Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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A detailed account of Behrens proof of Bott periodicity

Master's thesis in MLREAL
Supervisor: Gereon Quick
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Abstract

We give a detailed account of Behrens' proof of real and complex Bott periodicity theorem which is based on the ideas by McDuff. We include thorough computations and some intuition behind constructions and concepts. The proof is done by repeated constructions of certain quasifibrations with contractible total spaces in order to determine the iterated loop spaces of U and O.

Sammedrag

Vi gir en detaljert redegjørelse av Behrens' bevis for reell og kompleks Bott periodisitetsteorem som er basert på ideene til McDuff. Vi inkluderer fullstendige bergegninger og litt intuisjon bak konstruksjoner og konsepter. Beviset utføres gjennom repeterende konstruksjoner av visse kvasifibreringer med kontraktible totale rom for å bestemme de iterative løkkerommene til U og O.

1 Introduction

Bott periodicity theorem is a classic result in algebraic topology. This theorem played a key part in the development of K-theory, which is a generalized cohomology theory, that had a great impact on various fields of mathematics. There has been a lot of earlier proofs of this theorem, using a wide range of methods. Bott [6] used Morse theory in his original proof. Atiyah [3] used the index theorem and elliptic operators, and Atiyah, Bott and Shapiro [2] made use of Clifford modules, to name a few. The proof presented in this thesis is by Behrens [4], which is a simplification of the methods from the proof of complex Bott periodicity by Aguilar and Prieto [1], as well as an extension of the methods to prove real Bott periodicity. These proofs are based on the ideas of McDuff [13]. Behrens used quasifibration theory and linear algebra, along with some basic differential geometry, and constructed the iterated loopspace of O and U in order to prove the theorem. Compared to earlier proofs, this one is much simpler and more elementary, and therefore has the potential of being appreciated by a wider audience. This thesis will provide a detailed version of Behrens' proof, along with some extra inuition and explanations in hope of making it accessible to a broader audience. The thesis concludes with an analysis of the proof of the real case and complex case. In the analysis, a connection is drawn between the difference in complexity in the real and complex case and the additional constraint to the eigenvalues and eigenspaces of orthogonal matrices compared to unitary matrices.

The main goal of this thesis is to work towards making mathematics more available. This involves writing while being aware of how a reader would percieve the proof, and filling in additional explanations and intuition where a reader might need it. Having this focus is relevant for a mathematics didactical viewpoint, where one of the main points is how to commmunicate difficult mathematics in a comprehensive way.

2 Preliminary definitions and theorems

This section will consist of definitions of terms and some theorems that we will be using in the proof. These are divided into definitions from topology, definitions from linear algebra and Lie theory, some well known theorems stated for the readers convenience, and some theorems provided by Behrens' paper.

2.1 Topology

A **homotopy** between two maps is a continuous transformation from one map to the other. That is, two $f, g : X \to Y$ are homotopic if there is a continuous function $H : [0,1] \times X \to Y$ where $H(0,x) = f(x)$ and $H(1,x) = g(x)$. We denote $f \simeq g$ if they are homotopic.

A function $f : X \to Y$ is called a **homotopy equivalence** if there exists $g : Y \to X$ such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$. If this is the case, then we say X and Y are homotopy equivalent.

Given a pointed topological space (X, x_0), the **loop space**, denoted ΩX, is the space of all continuous pointed maps from the pointed circle (S^1, s_0) to X. That means all continuous maps $f : S^1 \to X$ such that $f(s_0) = x_0$. Intuitively, the loop space can be thought of as all closed loops in X that starts and ends at the base point x_0. The second iterated loop space is $\Omega(\Omega X)$, which we write as $\Omega^2 X$. We denote the n-th iterated loop space by $\Omega^n X$.

By modding out the homotopy equivalences in ΩX, we get the **fundamental group**, also known as the first homotopy group, denoted $\pi_1(X, x_0)$.

We denote by $\pi_n(X, x_0)$ the **higher order homotopy groups**, which is the collection of basepoint-preserving maps $f : S^n \to X$ modulo the homotopy equivalences. Note that we can equivalently define this using the map from I^n instead of S^n, with $f(\partial I^n) = x_0$. Where ∂I^n is the boundary of I^n

A **weak homotopy equivalence** between two spaces X and $Y$is a contionuous map $f : X \to Y$ that induces an isomorphism between the homotopy groups of all orders:

$$\pi_0(X) \cong \pi_0(Y)$$

$$\pi_n(X, x_0) \cong \pi_n(Y, y_0) \ \ \forall n.$$

The two spaces are in this case called weak homotopy equivalent.

The **relative homotopy groups** $\pi_n(X, A, x_0)$ where $x_0 \in A \subseteq X$, is the collection of all maps $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, where $J^{n-1} = \overline{\partial I^n - I^{n-1}}$, where $\overline{X}$ denotes the closure of X. Since ∂I^n is all the faces of I^n, and I^{n-1} is one of those faces, J^{n-1} is the union of all remaining faces.

We now define the notion of a **vector bundle** along with a chain of generalizations, each successor being a generalization of the previous ones.

In a vector bundle, we have a total space E, a base space B, a surjective continuous map called a projection map $p : E \to B$, as well as the fiber space F, where $F = p^{-1}(b)$ for a $b \in B$. This set $(E,\ B,\ p,\ F)$ is called a real vector bundle if $F = \mathbb{R}^k$ for $k \in \mathbb{N}$ for every $b \in B$, and for any point $b \in B$, there is a neigborhood U in B around b such that $p^{-1}(U) \cong U \times \mathbb{R}^k$. We call this last property the local trivialization property. A complex vector bundle and quaternionic vector bundle is defined the same way, with $\mathbb{R}$ replaced with $\mathbb{C}$ and $\mathbb{H}$ respectively. A basic, but important, example of a real vector bundle is called the trivial n-dimensional bundle, where $E = B \times \mathbb{R}^n$.

A generalization of a vector bundle is a **fiber bundle**. A fiber bundle is also the collection $(E,\ B,\ p,\ F)$, but F is instead a topological space. Such a collection is called a fiber bundle if every $b \in B$ has an open neighborhood U such that $p^{-1}(U) \cong U \times F$. In particular $p^{-1}(b) \cong F$ for all $b \in B$.

Instead of denoting this setup $(E,\ B,\ p,\ F)$, we will instead denote this as a **fiber sequence**:

$$F \to E \xrightarrow{p} B,$$

or simply

$$F \to E \to B.$$

A generalization of a fiber bundle is a **fibration**, also known as a Hurewicz fibration. Instead of having the local trivialization property, it instead has a property known as the **homotopy lifting property** with respect to any topological space X. This property says that a map ω exists in the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & E \\ {\scriptstyle \{0\}\times X}\downarrow & \overset{\omega}{\nearrow} & \downarrow{\scriptstyle p} \\ I \times X & \xrightarrow[\beta]{} & B \end{array}$$

In order to grasp the difference between fibrations and fiber bundles, we have that the fibers in a fibration need no longer be homeomorphic, but has to be homotopy equivalent to each other.

The next generalization is called a **Serre fibration**, which instead of having to satisfy the homotopy lifting property for all spaces X, only have to satisfy the homotopy lifting property for CW-complexes. This is equivalent to having to satisfy the homotopy lifting property for only all n-cubes I^n. That means that ω exists for all n in the following commutative diagram:

$$\begin{array}{ccc} I^n & \xrightarrow{\alpha} & E \\ {\scriptstyle \{0\}\times I^n}\downarrow & \overset{\omega}{\nearrow} & \downarrow{\scriptstyle p} \\ I^{n+1} & \xrightarrow[\beta]{} & B \end{array}$$

In a Serre fibration, the fibers don't have to be homotopy equivalent anymore.

The last generalization we will cover is called a **quasifibration**, and is in the heart of the proof. We will give two definitions, each of which will give different insights to the properties of a quasifibration. First of all, quasifibrations will not in general satisfy the homotopy lifting property for any space. Instead, we define a quasifibration as the collection $(E,\ B,\ p,\ p^{-1}(b))$ with one of the following equivalent properties, giving respectively definition 1 and 2:

1. Given $p : E \to B$, Then this is a quasifibration if the induced map $p_* : \pi_i(E, p^{-1}(b), x_0) \to \pi_i(B, b)$ is an isomorphism for all $b \in B$, $x_0 \in p^{-1}(b)$ and $i \geq 0$.

2. Given $p : E \to B$, and assume B path connected and that all fibers are CW-complexes. Then this is a quasifibration if all fibers $p^{-1}(b)$ are homotopy equivalent to the homotopy fiber of p over b.

Given $p : E \to B$, the **homotopy fiber** of a fixed point $b \in B$ is the collection of pairs (e, f), where $e \in E$, and $f : [0, 1] \to B$ is a path in B such that $f(0) = p(e)$ and $f(1) = b$. Therefore, the homotopy fiber consists of all fibers where the base of the fiber $p(e)$ in B is homotopic to b, and each path from $p(e)$ to b defines a distinct element in the homotopy fiber.

A common property between all the mentioned fibrations and bundles is that their fiber sequences induces a **long exact homotopy sequence**. If the fiber sequence is given by

$$F \to E \to B,$$

then the induced long exact homotopy sequence is given by

$$... \to \pi_{n+1}(B, b_0) \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to ...$$

We now present two important examples of fibrations. The first fibration exists for any pointed space (X, x_0), and is called the **path space fibration** of X. This is a fibration of the form

$$\Omega X \hookrightarrow PX \xrightarrow{p} X$$

Where ΩX is the loop space of X at x_0,

$$PX = \{\, f : I \to X \mid f \text{ continuous}, \ f(0) = x_0\}$$

is the path space of X, and $p(f) = f(1)$.

The second fibration is in fact a fiber bundle, and exists when the space is a topological group G. To get to the fiber bundle, we start by defining the **classifying space** BG. This is a pointed topological space such that the loop space of BG is homotopy equivalent to G, and the associated total space EGhas only trivial homotopy groups and makes the map $EG \to BG$ into a universal bundle over BG. The resulting fiber bundle is of this form:

$$G \to EG \to BG,$$

with $\Omega BG \simeq G$.

Let X be a compact Hausdorff topological space. The (complex) **K-theory** of X, denoted $K(X)$, is the set of all complex vector bundles over X under a certain equivalence relation. In fact, there are two different equivalence relations that is natural to consider, which yields K-theory and reduced K-theory, denoted $\widetilde{K}(X)$. First of all, in this definition, we use a broader definition of vector bundles than the one given earlier, which allows for two fibers to have different dimension if the base points are disconnected in the base space X. That way, the local trivialization property is still satisfied. Let ε^n be the trivial n-dimensional complex vector bundle. Define the equivalence relation $\approx$ between two vector bundles E_1 and E_2 as $E_1 \approx E_2$ if and only if $E_1 \oplus \varepsilon^n \cong E_2 \oplus \varepsilon^n$ for some n. Then

$$K(X) = \{\, E \mid p : E \to X \text{ is a complex vector bundle}\}/\approx .$$

Define another equivalence relation $\sim$ such that $E_1 \sim E_2$ if and only if $E_1 \oplus \varepsilon^n \cong E_1 \oplus \varepsilon^m$. Then

$$\widetilde{K}(X) = \{\, E \mid p : E \to X \text{ is a complex vector bundle}\}/\sim .$$

It can be shown that both $K(X)$ and $\widetilde{K}(X)$ form a ring with respect to the additive operation $\oplus$ and the multiplicative operation $\otimes$, and that $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$. One can similarly define real and quaternionic K-theory, also known as KO-theory and KSp-theory respectively, by considering real and quaternionic vector bundles rather than complex vector bundles.

2.2 Linear algebra and Lie theory

The basis of all constructions in the proof will be **inner product spaces**. An inner product space is a vector space V over a field F equipped with an inner product, that is, a map $\langle \cdot, \cdot \rangle : V \times V \to F$, that is conjugate symmetric, linear in the first term, and positive definite on nonzero vectors. In this proof, we will only consider the field to be $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

A linear map $\phi : V \to W$ between two inner procuct spaces V and W, is called a **linear isometry** if it preserves the inner product, i.e. $\langle v, w \rangle = \langle \phi(v), \phi(w) \rangle$. If ϕ in addition is bijective, then ϕ is an isomorphism between the two inner product spaces. When the inner product spaces are over $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$, then the set of all isomorphisms is denoted the orthogonal group, the unitary group, and the symplectic group, respectively. All of these sets form a group under matrix multiplication.

The **orthogonal group** of dimension n, denoted $O(n)$ or O_n, is the group consisting of all $n \times n$ orthogonal matrices, i.e. n-dimensional real matrices that satisfies $AA^T = A^T A = \mathrm{Id}$. When considering the orthogonal group of a space W, we write $O(W)$. Some properties of orthogonal matrices that are going to be of importance to us is that orthogonal matrices are normal, i.e. $AA^* = A^*A$ where * denotes conjugation transpose, and that the eigenvalues of orthogonal matrices are complex numbers with absolute value 1.

The **unitary group** of dimension n, denoted $U(n)$ or U_n, is the group of all $n \times n$ unitary matrices, i.e n-dimensional complex matrices that satisfy $AA^* = A^*A = \mathrm{Id}$, where * denotes complex conjugate transpose. When considering the unitary group of a space W, we write $U(W)$. Same as with orthogonal matrices, unitary matrices are normal and the eigenvalues has absolute value 1.

The **symplectic group** of dimension n, denoted $Sp(n)$ or Sp_n is the group of all symplectic matrices, which is in this paper taken to mean the $n \times n$ quaternionic matrices that satisfy $AA^* = A^*A = \mathrm{Id}$, where * denotes the quaternion conjugate transpose. When considering the symplectic group of a space W, we write $Sp(W)$. Symplectic matrices are normal and the eigenvalues are complex with absolute value 1.

When we have a normal matrix A that operates on an inner product space V, the **spectral teorem** tells us that we can write $A = U\Lambda U^*$, where U is a unitary matrix, and Λ is a diagonal matrix with its eigenvalues as entries. In particular, A has a decomposition called the **spectral decomposition** which is of the form $A = \sum_i \lambda_i \pi_{W_i}$, where π_{W_i} denotes the orthogonal projection onto the eigenspace W_i corresponding to the eigenvalue λ_i. It follows that $\bigoplus_i W_i = V$.

