Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

Ling Tan

Representation theory of Artin algebras and finite graded trees

Master's thesis in Mathematical Sciences
Supervisor: Sverre Olaf Smalø
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Sammendrag

Dette arbeidet diskuterer representasjonsteorien for artinske algebraer med fokus på de nesten-splitte sekvensene. Først introduserer vi Nakayama-algebraer, Auslander-algebraer og Auslander-Reiten-kogger. Deretter undersøker vi endeliggraderte representasjoner av et endelig tre; vi introduserer treet $\ddot{D}_n$ og beregner de endelige representasjonene av trærne $\ddot{D}_5$ og D_6. Til slutt introduseres Nakayama-endelige graderinger av et endelig tre, og vi gir den generelle formelen for Nakayama-endelige graderingen av trærne $\ddot{D}_n$ og D_n.

Abstract

This work discusses the representation theory of Artin algebras with a focus on the almost split sequences. First, we introduce the Nakayama algebras, Auslander algebras and Auslander-Reiten quivers. Second, we examine the representation finite gradings of a finite tree. We introduce the tree $\ddot{D}_n$ and calculate the representation finite gradings of the trees $\ddot{D}_5$ and D_6. Finally, we introduce the Nakayama finite gradings of a finite tree. We give the general formula for the number of the Nakayama finite gradings of the trees $\ddot{D}_n$ and D_n.

Introduction

In this thesis, we study the representation theory of artin algebras. In a broad sense, this is the study of the modules over artin algebras. When we study the theory of modules, category theory and homological algebra are useful. The property of artin algebras, that every finitely generated module admits finite length, gives us a good perspective when considering the category of finitely generated modules over an artin algebra. We concentrate on studying the theory of almost split sequences. The reason is that the results from the study of almost split sequences plays an important role in many recent work across several topics. We illustrate this point by looking at the Nakayama algebras and the representation finite gradings for a finite tree.

We are assuming the reader is familiar with the general concepts of rings and modules such as projective, and injective modules, and also some basic results from homological algebra.

This work is divided into six chapters. The first chapter contains the relevant background on artin algebras, quivers and path algebras. We discuss the duality and the transpose on module categories. In the second chapter, we focus on the almost split sequences and show the existence theorem of them. We also illustrate irreducible morphisms by giving an example from PIDs.

In chapter 3, we introduce the Nakayama algebras. We concentrate on the invariants of the indecomposable modules which are helpful to determine an indecomposable Nakayama algebra from a given admissible sequence. We show the general form of the almost split sequences of an indecomposble Nakayama algebra which are a helpful tool to understand the special structure of a Nakayama algebra.

Since it is useful to consider Auslander algebras while studying the artin algebras of representation finite type, we introduce the Auslander algebra and Auslander-Reiten quiver in chapter 4. We describe how to associate an Auslander-Reiten quiver to an artin algebra which is based on the almost split sequences.

In chapter 5, we introduce the representation finite gradings for a finite tree. We start by associating a translation quiver to a graded tree by defining the dimension map. We summarize the result from Bongartz and Gabriel in [3] showing that there is a bijection between the isomorphism classes of representation finite graded trees and the isomorphism classes of simply connected algebras. The theory studied in previous chapters is important here. We introduce the tree $\ddot{D}_n$. Last, we obtain the first result of this work by calculating the representation finite gradings for the trees $\ddot{D}_5$ and D_6.

In chapter 5, to show the existence of the representation finite gradings of a arbitrary finite tree, we introduce the result from Rohnes and Smalø in [5] which uses the corresponding Nakayama algebra of the tree. The final result of this thesis, is to give the general formula for the number of the Nakayama representation finite gradings of the trees $\ddot{D}_n$ and D_n respectively.

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Contents

1 Preliminary

In this chapter, we start by introducing the length of a module over an arbitrary ring referring to chapter 1-4 in [2]. After proving the Jordan–Hölder Theorem, we prove that for a left artin ring, every finitely generated module has finite length. We introduce the notion of a quiver and it's path algebra. Specifically, we illustrate how to associate a quiver to a finite dimensional basic algebra over an algebraically closed field. After that, we introduce the D-functor and the transpose. We also include the projectivization and the block decomposition of an artin algebra.

1.1 Modules

Let Λ be an arbitrary ring and let A be a Λ-module. If there is a finite filtration of submodules $F : A = A_0 \supset A_1 \supset \cdots \supset A_n = 0$ such that for $i \in \{0, \ldots, n\}$, A_i/A_{i+1} is simple, we call F a **composition series** and call the A_i/A_{i+1} the **composition factor** of F. The composition series is not unique. For example, $\mathbb{Z}_2 \times \mathbb{Z}_3$ has two composition series.

We use $m_S^F(A)$ to denote the number of composition factors of F which are isomorphic to S where S is a simple Λ-module. We use $l_F(A)$ to denote the sum of $m_{S_i}^F(A)$ where S_i ranges over all the isomorphism classes of simple Λ-modules. Further, we define the **length** of A denoted as $l(A)$ be the minimum of $l_{F_i}(A)$ and $m_S(A)$ be the minimum number of $m_S^{F_i}(A)$ where F_i ranges over all the composition series of A.

Jordan–Hölder Theorem state that $l_F(A)$ and $m_S^F(A)$ are actually independent from the choice of the composition series. The following proof is referring to Chapter 3 in [4].

Theorem 1.1. ***Jordan–Hölder Theorem.*** *Let M be a* Λ-module *of finite length. Let $F : 0 \subset M_1 \subset \cdots \subset M_n = M$ and $G : 0 \subset N_1 \subset \cdots \subset N_m = M$ be two composition series of M where $m \geq n$ then we have that $l_F(M) = l_G(M) = l(M)$ and $m_S^F(M) = m_S^G(M) = m_S(M)$ where S ranges over all the isomorphism classes simple modules of Λ.*

Proof. We prove it by induction on $l(M)$. If $l(M) = 0$, there is nothing to prove. If $l(M) = 1$, then M is simple and the only composition factor is itself. We assume when $l(M) \leq n - 1$, the hypothesis is satisfied. Suppose $l(M) = n$. Let $K = M_{n-1} \cap N_{m-1}$.

1. If $M_{n-1} = N_{m-1}$, we are done.

2. If $M_{n-1} \neq N_{m-1}$, $M_{n-1}+N_{m-1} = M$ and $M_{n-1}/K \cong (M_{n-1}+N_{m-1})/N_{m-1} = M/N_{m-1}$. Similarly, we have $N_{m-1}/K \cong M/M_{n-1}$. Again by M_{n-1}, N_{m-1}

being maximal, M_{n-1}/K and N_{m-1}/K are simple. K has composition series by taking the intersection of K with the composition series of M and deleting one zero factor. Let $H : 0 \subset K_1 \subset \cdots \subset K_r = K$ be a composition series of K. Then $F' : 0 \subset K_1 \subset \cdots \subset K_r = K \subset M_{n-1}$ and $G' : 0 \subset K_1 \subset \cdots \subset K_r = K \subset N_{m-1}$ are two composition series for M_{n-1} and N_{m-1} respectively. Since $l(K) \leq n-1$, we know that F' and J have the same length and composition factors, the same as G' and L. Then by $M_{n-1}/K \cong M/N_{m-1}$ and $N_{m-1}/K \cong M/M_{n-1}$, we have that $n = m$ and $m_S^F(M) = m_S^G(M)$.

□

Observation 1.2. *Modules are not uniquely determined by composition factors. For example, $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ have the same composition factors but they are not isomorphic.*

For a ring Λ, we define the **radical** r of Λ be the intersection of the maximal left ideals of Λ. We state Nakayama lemma without giving a proof.

Lemma 1.3. ***Nakayama lemma*** *Let Λ be a ring and let r be the radical of Λ. Let M be a finitely generated Λ-module. Then $rM = M$ if and only if $M = 0$.*

Proposition 1.4. *Let Λ be a left artin ring and r be the radical of Λ. Let A be a* Λ-module. *Then we have the following.*

1. *The radical r is nilpotent.*
2. *Λ/r is a semisimple ring.*
3. *A is semisimple if and only if $rA = 0$.*
4. *There is only a finite number of isomorphism classes of simple Λ-modules.*
5. *Λ is left noetherian.*

Proof.
1. We look at the radical filtration $\Lambda \supset r \supset r^2 \supset \cdots \supset r^n \supset \ldots$. There is a number $n \in \mathbb{N}$ such that $r^n = r^{n+1}$. By Nakayama's lemma, $r^n = 0$. Thus r is a nilpotent.
2. Since Λ is left artinian, Λ/r is left artinian. Since $rad(\Lambda/r) = rad(\Lambda)/r = 0$, Λ/r has no non-zero nilpotent ideals. So Λ/r is semisimple.
3. Obviously, when A is semisimple, then $rA = 0$. When $rA = 0$, the module A is also Λ/r-module. Thus A is semisimple.

4. Each non-isomorphic simple module of Λ is a Λ/r-module and occurs as a direct summand of Λ/r. Λ/r has only a finite number of isomorphism classes simple modules.

5. For the radical filtration $\Lambda \supset r \supset r^2 \supset \cdots \supset r^n = 0$, we have that $r(r^i/r^{i+1}) = 0, i \in \{0, \ldots, n\}$, then r^i/r^{i+1} is semisimple artinian. So r^i/r^{i+1} is noetherian. Thus Λ is neotherian.

□

Corollary 1.4.1. *Let Λ be a ring and r be the radical, the following are equivalent.*

1. *Every finitely generated* Λ-module *has finite length.*

2. *Λ is left artinian.*

3. *The radical r is a nilpotent and r^i/r^{i+1} is a finitely generated semisimple module for all $i \geq 0$.*

1. (1) $\Rightarrow$ (2). *Since Λ as a finitely generated module over itself, it has finite length, so Λ is left artin.*

2. (2) $\Rightarrow$ (3). *This is a direct consequence of the last proposition.*

3. (3) $\Rightarrow$ (1). *Let A be a finitely generated* Λ-module. *Since A is finitely generated, there is a surjective map $f : \Lambda^n \to A$, for some $n \in \mathbb{N}$. It is enough to show $l(\Lambda^n)$ has finite length. It is straightforward that Λ has finite length by* (3). *Then $l(\Lambda^n)$ has finite length , and then A has finite length.*

This corollary plays a very important role in the study of finitely generated modules of a left artin ring. In the rest of the thesis we use $\operatorname{mod}\Lambda$ to denote the category of **finitely generated modules** of Λ.

We state the Krull–Schmidt theorem without giving proof. The proof can be found in chapter 3 of [4] which is given by the induction on length.

Theorem 1.5. ***Krull–Schmidt theorem.*** *Let Λ be a left artin ring and let M be a finitely generated module. Then we have the following.*

1. *M can be written as a finite direct sum of indecomposable modules.*

2. *The decomposition of M into indecomposable modules are unique up to isomorphism.*

1.2 Path algebras

Definition 1.1. ***R-algebra***. *Let R be a commutative artin ring. An artin R-algebra is a ring Λ together with a ring homomorphism $\Phi : R \to \Lambda$, where $\operatorname{Im}\Phi$ is in the center of Λ, and such that Λ is a finitely generated R-module.*

Definition 1.2. ***K-algebra***. *Let K be a field. A K-algebra is a ring Λ together with a ring homomorphism $\Phi : K \to \Lambda$, where $\operatorname{Im}\Phi$ acts centrally in Λ, i.e. for $k \in K$ and $a, b \in \Lambda$, if we use ka to denote $\Phi(k)a$, then $k(ab) = (ak)b = a(kb) = (ab)k$.*

Definition 1.3. *Quiver. A quiver $\Gamma = (\Gamma_0, \Gamma_1)$ is an oriented graph. Γ_0 denotes the set of vertices and Γ_1 denotes the set of arrows between vertices.*

A quiver Γ is said to be finite if both Γ_0 and Γ_1 are finite. In the rest of this thesis, we assume Γ is a finite quiver. For each arrow α, we define **the starting vertex function** s such that $s(\alpha)$ is the starting vertex of the arrow α and define **the ending vertex function** e such that $e(\alpha)$ is the ending point of the arrow α.

A **path** in a quiver Γ is either a **trivial path** of a vertex i denoted as e_i with $s(e_i) = i$ and $e(e_i) = i$ or an ordered composition of arrows $q = a_1 a_2 \dots a_n$ where $e(a_i) = s(a_{i-1})$ for $i \in \{1, \dots, n\}$. We have $e(q) = e(a_1), s(q) = s(a_n)$. If q is non-trivial and $e(q) = s(q)$, we call it a **cycle**. We define the **length** l of a path as the number of arrows in the path, so $l(e_i) = 0$ and $l(q) = n$.

Example 1.1. *Let Γ be the quiver* $1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} 4 \xrightarrow{a_4} 5 \circlearrowleft a_5$.

So a_5 is a cycle. Hence e_1, e_2, e_3, e_4, e_5 are the trivial paths and $a_2 a_1$ is the path starting in 1 and ending in 3.

For a quiver Γ, we define the associated path algebra as following.

Definition 1.4. *Path algebra. Let k be a field and Γ be a quiver. The path algebra $k\Gamma$ is the k-vector space with all the paths of Γ as basis. The multiplication is given by juxtaposition of paths and then extended by bilinearity.*

We illustrate the multiplication as the following. Let Γ be a quiver. Let e_i, e_j be the trivial path of the vertex i and j respectively. Let a, b be arrows in Γ_1.

$$e_i e_j = \begin{cases} e_i & i = j \\ 0 & else \end{cases} \qquad e_i a = \begin{cases} a & e(a) = i \\ 0 & else \end{cases}$$

$$a e_i = \begin{cases} a & s(a) = i \\ 0 & else \end{cases} \qquad ab = \begin{cases} ab & s(a) = e(b) \\ 0 & else \end{cases}$$

Example 1.2. *Let k be a field. Let Γ be the quiver $1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3$. So $k\Gamma$ is the k-vector space with basis $\{e_1, e_2, e_3, a_1, a_2, a_2a_1\}$.*

Clearly, the **identity** of $k\Gamma$ is the sum of all **idempotents** e_i. We write it as $1 = e_1 + \cdots + e_n$. Since $e_ie_j = 0$ if $i \neq j$, it is a orthogonal decomposition of the identity.

Let J denote the ideal in $k\Gamma$ generated by all the arrows in Γ. When $k\Gamma$ is finite dimensional i.e. Γ has no cycle, $k\Gamma/J \cong ke_1 \times \cdots \times ke_n$ is semisimple, then J is the **radical** of $k\Gamma$.

In example 1.1, it is trivial that the associated path algebra of this quiver is an infinite dimension k-algebra since there is a circle which makes the basis infinite. Thus, $k\Gamma$ is finite if and only if there it no cycle in Γ.

Example 1.3.
$$\begin{tikzcd} & 2 \arrow[dl] \arrow[dr] & \\ 1 \arrow[dr] & \Gamma_1 & 3 \arrow[dl] \\ & 4 & \end{tikzcd} \qquad \begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] & \Gamma_2 & 3 \arrow[dl] \\ & 4 \arrow[ul] & \end{tikzcd}$$
Let k be a field. $k\Gamma_1$ is finite dimensional. $k\Gamma_2$ is infinite dimensional since there is a cycle in Γ_2.

It is natural to ask that for each k-algebra Λ, dose there exist a path algebra $k\Gamma$ such that $k\Gamma \cong \Lambda$? We give an counter example as following.

Example 1.4. *Let k be a field, $k[x]/(x^2)$ is the polynomial ring modulo the ideal generated by x^2. So $\{1, x\}$ is a basis of $k[x]/(x^2)$. If a path algebra $k\Gamma$ are isomorphic to $k[x]/(x^2)$, $k\Gamma$ has to satisfy the relation $1x = x1 = x$. The only quiver Γ we can find is* $\begin{tikzcd} 1 \arrow[loop right, "x"] \end{tikzcd}$. *But since it has a cycle, the path algebra $k\Gamma$ is not isomorphic to $k[x]/(x^2)$.*

Definition 1.5. *Relation of quiver. A relation ρ in quiver Γ over a field k is a k-linear combination of paths $\rho = k_1p_1 + \cdots + k_np_n$ where $e(p_1) = \cdots = e(p_n)$ and $s(p_1) = \cdots = s(p_n)$. We assume $l(p_i) \geq 2$ for all $i \in \{1, \ldots, n\}$.*

For a finite dimensional path algebra, we have the following observation.

Observation 1.6. *Let Γ be a finite quiver without cycles and let ρ be a relation in the path algebra $k\Gamma$. The ideal (ρ) generated by ρ satisfies that $\exists n \in \mathbb{N}, J^n \subseteq (\rho) \subseteq J^2$ where J is the ideal generated by all the paths in $k\Gamma$.*

Let ρ denote a set of relations in the quiver Γ over a field k, we use (Γ, ρ) to denote the **quiver with relations**. The associated path algebra is $k(\Gamma, \rho) =$

$k\Gamma/(\rho)$. In example 1.4, we can see $k[x]/(x^2) \cong k(\Gamma, \rho)$ where $\rho = x^2$ in the quiver

$$1 \circlearrowright x \,.$$

In the rest of this section, we will show how to associate a quiver to an basic finite dimensional algebra over an algebraically closed field. We will first introduce tensor ring and it's associate quiver since there is a natural connection between tensor ring and the associated path algebra.

Definition 1.6. ***Tensor ring***. *Let Σ be a ring and let V be a Σ-bimodule. $V^2 \cong V \otimes V$ and V^i is the i-fold tensor product of V. The tensor ring $T(\Sigma, V) = \Sigma \coprod V \coprod V^2 \coprod \ldots$.*

If we let $\Sigma = \prod_n(k)$ where k is a field and let V be a finite Σ-bimodule where k acts centrally. Then $\Phi : k \to \Sigma$ defined by $\phi(x) = (x, x, \ldots, x)$ gives the structure of $T(\Sigma, V)$ being a k-algebra. Then we define the **associated quiver** Γ for $T(\Sigma, V)$ as follows.

- The *ith*-vertex ϵ_i in Γ_0 is the idempotent in Σ of the form of $(0,\ldots,1,\ldots 0)$ where only ith coordinate is 1 and the rest is 0. Then we have $1 = \epsilon_1 + \cdots + \epsilon_n$.

- The number of arrows from the vertice j to the vertice i is the dimension of $\epsilon_j V \epsilon_i$ which is a k-subspace of V.

For a finite dimensional path algebra $k\Gamma$, we call a relation ρ **admissible** if it satisfies that there exists $n \in \mathbb{N}, J^n \subseteq (\rho) \subseteq J^2$ where J is the radical of $k\Gamma$ in observation 1.6. Motivated by that, we want to find a homomorphism which maps V_i to J_i for the tensor ring $T(\Sigma, V)$.

Proposition 1.7. *Let $\Sigma = \prod_n(k)$ and V be a finite dimensional Σ-bimodule where k acts centrally. Let Γ be the associated quiver for $T(\Sigma, V)$, then there is a k-algebra isomorphism $\Phi : T(\Sigma, V) \to k\Gamma$ such that $\Phi : (\coprod_{i \geq t} V^i) = J^t$ where J is the ideal generated by the paths in $k\Gamma$.*

Proof. We define a homomorphism $f : \Sigma \coprod V \to k\Gamma$ as following. For any $(a_1, \ldots, a_n) \in \Sigma$, $f(a_1, \ldots, a_n) = \sum_{i=1}^n a_i \epsilon_i$. The union of a chosen basis for each $\epsilon_i V \epsilon_j$ in $\{\epsilon_i V \epsilon_j\}_{i,j \in \{1,2,\ldots,n\}}$ are a basis of V. The map $f : \epsilon_i V \epsilon_j \to K\Gamma_1$ is defined by giving a bijection between the chosen basis of $\epsilon_i V \epsilon_j$ and the set of arrows from j to i. Clearly, f is a bijection of vector space $\Sigma \coprod V$ to $k^{\Gamma_0} \oplus k^{\Gamma_1}$. To extend f to $\tilde{f} : T(\Sigma, V) \to k\Gamma$ where $\tilde{f} \mid_{\Sigma \coprod V} = f$, we let $\tilde{f} \mid_{V^n} (V_1, \ldots, V_n) = f(V_1) f(V_2) \ldots f(V_n)$. So $\tilde{f}(a, w, w_1, \ldots, w_n) = f(a, w) + \sum_{i=1}^n \tilde{f} \mid_{V^n}$. Obviously, it is a ring homomorphism. Clearly, $\mathrm{Im}(f(V)) = J$. So $\tilde{f}(\coprod_{i \geq t} V^i) = J^t$. By observation 1.6, $\tilde{f}$ is surjective. Obviously, the kernel of $\tilde{f}$ is 0. So $\tilde{f}$ is the desired isomorphism. $\square$

Definition 1.7. ***Basic finite dimensional algebra***. *A finite dimensional algebra Λ is basic if and only if $\Lambda/r \cong \prod_{i=1}^{i=n}(M_i)$, where each M_i is a division rings.*

Definition 1.8. ***Elementary finite dimensional algebra***. *A finite dimensional algebra Λ over an a field k is elementary if and only if $\Lambda/r \cong \prod_{i=1}^{i=n}(k)$ as a k-algebra.*

Proposition 1.8. *A basic finite dimensional algebra Λ over an algebraically closed field k is an elementary k-algebra.*

Proof. Let $\Lambda/r \cong \prod_{i=1}^{i=n}(M_i)$ where M_i are division rings and r is the radical. Let $\phi : k \to \Lambda/r$ be the ring morphism making Λ a k-algebra. Then we have the projection $\phi^{M_i} : k \to M_i$. Thus M_i is a finite dimensional extension of k. Since k is algebraically closed, M_i is isomorphic to k. Thus $\Lambda/r \cong \prod_{i=1}^{i=n}(k)$. □

The associated quiver Γ of a finite dimensional elementary algebra Λ over field k is the associated quiver of the tensor ring $T(\Lambda/r, r/r^2)$. We will show that there is a path algebra with relation $k(\Gamma, \rho)$ such that $\Lambda \cong k(\Gamma, \rho)$.

Proposition 1.9. *Let Λ be an elementary finite dimensional algebra. Let $\{e_1, \dots, e_n\}$ be a set of primitive orthogonal idempotents in Λ such that the image in Λ/r generates Λ/r, and $\{r_1, \dots, r_t\}$ be the set of elements in r such the the image in r/r^2 is a basis of r/r^2 as Λ/r-module. Then $\{e_1, \dots, e_n, r_1, \dots, r_t\}$ generate Λ.*

Proof. We prove it by induction on the Loewy length ll of Λ. Λ is elementary that $\Lambda/r \cong \prod_{i=1}^{i=n}(k)$. So the idempotent $\overline{e_i}$ in Γ/r is of the form $(0, \dots, 1, \dots, 0)$ where the ith position is 1 and the rest is 0.

1. When $ll(\Lambda) = 1$, $r = 0$ and Λ is semisimple. Obviously Λ is generated by $\{e_1, \dots, e_n\}$.

2. When $ll(\Lambda) = 2$, $r^2 = 0$. Obviously Λ is generated by $\{e_1, \dots, e_n, r_1, \dots, r_t\}$.

3. We assume it is ture for $ll(\Lambda) = m$. When $ll(\Lambda) = m + 1$, let A denote the set $\{e_1, \dots, e_n, r_1, \dots, r_t\}$.

   Since $ll(\Lambda/r^m) = m$ and $(r/r^m)/(r^2/r^m) = r/r^2$, also $(\Lambda/r^m)/(r/r^m) = \Lambda/r$, then $\{e_1/(r^m), \dots, e_n/(r^m), r_1/(r^m), \dots, r_t/(r^m)\}$ is a generating set of $\Lambda/(r^m)$. So $\Lambda/r^m \cong < A > / < (A \cap r^m) >$. $\forall x \in \Lambda$, $\exists y \in A$ that $x - y \in r^m$. $\exists \alpha \in r^{m-1}$ and $\beta \in r$ that $\alpha\beta = x - y$. But $\exists \alpha' \in A$ and $\alpha'' \in r^m$ that $\alpha = \alpha' + \alpha''$. The same for β that $\beta = \beta' + \beta''$ where $\beta' \in A$ and $\beta'' \in r^m$. So $x - y = \alpha\beta = (\alpha' + \alpha'')(\beta' + \beta'')$. Since $ll(\Lambda) = m$, $\alpha''\beta, \alpha'\beta'', \alpha''\beta'' = 0$, so $x - y = \alpha'\beta' \in A$. Thus x is in A.

□

Corollary 1.9.1. *There is a surjective ring homomorphism* $\tilde{f} : T(\Lambda/r, r/r^2) \to \Lambda$ *such that* $\coprod_{i \geq ll(\Lambda)} (r/r^2)^i \subset ker\tilde{f} \subset \coprod_{i\geq 2}(r/r^2)^i$.

