we have a bijection

$$\mathbf{sPre}(\mathcal{F}_\bullet, \mathcal{F}') \cong \operatorname{ho}\mathbf{sPre}(\mathcal{F}_\bullet, \mathcal{F}').$$

It should be noted that the set $\mathbf{sPre}(\mathcal{F}_\bullet, \mathcal{F}')$ is equal to the equalizer of

$$\mathbf{Pre}(\mathcal{F}_0, \mathcal{F}') \rightrightarrows \mathbf{Pre}(\mathcal{F}_1, \mathcal{F}')$$

where the two maps are induced (under the Hom-functor $\mathbf{Pre}(-, \mathcal{F}')$) from the first face maps $\mathcal{F}_0 \leftleftarrows \mathcal{F}_1$.

We can also see that if we look at two constant simplicial presheaves $\mathcal{P}$ and $\mathcal{P}'$ coming from ordinary presheaves $\mathcal{P}, \mathcal{P}'$ in $\mathbf{Pre}$, then proposition 3.3.1 is saying

$$\mathbf{sPre}(\mathbf{a}\mathcal{P}, \mathbf{a}\mathcal{P}') \cong \operatorname{ho}\mathbf{sPre}(\mathcal{P}, \mathcal{P}').$$

With all of these bijections, we move on to the main results of [FH13].

3.3.2. The main results

Now that we know how to pass from $\mathbf{sPre}$ to $\operatorname{ho}\mathbf{sPre}$, we start by introducing the relevant definitions. We have seen, in section 2.4.2, that the differential forms on a "generalized object" is just the set of natural transformations from that object and into $\Omega^\bullet$. In our newest category, the definition is completely analogous.

Let $\mathcal{F}_\bullet$ be a simplicial presheaf. The set $\Omega^\bullet(\mathcal{F}_\bullet)$ of *differential forms* on $\mathcal{F}_\bullet$ is the collection $\operatorname{ho}\mathbf{sPre}(\mathcal{F}_\bullet, \Omega^\bullet)$ of all natural transformations $\mathcal{F}_\bullet \longrightarrow \Omega^\bullet$ from $\mathcal{F}_\bullet$ to the constant simplicial sheaf $\Omega^\bullet$ built from the sheaf $\Omega^\bullet$ of differential forms. Just as the sheaves Ω^k formed the universal de Rham complex

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^k \xrightarrow{d} \cdots$$

seen in construction 2.1.7, we form the *de Rham complex* of a simplicial sheaf $\mathcal{F}_\bullet$ as

$$\operatorname{ho}\mathbf{sPre}(\mathcal{F}_\bullet, \Omega^0) \xrightarrow{d} \operatorname{ho}\mathbf{sPre}(\mathcal{F}_\bullet, \Omega^1) \xrightarrow{d} \cdots \xrightarrow{d} \operatorname{ho}\mathbf{sPre}(\mathcal{F}_\bullet, \Omega^k) \xrightarrow{d} \cdots$$

In this new de Rham complex, the arrows d are induced (under the Hom-functors $\operatorname{ho}\mathbf{sPre}(\mathcal{F}_\bullet, -)$) from the differentials $d\colon \Omega^k \longrightarrow \Omega^{k+1}$. Recall that $\mathbf{Pre}(\mathcal{F}_n, \Omega^k) = \Omega^k(\mathcal{F}_n)$ by definition. Hence, by (the first remark under) proposition 3.3.1, since Ω^k is a constant simplicial sheaf, we see that at each term $\operatorname{ho}\mathbf{sPre}(\mathcal{F}_\bullet, \Omega^k)$ in the de Rham sequence can be computed as the equalizer of

$$\Omega^k(\mathcal{F}_0) \rightrightarrows \Omega^k(\mathcal{F}_1)$$

where the two maps are the pullbacks of the first face maps of $\mathcal{F}_\bullet$.

Observe that since we are in the homotopy category of $\mathbf{sPre}$, the de Rham complex of a simplicial sheaf is invariant under weak equivalence. This was the whole point of passing to the homotopy category. Another useful observation, which is perhaps not immediately clear, but still a tautology, is

PROPOSITION 3.3.2. *Let G be a Lie group, and suppose we are given a G-action on a sheaf $\mathcal{F}$. Then the de Rham complex of the simplicial sheaf*

$$\begin{tikzcd} \mathcal{F} \arrow[r, shift left=2, leftarrow] \arrow[r, dashed] \arrow[r, shift right=2, leftarrow] & \mathcal{F}_G \times \mathcal{F} \arrow[r, shift left=3, leftarrow] \arrow[r, shift left=1, dashed] \arrow[r, shift right=1, leftarrow] \arrow[r, shift right=3, dashed] \arrow[r, shift right=5, leftarrow] & \mathcal{F}_G \times \mathcal{F}_G \times \mathcal{F} \ \cdots \end{tikzcd}$$

from example 2.4.6 is the equalizer of

$$\Omega^\bullet(\mathcal{F}) \rightrightarrows \Omega^\bullet(\mathcal{F}_G \times \mathcal{F}).$$

This is a direct consequence of our definition. And so we see that the de Rham complex of $B_\nabla^{\mathrm{triv}} G$ is equal to the equalizer of the action of G on $\Omega^1 \otimes \mathfrak{g}$. And since we are working in the homotopy category, the simplicial sheaves $B_\nabla G$ and $B_\nabla^{\mathrm{triv}} G$ have isomorphic de Rham complexes. Thus computing the de Rham complex of $B_\nabla G$ is just an exercise in finding a certain equalizer. And it turns out that this equalizer is the set of *basic*[4] differential forms on $\Omega^1 \otimes \mathfrak{g}$. And the complex of these basic forms is isomorphic to the complex $I_2^\bullet G$ of

[4]We do not discuss what this means.

Ad-invariant polynomials on $\mathfrak{g}$, graded by twice the degree. Explicitly, it is the complex

$$I^0(G) \longrightarrow 0 \longrightarrow I^1(G) \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow I^k(G) \longrightarrow 0 \longrightarrow \cdots$$

where $I^k(G)$ is the ring of invariant polynomials of degree k, discussed in section 1.5.1. In other words, we have the following:

THEOREM 3.3.3. *The de Rham complex of $B_\nabla G$ is $(I_2^\bullet(G), d = 0)$.*

This gives a new perspective on the old Chern-Weil homomorphism

$$w(E; -)\colon I^k(G) \longrightarrow H^{2k}_{dR}(M),$$

which, if we recall, sends P injectively to the (cohomology class of the) characteristic form $P(F_\nabla^k)$ corresponding to P, see definition 1.5.4. Now, $E = E_\nabla G$ and $M = B_\nabla G$. A trivial consequence of theorem 3.3.3 is that $H^{2k}_{dR}(B_\nabla G) = I_2^{2k}(G) = I^k(G)$, and hence we have the map

$$w(E_\nabla G; -)\colon I^k(G) \longrightarrow H^{2k}_{dR}(B_\nabla G) = I^k(G).$$

Thus, given an invariant polynomial P of degree k on $\mathfrak{g}$, the process of applying it to the k-fold wedge product

$$F_\nabla^k = F_\nabla \wedge \cdots \wedge F_\nabla$$

of the curvature form $F_\nabla = d\nabla + \frac{1}{2}[\nabla, \nabla]$ of the universal connection $\nabla = \nabla^{\text{univ}} \in \Omega^1(E_\nabla G; \mathfrak{g})$ gives a $2k$-form $P(F_\nabla^k)$, and this construction is local and natural.

Furthermore, this determines the amount of invariants attached to connections on principal bundles. To understand this statement, we look at a bundle $E \longrightarrow X$. This induces the "bundle" $\mathcal{F}_E \longrightarrow \mathcal{F}_X$. All the cohomology classes $w(E_\nabla G; P)$ associated to $E_\nabla G$ induce characteristic classes $w(-; P)$. And so, by theorem 3.2.1, they pull back to elements in $H^{2k}_{dR}(\mathcal{F}_X) = \operatorname{ho}\mathbf{sPre}(\mathcal{F}_X, \Omega^{2k})$ via the map $\overline{f}$. By proposition 3.3.2, the set $\operatorname{ho}\mathbf{sPre}(\mathcal{F}_X, \Omega^{2k})$ is equal to the equalizer of

$$\mathbf{Pre}(\mathcal{F}_X, \Omega^{2k}) \underset{\text{id}}{\overset{\text{id}}{\rightrightarrows}} \mathbf{Pre}(\mathcal{F}_X, \Omega^{2k}).$$

The equalizer is in turn equal to $\mathbf{Pre}(\mathcal{F}_X, \Omega^{2k})$, and we in total have that

$$\operatorname{ho}\mathbf{sPre}(\mathcal{F}_X, \Omega^{2k}) = \mathbf{Pre}(\mathcal{F}_X, \Omega^{2k}) = \Omega^{2k}(X).$$

This implies that $\overline{f}^*\big(w(E_\nabla G; P)\big) = w(E; P)$. Thus any characteristic class c defined from connections must have $c(E) = \overline{f}^*\big(c(E_\nabla G)\big)$. But since the connection preserving map $\overline{f}$ is unique, this determines the invariant c. The upshot is, as mentioned, that there is a correspondence between invariants c and cohomology classes $H^\bullet(B_\nabla G)$. Conversely, note that because the Chern-Weil homomorphism is injective, it is an isomorphism, and thus all the invariant polynomial P defines all the invariants $c = w(-; P)$ possible.

Before ending this chapter, we mention the second of the main theorems of [FH13]. Recall, from definition 1.1.4, that the Koszul complex of a vector space V is the differential graded algebra

$$\operatorname{Kos}^\bullet V = \bigwedge^\bullet V \otimes \Sigma_2^\bullet V.$$

It can be shown that the de Rham complex of the constant simplicial sheaf $\Omega^1 \otimes \mathfrak{g}$ is isomorphic to the Koszul complex of $\mathfrak{g}^\vee$. Which means we have the following:

Theorem 3.3.4. *The de Rham complex of $E_\nabla G$ is* $(\operatorname{Kos}^\bullet \mathfrak{g}^\vee, d)$

We now move on to the holomorphic case.