For the next series of constructions, we will use the concept of a **direct limit**. Let $A_0, A_1, A_2, ...$, be a family of spaces, and define maps $f_{ij} : A_i \to A_j$ for $i \leq j$ that satisfy $f_{ik} = f_{jk} \circ f_{ij}$ for all $k \geq j \geq i$, and $f_{ii} = \mathrm{Id}$ for all i. The collection of all A_i and f_{ij} is called a direct system. Given a direct system, define the direct limit as $\varinjlim A_i = \coprod_i A_i / \sim$, where $\coprod$ is the disjoint union. The equivalence relation $\sim$ is defined the following way. Let $x_i \in A_i$ and $x_j \in A_j$. Then $x_i \sim x_j$ if and only if there exists a k with $i \leq k$ and $j \leq k$ such that $f_{ik}(x_i) = f_{jk}(x_j)$. Note that a more complete definition of a direct limit uses indexing from an arbitrary index category, but we will not need this in the proof.

We now have the necessary tool for defining the **infinite orthogonal group** $O(\infty)$, or simply O. We define a direct system given by $A_i = O(i)$ and define the maps f_{ij} by sending $X \in O(i)$ to $X \oplus I_{j-i} \in O(j)$. Define $O = \varinjlim_i O(i)$. Similarly, the infinite unitary group U and the infinite symplectic group Sp are defined as $U = \varinjlim_i U(i)$ and $Sp = \varinjlim_i Sp(i)$

We have an explicit way of constructing the classifying space for the infinite orthogonal, unitary and symplectic group. This is a construction using the **Grassmannian manifold**, shortened to simply Grassmannian, of real, complex and quaternionic vector spaces respectively. Given a vector space V of dimension k, the Grassmannian $Gr_n(k)$ is the space of n-dimensional linear subspaces of V.

We define the **classifying space of the infinite unitary group** as the following construction. Let V be a complex vector space of dimension k. It is therefore isomorphic to $\mathbb{C}^k$. Let $BU_n(V) = \{Y \mid Y \subseteq V, \dim_{\mathbb{C}} Y = n\} = Gr_n(k)$. Let $BU_n = \coprod_k BU_n(\mathbb{C}^k)$. Now, we have a family of spaces $BU_0, BU_1, BU_2, ...$ For $i \leq j$, define maps $f_{ij} : BU_i \to BU_j$ given by $Y \mapsto Y \oplus \mathbb{C}^{j-i}$. The classifying space BU is defined as the direct limit $\varinjlim_i (BU_i)$ under these maps. The classifying space of the infinite orthogonal and symplectic group are defined the same way, but with respectively real and quaternionic vector spaces instead of complex vector spaces.

Given a vector space V over any field F equipped with a symplectic bilinear form, i.e a map $\omega : V \times V \to F$ that is linear in both arguments, $\omega(v, v) = 0$ for all $v \in V$, and $\omega(u, v) = 0$ for all $v \in V$ implies $u = 0$. Let W be a subspace of

V. Define

$$W^{\perp} = \{ v \in V \mid \omega(v, w) = 0 \; \forall w \in W \}.$$

W is called a **Lagrangian subspace** if $W = W^{\perp}$.

When the vector space V is over $\mathbb{R}$ and $\mathbb{C}$, the set of all lagrangian subspaces is a smooth manifold, and a subspace of the Grassmannian of V. Over $\mathbb{R}$, this is called the **(real) Lagrangian Grassmannian**. and over $\mathbb{C}$, this is called the **complex Lagrangian Grassmannian**.

The orthogonal, unitary and symplectic group are examples of **Lie groups**. A Lie group is a group where the group operation and inversion are smooth maps, which gives the group the additional structure of a differentiable manifold.

Associated to the Lie group is the **Lie algebra**, which is an algebra generated by the commutator $XY - YX$, for X, Y in the Lie group. The Lie algebra represents the tangent space to the Lie group at identity. The Lie groups $O(n)$, $U(n)$, and $Sp(n)$ have the associated Lie algebras $\mathfrak{o}(n)$, $\mathfrak{u}(n)$, and $\mathfrak{sp}(n)$, respectively, where $\mathfrak{o}(n)$ consists of all $n \times n$ skew-symmetric matrices, $\mathfrak{u}(n)$ consists for all $n \times n$ skew-hermitian matrices, and $\mathfrak{sp}(n)$ consists of all skew-quaternionic-hermitian matrices, i.e all matrices A such that $A^* = -A$, where * denotes quaternion conjugate transpose.

In the construction of the quasifibrations in the proof, the projection maps are going to be **matrix exponentials**. Given a matrix A, the matrix exponential, which we denote e^A, is defined to be $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$. The matrix exponential satisfies a number of properties which we will use in the proof, which is that the matrix exponentials commute with the transpose and conjugate transpose, for Y invertible, $Ye^AY^{-1} = e^{YAY^{-1}}$, and $e^X e^Y = e^{X+Y}$ when X and Y commutes. Another important property of the matrix exponential is that it is a map from the Lie algebra to the Lie group when the Lie group is a matrix group, which includes the Lie groups mentioned above.

A **geodesic** in a Lie group is the shortest path between two elements p_0 and p_1 of the group, and is given by walking from the first element in the direction of the second element. We can choose a parametrization of this geodesic such that the path is traversed for a unit time. The geodesic γ must therefore satisfy $\gamma(0) = p_0$, and $\gamma'(0) = v$ for v in the tangent space, i.e the Lie algebra corresponding to the Lie group. The geodesic is therefore on the form $\gamma(t) = p_0 e^{tv}$.

2.3 Some important theorems

We now state a three theorems that we are going to use in the proof.

The first one is known as the **Whitehead theorem**. This says that if we have two spaces X and Y that are homotopy equivalent to CW-complexes, and are weakly homotopy equivalent, then they are homotopy equivalent.

The second theorem is the **orbit-stabilizer theorem**, and says that given a set A, and a fixed element $a \in A$, if a group G acts on a, and S is the stabilizer of that action, i.e, the subgroup of all elements s such that $sa = a$, then $G \cdot a \cong G/S$. In particular, if G acts transitively on A, i.e. $G \cdot a = A$, then $A \cong G/S$. If A is a Lie group, then A is called a homogeneous space.

The third theorem is called the **Heine-Borel theorem**. This theorem says that every closed and bounded subspace of $\mathbb{R}^n$ is compact. With an appropriate definition of boundedness, we will in the proof prove that $O(n), U(n)$ and $Sp(n)$ are compact.

2.4 More on quasifibrations

The rest of section 2 will be following section 2 in Behrens' paper.

In this section, we shall state some necessary results about quasifibrations. First, given a quasifibration sequence:

$$F \to E \to B,$$

There exists a corresponding long exact sequence of homotopy groups:

$$... \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to ...$$

for $b_0 \in B$ and $x_0 \in F$. If we additionally assume E contractible, we have a map from the quasifibration sequence to the path space fibration.

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ \Omega B & \longrightarrow & PB & \longrightarrow & B \end{array}$$

This induces a map between the long exact homotopy sequences.

$$\begin{array}{ccccccccc} ... \to \pi_{n+1}(E, x_0) & \longrightarrow & \pi_{n+1}(B, b_0) & \longrightarrow & \pi_n(F, x_0) & \longrightarrow & \pi_n(E, x_0) & \longrightarrow & \pi_n(B, b_0) \to ... \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ ... \to \pi_{n+1}(PB) & \longrightarrow & \pi_{n+1}(B, b_0) & \longrightarrow & \pi_n(\Omega B) & \longrightarrow & \pi_n(PB) & \longrightarrow & \pi_n(B, b_0) \to ... \end{array}$$

Since both E and PY are contractible, they are homotopy equivalent, and therefore have isomorphic homotopy groups. From the five-lemma, F and ΩB are weak homotopy equivalent. From the Whitehead theorem, if F is a CW-complex, then $F \simeq \Omega B$. All fibers F used in the proof will be CW-complexes, although we will not prove this. We summarize this in a lemma:

Lemma 2.1. *If $F \to E \to B$ is a quasifibration sequence with E contractible, and F a CW-complex, then $F \simeq \Omega B$, where ΩB is the loop space of B.*

A big part of the proof of Bott periodicity theorem is verifying whether a sequence is a quasifibration. The following theorem will provide us with a recipe for doing just that. For a map $p : E \to B$, and for a subset $S \subseteq B$, if the map $p^{-1}(U) \to U$ is a quasifibration for every open $U \subseteq S$, we say that S is distinguished.

Theorem 2.2. *Suppose $p : E \to B$ is surjective, and that E is equipped with an increasing filtration $\{F_iB\}$ such that the following conditions hold:*
(1) $F_nB - F_{n-1}B$ is distinguished for every n.
(2) For every n there exists a neighborhood N_n of $F_{n-1}B$ in F_nB along with a deformation $h : N_n \times I \to N_n$ such that $h_0(N_n) = Id$ and $h_1(N_n) \subseteq F_{n-1}B$.
(3) This deformation is covered by a deformation $H : p^{-1}(N_n) \times I \to p^{-1}(N_n)$ with $H_0 = Id$, and for every $y \in N_n$, the induced map $H_1 : p^{-1}(y) \to p^{-1}(h_1(y))$ is a weak homotopy equivalence.
Then p is a quasifibration.

2.5 Linear isometries and classical groups

This section is about a result that gives a correspondence between linear isometries and maps between finite linear automorphisms.

Let Λ be $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and let W and V be (possibly countably infinite dimensional) inner product spaces over Λ. Define the topology of W and V to be unions of their finite dimensional subspaces. Let $\mathcal{I}(W, V)$ denote the space of linear isometries from W to V. Let $G(W)$ be either $O(W), U(W)$ or $Sp(W)$, i.e $G(W)$ is the space of finite type linear automorphisms of W. Define a continuous map:

$$\Gamma_{W,V} : \mathcal{I}(W, V) \to \mathrm{Map}(G(W), G(V)).$$

Where the elements in $\mathrm{Map}(G(W), G(V))$ are linear continuous maps. Write $\Gamma_{W,V}(\phi) = \phi_*$. Let $X \in G(W)$. Because of the finite type assumption of G, there exists a finite dimensional subspace $W_0 \subseteq W$ along with a transformation $X' \in G(W_0)$ such that

$$X = X' \oplus I_{W_0^{\perp}}$$

under the ortogonal decomposition $W = W_0 \oplus W_0^{\perp}$. We can find an orthogonal decomposition on V: $V = \phi(W_0) \oplus \phi(W_0)^{\perp}$. Let $\phi_*(X)$ be determined componentwise on the orthogonal decomposition. For the $\phi(W_0)$ component, we want the definition of $\phi_*(X)$ to imply that the following diagram commutes

$$\begin{array}{ccc} W_0 & \xrightarrow{\phi_{W_0}} & \phi(W_0) \\ {\scriptstyle X'}\downarrow & & \downarrow{\scriptstyle \phi_*(X)|_{\phi(W_0)}} \\ W_0 & \xrightarrow[\phi_{W_0}]{} & \phi(W_0) \end{array}$$

Since ϕ_{W_0} is an isometry, it is injective, and since every map is surjective onto its image, ϕ_{W_0} is bijective onto its image, and possesses therefore an inverse. Therefore, define $\phi_*(X)|_{\phi(W_0)} = \phi_{W_0} X' \phi_{W_0}^{-1}$. As for the $\phi(W_0)^{\perp}$ component, we have that $\phi_*(X)|_{\phi(W_0)^{\perp}}$ is independent of ϕ_{W_0}. That means we may freely choose what $\phi_*(X)|_{\phi(W_0)^{\perp}}$ should be, as long as it is an automorphism on $\phi(W_0)^{\perp}$. The natural choice is $I_{\phi(W_0)^{\perp}}$. Therefore, define $\phi_*(X) : V \to V$ to be

$$\phi_*(X) = \phi_{W_0} X' \phi_{W_0}^{-1} \oplus I_{\phi(W_0)^{\perp}}$$

This definition is seen to be independent of the choice of W_0. Let $\mathcal{U}$ and $\mathcal{V}$ be countably infinite dimensional inner product spaces over Λ. In [12, II.1] it is proven that $\mathcal{I}(\mathcal{U},\mathcal{V})$ is contractible. We therefore have the following lemmas:

Lemma 2.3. *Let $\phi, \phi' \in \mathcal{I}(\mathcal{U},\mathcal{V})$. Then the induced maps $\phi_*, \phi'_* : G(\mathcal{U}) \to G(\mathcal{V})$ are homotopic*

Proof. Since $\mathcal{I}(\mathcal{U},\mathcal{V})$ is contractible, ϕ and ϕ' are homotopic, since they both are null-homotopic. Therefore, ϕ_{W_0} is homotopic to ϕ'_{W_0} with homotopy induced from the homotopy between ϕ and ϕ'.
Let H be a homotopy such that $H(0) = \phi_{W_0}$ and $H(1) = \phi'_{W_0}$. Then we define a homotopy $H'(X,t) = H(t)XH(t)^{-1} \oplus I_{\phi(W)^\perp}$ where $H(t)H(t)^{-1} = I_{W_0}$ and $H(t)^{-1}H(t) = I_{\phi(W_0)}$ for all $t \in [0,1]$. Then we have $H'(X,0) = \phi_*(X)$ and $H'(X,1) = \phi'_*(X)$, which makes ϕ_* homotopic to ϕ'_*. □

Lemma 2.4. *Let $\phi \in \mathcal{I}(\mathcal{U},\mathcal{V})$. Then the induced map ϕ_* is a homotopy equivalence*

Proof. Consider $\mathcal{I}(\mathcal{U},\mathcal{U})$. This space is contractible, so any two maps in $\mathcal{I}(\mathcal{U},\mathcal{U})$ are homotopic. We know that $I_\mathcal{U} \in \mathcal{I}(\mathcal{U},\mathcal{U})$ and that $\phi\phi^{-1} \in \mathcal{I}(\mathcal{U},\mathcal{U})$. Therefore $I_\mathcal{U} \simeq \phi\phi^{-1}$. Likewise, by considering $\mathcal{I}(\mathcal{V},\mathcal{V})$, we get that $I_\mathcal{V} \simeq \phi^{-1}\phi$. From lemma 2.3, we get that there exists a $(\phi^{-1})_*$ such that $I_{G(\mathcal{U})} \simeq \phi_*(\phi^{-1})_*$ and $I_{G(\mathcal{V})} \simeq (\phi^{-1})_*\phi_*$. ϕ_* is therefore a homotopy equivalence. □

3 Bott periodicity theorem

Bott periodicity theorem was proved first by Raoul Bott in 1959. It is a central theorem in homotopy theory, and has contributed a lot to the development of K-theory and stable homotopy theory of spheres.