Proof. Let $\{e_1, \dots, e_n\}$ be the primitive idempotents set of Λ such that the image $\{\overline{e_1}, \dots, \overline{e_n}\}$ in Λ/r is a basis of Λ/r. Let $\{r_1, \dots, r_t\}$ be the set of elements in r such that the image $\{\overline{r_1}, \dots, \overline{r_t}\}$ in r/r^2 is a basis of r/r^2. By proposition 1.9, $\{e_1, \dots, e_n, r_1, \dots, r_t\}$ is a generating set of Λ. We define a ring isomorphism $f : \Lambda/r \coprod r/r^2 \to \Lambda/r^2$ by letting $f(\overline{e_i}) = e_i$ and $\tilde{f}(\overline{r_i}) = r_i$. Let $\tilde{f} \mid_{(\Lambda/r \coprod r/r^2)} = f$. For each $x = x_1 \otimes \cdots \otimes x_i$ in $(r/r^2)^i$, we define that $\tilde{f}(x) = f(x_1)f(x_2)\dots f(x_i)$. Thus $\tilde{f} : T(\Lambda/r, r/r^2) \to \Lambda$ is a surjective ring homomorphism. Clearly, for a non-zero element x in $\Lambda/r \coprod r/r^2$, $\tilde{f}(x) \neq 0$. Then $ker\tilde{f} \subset \coprod_{i\geq 2}(r/r^2)^i$. Since $(r/r^2)^i = 0$ when $i \geq ll(\Lambda)$, $\coprod_{i\geq rl(\Lambda)} r^i \subset ker\tilde{f}$. Thus $\tilde{f}$ is the desired map. □

Corollary 1.9.2. *Let* Λ *be a finite dimensional elementary algebra over an algebraically closed field* k*, there is a path algebra with relation* $k(\Gamma, \rho)$*,* $J^n \subseteq (\rho) \subseteq J^2$ *such that* $k(\Gamma, \rho) \cong \Lambda$.

Proof. Let $\tilde{f} : T(\Lambda/r, r/r^2) \to \Lambda$ be the homomophism from corollary 1.9.1 and let $\tilde{h} : T(\Lambda/r, r/r^2) \to k\Gamma$ be the isomorphism from proposition 1.7. So a generating set of $\tilde{h}(ker^{-1}(\tilde{f}))$ is the desired relation ρ. Thus $k(\Gamma, \rho) \cong \Lambda$. □

We have seen a finite dimensional basic algebra Λ over an algebraically closed field k is elementary. So the associated quiver of Λ is the associated quiver Γ of tensor ring $T(\Lambda/r, r/r^2)$. Thus, there is a path algebra with relation $k(\Gamma, \rho)$ that is isomorphic to Λ.

Proposition 1.10. *Let* Λ *be a finite dimensional basic algebra over an algebraically closed field* k *and* $\{e_1, \dots, e_n\}$ *be the primitive idempotents decomposition set of identity such that* $1 = e_1 + \cdots + e_n$*. Then* $\Lambda = \Lambda e_1 + \cdots + \Lambda e_n$*. Let* P_i *denote* Λe_i *and* S_i *denote* P_i/rP_i*, so* $P_i \to S_i$ *is the projective cover. The following are equal.*

1. $dim_k(\mathrm{Ext}^1_\Lambda(S_i, S_j))$
2. *the multiplicity of* S_j *in* rP_i/r^2P_i
3. *the multiplicity of* P_j *in* P*, where* $P \to P_i \to S_i$ *is a minimal projective presentation of* S_i.
4. $dim_k(e_j(r/r^2)e_i)$

Proof. We have the exact sequence $0 \to rP_i \to P_i \to S_i \to 0$. Applying $\mathrm{Hom}_\Lambda(-, S_j)$, we have the exact sequence:

$$0 \to \mathrm{Hom}_\Lambda(S_i, S_j) \to \mathrm{Hom}_\Lambda(P_i, S_j) \xrightarrow{\mathrm{Hom}_\Lambda(h,S_j)} \mathrm{Hom}_\Lambda(rP_i, S_j) \to \mathrm{Ext}^1_\Lambda(S_i, S_j) \to 0$$

For $rP_i \hookrightarrow P_i \xrightarrow{h} S_j$, rP_i is in $ker(h)$. Since P is indecomposable, $\mathrm{Hom}_\Lambda(h, S_j) = 0$. Thus $dim_k(\mathrm{Ext}^1_\Lambda(S_i, S_j)) = dim_k \mathrm{Hom}_\Lambda(rP_i, S_j)$.

Since r^2P_i is in the kernel of all $f : rP_i \to S$ with S being simple, we have $\mathrm{Hom}_\Lambda(rP_i, S_j) \cong \mathrm{Hom}_\Lambda(rP_i/r^2P_i, S_j)$. Then the multiplicity of S_j in rP_i/r^2P_i is equivalent to $dim_k \mathrm{Hom}_\Lambda(rP_i, S_j)$ which is equal to $dim_k(\mathrm{Ext}^1_\Lambda(S_i, S_j))$.

Since P is the projective cover of rP_i, P is also the projective cover of rP_i/r^2P_i. Because projective cover is unique up to isomorphism, we have the multiplicity of P_j in P is equivalent to the multiplicity of S_j in rP_i/r^2P_i.

We have $\mathrm{Hom}_\Lambda(rP_i/r^2P_i, S_j) \cong \mathrm{Hom}_\Lambda(S_j, rP_i/r^2P_i)$ as vector space over k by rP_i/r^2P_i is semisimple and $\mathrm{Hom}_\Lambda(P_j, rP_i/r^2P_i) \cong \mathrm{Hom}_\Lambda(P_j/rP_j, rP_i/r^2P_i) \cong \mathrm{Hom}_\Lambda(S_j, rP_i/r^2P_i)$. But $\mathrm{Hom}_\Lambda(P_j, rP_i/r^2P_i) = \mathrm{Hom}_\Lambda(\Lambda e_j, re_i/r^2e_i)$. Since e_j is primitive idempotent and for all f in $\in \mathrm{Hom}_\Lambda(\Lambda e_j, re_i/r^2e_i)$, f is determined by $f(e_j)$, $\mathrm{Hom}_\Lambda(\Lambda e_j, re_i/r^2e_i)$ is isomorphic to $e_j(r/r^2)e_i$. Thus $dim_k(\mathrm{Ext}^1_\Lambda(S_i, S_j)) = dim_k \mathrm{Hom}_\Lambda(rP_i/r^2P_i, S_j) = dim_k(e_j(r/r^2)e_i)$.

□

Definition 1.9. ***Artin R-algebra***. *Let R be a commutative artin ring and let Λ be an R-algebra. Λ is said to be an artin R-algebra if Λ is finitely generated as an R-module.*

Definition 1.10. *Basic artin algebra. An artin algebra Λ is basic if $\Lambda = P_1 \oplus \cdots \oplus P_n$ where P_i is indecomposable projective module, and $P_i \ncong P_j$ for $i \neq j$.*

Clearly, if a quiver Γ over a field k has no cycles, the path algebra $k\Gamma$ is an artin k-algebra. In proposition 1.10, we have described the associated quiver for a basic finite dimensional algebra by using simples and $dim_k(\mathrm{Ext}^1_\Lambda(S_i, S_j))$. Motivated by that, we associate with any artin algebra Λ a quiver such that the vertices are simples and there is a arrow between vertices i and j if $\mathrm{Ext}^1_\Lambda(S_i, S_j) \neq 0$.

Example 1.5. *Let k be a field.* $T = \begin{bmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{bmatrix}$ *be the 3×3 matrix k-algebra.*

The associated quiver of T is the quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$ denoted as Γ and $k\Gamma \cong T$.

Proof. Let $e_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $a = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ then $ba = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

So $T \cong ke_1 + ke_2 + ke_3 + ka + kb + kba = k\Gamma$. □

A **representation** (V, f) of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ over a field k is a collection of finite dimensional vector spaces $\{V_i \mid i \in \Gamma_0\}$ together with a $k-$linear map $f : V_i \to V_j$ for each arrow $i \to j$.

We consider the category of finitely generated modules of $k\Gamma$ as the representation category of $k\Gamma$.

For a finite dimensional k-algebra Λ with k a field, We call it **finite representation type** if there is only a finite number of isomorphism classes of finitely generated indecomposable left Λ-modules.

1.3 Duality and transpose

1.3.1 D-functor

Let Λ be a ring and let $B \subset A$ where B, A are Λ-modules. We call A an **essential extension** of B if the intersection of each non-zero submodule of A with B is not zero. Let $f : A \to I$ be a monomorphism where I is injective. We call f an **injective envelop** if I is an essential extension of $\operatorname{Im} f$.

Let R be a commutative artin ring, so R has only a finite number of isomorphism classes simple modules denoted as $\{S_1, \ldots, S_n\}$. Let $S_i \to I_i$ be the injective envelop which exists and let $J = \oplus_{i=1}^{n} I_i$.

Proposition 1.11. *Let X be an R-module of finite length and let $D = \operatorname{Hom}_R(\ , J)$. Then we have the following.*

1. $\operatorname{Hom}_R(S_i, S_i) \cong D(S_i) \cong S_i, i \in \{1, \ldots, n\}$.
2. $m_{S_i}(D(X)) = m_{S_i}(X), i \in \{1, \ldots, n\}$
3. *D as a contravariant R-functor is a duality.*

Proof.

1. Let $S_i \cong R/m_i$, where m_i is the maximal ideal of R correspond to S_i. Then $\operatorname{Hom}_R(S_i, S_i) \cong \operatorname{Hom}_R(R/m_i, S_i)$. Since the morphism $R \to S_i$ maps m_i to zero, we have that $\operatorname{Hom}_R(R/m_i, S_i) \cong \operatorname{Hom}_R(R, S_i) \cong S_i$. Since the morphism $S_i \to J$ maps S_i to either zero or S_i, we have that $D(S_i) \cong \operatorname{Hom}_R(S_i, S_i)$. Thus we have that $\operatorname{Hom}_R(S_i, S_i) \cong D(S_i) \cong S_i$.

2. We prove it by induction on the the length of X. Obviously, when $l(X) = 0$ or $l(X) = 1$, the hypothesis is satisfied. We assume that when $l(X) \leq m-1$, the hypothesis is satisfied. Let $l(X) = m$, we consider the following exact sequence $0 \to X' \to X \to X'' \to 0$, where $l(X') = 1$. Applying the functor D, we have the exact seqence $0 \to D(X'') \to D(X) \to D(X') \to 0$ by J

being injective. Since the length of both X', X'' is less than m, we have that $m_{S_i}(D(X')) = m_{S_i}(X')$ and $m_{S_i}(D(X'')) = m_{S_i}(X'')$. Thus $m_{S_i}(D(X)) = m_{S_i}(X)$.

3. It is straight forward that D is an R-functor. From (2), we know that $l(X) = l(D^2(X))$. To prove D is a duality, it is enough to show $\phi : X \to D^2(X)$, given as $\phi(x)(f) = f(x)$ for $x \in X$ and $f \in D(X)$, is a monomorphism. For each $x \neq 0 \in X$, if $\phi(x) = 0$, then for all $f \in D(X)$, $f(x) = 0$. Let Rx be the submodule of X generated by x. Since Rx is not zero, $R/r(Rx) \neq 0$ by Nakayama's lemma where r is the radical of R. Then we have a map $h : R/r(Rx) \to J$ such that $h(x) \neq 0$, and we can extend h to a map $k : X \to J$ such that $k(x) \neq 0$ by J is injective. So x is not in the kernel of ϕ. Then ϕ is a monomorphism. Thus D is a duality on $\operatorname{mod} R$.

□

The following corollary is a direct result of the proposition.

Corollary 1.11.1. $l(D(X)) = l(X)$.

Let Λ be an artin R-algebra and let X be a module in $\operatorname{mod}\Lambda$ and $\lambda \in \Lambda^{op}$. $D(X)$ is considered as a Λ^{op}-module by defining for each f in $D(X)$, $(f\lambda)(x) = f(\lambda x)$. $D(X)$ is a finitely generated Λ^{op}-module, i.e. X is a finitely generated Λ-module. Thus $D : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{op}$ is a contravariant R-functor. And $\phi : X \to D^2(X)$ is still an isomorphism, since $\phi(\lambda x)(f) = f(\lambda x) = \phi(x)(f\lambda) = (\lambda\phi(x))f$ where $f \in D(X), \lambda \in \Lambda$. We have an isomorphism between $1_{\operatorname{mod}\Lambda}$ and D^2 and similarly an isomorphism between $1_{\operatorname{mod}\Lambda^{op}}$ and D^2. So we have proved the following proposition.

Proposition 1.12. *Let Λ be an artin R-algebra, $D : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{op}$ as an contravariant funtor is a duality, with the inverse $D : \operatorname{mod}\Lambda^{op} \to \operatorname{mod}\Lambda$.*

1.3.2 The functor $\operatorname{Hom}_\Lambda(-, \Lambda)$

Let Λ be an artin algebra and let A be a finitely generated Λ-module. We consider $\operatorname{Hom}_\Lambda(A, \Lambda)$ as a finitely generated Λ^{op}-module by defining $(f\lambda)(a) = f(a)\lambda$ where $f \in \operatorname{Hom}_\Lambda(A, \Lambda), \lambda \in \Lambda, a \in A$. We denote $\operatorname{Hom}_\Lambda(A, \Lambda)$ as A^*. It is straightforward that $\operatorname{Hom}_\Lambda(-, \Lambda)$ is a R-functor. Since $\operatorname{Hom}_\Lambda(\Lambda, \Lambda) \cong \Lambda_\Lambda$, so $\Lambda^{**} \cong \Lambda$. Thus $\phi_\Lambda : \Lambda \to \Lambda^{**}$ is an isomorphism in $\operatorname{mod}\Lambda$.

Proposition 1.13. *Let P be a indecomposable projective Λ-module, then P^* is projective in $\operatorname{mod}\Lambda^{op}$ and $P \cong P^{**}$*

Proof. We know that ${}_\Lambda\Lambda^* \cong \Lambda_\Lambda$ is projective in $\operatorname{mod}\Lambda^{op}$. Since P is a direct summand of Λ, P^* is a direct summand of Λ^*. Thus P^* is projective in $\operatorname{mod}\Lambda^{op}$. Similarly, since $\Lambda^{**} \cong \Lambda$ and P/rP is simple, we have that $P^{**} \cong P$. □

We use $\mathscr{P}(\Lambda)$ to denote the full subcategory of $\operatorname{mod}\Lambda$ such that the objects are all the projective modules. The following corollary is a immediate consequence of the proposition.

Corollary 1.13.1. *The functor* $\operatorname{Hom}_\Lambda(-,\Lambda) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{op}$ *restricted to* $\mathscr{P}(\Lambda)$ *is a duality* $\mathscr{P}(\Lambda) \to \mathscr{P}(\Lambda^{op})$*, with inverse* $\operatorname{Hom}_\Lambda(-,\Lambda) : \mathscr{P}(\Lambda^{op}) \to \mathscr{P}(\Lambda)$*.*

1.3.3 The transpose and the dual of the transpose

Let Λ be an artin algebra and C be a module in $\operatorname{mod}\Lambda$. Let $P_1 \xrightarrow{f} P \to C \to 0$ be A minimal projective presentation. Applying $\operatorname{Hom}_\Lambda(-,\Lambda)$, we get an exact sequence $0 \to C^* \to P_0^* \xrightarrow{f^*} P_1^* \to \operatorname{Tr} C \to 0$. TrC is the cokernel of f^*. We call $\operatorname{Tr} C$ the **transpose** of C. Obviously, $\operatorname{Tr} C$ is in $\operatorname{mod}\Lambda^{op}$. If C is projective, we have that the minimal projective presentation $0 \to P \to P \to 0$, by the definition of the transpose, $\operatorname{Tr} C = 0$. Similarly, we have that if $TrC = 0$, then C is projective in $\operatorname{mod}\Lambda^{op}$.

Proposition 1.14. *Let C be an indecomposable non-projective module in* $\operatorname{mod}\Lambda$ *and $P_1 \to P_0 \to C \to 0$ be a minimal projective presentation. Then $\sigma : P_0^* \to P_1^* \to \operatorname{Tr} C \to 0$ is a minimal projective presentation in* $\operatorname{mod}\Lambda^{op}$*.*

Proof. In the last section we have seen that $P_i^*, i \in 0,1$ are projective in $\operatorname{mod}\Lambda^{op}$ when P_i is projective in $\operatorname{mod}\Lambda$. If σ is not a minimal projective presentation, then we have $P_1^* \cong P \oplus E$ where $\pi : P \to \operatorname{Tr} C$ is a projective cover in $\operatorname{mod}\Lambda^{op}$. Let $F \to \ker\pi$ be a projective cover. Then $P_0^* = E \oplus F \oplus G$. Since $P_i^{**} = P_i, i \in 0,1$, it contradict that fact that $P_1 \to P_0 \to C \to 0$ being a minimal projective presentation. Thus $\sigma : P_0^* \to P_1^* \to \operatorname{Tr} C \to 0$ is a minimal projective presentation in $\operatorname{mod}\Lambda^{op}$. □

Corollary 1.14.1. *If A and C are indecomposable non-projective module in* $\operatorname{mod}\Lambda$*, we have the following*

1. $\operatorname{Tr}(\operatorname{Tr} C) = C$.
2. $\operatorname{Tr} A \cong \operatorname{Tr} C$ *if and only if* $A \cong C$.
3. $\operatorname{Tr} C$ *is indecomposable in* $\operatorname{mod}\Lambda^{op}$.

Proof.
1. It is a direct implementation from the last proposition and the duality of $\operatorname{Hom}_\Lambda(-,\Lambda)$ on $\mathscr{P}(\Lambda)$.
2. It is a trivial consequence of (1).
3. It is not hard to see that $\operatorname{Tr}(A \oplus B) = \operatorname{Tr}(A) \oplus \operatorname{Tr}(B)$. Since $\operatorname{Tr}(\operatorname{Tr} C) = C$ is indecomposable, $\operatorname{Tr} C$ is indecomposable.

□

We consider **the dual of the transpose DTr** which is applying the D-functor to the transpose. We know that $D(P)$ is injective when P is projective. The following proposition are direct consequence from above.

Proposition 1.15. *1.* $\mathrm{TrD}\, C = 0$*, if and only if* C *is injective in* $\mathrm{mod}\,\Lambda$*.*

2. $\mathrm{TrD}(\mathrm{DTr}\, C) \cong C$*, if* C *is an indecomposable non-projective module in* $\mathrm{mod}\,\Lambda$*.*

3. $\mathrm{DTr}(A_1 \oplus A_2) \cong \mathrm{DTr}\, A_1 \oplus \mathrm{DTr}\, A_2$ *where* $A_1, A_2 \in \mathrm{mod}\,\Lambda$*.*

4. For non-projective indecomposable modules A *and* B *in* $\mathrm{mod}\,\Lambda$*,* $\mathrm{DTr}\, A \cong \mathrm{DTr}\, B$ *if and only if* $A \cong B$*.*

1.4 Projectivization

In this section, we want to show the connection between path algebras and basic artin algebras. For an artin algebra Λ, we introduce the endomorphism algebra $\Gamma_A = End_\Lambda(A)^{op}$ where A is in $\mathrm{mod}\,\Lambda$. Clearly, $\mathrm{Hom}_\Lambda(A, -)$ is a functor between $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}\,\Gamma_A$. We denote $\mathrm{Hom}_\Lambda(A, -)$ as e_A. In addition, $\mathrm{add}\, A$ denote the full subcategory of $\mathrm{mod}\,\Lambda$ where the objects are $\{X \mid X \in \mathrm{mod}\,\Lambda, \exists Y \in \mathrm{mod}\,\Lambda, \exists n \in \mathbb{N}, A^n \cong X \oplus Y\}$.

Proposition 1.16. *Let* A *be a finitely generated module of an artin algebra* Λ*. For* $X \in \mathrm{add}\, A$ *and* $Y \in \mathrm{mod}\,\Lambda$*,* $e_A : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma_A$ *has the follwing properties.*

1. $e_A : \mathrm{Hom}_\Lambda(X, Y) \to \mathrm{Hom}_\Gamma(e_A(X), e_A(Y))$ *is an isomorphism.*

2. $e_A(X)$ *is in* $\mathscr{P}(\Gamma_A)$ *where* $\mathscr{P}(\Gamma_A)$ *is the full subcategory of* $\mathrm{mod}\,\Gamma_A$ *whose objects are all projective modules in* $\mathrm{mod}\,\Gamma_A$*.*

3. $e_A \mid_{\mathrm{add}\, A}: \mathrm{add}\, A \to \mathscr{P}(\Gamma_A)$ *is an equivalence of categories.*

Proof. 1. For each $f \in \mathrm{Hom}_\Lambda(X, Y)$, $e_A(f) = \mathrm{Hom}_\Gamma(A, f)$. Clearly, e_A is surjective. For a non-zero map f in $\mathrm{Hom}_\Lambda(X, Y)$, since $X \in \mathrm{add}\, A$, $e_A(f) \neq 0$. Then it is an isomorphism.

2. Clearly, $e_A(X)$ is a summand of $e_A(A^n)$ for some $n \in \mathbb{N}$. Since $e_A(A^n) = \mathrm{Hom}_\Lambda(A, A^n) \cong \mathrm{Hom}_\Lambda(A, A)^n \cong \Gamma_A^n$ is projective in $m\,\mathrm{mod}\,\Gamma_A$, then $e_A(A^n)$ is projective in $m\,\mathrm{mod}\,\Gamma_A$.

3. From (1), we have $e_A \mid_{\mathrm{add}\, A}$ is faithful and full. For any $P \in \mathscr{P}(\Gamma_A)$, we have $P \oplus Q \cong \Gamma_A^n$. So there is a idempotent $e_A(f) : e_A(A^n) \cong \Gamma_A^n \to e_A(A^n)$ that $ker(e_A(f)) = P$. Then, we have the left exact sequence $P \rightarrowtail e_A(A^n) \xrightarrow{e_A(f)} e_A(A^n)$. Because e_A preserve left exactness, we also have $ker f \to A^n \xrightarrow{f} A^n$, there $e_A(\ker f) = P$. Since $e_A(f)$ is idempotent, f is idempotent. So f

is split, $ker f$ is in $\operatorname{add} A$. Then $e_A \mid_{\operatorname{add} A}$ is dense. Thus $e_A \mid_{\operatorname{add} A}$ is an equivalence.

□

We use $\operatorname{mod} P$ to denote the full subcategory of $\operatorname{mod}\Gamma$ such that X is in $\operatorname{mod} P$ if and only if P_0, P_1 are in $\operatorname{add} P$ where $P_1 \to P_0 \to X$ is the minimal projective presentation of X.

Proposition 1.17. *Let P be a projective Γ-module, $e_P \mid_{\operatorname{mod} P}: \operatorname{mod} P \to \operatorname{mod}\Gamma_P$ is an equivalence of categories.*

Proof.

- Dense. For any $X \in \operatorname{mod}\Gamma_P$, there is a projective minimal presentation $P_1 \xrightarrow{g} P_0 \to X \to 0$. From proposition 1.16, we know there is a $Q_i \in \operatorname{add} P$ that $e_P(Q_i) = P_i$. So we have a right exact sequence $Q_1 \xrightarrow{f} Q_0 \twoheadrightarrow coker f$ where $e_P(f) = g$. Because P is projective, $\operatorname{Hom}_\Lambda(P, -)$ is exact functor. Then $X = e_P(coker f)$. Thus $e_P \mid_{\operatorname{mod} P}$ is dense.

- Faithful and full. For any A and B in $\operatorname{mod} P$, let $P_1 \to P_0 \to A \to 0$ be the minimal projective presentation of A. Since $\operatorname{Hom}_\Lambda(-, B)$ and e_P both preserve left exactness, we have following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \operatorname{Hom}_\Lambda(A,B) \arrow[r] \arrow[d, "e_p(1)"] & \operatorname{Hom}_\Lambda(P_0,B) \arrow[r] \arrow[d, "e_p(2)"] & \operatorname{Hom}_\Lambda(P_1,B) \arrow[d, "e_p(3)"] \\
0 \arrow[r] & \operatorname{Hom}_\Lambda(e_p(A),e_p(B)) \arrow[r] & \operatorname{Hom}_\Lambda(e_p(P_0),e_p(B)) \arrow[r] & \operatorname{Hom}_\Lambda(e_p(P_1),e_p(B))
\end{tikzcd}$$

  Since P_0 and P_1 is in $\operatorname{add} P$, by proposition 1.16, we know $e_p(2)$ and $e_p(3)$ is isomorphism. So $e_p(1)$ is also isomorphism. Thus $e_P \mid_{\operatorname{mod} P}$ is faithful and full.

□

Let P be the sum of all the indecomposable projective Λ-modules. We can see that $\operatorname{mod} P$ is the same as $\operatorname{mod}\Lambda$ since every Λ-module has minimal projective presentation. Thus we have the corollary as following.

Corollary 1.17.1. *Let P be the sum of all the indecomposable projective Λ-modules. Then $e_P : \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma_P$ is an equivalence of categories.*

Definition 1.11. *Morita equivalence. Let Γ, Λ be two artin algebra. They are said to be morita quivalent if and only if $\operatorname{mod}\Gamma \cong \operatorname{mod}\Lambda$.*

If we choose P as the sum of one from each isomorphic class of the indecomposable projective Λ-module. Then $\Gamma_P = End(P)^{op}$ is a basic artin algebra.

Observation 1.18. *By corollary 1.17.1, every artin algebra is morita equivalent to a basic artin algebra.*

Morita equivalence explains the connetion between an arbitrary artin algebra and a basic endomorphism algebra. We will use this property to construct the Auslander algebra of an artin algebra.

1.5 Block decomposition

For an artin algebra Λ, we could decompose it in to a product of indecomposable artin algebras. Let $1 = e_1 + e_2 + \cdots + e_n$ be the sum of primitive orthogonal idempotents of Λ. We can easily see that $\Lambda = e_1\Lambda \times e_2\Lambda \times \cdots \times e_n\Lambda$ is the product decomposition and e_i is the primitive idempotent in $e_i\Lambda$. $e_i\Lambda$ is indecomposable follows from $e_i\Lambda$ is primitive. We call $e_i\Lambda$ the **blocks** of Λ.