Part II

The Holomorphic Case

CHAPTER 4

Classifying Holomorphic Bundles

For the final chapter, we repeat the story from part I, just with holomorphic constructions. Thus, we need to know what "holomorphic" even means, and not only for maps of one complex variable. For example for complex manifolds, we need to know what is meant by a neighborhood being holomorphic to an open set of $\mathbb{C}^n$. We also need to know what complex and holomorphic differential forms are. This is what the first section is dedicated to.

Moving on, we explore what complex manifolds and complex Lie groups are. This allows us to define holomorphic vector bundles and holomorphic principal G-bundles. Finally, with all of the terminology in place, we explore what a definition of a holomorphic G-connections should be. When searching through the literature, there are several ways of going about it (see for example [Ati57] and [Bis10]). Inspired by our efforts in section 1.4, we give a slightly different definition than what is done in [Ati57]. This allows us to construct many of the same objects, like the Maurer-Cartan form, just in a holomorphic perspective. And more crucially, we can identify the holomorphic universal bundle $E_{\nabla,\mathbb{C}}G \longrightarrow B_{\nabla,\mathbb{C}}G$ with its universal connection. The novelty in this thesis is summarized in theorem 4.4.3.

4.1. Preliminaries

In this section. we recall what holomorphic and biholomorphic functions of several complex variables are. We then transfer our knowledge of tensors from section 1.1 to the complex world, and see what changes in this world. Lastly, we describe the local nature of complex manifolds.

All of the material can be found in Chapter 1 of [Huy05].

4.1.1. Local theory of complex functions

Locally, neighborhoods of smooth manifolds are diffeomorphic to open sets of real Euclidean space $\mathbb{R}^n$. We need to know what complex manifolds look like locally, and how they relate to open sets in $\mathbb{C}^n$.

The purpose of this subsection is to develop our language to describe this.

Recall that a function $f\colon U \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ of one complex variable variable is *holomorphic* if for any point z_0 in U, there exists a ball $B(\varepsilon; z_0) \subseteq U$ of radius $\varepsilon > 0$ centered at z_0 such that $f \restriction_{B(\varepsilon;z_0)}$ can be written as a convergent power series, i.e. for all $z \in B(\varepsilon; z_0)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

There are several equivalent definitions of holomorphicity. If we denote the real and imaginary parts of $z \in \mathbb{C}$ by x and y, respectively, then f can be regarded as a complex function $f(x, y)$. Hence, f can be written as the sum $f(x, y) = u(x, y) + iv(x, y)$, where u and v are functions of real numbers, equaling the real and imaginary parts of f, respectively. One can show that f is holomorphic if and only if u and v are continuously differentiable and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We call these the *Cauchy-Riemann equations.*

Recall that the complex conjugate of $z \in \mathbb{C}$ is $\overline{z} = \overline{x + iy} = x - iy$. Hence, if we introduce the the differenital operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

we see that $\frac{\partial}{\partial z}(z) = 1 = \frac{\partial}{\partial \overline{z}}(\overline{z})$, and $\frac{\partial}{\partial z}(\overline{z}) = 0 = \frac{\partial}{\partial \overline{z}}(z)$. But furthermore, we can calculate

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] + i\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]\right).$$

And thus, using the Cauchy-Riemann equations, we have the following theorem.

THEOREM 4.1.1. *A complex function f is holomorphic if and only if*

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

Inspired by this theorem, we an now define holomorphicity for complex functions of several variables.

Let $f\colon U \subseteq \mathbb{C}^n \longrightarrow \mathbb{C}$ be a complex function of several variables. If we, for all coordinates k, write $z_k = x_k + iy_k$, and introduce the analogous notation

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right), \quad \frac{\partial}{\partial \overline{z_k}} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right),$$

we can call f *holomorphic* if for all $k = 1, \ldots, n$ we have

$$\frac{\partial f}{\partial \overline{z_k}} = 0.$$

This is often written as $\overline{\partial} f = 0$, and the notation will soon[1] make more sense.

Lastly, for a function $f\colon U \subseteq \mathbb{C}^m \longrightarrow \mathbb{C}^n$, write $f = (f_1, \ldots, f_n)$. We call such a function f *holomorphic* if all $f_1, \ldots, f_n$ are holomorphic functions $U \subseteq \mathbb{C}^m \longrightarrow \mathbb{C}$. We call f *biholomorphic* if it is holomorphic and bijective, and the inverse map f^{-1} is holomorphic as well. This is the correct replacement for diffeomorphic.

We will not show it here (and the curious reader is referred to Proposition 1.1.13 in [Huy05]), but it turns out a bijective holomorphic map between open sets $U, V \subseteq \mathbb{C}^n$ are biholomorphic. This will be useful to know when we deal with complex manifolds.

4.1.2. Complex Structures

In section 1.1, we saw how to form tensor products over the real numbers $\mathbb{R}$, and studied some vector spaces related to these tensor products, e.g. the exterior powers. The construction can actually be done over any field, and hence we can form similar spaces. But there are some fundamental differences, as we will see.

For this subsection, let V be a finitie-dimensional real vector space.

Recall that an *orientation* on a finite dimensional vector space is an equivalence class on the set of all ordered bases of the vector space. A linear map $I\colon V \longrightarrow V$ such that $I^2 = -\operatorname{id}_V$ is called an *almost complex structure* on V. The most trivial example of an almost complex structure is when $V = \mathbb{C}^n$, considered as the vector space $\mathbb{R}^{2n}$ over $\mathbb{R}$. Then the $\mathbb{R}$-linear map $v \longmapsto iv$ is an almost complex structure on $\mathbb{C}^n$. And in general, if V is real vector space of an underlying complex vector space, then we can define a similar complex structure. Also note that the converse holds as well: If I is an almost complex structure on V, then V admits, in a natural way, the structure of a complex vector space. The $\mathbb{C}$-module structure on V is defined by $(a + ib)v = av + bI(v)$. Since $I^2 = -\operatorname{id}_V$ and I is linear, we get that $i(iv) = -v$. Thus, when looking at vector spaces,

[1] In definition 4.1.5.

the notions "almost complex" and "complex" are the same thing. In particular, every almost complex structure is on an even-dimensional vector space, and the vector space has a canonical orientation.

For a real vector space V, the complex vector space $V \otimes_{\mathbb{R}} \mathbb{C}$ is denoted $V_{\mathbb{C}}$. And so we see that V can be embedded into $V_{\mathbb{C}}$ via the map $v \longmapsto v \otimes 1$. This is also the part left invariant under complex conjugation (on $V_{\mathbb{C}}$), which is defined by $\overline{(v \otimes \lambda)} = v \otimes \overline{\lambda}$.

Assume that we have an almost complex structure I on V. Then we have a $\mathbb{C}$-linear extension of I on $V_{\mathbb{C}}$, defined by $v \otimes \lambda \longmapsto I(v) \otimes \lambda$. We denote this new map $V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}$ by I as well. A natural question is which eigenvalues I has. If we assume $I(v) = \lambda v$, we get that $(\lambda^2+1)v = 0$, hence the only eigenvalues of I are $+i$ and $-i$. We denote the eigenspaces of these eigenvalues by $V^{1,0}$ and $V^{0,1}$, respectively. That is,

$$V^{1,0} = \{v \in \mathbb{C} \mid I(v) = i \cdot v\}, \quad V^{0,1} = \{v \in \mathbb{C} \mid I(v) = -i \cdot v\}.$$

By basic linear algebra, we see that

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}.$$

Furthermore, complex conjugation on $V_{\mathbb{C}}$ induces an $\mathbb{R}$-linear isomorphism $V^{1,0} \cong V^{0,1}$.

Again, by elementary linear algebra, it is not too hard to see that the induced linear map on the dual space $V^{\vee}$ has the same eigenvalues $+i, -i$, and that we get a similar decomposition of $V^{\vee}$ into

$$(V^{\vee})^{1,0} = \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f\big(I(v)\big) = i \cdot f(v)\} = (V^{1,0})^{\vee},$$
$$(V^{\vee})^{0,1} = \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f\big(I(v)\big) = -i \cdot f(v)\} = (V^{0,1})^{\vee}.$$

Recall, from section 1.1.3, that given a real vector space V one can form the exterior algebra $\bigwedge V$, which decomposes as

$$\bigwedge V = \bigoplus_{k=0}^{\infty} \bigwedge^k V.$$

Similarly, we define the exterior algebra $\bigwedge V_{\mathbb{C}}$ of the complex vector space $V_{\mathbb{C}}$ as the algebra which decomposes as

$$\bigwedge V_{\mathbb{C}} = \bigoplus_{k=0}^{\infty} \bigwedge^k V_{\mathbb{C}}.$$

Furthermore, we can see that $\bigwedge V_{\mathbb{C}} = \big(\bigwedge V\big) \otimes_{\mathbb{R}} \mathbb{C}$, and so $\bigwedge V$ is the subspace of $\bigwedge V_{\mathbb{C}}$ which is left invariant under complex conjugation.

Note that if V has an almost complex structure I, then its dimension d is even, say $d = 2n$. And $V_{\mathbb{C}}$ decomposes into $V^{1,0} \oplus V^{0,1}$. Both of these can be regarded as complex vector spaces of dimension n. We define

$$\bigwedge^{p,q} V = \bigwedge^{p} V^{1,0} \otimes_{\mathbb{C}} \bigwedge^{q} V^{0,1},$$

where $V^{1,0}$ and $V^{0,1}$ are considered complex vector spaces, and the exterior products are taken as complex vector spaces. We note that we have the following results.