It is a well-known fact that homotopy groups are in general quite difficult to calculate. The usefulness of Bott periodicity theorem is evident in the fact that it simplifies the calculation for some important homotopy groups, for example the stable homotopy groups of spheres.

While the Bott periodicity theorem takes on a different form depending on which setting it is applied to, the theorem always exhibits a periodic structure. For example, a component of the stable homotopy groups of spheres varies periodically with period 8 when varying k in the expression $\pi_{n+k}(S^n)$. As for complex K-theory, if we define $K^n(X) = K(\Sigma^n X)$, where Σ^n denotes the n-times iterated reduced suspension, then Bott periodicity says that $K^n(X) \cong K^{n+2}(X)$. KO-theory and KSp-theory will exhibit the same pattern, but with a period of 8 instead of 2.

We are going to use the following form of Bott periodicity in our proof. This form considers the homotopy groups of the infinite orthogonal group and unitary group. Bott periodicity says the following:

$$\pi_n(U) \cong \pi_{n+2}(U)$$

and

$$\pi_n(O) \cong \pi_{n+8}(O).$$

Equivalently, since $\pi_{n+k}(X) \cong \pi_n(\Omega^k X)$, Bott periodicity states that $\Omega^2 U \simeq U$ and $\Omega^8 O \simeq O$.

4 Proof of complex Bott periodicity

To start with, the following fiber sequence exists:

$$U \to EU \to BU$$

From which, lemma 2.1 yields that $\Omega BU \simeq U$. To prove the two-periodicity, we want to show that $\pi_n(\Omega^2 BU) \cong \pi_n(BU)$. In practice, we are going to show that $\Omega^2 BU \simeq BU \times \mathbb{Z}$. This gives us what we want since the homotopy group of a product of spaces is isomorphic to the product of the homotopy group of each space, and since all homotopy groups of $\mathbb{Z}$ is trivial because $\mathbb{Z}$ is discrete. This means that proving the following theorem will be all we need to prove complex Bott periodicity.

Theorem 4.1. *Let U denote the infinite unitary group. The following quasifibration sequence exists*

$$BU \times \mathbb{Z} \to E \to U$$

where E is contractible. Consequently, $\Omega U \simeq BU \times \mathbb{Z}$

The strategy for proving this is to construct E and U from certain linear isometries and hermitian linear transformations, along with a suitable map $p : E \to U$ and prove that it is a quasifibration using theorem 2.2, with $BU \times \mathbb{Z}$ as the fiber.

Let $\mathcal{U} \cong C^\infty$ be a fixed infinite dimensional complex inner product space, and $W \subset \mathcal{U}$ a finite complex subspace. Define $U(W \oplus W)$ to be the complex linear isometries of $W \oplus W$.

For $V \subseteq W$, we may find a basis β for W where $\beta = (v_1, ..., v_n, w_1, ...w_m)$ such that $\alpha = (v_1, ..., v_n)$ is a basis for V. Denote by $W - V$, or $V^\perp$ if it is clear from the context what W is, the space determined by the basis $(w_1, ..., w_m)$. This is called the orthogonal complement of V in W. It is now easy to see that $W = V \times (W - V)$. Therefore we may write:

$$W \oplus W = (V \times (W - V)) \oplus (V \times (W - V)) = (V \oplus V) \times ((W - V) \oplus (W - V)),$$

Where the last equality can be found by a rearrangement of the basis for $W \oplus W$.

We now wish to construct

$$i_{V,W} : U(V \oplus V) \to U(W \oplus W)$$

that is the identity map when $V = W$. In other words, we want to find a unitary matrix $A \in U(W \oplus W)$ that is still a unitary matrix when restricted to the subspace $V \oplus V$. One can verify that a unitary matrix of the form

$$A = \left[\begin{array}{c|c} X & 0 \\ \hline 0 & C \end{array}\right]$$

where $X \in U(V \oplus V)$ and $C \in U((W-V) \oplus (W-V))$ will fulfill the requirement. A natural choice for C lets us define

$$i_{V,W}(X) = \left[\begin{array}{c|c} X & 0 \\ \hline 0 & I_{(W-V)\oplus(W-V)} \end{array}\right] = X \oplus I_{(W-V)\oplus(W-V)}$$

Where $I_{(W-V)\oplus(W-V)}$ is the identity map on $(W-V) \oplus (W-V)$. These maps gives us a direct system, which means that we can take the direct limit, which is the set of equivalence classes of the disjoint union of $U(W \oplus W)$ over all W, where two elements X, Y belong to the same equivalence class if $i_{v,w}(X) = Y$ or $i_{v,w}(Y) = X$. This gives us the infinite unitary group, i.e

$$U = \varinjlim{}_W U(W \oplus W).$$

To see this, let us compare this direct limit with the canonical expression for U, which is $U = \varinjlim{}_{W'} U(W')$, with the map $i'_{V',W'} : U(V') \to U(W')$ given by $i'_{V',W'}(X) = X \oplus I_{W'-V'}$. We will show that these two direct limits are isomorphic by showing mutual inclusion. We first show that $\varinjlim{}_W U(W \oplus W) \subseteq \varinjlim{}_{W'} U(W')$. Since two vector spaces are isomorphic if they have the same dimension, $W \oplus W \cong W'$ when $\dim W' = 2 \dim W$. If $W' \cong W \oplus W$, then certainly, $U(W') \cong U(W \oplus W)$. So given $U(W \oplus W)$, we can find a W' such that $U(W \oplus W) \cong U(W')$. It follows that $\varinjlim{}_W U(W \oplus W) \subseteq \varinjlim{}_{W'} U(W')$.

We now show that $\varinjlim{}_{W'} U(W') \subseteq \varinjlim{}_W U(W \oplus W)$. Let $X' \in U(W')$. Then X is either even-dimensional or odd-dimensional. If X' is even, then by the previous argument, since we can find a W such that $U(W') \cong U(W \oplus W)$, we can find an $X \in U(W \oplus W)$ such that $X \cong X'$. If X' is odd, then $X' \oplus I$ is even. In the direct limit, these respresent the same element, so we can take $X' \oplus I$ to be the representative. But since $X' \oplus I$ is even, we can repeat the same argument to find an X in $U(W \oplus W)$ for some W such that $X' \oplus I \cong X$. Therefore $\varinjlim{}_{W'} U(W') \subseteq \varinjlim{}_W U(W \oplus W)$. Since we have mutual inclusions, they are equal.

Let $H(W \oplus W)$ denote the hermitian linear transformations of $W \oplus W$, that is, all matrices A such that $A^H = A$ where A^H denotes the complex conjugate transpose of A, and let

$$E(W) = \{\, A \subseteq H(W \oplus W) \mid \mu_i \in [0,1]\ \forall i\},$$

where μ denotes the eigenvalues of the matrix. Define

$$p_W : E(W) \to U(W \oplus W)$$

with $p_W(A) = e^{2\pi i A}$. This map makes sense because since A is hermitian, and therefore normal, it has a diagonalization $A = U\Lambda_{\mu_i}U^T$, where U is unitary, and Λ_{μ_i} is the diagonal matrix with the eigenvalues μ_i of A as entries. Therefore $e^{2\pi i A} = e^{U\cdot(2\pi i\Lambda_{\mu_i})U^T} = U \cdot e^{2\pi i\Lambda_{\mu_i}}U^T = U\Lambda_{\lambda_i}U^T$, where Λ_{λ_i} is the diagonal matrix with $\lambda_i = e^{2\pi i\mu_i}$ as entries. Since all hermitian matrices have real eigenvalues, $|\lambda_i| = 1$. Then $U\Lambda_{\lambda_i}U^T$ equals a complex normal matrix with eigenvalues that all has length 1. This is precisely the unitary matrices.

Same as with U, we define a map $E(V) \to E(W), V \subseteq W$ by sending A to $A \oplus \pi_{(W-V)\oplus 0}$, where π_Y is the orthogonal projection onto the subspace Y of W. Let us take a look at what this means. First of all, we may decompose $W \oplus W$ into $(V\oplus V)\oplus((W-V)\oplus 0)\oplus(0\oplus(W-V))$, where these subspaces are orthogonal to each other. Due to the spectral theorem for hermitian matrices, given an orthogonal decomposition of $W \oplus W$ and an eigenvalue assigned to each component, there is a unique hermitian matrix with each component being the eigenspace of the matrix corresponding to the assigned eigenvalue. Since A have given us such a decomposition of $V \oplus V$, all that remains is to choose eigenvalues correponding to $(W - V) \oplus 0$ and $0 \oplus (W - V)$, in which we will choose eigenvalue 1 and 0 respectively. The resulting hermitian matrix is then $A \oplus \pi_{(W-V)\oplus 0}$. It follows that the square

$$\begin{array}{ccc} E(V) & \longrightarrow & E(W) \\ \downarrow{\scriptstyle p_V} & & \downarrow{\scriptstyle p_W} \\ U(V\oplus V) & \longrightarrow & U(W\oplus W) \end{array}$$

commutes. We define $E = \varinjlim_W E(W)$. Thus we get an induced map on the direct limits:

$$p : E \to U$$

We wish to show that this map is a quasifibration, and that the fibers are $BU \times \mathbb{Z}$.

First we construct BU as the direct limit of complex Grassmannian n-planes. Define

$$BU_n(Y) = \{\, V \mid V \subseteq Y,\ \dim_{\mathbb{C}} V = n\}$$

for any $Y \subset \mathcal{U} \oplus \mathcal{U}$.

Let W_n be a subspace of $\mathcal{U}$ with dimension n, and let $W_n \subseteq W_{n+k}$ for $k \geq 0$. We choose a map $\phi^k_{m,n} : BU_m(W_n \oplus W_n) \to BU_{m+k}(W_{n+k} \oplus W_{n+k})$, given by sending V to $V \oplus (W_{n+k} - W_n) \oplus 0$ where 0 is taken to be the zero matrix of dimension k. Notice that since $W_{n+k} - W_n \cong \mathbb{C}^k$, we may say that $\phi^k_{m,n}$ sends V to $\mathbb{C}^k \oplus V \oplus 0$, where we have $\mathbb{C}^k$ on the left in order to apply a convenient illustration of $BU_n(W \oplus W)$ in the following proof.

Let $BU(Y) = \amalg_n BU_n(Y)$, where $\amalg$ denotes the disjoint union. Consider the following expression: $\varinjlim_W BU(W \oplus W)$ under the map $\phi^k_{m,n}$. We wish to show that this is isomorphic to $BU \times \mathbb{Z}$. In fact, for future references, by changing W from a complex space to a real space and a symplectic space, the following proof will also prove that $\lim_{\rightarrow W} BO(W \oplus W) \cong BO \times \mathbb{Z}$ and $\lim_{\rightarrow W} BSp(W \oplus W) \cong BSp \times \mathbb{Z}$ respectively. For convenience sake, we state the statement as a lemma

Lemma 4.2. *Let $W \subset \mathcal{U}$ be a complex space. Then $\lim_{\rightarrow W} BU(W \oplus W) \cong BU \times \mathbb{Z}$*

Proof. First we note that since $BU(Y) = \amalg_n BU_n(Y)$, all cosets in $BU(Y)$ are trivial, so each coset is therefore equal to the element it consists of. Therefore, the cosets of $\lim_{\rightarrow W} BU(W \oplus W)$ is only determined by the maps $\phi^k_{m,n}$, and two elements are in the same coset if and only if $n - m$ for both element are equal. This means that we can sort $\phi^k_{m,n}$ into families of maps: $\phi^k_{n-i,n}$ for each $i \in \mathbb{Z}$, and each of these families produce a direct system. We show that the direct limit for each of these direct systems is isomorphic to BU.

Recall the construction of BU. We have a map $\psi^k_n : BU_n \to BU_{n+k}$, $V' \mapsto V' \oplus \mathbb{C}^k$. Under this map, we define $BU = \lim_{\rightarrow n} BU_n$.

Let $\lim_{\rightarrow W} BU(W \oplus W)^i$ denote the direct limit of $BU(W \oplus W)$ under the map $\phi^k_{n-i,n}$ We are now going to give a labeled basis for $W \oplus W$. The idea is for the first W to have negative indexed basis vectors, while the second W has positive and zero indexed basis vectors. If W has dimension n, then a basis for $W \oplus W$ look like this: $\{b_{-n}, b_{-(n-1)}, ..., b_{-2}, b_{-1}, b_0, b_1, b_2, ..., b_{n-1}\}$. A basis for W when considering BU will be indexed with zero and positive integers.

Let us define a relabeling function.

$$N_t : W \oplus W \to W \oplus W$$

$$\{b_{-n}, b_{-(n-1)}, ..., b_0, ..., b_{n-1}\} \mapsto \{b_{-n+t}, b_{-(n-1)+t}, ..., b_{0+t}, ..., b_{(n-1)+t}\}.$$

Note that we define the function to simply be a relabeling of indices, and is therefore nothing but the identity map on the space itself. N_t is an isomorphism, as it has N_{-t} as an inverse. From this map, we can find an orthogonal decomposition of $W \oplus W$, given by $W_t^\perp \oplus W_t$, where $\{b_{-n+t}, b_{-(n-1)+t}, ..., b_{-1}\}$ is a basis for $W_t^\perp$, and $\{b_0, b_1, ..., b_{(n-1)+t}\}$ is a basis for W_t. If $-n + t \geq 0$, then $W_t^\perp$ is empty, and if $n - 1 + t < 0$, then $W_t^\perp$ is empty.