Example 1.6. *In quiver $\Lambda : \cdot \to \cdot \to \cdot$, the identity is the sum of all the vertices, $1 = e_1 + e_2 + e_3$. So the block decomposition is $\Lambda = e_1\Lambda \times e_2\Lambda \times e_3\Lambda$. Each component of the decomposition is the natural indecomposable projective module.*

As an artin algebra could be written as a direct sum of finite copies of indecomposable projective modules, we want to investigate how to decompose it to projective blocks.

Definition 1.12. *Block partition. Let $\mathscr{P}$ be the set of all indecomposable projective modules of aritin algebra Λ. The $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2 \cup \cdots \cup \mathscr{P}_n$ is block partition if*

1. *Let $P \in \mathscr{P}_i and P \in \mathscr{P}_j, i \neq j$, then* $\mathrm{Hom}_\Lambda(P, Q) = 0$.
2. *If P and Q are in the same $\mathscr{P}_i$, there is a chain $P = Q_1 - Q_2 - \cdots - Q_n = Q$ in $\mathscr{P}_i$ with nozero map from Q_i to Q_{i+1} or Q_{i+1} to Q_i.*

We will prove the block partition actually give the block decomposition of an artin algebra Λ.

Proposition 1.19. *Let $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2 \cup \cdots \cup \mathscr{P}_n$ be the block partition of indecomposable projective modules for an artin algebra Λ. Let $\Lambda = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ where P_i is the sum of the indecomposable modules in $\mathscr{P}$. The $\Lambda \cong End_\Lambda(\Lambda)^{op} = End_\Lambda(P_1)^{op} \times End_\Lambda(P_2)^{op} \times \cdots \times End_\Lambda(P_n)^{op}$ is the block decomposition of Λ.*

Proof. Λ is isomorphic to $End_\Lambda(\Lambda)^{op}$ since all f in $End_\Lambda(\Lambda)^{op}$ are determined by $f(1_\Lambda)$. Suppose $End_\Lambda(P_i)^{op}$ is decomposable, let $End_\Lambda(P_i)^{op} = End_\Lambda(P_i')^{op} \times End_\Lambda(P_i'')^{op}$. So $\mathrm{Hom}_\Lambda(P_i', P_i'') = 0$ and $\mathrm{Hom}_\Lambda(P_i'', P_i') = 0$, then P_i' and P_i'' are in the different block partition which contradicts the assumption. Then $1_{End_\Lambda(\Lambda)^{op}} = 1_{End_\Lambda(P_1)^{op}} + \cdots + 1_{End_\Lambda(P_n)^{op}}$, is the decompostion of primitive orthogonal idempotents. Thus the $\Lambda \cong End_\Lambda(\Lambda)^{op} = End_\Lambda(P_1)^{op} \times End_\Lambda(P_2)^{op} \times \cdots \times End_\Lambda(P_n)^{op}$ is the block decomposition. □

Observation 1.20. *Let Λ be an indecomposable artin algebra, the block partition of Λ only contains one component formed by all the indecomposable projective modules up to isomorphism.*

2 Almost split sequences

In this chapter, we introduce almost split sequences and irreducible morphisms referring to chapter 4 and 5 in [2]. We first look at the connection between the covariant defect and the contravariant defect of a exact sequence. Based on this, we present the proof of the existence theorem of almost split sequences. We also give an example for PID to illustrate the irreducible morphisms.

2.1 Defects of exact sequences

Definition 2.1. *Let Λ be an artin R-algebra and let $\delta : 0 \to A \to B \to C \to 0$ be an exact sequence in* $\operatorname{mod}\Lambda$*. We define the covariant defect δ_* of the exact sequence and the contravariant defect δ^* of the exact sequence by the following.*

$$0 \to \mathrm{Hom}_\Lambda(C,\ ) \to \mathrm{Hom}_\Lambda(B,\ ) \to \mathrm{Hom}_\Lambda(A,\ ) \to \delta_* \to 0$$
$$0 \to \mathrm{Hom}_\Lambda(\ ,A) \to \mathrm{Hom}_\Lambda(\ ,B) \to \mathrm{Hom}_\Lambda(\ ,C) \to \delta^* \to 0$$

Clearly, both $\delta_*(X)$ and $\delta^*(X)$ for each $X \in \operatorname{mod}\Lambda$ are finitely generated R-module. For an R-module M, we use $< M >$ to denote the length of M.

Theorem 2.1. *Let $\delta : 0 \to A \to B \to C \to 0$ be an exact sequence in* $\operatorname{mod}\Lambda$ *where Λ is an artin R-algebra. We have $< \delta_*(DTrX) >=< \delta^*(X) >$.*

Proof. Let $P_1 \to P_0 \to X \to 0$ be a minimal projective presentation of X and let Z be a Λ-module. Since the funtor $- \otimes_\Lambda Z$ preserves right exactness and the functor $\mathrm{Hom}_\Lambda(-, Z)$ preserves left exactness. We use $-^*$ to denote $\mathrm{Hom}_\Lambda(-, \Lambda)$. We have the following exact sequences.

$$\begin{tikzcd}
 & & P_0^* \otimes_\Lambda Z \arrow[r] \arrow[d, "\cong"] & P_1^* \otimes_\Lambda Z \arrow[r] \arrow[d, "\cong"] & TrX \otimes_\Lambda Z \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Hom}_\Lambda(X,Z) \arrow[r] & \mathrm{Hom}_\Lambda(P_0,Z) \arrow[r] & \mathrm{Hom}_\Lambda(P_1,Z) & &
\end{tikzcd}$$

We define $\phi : \Lambda^* \otimes_\Lambda Z \to \mathrm{Hom}_\Lambda(\Lambda, Z)$ by $\phi(f \otimes z)(\lambda) = f(\lambda)z$ where $f \in \Lambda^*, z \in Z, \lambda \in \Lambda$. Then ϕ is an isomorphism. $\phi : P_i^* \otimes_\Lambda Z \to \mathrm{Hom}_\Lambda(P_i, Z)$ is defined by $\phi(f \otimes z)(x) = f(x)z$ where $f \in P_i^*, z \in Z, x \in P_i$. Since P_i is projective in $\operatorname{mod}\Lambda$, we have $P_i^* \otimes_\Lambda Z \cong \mathrm{Hom}_\Lambda(P_i, Z), i \in \{0, 1\}$. Then we have the following exact sequence.

$$0 \to \mathrm{Hom}_\Lambda(X,Z) \to \mathrm{Hom}_\Lambda(P_0,Z) \to \mathrm{Hom}_\Lambda(P_1,Z) \to TrX \otimes_\Lambda Z \to 0$$

We use $< -, - >$ to denote $< \mathrm{Hom}_\Lambda(-,-) >$. Since $\mathrm{Hom}_\Lambda(Z, DTrX) \cong D(TrX \otimes_\Lambda Z)$, we have $< P_1, Z > - < P_0, Z > + < X, Z > - < Z, DTrX >= 0$ since the module length is an invariant of the functor D.

By the definition of defects, we have that

$$< \delta_*(DTrX) >=< A, DTrX > - < B, DTrX > + < C, DTrX >$$
$$< \delta^*(X) >=< X, C > - < X, B > + < X, A >$$

Then we have$< \delta^*(X) > - < \delta_*(DTrX) >=< P_0, C > - < P_1, C > + < P_0, A > - < P_1, A > + < P_1, B > - < P_0, B >=< \delta^*(P_0) > - < \delta^*(P_1) >$ Since P_i is projective, $\mathrm{Hom}_\Lambda(P_i, -)$ is exact. So $< \delta^*(P_i) = 0 >$. Thus $< \delta^*(X) > - < \delta_*(DTrX) >=0$ □

Corollary 2.1.1. *Let $\delta : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence in* $\mathrm{mod}\,\Lambda$. *Then for each $X \in \mathrm{mod}\,\Lambda$, the following are equivalent.*

1. *Every morphism $h : X \to C$ factors through $g : B \to C$.*

2. *Every morphism $t : A \to DTrX$ factors through $f : A \to B$.*

Proof. (1) implies $< \delta^*(X) >= 0$. Thus $< \delta_*(DTrX) >= 0$ which implies (2). Similarly, we have that (2) implies (1). □

By duality we have the following corollary.

Corollary 2.1.2. *Let $\delta : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence in* $\mathrm{mod}\,\Lambda$. *Then for each $X \in \mathrm{mod}\,\Lambda$, the following are equivalent.*

1. *Every morphism $h : \mathrm{TrD}\,X \to C$ factors through $g : B \to C$.*

2. *Every morphism $t : A \to X$ factors through $f : A \to B$.*

2.2 Almost split sequences

Let $A \xrightarrow{f} B$ be a monomorphism. If there is $h : B \to A$ such that $hf = 1_A$, then f is a **split monomorphism**. Similarly, let $B \xrightarrow{g} C$ be an epimorphism. If there is $k : C \to B$ such that $gk = 1_C$, then g is a **split epimorphism**.

Let $\sigma : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence. If f or g split, then they both splits and we call σ a **split exact sequence**.

If $A \xrightarrow{f} B$ is not a split monomorphism and for each morphism $h : A \to Y$ which is not a split monomorphism, h factors through f, we call f **left almost split**. Similarly, if $B \xrightarrow{g} C$ is not a split epimorphism and for each morphism $h' : Y' \to C$ which is not a split epimorphism, h' factors through g, we call g **right almost split**.

For a morphism $A \xrightarrow{f} B$, if every $g : A \to A$ which makes $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ A \arrow[ur, "f"'] & \end{tikzcd}$ commute is an automorphsim, we say f is **right minimal**. Similarly, for a morphism $B \xrightarrow{f} C$, if every $g : C \to C$ which makes $\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "f"'] & C \arrow[d, "g"] \\ & C \end{tikzcd}$ commute is an automorphsim, we say f is **left minimal**.

Observation 2.2. *Monomorphism are right minimal.*

Example 2.1. *For an artin algbra Λ, P is a indecomposable projective module, then $i : rP \hookrightarrow P$ is right almost split morphism. The map i is a natural inclusion, so it is non-split epimorphism. For each morphism $A \xrightarrow{g} P$ which is not a split epimorphism, $Im(g)$ is in or equal to rP since P is indecomposable. Then,* $\begin{tikzcd} rP \arrow[r, hook, "i"] & P \\ A \arrow[u, "f"] \arrow[ur, "g"'] & \end{tikzcd}$ *commutes. Thus i is right almost split.*

So we have determined $i : rP \hookrightarrow P$ is right almost split for each indecomposable projective module P. But is it the unique right almost split morphism to P? For a morphism $g : A \to rP$, the induced morphism $A \oplus rP \to P$ is also right almost split. If a morphism $f : A \to P$ is right almost split, Imf must be equal to rP. Obviously, $i : rP \hookrightarrow P$ is right minimal.

We call a morphism **minimal right almost split** if it is both right minimal and right almost split. Similarly, We call a moprhism **minimal left almost split** if it is both left minimal and left almost split. The morphism $i : rP \hookrightarrow P$ is minimal right almost split.

There are some straightforward observations from the definition of almost split morphism.

Lemma 2.3. 1. *Let $f : A \to B$ be right almost split, then B is an indecomposable module.*

2. *Let $g : B \to C$ be left almost split, then B is an indecomposable module.*

Proof. 1. Assume B is decomposable and $B \cong B_1 \oplus B_2$ where B_1 and B_2 are both non-zero. Since the natural inclusion $B_1 \to B$ and $B_2 \to B$ is not split epimorphism so they factor through f. So 1_B factors through f which implies f is a split epimorphism. Thus, f is not right almost split.

2. It follows by duality.

$\square$

Proposition 2.4. *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be a exact sequence.*

1. *If g is not a split epimorphism and C is indecomposable, then f is left minimal.*

2. *If f is not a split monomorphism and A is indecomposable, then g is right minimal.*

Proof. 1. Assuming f is not left minimal, then there is a non-isomorphic endomorphism $i : B \to B$ such that $if = f$. But since C is indecomposable, $\mathrm{End}_\Lambda(C)$ is local. Then there are some $n \in \mathbb{N}$ such that $i^n = 0$. So $f = i^n f = 0$ which contradicts the hypothesis. Thus g is left minimal.

2. It follows by duality.

$\square$

Proposition 2.5. *Let $f : B \to C$ be a minimal right almost split morphism such that C is not projective. Then we have the following.*

1. *f is surjective.*

2. *For the exact sequence $0 \to \ker f \xrightarrow{g} B \xrightarrow{f} C \to 0$, we have that $\ker f \cong \mathrm{DTr}\, C$ and g is a minimal left almost split morphism.*

Proof. 1. The map f being surjective follows by that the projective cover of C, which is not a split epimorphism, factors through f.

2. C is indecomposable by f is right almost split. By proposition 2.4, the map g is left minimal.

   We now show $\ker f$ is indecompsable. We assume $\ker f = A_1 \oplus \cdots \oplus A_n, n \in \mathbb{N}$ with A_i indecomposable and non-zero. Since f is non-split, g is non-split monomorphism. Then there is an $A_k \in \{A_1, \ldots, A_n\}$ such that $j : ker f \to A_k$ dose not fact through g. Then we consider the following pushout diagram.

$$\begin{tikzcd}
0 \arrow[r] & \ker f \arrow[r, "g"] \arrow[d, "j"] & B \arrow[r, "f"] \arrow[d, "i"] & C \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & A_k \arrow[r, "g^*"] & PO \arrow[r, "f^*"] & C \arrow[r] & 0
\end{tikzcd}$$

   The map g^* being non-split follows by that j dose not fact through g. Since A_k is indecomposable, by proposition 2.4, f^* is also right minimal. So PO is isomorphic to B. Thus $\ker f$ is indecomposable.

We now show that $\ker f \cong \mathrm{DTr}\, C$. Since g is not a split monomorphism, we know that $\ker f$ is not injective. Let Y be a non-injective indecomposable module such that $Y \ncong \mathrm{DTr}\, C$. So $\mathrm{TrD}\, Y$ exists and is not isomorphic to C. Since $\mathrm{TrD}\, Y$ is indecomposable and f is right almost split, all morphisms $\mathrm{TrD}\, Y \to C$ factor through f. By corollary 2.1.2, we know that all morphisms $\ker f \to Y$ factor through g. Thus $\ker f \ncong Y$ since g is non-split. So $\ker f \cong \mathrm{DTr}\, C$. Thus to prove that g is left almost split, we now only need to show that each non-isomorphism $h : \ker f \to \ker f$ factors through g.

We know $\mathrm{Im}\, h$ is a proper submodule of $\ker f$, so $\ker f \to \mathrm{Im}\, h$ factors through g, thus h factors through g.

□

We call an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ **almost split** if f is left almost split and g is right almost split.

Proposition 2.6. *The following are equivalent for an exact sequence* $\sigma : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$.

1. *The sequence σ is almost split.*
2. *The map g is minimal right almost split.*
3. *The map f is minimal left almost split.*
4. *A is indecomposable and g is right almost split.*
5. *C is indecomposable and f is left almost split.*
6. *$A \cong \mathrm{DTr}\, C$ and g is right almost split.*
7. *$C \cong \mathrm{TrD}\, A$ and f is left almost split.*

Proof. In the proof of the last proposition, we have seen the equivalence between (1),(2),(4) and also have seen (2) implies (6).

(6) ⇒ (2), since g is right almost split which implies C is indecomposable and $A \cong \mathrm{DTr}\, C$, A is indecomposable. So g is minimal right almost split.

The rest follows by duality. □

The following is **the existence theorem of almost split sequence.** In [1], Auslander illustrates the idea in multiple perspectives. The proof is referring to chapter 5 in [2].

Theorem 2.7. *Let Λ be an artin algebra with C in* $\mathrm{mod}\, \Lambda$. *There exists an almost split sequence* $\sigma : 0 \to \mathrm{DTr}\, C \xrightarrow{e} D \xrightarrow{r} C \to 0$.

Proof. By proposition above, it is enough to show r is right almost split then σ is almost split.

As $\operatorname{DTr} C$ is not injective, we can find an non-split exact sequence $0 \to \operatorname{DTr} C \xrightarrow{q} A \xrightarrow{w} B \to 0$.

We have seen that if all $C \to B$ factors through w, then all endomorphisms of $\operatorname{DTr} C$ factor through q in corollary 2.1.1. Since q is non-split, there exists some morphisms from C to B does not factor through w. Thus $\operatorname{Ext}^1_\Lambda(C, \operatorname{DTr} C)$ is non-zero. Let $\Gamma = End_\Lambda(C)^{op}$. $\operatorname{Ext}^1_\Lambda(C, \operatorname{DTr} C)$ is an Γ-module of finte length. We choose the morphism $j : C \to B$ such that in the following pullback diagram, $0 \to \operatorname{DTr} C \to D \to C \to 0$ is in the socle of $\operatorname{Ext}^1_\Lambda(C, \operatorname{DTr} C)$ as a Γ-module.

We consider

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{DTr} C \arrow[r, "e"] \arrow[d, "\cong"] & D \arrow[r, "r"] \arrow[d, "i"] & C \arrow[r] \arrow[d, "j"] & 0 \\
0 \arrow[r] & \operatorname{DTr} C \arrow[r, "q"] & A \arrow[r, "w"] & B \arrow[r] & 0
\end{tikzcd}$$

Claim: $0 \to \operatorname{DTr} C \xrightarrow{e} D \xrightarrow{r} C \to 0$ is an almost split sequence.

1. We first prove $r : D \to C$ is not split. If r splits, we have that $j = yi \mid_C$ which contradicts that j does not factor through w. So r is not split.

2. We want to prove for each non-split epimorphism $h : E \to C$, h factors through r.

   We assume that h factors through r i.e. there is a morphism $k : E \to D$ such that $rk = h$. Then we have the following pullback diagram.

$$\begin{tikzcd}
E \arrow[drr, bend left, "1_C"] \arrow[dr, dashed, "\exists ! q"] \arrow[ddr, bend right] & & \\
& PB \arrow[r, two heads, "t"] \arrow[d, "y"] & E \arrow[d, "h"] \arrow[dl, dashed, "k"] \\
& D \arrow[r, two heads, "r"] & C
\end{tikzcd}$$

   Claim: h factors through r if and only if $t : PB \to E$ splits.

   There is an unique $q : E \to PB$ such that $1_C = tq$. Then t is a split epimorphism. It is straightforward that when t splits, h factors through r. Thus h factors through r if and only if t splits.

   So we have a split exact seuqence $0 \to \operatorname{DTr} C \xrightarrow{l} PB \xrightarrow{t} E \to 0$. The map l splits if and only if all endomorphisms of $\operatorname{DTr} C$ factor through l, which is equivalent to that each $C \xrightarrow{s} E$ factors through t by corollary 2.1.1. Thus it is enough to show $C \xrightarrow{s} E$ factors through t, then h factors through r.

We consider the following pullback diagram.

$$
\begin{tikzcd}
C \arrow[dr, dashed, "z"] \arrow[drr, bend left, "s"] \arrow[ddr, dashed, bend right, "v"'] \arrow[dddr, dashed, bend right, "m"'] & & \\
& PB \arrow[r, two heads, "t"] \arrow[d, "y"] & E \arrow[d, "h"] \\
& D \arrow[r, two heads, "r"] \arrow[d, "i"] & C \arrow[d, "j"] \\
& A \arrow[r, two heads, "w"] & B
\end{tikzcd}
$$

So the image of jhs in $\operatorname{coker}\operatorname{Hom}_\Lambda(C, w)$ as a Γ-module is a proper submodule of the image of j in $\operatorname{coker}\operatorname{Hom}_\Lambda(C, w)$. Since the image of j in $\operatorname{coker}\operatorname{Hom}_\Lambda(C, w)$ is a simple Γ-module by our choice, the image of jhs in $\operatorname{coker}\operatorname{Hom}_\Lambda(C, w)$ is zero. Thus there is $m : C \to A$ such that $wm = jhs$. Since D is a pullback, there is an unique $v : C \to D$ such that $rv = hs$. Again since PB is a pullback, there is an unique $z : C \to PB$ such that $tz = h$. So s factors through t.

Thus $0 \to \operatorname{DTr} C \xrightarrow{e} D \xrightarrow{r} C \to 0$ is our desired almost split sequence. □

2.3 Irreducible morphisms

Definition 2.2. *Let Λ be an artin algebra. A morphism $f : A \to B$ in* $\operatorname{mod}\Lambda$ *is called irreducible if f satisfies the following.*

1. *f is not a split monomorphism.*
2. *f is not a split epimorphism.*
3. *If there are $s : A \to X, t : X \to B$ such that $ts = f$, then s is a split monomorphism or t is a split epimorphism.*

Proposition 2.8. *Let $f : A \to B$ be an irreducible morphism in* $\operatorname{mod}\Lambda$. *Then f is either injective or surjective.*

Proof. We consider the induced map $A \xrightarrow{s} A/\ker f \xrightarrow{t} B$. Obviously, $ts = f$. If s is a split monomorphism then f is injective. If t is a split monomorphism then f is surjective. □

Example 2.2. *Let P be a Principal Integral Domain. Then the irreducible morphism $P \to P$ of in the form $P \xrightarrow{[p]} P$ where p is a prime element in P and $[p]$ is the 1×1 matrix.*

Proof. We consider the following communicate diagram such that $ts = f$ and f is irreducible. Let $f = [k], k \in P$.

$$\begin{tikzcd} & M \arrow[dr, "t"] & \\ P \arrow[ur, "s"] \arrow[rr, "f"] & & P \end{tikzcd}$$

If k is not prime, assuming $k = ab, a, b \in P$, let $s : P \xrightarrow{b} P, t : P \xrightarrow{a} P$, then we have that s is not a split monomorphism and t is not a split epimorphism. But then $ts = f$ which contradicts the fact that f is irreducible.

Let k be prime. Assuming t is surjective, since P is free, t is split epimorphism. Assuming t is not surjective, we have that $kP = \operatorname{Im} f$ where kP is a maximal ideal of P. Then we have $P \to M \to kP \to 0$. Since f is injective, s is injective. So $t(\operatorname{Im} s) = kP$, then $M \cong P \oplus \ker t$. Thus s is a split monomorphism. □

Proposition 2.9. *Let Λ be an artin algebra and let B be an indecomposable module in $\operatorname{mod} \Lambda$. The following are equivalent.*

1. *The morphism $f : A \to B$ in $\operatorname{mod} \Lambda$ is irreducible.*
2. *There exists a morphism $f' : A' \to B$ such that $(f, f') : A \oplus A' \to B, A' \in \operatorname{mod} \Lambda$ is a minimal right almost split morphism.*

Proof.

1. (1) $\Rightarrow$ (2). In theorem 2.7 and example 2.1, we have proved the existence of a minimal right almost split morphism for an indecomposable module. Then let $g : E \to B$ be right minimal almost split. Since f is not a split epimorphism, so f factors through g denoted as $f = gh$. Since g is not a split epimorphism, h is a split monomorphism. Then $E = A \oplus A'$ where A' is $\operatorname{coker} h$. Thus $(f, f') = (f, g \mid_{A'})$.
2. (2) $\Rightarrow$ (1). f is not a split monomorphism by (f, f') is right minimal. f is not a split epimorphism by (f, f') is not a split epimorphism. For each $g : A \to M, t : M \to B$ such that $tg = f$, it is enough to show that when t is not a split epimorphism, g is a split monomorphism, then f is irreducible.

Supose t is not a split epimorphsim, there is $k : M \to A$ such that $fk = t$. We consider the following diagram.

$$\begin{tikzcd}
A \oplus A' \arrow[r, "\left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \arrow[dr, bend right, "{(f, f')}"'] & M \oplus A' \arrow[r, "\left(\begin{smallmatrix} k & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "{(t, f')}"] & A \oplus A' \arrow[dl, bend left, "{(f, f')}"] \\
& B &
\end{tikzcd}$$

So $(fkg, f') \cong (tg, f') \cong (f, f')$. Since (f, f') is right minimal, $(kg, 1)$ is an isomorphism, thus g is a split monomorphism.

$\square$

3 Nakayama Algebras

In this chapter, we introduce the Nakayama Algebras, referring to chapter 4 in [2]. We look at the general form of an almost split sequence of a Nakayama algebra which gives further understanding of what we studied in the last chapter. Later this is useful for studying the representation finite graded trees. We show that every indecomposable module of a Nakayama algebra is uniserial. We prove that the length of a non-projective module is an invariant of DTr. We introduce the Kupisch series of a Nakayama algebra and how to construct a Nakayama algebra from a given admissible sequence.

Definition 3.1. *Uniserials. Let M be a module of an algebra Λ. M is uniserial if it's submodules are totally ordered by inclusion.*

Proposition 3.1. *Let M be a non-zero a finite length module of an algebra Λ. Then The following are equivalent.*

1. *M is uniserial*
2. *M only has one composition series*
3. *The radical filtration, $M \supseteq rM \supseteq r^2M \supseteq \cdots \supseteq r^nM$, is a composition series of M*

Proof. Obviously, the submodules of a uniserial module are uniserial.

- (1) $\Rightarrow$ (2) Assume M is uniserial. Let $M \supseteq F_1 \supseteq \cdots \supseteq F_n$ and $M \supseteq G_1 \supseteq \cdots \supseteq G_n$ be two different composition series of M. Then F_1 and G_1 are both maximal submodules. We assume $F_1 \neq G_1$. But we have $G_1 \subseteq F_1$, or $F_1 \subseteq G_1$ by (1) which contradicts the fact that F_1, G_1 both are maximal submodules. Thus $F_1 = G_1$.

- (2) $\Rightarrow$ (3) From (2), we know there is only one maximal submodule of M which is equivalent to the radical, so the radical filtration is the composition series.