Proposition 4.1.2. *For a real vector space V endowed with an almost complex structure I one has:*

(1) $\bigwedge^{p,q} V$ is (canonically) a subspace of $\bigwedge^{p+q} V_{\mathbb{C}}$;
(2) $\bigwedge^{k} V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V$; and
(3) $\bigwedge^{p,q} V \cong \bigwedge^{q,p} V$.

Proof. See Proposition 1.2.8 in [Huy05]. □

With these relations of vector spaces, we define the projections

$$\Pi^k \colon \bigwedge^{*} V_{\mathbb{C}} \longrightarrow \bigwedge^{k} V_{\mathbb{C}}, \quad \Pi^{p,q} \colon \bigwedge^{*} V_{\mathbb{C}} \longrightarrow \bigwedge^{p,q} V.$$

Now we are ready to study complex differential forms.

4.1.3. Local tangent spaces of complex manifolds

A smooth manifold M can be studied through its tangent bundle TM, the collection of all tangent spaces T_pM for $p \in M$. But it can also be studied by means of $\bigwedge^k T^*M$, as we saw in section 1.3.1. In this section, we let $M = U \subseteq \mathbb{C}^n$ be some open subset, considered both as a neighborhood of $\mathbb{C}^n$, and a smooth manifold of dimension $2n$. Thus, for $x \in U$, there is a real tangent space T_xU of tangent vectors at x. If $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2, \ldots, z_n = x_n + iy_n$ are the standard coordinates on $\mathbb{C}^n$, there is a canonical basis of T_xU, namely

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}.$$

The dual basis of $(T_xU)^\vee$ is denoted by $dx_1, \ldots, dx_n, dy_1, \ldots, dy_n$.

Each of these tangent spaces has a natural almost complex structure, defined by

$$I_x \colon T_xU \longrightarrow T_xU, \qquad \frac{\partial}{\partial x_k} \longmapsto \frac{\partial}{\partial y_k}, \quad \frac{\partial}{\partial y_k} \longmapsto -\frac{\partial}{\partial x_k}.$$

Just as we can complexify a real vector space, we can turn T_xU into a complex vector space as well. If we use $\mathbb{C}$ to denote the smooth trivial vector bundle $U \times \mathbb{C}$ (which is equal to $U \times \mathbb{R}^2$), we see that we can complexify the whole tangent bundle TU. This new tangent bundle, denoted $T_{\mathbb{C}}U$, is bundle tensor product

$$T_{\mathbb{C}}U = TU \otimes \mathbb{C},$$

and we call it the *complexified tangent bundle*. Using the observations from the previous subsection, we note that the complexified tangent bundle $T_{\mathbb{C}}U$ globally decomposes into a direct sum of complex vector bundles

$$T_{\mathbb{C}}U = T^{1,0}U \oplus T^{0,1}U.$$

We call the $(1,0)$-part $T^{1,0}U$ the *holomorphic tangent bundle* over U. The complexified cotangent bundle $T^*_{\mathbb{C}}U = T^*U \otimes \mathbb{C}$ has a similar decomposition, and its $(1,0)$-part is called the *holomorphic cotangent bundle* over U.

Let $f\colon U \longrightarrow V$ be a holomorphic map between subsets $U \subseteq \mathbb{C}^m, V \subseteq \mathbb{C}^n$. The differential $f_{*,x}\colon T_xU \longrightarrow T_{f(x)}V$ extends to a $\mathbb{C}$-linear map $f_{*,x}\colon T_xU \otimes \mathbb{C} \longrightarrow T_xV \otimes \mathbb{C}$, and this extension respects the decomposition. Thus we have the following definition.

Definition 4.1.3. Let $U \subseteq \mathbb{C}^n$ be an open set. Then we have the complex vector bundles

$$\bigwedge^k T^*_{\mathbb{C}}U, \quad \bigwedge^{p,q} T^*U = \left(\bigwedge^p (T^*U)^{1,0}\right) \otimes \left(\bigwedge^q (T^*U)^{0,1}\right),$$

over U. The space of sections of these bundles are denoted $\Omega^k_{\mathbb{C}}(U)$ and $\Omega^{p,q}(U)$, respectively. We call these sections for *complex differential forms*.

A natural consequence of proposition 4.1.2 is

Corollary 4.1.4. *For an open set $U \subseteq \mathbb{C}^n$, there are natural decompositions*

$$\bigwedge^k T^*_{\mathbb{C}}U = \bigoplus_{p+q=k} \bigwedge^{p,q} T^*U, \quad \Omega^k(U;\mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(U).$$

As before, we have a natural projection map

$$\Pi^{p,q}\colon \Omega^k_{\mathbb{C}}(U) \to \Omega^{p,q}(U).$$

Definition 4.1.5. Let $d\colon \Omega^k_{\mathbb{C}}(U) \longrightarrow \Omega^k_{\mathbb{C}}(U)$ be the $\mathbb{C}$-linear extension of the usual exterior derivative $d\colon \Omega^k(U) \longrightarrow \Omega^k(U)$ from definition 1.3.3. Then we define

$$\partial\colon \Omega^{p,q}(U) \longrightarrow \Omega^{p+1,q}(U), \quad \overline{\partial}\colon \Omega^{p,q}(U) \longrightarrow \Omega^{p,q+1}(U),$$

where $\partial = \Pi^{p+1,q} \circ d$ and $\overline{\partial} = \Pi^{p,q+1} \circ d$. (This makes sense as $\Omega^{p,q}(U) \subseteq \Omega^{p+q}_{\mathbb{C}}(U)$.)

Just as we, for $f \in C^\infty(M, \mathbb{R})$, in the smooth case had that $df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$, we similarly, for $f = C^\infty(U, \mathbb{C})$, get that

$$\begin{aligned} df &= \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n + \frac{\partial f}{\partial y_1} dy_1 + \cdots + \frac{\partial f}{\partial y_n} dy_n \\ &= \sum_{k=1}^n \frac{\partial f}{\partial x_k} dx_k + \sum_{k=1}^n \frac{\partial f}{\partial y_k} dy_k = \sum_{k=1}^n \frac{\partial f}{\partial z_k} dz_k + \sum_{k=1}^n \frac{\partial f}{\partial \overline{z_k}} d\overline{z_k}. \end{aligned}$$

And so we see the meaning of the notation $\overline{\partial} f = 0$ of a holomorphic function $f \colon \mathbb{C}^n \longrightarrow \mathbb{C}$ from section 4.1.1. Moreover, the complex extension of d respects the decomposition, i.e.

$$d(\Omega^{p+q}(U)) = \Omega^{p+1,q}(U) \oplus \Omega^{p,q+1}(U) \subseteq \Omega^{p+q+1}(U).$$

Thus, it is not hard to see that

(1) $d = \partial + \bar{\partial}$;
(2) $\partial^2 = 0 = \bar{\partial}^2$; and
(3) $\partial\bar{\partial} = -\bar{\partial}\partial$.

The operators ∂ and $\overline{\partial}$ also satisfy the Leibniz rule. This means that we have a a web

$$\begin{tikzcd}
 & \vdots \arrow[d, "\bar\partial"] & \vdots \arrow[d, "\bar\partial"] & \vdots \arrow[d, "\bar\partial"] & \\
\cdots \arrow[r, "\partial"] & \Omega^{p-1,q-1}(U) \arrow[r, "\partial"] \arrow[d, "\bar\partial"] & \Omega^{p,q-1}(U) \arrow[r, "\partial"] \arrow[d, "\bar\partial"] & \Omega^{p+1,q-1}(U) \arrow[r, "\partial"] \arrow[d, "\bar\partial"] & \cdots \\
\cdots \arrow[r, "\partial"] & \Omega^{p-1,q}(U) \arrow[r, "\partial"] \arrow[d, "\bar\partial"] & \Omega^{p,q}(U) \arrow[r, "\partial"] \arrow[d, "\bar\partial"] & \Omega^{p+1,q}(U) \arrow[r, "\partial"] \arrow[d, "\bar\partial"] & \cdots \\
\cdots \arrow[r, "\partial"] & \Omega^{p-1,q+1}(U) \arrow[r, "\partial"] \arrow[d, "\bar\partial"] & \Omega^{p,q+1}(U) \arrow[r, "\partial"] \arrow[d, "\bar\partial"] & \Omega^{p+1,q+1}(U) \arrow[r, "\partial"] \arrow[d, "\bar\partial"] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

of complexes. And in particular, the complex $\Omega^{\bullet,0}$ has only holomorphic sections. For example, the holomorphic cotangent bundle $\Omega^{1,0}(U)$

has complex 1-forms df such that

$$df = \sum_{k=1}^{n} \frac{\partial f}{\partial z_k} dz_k + \underbrace{\sum_{k=1}^{n} \frac{\partial f}{\partial \overline{z_k}} d\overline{z_k}}_{=0} = \sum_{k=1}^{n} \frac{\partial f}{\partial z_k} dz_k.$$

Thus, since $\overline{\partial} f = 0$, all the sections are holomorphic. And in general, for any $\Omega^{p,0}(U)$, we force the $\overline{\partial}$-component of any complex differential form to be 0, which is how we defined holomorphicity in the first place. So we define any section in $\Omega^{p,0}(U)$ as a *holomorphic p-form* on U.

4.2. Holomorphic bundles

In this section, we see how to transport all the theory from section 1.2 to the holomorphic world. This includes complex manifolds, complex Lie groups, holomorphic local trivializations, holomorphic vector bundles, and holomorphic principal bundles.

The definitions and results can be found in [Huy05] and [Ati57].

4.2.1. Complex manifolds

Recall, from section 1.2.1, that an atlas A on a topological manifold X is just a collection of charts $(U_\alpha, \varphi_\alpha)$ that cover X. Just as an atlas is called smooth if all the transition maps are smooth maps on $\mathbb{R}^n$, we call an atlas *holomorphic* if all the transition maps are holomorphic maps on $\mathbb{C}^n$. A union $A \cup A'$ of two holomorphic atlases is not always holomorphic, but when it is, we call the two atlases *holomorphicly equivalent*. This relation is an equivalence relation of atlases. Thus, we call such an equivalence class $\mathscr{A} = \{A, A', \dots\}$ of holomorphicly equivalent atlases a *holomorphic structure*.