We are now going to define a map from $\lim_{\rightarrow W} BU(W \oplus W)^i$ to BU where the image of $V \in \lim_{\rightarrow W} BU(W \oplus W)^i$ is determined by the following process. First, we assume there is no V' such that $\phi^k_{n-i,n}(V') = V$. Let $\dim_C(V) = m$. We then know that $W \oplus W \supseteq V$ has dimension $2m + 2i = 2n$. Let V be the span of $\{b_{-n}, b_{-(n-1)}, ..., b_0, ..., b_{n-1}\}$, where each of the vectors are either the basis vector in $W \oplus W$ of the same index or the zero-vector. Let $-t$ be the index of the first non-zero basis vector in V. Then, we get a map $V \to V \cap W_t$ which is a projection. $V \cap W_t$, being a subspace of W_t, is spanned by $\{b_{-t}, b_{1-t}, ..., b_{((m+i)-1)}\}$ where each of the vectors are either the basis vector in W_t with the same index, or the zero-vector. Note that in this construction we have removed $n - t$ zero vectors from W on the left. We look at the relabeled space $N_t(V \cap W_t)$ which has basis $\{b_0, b_1, ..., b_{(m+i+t)-1}\}$. Therefore W_t is an $(m+i+t)$ dimensional complex space, which means $W_t \subset \mathcal{U}$. We therefore have $V \cap W_t \in BU$.

We now construct an inverse to this map.

Let $V' \in BU$. Assume there is no V'' such that $\psi_n^k(V'') = V'$, and assume the span of V' has a non-zero vector in the zeroeth spot. If it has not, we may add a non-zero vector on the left, and apply N_1, since the resulting subspace is in the same coset as V'. Let $\dim V' = m$, and let $W' \supseteq V'$ be the smallest space that has V' as subspace. Let $\dim W' = n$. The number of zero vectors in the span of V' is therefore $n - m$.

Compute $s = 2i - (n - m)$. Add s lots of zero-vectors on the left in the span of V', and apply N_s. If s is negative, add $|s|$ lots of non-zero basis vectors on the right. Let the resulting space be called V, and let the smallest space that contains V be called $W \oplus W$. Now, the dimension of $W \oplus W$ is $m+s$, and the dimension of V is either n or $n+s$. Find t such that $\dim(W \oplus W) = \dim(V) + i + t$. Apply N_{-t} to V. We now have $V \in \varinjlim_W BU(W \oplus W)^i$.

Running through the steps, one can verify that these maps are indeed mutually inverse to each other, and that two representatives of the same coset gets mapped to the same coset. This means that $\varinjlim_W BU(W \oplus W)^i \cong BU \forall i \in \mathbb{Z}$. Taking the disjoint union, we get $\varinjlim_W BU(W \oplus W) \cong \coprod_i \varinjlim_W BU(W \oplus W)^i \cong \coprod_i BU \cong BU \times \mathbb{Z}$ □

We will now prove the following lemma, which tells us about the structure of the fiber of the map p_W.

Lemma 4.3. *Let $X \in U(W \oplus W)$. Then $p_W^{-1}(X) \cong BU(\ker(X - I))$*

Proof. Define $\phi : p_W^{-1}(X) \to BU(\ker(X - I))$ by sending A to $\ker(A - I)$. First we need to make sure the map makes sense. We need to make sure that $\ker(A-I) \in BU(\ker(X-I))$. In other words, we have to check that $\ker(A-I) \subseteq \ker(X - I)$.

Suppose $v \in \ker(A - I)$. That means $Av = v$. Remember that A and X are related by $X = e^{2\pi i A}$. Then

$$Xv = e^{2\pi i A}v = \sum_n \frac{(2\pi i)^n}{n!} A^n v = e^{2\pi i} v = v$$

so $v \in \ker(X - I)$.

Since X is unitary, it has a spectral decomposition, which we assume is

$$X = \pi_V + \sum_i \lambda_i \pi_{V_i}$$

Where V_i denote the eigenspaces corresponding to the eigenvalue λ_i, and V corresponds to $\lambda = 1$, which means $V = \ker(A - I)$. We also have $\lambda_i \neq \lambda_j$ when $i \neq j$, and $\lambda_i \neq 1$.

Since X is unitary, $|\lambda_i| = 1$, and $W \oplus W = V \oplus \bigoplus_i V_i$. Suppose $A \in p_W^{-1}(X)$. Since A is hermitian, it also has a spectral decomposition:

$$A = \pi_{V'} + 0 \cdot \pi_{V''} + \sum_i \mu_i \pi_{W_i},$$

where V'' is the eigenspace corresponding to the eigenvalue $\mu = 0$. We now have $W \oplus W = V' \oplus V'' \oplus \bigoplus_i W_i$. But by the relation $X = e^{2\pi i A}$, we get another spectral decomposition for X, namely

$$X = e^{2\pi i A} = \pi_{V' \oplus V''} + \sum_i e^{2\pi i \mu_i} \pi_{W_i},$$

where we have used that $\pi_Y \circ \pi_Y = \pi_Y$ and that $\pi_{V'} + \pi_{V''} = \pi_{V' \oplus V''}$. However, the spectral decomposition is unique, so we get $V' \oplus V'' = V$, $V_i = W_i$ and $\lambda_i = e^{2\pi i \mu_i}$. Since $\mu_i \in (0, 1)$, μ_i is completely determined by the non-unital eigenvalues λ_i of X. In particular, since X and its spectral decomposition is assumed known, and A is completely determined by X and V', we get that $\phi(A) = V'$ has an inverse $\psi : BU(V) \to p_W^{-1}(X)$ given by

$$\psi(V') = \pi_{V'} + \sum_i \mu_i \pi_{V_i}.$$

We verify this:

$$\phi \circ \psi(V') = \phi(\pi_{V'} + \sum_i \mu_i \pi_{V_i}) = \ker((\pi_{V'} + \sum_i \mu_i \pi_{V_i}) - I) = V'$$

$$\psi \circ \phi(A) = \psi(\ker(A - I)) = \pi_{\ker(A-I)} + \sum_i \mu_i \pi_{V_i} = A$$

□

We will now proceed to prove that $p : E \to U$ is a quasifibration. We will be using theorem 2.2 to prove it, so we need an expression for $p^{-1}(X)$.

Lemma 4.4. *$p^{-1}(X) \cong \varinjlim_{W' \geq W} BU(\ker(X - I) \oplus (W' - W) \oplus (W' - W))$, and $\varinjlim_{W' \geq W} BU(V \oplus (W' - W) \oplus (W' - W)) \cong BU \times \mathbb{Z} \quad \forall V \subseteq W$.*

Proof. Define

$$\overline{BU}_{V,W} = \varinjlim_{W' \geq W} BU(V \oplus (W' - W) \oplus (W' - W))$$

for W finite dimensional and $V \subseteq W \oplus W$. First of all, if $W^\perp$ is the orthogonal complement of W in $\mathcal{U}$, then we have that $V \oplus W^\perp \oplus W^\perp \cong \mathcal{U} \oplus \mathcal{U}$, by a suitable isometry. For example, assume we have the following basis $\{v_1, v_2, ..., v_n\}$ for V, $\{v_1, v_2, ..., v_n, w_{n+1}, ...W_{2m}\}$ for $W \oplus W$, and $\{w_k, w_{k+1}, ...\}$ for $W^\perp \oplus W^\perp$. Note that $\{w_{n+1}, ..., w_{2m}\}$ is a basis for $(W \oplus W) - V$. We can map the basis vectors in $V \oplus W^\perp \oplus W^\perp$ to the basis vectors in $W \oplus W \oplus W^\perp \oplus W^\perp = \mathcal{U} \oplus \mathcal{U}$ by

$v_n \mapsto v_n$, $w_{k+i} \mapsto w_{n+i+1}$ for $0 \leq i \leq (2m-(n+1))$ and $w_{k+2m-n+j} \mapsto w_{k+j}$ for all $j \in \mathbb{N}$. This reveals a one-to-one correspondence between basis vectors, so $V \oplus W^{\perp} \oplus W^{\perp} \cong \mathcal{U} \oplus \mathcal{U}$.

Let us compare $\varinjlim_{W' \geq W} BU(V \oplus (W'-W) \oplus (W'-W))$ and $\varinjlim_{W''} BU(W'' \oplus W'') \cong BU \times \mathbb{Z}$. Certainly, for any choice of W'', we can find W' such that $W'' \oplus W'' \subseteq V \oplus (W'-W) \oplus (W'-W)$, for example $W' = W'' \oplus W$. Therefore $\varinjlim_{W' \geq W} BU(V \oplus (W'-W) \oplus (W'-W)) \subseteq \varinjlim_{W''} BU(W'' \oplus W'')$. Likwise, for any choice of W', we can find W'' such that $V \oplus (W'-W) \oplus (W'-W) \subseteq W'' \oplus W''$, for example $W'' = V \oplus (W'-W)$. Therefore $\varinjlim_{W''} BU(W'' \oplus W'') \subseteq \varinjlim_{W' \geq W} BU(V \oplus (W'-W) \oplus (W'-W))$. Therefore, we have that $\overline{BU}_{V,W} \cong \varinjlim_{W''} BU(W'' \oplus W'') \cong BU \times \mathbb{Z}$. Since we have map $U(W \oplus W) \to U(W' \oplus W')$ for $W \subseteq W' \subset \mathcal{U}$, we may also consider $p_{W'}^{-1}(X)$. From lemma 4.3, we may write $p_{W'}^{-1}(X) = BU(\ker(X-I) \oplus (W'-W) \oplus (W'-W))$. Along with the induced maps $p_W^{-1}(X) \to p_{W'}^{-1}(X)$, we may take the direct limit, which gives us $p^{-1}(X) = \varinjlim_{W' \geq W} BU(\ker(X-I) \oplus (W'-W) \oplus (W'-W)) = \overline{BU}_{\ker(X-I),W} \cong BU \times \mathbb{Z}$ □

We will now find a suitable filtration of U. From the spectral theorem of X, we have that $W \oplus W = \ker(X-I) \oplus \bigoplus_i V_i$, and by orthogonal decomposition, $\bigoplus_i V_i = \ker(X-I)^{\perp}$. Using this calculation as inspiration, define the filtration:

$$F_n U = \{\, X \mid \dim_{\mathbb{C}}(\ker(X-I)^{\perp}) \leq n\} \subseteq U$$

Let $B_n := F_n U - F_{n-1} U$. We start by proving that B_n is distinguished.

We have that every Serre fibration is a quasifibration. By using definition 2 of quasifibration, one can easily see that given an open subset U of B, if $p : E \to B$ is a quasifibration, so is $p : p^{-1}(U) \to U$, since the homotopy fiber on each element in U is a subspace of the homtopy fiber on the same element in B. Therefore, the following lemma proves that B_n is distinguished.

Lemma 4.5. *$p^{-1}(B_n) \to B_n$ is a Serre fibration.*

Proof. We start with the following commutative square

$$\begin{array}{ccc} \{0\} \times I^k & \xrightarrow{\alpha} & p^{-1}(B_n) \\ \downarrow & & \downarrow \\ I^{k+1} & \xrightarrow[\beta]{} & B_n \end{array}$$

We wish to find a lift of this diagram, i.e. a map $\gamma : I^{k+1} \to p^{-1}(B_n)$ making the triangles in the diagram commute.

Since all the unit k-cubes are compact, their image must also be compact. However, neither B_n nor $p^{-1}(B_n)$ are compact. This is because we can define a cover of B_n as $\{\, C_k \mid k \in \mathbb{N}\}$ where $C_k = \{X \in B_n \mid \dim_{\mathbb{C}} X = k\}$. This cover does not have a finite subcover. $p^{-1}(B_n)$ is not compact because had it been, its image under p would have been compact, which we showed is not. However, we have that $E(W)$ and $U(W \oplus W)$ is compact for W finite dimensional. They are closed since they are the preimage of the closed spaces S^1 and I respectively under the determinant map. They are bounded because $||Aw|| \leq ||w||$ for all $A \in E(W)$ and $w \in W \oplus W$, and $||Xw|| \leq ||w||$ for all $X \in U(W \oplus W)$ and $w \in W \oplus W$, which is in accordance with the defintion of boundedness given in [10]. By the Heine-Borel theorem, they are therefore both compact.

This means there exist a finite dimensional $W \subset \mathcal{U}$ such that the diagram factors as

$$\begin{array}{ccccc} \{0\} \times I^k & \xrightarrow{\alpha'} & E(W) \cap p^{-1}(B_n) & \longrightarrow & p^{-1}(B_n) \\ \downarrow & & \downarrow & & \downarrow \\ I^{k+1} & \xrightarrow[\beta']{} & U(W \oplus W) \cap B_n & \longrightarrow & B_n \end{array}$$

Let $A(0, t_1, ..., t_k) = \alpha'(t_1, ..., t_k)$ and $X(t_0, t_1, ..., t_k) = \beta'(t_0, ..., t_k)$. For $t \in I^k, I^{k+1}$ respectively, we may write the spectral decomposition of A and X as

$$A(t) = \pi_{V'(t)} + \sum_l \mu_l(t) \pi_{W_l(t)},$$

$$X(t) = \pi_{V(t)} + \sum_l \lambda_l(t) \pi_{V_l(t)},$$

where $e^{2\pi i \mu_l(t)} = \lambda_l(t)$, $V'(t) \subseteq V(t)$, and $W_l(t) = V_l(t)$ when $t \in I^k$. Consider the following space for an n-dimensional complex subspace W of $\mathcal{U}$:

$$\text{Perp}_{i,j}(W \oplus W) = \{\, (V', V'') \mid V', V'' \subseteq W \oplus W,\ V' \perp V'',$$
$$\dim_{\mathbb{C}} V' = i,\ \dim_{\mathbb{C}} V'' = j\}$$

We may characterize this space by considering the unitary group over $W \oplus W$. We get all possible V' and V'' by applying all unitary transformations on one pair of (V', V''). In other words, $U(W \oplus W)$ acts transitively on (V', V''). We wish to identify all the unique pairs (V', V''), which we will do by finding out which transformations in $U(W \oplus W)$ induces automorphisms on V' and V'', and therefore also on $W \oplus W - (V' \oplus V'')$, simultaneously. Note that the automorphisms of V' are exactly the elements in $U(V')$, which we will denote U_i since we know the dimension of V' to be i. Let

$$W \oplus W = V' \times V'' \times ((W \oplus W) - (V' \times V'')).$$

Let $T \in U(W \oplus W)$ be of the form $T = A \oplus B \oplus C$, where $A \in U_i, B \in U_j, C \in U_{2n-(i+j)}$. We then get

$$T(W \oplus W) = A(V') \times B(V'') \times C((W \oplus W) - (V' \times V''))$$
$$\cong V' \times V'' \times (W \oplus W - (V' \times V''))$$