- (3) $\Rightarrow$ (2) The radical rM is a maximal submodule of M and if $rM \neq 0$, it's submodule also only has one maximal submodule. So The radical filtration is a composition series.

□

Observation 3.2. *Let M be an uniserial module of an algebra Λ and $l(M) = n$.*

- *Any submodule is of the form of $r^iM, i \in \{0, 1, \ldots, n\}$.*

- *Let $f : P \to M$ be a projective cover. If P is uniserial, there exist $j \in \{0, \dots, l(P)\}$ such that $ker f \cong r^j P$.*

Definition 3.2. *Nakayama Algebra. Let Λ be an artin algebra. Then Λ is called Nakayama algebra, if all the indecomposable projective modules and all the injective modules are uniserial.*

By duality, an artin algebra Λ is a Nakayama algebra if and only if all the indecomposable projective modules of Λ and Λ^{op} are uniserial.

Example 3.1. *Let $K[x]$ be a polynomial ring over a field K. Then $K[x]/(x^n)$ is a Nakayama algebra when $n \geq 1$. It's composition series is $K[x]/(x^n) \supseteq (x)/(x^n) \supseteq (x^2)/(x^n) \supseteq \cdots \supseteq (x^{n-1})/(x^n) \supseteq 0$. The only simple submodule up to isomorphism is $K[x]/(x)$ and the composition series is also the radical filtration. So $K[x]/(x^n)$ is a uniserial projective module.*

Proposition 3.3. *Let M be an indecomposable module of a Nakayama algebra Λ. The follwoing are equivalent.*

1. *M is uniserial.*
2. *M/rM is simple*
3. *If $f : P \to M$ is a projective cover, then P is uniserial.*

Proof. (1) $\Rightarrow$ (2). M is uniserial implies rM is maximal submodule, so M/rM is simple. (2) $\Rightarrow$ (3). Since $P/rP \cong M/rM$, M/rM is simple implies P is indecomposable then uniserial. (3) $\Rightarrow$ (1) $r^n M \cong r^n Im(f)$. The radical filtration is a composition series of M. Thus M is uniserial. □

Proposition 3.4. *Let M, N be uniserial modules of the Nakayma algebra Λ. Then $N \cong M$ if and only if $l(N) = l(M), N/rN \cong M/rM$.*

Proof. From left side to right side is obvious. Suppose $P \to N$ is a projective cover, then $P \to N/rN$ is also projective cover. We do the same thing to M, so by the uniqueness of projective cover, we have and $P \to N$ and $P \to M$ are projective cover. Then, by observation 3.2, the kernel of $P \to N/rN$ and $P \to M/rM$ are both $r^n P$, where N and M both have the length n. Thus, $N \cong P/r^n P \cong M$. □

Corollary 3.4.1. *Let M, N be uniserial modules of the Nakayma algebra Λ. Then $N \cong M$ if and only if $l(N) = l(M)$ and $soc(M) \cong soc(N)$*

Proof. By duality, $D(soc(N)) \cong DD(D(N)/rD(N)) \cong D(N)/rD(N)$. Then we have $D(M)/rD(M) \cong D(N)/rD(N)$. Since the funcor D preserves the length of modules, we have $l(D(M)) = l(D(N))$. By proposition 3.4, D(N) is isomorphic to D(M). Thus, N is isomorphic to M. □

For artin algebras, in general, the transpose does not always preserve the length of nonprojective module. But for Nakayama algebras, the transpose preserves the length of nonprojective module and also the property of being uniserial.

Proposition 3.5. *Let C be a nonprojective uniserial module of a Nakayama algebra Λ where $l(C) = n$. Then, TrC and $DTrC$ are both uniserial and $l(TrC) = l(DTrC) = l(C)$.*

Proof.
- Suppose $P_1 \to P_0 \to C$ is the minimal projective presentation. P_0 is indecomposable and uniseiral since $P_0 \to C$ is projective cover and C is uniserial. So P_1^* is indecomposable and uniserial since $P_1 \to ker(P_0 \to C)$ is a projective cover. Since $P_0^* \to P_1^* \to TrC$ is a minimal projective presentation, TrC is uniserial. Thus $DTrC$ is uniserial.

- $l(C) = n$ describes the maximal length of the chain $P_1 \to Q_{n-1} \to \cdots \to Q_1 \to P_0$, where Q_i is indecomposable projective and the maps are nonisomorphism. The kernel of $P_0 \to C$ is $r^n P_0$ since C is uniserial. So $P_1 \to r^n P_0$ is a projective cover. Let Q be a projective module. As P_0 is uniserial, $Im(Q \to P_0)$ is of the form of $r^i P_0$. Since Q is projective, $Q \to r^i P_0$ is a projective cover. We choose $Q_i \to r^i P_0$ to be a projective cover. Then we have the chain above. By the uniqueness of projective cover, the maximal length of the chain is $l(C)$.
  Applying the transpose to the chain, we have the chain $\sigma : P_0^* \to Q_1^* \to \cdots \to Q_{n-1}^* \to P_1^* \to TrC$. Similarly, we get $l(TrC)$ is the same as the maximal length of the chain σ which by duality is equal to $l(C)$. Thus, $l(DTrC) = l(TrC) = l(C)$

□

Corollary 3.5.1. *Let C be a nonprojective uniserial module of a Nakayama algebra Λ where $l(C) = n$. If $P \to C$ is a projective cover, then $DTrC \cong rP/r^{n+1}P$.*

Proof. Let $P_1 \to P_0 \xrightarrow{f} C$ be the minimal presentation, then $soc(DTrC) \cong P_1/rP_1$. Since P is uniserial, $soc(rP/r^{n+1}P) \cong r^nP/r^{n+1}P$. We also have $ker(f) \cong r^nP_0$, so $P_1 \to r^nP_0$ is projctive cover. Since both $P \to C$ and $P_0 \to C$ are projective cover, by uniqueness, $r^nP/r^{n+1}P \cong r^nP_0/r^{n+1}P_0 \cong P_1/rP_1$. Thus $soc(DTrC) \cong soc(rP/r^{n+1}P)$. Obviously, $l(rP/r^{n+1}P) = n = l(DTrC)$. By corollary 3.4.1, $DTrC$ is isomorphic to $rP/r^{n+1}P$. □

Proposition 3.6. *Let P be an indecomposable projective module of a Nakayama algebra Λ. Then P/r^nP where $n \leq l(p)$ are uniserial.*

Proof. We prove P/r^nP is uniserial by induction. Since P is indecomposable, P/rP is simple. When $n \leq 2$, it is obvious that P/r^nP is uniserial. Assume $n \geq 3$, $P/r^{n-1}P$ is uniserial whose composition series is $P/r^{n-1} \supseteq rP/r^{n-1} \supseteq r^2P/r^{n-1} \supseteq$

$\cdots \supseteq r^{n-2}P/r^{n-1}P \supseteq 0$. $r^iP/r^{i+1}P$, where $i \leq n-2$, as a composition factor is simple. The compostion series of P/r^nP is that $P/r^n \supseteq rP/r^n \supseteq r^2P/r^n \supseteq \cdots \supseteq r^{n-1}P/r^nP \supseteq 0$. To show P/r^nP is uniserial, it is enough to prove $r^{n-1}P/r^nP$ is simple. There exist a projective module Q such that $Q \to r^2P$ is the projective cover of r^2P. Since $r^{n-2}P/r^{n-1}P$ is simple, Q is indecomposable and uniserial. rQ/r^2Q is simple. But $rQ/r^2Q \to r^{n-1}P/r^nP$ is an epimorphism, so $r^{n-1}P/r^nP$ is simple. $\square$

Observation 3.7. *By uniserial, when $n \leq l(p)$, $l(P/r^nP) = n$, where P is an idecomposable projective module.*

Proposition 3.8. *All indecomposable modules of Nakayama algebra are uniserial.*

Proof. Let Λ be a Nakayama algebra and let M be an arbitrary indecomposable Λ-module. There are $p : P \twoheadrightarrow M$ and $i : M \hookrightarrow I$ where p is a projective cover and i is a injective envelop. Let $I = I_1 \oplus \cdots \oplus I_n, n \in \mathbb{N}$ where $I_i, i \in \{1, \ldots, n\}$ is non-zero indecomposable and $I_i \ncong I_j$ when $i \neq j$. Let $\rho_i : I_1 \oplus \cdots \oplus I_n \to I_i$ be the projection. Let j be the index with maximal length of $\rho_j i(M)$. Let $P = P_1 \oplus \cdots \oplus P_m, m \in \mathbb{N}$ where $P_i, i \in \{1, \ldots, m\}$ is non-zero indecomposable and $P_i \ncong P_j$ when $i \neq j$. Let $l_i : P_i \to P_1 \oplus \cdots \oplus P_m$ be the natural inclusion. Then there is $P_k, k \in \{1, \ldots, m\}$ such that $\rho_j i p l_k : P_k \to \rho_j i(M)$ is a projective cover. By corollary 3.7, $\rho_j i(M) \cong P_k/r^{(l(\rho_j i(M)))}$. Then $P_k/r^{(l(\rho_j i(M)))}$ is also a submodule of M, and in fact a direct summand of M. Since M is indecomposable, M is $P_k/r^{(l(\rho_j i(M)))}$ which is uniserial by proposation 3.6. Thus M is uniserial.

$\square$

Corollary 3.8.1. *Any indecomposable module of a Nakayama algebra is of the form P/r^nP where $n \leq l(p)$, where P is a indecomposable projective module in the algebra.*

Proof. Let M be an indecomposable module of a Nakayama algebra Λ. Then M is uniserial by proposition 3.8. Let $P \xrightarrow{f} M$ be a projective cover, then P is indecomposable by proposition 3.3. Thus there is $n \leq l(p)$ such that $ker f = r^nP$. So $M \cong P/r^nP$. $\square$

For an artin algebra Λ, we define the **top** of a module M to be $M/rad(M)$. By proposition3.4, for a Nakayama algebra, an indecomposable module is determined up to isomorphism by the length and the top, or by the length and the socle.

3.1 Kupisch series

Definition 3.3. *DTr-orbit. Let M be an indecomposable Λ-module where Λ is an artin algebra. The DTr-orbit of M is the collection $\{(DTr)^iM\}_{i\in\mathbf{N}}$ of modules.*

If Λ is an artin algebra of finite representation type, the DTr-orbit is finite. Let $\{(DTr)^0M = M, \dots, (DTr)^nM\}$ be the DTr-orbit of M, then $(DTr)^nM$ is projective or $(DTr)^{n+1}M = (DTr)^0M$.

Definition 3.4. *Kupisch series. Let S be an indecomposable module of a Nakayama algebra Λ. Then the Kupisch series of S is the DTr-orbit of S in the order $\{(DTr)^iS\}_{i\in\mathbf{N}}$ where $DTr^0S = S$.*

Let $\mathfrak{o}_\mathfrak{i}$ denote the Kupisch series of S_i in a Nakayma algebra Λ where S_i is a simple module in $\operatorname{mod}\Lambda$. The correspond projective module set $\{P_i\}_{i\in\mathbb{N}}$ where $P_i \to DTr^iS$ is projective covers is called the **induced kupisch series** of S.

Proposition 3.9. *Let Λ be an indecomposable Nakayama algebra and $\{P_i, \dots, P_n\}$ be the induced Kupisch series of S where S is a simple module in* $\operatorname{mod}\Lambda$*. Then $h : P_{i+1} \to rP_i$ is a projective cover and there is a morphism $f : P_{i+1} \to P_i$.*

Proof. We have $DTr^i(S) \cong rP_i/r^2P_i$ by corollary 3.5.1. Since $P_{i+1} \to DTr^i(S)$ is a projective cover, we have $P_{i+1} \xrightarrow{h} rP_i$ is also a projective cover. Let $P_i \xrightarrow{s} rP_i$ be the natural surjection, then there is a morphism $f : P_{i+1} \to P_i$ that $sf = h$. $\square$

Proposition 3.10. *Let Λ be an indecomposable Nakayama algebra, all the simple Λ-modules are in the same DTr-orbit.*

Proof. We assume that the simple Λ-modules are in two different Kupisch series $\mathfrak{o} = \{S_1, \dots, S_n\}$ and $\mathfrak{o}' = \{S_1', \dots, S_n'\}$. Let $\tilde{\mathfrak{o}} = \{P_1, \dots, P_n\}$ and $\tilde{\mathfrak{o}}' = \{P_1', \dots, P_n'\}$ be the correspond induced Kupisch series. Suppose $\operatorname{Hom}_\Lambda(P_i, P_j') \neq 0$ where P_i is in $\tilde{\mathfrak{o}}$ and P_j' is in $\tilde{\mathfrak{o}}'$. For $\forall f \in \operatorname{Hom}_\Lambda(P_i, P_j')$, $\exists k \in \{0, \dots, l(P_j')\}$ that $Imf = r^kP_j'$ since P_j' is uniserial. So $P_i \to r^kP_j'$ is a projective cover thus $P_i \to r^kP_j'/r^{(k+1)}P_j'$ is a projective cover. But $r^kP_j'/r^{(k+1)}P_j'$ is in $\mathfrak{o}'$ by P_j'/rP_j' is in $\mathfrak{o}'$ and corollary 3.5.1, so P_i is in $\tilde{\mathfrak{o}}'$. Thus if $\mathfrak{o}$ and $\mathfrak{o}'$ are different DTr-orbits, $\operatorname{Hom}_\Lambda(P, P') = 0$ and $\operatorname{Hom}_\Lambda(P', P) = 0$ for $\forall P \in \tilde{\mathfrak{o}}$ and $\forall P' \in \tilde{\mathfrak{o}}'$. But that means $\{P_1, \dots, P_n\}$ and $\{P_1', \dots, P_n'\}$ are in different block partitions. Then Λ is decomposable by propostion 1.19 which contradicts the assumption. Thus all the simple Λ-modules are in the same DTr-orbit. $\square$

An indecomposable projective module P is determined up to isomorphism by the simple module P/rP. Thus for an indecomposable Nakayama algebra, all indecomposable projective modules up to isomorphism are in the same induced Kupisch series. Let $\{P_1, \dots, P_n\}$ be the induced Kupisch series. We have $P_{i+1} \to rP_i$ is a projective cover when $i \in \{1, \dots, n-1\}$. So $l(P_{i+1}) \geq l(P_i) - 1$ when $i \in \{1, \dots, n-1\}$. Since either P_n is simple projective or $P_1 \to rP_n$ is projective cover, we have $l(P_1) \geq l(P_n) - 1$.

Definition 3.5. *Admissible sequence. The positive integers sequence* $\{a_0, \dots, a_n\}$ *is called an admissible sequence if* $a_{i+1} \geq a_i - 1$ *for* $i \in \{0, \dots, n-1\}$ *and* $a_0 \geq a_n - 1$.

Obviously, the sequence of the length of projective modules in the induced Kupisch series of a Nakayama algebra Λ is a admissible sequence. We call it the admissible sequence of Λ.

Proposition 3.11. *Given any admissible sequence* $\{a_0, \dots, a_n\}$ *over a field* k, *there is a*
Nakayama algebra whose admissible sequence is $\{a_0, \dots, a_n\}$.

Proof. If $a_n = 1$, we associate a quiver Γ to the admissible sequence as following.

$$1 \xrightarrow{b_1} 2 \xrightarrow{b_2} \dots \xrightarrow{b_{n-1}} n.$$

If $a_n \neq 1$, then $a_0 \geq a_n - 1$, we associate a quiver Γ to the admissible sequence as following.

$$\begin{tikzcd} 1 \arrow[r, "b_1"] & 2 \arrow[r, "b_2"] & \cdots \arrow[r, "b_{n-1}"] & \text{n} \arrow[lll, bend right, "b_n"'] \end{tikzcd}$$

For ith vertex, there is a unique path p_i started from i such that $l(p_i) = a_i - 1$. Let I be the ideal generated by $\{p_1, \dots, p_n\}$. Let T denote the path algebra with relation $(k\Gamma, I)$. Let Λ denote $k\Gamma$. Let e_i denote the correspond primitive idempotent of ith vertice in Λ, then the correspond indecomposable projective module is Λe_i. Obviously, T is Nakayama algebra. In addition, $l(\Lambda e_i) = a_i$ and $r\Lambda e_i / r^2 \Lambda e_i = \Lambda e_{i+1} / r\Lambda e_{i+1}$ for all $i \leq n-1$. In the first quiver, we have $l(\Lambda e_n) = 1$. In the second quiver, we have $r\Lambda e_n / r^2 \Lambda e_n = \Lambda e_1 / r\Lambda e_1$. So the admssible sequence of $k\Gamma/I$ coincide with $\{a_0, \dots, a_n\}$. Thus T is the desired Nakayama algebra.

□

Example 3.2. *In example 3.1, the Nakayama algebra* $K[x]/(x^n)$ *is introduced. Now we can look at it's Kupisch series and also admissible sequence. We know that it's only simple module is the field* $K[x]/(x)$ *and* $K[x]/(x^n) \to K[x]/(x)$ *is a projective cover. So the Kupisch series are* $\{K\}$ *and* $\{K[x]/(x^n)\}$ *where the correspond admissible sequence is* $\{n\}$.

At corollary 3.5.1, we have looked at the form of $DTrC$ where C is a nonprojective uniserial module. Since we have also proved that all the indecomposable module are actually uniserial and of the form $P/r^n P$. It is interested to look at the uniform form of $DTrC$ in a Kupisch series.

Proposition 3.12. P_i *is in the projective cover Kupisch series of an indecomposable Nakayama algebra. Then* $DTr(P_i/r^n P_i) \cong P_{i+1}/r^n P_{i+1}$, *when* $n \leq l(P_i) - 1$.

Proof. Trivially, $l(P_i/r^nP_i) = n$. P_i/r^nP_i is uniserial by proposition 3.8 and $f : P_i \to P_i/r^nP_i$ is a projective cover, so $DTr(P_i/r^nP_i)$ is isomorphic to $rP_i/r^{n+1}P_i$ if $n \leq l(P_i)-1$ by corollary 3.5.1. Clearly, $rP_i/r^{n+1}P_i$ is uniserial and $l(rP_i/r^{n+1}P_i) = n$. In addition, the top of $rP_i/r^{n+1}P_i$ is rP_i/r^2P_i. $P_{i+1} \to rP_i$ is projective cover, so $top(P_{i+1}/r^nP_{i+1}) = P_{i+1}/rP_{i+1} \cong rP_i/r^2P_i$. Since $l(P_{i+1}/r^nP_{i+1}) = n$, P_{i+1}/r^nP_{i+1} and $rP_i/r^{n+1}P_i$ have the same top and length. Thus P_{i+1}/r^nP_{i+1} is isomorphic to $rP_i/r^{n+1}P_i$ by proposition 3.4. Therefore, $DTr(P_i/r^nP_i) \cong P_{i+1}/r^nP_{i+1}$. $\square$

Since we now know the form of $DTr(P_i/r^nP_i)$, if we could find a non-split exact sequence whose left and right side are $DTr(P_i/r^nP_i)$ and P_i/r^nP_i respectively, we can easily tell if it is almost split or not.

3.2 The general form of almost split sequences

Proposition 3.13. *Let $\{P_1, \ldots, P_n\}$ be the induced Kupisch series in an indecomposable Nakayama algebra Λ, then*

$$0 \to P_{i+1}/r^nP_{i+1} \to P_{i+1}/r^{n-1}P_{i+1} \oplus P_i/r^{n+1}P_i \to P_i/r^nP_i \to 0$$

is an almost split sequence when $n \leq l(P_i) - 1$.

Proof. There are a natural surjection $P_{i+1}/r^nP_{i+1} \to P_{i+1}/r^{n-1}P_{i+1}$ and a natural injection $P_i/r^{n+1}P_i \to P_i/r^nP_i$. Since $P_{i+1} \to rP_i$ is a projective cover, we haveP_{i+1}/rP_{i+1} is isomorphic to rP_i/r^2P_i. Consequently, there is a natural inclusion $P_{i+1}/r^{n-1}P_{i+1} \to P_i/r^nP_i$. Since P_i and P_{i+1} are indecomposable, we have $l(P_{i+1}/r^{n-1}P_{i+1} \oplus P_i/r^{n+1}P_i) = l(P_{i+1}/r^{n-1}P_{i+1}) + l(P_i/r^{n+1}P_i)) = n+1+n-1 = 2n$. So the sequence is exact. By proposition 3.12, $DTr(P_i/r^nP_i) \cong P_{i+1}/r^nP_{i+1}$, to prove the sequence is almost split, it is enough to show $f : P_{i+1}/r^{n-1}P_{i+1} \oplus P_i/r^{n+1}P_i \to P_i/r^nP_i$ is right almost split. Since P_i and P_{i+1} are both indecomposable, f is a nonsplit epimorphism. In addition, each non-split epimorphism $X \to P_i/r^nP_i$ can factor through $P_i/r^{n+1}P_i \to P_i/r^nP_i$. Thus, f is right almost split. $\square$

4 Auslander-reiten quiver

In this chapter, we start by introducing the Auslander algebra and the Auslander-Reiten quiver, referring to chapter 6 and 7 in [2]. We show that for an artin algebra Λ of finite representation type with M as an additive generator, the algebra $End_\Lambda(M)^{op}$ is an Auslander algebra. This helps us to associate the Auslander-Reiten quiver to an artin algebra. This is an implementation of the almost split sequences and the irreducible morphisms. We will study the gradings for a finite tree based on the result in this chapter.

4.1 Auslander algebras

Definition 4.1. *Finite representation type. An artin algebra Λ is of finite representation type if there is only a finite number of finitely generated isomorphism classes of indecomposable left Λ-modules.*

To study artin algebra of finite representation type, it is helpful to look at the Auslander algebra. In this section, we will introduce the associate Auslander algebra for an artin algebra of finite representation type and discuss some important homological facts of it. Motivated from the associated quiver of an artin algebra, we will also associate a quiver to an Auslander algebra.

Definition 4.2. ***Aslender algebra***. *An artin algebra Γ is said to be an Auslander algebra if and only if $gl.dim\Gamma \leq 2$ and if $0 \to \Gamma \to I_0 \to I_1 \to I_2 \to 0$ is a minimal injective resolution of Γ, then I_0, and I_1 are projective Γ-modules.*

Definition 4.3. *Additive generator. Let M be a module of an artin algebra Λ. M is called an additive generator of Λ if $\operatorname{add} M = \operatorname{mod} \Lambda$.*

Observation 4.1. *An artin algebra Λ is of finite representation type if and only if there existes an additive generator of Λ.*

Proof. Let M be an additive generator of Λ, then all the indecomposable modules in $\operatorname{mod} \Lambda$ are up to isomorphism summands of M. And also we know M is finitely generated if M exists. So the existence of M implies that Λ is of finite representation type. And if Λ is of finite representation type, the direct sum of one copy from each isomorphism class of the indecomposable module is an additive generator of Λ. □

In addtion, the additive generator of Λ is not unique. Let M be an additive generator of Λ. Then any finitely generated module with M as a summand is also an additive generator.

Let M be an additive generator of an artin algebra Λ of finite representation type. We associate the algebra $\Gamma_M = End_\Lambda(M)^{op}$ to Λ. As we discussed in proposition 1.16, the functor $\mathrm{Hom}_\Lambda(M, -)$ introduces an equivalence between the category $\mathrm{add}\, M$ and the full subcategory $\mathscr{P}(\Gamma_M)$ of $\mathrm{mod}\, \Gamma_M$ that consists of projective modules of $\mathrm{mod}\, \Gamma_M$. Then we have that $\mathrm{mod}\, \Lambda$ is equivalent to $\mathscr{P}(\Gamma_M)$ since $\mathrm{mod}\, \Lambda = \mathrm{add}\, M$. Thus, if M' is also an additive generator of $\mathrm{mod}\, \Lambda$, we have that $\mathscr{P}(\Gamma_M)$ is equivalent to $\mathscr{P}(\Gamma_{M'})$.

We want to show that the associate algebra Γ_M of Λ is actually an Auslander algebra. To prove that, we need first to introduce some important homological facts.

Proposition 4.2. *Let Λ be an artin algebra. Then we have following.*

1. *Let M be a finitely generated Λ-module with $pd_\Lambda M = n$, then* $\mathrm{Ext}^n_\Lambda(M, \Lambda) \neq 0$

2. *Assume $gl.dim\Lambda = n$ where n is a finite number. Then we have the following.*

   (a) *$id_\Lambda \Lambda = n$*

   (b) *Let $0 \to \Lambda \to I_0 \to \cdots \to I_n \to 0$ be a minimal injective resolution of Λ in* $\mathrm{mod}\, \Lambda$. *Then any indecomposable injective Λ-module is isomophic to a summand of I_i, $i \in \{0, 1, \ldots, n\}$.*

Proof. 1. Let $0 \to P_n \xrightarrow{i} P_{n-1} \to \cdots \to P_0 \to M$ be a minimal projective resolution of M. Since each indecomposable projective module up to isomorphism is a summand of Λ, we have that if $\mathrm{Ext}^n_\Lambda(M, \Lambda) = 0$, then $\mathrm{Ext}^n_\Lambda(M, P_n) = 0$. So $\mathrm{Hom}_\Lambda(P_{n-1}, P_n) \to \mathrm{Hom}_\Lambda(P_n, P_n)$ is an epimorphism. Then $\exists g \in \mathrm{Hom}_\Lambda(P_{n-1}, P_n)$ and $\exists f \in \mathrm{Hom}_\Lambda(P_n, P_{n-1})$, $gf = 1_{P_n}$. Thus i is a split monomorphism which contradicts $pd_\Lambda M = n$. So $\mathrm{Ext}^n_\Lambda(M, \Lambda) \neq 0$.