A *complex manifold* X of dimension n is a smooth manifold of (real) dimension $2n$ with a holomorphic structure. We can, completely analogous to the differential case, define a *maximal holomorphic atlas* $\mathscr{A}_{\max}$ on a complex manifold X. This atlas is just the union of all the holomorphic atlases in the holomorphic structure, and hence, by definition, it is a holomorphic atlas on X. When we talk about a holomorphic chart, we always assume it is chosen from this maximal atlas.

We call a map $F\colon X_1 \longrightarrow X_2$ between complex manifolds X_1 and X_2 *holomorphic* if for every holomorphic chart on X_1 and X_2, the induced map between open sets of complex vector spaces is holomorphic. Since bijective holomorphic maps between open sets are biholomorphic, we say $F\colon X_1 \longrightarrow X_2$ is *biholomorphic* if it is a holomorphic homeomorphism. Thus, all the homeomorphisms in $\mathscr{A}_{\max}$ are biholomorphic. The identity map $\mathrm{id}_X : X \longrightarrow X$ is also biholomorphic. And as the composition of holomorphic maps is holomorphic, we have a category $\mathbf{Man}_{\mathbb{C}}$ of finite dimensional smooth manifolds.

4.2.2. Holomorphic vector bundles

Recall that a smooth fibre bundle is a quadruple (E, X, π, F) where E, X, F are smooth manifolds and $\pi\colon E \longrightarrow X$ is a smooth surjection which has a local trivialization with fiber F. If we require the manifolds E, X, F to be complex, and the map π holomorphic, we call the fibre bundle a *holomorphic fibre bundle* if, in addition, all the fiber preserving diffeomorphisms $\varphi\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$ of the local trivialization are homeomorphic as well.

DEFINITION 4.2.1. An *n-dimensional holomorphic vector bundle*, denoted $\pi\colon V \longrightarrow M$, is a holomorphic fibre bundle $(E, X, \pi, \mathbb{C}^n)$ with the follow extra condition:

- Each fibre preserving map $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n$ restricts to an isomorphism $\varphi_\alpha \restriction_{V_p}$ of vector spaces $V_p = \pi^{-1}(\{p\}) \longrightarrow \{p\} \times \mathbb{C}^n \cong \mathbb{C}^n$.

As any complex manifold is a smooth manifold, any holomorphic vector bundle is in particular also a smooth vector bundle. But holomorphic vector bundles should not be confused with *complex vector bundles*, as complex vector bundles are just smooth vector bundles whose fibres are complex vector spaces with $\mathbb{C}$-linear transition maps.

Most of the examples we have of smooth vector bundles have holomorphic analogues. This includes the trivial bundle, the tangent bundle and the pullback bundle. The notion of a map is also completely parallel. This means that a *holomorphic vector bundle map* is a smooth fibre bundle map that is holomorphic, and the restriction to fibres gives a $\mathbb{C}$-linear map. An *isomorphism* is a biholomorphic map covering the identity.

Just as we had meta theorem 1.2.14 for smooth vector bundles, we have

META-THEOREM 4.2.2. *Any canonical construction in linear algebra gives rise to a geometric version for holomorphic vector bundles.*

We do not bother listing examples, as it would be almost an exact copy of the list in example 1.2.15.

4.2.3. Holomorphic principal bundles

Before defining holomorphic principal G-bundles, we need to know what complex Lie groups are. But as one might expect, they are analogously defined to real Lie groups. A *complex Lie group* is a group that is also a complex manifold, and the equipped maps $* \longmapsto e, g \longmapsto g^{-1}$, and $(g, h) \longmapsto gh$ are holomorphic.

DEFINITION 4.2.3. Let G be a complex Lie group. A *holomorphic principal G-bundle* $\pi\colon E \longrightarrow M$ is a holomorphic fibre bundle (E, M, π, G) with the following extra conditions:

- E is a G-space such that the right action is holomorphic, and for each point $p \in M$ the action restricted to each space $E_p = \pi^{-1}(p)$ is regular; and
- each fibre preserving map $\phi_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times G$ restricts to a G-equivariant and holomorphic map $\phi_\alpha \restriction_{E_p}$, i.e. for all $g \in G$, and for all $x \in \pi^{-1}(U_\alpha)$, we have
$$\phi_\alpha(x \cdot g) = \phi_\alpha(x) \cdot g,$$
where the action $\phi_\alpha(x) \cdot g$ is just group multiplication on the right coordinate in $U_\alpha \times G$.

We do not construct any examples here, but quickly mention that the trivial bundle and the pullback bundle also come in holomorphic versions.

A *holomorphic principal bundle map* is just a fibre bundle map which is holomorphic and such that the map of total spaces is equivariant. A *holomorphic principal bundle isomorphism* is a biholomorphic equivariant map covering the identity such that the inverse is also G-equivariant.

4.3. Holomorphic connections on holomorphic bundles

Now that we know what holomorphic bundles are, we need a notion of holomorphic principal connection. It is not immediately clear how one defines this, so we use look at the simpler case of holomorphic linear connections first.

4.3.1. Connections on holomorphic vector bundles

As holomorphic vector bundles are smooth vector bundles, we do not necessarily need to define *holomorphic* connections on holomorphic vector bundles, as we have a perfect notion of linear connections on smooth vector bundles. Or, we could even define holomorphic linear connections as the linear connections $\nabla\colon \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$ that satisfy the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla(s),$$

(from section 1.4.2) and are also compatible with the holomorphic structure. And these are interesting in their own right. But, as Serre showed in his seminal paper, colloquially referred to as *GAGA*[2] ([Ser56]), purely holomorphic constructions carry over to the algebraic setting. Hence we might expect a more applicable notion if we restrict to the purely holomorphic setting as well. As explained in [Huy05] (p. 179), this choice lets us get a purely algebraic definition, which now follows.

DEFINITION 4.3.1. Let $E \to X$ be a holomorphic vector bundle on a complex manifold X. A *holomorphic linear connection* on E is a $\mathbb{C}$-linear map (of sheaves) $\nabla\colon \Gamma(E) \to \Gamma(\Omega_X \otimes E)$ with

$$\nabla(fs) = \partial f \otimes s + f\nabla s,$$

for any local holomorphic function f on X and any local holomorphic section s of E.

We see this mimics the linear connections on smooth vector bundles, as $d = \partial + \overline{\partial}$, and thus for a holomorphic function, we have $df = \partial f$.

Every smooth vector bundle admits a smooth linear connection. But not every holomorphic vector bundle admits a holomorphic connection. The degree of failure is measured by the Atiyah class, which is an element

$$A(E) \in H^1\big(X; \Omega^{1,0} \otimes \operatorname{End}(E)\big),$$

in the Čeck cohomology of the tensor product bundle $\Omega^{1,0} \otimes \operatorname{End}(E)$. Explicitly, it is given by the Čeck cocycle

$$A(E) = \{U_{ij}, \varphi_j^{-1} \circ (\varphi_{ij}^{-1} d\varphi_{ij}) \circ \varphi_j\}.$$

Proposition 4.2.19 in [Huy05] states that a holomorphic vector bundle has a holomorphic connection if and only if the Atiyah class is trivial. We have a similar result for holomorphic connections on holomorphic principal bundles.

[2] Short for *Géométrie algébrique et géométrie analytique*, which translates to *Algebraic geometry and analytic geometry*.

4.3.2. Connections on holomorphic principal bundles

From section 1.4, we know that connections come from horizontal distributions. Meaning a connection is equivalent to a splitting of the short exact sequence

$$0 \longrightarrow VE \longrightarrow TE \longrightarrow HE \longrightarrow 0$$

of vector bundles. And from the previous subsection, we saw that we should define holomorphic connections as operators not only *compatible* with the holomorphic structure, but holomorphic operators in their own right. However, when settling for a definition of a *holomorphic G-connection* on a holomorphic principal G-bundle, it is usually done in a complicated manner. For example, in [Ati57], a *holomorphic G-connection* is defined as a splitting of the short exact[3] sequence

$$0 \longrightarrow \mathrm{Ad}(E) \longrightarrow \mathrm{At}(E) \xrightarrow{\pi_*} TX \longrightarrow 0$$

now know as the *Atiyah sequence*. The middle bundle $\mathrm{At}(E)$ is called the *Atiyah bundle*, and is just a fancy way of writing TE/G. The action on TE comes from the differential of the action on E, hence we can sensibly talk about the quotient TE/G. As the projection map $TE/G \longrightarrow E/G$ is G-equivariant, and $E/G = X$, the Atiyah bundle is a bundle over X. The *adjoint bundle* $\mathrm{Ad}(E)$ over X is, as the name implies, just the quotient bundle $E \times \mathfrak{g}/G$, where the action on $E \times \mathfrak{g}$ is by G and defined, for any $g \in G$ as

$$(x, X) \cdot g = \big(xg, \mathrm{Ad}_g(X)\big).$$

In other words, $\mathrm{Ad}(E) = E \times_{\mathrm{Ad}} \mathfrak{g}$. Theorem 2 in [Ati57] states that a holomorphic principal bundle has a holomorphic bundle if and only if the Atiyah class (which is defined as a certain element $A(E) \in H^1(X; \Omega^{1,0} \otimes \mathrm{End}(E))$ analogously to the vector bundle case) vanishes.