Thus $U_i \oplus U_j \oplus U_{2n-(i+j)} \cong U_i \times U_j \times U_{2n-(i+j)}$ is the stabilizer of the transitive action of U_{2n} on $\mathrm{Perp}_{i,j}$ by left multiplication, which by the orbit-stabilizer theorem, gives

$$\mathrm{Perp}_{i,j}(W \oplus W) \cong U_{2n}/(U_i \times U_j \times U_{2n-(i+j)}).$$

We have a natural map

$$P : \mathrm{Perp}_{i,j} \to BU_{i+j}(W \oplus W)$$

by $P(V', V'') = V' \oplus V''$. By the same procedure, since U_{2n} acts transitively on $BU_{i+j}(W \oplus W)$ and the stabilizer consists of transformations that induces automorphisms on $V' \oplus V''$ and $W \oplus W - (V' \oplus V'')$ simultaneously, we get that $BU_{i+j} \cong U_{2n}/(U_{i+j} \times U_{2n-(i+j)})$. We observe that $\mathrm{Perp}_{i+j,0} \cong BU_{i+j}$, so $BU_{i+j} \subseteq \mathrm{Perp}_{i,j}$, and that $P_{|\mathrm{Perp}_{i+j,0}} \cong I$, we therefore have that P is a projection, which makes it a fibration. That means we can find a lift ω'' to the following commutative diagram.

$$\begin{array}{ccc} \{0\} \times I^k & \xrightarrow{\alpha''} & \mathrm{Perp}_{i,j}(W \oplus W) \\ \downarrow & \overset{\omega''}{\nearrow} & \downarrow P \\ I^{k+1} & \xrightarrow[\beta'']{} & BU_{i+j}(W \oplus W) \end{array}$$

Let $i = \dim V'(0)$ and $j = \dim(V(0) - V'(0))$, i.e the dimension of the eigenspaces of A(0) corresponding to $\mu = 1$ and $\mu = 0$ respectively. Let $\alpha'' : I^k \to \mathrm{Perp}_{i,j}(W \oplus W)$ be given by $\alpha''(t) = (V'(t), V(t) - V'(t))$ and let $\beta : I^{k+1} \to BU_{i+j}(W \oplus W)$ be given by $\beta''(t) = V(t)$. Since $V(t) \in BU_{i+j}\ \forall t$, we have that $V(t)$ has constant dimension for all t. Therefore, we may define $\omega''(t) = (W'(t), V(t) - W'(t))$, where $W'(t)$ is obtained from $V'(t)$ by a homotopy. Let $\mu_l(t) \in (0, 1)$ be the unique solution to $e^{2\pi i \mu_l(t)} = \lambda_l(t)$. We can now define $\omega' : I^{k+1} \to E(W) \cap p^{-1}(B_n)$ by

$$\omega'(t) = \pi_{W'(t)} + \sum_l \mu_l(t) \pi_{V_l(t)}$$

and by inclusion, we obtain a lift ω to our original diagram. □

We now aim to prove (2) and (3) of theorem 2.2.

Define a neighborhood N_n of F_{n-1} in F_n as

$$N_n = \{\, X \in F_n U \mid \dim_{\mathbb{C}} \mathrm{Eig}_{e^{2\pi i [1/3, 2/3]}} X < n\} \subseteq F_n U$$

Where $\mathrm{Eig}_S X$ denote the direct sum of eigenspaces corresponding to eigenvalues in S.

Certainly, any matrix in $F_{n-1}U$ is also in N_n. This is because the eigenspace corresponding to all eigenvalues of $X \in F_{n-1}U$ of the form $e^{2\pi i a}$ where $a \in [1/3, 2/3]$ has dimension less than n, since the entire eigenspace corresponding to all non-unital eigenvalues has dimension less than n. In addition to N_n containing $F_{n-1}U$, it contains some matrices in which the sum of the eigenspaces corresponding to the non-unit eigenvalues have dimension n. In some sense, these matrices have "too many" non-unit eigenvalues.

We are going to deform the matrices in N_n such that the eigenvalues inside the range $e^{2\pi i[1/3,2/3]}$ will correspond to all non-unit eigenvalues of the deformed matrix, while the remaining eigenvalues will correspond to eigenvalue 1. That way, every deformed matrix will be in F_{n-1} as theorem 2.2 require. Define $f : I \to I$ by

$$f(x) = \begin{cases} 1, & x \geq \frac{2}{3} \\ 3x - 1, & \frac{1}{3} \leq x \leq \frac{2}{3} \\ 0, & x \leq \frac{1}{3} \end{cases}$$

We note that $f \simeq \mathrm{Id}$ rel ∂I. That is, there exists a homotopy between the two functions that is constant on the endpoints. In our case, this ensures that the eigenvalue 1 will not be deformed to a non-unit eigenvalue. Let $H(x, t)$ be such a homotopy, for example by $H(x, t) = t(f(x)) + (1 - t)x$. It follows that there exists an $h : S^1 \times I \to S^1$ that makes the following diagram commute:

$$\begin{array}{ccc} I & \xrightarrow{H_t} & I \\ {\scriptstyle e^{2\pi i(\cdot)}}\downarrow & & \downarrow{\scriptstyle e^{2\pi i(\cdot)}} \\ S^1 & \xrightarrow[h_t]{} & S^1 \end{array}$$

For $A \in E$ of the form $A = \sum_i \mu_i \pi_{W_i}$, we define a new hermitian matrix $H_t(A)$ where $t \in I$

$$H_t(A) = \sum_i H_t(\mu_i)\pi_{W_i}.$$

similarly, we define $h_t : U \to U$ by

$$h_t(X) = \sum_i h_t(\lambda_i)\pi_{W_i} = \sum e^{2\pi i H_t(\mu_i)}\pi_{W_i}.$$

Note that $h_t : N_n \to N_n$ satisfy $h_0 = \mathrm{Id}$ and $h_1(N_n) \subseteq F_{n-1}U$. In addition, h_t is covered by $H_t : p^{-1}(N_n) \to p^{-1}(N_n)$, where $H_0 = \mathrm{Id}$. What remains, is to show that the induced map on the fibers: $H_1 : p^{-1}(X) \to p^{-1}(h_0(X))$ is a weak equivalence, and we have proven that p is a quasifibration.

By the construction of H_t, $\ker(X - I) \subseteq \ker(H_1(X) - I)$. This means that proving that p is a quasifibration is reduced to proving the following lemma:

Lemma 4.6. *Suppose $V \subseteq V' \subseteq W \oplus W$, and $V'' \subseteq V' - V$. Then the map $\overline{BU}_{V,W} \to \overline{BU}_{V',W}$ given by sending Y to $Y \oplus V''$ is a weak equivalence.*

Proof. We have that both $\overline{BU}_{V,W}$ and $\overline{BU}_{V',W}$ are congruent to $BU \times \mathbb{Z}$. We know that $\widetilde{K}_{\mathbb{C}}(C) \cong [C, BU \times \mathbb{Z}]$ (See [8], 1.2), for any pointed compact space

C. Where $[A, B]$ denotes the homotopy classes of maps from A to B.

We therefore get an induced map

$$\widetilde{K}_{\mathbb{C}}(C) \cong [C, \overline{BU}_{V,W}] \to [C, \overline{BU}_{V',W}] \cong \widetilde{K}_{\mathbb{C}}(C).$$

This means any coset representative is mapped to another representative of the same coset in $K_{\mathbb{C}}(C)$. Since $V \subseteq V'$, the map has to be addition of a trivial bundle, so the map is an isomorphism. In particular, for $C = S^i$, we get an isomorphism of homotopy groups. □

5 Proof of real Bott periodicity

Theorem 5.1. *The iterated loop spaces of BO are as follows:*

$$\Omega BO \simeq O$$
$$\Omega O \simeq O/U$$
$$\Omega(O/U) \simeq U/Sp$$
$$\Omega(U/Sp) \simeq BSP \times \mathbb{Z}$$
$$\Omega BSp \simeq Sp$$
$$\Omega Sp \simeq Sp/U$$
$$\Omega(Sp/U) \simeq U/O$$
$$\Omega(U/O) \simeq BO \times \mathbb{Z}$$

We will be proving this one loop at a time by constructing quasifibrations with contractible spaces using the same procedure as in the complex case. We already know that $\Omega BO \simeq O$ and $\Omega BSp \simeq Sp$.

5.1 $\Omega O \simeq O/U$

Let $\mathcal{U} \cong \mathbb{C}^\infty$ be an infinite dimensional complex inner product space. For $W \subset \mathcal{U}$ finite complex subspace, let O(W) denote the real linear isometries of W. When considering the real transformations of W, we will view W as a real vector space with twice the dimension of its complex counterpart. Define

$$E(W) = \{\, A \subseteq \mathfrak{o}(W) \mid \mu_j \in [-i, i] \ \forall j\}$$

where $\mathfrak{o}$ denote the Lie algebra of $O(W)$ known to consist of all skew symmetric linear transformations.

E(W) is contractible by the contracting homotopy. Define

$$p_W : E(W) \to O(W)$$

by $p_W(A) = -e^{\pi A}$. For $V \subseteq W$, we have maps $O(V) \to O(W)$ given by sending X to $X \oplus I_{W-V}$, and $E(V) \to E(W)$ by sending A to $A \oplus i$, where i, the imaginary unit, is thought of as a skew-symmetric real transformation of $W - V$. We get by taking direct limits over all finite subspaces of $\mathcal{U}$ a map $p : E \to O$.

Since E is contractible, it is path connected. Its image under p is therefore also path connected. However, O consists of matrices with determinant $+1$ and -1. There is no path that connect these two sets of matrices, so O isn't path-connected. Since SO, the subspace of O consisting of matrices with determinant $+1$, is the path-component of O that includes the identity matrix, the image of p is SO. We will therefore show that this map is a quasifibration onto SO, with fiber O/U.

First of all, we make sure the quotient O/U makes sense. Consider $O/U(W)$.

$O(W)$ is the orthogonal group O_{2n} dimensions when regarding W as a $2n$ dimensional real vector space. Meanwhile, $U(W)$ is the unitary group U_n when regarding W as a n dimensional complex vector space. While $O(2n)$ preserve the real structure on W, it does not need to preserve the complex structure, that is, the relation between the real and imaginary part of the corresponding complex vector space. So U_n preserve more structure than O_{2n}, which means $U_n \subseteq O_{2n}$, so taking the quotient makes sense.

To construct O/U, we take the direct limit $O/U = \varinjlim_W O/U(W)$, with the maps $O/U(V) \to O/U(W)$ for $V \subseteq W \subset \mathcal{U}$ defined as $[X] \mapsto [X \oplus I_{W-V}]$.

We are going to need a nice representation of $O/U(W)$. For $W \subset \mathcal{U}$ finite dimensional, let $CX(W)$ denote the space of complex tructures on W, that is, the space of linear isometries $J : W \to W$ that satisfy $J^2 = -I$. Intuitively, this is the matrix analogy of multiplying by i, and identifying multiplication by J with multiplication by i is in fact a way to construct a complex vector space from a real one.

Lemma 5.2. *Let $W \subset \mathcal{U}$ be finite dimensional. Then $O/U(W) \cong CX(W)$*

Proof. Let $O(W)$ act on $CX(W)$ by conjugation. This is indeed an action because for $A \in O(W)$ and $J \in CX(W)$

$$AJA^{-1} = X => X^2 = (AJA^{-1})(AJA^{-1}) = A(-I)A^{-1} = -IAA^{-1} = -I.$$

Which means $X \in CX(W)$

We can define the complex space of W induced by $J \in CX(W)$ given by defining $Jw = iw$ for all $w \in W$, and with i being the imaginary unit. Denote the induced complex space by (W, J). Consider two induced complex spaces (W, J_1) and (W, J_2). We know the two spaces has equal dimension, and J_1 and J_2, being orthogonal matrices, both preserve the dot product. We therefore have $(W, J_1) \cong (W, J_2)$. Any isometry $U : (W, J_1) \to (W, J_2)$ must satisfy

$$U(J_1(w)) = U(iw) = i(U(w)) = J_2(U(w)).$$

We may view U as an isometry on W as a real vector space, which makes it an orthogonal matrix, and in particular, invertible. We therefore have $UJ_1U^{-1} = J_2, U \in O(W)$, which proves that the action on $CX(W)$ is transitive.

We will now find the stabilizer of this action. That is, we will find all $U \in O(W)$ such that $UJ = JU \ \ \forall J \in CX(W)$.

Assume $UJ(w) = JU(w)$ for all $w \in W$. Then, on the induced complex space, we have $U(iw) = i(U(W))$. That means if U is in the stabilizer, then it is a complex isometry. Now, assume $UJ(w) \neq JU(w)$. Then, on the induced complex space, we have $U(iw) \neq i(U(w))$. Contrapositively, that means that if U is a complex isometry, then it is in the stabilizer. We can conclude that the stabilizer is all complex isometries on W, which is $U(W)$. By the orbit-stabilizer theorem, we have

$$O/U(W) \cong CX(W)$$

□

We now proceed to identify the fiber of p_W.

Lemma 5.3. *For $x \in SO(W)$, $p_W^{-1}(X) \cong CX(\ker(X-I))$*

Proof. Let $A \in p_W^{-1}(X)$.

Let $W \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of W that extends the underlying field of scalars from $\mathbb{R}$ to $\mathbb{C}$ We have that $\mathfrak{o}(W) \subseteq \mathfrak{u}(W \otimes_{\mathbb{R}} \mathbb{C})$, since a skew-symmetric matrix is merely a skew-hermitian matrix with only real entries. That means we can find a spectral decomposition for A:

$$A = i\pi_{V'} - i\pi_{V''} + \sum_j \mu_j \pi_{W_j},$$

Where $\mu_j \in (-i, i)$.

For a skew-symmetric matrix, we have that if μ is an eigenvalue, so is $-\mu$, and their corresponding eigenspaces have the same dimension. This is because since $A^T = -A$, and since A^T has the exact same eigenvalues as A, $\mu \in A \implies \mu \in A^T \implies \mu \in -A$. However, we can see from the spectral decomposition that multiplying A by -1 has the effect of multiplying every eigenvalue by -1. Therefore $\mu \in -A \implies -\mu \in A$. It follows that if μ is an eigenvalue of A, so is $-\mu$. But since μ is a pure imaginary number, $-\mu$ is its complex conjugate. Therefore, the eigenspaces have the same dimension.