2. (a) Since $gl.dim\Lambda = n$, there is a Λ-module M such that $pd_\Lambda M = n$. Then $\mathrm{Ext}^n_\Lambda(M, \Lambda) \neq 0$ which implies that $id_\Lambda \Lambda \geq n$. But $id_\Lambda \Lambda \leq gl.dim\Lambda = n$. Thus $id_\Lambda \Lambda = n$.

   (b) For each indecompodable injective Λ-module, there is an injective envelop $S \to I$, where S is a simple Λ-module. Suppose $pd_\Lambda S = m$. Then $\mathrm{Ext}^m_\Lambda(S, \Lambda) \neq 0$ which implies that there is $m \in 0, 1, \ldots, n$ such that $\mathrm{Hom}_\Lambda(S, I_m) \neq 0$. So there is an indecomposable summand I' of I_m such that $\mathrm{Hom}_\Lambda(S, I) \neq 0$. Then $S \to I$ is an injective envelop. So $I \cong I'$.

□

Proposition 4.3. *Let Λ be an artin algebra of finite represetation type and M be an additive generator of Λ. Then $gl.dim\Gamma_M \leq 2$.*

Proof. Let X be a Γ_M-module and $P_1 \xrightarrow{h} P_0 \to X$ be part of the minimal resolution of X. Then there is $0 \to \ker f \to A_1 \xrightarrow{f} A_0$ where $\ker f, A_1, A_0 \in \operatorname{mod} \Lambda$ such that

$$0 \to \mathrm{Hom}_\Lambda(M, kerf) \to \mathrm{Hom}_\Lambda(M, A_1) \xrightarrow{\mathrm{Hom}_\Lambda(M,f)} \mathrm{Hom}_\Lambda(M, A_0) \to X$$

is a minimal projective resolution, where $\mathrm{Hom}_\Lambda(M, f) = h$, $P_1 \cong \mathrm{Hom}_\Lambda(M, A_1)$ and $P_0 \cong \mathrm{Hom}_\Lambda(M, A_0)$ by that $\mathrm{Hom}_\Lambda(M, -) : \operatorname{mod} \Lambda \to \Gamma_M$ introduces an equivalence between $\operatorname{mod} \Lambda$ and $\mathscr{P}(\Gamma_M)$. So we have $pd_{\Gamma_M} X \leq 2$. Thus $gl.dim\Gamma_M \leq 2$. □

Proposition 4.4. *Let Λ to be an artin algebra of finite representation type and M be an additive generator of Λ. Then we have following.*

1. *Let I be an injective module in Λ, then $\mathrm{Hom}_\Lambda(M, I)$ is also an injective module in Γ_M.*
2. *Let $0 \to A \xrightarrow{f} I_0 \to I_1$ be a minimal injective copresentation of A where $A \in \operatorname{mod} \Lambda$. Then $0 \to \mathrm{Hom}_\Lambda(M, A) \xrightarrow{h} \mathrm{Hom}_\Lambda(M, I_0) \to \mathrm{Hom}_\Lambda(M, I_1)$ is also a minimal injective copresentation of $\mathrm{Hom}_\Lambda(M, A)$ where $\mathrm{Hom}_\Lambda(M, A)$ is a projective Γ_M module.*
3. *Let N be a Γ_M-module. N is both projective and injective if and only if there exists a injective Λ-module I such that N is isomorphic to $\mathrm{Hom}_\Lambda(M, I)$.*
4. *The functor $\mathrm{Hom}_\Lambda(M, -) : \operatorname{mod} \Lambda \to \operatorname{mod} \Gamma_M$ introduces an equivalence between the full subcategory $\mathscr{I}(\Lambda)$ of $\operatorname{mod} \Lambda$ that consists of the injective modules of $\operatorname{mod} \Lambda$ and the full subcategory of $\operatorname{mod} \Gamma_M$ that consists of the modules of $\operatorname{mod} \Gamma_M$ being both projective and injective.*

Proof. 1. Let I' denote $\mathrm{Hom}_\Lambda(M, I)$. To prove I' is injective, we shall show that for any X in Γ_M-modules, $\mathrm{Ext}^1_{\Gamma_M}(X, I') = 0$. We have seen in proposition 4.3 that there is a minimal projective resolution of X, $0 \to \mathrm{Hom}_\Lambda(M, A) \to \mathrm{Hom}_\Lambda(M, B) \to \mathrm{Hom}_\Lambda(M, C) \to X$ such that $0 \to A \to B \to C \to 0$ is exact where $A, B, C \in \operatorname{mod} \Lambda$. Applying $\mathrm{Hom}_{\Gamma_M}(-, I')$, we have

$$\begin{aligned} \mathrm{Hom}_{\Gamma_M}(\mathrm{Hom}_\Lambda(M, C), I') \to \mathrm{Hom}_{\Gamma_M}(\mathrm{Hom}_\Lambda(M, B), I') \to \\ \mathrm{Hom}_{\Gamma_M}(\mathrm{Hom}_\Lambda(M, A), I') \to 0 \end{aligned} \tag{1}$$

Since $\operatorname{add} M = \operatorname{mod} \Lambda$ and I is injective Λ-module, we have (1) is isomorphic to the exact sequence

$$0 \to \mathrm{Hom}_\Lambda(C, I) \to \mathrm{Hom}_\Lambda(B, I) \to \mathrm{Hom}_\Lambda(A, I) \to 0$$

Thus (1) is exact. So $\mathrm{Ext}^1_{\Gamma_M}(X, I') = 0$.

2. Obviously, $0 \to \mathrm{Hom}_\Lambda(M,A) \to \mathrm{Hom}_\Lambda(M,I_0) \to \mathrm{Hom}_\Lambda(M,I_1)$ is an injective copresentaion. That is minimal follows from the fact that $I_0/Imf \to I_1$ is an injective envelop, so $\mathrm{Hom}_\Lambda(M,I_0)/Imh \to \mathrm{Hom}_\Lambda(M,I_1)$ is also an injective envelop.

3. Obviously, $\mathrm{Hom}_\Lambda(M,I)$ is both projective and injective. Since N is a projective Γ_M-module, there is a Λ-module A such that $\mathrm{Hom}_\Lambda(M,A) \cong N$. Let $A \to I$ be an injective envelop, then $\mathrm{Hom}_\Lambda(M,A) \to \mathrm{Hom}_\Lambda(M,I)$ is also an injective envelop. But $\mathrm{Hom}_\Lambda(M,A)$ is injective, so $\mathrm{Hom}_\Lambda(M,A) \to \mathrm{Hom}_\Lambda(M,I)$ is a split monomorphism. Then $A \to I$ splits, so $A \cong I$. Thus $\mathrm{Hom}_\Lambda(M,A) \cong \mathrm{Hom}_\Lambda(M,I)$ and then $N \cong \mathrm{Hom}_\Lambda(M,I)$.

4. It directly follows from 3.

$\square$

Right now we are ready to prove the associate algebra Γ_M of Λ is an Auslander algebra.

Proposition 4.5. *Let M be an additive generator of an artin algebra Λ, then Γ_M is an Auslander algebra.*

Proof. In proposition 4.3, we have seen that $gl.dim\Gamma_M \leq 2$. So it is enough to show if $\mathrm{Hom}_\Lambda(M,M) \to I'_0 \to I'_1 \to I'_2 \to 0$ is minimal injective resolution for $\mathrm{Hom}_\Lambda(M,M)$ in $\mathrm{mod}\,\Gamma_M$, then I'_0 and I'_1 are projective. Let $M \to I_0 \to I_1$ be minimal injective copresentation of M in $\mathrm{mod}\,\Lambda$. From proposition 4.4, we know that

$$0 \to \mathrm{Hom}_\Lambda(M,M) \to \mathrm{Hom}_\Lambda(M,I_0) \xrightarrow{f} \mathrm{Hom}_\Lambda(M,I_1) \xrightarrow{s} cokerf \to 0 \qquad (2)$$

is the minimal injective resolution. In addition, $\mathrm{Hom}_\Lambda(M,I_0)$ and $\mathrm{Hom}_\Lambda(M,I_1)$ are projective. And $cokerf$ is injective since $gl.dim\Gamma_M \leq 2$. In addition, if $cokerf$ is projective then s splits which contradicts (2) being minimal. So $cokerf$ is not projective. Thus Γ_M is an Auslander algebra. $\square$

Observation 4.6. *Let Λ be a semisimple artin algebra and M be a additive generator of Λ. Then Λ is morita equivalent to Γ_M and $gl.dim\Lambda = gl.dim\Gamma_M = 0$.*

Proof. If Λ is semisimple, then all the modules are semisimple and projective. So M is semisimple. Then Γ_M is semisimple and all Γ_M-modules are projective. So we have that $\mathrm{mod}\,\Lambda \cong \mathscr{P}(\Lambda) \cong \mathscr{P}(\Gamma_M) \cong \mathrm{mod}\,\Gamma_M$ and $gl.dim\Lambda = gl.dim\Gamma_M = 0$. $\square$

Let $\Gamma_M \to I_0 \to I_1 \to \cdots \to I_n \to 0$ be a minimal injective resolution. We introduce **dominant dimension** to describe the maximal number n in an minimal injective resolution such that when $i < n$, I_i is projective. Thus, if Γ_M is an Aslender algebra and $\Gamma_M \to I_0 \to I_1 \to I_2 \to 0$ is the minimal injective resolution of Γ_M, we have $dom.dim\Gamma_M = 2$.

Proposition 4.7. *Let Λ be a non-semisimple artin algebra of finite representation type and M be an additive generator of Λ, then we have the following.*

1. $id_{\Gamma_M}\Gamma_M = gl.dim\Gamma_M = 2$

2. $dom.dim\Gamma_M = 2$.

3. *Let Q be a projective injective Γ_M-module such that* $\operatorname{add} Q$ *is the full subcategory of* $\operatorname{mod}\Gamma_M$ *that consists of the modules that are both projective and injective. Then* $\mathscr{P}(End\Gamma_M(Q)^{op}) \cong \mathscr{P}(\Lambda)$.

Proof.

1. Since Λ is non-semisimple, we know that there is a simple module S such that S is not projective. Let $P \to S$ be a projecive cover, then $\mathrm{Hom}_\Lambda(M, P) \xrightarrow{f} \mathrm{Hom}_\Lambda(M, S) \to coker f \to 0$ is part of a minimal projective resolution of $coker f$ since $\mathrm{Hom}_\Lambda(M, P)$ and $\mathrm{Hom}_\Lambda(M, S)$ are both idecomposable and projective. Then $pd_{\Gamma_M} coker f \geq 2$. From proposition above we have that $gl.dim\Gamma_M \leq 2$. Thus $gl.dim\Gamma_M = 2$.

2. Since Γ_M is an artin algebra, we have $id_{\Gamma_M}\Gamma_M = gl.dim\Gamma_M = 2$ by proposition 4.2. Since Λ is not semisimple, M is not injective. Let $M \to I_0 \to I_1$ be part of the minimal injective resolution of M in $\operatorname{mod}\Lambda$. Then $\mathrm{Hom}_\Lambda(M, M) \to \mathrm{Hom}_\Lambda(M, I_0) \xrightarrow{f} \mathrm{Hom}_\Lambda(M, I_1) \to coker f \to 0$ is the minimal injective resolution of $\mathrm{Hom}_\Lambda(M, M)$ in $\operatorname{mod}\Gamma_M$. $coker f$ is injective since $id_{\Gamma_M}\Gamma_M = 2$. Since $\operatorname{mod}\Lambda \cong \mathscr{P}(\Gamma_M)$, $\mathrm{Hom}_\Lambda(M, I_0)$ and $\mathrm{Hom}_\Lambda(M, I_1)$ are projective. In addition, If $coker f$ is projective, then $\mathrm{Hom}_\Lambda(M, I_1) \to coker f$ is a split epimorphism. It contradict that the injective resolution is minimal. So we have $coker f$ is not projective. Thus $dom.dim\Gamma_M = 2$.

3. Since for each Λ-module N, $\mathrm{Hom}_\Lambda(\Lambda, N) \cong N$, then $\operatorname{add} D(\Lambda)$ is equivalent to the full subcategory of injectives in $\operatorname{mod}\Lambda$.

   So $\mathrm{Hom}_\Lambda(M, \operatorname{add}(D(\Lambda))) = \operatorname{add}\mathrm{Hom}_\Lambda(M, D(\Lambda))$ is the full subcategory of $\operatorname{mod}\Gamma_M$ that consists of the modules that are both projective and injective by proposition 4.4. Then we have $\operatorname{add} Q = \operatorname{add}\mathrm{Hom}_\Lambda(M, D(\Lambda))$. So $\mathscr{P}(End_{\Gamma_M}(Q)) \cong \mathscr{P}(End_{\Gamma_M}(\mathrm{Hom}_\Lambda(M, D(\Lambda))))$.

   But $End_{\Gamma_M}(\mathrm{Hom}_\Lambda(M, D(\Lambda))) \cong End_\Lambda(D(\Lambda)) \cong \Lambda^{op}$ by $D(\Lambda) \in \operatorname{add} M$. Thus $\mathscr{P}(End_{\Gamma_M}(Q)^{op}) \cong \mathscr{P}(\Lambda)$.

□

We will use almost split sequence to associate a quiver to $\operatorname{mod}\Gamma_M$. But we have seen that for a semisimple artin algebra Λ with additive generator M, the associated algebra Γ_M is also semisimple. Then all simple modules in $\operatorname{mod}\Gamma_M$ are projective. So there is no almost split sequence in $\operatorname{mod}\Gamma_M$. Thus, we will mainly look at non-semisimple artin algebras.

Firstly, we will study right almost split morphisms in $\mathscr{P}(\Gamma_M)$ by using $\operatorname{mod}\Lambda \cong \mathscr{P}(\Gamma_M)$.

Proposition 4.8. *Let Λ be a artin algebra. Then for $f : P \to Q$ where P and Q are in $\mathscr{P}(\Lambda)$, the following are equivalent.*

1. *f is right almost split in $\mathscr{P}(\Lambda)$*
2. *Q is indecomposable and $\operatorname{Im} f = rQ$.*

Proof.

1. ($1 \Longrightarrow 2$) Since f is right almost split, we have Q is indecomposable by lemma 2.3. If f is an epimorphism, then f is split since Q is projective. Thus f is not an epimorphism. Then $Imf \subseteq rQ$ since Q is indecomposable. There exists a projective module P' such that $g : P' \to rQ$ is surjective. Let $i : rQ \hookrightarrow Q$ be the natural inclusion, then we have $ig : P' \to Q$ where $\operatorname{Im} ig = rQ$. Then ig factors through f. Thus $\operatorname{Im} f \supseteq \operatorname{Im} ig = rQ$. So we have $\operatorname{Im} f = rQ$.
2. ($2 \Longrightarrow 1$) Obviously, f is not a split epimorphism. Let $g : M \to Q$ be a non-split epimorphism in $\mathscr{P}(\Lambda)$. So $\operatorname{Im} g \subseteq rQ$. But we have that $P \xrightarrow{f} rQ$ is surjective. Since M is projective, there is a morphism $k : M \to P$ such that $fk = g$. Thus f is right almost split in $\mathscr{P}(\Lambda)$.

□

Proposition 4.9. *Let Λ be a non-semisimple artin algebra with finite representation type and M be an additive generator of Λ. Then the following are equivalent for a morphism $f : P \to Q$ in $\operatorname{mod}\Lambda$.*

1. *f is right almost split*
2. $\operatorname{Hom}_\Lambda(M, -) : \operatorname{Hom}_\Lambda(M, P) \to \operatorname{Hom}_\Lambda(M, Q)$ *is right almost split in $\mathscr{P}(\Gamma_M)$*
3. $\operatorname{Hom}_\Lambda(M, P)$ *is an indecomposable projective module in* $\operatorname{mod}\Gamma_M$*. In addition,* $\operatorname{Im}\operatorname{Hom}_\Lambda(M, f) = r\operatorname{Hom}_\Lambda(M, Q)$.

Proof. The equivalence between (1) and (2) comes from the fact that the functor $\operatorname{Hom}_\Lambda(M, -)$ introduces an equivalence between $\operatorname{mod}\Lambda$ and $\mathscr{P}(\Gamma_M)$.

The equivalence between (2) and (3) is a simple implementation of proposition 4.8. □

Proposition 4.10. *Let Λ be an artin algebra and let S be an* Λ-module*. The following are equivalent for S.*

1. *Each non-zero homomorphism $f : M \to S$ in* $\operatorname{mod}\Lambda$ *is a split epimorphism.*
2. *S is a simple projective* Λ-module.

Proof.

($1 \Longrightarrow 2$) Since $f : P \to S$ is a spilt epimorphism for all projective Λ-modules, we have that S is simple. Suppose S is not projective, then there is a projective cover $P' \xrightarrow{h} S$ such that P' is indecomposable which contradicts that h splits. Thus S is simple projective Λ-module.

($2 \Longrightarrow 1$) Since S is simple, all non-zero morphisms to S are surjective. Since S is projective, f splits.

$\square$

Right now we are ready to show some homological facts of $\operatorname{mod}\Gamma_M$ which is crucial for associating a quiver to the Auslander algebra Γ_M.

Proposition 4.11. *Let Λ be a non-semisimple artin algebra of finite representation type and let M be an additive generator of Λ. Let S be a simple Γ_M-module and let C be the Λ-module up to isomorphism such that $\operatorname{Hom}_\Lambda(M, C) \to S$ is a projective cover. Then we have the following.*

1. *The following are equivalent.*
   - (a) $pd_{\Gamma_M} S = 0$
   - (b) $\operatorname{Hom}_\Lambda(M, C) = S$
   - (c) *C is a simple projective Λ-module*
2. *The following are equivalent.*
   - (a) $pd_{\Gamma_M} S = 1$
   - (b) *C is a nonsimple projective Λ-module*
   - (c) *$0 \to \operatorname{Hom}_\Lambda(M, rC) \to \operatorname{Hom}_\Lambda(M, C) \to S \to 0$ is a minimal projective resolution of S*
3. *The following are equivalent.*
   - (a) $pd_{\Gamma_M} S = 2$
   - (b) *C is not a projective Λ-module*

Proof. 1. • ($a \Longrightarrow b$) Since $pd_{\Gamma_M} S = 0$, S is a simple projective module. Thus $\operatorname{Hom}_\Lambda(M, C) = S$.

• ($b \Longrightarrow c$) For any non-zero homomorphism $f : B \to C$ in $\operatorname{mod}\Lambda$, $\operatorname{Hom}_\Lambda(M, f) : \operatorname{Hom}_\Lambda(M, B) \to \operatorname{Hom}_\Lambda(M, C)$ is a split epimorphism since $\operatorname{Hom}_\Lambda(M, C)$ is simple and projective. But then f is also a split epimorphism. Thus C is a simple projective Λ-module by proposition 4.10.

- ($c \Longrightarrow a$) For each projective Γ_M-module $\mathrm{Hom}_\Lambda(M,B)$, if there is a non-zero homomorphism $\mathrm{Hom}_\Lambda(M,h) : \mathrm{Hom}_\Lambda(M,B) \to \mathrm{Hom}_\Lambda(M,C)$, since $h : B \to C$ is a split epimorphism, we have that $\mathrm{Hom}_\Lambda(M,h)$ is split epimorphism. Then $\mathrm{Hom}_\Lambda(M,C)$ is simple projective Γ_M-module by proposition 4.10. Thus $pd_{\Gamma_M} S = 0$.

2. - ($a \Longrightarrow b$) we will prove it at the end.
   - ($b \Longrightarrow c$) From the assumption, we have that C is indecomposable. We have seen that $i : rC \to C$ is right almost split in example 2.1. By proposition 4.9, we have that $\mathrm{Hom}_\Lambda(M,i) : \mathrm{Hom}_\Lambda(M,rC) \to \mathrm{Hom}_\Lambda(M,C)$ is right almost split and $\mathrm{Im}\,\mathrm{Hom}_\Lambda(M,i) = r\,\mathrm{Hom}_\Lambda(M,C)$. Consequently, *coker* $\mathrm{Hom}_\Lambda(M,i) = \mathrm{Hom}_\Lambda(M,C)/r\,\mathrm{Hom}_\Lambda(M,C) = S$. Thus $0 \to \mathrm{Hom}_\Lambda(M,rC) \to \mathrm{Hom}_\Lambda(M,C) \to S$ is a minimal projective reslotion.
   - ($c \Longrightarrow a$) It is trivial.

3. - ($a \Longrightarrow b$) We have seen above that if C is a projective Λ-module, then $pd_{\Gamma_M} S = 1$ or $pd_{\Gamma_M} S = 0$. We also know that $pd_{\Gamma_M} S \leq 2$ since Γ_M is an Auslander algebra. Thus if $pd_{\Gamma_M} S = 2$, C is not a projective Λ-module.
   - ($b \Longrightarrow a$) Since C is not projective and is indecomposable, there exists an almost split sequence $0 \to A \to B \to C \to 0$. So $B \to C$ is a right almost split. Then $g : \mathrm{Hom}_\Lambda(M,B) \to \mathrm{Hom}_\Lambda(M,C)$ is right almost split and $\mathrm{Im}\, g = r\,\mathrm{Hom}_\Lambda(M,C)$ by proposition 4.9. Thus $0 \to \mathrm{Hom}_\Lambda(M,A) \to \mathrm{Hom}_\Lambda(M,B) \to \mathrm{Hom}_\Lambda(M,C) \to S$ is a projective resolution of S. If it is not minimal, then $\mathrm{Hom}_\Lambda(M,B)$ splits and $\mathrm{Hom}_\Lambda(M,B) \cong \mathrm{Hom}_\Lambda(M,A) \oplus r\,\mathrm{Hom}_\Lambda(M,C)$. But then B splits and $B = A \oplus C$ which contradicts $0 \to A \to B \to C \to 0$ is almost split. So $0 \to \mathrm{Hom}_\Lambda(M,A) \to \mathrm{Hom}_\Lambda(M,B) \to \mathrm{Hom}_\Lambda(M,C) \to S$ is a minimal projective resolution of S. Thus $pd_{\Gamma_M} S = 2$.

For (2) ($a \Longrightarrow b$), we now can conclude that $pd_{\Gamma_M} S = 1$ if and only is C is a nonsimple projective Λ-module.

□

From the proposition above, we know that the simple modules in $\mathrm{mod}\,\Gamma_M$ is one to one correspond with the isomorphism classes of indecomposable modules in $\mathrm{mod}\,\Lambda$. We use $[X]$ to denote the isomorphism class in $\mathrm{mod}\,\Lambda$ of X where X is the indecomposable Λ-module such that $\mathrm{Hom}_\Lambda(M,X) \to S$ is a projective cover. Let S_X denote the correspond simple module of $[X]$ in $\mathrm{mod}\,\Gamma_M$.

We have introduced how to construct a quiver for an artin algebra. Motivated by that, we let the isomorphism classes of indecompoable Λ-module be the vertices of the quiver of Γ_M. There is an arrow from vertices $[X]$ to $[Y]$ if $\mathrm{Ext}^1_{\Gamma_M}(S_X, S_Y) \neq 0$. Let $P \to \mathrm{Hom}_\Lambda(M, X) \to S_X$ be the minimal projective presentation of S_X in $\mathrm{mod}\,\Gamma_M$. We have seen in proposition 1.10 that $\mathrm{Ext}^1_{\Gamma_M}(S_X, S_Y) \neq 0$ if and only if $\mathrm{Hom}_\Lambda(M, Y)$ is a summand of P.

By proposition 4.11, let $0 \to A \to B \to X \to 0$ be the almost split sequence of X where $0 \to \mathrm{Hom}_\Lambda(M, A) \to \mathrm{Hom}_\Lambda(M, B) \to \mathrm{Hom}_\Lambda(M, X) \to S_X$ is a minimal projective resolution of S_X. We can easily see that $\mathrm{Hom}_\Lambda(M, Y)$ is a summand of $\mathrm{Hom}_\Lambda(M, B)$ if and only if Y is a summand of B where $B \to X$ is minimal right almost split. Thus there is an arrow from vertices $[X]$ to $[Y]$ if and only if there is an irreducible morphism $Y \to X$.

We associate a valuation (a, b) to the arrow from $[X]$ to $[Y]$ such that b is the multiplicity of $\mathrm{Hom}_\Lambda(M, Y)$ in P. Since $B \to X$ is minimal right almost split, then b is the multiplicity of Y in B. Similarly, if $Y \to Q$ is minimal left almost split, a is the multiplicity of X in Q. In general, a is not equal to b. But for Nakayama algebras, the valuation based on the minimal right and left split morphisms are always $(1, 1)$.

We will look at some examples for indecomposable Nakayama algebras. In proposition 3.11, we have seen that we can associate an indecomposable Nakayama algebra to a given admissible sequence. In addition, we have investigated in proposition 3.13 that for an indecomposable Nakayama algebra Λ, the almost split sequences are of the form

$$0 \to P_{i+1}/r^n P_{i+1} \to P_{i+1}/r^{n-1} P_{i+1} \oplus P_i/r^{n+1} P_i \to P_i/r^n P_i \to 0$$

Where $\{P_1, \ldots, P_n\}$ is the induced Kupisch series of Λ. We use S_i^j denote $P_i/r^j P_i$. We will look at two examples with different admissible sequence as following.