As we see, these two sequences barely resemble one another. In particular, a splitting of the Atiyah sequence does not immediately yield a $\mathfrak{g}$-valued differential form on E. And hence, we can not use our constructions from part I. But we can sidestep this problem if we look at how the Atiyah sequence is constructed. The sequence is actually the quotient of the following short exact sequence (over E and not X):

$$0 \longrightarrow VE \longrightarrow TE \xrightarrow{\pi_*} \pi^* TX \longrightarrow 0$$

The quotient ensures that any splitting is G-equivariant, and changes the base space from E to X. But we, as seen in section 1.4.3, can

[3] It is exact by Theorem 1 in [Ati57].

force the equivariance by other means than quotients, using the right multiplication map $R_g\colon E \longrightarrow E$. And so we define a *holomorphic G-connection* as a splitting of

$$0 \longrightarrow V_xE \longrightarrow T_xE \xrightarrow{\pi_*} (\pi^*TX)_x \longrightarrow 0$$

such that it is holomorphic and equivariant in x. Concretely this means that we have a map $\nabla_x\colon T_xE \longrightarrow V_xE$ that varies holomorphically with x, and such that the pullback $R_g^*\nabla = \mathrm{Ad}_{g^{-1}} \circ \nabla$. As V_xE is isomorphic to $\mathfrak{g}$, we see that such a connection ∇ really is a $\mathfrak{g}$-valued differential 1-form on E.

These types of connections are also preserved by pullback. Meaning for two holomorphic principal G-bundles E_1 and E_2, with a holomorphic bundle map $\varphi\colon E_1 \longrightarrow E_2$, then a connection ∇ on E_2 induces a connection $\varphi^*\nabla$ on E_1. In addition, for exactly the same reasons as in the smooth case, we can construct the *holomorphic Maurer-Cartan form* on any trivial holomorphic G-bundle $X \times G$. A fundamental difference from the smooth case is that not all holomorphic principal G-bundles have holomorphic connections. This is because we do not have (an appropriate analogue of) partition of unity.

4.4. Generalized complex manifolds

In the last section of this thesis, we use all of the theory developed throughout this thesis, and construct the universal holomorphic bundle with universal connection. We end with an original result, namely theorem 4.4.3.

4.4.1. The universal holomorphic bundle

This subsection summarizes how to get from complex manifolds to simplicial presheaves on complex manifolds, and serves the same role as chapter 2 did to part I. We also explicitly construct the holomorphic parallels of $B_\nabla G$ and $E_\nabla G$. Most of the analogue definitions and examples, like complex differential forms on presheaves, are omitted here, but used later. The reason we exclude them is to simplify the text, as all unmentioned definitions are of lesser importance and also almost exact copies of the corresponding ones from part I.

A *presheaf on complex manifolds* is a functor

$$\mathcal{F}\colon \mathbf{Man}_{\mathbb{C}}^{\mathrm{op}} \longrightarrow \mathbf{Set}\,.$$

Still, the standard example is the *associated presheaf* $\mathcal{F}_X$ of a complex manifold X, which of course is the Hom-functor $\mathbf{Man}_{\mathbb{C}}(-, X)$. Maps of presheaves on complex manifolds are natural transformations, so we get a category $\mathbf{Pre}_{\mathbb{C}}$ of presheaves on complex manifolds, and the Yoneda embedding ensures that it makes sense to talk about holomorphic maps of presheaves. Sheaves are, yet again, just presheaves that satisfy the equalizer definition. We have the same sheafification functor $\mathbf{a}\colon \mathbf{Pre}_{\mathbb{C}} \longrightarrow \mathbf{Sh}$.

A *simplicial presheaf on complex manifolds* is a functor

$$\mathcal{F}_{\bullet}\colon \mathbf{Man}_{\mathbb{C}}^{\mathrm{op}} \longrightarrow \mathbf{Set}_{\Delta},$$

and maps of simplicial presheaves are natural transformations. This determines the category $\mathbf{sPre}_{\mathbb{C}}$. Weak equivalences of simplicial presheaves on complex manifolds are defined as the maps which induce weak equivalences of simplicial sets on all the stalks.

We now construct the complex analogues of $B_{\nabla}G$ and $E_{\nabla}G$.

Construction 4.4.1 ($B_{\nabla,\mathbb{C}}G$). Just as $B_{\nabla}G(M)$ is the simplicial set determined from the groupoid $G\,\mathbf{Bund}_{\nabla}(M)$, we have a category $G\,\mathbf{Bund}_{\nabla,\mathbb{C}}(X)$ of holomorphic principal G-bundles with connection. The arrows are holomorphic principal G-bundles which preserve connection. This groupoid determines a simplicial set

$$B_{\nabla,\mathbb{C}}G(M)_0 \;\substack{\longleftarrow\\ \dashrightarrow\\ \longleftarrow}\; B_{\nabla,\mathbb{C}}G(M)_1 \;\substack{\longleftarrow\\ \dashrightarrow\\ \longleftarrow\\ \dashrightarrow\\ \longleftarrow}\; \cdots$$

which we denote by $B_{\nabla,\mathbb{C}}G(M)$. The construction is functorial, and we thus have a simplicial sheaf

$$B_{\nabla}G\colon \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Set}_{\Delta}\,.$$

One might worry that since we have no partition of unity in the holomorphic case, not every holomorphic principal G-bundle has a connection, and perhaps there might exist some values of $B_{\nabla,\mathbb{C}}G$ such that $B_{\nabla,\mathbb{C}}G(M) = \varnothing$. But as we always can construct the trivial bundle $M \times G$ over M, and this can be equipped with the holomorphic Maurer-Cartan connection, we should not have any worries.

Construction 4.4.2 ($E_{\nabla,\mathbb{C}}G$). Again, the construction is completely paralell to that of the smooth world. The category $G\,\mathbf{Bund}^{\mathrm{triv}}_{\nabla,\mathbb{C}}(M)$ of holomorphic and trivial principal G-bundles is a groupoid. The morphisms in this category are morphisms in $G\,\mathbf{Bund}_{\nabla,\mathbb{C}}(M)$ that also preserve trivializations. This groupoid determines the simplicial set

$$E_{\nabla,\mathbb{C}}G(M)_0 \;\substack{\longleftarrow\\ \dashrightarrow\\ \longleftarrow}\; E_{\nabla,\mathbb{C}}G(M)_1 \;\substack{\longleftarrow\\ \dashrightarrow\\ \longleftarrow\\ \dashrightarrow\\ \longleftarrow}\; \cdots$$

and we of course denote it by $E_{\nabla,\mathbb{C}}G(M)$. Thus we have the simplicial sheaf $E_{\nabla,\mathbb{C}}G$.

4.4.2. The classification theorem

Now that we have constructed the holomorphic universal bundle, we need to find a universal connection. Just as in the differential case, it will be the weak equivalence $\nabla^{\text{univ}} = \psi$, which, component wise, looks like

$$\psi_M \colon E_{\nabla.\mathbb{C}}G(M) \longrightarrow \Omega^{1,0} \otimes \mathfrak{g}(M), \quad (E, \pi, \nabla, s) \longmapsto s^*\nabla,$$

from proposition 3.1.3. Here we see our first subtle difference compared to the differential case. We have to pick the constant simplicial sheaf coming from the holomorphic cotangent bundle $\Omega^{1,0}$ instead of the complexified tangent bundle $\Omega^1_{\mathbb{C}}$ because if not, then the pullback $s^*\nabla$ could be a connection which is only compatible with the holomorphic structure, and not a holomorphic connection. Still, to prove that ψ is a weak equivalence would result in an almost word-for-word copy of the proof of proposition 3.1.3, so we omit it.

Just as we had a well defined smooth action on $\Omega^1 \otimes \mathfrak{g}$, we have a holomorphic action on the sheaf $\Omega^{1,0}\otimes\mathfrak{g}$, defined completely analogous. We do not prove this fact, but a corollary is that $B_{\nabla,\mathbb{C}}G$ is weakly equivalent to the simplicial sheaf induced from this action. The latter sheaf is denoted $B^{\text{triv}}_{\nabla,\mathbb{C}}G$. And so we see that $\Omega^{1,0}\otimes\mathfrak{g}/G$ is $B^{\text{triv}}_{\nabla,\mathbb{C}}G$. Thus, in the homotopy category $\operatorname{ho}\mathbf{sPre}_{\mathbb{C}}$ it makes sense to talk about the universal bundle $E_{\nabla,\mathbb{C}}G \longrightarrow B_{\nabla,\mathbb{C}}G$.

And we have the following:

THEOREM 4.4.3. *Let $\pi\colon E \longrightarrow M$ be a holomorphic principal G-bundle with a holomorphic G-connection $\nabla \in \Omega^1(E;\mathfrak{g})$, and denote the induced discrete simplicial sheaves by $\mathcal{F}_E$ and $\mathcal{F}_M$ respectively. Then there is a unique classifying map*

$$\begin{array}{ccc}
\mathcal{F}_E & \xrightarrow{f} & E_\nabla G \\
\big\downarrow & & \big\downarrow \\
\mathcal{F}_X & \xrightarrow{\bar{f}} & B_\nabla G
\end{array}$$

such that $f^*(\nabla^{univ}) = \nabla$.

To conclude, we should note that the holomorphic version of theorem 3.2.1 is slightly weaker than its differential sibling. This is because we assumed that the holomorphic bundle was equipped with a holomorphic connection. We cannot find the classification maps if this is not the case (because if we could, then all holomorphic bundles would have a connection). As mentioned, this is not always the case, as the Atiyah class is not necessarily 0. But for a great deal of complex manifolds, it actually vanishes. For example, as mentioned in [Ati57], all Stein manifolds have trivial Atiyah class. As a fact check, we can also notice that the Atiyah class $A(E_{\nabla,\mathbb{C}}G)$ of the classifying space $E_{\nabla,\mathbb{C}}G$ is 0 because $\mathrm{End}(E_{\nabla,\mathbb{C}}G)$ is trivial, which follows from the fact that it is a discrete groupoid. So it makes sense that $E_{\nabla,\mathbb{C}}G$ has a holomorphic connection.

To proceed from here, we should probably study the relationship between Atiyah's definition of a holomorphic connection, and the one used in this thesis, as it is not immediately apparent if they always are equivalent (although we do not rule out the possibility). Sadly, we did not find enough time to do this, but it is certainly worth doing if one wants to explore the subject further.