We also have $O(W) \subseteq U(W \otimes_{\mathbb{R}} \mathbb{C})$ since an orthogonal matrix is a unitary matrix with only real entries. Therefore we can find a spectral decomposition for X

$$X = \pi_V + \sum_j \lambda_j \pi_{V_j},$$

where $|\lambda_j| = 1$ and $\lambda_j \neq 1$. Since $X = -e^{\pi A}$ and by uniqueness of spectral decomposition, we get that $V = V' \oplus V'' = \ker(X-I) \otimes_{\mathbb{R}} \mathbb{C}$, $V_j = W_j$ and μ_j is completely determined by λ_j for all j. We wish to show that $A^2|_{\ker(X-I)} = -I_{\ker(X-I)}$. First we have to show that $A(\ker(X-I)) \subseteq \ker(X-I)$. To do that, we check that for a vector $v \in \ker(X-I)$, $A(v) \in \ker(X-I)$. Note that $v = \pi_{V'}(v) + \pi_{V''}(v)$ since $v \in V' \oplus V''$

$$A(v) = i\pi_{V'}(v) - i\pi_{V''}(v) + \sum_j \mu_j \pi_{W_j}(v) = i\pi_{V'}(v) - i\pi_{V''}(v) \in V' \oplus V'' = V.$$

So $A(v) \in \ker(X-I)$. We used here that v is orthogonal onto every eigenspace other than V' and V'', so the orthogonal projection of v onto W_j vanishes. We use this fact in the following calculation as well. Now,

$$A^2(v) = A(i\pi_{V'}(v) - i\pi_{V''}(v))$$
$$= (i \cdot i\pi_{V'}^2(v) + i \cdot (-i)\pi_{V'}\pi_{V''}(v) + i \cdot (-i)\pi_{V''}\pi_{V'}(v) + (-i) \cdot (-i)\pi_{V''}^2(v)$$
$$= -\pi_{V'}(v) - \pi_{V''}(v) = -v$$

Which means $A^2|_{\ker(X-I)} = -I_{\ker(X-I)}$. Therefore, $A \in CX(\ker(X-I))$. We used that projecting v orthogonally to two vector spaces orthogonal to each other successively yields 0, as well as the fact that $\pi_Y^2 = \pi_Y$.

Conversely, given $J \in CX(\ker(X-I))$, let $A = J + \sum_j \mu_j \pi_{V_j}$ From the above calculation, $J_0 = i\pi_{V'} - i\pi_{V''}$ satisfies $J_0^2 = -I$, and since all $J \in CX(\ker(X-I))$ are similar, they have the same eigenvectors and eigenspace. That means $J = i\pi_{V'} - i\pi_{V''}$ for all J. Consequently, $A = J + \sum_j \mu_j \pi_{V_j} \in p_W^{-1}(X)$. □

Define

$$\overline{O/U}_{V,W} = \lim_{W' \geq W} O/U(V \oplus (W' - W))$$

for $V \subseteq W \subset \mathcal{U}$ where W is a complex space and V is a real even dimensional space. By a similar argument as in the previous section, for a choice of isometry, we have $V \oplus W^\perp \cong \mathcal{U}$, and we get $\overline{O/U}_{V,W} \cong O/U$. In addition, for $X \in SO(W)$, we have $p^{-1}(X) \cong \overline{O/U}_{\ker(X-I),W}$

We will now show that p is a quasifibration. We start by defining a filtration on SO analogously to the previous filtration:

$$F_n SO = \{\, X \in SO \mid \dim_{\mathbb{R}} \ker(X-I)^\perp \leq 2n\}.$$

Note that since W is a complex subspace, any complex subspace of W, in particular $\ker(X-I)$ has an even real dimension. Therefore $B_n = F_n SO - F_{n-1} SO$ is the set of all $X \in SO$ such that $\dim_{\mathbb{R}} \ker(X-I)^\perp = 2n$. We will now prove that $p^{-1}(B_n) \to B_n$ is distinguished.

Lemma 5.4. *$p^{-1}(B_n) \to B_n$ is a Serre fibration*

Proof. This proof will be completely analogous to lemma 4.5. We start with the following commutative square

$$\begin{array}{ccc} \{0\} \times I^k & \xrightarrow{\alpha} & p^{-1}(B_n) \\ \downarrow & & \downarrow \\ I^{k+1} & \xrightarrow[\beta]{} & B_n \end{array}$$

We wish to find a lift of this diagram, i.e. a map $\gamma : I^{k+1} \to p^{-1}(B_n)$ making the triangles in the diagram commute.

By compactness, there exist a finite dimensional $W \subset \mathcal{U}$ such that the diagram factors as

$$\begin{array}{ccccc} \{0\} \times I^k & \xrightarrow{\alpha'} & E(W) \cap p^{-1}(B_n) & \longrightarrow & p^{-1}(B_n) \\ \downarrow & & \downarrow & & \downarrow \\ I^{k+1} & \xrightarrow[\beta']{} & SO(W) \cap B_n & \longrightarrow & B_n \end{array}$$

Let $A(0, t_1, ..., t_k) = \alpha'(t_1, ..., t_k)$ and $X(t_0, t_1, ..., t_k) = \beta'(t_0, ..., t_k)$. For $t \in I^k, I^{k+1}$ respectively, we may write the spectral decomposition of A and X as

$$A(t) = i\pi_{V'(t)} - i\pi_{V''(t)} + \sum_l \mu_l(t) \pi_{W_l(t)},$$

$$X(t) = \pi_{V(t)} + \sum_l \lambda_l(t) \pi_{V_l(t)},$$

where $-e^{\pi \mu_l(t)} = \lambda_l(t)$, $W_0(t) \oplus W_1(t) = V(t)$, and $W_l(t) = V_l(t)$ when $t \in I^k$.

Consider the following subspace of an m-dimensional real subspace W of $\mathcal{U}$:

$$\begin{aligned} \mathrm{Perp}_n(W) = \{ (V', V'') \mid V', V'' \subseteq W,\ V' \perp V'', \\ i\pi_{V'} - i\pi_{V''} \in CX(V' \oplus V'')\} \end{aligned}$$

Let $J = i\pi_{V'} - i\pi_{V''}$. Since J has real entries, and complex conjugate eigenvalues, the eigenspace for each eigenvalue is complex conjugate of each other, which means they have the same dimension. Therefore we may think of $span\{V', V''\}$ as a complex n-dimensional complex space. Note that a unitary transformation of J is isomorphic to J.

We may characterize this space by considering the orthogonal group over W. We get all possible V' and V'' by letting $O(W)$ act on one pair (V', V'') by conjugation. That means $O(W)$ acts transitively on $\mathrm{Perp}_n(W)$. The stabilizer is given by the orthogonal matrices that can be decomposed into a unitary matrix that acts on $V' \oplus V''$, and an orthogonal matrix that acts on $(W - (V' \times V''))$. Thus the stabilizer is $U_n \oplus O_{m-2n} \cong U_n \times O_{m-2n}$. We therefore have

$$\mathrm{Perp}_n(W) \cong O_m/(U_n \times O_{m-2n})$$

Define $BO_n(Y) = \{ V \mid V \subseteq Y,\ \dim_{\mathbb{R}} V = n\}$

We have a natural map $P : \mathrm{Perp}_n(W) \to BO_{2n}(W)$ given by $P(V', V'') = V' \oplus V''$. We characterize $BO_{2n}(W)$ in a similar way. We have that O_m acts transitively on $BO_{2n}(W)$, with stabilizer the orthogonal matrices that can be decomposed into the direct sum of an orthogonal matrix that acts on $V \in BO_{2n}(W)$, and an orthogonal matrix that acts on $W - V$. That is, an orthogonal matrix of the form $O_{2n} \oplus O_{m-2n} \cong O_{2n} \times O_{m-2n}$. Thus $BO_{2n}(W) \cong O_m/(O_{2n} \times O_{m-2n})$. We have that $U_n \subseteq O_{2n}$, which means $O_m/(O_{2n} \times O_{m-2n}) \subseteq O_m/(U_n \times O_{m-2n})$. It follows that P is a projection, so $P : \mathrm{Perp}_n(W) \to BO_{2n}(W)$ is a fibration. That means we can find a lift ω'' to the following commutative diagram.

$$\begin{array}{ccc} \{0\} \times I^k & \xrightarrow{\alpha''} & \mathrm{Perp}_n(W) \\ \downarrow & \nearrow^{\omega''} & \downarrow P \\ I^{k+1} & \xrightarrow[\beta'']{} & BO_{2n}(W) \end{array}$$

Let $n = \dim W_0(0)$, i.e the dimension of the eigenspaces of A(0) corresponding to $\mu = i$. Let $\alpha'' : I^k \to \mathrm{Perp}_n(W)$ be given by $\alpha''(t) = (W_0(t), W_1(t))$ and let $\beta : I^{k+1} \to BU_{i+j}(W \oplus W)$ be given by $\beta''(t) = V(t)$, where we have $W_0(t) \oplus W_1(t) = V(t)$. Define $\omega''(t) = (W'(t), V - W'(t))$, where $W'(t)$ is obtained from $W_0(t)$ by a homotopy. Let $\mu_l(t) \in (0, 1)$ be the unique solution to $-e^{\pi \mu_l(t)} = \lambda_l(t)$. We can now define $\omega' : I^{k+1} \to E(W) \cap p^{-1}(B_n)$ by

$$\omega'(t) = i\pi_{W'(t)} - i\pi_{(V(t) - W'(t))} + \sum_l \mu_l(t) \pi_{V_l(t)}$$

and by inclusion, we obtain a lift ω to our original diagram. $\square$

We define a neighborhood N_n of $F_{n-1}SO$ in F_nSO given by

$$N_n = \{ X \mid \dim_{\mathbb{R}} Eig_{e^{2\pi i[1/4,3/4]}} X < 2n\} \subseteq F_nSO.$$

To find a homotopy deforming N_n so that it lies inside $F_{n-1}SO$, we define a function $f : [-i, i] \to [-i, i]$

$$f(x) = \begin{cases} -i, & im(x) \leq -\frac{1}{2} \\ 2x, & -\frac{1}{2} \leq im(x) \leq \frac{1}{2} \\ i, & im(x) \geq \frac{1}{2} \end{cases}$$

Then $f \simeq \mathrm{Id}\ \mathrm{rel}\{-i, i\}$. $H(x,t) = t \cdot f + (1-t) \cdot x$ is a homotopy that satisfies this. Let $h : S^1 \times I \to S^1$ be defined so that the following square commutes for all $t \in I$:

$$\begin{array}{ccc} [-i,i] & \xrightarrow{H_t} & [-i,i] \\ \downarrow{\scriptstyle -e^{\pi(\cdot)}} & & \downarrow{\scriptstyle -e^{\pi(\cdot)}} \\ S^1 & \xrightarrow[h_t]{} & S^1 \end{array}$$

The induced homotopy $h : N \times I \to N$ deforms N_n into $F_{n-1}SO$ for all n, and is covered by a homotopy $H : p^{-1}(N) \times I \to p^{-1}(N)$, as was required by theorem 2.2. Both induced homotopies are defined similarly to the previous section.

It remains to check that H_1 induces weak equivalences on fibers, that is, that $H_1 : p^{-1}(X) \to p^{-1}(h_1(X))$ is a weak equivalence. Following the same argumentation leading up to lemma 4.6, this boils down to proving the following lemma.

Lemma 5.5. *Let $V \subseteq V'$ be even dimensional suspaces of a finite dimensional subspace $W \subset \mathcal{U}$. Then the map $f : \overline{O/U}_{V,W} \to \overline{O/U}_{V',W}$ given by sending A to $A \oplus J$ for a fixed complex structure J on $V' - V$ is a homotopy equivalence.*

Proof. Since $CX(W) \cong O/U(W)$, there is a correspondence between an element in CX and a coset in O/U which has a representative in O. Let S in O correspond to $A \in CX$, and define J such that its corresponding element in O is $I_{V'-V}$. Consider the following commutative square:

$$\begin{array}{ccc} \varinjlim_{W' \geq W} (V \oplus (W' - W)) & \xrightarrow{\phi} & \varinjlim_{W' \geq W} (V' \oplus (W' - W)) \\ \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S \oplus I_{V'-V}} \\ \varinjlim_{W' \geq W} (V \oplus (W' - W)) & \xrightarrow[\phi]{} & \varinjlim_{W' \geq W} (V' \oplus (W' - W)) \end{array}$$

Connecting the notation to lemma 2.4, we have $f = \phi_*$ and $S \oplus I_{V'-V} = \phi S \phi^{-1} \oplus I_{V'-V} = f(A) = \phi_*(A)$. From the second equality, we get that ϕ is an orthogonal matrix matrix and therefore an isometry. Since both S and $I_{V'-V} \in O$, we can use lemma 2.4 to conclude that f is a homotopy equivalence. In particular, f is a weak homotopy equivalence. $\square$

5.2 $\Omega(O/U) \simeq U/Sp$

Let $\mathcal{U} \cong \mathbb{H}^\infty$ be an infinite dimensional quaternionic inner product space. For $W \subset \mathcal{U}$ finite quaternionic subspace, let $O(W)$ denote the space of real linear isometries of W, where W is viewed as a real subspace of four times the dimension of its quaternionic counterpart. Let $U(W)$ denote the space of complex isometries of W where W is viewed as a complex subspace of twice the dimension of its quaternionic counterpart. Note that $U(W) \subseteq O(W)$. Let $O/U = \varinjlim_W O/U(W)$, under the map $i_{V,W} : O/U(V) \to O/U(W)$, $i_{V,W}([X]) = [X \oplus I_{W-V}]$ for $V \subseteq W$.