Example 4.1. *Given an admissible sequence $\{3, 4, 3\}$, the associated quiver Γ is*

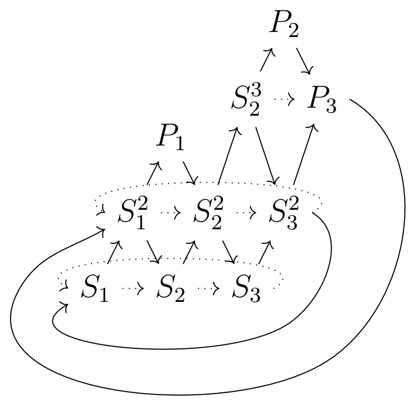
$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[ll, bend right] \end{tikzcd}$$

Let p_i be the path starting from the ith vertex such that $l(p_i) = v(i)$ where $v(i)$ is the ith item in the admissible sequence. Let k be a field, the associated Nakayama algebra Λ of this admissible sequence is the path algebra $k\Gamma$ modulo the ideal generated by $\{p_1, p_2, p_3\}$. Thus the almost split sequence of Λ is the following.

$$\begin{aligned}
S_2 \to S_1^2 &\to S_1 \\
S_2^2 \to S_2 \oplus P_1 &\to S_1^2 \\
S_3 \to S_2^2 &\to S_2 \\
S_3^2 \to S_3 \oplus S_2^3 &\to S_2^2 \\
S_1 \to S_3^2 &\to S_3 \\
S_1^2 \to S_1 \oplus P_3 &\to S_3^2 \\
P_3 \to S_3^2 \oplus P_2 &\to S_2^3
\end{aligned}$$

For $[S_1] \to [S_1^2]$*, the valuation is* $(1,1)$ *and for* $[S_2^3] \to [P_2]$*, the valuation is also* $(1,1)$*.*

The quiver of the associated Auslander algebra of Λ *is as following.*



The dotted arrow is the translation DTr*. We can see from the quiver that the going-up arrows are irreducible epimorphisms and the going-down arrows are irreducible monomorphisms. And because the admissible sequence is not end up by 1,* Λ *does not contain simple projective module, so the quiver is periodic. In addition, by the definition of* DTr*, we know that the modules in the quiver without going-out dotted arrows are projective and without coming-in dotted arrows are injective. Thus, we have that* P_3 *are projective and* P_1, P_2 *are projective injective, also* S_2^3 *are injective.*

Example 4.2. *Given the admissible sequence* $\{3,5,4,3,2,1\}$*, the associated quiver* Γ *is* $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6$ *. Let* k *be a field. The associated Nakayama algebra* Λ *is* $k\Gamma/(p_1,p_2,p_3,p_4,p_5,p_6)$*. Thus we have the following almost split sequences.*

$$\begin{array}{rl}
S_2 \to S_1^2 \to S_1 & S_2^2 \to S_2 \oplus P_1 \to S_1^2 \\
S_3 \to S_2^2 \to S_2 & S_3^2 \to S_3 \oplus S_2^3 \to S_2^2 \\
S_4 \to S_3^2 \to S_3 & S_4^2 \to S_4 \oplus S_3^3 \to S_3^2 \\
S_5 \to S_4^2 \to S_4 & P_4 \to S_4^2 \oplus P_3 \to S_3^3 \\
P_6 \to P_5 \to S_5 & P_5 \to S_5 \oplus P_4 \to S_4^2 \\
S_3^3 \to S_3^2 \oplus S_2^4 \to S_2^3 & P_3 \to S_3^3 \oplus P_2 \to S_2^4
\end{array}$$

For $[P_5] \to [P_6]$*, the valuation is* $(1,1)$*. For* $[S_1^2] \to [S_2]$*, the valuation is* $(1,1)$*.*
The quiver of the associated Auslander algebra is as following. The dotted arrows

are the translation DTr.

$$
\begin{tikzcd}
& & & & & & P_2 \arrow[dr] & & & & \\
& & & & & S_2^4 \arrow[ur] \arrow[rr, dashed] \arrow[dr] & & P_3 \arrow[dr] & & & \\
& & & & S_2^3 \arrow[ur] \arrow[rr, dashed] \arrow[ddr] & & S_3^3 \arrow[ur] \arrow[rr, dashed] \arrow[ddr] & & P_4 \arrow[ddr] & & \\
& & P_1 \arrow[dr] & & & & & & & & \\
& S_1^2 \arrow[ur] \arrow[rr, dashed] \arrow[dr] & & S_2^2 \arrow[uur] \arrow[rr, dashed] \arrow[dr] & & S_3^2 \arrow[uur] \arrow[rr, dashed] \arrow[dr] & & S_4^2 \arrow[uur] \arrow[rr, dashed] \arrow[dr] & & P_5 \arrow[dr] & \\
S_1 \arrow[ur] \arrow[rr, dashed] & & S_2 \arrow[ur] \arrow[rr, dashed] & & S_3 \arrow[ur] \arrow[rr, dashed] & & S_4 \arrow[ur] \arrow[rr, dashed] & & S_5 \arrow[ur] \arrow[rr, dashed] & & P_6
\end{tikzcd}
$$

We can see from the quiver that P_3, P_4, P_5, P_6 are projective and P_1, P_2 are projective injective, also S_1, S_1^2, S_2^3, S_2^4 are injective. The going-up arrows are irreducible epimorphisms and the going-down arrows are irreducible monomorphisms. The quiver is not periodic because $S_6 = P_6$ is a simple projective Λ-module.

4.2 Auslander-Reiten-quivers

Motivated by the last section, for any artin algebra Λ, we associate to Λ a valued quiver Γ_Λ such that the vertices of Γ_Λ are in one to one correspond with the isomorphism classes of indecomposable modules in $\operatorname{mod}\Lambda$. We use $[X]$ to denote the isomorphism class of X in $\operatorname{mod}\Lambda$. There is an arrow between vertices $[X]$ and $[Y]$ if and only there is an irreducible morphism from X to Y. The arrow between $[X]$ and $[Y]$ has valuation (a, b) if $X^a \oplus M \to Y$, where X is not a summand of M, is minimal right almost split and if $X \to Y^b \oplus N$, where Y is not a summand of N, is minimal left almost split. The vertices correspond to projective isomorphism classses are called **projective vertices**. The vertices correspond to injective isomorphism classes are called **injective vertices**. Moreover, we define the **translation** of Γ_Λ to be the correspondence DTr which induces a map from the nonprojective vertices to the noninjective vertices.

Definition 4.4. *Auslander-Reiten-quiver (AR-quiver). For any artin algebra Λ, the AR-quiver of Λ is the associated quiver Γ_Λ together with the translation τ.*

Example 4.3. *Let k be a field and let Γ be quiver $1 \to 2 \leftarrow 3$. For path algebra $k\Gamma$, we have the following.*

1. *Projective modules:*

$$P_1 : k \to k \leftarrow 0 \qquad P_2 : 0 \to k \leftarrow 0 \qquad P_3 : 0 \to k \leftarrow k$$

2. Injective modules:

$$I_1 : k \to 0 \leftarrow 0 \qquad I_2 : k \to k \leftarrow k \qquad I_3 : 0 \to 0 \leftarrow k$$

3. Applying $\mathrm{Hom}_\Lambda(-, \Lambda)$ *to the projective module* P_1, P_2 *and* P_3 *respectively, we have:*

$$P_1^* : k \leftarrow 0 \to 0 \qquad P_2^* : k \leftarrow k \to k \qquad P_3^* : 0 \leftarrow 0 \to k$$

I_1, P_2, I_3 *are simple modules.*

1. $P_2 \to P_1 \to I_1$ *is a minimal projective presentation. Then we have* $P_1^* \to P_2^* \to TrI_1$. *Thus* $DTrI_1 = P_3$
2. $P_2 \to P_1 \oplus P_3 \to I_2$ *is a minimal projective presentation. Then we have* $P_1^* \oplus P_3^* \to P_2^* \to TrI_2$. *Thus* $DTrI_1 = P_2$
3. $P_2 \to P_3 \to I_3$ *is a minimal projective presentation. Then we have* $P_3^* \to P_2^* \to TrI_3$. *Thus* $DTrI_3 = P_1$

So we have almost split sequence as following.

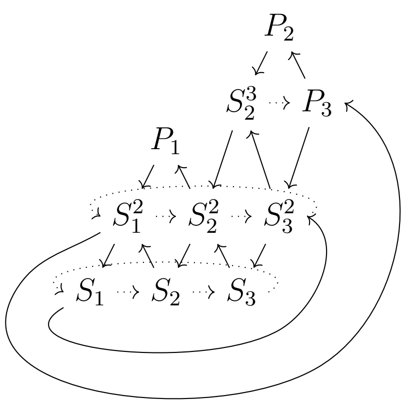
1. $P_3 \to I_2 \to I_1$
2. $P_2 \to P_1 \oplus P_3 \to I_2$
3. $P_1 \to I_2 \to I_3$

Thus the AR-quiver of Λ *is the following. The dotted arrow is the translation.*

$$\begin{tikzcd}
 & P_1 \arrow[dr] & & I_3 \arrow[ll, dotted] \\
P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] \arrow[ll, dotted] & \\
 & P_3 \arrow[ur] & & I_1 \arrow[ll, dotted]
\end{tikzcd}$$

We will also give the AR-quiver for examples 4.1 and 4.2.

AR-quiver of example 4.1.



***AR*-quiver of example 4.2.**

$$
\begin{tikzcd}[column sep=tiny]
 & & & & & & P_2 \arrow[dl] & & & & \\
 & & & & & S_2^4 \arrow[dl] \arrow[rr, dotted] & & P_3 \arrow[ul] \arrow[dl] & & & \\
 & & & & S_2^3 \arrow[ddl] \arrow[rr, dotted] & & S_3^3 \arrow[ul] \arrow[ddl] \arrow[rr, dotted] & & P_4 \arrow[ul] \arrow[ddl] & & \\
 & & P_1 \arrow[dl] & & & & & & & & \\
 & S_1^2 \arrow[dl] \arrow[rr, dotted] & & S_2^2 \arrow[ul] \arrow[dl] \arrow[rr, dotted] & & S_3^2 \arrow[uul] \arrow[dl] \arrow[rr, dotted] & & S_4^2 \arrow[uul] \arrow[dl] \arrow[rr, dotted] & & P_5 \arrow[uul] \arrow[dl] & \\
S_1 \arrow[rr, dotted] & & S_2 \arrow[ul] \arrow[rr, dotted] & & S_3 \arrow[ul] \arrow[rr, dotted] & & S_4 \arrow[ul] \arrow[rr, dotted] & & S_5 \arrow[ul] \arrow[rr, dotted] & & P_6 \arrow[ul]
\end{tikzcd}
$$

We have observed that in the AR-quiver, the going-up arrows are irreducible monomorphisms and the going-down arrows are irreducible epimorphisms. In addition, the vertices without dotted arrow going-out are projective and the vertices without dotted arrow coming-in are injective.

5 The representation finite graded trees

In this chapter, we introduce the translation quiver, grading tree and simply connected algebra. We mainly discuss the representation finite gradings for a finite tree. In [3], Bongartz and Gabriel showed that there is a bijection between the isomorphism classes of representation finite graded trees and the isomorphism classes of simply connected algebras. We summarize the result here. We introduce the tree $\ddot{D}_n$. We calculate and list all the representation finite gradings for $\ddot{D}_5$ and D_6.

5.1 Translation quivers

Let Γ be a quiver. We call Γ **locally finite** if for each vertex x in Γ_0 has only a finite number of arrows which are ending in x and starting from x. We use x^- denote the set $\{y \in \Gamma_0 \mid \exists\ (y \to x) \in \Gamma_1\}$ and x^+ denote the set $\{y \in \Gamma_0 \mid \exists\ (x \to y) \in \Gamma_1\}$.

Definition 5.1. *Translation quiver. Let Γ be a quiver and let $\tau : \Gamma_0 \to \Gamma_0$ be partially defined. (Γ, τ) is called a translation quiver if it satisfies the following conditions.*

1. *Γ has no loop $\circlearrowleft$*
2. *If two vertices in Γ are connected, there is only one arrow between these two vertices $\cdot \longrightarrow \cdot$*
3. *If $\tau(x)$ is defined where $x \in \Gamma_0$, then $x^- = (\tau(x))^+$.*

Let (Γ, τ) be a translation quiver. A vertex is called **projective** if τ is not defined on it. A vertex is called **injective** if τ^{-1} is not defined on it.

Since $x^- = (\tau(x))^+$, for a non-projective vertex x, the **mesh** of x is the full sub-quiver of Γ formed by $x, \tau(x)$ and x^-. We denote the mesh of x as m_x. If there is a arrow $y \xrightarrow{\alpha} x$ in m_x, there is a unique arrow $\tau(x) \xrightarrow{\beta} y$ such that $\alpha\beta$ is a arrow from $\tau(x)$ to x. We define map $\sigma : \Gamma_1 \to \Gamma_1$ such that $\sigma(\alpha) = \beta$.

Example 5.1. *Let the map DTr be the translation denoted as τ and let Λ be any artin algebra. Then the AR-quiver of Λ together with τ is a translation quiver. τ is not defined on projective vertices and τ^{-1} is not defined on injective vertices.*

In the rest of this thesis, we use dotted arrow to illustrate the translation.

We call a translation quiver **stable** if τ and it's inverse is defined everywhere.

Let (Γ, τ) be a translation quiver. We define $\tilde{\Gamma}$ as the **extended quiver** of Γ such that

1. $\tilde{\Gamma}_0 = \Gamma_0$.

2. There is two different type of arrows in $\tilde{\Gamma}$.
   (a) The first type of arrows are the arrows and the inverse arrows in Γ_1. Let α be a arrow in Γ_1, we use α^{-1} to denote the inverse arrow.
   (b) The second type of arrows are the translations and the inverse translations in Γ. Let τ_x denote $x \to \tau(x)$ where τ is the translation in Γ. We denote the inverse of τ_x as τ_x^{-1}.

We illustrate the extension quiver in the following example. We use dotted arrows to represent the second type of arrows.

Example 5.2. *The quiver Γ on the left side is a translation quiver. The extension quiver $\tilde{\Gamma}$ is the one on the right side.*

$$\begin{tikzcd} & y \arrow[dr, "\beta"] & \\ \tau(x) \arrow[ur, "\alpha"] & & x \arrow[ll, dotted, "\tau"] \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[dr, shift left, "\beta"] \arrow[dl, shift left, "\alpha^{-1}"] & \\ \tau(x) \arrow[ur, shift left, "\alpha"] \arrow[rr, dotted, shift left, "\tau^{-1}"'] & & x \arrow[ul, shift left, "\beta^{-1}"] \arrow[ll, dotted, shift left, "\tau"] \end{tikzcd}$$

Let x, y be vertices in Γ, we define a **walk** from x to y denoted as $w = (x \mid a_n a_{n-1} \dots a_1 \mid y)$ of Γ is a path of $\tilde{\Gamma}$ such that $a_n a_{n-1} \dots a_1$ is a composition of arrows in $\tilde{\Gamma}$ where $s(a_n a_{n-1} \dots a_1) = x$ and $e(a_n a_{n-1} \dots a_1) = y$. The composition of two walks is still a walk. Let v be a walk form y to z and k be a walk from x to y, then the composition vk is a walk form x to z.

We define the **homotopy** for walks by giving a equivalent relation as following.

1. $(x \mid aa^{-1} \mid x) \sim (x \mid b^{-1}b \mid x) \sim= (x \mid \mid x)$ where $(x \mid \mid x)$ is the trivial path of x in $\tilde{\Gamma}_1$ and a, b is in $\tilde{\Gamma}_1$.
2. $(x \mid (\sigma(\alpha))^{-1}\alpha^{-1} \mid \tau(x)) \sim (x \mid \tau_x \mid \tau(x))$ where x is non-projective.
3. $(\tau(x) \mid \alpha\sigma(\alpha) \mid x) \sim (\tau(x) \mid \tau_x^{-1} \mid x)$ where x is non-projective
4. Let v, w, w', v' be walks and the index be the starting and end points of the walk. If $w_{x,y} \sim w'_{x,y}$, then $v_{y,z}w_{x,y} \sim v_{y,z}w'_{x,y}$ and $w_{x,y}v'_{z,x} \sim w'_{x,y}v'_{z,x}$

Let $\Pi(\Gamma, x)$ be the homotopy classes defined above of walks from x to x where x is a vertex in Γ_0. It is obviously that $\Pi(\Gamma, x)$ forms a group. We call $\Pi(\Gamma, x)$ **the fundamental group** of Γ in x .

Example 5.3. *Let Γ be the following translation quiver.*

$$\begin{tikzcd} & & m \arrow[dr, "\alpha"] & \\ y & \tau(x) \arrow[l] \arrow[ur, "\sigma(\alpha)"] \arrow[dr, "\sigma(\beta)"'] & & x \arrow[ll, dotted, "\tau_x"'] \\ & & n \arrow[ur, "\beta"'] & \end{tikzcd}$$

We have all the walks from x to x is equivalent to $(x \mid \mid x)$ thus the fundamental group of Γ in x is trivial. Similarly, the fundamental group of Γ in y and $\tau(x)$ are also trivial. For vertex m, we have that $(m \mid \sigma(\alpha)\tau_x\alpha \mid m) \sim (m \mid \sigma(\alpha)(\sigma(\alpha))^{-1}\alpha^{-1}\alpha \mid m) \sim (m \mid \mid m)$, so the fundamental group of Γ in m is also trivial.

Observation 5.1. *Let Γ be a connected translation quiver. It is straightforward that the fundamental group $\Pi(\Gamma, x)$ dose not depend on the choice of x.*

Thus if Γ is a connected quiver, we define the fundamental group of Γ denoted as $\Pi(\Gamma)$ to be $\Pi(\Gamma, x)$ for any $x \in \Gamma_0$.

Definition 5.2. ***Simply connected translation quiver***. *A connected translation quiver Γ is simply connected if it's fundamental group is trivial.*

That definition is equivalent to that a translation quiver Γ is called simply connected if there exist x in Γ_0 such that $\Pi(\Gamma, x)$ is trivial.

Observation 5.2. *Let Γ be a simply connected translation quiver, then there is only one homotopy class of the walk from x to y where $x, y \in \Gamma_0$.*

We now consider the map between two translation quiver.

Definition 5.3. *Translation quiver morphism. A morphism $f : (\Gamma, \tau) \to (\Gamma', \tau')$ is called a translation quiver morphism if the following conditions are satisfied.*

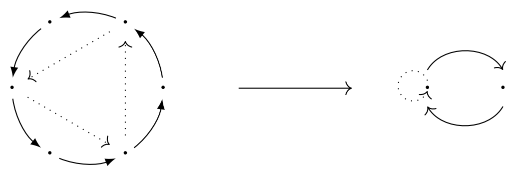
1. *$f \mid_{\Gamma_0}: \Gamma_0 \to \Gamma'_0$ and $f \mid_{\Gamma_1}: \Gamma_1 \to \Gamma'_1$.*
2. *Let $\alpha : x \to y$ be a arrow in (Γ, τ), then $f(\alpha)$ is a arrow $f(x) \to f(y)$ in (Γ', τ').*
3. *$f(\tau(x)) = \tau'(f(x))$ for all non-projective vertices $x \in \Gamma_0$.*

Further, we consider the onto translation quiver morphism.

Definition 5.4. *Covering. A translation quiver morphism $f : (\Gamma, \tau) \to (\Gamma', \tau')$ is called a covering if the following conditions are satisfied.*

1. *f is onto.*
2. *If $x \in \Gamma_0$ is projective, then $f(x)$ is projective in Γ'_0.*
3. *If $x \in \Gamma_0$ is injective, then $f(x)$ is injective in Γ'_0.*
4. *For each $x \in \Gamma_0$, f introduces a bijection from x^- to $f(x)^-$ and from x^+ to $f(x)^+$ respectively.*

Example 5.4. *The following translation quiver morphism $f : (\Gamma, \tau) \to (\Gamma', \tau')$ is a covering. Dotted arrow represents translation.*



We now consider the quiver whose objects is the homotopy class of walks of Γ denoted by $[w]$ where w is a walk in Γ.

Definition 5.5. *Universal cover. Let (Γ, τ) be a translation quiver. The universal cover $(\hat{\Gamma}, \hat{\tau})$ of Γ at the point $x \in \Gamma$ is a translation quiver defined in the following way.*

1. *The vertices are the homotopy class of walks of Γ which is starting from x.*
2. *There is an arrow between $[w]$ and $[u]$ if there is an arrow in Γ_1 from the endpoint of $[w]$ to the endpoint of $[u]$.*
3. *Let y be the endpoint of $[w]$, if y is a non-projective vertex in Γ_0, then $\hat{\tau}([w])$ is the homotopy class of the composition $[(y \mid \tau_y \mid \tau(y))w]$.*

We now introduce a natural projection $\pi : (\hat{\Gamma}, \hat{\tau}) \to (\Gamma, \tau)$. We use $\dot{w}$ to denote the endpoint of $[w] \in \hat{\Gamma}_0$. π maps $[w] \in \hat{\Gamma}_0$ to $\dot{w}$ which is in Γ_0. Let α be an arrow between $[w]$ and $[v]$, then π maps α to the arrow in Γ from $\dot{w}$ to $\dot{v}$. Obviously, the natural projection π is a covering.

Proposition 5.3. *Let Γ be a simply connected translation quiver, then each connected covering $f : \delta \to \Gamma$ is an isomorphism.*

Proof. f is onto by the hypothesis. Since f is a connected covering, we know for each $x \in \delta_0$, f introduce an isomorphism from x^- to $f(x)^-$ and from x^+ to $f(x)^+$ respectively. Let m be any vertex in Γ_0, if the number of $f^{-1}(m)$ is more then one, then it contradicts the fact we stated above. So the number of $f^{-1}(m)$ is one, then f is injective. Thus, f is an isomorphism. $\square$

Corollary 5.3.1. *Let Γ be a simply connected translation quiver. The universal cover $\hat{\Gamma}$ is equivalent to Γ.*

Proof. We have defined the natural projection $\pi : \hat{\Gamma} \to \Gamma$ which is a connected covering. By the proposition above, we know π is an isomorphism. Thus $\hat{\Gamma}$ is equivalent to Γ. $\square$

Proposition 5.4. *Let Γ be a simply connected translation quiver and $x_0 \in \Gamma_0$. There is one and only one translation quiver morphism $f : \Gamma \to \mathbb{Z}A_2$ such the $f(x_0) = 0$.*

$$\begin{tikzcd}
{} & \mathit{-3} \arrow[l, dotted] \arrow[dr] & & \mathit{-1} \arrow[ll, dotted] \arrow[dr] & & \mathit{1} \arrow[ll, dotted] \arrow[dr] & & {} \arrow[ll, dotted] & \\
& {} & \mathit{-2} \arrow[l, dotted] \arrow[ur] & & \mathit{0} \arrow[ll, dotted] \arrow[ur] & & \mathit{2} \arrow[ll, dotted] & & {} \arrow[ll, dotted]
\end{tikzcd}$$

$\mathbb{Z}A_2$

Proof. We define the length l of the homotopy class of a walk in Γ as the following way.

1. $l(x \mid \mid x) = 0$.
2. Let α be an arrow $x \to y$ in Γ, then $l(x \mid \alpha \mid y) = 1$ and $(y \mid \alpha^{-1} \mid x) = -1$.
3. Let τ_x be the translation from x to $\tau(x)$, then $l(x \mid \tau_x \mid \tau(x)) = -2$ and $l(\tau(x) \mid (\tau_x)^{-1} \mid x) = 2$.
4. $l(x_n \mid a_n \ldots a_1 \mid x_0) = l(x_0 \mid a_1 \mid x_1) + \cdots + l(x_{n-1} \mid a_n \mid x_n)$.

From observation 5.2, we know that all walks from x to y are in the same homotopy class where $x, y \in \Gamma_0$. Let $f(x) = l(x_0 \mid \cdots \mid x)$, then we have that $f(x_0) = l(x_0 \mid \mid x_0) = 0$. Thus we get our desired map.

□

Corollary 5.4.1. *If Γ is a finite simply connected translation quiver, there is one and only one translation quiver morphism $f : \Gamma \to \mathbb{Z}A_2$ such that $\min_{\forall x \in \Gamma_0} f(x) = 0$.*

Proof. Pick arbitrary $x_0 \in \Gamma_0$ as the fixed point. Let $h : \Gamma \to \mathbb{Z}A_2$ be the map introduced in proposition 5.4 such that $h(x_0) = 0$. Since Γ is finite, there is $a \in \Gamma_0$ such that $h(a) \leq h(x)$ for all $x \in \Gamma_0$. We define f as in proposition 5.4 and by letting $f(a) = 0$. Thus we get our desired map. □

5.2 Grading Trees

Definition 5.6. *Tree. Let T_0 denote the set of vertices and let T_1 denote the set of path between vertices. A tree $T = (T_0, T_1)$ is a non-oriented graph which satisfies the following.*

1. *There is no circle path .*
2. *If two vertices are connected, there is exactly one simple path .*

We call a tree **finite** if the number of vertices is finite. Two vertices are **neighbours** in a tree if they are connected by an edge. To study the simply connected algebras, K.Bongartz and P.Gabriel introduced graded trees in [3]. A **grading** of a tree T is a function $g : T_0 \to \mathbb{N}$ which satisfies the following.

1. If x and y are neighbours in T, then $g(x) - g(y)$ is odd.
2. $\exists x \in T_0, g(x) = 0$.

Definition 5.7. *Graded Tree. A graded tree (T, g) is a tree T together with a grading g.*

We will define a representation-finite graded tree by giving the associated translation quiver and a dimension map to this quiver.