APPENDIX A

Manifolds and Tangent Spaces

A.1. Smooth manifolds

Unless otherwise stated, $\mathbb{R}^n$ will always be assumed equipped with the standard topology, i.e. $U \in \mathscr{T}_{\mathbb{R}^n}$ is open if $U = \bigcup B_r(x)$.

The prototypical examples of manifolds are the surfaces, like the sphere and the torus, and curves, like the unit interval and the circle, in $\mathbb{R}^3$. The generalized notion of this kind of space is a manifold.

All the material can be found in [Lee13] and [Tu17], we do not prove every proposition. We follow the notation used by Lee. This means in particular that the differential of a map f is denoted df instead of f_*, which differs from what is used in the main matter of this thesis.

A.1.1. Topological manifolds and atlases

Definition A.1.1. An *n-dimensional topological manifold* (or *n-manifold*) is a topological space $(M, \mathscr{T})$ with the following properties:

(1) $(M, \mathscr{T})$ is a Hausdorff space,
(2) There exist a second-countable basis for $(M, \mathscr{T})$, and
(3) M is locally Euclidean of dimension n, i.e. for any point $p \in M$ there exists a neighborhood $U \in \mathscr{T}$ of p, an open set $V \subseteq \mathbb{R}^n$ and a homeomorphism $\varphi : U \longrightarrow V = \varphi(U)$.

Definition A.1.2. Let $(M, \mathscr{T})$ be an n-dimensional manifold. The pair (U, φ), where $U \in \mathscr{T}$ and $\varphi : U \longrightarrow V$ is called a *chart*. For a point p in the manifold, the tuple $\varphi(p) = (\varphi_1(p), \ldots, \varphi_n(p))$ is called the *coordinates of* p. An *atlas* for the manifold is a collection $A = \{(U_i, \varphi_i)\}_{i\in\Lambda}$ of charts such that

$$\bigcup_{i\in\Lambda} U_\alpha = M.$$

If (U_i, φ_i) and (U_j, φ_j) are two charts in the atlas, we call the composite map $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi(U_i \cap U_j)$ the *transition map* from φ_j to φ_i.

A.1.2. Smooth manifolds and maps

DEFINITION A.1.3. Given two charts (U_i, φ_i) and (U_j, φ_j) in an atlas A, we call them *smoothly compatible* if either

(1) $U \cap V = \varnothing$; or
(2) the transition map is a smooth, i.e. $\varphi_{ij} \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

The atlas A is called *smooth* if all maps are smoothly compatible.

Given two smooth atlases A and A' of a manifold M, then the union $A \cup A'$ is an atlas of M. If this is smooth, we say that the atlases are *smoothly equivalent.*

PROPOSITION A.1.4. *Given a manifold M, the notion of smoothly equivalent atlases is an equivalence relation on the set of smooth atlases.*

DEFINITION A.1.5. A *smooth structure on M* is an equivalnece class $\mathscr{A}$ of smooth atlases on M

DEFINITION A.1.6. An *n-dimensional smooth manifold* $(M, \mathscr{T}, \mathscr{A})$ is a collection of data, where

(1) $(M, \mathscr{T})$ is a topological manifold of dimension n,
(2) $\mathscr{A}$ is an smooth structure on the manifold.

We usually omit mentioning $\mathscr{T}$ and $\mathscr{A}$ when these specifications are not important, saying only M, and not the tuple $(M, \mathscr{T}, \mathscr{A})$, is smooth.

There are, in general, many smooth structures to give to any manifold. For example, $\mathbb{R}^n$ has been equipped with the *standard smooth structure* used in standard analysis courses.

If we have some map f between smooth manifolds M and N, we would like to have a description of smoothness for the map as well, taking into account the smooth structures of each manifold. We can lend our description of smoothness in Euclidean space to the abstract manifold using the charts.

DEFINITION A.1.7. Let M and N be smooth manifolds, and $f\colon M \longrightarrow N$ a continuous map. We say *f is smooth at $m \in M$* if there exits charts $\varphi\colon U_1 \longrightarrow V_1$ and $\psi\colon U_2 \longrightarrow V_2$ on M and N respectively, where $x \in U_1$ and $f(x) \in U_2$, such that the dashed arrow

$$\begin{tikzcd}
U_1 \arrow[r, "f"] & U_2 \arrow[d, "\psi"] \\
\varphi\big(f^{-1}(U_2)\big) \arrow[u, "\varphi^{-1}"] \arrow[r, dashed, "\psi\circ f\circ\varphi^{-1}"] & V_2
\end{tikzcd}$$

is a smooth map between Euclidean spaces. The collection of all smooth maps $f\colon M \longrightarrow N$ is denoted $C^\infty(M,N)$.

If, for all $m \in M$, we have that f is smooth at x, we say f *is smooth*. If f is smooth, bijective and has a smooth inverse, then we call f a *diffeomorphism*, and we say that the smooth manifolds M and N are *diffeomorphic*.

As soon as we have chosen a smooth structure $\mathscr{A}$ on M, we know which maps on M are smooth. More specifically, we know when a chart map $U \subseteq M \longrightarrow V \subseteq \mathbb{R}^n$ is a diffeomorphism. We can therefore define a new atlas $\mathscr{A}_{\max}$, called the *maximal atlas associated to* $\mathscr{A}$, which is the atlas

$$\mathscr{A}_{\max} = \{f\colon U \longrightarrow V \mid U \in \mathscr{T}_M, V \in \mathscr{T}_{\mathbb{R}^n}, f \in C^\infty(U,V)\}.$$

This is an atlas because diffeomorphisms are homeomorphisms (see Proposition 2.4 in [Lee13]), and it is smooth because all the maps are smooth. From now on, a chart will mean a chart in the maximal atlas.

Smooth manifolds and smooth maps indeed form a category, usually denoted **Man**:

We also have the following:

PROPOSITION A.1.8. *The constant map and the inclusion map is smooth.*

A.1.3. Partition of unity

DEFINITION A.1.9. Let M be a topological space, and $\mathscr{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ an open cover of M. Then a *partition of unity subordinate to* M is an indexed family $\{\psi_\alpha\}_{\alpha\in\Lambda}$ of continuous functions $\psi\colon M \longrightarrow \mathbb{R}$ with the following properties:

(1) For each $\alpha \in \Lambda$, and each $p \in M$, we have $0 \le \psi_\alpha(p) \le 1$;
(2) For each $\alpha \in \Lambda$, we have $\operatorname{supp}(\psi_\alpha) \subseteq U_\alpha$;
(3) The family of supports $\{\operatorname{supp}(\psi_\alpha)\}_\alpha$ is locally finite, meaning that every point has a neighborhood that intersects $\operatorname{supp}(\psi_\alpha)$ for only finitely many values of α; and (the reason we call it*unity*)
(4) At each point $p \in M$, we have $\sum_{\alpha\in\Lambda} \psi_\alpha(p) = 1$.

If M is additionally a smooth manifold, a *smooth partition of unity* is a partition of unity where each ψ_α is smooth.

REMARK A.1.10. The sum $\sum_{\alpha\in\Lambda} \psi_\alpha(p)$ actually only has finitely many non-zero terms because of condition (3), so there is no issue of convergence.

We need partition of unity later when we create characteristic classes. Also partition of unity does not hold for $\mathbb{C}$.

THEOREM A.1.11. *Suppose M is a smooth manifold, and $\mathscr{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ is any open cover of M. Then there exists a smooth partition of unity subordinate to $\mathscr{U}$.*

A.1.4. Tangent vectors

In this subsection, we construct a functor $T_*\colon \mathbf{Man}_* \longrightarrow \mathbf{Vect}$, which sends a pointed smooth pointed manifold (M, p) to the tangent space $T_*(M, p) = T_pM$ at $p \in M$, which is a subset of

$$\operatorname{Hom}_{\mathbb{R}}\big(C^\infty(M, \mathbb{R}), \mathbb{R}\big).$$

DEFINITION A.1.12. Let M be a smooth manifold, and $p \in M$. A linear map $v\colon C^\infty(M, \mathbb{R}) \longrightarrow \mathbb{R}$ is called a *point-derivation at p* if it satisfies the *product rule*, i.e., for any $f, g \in C^\infty(M, \mathbb{R})$, we have

$$v(fg) = v(f)g(p) + f(p)v(g).$$

The set of all point-derivations at p, denoted T_pM, is called the *tangent space to M at p*, and an element is called a *tangent vector at p*.

REMARK A.1.13. We usually omit the parenthesis surrounding the argument of a derivation at a point, just as we do for regular derivation in $\mathbb{R}$. We also tend to write the scalars on the left, since T_pM will become a vector space over $\mathbb{R}$. E.g. $v(f)g(p) + f(p)v(g)$ becomes $f(p)vg + g(p)vf$.

EXAMPLE A.1.14 (The partial derivative). If we let $M = \mathbb{R}^n$, and let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be smooth, then the standard coordinates $x_1, \dots, x_n$ gives rise to the following derivatives:

$$\frac{\partial}{\partial x_i}\bigg|_p f = \frac{\partial f}{\partial x_i}\bigg|_p = \lim_{h\longrightarrow 0} \frac{f(p_1, \dots, p_i + h, \dots, p_n) - f(x_1, \dots, x_n)}{h},$$

namely the partial derivatives at $p = (p_1, \dots, p_n)$. As they satisfy the product rule, we can consider $\partial/\partial x_i$ as elements of $T_p\mathbb{R}^n$.