Define:

$$E(W) = \{ A \subseteq \mathfrak{o}(W) \mid A \text{ is conjugate linear and } \mu_m \in [-i, i] \ \forall m\}$$

Where λ_m denotes the eigenvalues in A. Demanding that A is conjugate linear (i.e. $A(bw_1 + cw_2) = \bar{b}A(w_1) + \bar{c}A(w_2), w_1, w_2 \in W$ where the bar denotes complex conjugation), corresponds intuitively to the fact that for quaternions we have $i \cdot j = -j \cdot i$. We have a map $E(V) \to E(W)$ given by sending A to $A \oplus j$ where j, which is one of the imaginary units of the quaternions, is viewed as a conjugate linear skew symmetric tranformation on $W - V$.

We have that a skew-symmetric matrix $A \in \mathfrak{o}(W)$ has to be either linear or conjugate linear, depending on whether we consider W to be a complex vector space or a quaternionic vector space. We then have that $\mathfrak{u}(W)$ is the subspace of linear skew-symmetric transformations of W. We therefore have that the orthogonal complement $\mathfrak{u}(W)^\perp$ is the space of conjugate linear skew-symmetric transformations of W. Consequently, $\mathfrak{u}(W)^\perp$ is the Lie algebra corresponding to the Lie group $O/U(W)$.

We also have that $E(W)$ is contractible by the contracting homotopy. That can be verified by noting that a scalar multiple of a conjugate linear map is still conjugate linear. Define

$$p_W : E(W) \to O/U(W)$$

by $p_W(A) = [je^(\frac{\pi}{2}A)]$. By taking direct limits over finite dimensional quaternionic subspaces of $\mathcal{U}$, we get a map $p : E \to O/U$. We wish to show that this map determines a quasifibration over SO/U with U/Sp as the fiber, where Sp denotes the infinite group of quaternionic linear isometries. The reason we restrict the base space to SO/U is because O/U is disconnected. This is because since $O(2n)$ is disconnected, and $U(n)$ is a connected subspace, $U(n)$ must lie in one of the connected components of $O(2n)$, and since $U(n)$ contains the identity transformation, it must lie in $SO(2n)$. O/U may therefore be decomposed into $SO/U \oplus SO^\perp$. From the last section, we concluded that the base space must be connected, so we have that $p : E \to O/U$ restricts to $E \to SO/U$.

We start by finding a nice representation of $U/Sp(W)$. Define $QS(W)$ denote the space of quaternionic structures on W viewed as a complex space. These are the conjugate linear isometries J of W that satisfies $J^2 = -I$.

To get an intuition of how J operates, consider the complex number $a + ib$.

The following expression gives an orthogonal basis for $\mathbb{C}^2$: $(a+bi, J(a+ib))$. We can make this into a quaternionic vector space by defining $J(a+ib) = ja + ijb$. We have a corresponding basis given by $a + bi + ja + ijb = a + ib + ja + kb$ where $k := ij$.

Lemma 5.6. $U/Sp(W) \cong QS(W)$

Proof. Let $U(W)$ act on $QS(W)$ by conjugation. This is indeed an action because for $A \in U(W)$ and $J \in QS(W)$

$$AJA^{-1} = X => X^2 = (AJA^{-1})(AJA^{-1}) = A(-I)A^{-1} = -IAA^{-1} = -I.$$

Which means $X \in QS(W)$

Given W as a complex vector space, with $\{a + ix, J(a + ix)\}$ as the first two components of the basis, the other pairs being defined a similar way, we can define the quaternionic vector space of W induced by $J \in QS(W)$ given by defining $J(x + iy) = jx + ijy = jx + ky$ for all $x + iy \in W$, x and y being real vectors, and with j being the second imaginary unit with $ij = k$. The first basis vector in the induced basis for the quaternionic space is therefore $\{(x + iy) + J(x + iy)\} = \{x + iy + jx + ky\}$. Denote the induced quaternionic space by (W, J). Consider two induced quaternionic spaces (W, J_1) and (W, J_2). We know the two spaces has equal dimension, and J_1 and J_2, being unitary matrices, both preserve the dot product. We therefore have $(W, J_1) \cong (W, J_2)$. Any isometry $S : (W, J_1) \to (W, J_2)$ must satisfy

$$S(J_1(w)) = S(jw) = j(S(w)) = J_2(S(w)).$$

We may view S as an isometry on W as a complex vector space, which makes it a unitary matrix, and in particular, invertible. We therefore have $SJ_1S^{-1} = J_2, S \in U(W)$, which proves that the action on $QS(W)$ is transitive.

We will now find the stabilizer of this action. That is, we will find all $S \in U(W)$ such that $SJ = JS \;\; \forall J \in CX(W)$.

Assume $SJ(w) = JS(w)$ for any $w \in W$. Then, on the induced quaternionic space, we have $S(jw) = j(S(W))$. That means if S is in the stabilizer, then it is a quaternionic linear isometry. Now, assume $SJ(w) \neq JS(w)$. Then, on the induced quaternionic space, we have $S(jw) \neq j(S(w))$. Contrapositively, that means that if S is a quaternionic linear isometry, then it is in the stabilizer. We can conclude that the stabilizer is all quaternionic isometries on W, which is $Sp(W)$. By the orbit-stabilizer theorem, we have

$$U/Sp(W) \cong QS(W)$$

□

For $V \subseteq W \subset \mathcal{U}$, we define a map: $U/Sp(V) \to U/Sp(W)$ given by sending $[X]$ to $[X + I_{W-V}]$. We define $U/Sp = \lim_W(U/Sp(W)$.

In order to do computations with elements of O/U, we need to understand their coset representatives. The following lemmas provide us with some insight.

Lemma 5.7. *Suppose $Y = e^A$, where $A \in \mathfrak{o}(W)$ is conjugate linear. Then $Yi = iY^{-1}$.*

Proof.

$$-iYi = -ie^A i = e^{-iAi} = e^{(-i)\cdot(-i)A} = e^{-A} = Y^{-1}$$

Where we have applied the conjugate linear criterion in the third equality. □

Lemma 5.8. *Suppose that $Y, Z \in O(W)$ satisfy $-iYi = Y^{-1}$ and $-iZi = Z^{-1}$. Then there is an $X \in U(W)$ such that $jY = XZ$ if and only if $-Y^2 = Z^2$.*

Proof. Assume there is an $X \in U(W)$ such that $jY = XZ$. We then have

$$j(Y^{-1}i) = Z^{-1}X^{-1}i$$

$$j(iY) = -ijY = -iXZ = -XiZ = -XZ^{-1}i$$

Where we have used that $Xi = iX$ for $X \in U(W)$. Since $iY = Y^{-1}i$, we get that $Z^{-1}X^{-1}i = -XZ^{-1}i$ which yields $XZX = -Z$. This means that $Y^2 = (XZ)(XZ) = (XZX)Z = -Z^2$.

Conversely, assume $-Y^2 = Z^2$. Then $Y = -(Y^{-1}Z)Z$, which means $jY = (-jY^{-1}Z)Z$. We therefore need to show that $-jY^{-1}Z \in U(W)$. $jY = (-jY^{-1}Z)Z \implies jYZ^{-1} = -jY^{-1}Z$. So we get $-jY^{-1}Z \cdot i = -jY^{-1}iZ^{-1} = -jiYZ^{-1} = ijYZ^{-1} = i(-jY^{-1}Z)$, so $-jY^{-1}Z \in U(W)$. □

We will call $X \in SO(W)$ such that $X = e^A$ for a conjugate linear $A \in \mathfrak{o}(W)$ a special representative of the coset $[X] \in SO/U(W)$.

Assume $Y, Z \in O(W)$ have the property as in lemma 5.8. Then $-i(jY)i = -j(-iYi) = -j(Y^{-1}) = (jY)^{-1}$, so jY also have that property. Therefore $j(jY) = XZ$ if and only if $-(jY)^2 = Z^2$. Therefore $-Y = XZ$ if and only if $Y^2 = Z^2$. But we therefore have $Y = (-X)Z$. Since $-X \in U(W)$, we get that $Y = XZ$ if and only if $Y^2 = Z^2$.

Therefore lemma 5.7 and 5.8 says that two special representatives belong to the same coset if and only if they have equal squares.We have, however, not yet determined whether such an X exists for every coset. We will prove this in the following lemma:

Lemma 5.9. *Every $[X] \in SO/U(W)$ has a special representative*

Proof. First, we have that $SO/U(W)$ is geodesically complete. We show this by showing that the riemannian exponential map is defined on the entire tangent space of each element in SO/U. Now, SO/U is a compact Lie group, since it is a quotient of a compact Lie group SO by a normal Lie subgroup U. Therefore, the riemannian exponential map coincides with matrix exponentiation, and the tangent space at the identity is the corresponding Lie algebra. At any other point, the tangent space is that point multiplied by the Lie algebra [11]. Therefore the exponential map is defined everywhere, so $SO/U(W)$ is geodesically complete. All geodesics in $SO/U(W)$ take on the form $[Ye^{tB}]$ for $Y \in SO(W)$ and $B \in \mathfrak{u}(W)^{\perp}$.

Now, we either have $Y \in U(W)$ or $Y \in SO(W) - U(W)$. But if $Y \in U(W)$, then $[Ye^{tB}] = [e^{tB}]$ since Y lies in the same coset as I. These cosets contain the special representative $e^{tB} \in SO(W)$ for all t. We are therefore left with $Y \in SO(W) - U(W)$. But $e^{tB} \in SO(W) - U(W)$, so $Ye^{tB} \in SO(W) - U(W)$. Therefore there exists a $C \in \mathfrak{u}(W)^{\perp}$ such that $Ye^{tB} = e^{tC}$, which makes $Ye^{tB} \in SO(W)$ a special representative for the coset $[Ye^{tB}]$ for all t. □

We proceed to find an expression for the fiber of p_W.

Lemma 5.10. *Suppose that $W \subset \mathcal{U}$ is a finite dimensional quaternionic space. Let X be a special representative for the coset $[X] \in SO/U(W)$. Then* $p_W^{-1}([X]) = U/Sp(\ker(X^2 - I))$.

Proof. Suppose $A \in p_W^{-1}([X])$. We wish to show that A defines a quaternionic structure on $\ker(X^2 - I)$. That means that $A(\ker(X^2 - I)) \subseteq \ker(X^2 - I)$ and $A^2|_{\ker(X^2-I)} = -I_{\ker(X^2-I)}$. Just like the last section, we can view $A \in E(W)$ as an element of $\mathfrak{u}(W \otimes_{\mathbb{R}} \mathbb{C})$. This gives us a spectral decomposition:

$$A = i\pi_{W'} - i\pi_{W''} + \sum_l \mu_l \pi_{W_l},$$

where $\mu_l \in (-i, i)$ are the eigenvalues of A not equal to $\pm i$. We may also regard $X \in SO(W)$ as an element of $U(W \otimes_{\mathbb{R}} \mathbb{C})$, so we can write its spectral decomposition as

$$X = \pi_{V'} - \pi_{V''} + \sum_l (\lambda_l \pi_{V'_l} - \lambda_l \pi_{V''_l}),$$

Where $|\lambda_l| = 1$ and $\mathrm{Im}(\lambda_l) < 0$. We will then get

$$X^2 = \pi_{V' \oplus V''} + \sum_l \lambda_l^2 \pi_{V'_l \oplus V''_l}$$

Since $A \in p_w^{-1}([X])$, $p_w(A) = [je^{\frac{1}{2}\pi A}] = [X]$, so $X^2 = (je^{\frac{1}{2}\pi A})^2 = -e^{\pi A}$. It follows that $V' \oplus V'' = W' \oplus W'' = \ker(X^2 - I) \otimes_{\mathbb{R}} \mathbb{C}$. By a computation completely similar to that in lemma 5.3, we get that $A \in QS(\ker(X^2 - I))$. Conversely, let J be a quaternionic structure on $\ker(X^2 - I) \otimes_{\mathbb{R}} \mathbb{C}$. We will show that $A = J + \sum_l \mu_l \pi_{V' \oplus V''} \in p_W^{-1}([X])$, where $\mu_l \in (-i, i)$ are the unique solutions to $-e^{\pi \mu_l} = \lambda_l^2$. Since $J_0 = i\pi_{W'} - i\pi_{W''}$ satisfies this, and since all J are similar to each other, they all have the same spectral decomposition. Therefore $A = J + \sum_l \mu_l \pi_{V' \oplus V''} \in p_W^{-1}([X])$. □

For $V \subseteq W \subset \mathcal{U}$, define

$$\overline{U/Sp}_{V,W} = \lim_{\rightarrow} {}_{W' \geq W} U/Sp(V \oplus (W' - W))$$

Following the same line of reasoning as in lemma 4.4, with the isometry $V \oplus W^{\perp} \cong \mathcal{U}$, we get that $\overline{U/Sp}_{V,W} \cong U/Sp$. In addition,
$p^{-1}([X]) \cong \overline{U/Sp}_{\ker(X^2-I),W}$, which gives us $p^{-1}(X) \cong U/Sp$.

We will now prove that $p : E \to SO/U$ is a quasifibration. We start by defining a filtration of SO/U as the following:

$$F_nSO/U = \{\,[X] \mid X \text{ is a special representative, } \dim_{\mathbb{C}} \ker(X^2 - I)^{\perp} \leq 2n\}.$$

Note that since any two special representatives of the same coset have equal squares, the filtration is independent of the choice of special representative.

We define $B_nSO/U = F_nSO/U - F_{n-1}SO/U$. So

$$B_nSO/U = \{\,[X] \mid X \text{ is a special representative, } \dim_{\mathbb{C}} \ker(X^2 - I)^{\perp} = 2n\}$$

. The proof that this is a quasifibration follow the same arguments as the previous sections. First we prove the following:

Lemma 5.11. *$p^{-1}(B_nSO/U) \to B_nSO/U$ is a Serre fibration.*

Proof. This follow the same procedure as in the preceding sections. We start with the following commutative square

$$\begin{array}{ccc} \{0\} \times I^k & \xrightarrow{\alpha} & p^{-1}(B_nSO/U) \\ \downarrow & & \downarrow \\ I^{k+1} & \xrightarrow[\beta]{} & B_nSO/U \end{array}$$

We wish to find a lift of this diagram, i.e. a map $\gamma : I^{k+1} \to p^{-1}(B_nSO/U)$ making the triangles in the diagram commute.