Definition 5.8. *Associated translation quiver of a tree. Let Q_T be the associated translation quiver of a tree T. We define Q_T in the following way.*

1. *The vertices in Q_T are the collection of (n, t) where t is a vertex in T and $n - g(t) \in 2\mathbb{N}$.*
2. *There is a arrow from (m, s) to (n, t) if $n - 1 = m$ and s, t are neighbours in T.*
3. *Projective vertices are $(g(t), t)$.*
4. *Let τ denote the translation, then $\tau(n, t) = (n-2, t)$ if (n, t) is non-projective.*

Example 5.5. *For the graded tree* $T = \begin{tikzcd}[column sep=small, row sep=small] 6 \arrow[dr, dash] & & 8 \arrow[dl, dash] \\ & 3 \arrow[d, dash] & \\ & 0 & \end{tikzcd}$ *, we have the associated translation quiver Q_T as following.*

$$\begin{tikzcd}[column sep=tiny]
 & & & & & & & & (8,n) \arrow[ddr] & & (10,n) \arrow[ll, dotted] \arrow[ddr] & & \cdots \arrow[ll, dotted] & \\
 & & & & & & (6,m) \arrow[dr] & & (8,m) \arrow[ll, dotted] \arrow[dr] & & (10,m) \arrow[ll, dotted] \arrow[dr] & & \cdots \arrow[ll, dotted] & \\
 & & & (3,t) \arrow[dr] & & (5,t) \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & (7,t) \arrow[ll, dotted] \arrow[ur] \arrow[uur] \arrow[dr] & & (9,t) \arrow[ll, dotted] \arrow[ur] \arrow[uur] \arrow[dr] & & (11,t) \arrow[ll, dotted] \arrow[ur] \arrow[uur] \arrow[dr] & & \cdots \arrow[ll, dotted] \\
(0,s) & & (2,s) \arrow[ll, dotted] \arrow[ur] & & (4,s) \arrow[ll, dotted] \arrow[ur] & & (6,s) \arrow[ll, dotted] \arrow[ur] & & (8,s) \arrow[ll, dotted] \arrow[ur] & & (10,s) \arrow[ll, dotted] \arrow[ur] & & \cdots \arrow[ll, dotted] &
\end{tikzcd}$$

where dotted arrow is translation τ. $(0, s), (3, t), (6, m), (8, n)$ are projective.

The **dimension map** $d : Q_T \to \mathbb{N}^{(Q_T)_0}$ is defined as following.

1. For a projective vertice $(g(t), t)$, $d(g(t), t) = \delta(t) + \sum_s d(g(t) - 1, s)$ where s is the neighbour of t such that $g(s) < g(t)$. $\delta(t)$ is the vector having value 1 at t-th position and having zero at the rest place.

2. For a non-projective vertices (n, t), $d(n, t) = \sum_s d(n-1, s) - d(n-2, t)$ if $d(n-2, t) > 0$ and $\sum_s d(n-1, s) - d(n-2, t) > 0$ where s is the neighbour of t such that $g(s) < n$.

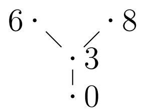
3. For any other vetices, we have $d((n, t)) = 0$.

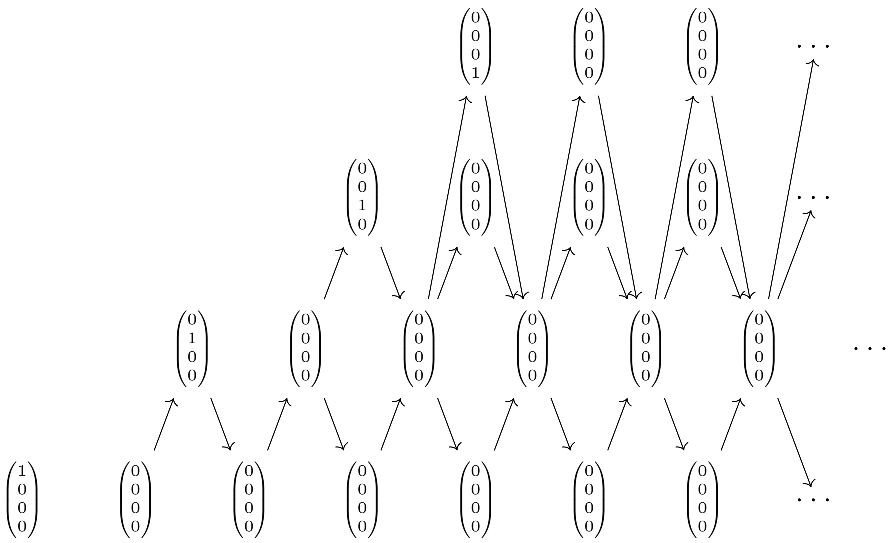
Let R_T denote the full sub-quiver of Q_T such that if the vertex (n, t) is in R_T, then $d(n, t) \neq 0$. We call the graded tree (T, g) **admissible** if R_T is a connected sub-quiver of Q_T. Then T is called **admissible graded tree**. The grading g is called **representation-finite** if (T, g) is admissible and R_T is finite. Then T is called a **representation-finite graded tree**.

Observation 5.5. *Apparently, T is admissible if and only if R_T is a component which contains all the projective vertices $(g(t), t)$.*

We have looked at the associated quiver for example 5.5. In the following we will look at the correspond dimension map and whether it is a representation-finite tree.

Example 5.6. *Dimension map of $T =$*





The sub-quiver R_T is finite but not connected, so T is not admissible also not representation-finite.

Since we have introduced how to associate a translation quiver to a tree, we are also interested in how to find an associated tree for a given translation quiver.

Let (Γ, τ) be a locally finite translation quiver. Let x be an arbitrary vertex in Γ_0 where τ is defined. We call the set $x^\tau = \{\tau^n(x) : n \in \mathbb{Z}\}$ the **τ-orbit** of x. We call x^τ **stable** if $\tau^n(x) \neq 0$ for all $n \in \mathbb{Z}$. If x^τ is stable and the cardinality is a finite number, then we say x^τ is **periodic**. We have the following straightforward observation.

Observation 5.6. *Let x and y be two connected stable vertices in Γ_0, if one of them are periodic then both of them are periodic.*

We say a component is **a periodic component** if it is formed by connected periodic τ-orbits.

Let x be a vertex in Γ_0 where τ is defined and let y be a vertex in Γ_0 such that there is an arrow $y \xrightarrow{\alpha} x$. Then there is an arrow $\tau(x) \to y$ denoted as $\sigma(\alpha)$. The **σ-orbit** of α denoted as α^σ is the set of all arrows in Γ_1 of the form $\sigma^n(\alpha)$ where $n \in \mathbb{Z}$. Two τ-orbits are connected if they are connected by a σ-orbit.

We define **the associated graph** G_Γ of a quiver Γ as following.

1. The vertice of G_Γ are the periodic components and the τ-orbits of Γ.

2. If the vertex of G_Γ is the periodic components of Γ, we associate a loop to it $\circlearrowleft$.

3. If x^τ and y^τ are two connected τ-orbits by α^σ and they are not in the same periodic component, then the correspond vertices of x^τ and y^τ in G_Γ are also connected.

Observation 5.7. *Let Λ be an algebra over an algebraically closed field k such that it has a simply connected Auslander-Reiten quiver Γ_Λ. Then the associated graph G_{Γ_Λ} is a tree, since simply connected translation quivers do not admit periodic τ-orbit.*

Observation 5.8. *Let Γ be a simply connected translation quiver. Let $f : \Gamma \to \mathbb{Z}A_2$ be the map we defined in corollary 5.4.1 such that $\min_{\forall x \in \Gamma_0} f(x) = 0$. We use this result to define **the grading** of G_Γ.*

We use (G_Γ, g_Γ) to denote the graded associated graph of quiver Γ. By the construction of G_Γ, each vertex y in G_Γ is correspond with an τ-orbit in Γ denoted as y^τ. There is only one the projective vertex P in y^τ. We define g by letting $g_\Gamma(y) = f(P)$. Since $g(y) - f(x)$ is odd when y and x are neighbours in G_Γ and $g_\Gamma^{-1}(0)$ is not empty, g_Γ is a grading function.

5.3 Simply connected algebras

Let k be an algebraically closed field and Λ be a finite-dimensional basic $k-algebra$. Let Γ be the quiver such that $k\Gamma/I$ is isomorphisc to Λ where I is admissible. We call Λ connected if Γ is connected, i.e. Λ is indecomposable as an algebra.

Definition 5.9. ***Simply connected algebras.*** *An algebra Λ over an algebraically closed field k is simply connected if Λ is representation-finite, connected, basic, finite-dimensional and having simply connected Auslander-Reiten quiver Γ_Λ.*

We use G_Λ to denote the associated graph of the Auslander-Reiten quiver Γ_Λ. From observation 5.7, we know that Γ_Λ is a finite tree. It is natural to ask the relation between finite trees and simple connected algebras. Bongartz and Gabriel showed the following statements in [3].

1. The number of isomorphism classes of simply connected algebras Λ such that G_Λ is isomorphic to a finite tree is finite.

2. Each finite tree admits only a finite number of representation-finite gradings.

This is proved by induction on the size of the tree.

We will use **mesh category** to transfer the studying of indecomposable modules to the study of homomorphism space.

Definition 5.10. ***Mesh category***. *Let Γ be a translation quiver. A mesh on $x \in \Gamma_0$ is the full subquiver of Γ whose vertices are the same as Γ_0. The mesh relation m_x of Γ on x where $x \in \Gamma_0$ is defined by $m_x = \Sigma_{\{\alpha\in\Gamma_1 | e(\alpha)=x\}}\alpha\sigma(\alpha)$. The mesh ideal is the ideal I generated by $\{m_x\}$ where x rangs over all vertices in Γ_0. The mesh category of Γ is the residue category $k\Gamma/I$ denoted as $k(\Gamma)$.*

Example 5.7. *For $\mathbb{Z}A_2$, the meshes are of the form* $\begin{tikzcd}[sep=small] \cdot \arrow[dr] & & \cdot \\ & \cdot \arrow[ur] & \end{tikzcd}$ *or* $\begin{tikzcd}[sep=small] & \cdot \arrow[dr] & \\ \cdot \arrow[ur] & & \cdot \end{tikzcd}$ *The objects of the mesh category $k(\mathbb{Z}A_2)$ are the vertices in $(\mathbb{Z}A_2)_0$ and the morphisms are the arrows in $(\mathbb{Z}A_2)_1$.*

$$\begin{tikzcd}[column sep=small]
{} & & -3 \arrow[ll, dotted] \arrow[dr] & & -1 \arrow[ll, dotted] \arrow[dr] & & 1 \arrow[ll, dotted] \arrow[dr] & & {} \arrow[ll, dotted] & \\
& {} & & -2 \arrow[ll, dotted] \arrow[ur] & & 0 \arrow[ll, dotted] \arrow[ur] & & 2 \arrow[ll, dotted] & & {} \arrow[ll, dotted]
\end{tikzcd}$$

$\mathbb{Z}A_2$

We use $\operatorname{ind}\Lambda$ to denote the full sub-category of Λ whose objects are a chosen set of representative of the indecomposable modules.

Proposition 5.9. *For a simply connected algebra* Λ*,* $k(\Gamma_\Lambda)$ *is isomorphic to* $\operatorname{ind}\Lambda$ *and* Λ *is isomorphic to* $\underset{p,q}{\oplus} k(\Gamma_\Lambda)(p,q)$ *where* p,q *ranges over all the projective indecomposable modules of* Λ*.*

Proof. By the construction of Γ_Λ, we know the objects in $k(\Gamma_\Lambda)$ are indecomposable modules in Λ. Since Λ is representation finite, the dimension of the homomorphism space between two indecomposable modules of Λ is less than two. Thus, $\operatorname{ind}\Lambda \cong k(\Gamma_\Lambda)$. $\Lambda \cong \underset{p,q}{\oplus} k(\Gamma_\Lambda)(p,q)$ is coming from $\Lambda \cong \underset{p,q \in \operatorname{ind}\Lambda}{\oplus} \operatorname{Hom}_\Lambda(p,q)$. □

Let (T,g) be an admissible graded tree and let R_T be the full sub-quiver of the associated translation quiver of (T,g) such that $d(n,t) \neq 0$ where d is the dimension map of (T,g). We use A_T to denote the algebra $\underset{p,q}{\oplus} k(R_T)(p,q)$, where q,p ranges over all projective vertices of R_T.

We associate $\underset{p}{\oplus} k(R_T)(p,x)$ to $x \in (R_T)_0$. It is obvious that $\underset{p}{\oplus} k(R_T)(p,x)$ becomes a left module of ${A_T}^{op}$. There are morphisms from $\underset{p}{\oplus} k(R_T)(p,x)$ to $\underset{p}{\oplus} k(R_T)(p,y)$ in $\operatorname{mod} A_T$ if there are some paths from x to y in R_T. It yields a functor $M : k(R_T) \to \operatorname{mod} {A_T}^{op}$.

Proposition 5.10. *Let* p *be projective in* $k(R_T)$*. For* $M : k(R_T) \to \operatorname{mod} {A_T}^{op}$*, we have the following.*

1. $dim_k(End_{A_T}(M(p))) = 1$*. Equivalently,* $End_{A_T}(M(p)) = k$*.*
2. $\underset{\alpha}{\oplus} M(x)$ *is isomorphic to the radical of* $M(p)$ *where* $x \to P$ *ranges over all the arrows stoping at* p*.*

Proof.
1. Since there is no cycle in $k(R_T)_1$, the only path from p to p in $k(R_T)$ is the identity. Then the identity map is a generator of $End_{A_T}(M(p))$. Thus $dim_k(End_{A_T}(M(p))) = 1$. Consequently, $End_{A_T}(M(p)) = k$.
2. Since $M(p)$ is isomorphic to $\operatorname{Hom}_{A_T}(A_T, M(p))$, $rad_{A_T}(M(p))$ is isomorphic to $rad_{A_T}(\operatorname{Hom}_{A_T}(A_T, M(p)))$ where $A_T \cong \underset{q}{\oplus} M(q)$ that q ranges over all the indecomposable projective modules in $k(R_T)$.

   Then $\underset{q}{\oplus} \operatorname{Hom}_{A_T}(q,p)$ where q ranges over all the projective vertice except p in R_T is the radical of $\operatorname{Hom}_{A_T}(A_T, M(p))$ since there is no cycle in R_T.

   If there is a path from the projective vertex q to p, the path must pass through a vertex x such that there is a arrow from x to p. Since each vertex in $(R_T)_0$ only belongs to one τ-orbits and each τ-orbits of R_T only contains one projective vertex, we have that $\bigoplus\limits_{\alpha} M(x)$ is isomorphic to $\bigoplus\limits_{q} \mathrm{Hom}_{A_T}(q,p)$. Thus $\bigoplus\limits_{\alpha} M(x)$ is isomorphic to the radical of $M(p)$

□

Corollary 5.10.1. *Let p be projective vertex in $k(R_T)$, then $M(p)$ is indecomposable projecitve module in* $\mathrm{mod}\, {A_T}^{op}$.

Proof. We know that $M(p)$ is indecomposable in R_T if and only if $End_{A_T}(M(p))$ only admits 0 and 1 as idempotents. Thus $M(p)$ is indecomposable in $\mathrm{mod}\, {A_T}^{op}$. Since ${A_T}^{op} = \bigoplus\limits_{q} M(q)$ where q ranges over all projective vertices in R_T, $M(p)$ is projective. □

Proposition 5.11. *Let (n,t) be a non-projective vertex in $k(R_T)$, then $M(n-2,t) \xrightarrow{M(\tilde{f})} \bigoplus\limits_{s} M(n-1,s) \xrightarrow{M(\tilde{g})} M(n,t)$ where s ranges over all the neighbours of t is an Auslander Reiten sequence.*

Proof. In mesh category, the mesh relation is isomorphisc to zero, so $M(\tilde{g}) \circ M(\tilde{f}) = M(\tilde{g}\tilde{f}) = 0$. The kernel of $M(\tilde{f})$ is zero in $M(n-2,t)$ since there is an arrow from $(n-2,t)$ to $(n-1,s)$ in $k(R_T)$. Then $M(\tilde{f})$ is injective.

By that for each path from a projective vertex p to (n,t) in $k(R_T)$, the path must pass through one of the $(n-1,s)$, we know $M(\tilde{g})$ is minimal almost right split which implies $M(\tilde{g})$ is surjective. Thus $M(n-2,t) \xrightarrow{M(\tilde{f})} \bigoplus\limits_{s} M(n-1,s) \xrightarrow{M(\tilde{g})} M(n,t)$ is an almost split sequence then an Auslander Reiten sequence. □

Corollary 5.11.1. *Let (n,t) be a non-projective vertex in $(k(R_T))$, then $M(n,t)$ is indecomposable in* $\mathrm{mod}\, {A_T}^{op}$.

Proof. From proposition 5.11, we know $\bigoplus\limits_{s} M(n-1,s) \xrightarrow{M(\tilde{g})} M(n,t)$ is minimal right almost split which implies $M(n,t)$ is indecomposable. □

Summarizing corollary 5.10.1 and 5.11.1, we proved the following proposition.

Proposition 5.12.
1. *For each vertex (n,t) in $(k(R_T))_0$, $M(n,t)$ is indecomposable in* $\mathrm{mod}\, {A_T}^{op}$.
2. *Let* $\mathrm{ind}\, {A_T}^{op}$ *be the full sub-category of* $\mathrm{mod}\, {A_T}^{op}$ *which consists of the indecomposable modules. $M : k(R_T) \to \mathrm{mod}\, {A_T}^{op}$ introduces an equivalence between $k(R_T)$ and a full sub-category of* $\mathrm{ind}\, {A_T}^{op}$. *It also introduces an translation quiver isomorphism between R_T and a component of $\Gamma_{{A_T}^{op}}$.*

For a finite graded tree (T,g), the second part of the proposition describes there is an Auslander reiten quiver such that it is isomorphic to R_T.

Proposition 5.13. *Let (T, g) be a representation-finite graded tree, then (T, g) is isomorpic to the associated graded graph (G_{A_T}, g_{A_T}) of A_T defined in observation 5.8.*

Proof. Since $M(p)$ where p ranges over all projective vertexs in $k(R_T)$ completes the set of indecomposable projective modules in A_T, we have that $(G_{A_T})_0 \cong T_0$. If two vertices are connected in T, then the correspond vertice are connected in G_{A_T} by proposition 5.12.

Let Γ_{A_T} be the Auslander reiten quiver of A_T. In observation 5.8, we illustrated how to define g_{A_T} through a specific map $f : \Gamma_{A_T} \to \mathbb{Z}A_2$ such that $\min_{\forall x \in (\Gamma_{A_T})_0} f(x) = 0$. Clearly x is a projective vertex in Γ_{A_T}. Then for each projective vertex p_i in Γ_{A_T}, $f((x \mid a_n \dots a_0 \mid p_i))$ is equal to the grading of the correspond vertex of p_i in T. Thus $(T, g) \cong (G_{A_T}, g_{A_T})$ □

Proposition 5.14. *There is a bijection between the isomorphism classes of representation finite graded trees and the isomorphism classes of simply connected algebras.*

Proof. Let Λ be a simply connected algebras, then $\bigoplus_{p,q} k(\Gamma_\Lambda)(p, q)$ where p, q ranges over all the projective indecomposable modules of Λ by proposition 5.9. By proposition 5.13, for a representation finite graded tree, we have that $(T, g) \cong (G_{A_T}, g_{A_T})$ where A_T is in the form of $\bigoplus_{p,q} k(R_T)(p, q)$. Thus there is a bijection between the isomorphic classes of representation finite graded tree and simply connected algebra. □

Proposition 5.15. *Each finite tree T only admits a finite number of representation finite gradings.*

Proof. We will prove it by induction. Let N_T denote the number of vertices of T. When $N_T = 1$, there is only one grading g such that $g = 0$. We assume when $N_T \leq m - 1$, the hypothesis is satisfied.

When $N_T = m$, for each representation finite grading g of T, there is a vertex $x \in T$ such that the sub-tree of T containing all vertices except x in T is still representation finite.

Since there is only a finite number of trees having $m - 1$ vertices and they all only admits a finite number of representation finite gradings, there is $N \in \mathbb{N}$ such that N is the maximum of all the grades in all representation finite graded trees which has $m - 1$ vertices. Then for $x \in T_0$, $g(x) \leq M + 2$. Thus for $N_T = n$, T admits only finite number of representation finite gradings.

□

For each vertex x in $k(R_T)$, $M(x)$ is isomorphic to $\underset{p}{\oplus} k(R_T)(M(p), M(x))$ where p ranges over all the indecomposable vertices in $k(R_T)$. Thus $dim_{A_T} M(x)$ is equal to $\underset{p}{\oplus} dim_{A_T} k(R_T)(M(p), M(x))$.

Definition 5.11. *Dimension vector. Let* Λ *be an artin ring and* A *be a finite length* Λ-module. *Let* $\{S_1, \dots, S_n\}, n \in \mathbb{N}$ *be be a chosen set of representative of the simple modules in* $\operatorname{mod}\Lambda$. *The dimension vector* d *of* A *is defined as the* n *dimensional vector* $(d_1, \dots, d_n)$ *such that* $d_i = m_{S_i}(A)$.

Proposition 5.16. *Let* Λ *be an elementary artin algebra and* A *be a finitely generated* Λ-module. *Let* $\{S_1, \dots, S_n\}$ *be a chosen set of representative of the simple modules in* $\operatorname{mod}\Lambda$. *Let* $\{P_1, \dots, P_n\}$ *be the set of the indecomposable projective modules such that* $P_i \to S_i$ *is a projective cover. Let* $(d_1, \dots, d_n)$ *be the dimension vector of* A. *Then* $d_i = l_{End(P_i)^{op}} \operatorname{Hom}_\Lambda(P_i, A) = dim_{End(P_i)^{op}} \operatorname{Hom}_\Lambda(P_i, A)$.

Proof. We will prove it by induction on the length of A. When $l(A) = 1$, if $A \cong S_1$, we have $d_i = 1$ and $dim_{End(P_i)^{op}} \operatorname{Hom}_\Lambda(P_i, A) = 1$. Otherwise, $d_i = dim_{End(P_i)^{op}} \operatorname{Hom}_\Lambda(P_i, A) = 0$.

We assume that when $l(A) \leq n - 1$, $d_i = dim_{End(P_i)^{op}} \operatorname{Hom}_\Lambda(P_i, A)$.

When $l(A) = n$, let $A \supset N \supset \cdots \supset 0$ be composition series of A. We have the exact sequence $0 \to A/N \to A \to N \to$ where A/N is simple. Applying $\operatorname{Hom}_\Lambda(P_i, -)$, $0 \to \operatorname{Hom}_\Lambda(P_i, A/N) \to \operatorname{Hom}_\Lambda(P_i, A) \to \operatorname{Hom}_\Lambda(P_i, N) \to 0$ is exact. Then we have that $dim_{End(P_i)^{op}} \operatorname{Hom}(P_i, A) = dim_{End(P_i)^{op}} \operatorname{Hom}(P_i, N) + dim_{End(P_i)^{op}} \operatorname{Hom}(P_i, A/N)$. Since $l(N)$ and $l(A/N)$ both less then n, we have that $dim_{End(P_i)^{op}} \operatorname{Hom}_\Lambda(P_i, A) = d_i(N) + d_i(A/N) = d_i(A)$. □

Proposition 5.17. *For each vertex* (n, t) *in* $k(R_T)$, *the dimension vector of* $M(n, t)$ *is* $d(n, t)$.

Proof. Let $d(n, t) = (d_1, \dots, d_n)$. We showed in proposition 5.16 that $d_i = dim_{A_T} k(R_T)(M(p_i), M(n, t))$ where P_i is the correspond indecomposable projective vertex in $k(R_T)$.

We will prove it by induction. We use N_t to denote the set of neighbors of t in T.

For each $(0, s)$ in $k(R_T)$, $d_i(0, s) = dim_{A_T} k(R_T)(M(p_i), M(n, t)) = 1$ if $M(p_i) = M(n, t)$. Otherwise, $d_i(0, s) = 0$. Thus for $n = 0$, the hypothesis is satisfied.

We assume when $n \leq m - 1$, the hypothesis is satisfied.

When $n = m$, there is a morphism from $M(P_i)$ to $M(m, t)$ if there are some paths from P_i to $(m - 1, s)$ where s is an arbitrary neighbor of t.

When (m, t) is projective, $End_{A_T}(M(m, t))$ is generated by the identity map. Then $dim_{A_T} k(R_T)(M(p_i), M(m, t)) = 1$, if $p_i = (m, t)$. Otherwise, we have that $dim_{A_T} k(R_T)(M(p_i), M(m, t)) = \underset{s}{\Sigma} dim_{A_T} k(R_T)(M(p_i), M(m-1, s))$ where $s \in N_t$. Thus $dim_{A_T} k(R_T)(M(p_i), M(m, t)) = d_i(m, t)$.

We use I to denote the cardinality of the set that consists of the independent relations from P_i to (m,t) which contains the mesh relation on (m,t).

For non-projective vertex (m,t), $I = dim_{A_T}k(R_T)(M(p_i), M(m-2,t))$. Then $dim_{A_T}k(R_T)(M(p_i), M(m-2,t)) = d_i(m-2,t)$ since $m-2 < m$.