Now let M be an arbitrary smooth n-dimensional manifold, and let $f \in C^\infty(M, \mathbb{R})$. We define $x_i\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ to be the projection onto the i-th coordinate. Let p be a point in a coordinate chart (U, φ), and define x'_i to be the i-th coordinate of φ, i.e. $x'_i = x_i \circ \varphi$. If we define

$$\frac{\partial}{\partial x'_i}\bigg|_p f = \frac{\partial}{\partial x_i}\bigg|_p f \circ \varphi^{-1},$$

where the latter derivation is just ordinary partial derivation on $\mathbb{R}^n$, we get an induced "partial" point-derivation at p on M, i.e. $\frac{\partial}{\partial x_i'}\big|_p \in T_pM$. These new partial derivatives are are dependent on the choice of chart (U, φ). But in a new chart (U', ψ) containing p, we can move back to the former chart by the transition function $\varphi^{-1} \circ \psi \colon U' \longrightarrow U$, which is smooth since M is a manifold.

$$\begin{tikzcd}
M \supseteq U \arrow[d, "\varphi"'] \arrow[dr, bend left, "f"] & \\
\mathbb{R}^n \supseteq V \arrow[r] & \mathbb{R} \\
M \supseteq U' \arrow[u, "\psi"] \arrow[ur, bend right, "f"'] &
\end{tikzcd}$$

The terminology suggests that T_pM is a vector space.

Proposition A.1.15. *If M is a smooth manifold, and $p \in M$ is any point, then T_pM is a vector space with the pointwise operations from* $\operatorname{Hom}\big(C^\infty(M, \mathbb{R}), \mathbb{R}\big)$.

Example A.1.16 (The partial derivative). As we saw in example A.1.14, we have partial derivatives

$$\frac{\partial}{\partial x_i}\bigg|_p \in T_p\mathbb{R}^n, \quad \frac{\partial}{\partial x_i'}\bigg|_p \in T_pM$$

namely the partial derivatives at $p = (p_1, \ldots, p_n)$. These are linearly independent as well, because for any linear combination $\sum_{i=0}^n a_i \frac{\partial}{\partial x_i}$ summing to 0, one can set f to be the projection $x_j \colon \mathbb{R}^n \longrightarrow \mathbb{R}$ to the j-th coordinate, and so we get

$$0 = \sum_{i=1}^n a_i \frac{\partial x_j}{\partial x_i}\bigg|_p = \sum_{i=1}^n a_i \delta_i^j = a_j,$$

where $\partial x_j / \partial x_i = \delta_i^j$ is the Kronecker delta symbol. So the j-th coefficient is 0, for any $j = 1, \ldots, n$. Finally, we can use Taylor's theorem to see that any point-derivation at p is a linear combination of the partial derivatives.

If $F \colon M \longrightarrow N$ is a smooth map, and $p \in M$, then $F(p)$ would be some point in N. We could then ask if there is some relationship between T_pM and $T_{F(p)}N$ dependent on F. It turns out there is an induced function $T_pM \longrightarrow T_{F(p)}M$. This must, in particular, send a derivation v on $C^\infty(M, \mathbb{R})$ to a derivation on $C^\infty(N, \mathbb{R})$, the latter depending on v. This can naturally be done as follows:

DEFINITION A.1.17. If M and N are smooth manifolds, $p \in M$ a point, and $F\colon M \longrightarrow N$ is a smooth map, we can define a map

$$\begin{aligned} dF_p\colon T_pM &\longrightarrow T_{F(p)}N, \\ v &\longmapsto v(- \circ F). \end{aligned}$$

This is called the *differential of F at p*. More concretely, for some $f\colon N \longrightarrow \mathbb{R}$, and $v \in T_pM$, we have $[dF_p(v)](f) = v(f \circ F)$. (This makes sense as $f \circ F$ is a smooth map $M \longrightarrow \mathbb{R}$.)

REMARK A.1.18. It is not immediately clear that $dF_p(v)$ is a derivation at $F(p)$. But quick calculation shows

$$\begin{aligned} [dF_p(v)](fg) &= v\big((fg \circ F\big) = v\big((f \circ F)(g \circ F)\big) \\ &= f \circ F(p) v(g \circ F) + g \circ F(p) v(f \circ F) \\ &= f\big(F(p)\big)[dF_p(v)](g) + g\big(F(p)\big)[dF_p(v)](f), \end{aligned}$$

so satisfy the product rule, hence $dF_p(v) \in T_{F(p)}N$.

PROPOSITION A.1.19. *Let everything in the sequence $M \xrightarrow{F} N \xrightarrow{G} P$ be smooth.*

(1) The map $dF_p\colon T_pM \longrightarrow T_{F(p)}N$ is linear.
(2) We have $d(G \circ F)_p = dG_{F(p)} \circ dF_p\colon T_pM \longrightarrow T_{F(p)}M \longrightarrow T_{G\circ F(p)}P$.
(3) The differential of the identity is the identity, i.e.

$$d(\mathrm{id}_M)_p = \mathrm{id}_{T_pM}\colon T_pM \longrightarrow T_pM.$$

(4) If F is a diffeomorphism, then $dF_p\colon T_pM \longrightarrow T_pN$ an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

COROLLARY A.1.20. *We have a functor*

$$\begin{aligned} T_*\colon\ \mathbf{Man}_* &\longrightarrow \mathbf{Vect}, \\ (M, p) &\longmapsto T_pM, \\ (F\colon M \longrightarrow N) &\longmapsto (dF_p\colon T_pM \longrightarrow T_{F(p)}M). \end{aligned}$$

LEMMA A.1.21. *Let M be a smooth manifold, $p \in M$, and $v \in T_pM$. If $f, g\colon M \longrightarrow \mathbb{R}$ are smooth functions such that $f \restriction_U = g \restriction_U$ for some neighborhood U of p, then $vf = vg$.*

PROOF. Assume $f \restriction_U = g \restriction_U$, and let $h = f - g$. Then, h is smooth, and $h(U) = 0$. In particular $h(p) = 0$. We want to show that $vh = 0$, because, by linearity of v, we would have $vh = vf - vg$ and so we would have $vf = vg$. The proof goes by existence of technical tools.

Without going into details, there exists a smooth function on M[1], say $\psi_U\colon M \longrightarrow \mathbb{R}$, such that for every point x outside of U, the value $\psi_U(x)$ is constant and equal to 1, and $\psi_U(p) = 0$. Because the product ψh is equal to h, we have

$$v(h) = v(\psi_U h) = \underbrace{\psi_U(p)}_{=0} v(h) + \underbrace{h(p)}_{=0} \psi_U(p) = 0,$$

and this completes the proof. $\square$

Proposition A.1.22. *The derivative of the inclusion map $i\colon U \longrightarrow M$ is an isomorphism, i.e.*

$$T_pU \cong T_pM$$

Proof. We show that, for any $p \in U \subseteq M$, the map $di_p\colon T_p \longrightarrow T_pM$ is injective and surjective, using lemma A.1.21 and some technical tools.

For the injectivity, we verify that the kernel is trivial, which is equivalent to injectivity. Assume that v is in the kernel of di_p, i.e. $di_p(v) = 0$. We want to check that $v = 0$, meaning that for every $f \in C^\infty(U, \mathbb{R})$, we have $v(f) = 0$. Let $C \subseteq U$ be a closed neighborhood of p. Without going into details, it can be shown that for each such $f\colon U \longrightarrow \mathbb{R}$, there exits a smooth function[2] on M, say $\widetilde{f}$, such that $f \restriction_C = \widetilde{f} \restriction_C$. Thus, by lemma A.1.21, we have

$$vf = v(\widetilde{f} \restriction_U) = v(f \circ i),$$

and, as $di_p(v) = v(- \circ i)$, we get that $v(\widetilde{f} \circ i) = [di_p(v)](\widetilde{f})$, which is equal to 0, as we assumed that $v \in \ker(di_p)$. Thus $vf = 0$ for any f, and so $v = 0$.

For the surjectivity, let $w \in T_pM$ be any tangent vector. We want to find a $v \in T_pU$ mapping to w. As in the previous step, given a closed $C \subseteq U$, we can extend any $f \in C^\infty(U, \mathbb{R})$ to a smooth map $\widetilde{f}$ agreeing with f on C. If we define $vf = w\widetilde{f}$, we get, for any $g \in C^\infty(M)$, that

$$[di_p(v)](g) = v(g \circ i) = w(\widetilde{g \circ i}) = w(g \circ i) = wg,$$

where the the last two equalities follow from (lemma A.1.21 and) the fact that $\widetilde{g \circ i} \restriction_C = g \circ i \restriction_C = g \restriction_C$. And so, to any $w \in T_pM$ there is some $v \in T_pU$ mapping to w.

$\square$

[1]Called a *smooth bump function.*

[2]By the *extension lemma for smooth functions*

COROLLARY A.1.23. *The dimension of M and T_pM agree.*

PROOF. As charts have diffeormorphisms between neighborhoods of M and $\mathbb{R}^n$, the tangent spaces of these neighborhoods are isomorphic (by proposition A.1.19). And as these "local" tangent spaces are isomorphic to "global" ones, we have that T_pM has the same dimension as M. $\square$

COROLLARY A.1.24. *For any chart $\big(U, (x_1 \ldots, x_n)\big)$, the partial derivatives $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ form a basis of T_pM.*

A.1.5. Tangent bundles

In this subsection, we expand on the idea of $T_*\colon \mathbf{Man}_* \longrightarrow \mathbf{Vect}$ and obtain a functor $T\colon \mathbf{Man} \longrightarrow \mathbf{Man}$, sending a manifold M to its tangent bundle TM.

DEFINITION A.1.25. Let M be a smooth manifold. Then the *tangent bundle of M*, denoted TM, is the disjoint union of all the tangent spaces of M. More concretely,

$$TM = \bigsqcup_{p \in M} T_pM.$$

The *projection map of TM* is the surjective map

$$\pi\colon TM \longrightarrow M, \quad (p, v) \longmapsto p,$$

sending a tangent vector v at p to the base point p.