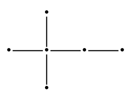
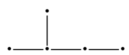
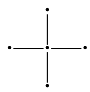
Since $dim_{A_T}k(R_T)(M(p_i), M(m,t)) = (\sum\limits_{s\in N_t} dim_{A_T}k(R_T)(M(p_i), M(m-1,s))) - I$, we have that $\sum\limits_{s\in N_t} dim_{A_T}k(R_T)(M(P_i), M(m-1,s)) = \sum\limits_{s\in N_t} d_i(M(m-1,s))$. Thus, $dim_{A_T}k(R_T)(M(p_i), M(m,t)) = \sum\limits_{s\in N_t} d_i(M(m-1,s)) - d_i(m-2,t) = d_i(n,t)$. □

5.4 Representation finite gradings of $\ddot{D}_5$ and D_6

We define $\ddot{D}_n$ be the tree with $n+1$ vertices and of the form

$$\begin{array}{ccccccccccc} & & \text{n} & & & & & & & & \\ & & | & & & & & & & & \\ 1 & — & 2 & — & 3 & — & 4 & — & \cdots & — & \text{n-1} \\ & & | & & & & & & & & \\ & & \text{n+1} & & & & & & & & \end{array}$$

5.4.1 Representation finite gradings of $\ddot{D}_5$

In the following, we will calculate all the representation finite gradings for the tree $\ddot{D}_5$ . For each represenation finite gradings g, there is a vertex t in the tree such that the sub-tree T of $\ddot{D}_5$ by removing t is still a tree and the correspond grading for T is generated by $g|_T - \min\limits_{x\in T_o} g(x)$ where $g|_T$ is g confined in T is still representation finite. The connected sub-tree with 5 vertices of $\ddot{D}_5$ are of the form D_5  or C_5 . Thus we can find the representation finite gradings for $\ddot{D}_5$ by extending the representation finite gradings of D_5 and C_5. For example for $\begin{array}{cccc} & 3 & & \\ & | & & \\ 3 - & 0 & - 1 - & 0 \end{array}$, we have the none-zero dimension quiver

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & & 01101 \arrow[ddr] & & 00110 \arrow[ddr] & & 00001 \\
 & & & 01110 \arrow[dr] & & 00101 \arrow[dr] & & 00010 \\
10000 \arrow[dr] & & 01100 \arrow[uur] \arrow[ur] \arrow[dr] & & 01211 \arrow[uur] \arrow[ur] \arrow[dr] & & 00111 \arrow[uur] \arrow[ur] & \\
 & 11100 \arrow[ur] \arrow[dr] & & 00100 \arrow[ur] & & 01111 \arrow[ur] & & \\
01000 \arrow[ur] & & 10100 \arrow[ur] & & & & &
\end{tikzcd}$$

Then we have extensions $\begin{array}{cccc} & 3 & & \\ & | & & \\ 3 - & 0 & - 1 - & 0 \\ & | & & \\ & 1 & & \end{array}$, $\begin{array}{cccc} & 3 & & \\ & | & & \\ 3 - & 0 & - 1 - & 0 \\ & | & & \\ & 3 & & \end{array}$, $\begin{array}{cccc} & 3 & & \\ & | & & \\ 3 - & 0 & - 1 - & 0 \\ & | & & \\ & 5 & & \end{array}$, $\begin{array}{cccc} & 3 & & \\ & | & & \\ 3 - & 0 & - 1 - & 0 \\ & | & & \\ & 7 & & \end{array}$.

Specially,let t be the vertex in $\ddot{D}_5$ which only have one neighbour and let the grade of t be the only zero in $\ddot{D}_5$. Let g^t be the grading of D_5 or C_5 such that the correspond vertex which will be the neighbour of t in $\ddot{D}_5$ has grade 0, then the grading of the vertices in $\ddot{D}_5$ except t is defined by $g^t + 1$. For example, we can extend $\begin{array}{cccc} & 3 & & \\ & | & & \\ 3 - & 0 & - 1 - & 0 \end{array}$ to $\begin{array}{cccc} & 4 & & \\ & | & & \\ 4 - & 1 & - 2 - & 1 \\ & | & & \\ & 0 & & \end{array}$. Bongartz and Gabriel have calculated the representation finite gradings of D_5 and C_5 in [3]. Based on their result, we calculated the the representation finite gradings for $\ddot{D}_5$. The gradings are listed as following in the order of

$$\begin{array}{ccccccc} & & 5 & & & & \\ & & | & & & & \\ 1 & — & 2 & — & 3 & — & 4 \\ & & | & & & & \\ & & 6 & & & & \end{array}$$

Remark 5.1. *By the rotation of the vertices in $\ddot{D}_5$, the gradings with the form of The gradings in the form of $E---BC, E---CB, B---EC, B---CE, C---BE$ and $C---EB$ are considered as the same. are considered as the same. $'A'$ denotes number* 10.

101415	101615	101815	101A15	101215	101015	105611
105411	103215	103415	501013	501213	501413	501613
501813	105431	105631	105831	105A31	103217	103417
107631	107831	301017	301217	301417	301617	301033
301233	103233	103433	103633	103833	301035	301235
103235	103435	103635	103835	305431	305631	501035
501235	305451	305651	305851	501037	501237	703251

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703451	105637	105437	307815	307615	010106	010506
016500	016700	010300	210106	210306	210506	012106
012306	012506	012706	012906	016520	016720	410104
410304	014304	014504	014704	410106	410306	014306
014506	014706	210126	210326	210526	121620	321620
521620	721620	921620	216520	216720	410124	410324
410524	410724	412140	412340	014324	014524	014724
014924	410126	410326	410526	410726	410926	612140
612340	014326	014526	216540	216740	121017	321017
521015	521017	321037	521035	521037	101011	101013
101015	101031	101033	101035	101037	101039	101051
101053	101055	101057	301031	301033	301035	301037
301051	301053	301055	301057	121011	121015	121013
121017	121019	121031	121033	121035	121037	121039
121051	121053	121055	121057	321033	321031	321035
321037	321039	321051	321053	321055	321057	010100
010102	010104	010106	010108	010120	010122	010124
010126	010140	010142	010144	210120	210122	210124
210126	210128	210140	210142	210144	210146	210148
410140	410142	410144	410146	410160	410162	410164
410166	410168	230122	230124	230126	230128	230142
230144	230146	230148	230162	230164	230166	230168
430142	430144	430146	430148	430126	430146	430166
430186	012100	012102	012104	012106	012120	012122
012124	012126	012140	012142	012144	012146	012148
012160	012162	012164	012166	012168	101211	101213
101215	101231	101233	101235	210151	210153	210155
210157	301231	301233	301235	301237	301251	301253
301255	301257	301259	501251	501253	501255	501257
501271	501273	501275	501277	501279	103211	103213
103215	103217	103219	103231	103233	103235	103237
103251	103253	103255	103257	010300	010302	010304
010306	010308	010320	010322	010324	010326	010328
010340	010342	010344	010346	010360	010362	010364
010366	010368	210320	210322	210324	210326	210328
210340	210342	201344	210346	210360	210362	210364
012320	012322	012324	012326	012328	012340	012342
012344	012346	012348	012360	012362	012364	012366
012368	014300	014302	014304	014306	014320	014322
014324	014326	014328	101411	101413	101415	101417

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101419	101431	101433	101435	101437	103411	103413
103415	103417	103419	103431	103433	103435	103437
103451	103453	103455	103457	103471	103473	103475
103477	103479	105431	105433	105435	105437	010500
010502	010504	010506	010508	010520	010522	010524
010526	010528	210520	210522	210524	210526	210528
21052A	210540	210542	210544	210546	210548	012500
012502	012504	012506	012508	012520	012522	012524
012526	012528	014500	014502	014504	014506	014508
014520	014522	014524	014526	014528	101611	101613
101615	101631	101633	101635	101637	101639	103631
103633	103635	105631	105633	105635	105637	105639
210740	210742	210744	012700	012702	012704	012706
012720	012722	012724	012726	212120	212140	212160
412140	412160	212320	212340	214360	212520	212540
214520	214540	214560	214580	216540	212710	212740
214740	216740	212360	412340	412360	612360	612380
214320	214340	010146	010148	012128		

Table 2: Representation finite gradings for $\ddot{D}_5$

5.4.2 Representation finite gradings of D_6

We first calculated the representation finite gradings for A_5 through extending A_3. We listed the gadings as following in the order of $1 - 2 - 3 - 4 - 5$.

01212	01214	01232	01234	30121	50121	01210
21012	41012	61012	10121	10123	10125	01010
01012	01014	10101	30101	50101	21014	41014
10143	10145	30103	01032	01034	21034	10343
10345	30123	50123	70123	21032	21036	21056
21054	10321	10323	10325			

Table 3: Representation finite gradings for A_5

We calculated the representation finite gradings for D_6 through extending A_5 and D_5. The gradings are listed in the order of

$$\begin{array}{ccccccccc} & & 6 & & & & & & \\ & & | & & & & & & \\ 1 & — & 2 & — & 3 & — & 4 & — & 5 \end{array}$$

Remark 5.2. *By the rotation of the vertices in D_6, the gradings with the form of $B - - - - E$ and $E - - - - B$ are considered as the same. $'A'$ denotes number 10.*

012120	012122	012124	012126	212100	212102	212104
212106	412100	412102	412104	412106	012140	012142
012144	012146	012148	012320	012322	012324	012326
012328	232102	232104	232106	012340	012342	012344
012346	012348	432102	432104	432106	301211	301213
301215	121031	121033	121035	121037	501211	501213
501215	501217	121051	121053	121055	012100	012102
012104	012106	210120	210122	210124	210126	410120
410122	410124	410126	210140	210142	210144	610120
610122	610124	610126	610128	210160	210162	210164
101211	101213	101215	101217	121011	121013	121015
101231	101233	101235	101237	321011	321013	321015
101251	101253	101255	521011	521013	521015	521017
010100	010102	010104	010120	010122	010124	010126
210100	210102	210104	210106	010140	010142	010144
410100	410102	410104	410106	101011	101013	101015
301011	301013	301015	101031	101033	103015	103017
501011	501013	501015	501017	101051	101053	101055
210140	210143	210145	410120	410123	410125	410127
410140	410142	410144	410146	101431	101433	341013
341015	341017	101451	101453	541013	541015	541017
301031	301033	301035	301037	010320	010322	010324
010326	230102	230104	230106	010340	010342	010344
010346	010348	430102	430104	430106	301231	301233
301235	301237	321031	321033	321035	321037	501231
501233	501235	501237	321051	321053	321055	701231
701233	701235	701237	701239	321071	321073	321075
210340	210342	210344	210346	430122	430124	430126
103431	103433	103435	103437	343013	343015	343017
103451	103453	103455	103457	103459	543013	543015
543017	121032	321032	521032	721032	230122	230124
230126	210360	210362	210364	210366	630122	630124
630126	630128	210560	210562	210564	650124	650126
650128	210540	210542	210544	450124	450126	450128
103211	103213	103215	123011	123013	123015	123017
103231	103233	103235	103237	323011	323013	323015
323017	103251	103253	103255	523011	523013	523015

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523017	412320	612320	212320	212340	212360	212120
412120	212140	612120	212160	212540	212560	412140
412340	612340	812340	214540	214560	214320	214340
214360	101011	101031	101051	101071	101013	101033
101053	101073	101015	101035	101055	301013	301033
301015	301035	121011	121031	121051	121071	121013
121033	121053	121073	121015	121035	121055	321012
321033	321053	321073	321015	321035	321055	321075
321095	010100	010120	010140	010160	010180	010102
010122	010142	010162	010182	010104	010124	010144
210102	210122	210142	210162	210182	210104	210124
210144	210164	210184	410104	410124	410144	410106
410126	410146	230102	230122	230104	230124	230106
230126	430104	430124	430106	430126	012100	012120
012140	012160	012102	012122	012142	012162	012182
012104	012124	012144	012106	012126	012146	101211
101231	101251	101213	101233	101253	101215	101235
101255	301215	301235	301213	301233	501215	501235
501217	501237	103211	103231	103251	103213	103233
103253	103273	103215	103235	103255	010320	010340
010360	010322	010342	010362	010324	010344	010326
010346	210322	210342	210362	210324	210344	210364
210384	210326	210346	210366	012320	012340	012360
012380	0123A0	012322	012342	012362	012382	0123A2
012324	012344	012326	012346	014320	014340	014360
014322	014342	014362	101431	101451	101471	101433
101453	101473	103431	103451	103433	103453	103473
103435	103455	103437	103457	105433	105453	105473
010540	010560	010542	010562	210542	210562	210544
210564	210584	012540	012560	012580	012542	012562
012582	014540	014560	014542	014562	101651	101671
101653	101673	103653	103673	105653	105673	210764
210784	012760	012780	012762	012782	212102	212104
212106	412104	412106	232102	232104	232106	432104
432106						

Table 5: Representation finite gradings for D_6

6 Nakayama algebras and graded trees

In this chapter, referring to [2], we show that each finite tree admits some representation finite gradings by looking at the Nakyama algebras related to the walks around the tree. We calculate and list the Nakayama representation finite gradings for the trees $\ddot{D}_5$ and D_6. We give the formula for the number of the Nakayama representation finite gradings of $\ddot{D}_n$ and D_n respectively.

6.1 Nakayama algebras and finite trees

In [5], Rohnes and Smalø showed that for each finite tree T, there is a representation finite grading g such that $T \cong G_\Lambda$ where Λ is an indecomposable Nakayama algebra.

In proposition 3.11, we have showed how to associate an indecomposable Nakayama algebra Λ to the quiver $1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow \mathrm{n}$ by given an admissible sequence. In addition, in proposition 3.13, we have seen the form of almost split sequence by the induced projective Kupisch series.

$$0 \to P_{i+1}/r^n P_{i+1} \to P_{i+1}/r^{n-1} P_{i+1} \oplus P_i/r^{n+1} P_i \to P_i/r^n P_i \to 0 \tag{3}$$

Obviously, the fundamental group of Λ is trivial. Since Λ is an artin algebra, the τ-orbit are finite. By the form of it's almost split sequences, Γ_Λ is representation finite. Thus Λ is a simply connected algebra.

In proposition 5.13 and proposition 5.14, we have seen that the bijection between the isomorphic classes of simply connected algebra and the isomorphic classes of representation finite graded tree are introduced by $(T', g') \cong (G_{A'}, g_{A'})$ where (T', g') is a representation finite graded tree and A' is a simply connected algebra. The grading $g_{A'}$ is defined as in observation 5.8. Since $g_{A'}$ is unique by construction, for each finite tree T, if we can find a simply connected algebra A such that $G_A \cong T$, then T has a representation finite grading.

6.1.1 Admissible sequences of a finite tree

Let $\{t_1, t_2 \dots t_n\}$ be the vertices of T. Let w be a walk from t_i to t_j passed through k edges in T. Then we define the length of w that $l(w) = k$. It is not hard to see that the shortest walk between two vertices is the walk that does not pass through any vertex twice. We use $L(t_i, t_j)$ to denote the length of the shortest walk from t_i to t_j.

In the this section, we will illustrate an admissible sequence in the reverse order of definition 3.5 such that $\{a_0, \dots, a_n\}$ is a admissible sequence if $a_n \geq a_0 - 1$ and $a_{i-1} \geq a_i - 1$.

We associate an admissible sequence S to T by the following steps.

1. Fixing an vertex x in T.

2. Finding the walk w from x to x which exactly passes trough each edge of T twice.

3. Ordering the vertices of T by the first time w passing through them.

4. Ordering the sequence of $L(x, -)$ in the order of step 3. This sequence is admissible by construction.

We use the following example to illustrate how to associate an admissible sequence to a tree.

Example 6.1.

$$\begin{tikzcd}
H \arrow[dr, dash] & & G \arrow[dl, dash] & \\
& E \arrow[dr, dash] & & F \arrow[dl, dash] \\
C \arrow[dr, dash] & & D \arrow[dl, dash] & \\
& B \arrow[d, dash] & & \\
& A & &
\end{tikzcd}$$

Fixing A, then $A-B-D-F-D-E-G-E-H-E-D-B-C-B-A$ is the walk which passes through each edge in the tree exactly twice. We order the vertices by the ordering of the walk passing through each vertex the first time. Then we have the sequence $\tilde{T} = \{A, B, D, F, E, G, H, C\}$. The associated admissible sequence of the tree is the correspond length of the shortest walk from A to the vertices in order of $\tilde{T}$ which is $\{0, 1, 2, 3, 3, 4, 4, 2\}$.

Observation 6.1. *Obviously, the associated admissible sequence for a tree is not unique. It could varies from the choice of the fixing vertex and also the choice of the walk.*

We are ready to prove that each finite trees admits at least one representation finite grading.

So for a finite tree T with the vertices $\{t_i, \ldots, t_n\}, n \in \mathbb{N}$, we associate an admissible sequence $S = \{s_1, \ldots, s_n\}$ to it. Then $K = \{k_1 = s_1 + 1, \ldots, k_n = s_n + 1\}$ becomes a Kupisch series. We construct the correspond Nakayama algebra Λ for K in terms of proposition 3.11. Obviously, the number of vertices in G_Λ is the same as the number of vertices in T. From observation 5.7, we have that G_Λ is a tree since Λ is simply connected. To prove $T \cong \Gamma_\Lambda$, it is enough to show the correspond vertices in Γ_Λ of two connected vertices in T are also connected.

Let $\{P_i, \ldots, P_j\}$ be the correspond projective Kupisch series of K. We also use $\{P_i, \ldots, P_j\}$ to denote the vertices of G_Λ since the vertices in G_Λ are one to one correspond to $\{P_i, \ldots, P_j\}$ by observation 5.8. Let t_i and t_j be two connected vertices in T. By construction, if we assume $k_j > k_i$, we know that $k_j - k_i = 1$ and

$k_m > k_i$ when $j > m > i$. Let P_i and P_j be the correspond vertices in G_Λ of t_i and t_j. Then $l(P_i) = k_i$ and $l(P_j) = k_j$. If P_i and P_j are connected, then there is an irreducible morphism from an element in DTr-orbit of P_i to P_j. By the equation 3, we have that the almost split sequence containing P_j is the following.

$$0 \to P_{j-1}/r^{k_i}P_{j-1} \to P_{j-1}/r^{k_i-1}P_{j-1} \oplus P_j \to P_j/r^{k_i}P_j \to 0$$

If $P_{j-1}/r^{k_i}P_{j-1} \cong DTr(P_j/r^{k_i}P_j) \cong P_i$, then P_i and P_j are connected.

If $P_{j-1}/r^{k_i}P_{j-1} \neq P_i$, we calculate $DTr^2(P_j/r^{k_i}P_j)$ in the same way. We repeat this process until that for $q \in \mathbb{N}$, $DTr^q(P_j/r^{k_i}P_j) = P_{j-q}/r^{k_i}P_{j-q}$ is projective. From observation 3.7, we know that $l(P_{j-q}/r^{k_i}P_{j-q}) = k_i$. Since $k_m > k_i$ when $j > m > i$, $P_{j-q}/r^{k_i}P_{j-q} = P_i/r^{k_i}P_i = P_i$. Then $DTr(P_j/r^{k_i}P_j)$ belongs to the DTr-orbit of P_i which implies there is an irreducible morphism from an element in the DTr-orbit of P_i to P_j. Thus P_i and P_j are connected.

Summarizing all results above, we have proved the following theorem which is the main result in [5].

Theorem 6.2. *If T is a finite tree, then there is a grading g such that (T, g) is representation-finite, and such that the corresponding simply connected algebra Λ is a Nakayama algebra.*

Observation 6.3. *Let $\{t_1, \ldots, t_n\}, n \in \mathbb{N}$ be the vertices of a finite tree T and $S = \{s_1, \ldots, s_n\}$ be the associated admissible sequence. The admissible sequence is a grading but not always is representation finite. For example, the following tree has grading (01111) which coincides one of the admissible sequence of the tree but the grading is not finite.*

$$\begin{array}{ccccc} & & 1 & & \\ & & | & & \\ 1 & — & 0 & — & 1 \\ & & | & & \\ & & 1 & & \end{array}$$

We will show how to find a Nakayama representation finite grading for D_6 in detail.

Example 6.2. *Let denote the vertices of D_6 in the following way.*

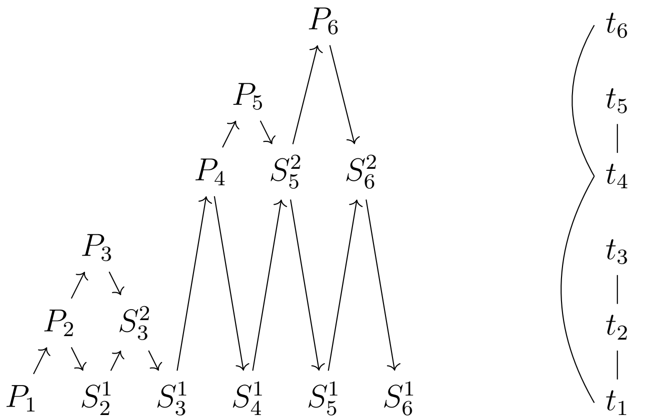
$$\begin{array}{ccccccccc} & & F & & & & & & \\ & & | & & & & & & \\ A & — & B & — & C & — & D & — & E \end{array}$$

The walk $C - D - E - D - C - B - F - B - A - B - C$ gives the admissible sequence $S = \{0, 1, 2, 1, 2, 2\}$. Let Λ be the correspond Nakayama algebra of S and

$\{P_1, P_2, P_3, P_4, P_5, P_6\}$ *be the correspond projective Kupisch series. Then we list all the almost split sequences of* Λ.

$$\begin{aligned}
P_5/r^2P_5 \to P_5/rP_5 \oplus P_6 \to P_6/r^2P_6\\
P_5/rP_5 \to P_6/r^2P_6 \to P_6/rP_6\\
P_4 \to P_4/rP_4 \oplus P_5 \to P_5/r^2P_5\\
P_4/rP_4 \to P_5/r^2P_5 \to P_5/rP_5\\
P_3/rP_3 \to P_4 \to P_4/rP_4\\
P_2 \to P_2/rP_2 \oplus P_3 \to P_3/r^2P_3\\
P_2/rP_2 \to P_3/r^2P_3 \to P_3/rP_3\\
P_1 \to P_2 \to P_2/rP_2
\end{aligned}$$

We use S_n^m *to denote* P_n/r^mP_n. *Then we have the Auslander Reiten quiver for* Λ *as following.*



The right side is G_Λ *which has grading* $\{0, 1, 2, 5, 6, 8\}$. *Thus* $G_\Lambda \cong D_6$ *and* $\{850126\}$ *in the same order of table 7 is a Nakayama representation finite grading of* T.

6.2 The Nakayama representation finite gradings of $\ddot{D}_n$ and D_n

A representation finite grading g of a finite tree T is said to be Nakayama representation finite if g is correspond to a Nakayama algebra in the way described above.

We have seen all the representation finite grading of $\ddot{D}_5$ and D_6. By analysing the walk of $\ddot{D}_5$ and D_6, we listed all Nakayama representation finite gradings for them.

Nakayama representation finite grading of $\ddot{D}_5$

We give the gradings in the order of

$$\begin{array}{ccccccc} & & 5 & & & & \\ & & | & & & & \\ 1 & — & 2 & — & 3 & — & 4 \\ & & | & & & & \\ & & 6 & & & & \end{array}$$

Remark 6.1. *The gradings in the form of $E---BC, E---CB, B---EC, B---CE, C---BE$ and $C---EB$ are considered as the same.*

Observation 6.4. *By looking at the number of different walks on $\ddot{D}_n$, the number of Nakayama representation finite gradings of $\ddot{D}_n$ is $2n$.*

By the formula above, the number of Nakayama representation finite gradings of $\ddot{D}_5$ is 10.

012368	016724	014528	307815	703419
105639	701259	630148	410926	721035

Table 6: Nakayama representation finite gradings for $\ddot{D}_5$

Nakayama representation finite grading of D_6
We give gradings in the order of

$$\begin{array}{ccccccccc} & & 6 & & & & & & \\ & & | & & & & & & \\ 1 & — & 2 & — & 3 & — & 4 & — & 5 \end{array}$$

Remark 6.2. *The gradings in the form of $B----E$ and $E----B$ are considered as the same.*

Observation 6.5. *By looking at the number of different walks on D_n, the number of Nakayama representation finite gradings of D_n is $2n-2$.*

By the formula above, the number of Nakayama representation finite gradings of D_6 is 10.

014562	012348	105673	103459	701239
210784	650128	543017	321095	432106

Table 7: Nakayama representation finite gradings for D_6

7 Conclusion

We have studied almost split sequences, the Auslander algebra and the representation finite graded trees. Specifically, we have calculated the representation finite gradings for $\ddot{D}_5$ and D_6. We also give the general formula for the number of the Nakayama representation finite gradings of $\ddot{D}_n$ and D_n.

Bongartz and Gabriel have given the general formula for the number of the representation finite gradings in [3] by looking at the binary tree on lexicographically form. An interesting topic for future work would be to try to find the general formula for the number of the representation finite gradings of D_n. Also of interest would be to look at other relevant topics such as covering spaces, tilting theory and homologically finite subcategories.

References

[1] Maurice Auslander. *A survey of existence theorems for almost split sequences*, page 81–90. London Mathematical Society Lecture Note Series. Cambridge University Press, 1986.

[2] Maurice Auslander, Idun Reiten, and Sverre O. Smalo. *Representation Theory of Artin Algebras*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1995.

[3] Klaus Bongartz and Peter Gabriel. Covering spaces in representation-theory. *Inventiones mathematicae*, 65(3):331–378, 1982.

[4] Karin Erdmann and Thorsten Holm. *Simple Modules and the Jordan–Hölder Theorem*, page 61–84. Springer International Publishing, Cham, 2018.

[5] Brit Rohnes and Sverre O Smalø. Nakayama algebras and graded trees. *Transactions of the American Mathematical Society*, 279(1):249–256, 1983.

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