PROPOSITION A.1.26. *The tangent bundle TM of a smooth manifold has a natural smooth structure, making it into a smooth manifold. With this structure, the map $\pi\colon TM \longrightarrow M$ is smooth. The dimension of TM is twice the dimension of M.*

PROPOSITION A.1.27. *The tangent bundle har local trivializations*

Recall that if we have a smooth map $F\colon M \longrightarrow N$, then we can create a (linear) map $dF_p\colon T_pM \longrightarrow T_{F(p)}N$. (See definition A.1.17.) These gives ut a "global" differential $dF\colon TM \longrightarrow TN$. In essence, we can consider

PROPOSITION A.1.28. *If $F\colon M \longrightarrow N$ is a smooth map, then its global differential*

$$dF\colon TM \longrightarrow TN, \quad (p, v) \longmapsto \big(F(p), dF_p(v)\big),$$

is a smooth map.

Corollary A.1.29. *We have a functor*

$$\begin{aligned} T\colon \mathbf{Man} &\longrightarrow \mathbf{Man}, \\ M &\longmapsto TM, \\ (F\colon M \longrightarrow N) &\longmapsto (dF\colon TM \longrightarrow TN) \end{aligned}$$

Vector fields are 1st order differential operators, vectorfields, and infinitesimal automorphisms

Definition A.1.30. Let M be a smooth manifold. Then a *vector field on* M is a smooth map $X\colon M \longrightarrow TM$, usually written $p \longmapsto X_p$, such that

$$\pi \circ X = \mathrm{id}_M\colon M \longrightarrow TM \longrightarrow M.$$

The set of all vector fields on M is usually denoted $\mathfrak{X}(M)$ or $\Gamma(M)$, depending on the perspective.

Remark A.1.31. The criterion $\pi \circ X = \mathrm{id}_M$ is equivalent to specifying $X_p \in T_pM$.

Proposition A.1.32. *The set $\mathfrak{X}(M)$ of all vector fields on M is a vector space.*

A.2. Lie groups and equivariant maps

Lie groups are special types of manifolds. We study them now. All the material can be found in [Lee13] and [Tu17].

A.2.1. Lie groups

Definition A.2.1. A *Lie group* is a smooth manifold G that is also a group (in the algebraic sense), with the property that the maps

$$\begin{aligned} e\colon G^0 &\longrightarrow G, \quad 0 \longmapsto e_G, \\ i\colon G^1 &\longrightarrow G, \quad g \longmapsto g^{-1}, \\ m\colon G^2 &\longrightarrow G, \quad (g,h) \longmapsto gh, \end{aligned}$$

are all smooth. (The space G^2 is a smooth manifold by meta theorem 1.2.1.)

Example A.2.2. If G is a Lie group, any element $g \in G$ defines two maps,

$$L_g\colon G \longrightarrow G, \quad h \longmapsto gh, \quad \text{and} \quad R_g\colon G \longrightarrow G, \quad h \longmapsto hg.$$

These are called *left translation* and *right translation*, respectively, as they translate elements around the manifold. It should be noted that these maps are both smooth, as they are the composition of two smooth maps. For example, L_g is the composition of the injection map $h \longmapsto (g,h)$ and the multiplication map $(g,h) \longmapsto gh$. They are actually diffeomorphisms, as the maps $L_{g^{-1}}$ and $R_{g^{-1}}$ are smooth inverses.

EXAMPLE A.2.3. Let G be a Lie group, and $g \in G$ be any point in G. Similarly to the previous example, we have a map

$$\psi_g \colon G \longrightarrow G, \quad h \longmapsto ghg^{-1},$$

called the *conjugation map*, as it conjugate h with g. There are several ways of proving this is smooth: for example, it is the composition $\psi_g(h) = ghg^{-1} = L_g(hg^{-1}) = L_g\big(R_{g^{-1}}(h)\big) = L_g \circ R_{g^{-1}}(h)$ of the left translation map L_g and the right translation map $R_{g^{-1}}$. This also shows that it is a diffeomorphism, as it is the composition of two diffeomorphisms. The inverse, as can easily be checked, is $\psi_{g^{-1}}$.

A.2.2. Group actions and equivariant maps

In this subsection we discuss left actions, but we could just as easily talked about right actions.

Recall that a (left) action of a group G on a set X is a map $G \times X \longrightarrow X$, often written as $(g,x) \longmapsto g \cdot x$, that satisfies

$$e \cdot x = x, \quad g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x.$$

DEFINITION A.2.4. Let G be a Lie group, and M a smooth manifold. We call M a *(left) G-space* if $G \times M \longrightarrow M$ is a continuous map. If this is a smooth map, we say we have a *smooth (left) action*.

DEFINITION A.2.5. Let M be a G-space.

(1) For each $p \in M$, the *orbit of* p, $G \cdot p$, is the set of all images of p under the action by G:

$$G \cdot p = \{g \cdot p \mid g \in G\} \subseteq G.$$

(2) For each $p \in M$, the *stabilizer of* p, G_p is the set of all elements of G that fixes p:

$$G_p = \{g \mid g \cdot p = p \in G\} \subseteq G.$$

(3) The action is called *transitive* if, for every $p, q \in M$, there exist some $g \in G$ such that $g \cdot p = q$.

(4) The action is called *free* if, for every $p \in M$ such that $g \cdot p = p$, we must have $g = e$.

REMARK A.2.6. We collect a few remarks from the above definiton.

(1) Observe that the stabilizer G_p of p is always a subgroup $G_p \leq G$ of G.
(2) A transitive action is equivalent to every orbit space being M.
(3) A transitive action is equivalent to to every stabilizer group being trivial.

Definition A.2.7. Let M and N be G-spaces with smooth actions

$$M \times G \longrightarrow M, \quad (m, g) \longmapsto m \cdot g,$$
$$N \times G \longrightarrow N, \quad (n, g) \longmapsto n \cdot g,$$

and let $f\colon M \longrightarrow N$. We call f *equivariant* if, for all $g \in G$, we have $f(m \cdot g) = f(m) \cdot g$, or equivalently, the following diagram commutes.

$$\begin{tikzcd} M \arrow[r, "\cdot g"] \arrow[d, "f"'] & M \arrow[d, "f"] \\ N \arrow[r, "\cdot g"] & N \end{tikzcd}$$

A.2.3. Lie algebras

Recall that a vector field is a map $X\colon M \longrightarrow TM$ such that $X_p \in T_pM$. (See definition A.1.30.) Let $f\colon M \longrightarrow \mathbb{R}$ be a smooth function. Because X_p is a derivation at p, then Xf will be another smooth function. IF we apply the vector field Y to this new function Xf, we get yet another smooth function $YXf = Y(Xf)$. But the derivation $f \longmapsto YX$ is not, in general, a derivation at p as it does not always satisfy the product rule. This is because the product YX of two 1st order differential operators is a 2nd order differential operator. The same problem applies to $f \longmapsto XY$. But what is also true is the fact that all the 2nd degree terms of YX and YX commute. So if we subtract them, $YX - XY$, the 2nd degree parts die, and we are left with a 1st degree term only. And this satisfies the product rule, or, put differently, $YXf - XYf$ is a differential operator.

Definition A.2.8. Given two vector fields $X, Y\colon M \longrightarrow TM$, the operator $[X, Y]\colon C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$, defined by

$$[X, Y]f = XYf - YXf,$$

is called the *Lie bracket of X and Y*.

Proposition A.2.9. *The Lie bracket of X and Y at p is a derivation at p.*

Proof. Just calculate $[X, Y]_p = X_p(Yf) - Y_p(Xf)$. $\square$

The Lie bracket is bilinear and antisymmetric.

Recall that a Lie group acts smoothly on itself via left translation: $L_g(h) = gh$. (See example A.2.2.)

DEFINITION A.2.10. Let G be a Lie group. A vector field X on G is said to be *left-invariant* if it is invariant under all left translations, meaning, for any $g \in G$, derivation of the vector field X at any point g' is the same as translating the vector field by g from the left. More explicitly,

$$d(L_g)_{g'}(X_{g'}) = X_{gg'}.$$

REMARK A.2.11. Since L_g is a diffeomorphism, we can abbreviate the equation in definition A.2.10 to $(L_g)_* = X$, which also makes the terminology more clear.

One can observe the fact that the set of all left-invariant vector fields $X \in \mathfrak{X}(M)$ form a linear subspace of $\mathfrak{X}(M)$. This follows directly from the fact that $(L_g)_*$ is a linear transformation. But, slightly less obvious,, but far more important, is the fact that taking the Lie bracket of two left-invariant vector fields gives a left-invariant vector field.

PROPOSITION A.2.12. *Let G be a Lie group, and suppose X and Y are smooth left-invariant vector fields on G. Then $[X, Y]$ is also left-invariant.*

DEFINITION A.2.13. Let G be a Lie group. The set of all smooth left-invariant vector fields on G, denoted $\mathrm{Lie}(G)$, is called the *Lie algebra of G*.

Recall that we use the notation $\mathfrak{g}$ for the tangent space T_eG at the identity of a Lie group G.

THEOREM A.2.14. *Let G be a Lie group. The evaluation map*

$$\varepsilon(X)\colon \mathrm{Lie}(G) \longrightarrow \mathfrak{g}$$

given by $\varepsilon(X) = X_e$ is a vector space isomorphism. Thus, $\mathrm{Lie}(G)$ *is finite-dimensional, with dimension equal to* $\dim(G)$.

Recall that $\psi_g(h) = ghg^{-1}$ is a diffeomorphism, called the conjugation map. (See example A.2.3.) Observe that it sends $e \longmapsto e$, so the derivative at e would be a map $\mathfrak{g} = T_eG \longrightarrow T_{\psi_g(e)}G = T_eG = \mathfrak{g}$. This map,

$$\mathfrak{g} \longrightarrow \mathfrak{g}, \quad X \longmapsto d(\psi_g)_e(X),$$

has the name Ad_g.

Definition A.2.15. Let G be a Lie group. The *adjoint representation of* G, denoted Ad is the map

$$\mathrm{Ad}\colon G \longrightarrow \mathrm{Aut}(\mathfrak{g}), \quad g \longmapsto \mathrm{Ad}_g\,.$$

The reason for the name is the following:

Proposition A.2.16. *The adjoint representation of G is a (group) representation of the Lie group G.*

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