By compactness, there exist a finite dimensional $W \subset \mathcal{U}$ such that the diagram factors as

$$\begin{array}{ccccc} \{0\} \times I^k & \xrightarrow{\alpha'} & E(W) \cap p^{-1}(B_n) & \longrightarrow & p^{-1}(B_n) \\ \downarrow & & \downarrow & & \downarrow \\ I^{k+1} & \xrightarrow[\beta']{} & SO/U(W) \cap B_n & \longrightarrow & B_n \end{array}$$

Let $A(0, t_1, ..., t_k) = \alpha'(t_1, ..., t_k)$ and $X(t_0, t_1, ..., t_k) = \beta'(t_0, ..., t_k)$ for a special representative X of $SO/U(W)$. For $t \in I^k, I^{k+1}$ respectively, we may write the spectral decomposition of A and X as

$$A(t) = i\pi_{W_0(t)} - i\pi_{W_1(t)} + \sum_l \mu_l(t)\pi_{W_l(t)},$$

$$X(t) = \pi_{V(t)} - \pi_{V'(t)} + \sum_l (\lambda_l(t)\pi_{V_l(t)} - \lambda_l(t)\pi_{V_l'(t)}),$$

where $-e^{\pi\mu_l(t)} = \lambda_l^2(t)$, $W_0(t) \oplus W_1(t) = V(t) \oplus V'(t)$, and $W_l(t) = V_l(t) \oplus V_l'(t)$ when $t \in I^k$.

Consider the following subspace of an m-dimensional complex subspace W of $\mathcal{U}$ (note: m is even):

$$\begin{aligned}\operatorname{Perp}_n(W) = \{\,(V', V'') \mid V', V'' \subseteq W,\ V' \perp V'', \\ i\pi_{V'} - i\pi_{V''} = J \in QS(W)\}\end{aligned}$$

By [9], the eigenspace for each eigenvalue in J is complex conjugate of each other, which means they have the same complex dimension n. Therefore we may think of $\operatorname{span}\{V', V''\}$ as a quaternionic n-dimensional space. Note that a transformation of J by a symplectic matrix is isomorphic to J.

We may characterize this space by considering the unitary group over W. We get all possible V' and V'' by letting $U(W)$ act on one pair (V', V'') by conjugation. That means $U(W)$ acts transitively on $\operatorname{Perp}_n(W)$. The stabilizer is given by the unitary matrices that can be decomposed into a symplectic matrix that acts on $V' \oplus V''$, and a unitary matrix that acts on $(W - (V' \times V''))$. Thus the stabilizer is $Sp_n \oplus U_{m-2n} \cong Sp_n \times U_{m-2n}$. We therefore have

$$\operatorname{Perp}_n(W) \cong U_m/(Sp_n \times U_{m-2n})$$

Define $BU_n(Y) = \{\, V \mid V \subseteq Y,\ \dim_{\mathbb{C}} V = n\}$

We have a natural map $P : \operatorname{Perp}_n(W) \to BU_{2n}(W)$ given by $P(V', V'') = V' \oplus V''$. We characterize $BU_{2n}(W)$ in a similar way. We have that U_m acts transitively on $BU_{2n}(W)$, with stabilizer the unitary matrices that can be decomposed into the direct sum of a unitary matrix that acts on $V \in BU_{2n}(W)$, and a unitary matrix that acts on $W - V$. That is, a unitary matrix of the form $U_{2n} \oplus U_{m-2n} \cong U_{2n} \times U_{m-2n}$. Thus $BU_{2n}(W) \cong U_m/(U_{2n} \times U_{m-2n})$. We have that $Sp_n \subseteq U_{2n}$, which means $U_m/(U_{2n} \times U_{m-2n}) \subseteq U_m/(Sp_n \times U_{m-2n})$. It follows that P is a projection, so $P : \operatorname{Perp}_n(W) \to BU_{2n}(W)$ is a fibration. That means we can find a lift ω'' to the following commutative diagram.

$$\begin{array}{ccc} \{0\} \times I^k & \xrightarrow{\alpha''} & \operatorname{Perp}_n(W) \\ \downarrow & \nearrow^{\omega''} & \downarrow P \\ I^{k+1} & \xrightarrow[\beta'']{} & BU_{2n}(W) \end{array}$$

Let $n = \dim W_0(0)$, i.e the dimension of the eigenspaces of $A(0)$ corresponding to $\mu = i$. Let $\alpha'' : I^k \to \operatorname{Perp}_n(W)$ be given by $\alpha''(t) = (W_0(t), W_1(t))$ and let $\beta : I^{k+1} \to BU_{2n}(W \oplus W)$ be given by $\beta''(t) = V(t) \oplus V'(t)$, where we have $W_0(t) \oplus W_1(t) = V(t) \oplus V'(t)$. Here we consider $V(t) \oplus V'(t)$ as a single subspace of W, "forgetting" its decomposition. We can do this since the coset is only dependent on the square of a special representative, which means any decomposition of $V(t) \oplus V'(t)$ will result in a special representative of the same coset. Define $\omega''(t) = (W'(t), (V(t) \oplus V'(t)) - W'(t))$, where $W'(t)$ is obtained from $W_0(t)$ by a homotopy. Let $\mu_l(t) \in (0, 1)$ be the unique solution to $-e^{\pi\mu_l(t)} = \lambda_l^2(t)$. We can now define $\omega' : I^{k+1} \to E(W) \cap p^{-1}(B_n)$ by

$$\omega'(t) = i\pi_{W'(t)} - i\pi_{(V(t)\oplus V'(t))-W'(t)} + \sum_l \mu_l(t)\pi_{V_l(t)}$$

and by inclusion, we obtain a lift ω to our original diagram. $\square$

We define a neighborhood N_n of $F_{n-1}SO/U$ in F_nSO/U by:

$$N_n = \{ [X] \mid X \text{ is a special representative, } \dim_{\mathbb{C}} \mathrm{Eig}_{e^{\pi i[1/2,3/2]}} X^2 < 2n\}.$$

To find a homotopy deforming N_n so that it lies inside $F_{n-1}SO/U$, we define a function $f : [-i, i] \to [-i, i]$

$$f(x) = \begin{cases} -i, & im(x) \le -\frac{1}{2} \\ 2x, & -\frac{1}{2} \le im(x) \le \frac{1}{2} \\ i, & im(x) \ge \frac{1}{2} \end{cases}$$

Then $f \simeq \mathrm{Id}\ \mathrm{rel}\{-i, i\}$. $H(x, t) = t \cdot f + (1 - t) \cdot x$ is a homotopy that satisfies this. Let $h : S^1 \times I \to S^1$ be defined so that the following square commutes for all $t \in I$:

$$\begin{array}{ccc} [-i,i] & \xrightarrow{H_t} & [-i,i] \\ {\scriptstyle -e^{\pi(\cdot)}}\downarrow & & \downarrow{\scriptstyle -e^{\pi(\cdot)}} \\ S^1 & \xrightarrow[h_t]{} & S^1 \end{array}$$

The induced homotopy $h : N \times I \to N$ deforms N_n into $F_{n-1}SO/U$ for all n, and is covered by a homotopy $H : p^{-1}(N) \times I \to p^{-1}(N)$, as was required by theorem 2.2.

It remains to check that H_1 induces weak equivalences on fibers, that is, that $H_1 : p^{-1}(X) \to p^{-1}(h_1(X))$ is a weak equivalence. This boils down to proving the following lemma.

Lemma 5.12. *Let $V \subseteq V'$ be even dimensional complex subspaces of a finite dimensional complex subspace $W \subset \mathcal{U}$. Then the map $f : \overline{U/Sp}_{V,W} \to \overline{U/Sp}_{V',W}$ given by sending A to $A \oplus J$ for a fixed quaternionic structure J on $V' - V$ is a homotopy equivalence.*

Proof. Since $QS(W) \cong U/Sp(W)$, there is a correspondence between an element in QS and a coset in U/Sp which has a representative in U. Let S in U correspond to $A \in QS$, and define J such that its corresponding element in U is $I_{V'-V}$. Consider the following commutative square:

$$\begin{array}{ccc} \varinjlim_{W' \ge W}(V \oplus (W' - W)) & \xrightarrow{\phi} & \varinjlim_{W' \ge W}(V' \oplus (W' - W)) \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S \oplus I_{V'-V}} \\ \varinjlim_{W' \ge W}(V \oplus (W' - W)) & \xrightarrow[\phi]{} & \varinjlim_{W' \ge W}(V' \oplus (W' - W)) \end{array}$$

Connecting the notation to lemma 2.4, we have $f = \phi_*$ and $S \oplus I_{V'-V} = \phi S \phi^{-1} \oplus I_{V'-V} = f(A) = \phi_*(A)$. From the second equality, we get that ϕ is a unitary matrix and therefore an isometry. Since both S and $I_{V'-V} \in U$, we can use lemma 2.4 to conclude that f is a homotopy equivalence. In particular, f is a weak homotopy equivalence. $\square$

5.3 $\Omega U/Sp \simeq BSp \times \mathbb{Z}$

Let $\mathcal{U} \cong \mathbb{H}^\infty$ be a countably infinite quaternionic inner product space. For finite dimensional $W \subset \mathcal{U}$, $U(W \oplus W)$ is the group of complex linear isometries of $W \oplus W$, and $Sp(W \oplus W)$ is the subgroup of quaternionic linear isometries of $W \oplus W$. Then, along with the map from $U/Sp(V \oplus V) \to U/Sp(W \oplus W)$ given by $[X] \to [X + I_{W-V,W-V}]$, we have $U/Sp = \varinjlim_W U/Sp(W \oplus W)$ following the same argument as section 4. Define:

$$E(W) = \{ A \in H(W \oplus W) \mid jA = Aj,\ \mu_m \in I\ \forall m\}$$

Where $H(W \oplus W)$ is the set of all hermitian complex linear transformations of $W \oplus W$. Note that for $A \in H(W \oplus W)$, $iA \in \mathfrak{u}(W \oplus W)$, where $\mathfrak{u}(W \oplus W)$ consists of skew-hermitian complex linear transformations of $W \oplus W$. In addition, we have that $\mathfrak{sp}(W \oplus W)^\perp = \{ A \in \mathfrak{u}(W \oplus W) \mid Aj = -jA\}$. Define a map $p_W : E(W) \to U/Sp(W \oplus W)$ by sending A to $[e^{\pi i A}]$. We have a map $E(V) \to E(W)$ given by sending A to $A \oplus \pi_{(W-V)\oplus 0}$. By taking direct limits, we get a map $p : E \to U/Sp$.

Analogously to the previous section, we have the following two lemmas to understand the coset representatives of U/Sp.

Lemma 5.13. *Let $W \subset \mathcal{U}$ be finite dimensional. If $A \in H(W \oplus W)$ satisfies $jA = Aj$, then $X = e^{iA}$ has the property $Xj = jX^{-1}$*

Proof.

$$-jXj = -je^{iA}j = e^{-jiAj} = e^{-jijA} = e^{-(-j^2 i)A} = e^{-iA} = X^{-1}$$

□

Lemma 5.14. *Suppose that $Y, Z \in U(W \oplus W)$ have the property $-jYj = Y^{-1}$ and $-jZj = Z^{-1}$. Then there exists an $X \in Sp(W \oplus W)$ such that $Y = XZ$ if and only if $Y^2 = Z^2$.*

Proof. Assume there is an $X \in Sp(W \oplus W)$ such that $Y = XZ$. We then have

$$jY^{-1} = jZ^{-1}X^{-1}$$

$$Yj = XZj = XjZ^{-1} = jXZ^{-1}$$

Where we have used that $Xj = jX$ for $X \in Sp(W \oplus W)$. Since $Yj = jY^{-1}$, we get that $jZ^{-1}X^{-1} = jXZ^{-1}$ which yields $XZX = Z$. This means that $Y^2 = (XZ)(XZ) = (XZX)Z = Z^2$.

Conversely, assume $Y^2 = Z^2$. Then $Y = (Y^{-1}Z)Z$. We therefore need to show that $Y^{-1}Z \in Sp(W)$. $Y = (Y^{-1}Z)Z \implies YZ^{-1} = Y^{-1}Z$. So we get $Y^{-1}Zj = Y^{-1}jZ^{-1} = jYZ^{-1} = jY^{-1}Z$, so $Y^{-1}Z \in Sp(W \oplus W)$. □

We call $X \in U(W \oplus W)$ such that $X = e^{\pi i A}$ for an $A \in E(W)$ a special representative for the coset $[X] \in U/Sp(W \oplus W)$. By the previous lemmas, we have that any two special representatives belong to the same coset if and only if they have equal squares. The following lemma ensures that every coset has a special representative.

Lemma 5.15. *Every coset $[X] \in U/Sp(W \oplus W)$ has a special representative.*

Proof. Notice that since $iA \in \mathfrak{u}(W \oplus W)$, a special representative also has the property that $X = e^{\pi B}$ for $B \in \mathfrak{u}(W \oplus W)$. Using the exact same argument as in lemma 5.9, but with U instead of SO, and Sp instead of U, we arrive at the conclusion that $U/Sp(W \oplus W)$ is geodesically complete, with the geodesics on the form $\gamma_t = [Ye^{tB}]$ for $Y \in U(W)$ and $B \in \mathfrak{sp}(W \oplus W)^{\perp}$, which gives us the special representatives of $[X]$. $\square$

For a quaternionic space Y, define

$$BSp(Y) = \coprod_n \{ V \mid V \text{ is a quaternionic subspace of } Y, \dim_{\mathbb{H}} V = n \}.$$

For $V \subseteq W \subset \mathcal{U}$, we have a map $BSp(V \oplus V) \to BSp(W \oplus W)$ given by sending Y to $Y \oplus (W - V) \oplus 0$. By the quaternionic version of lemma 4.2, we get by taking direct limits that $BSp \times \mathbb{Z} = \varinjlim_W BSp(W \oplus W)$. We proceed to identify the fiber of p_W.

Lemma 5.16. *Let $W \subset \mathcal{U}$ be finite dimensional. If X is a special representative for $[X] \in U/Sp(W \oplus W)$, then $p_W^{-1}([X]) \cong BSp(\ker(X^2 - I))$.*

Proof. Let $A \in E(W)$. Define a map $p^{-1}([X]) \to BSp(\ker(X^2 - I))$ given by sending A to $\ker(A - I)$. Let us make sure that this map makes sense, that is, check that $\ker(A - I) \subseteq \ker(X^2 - I)$. First, A has a spectral decomposition

$$A = \pi_{W_0} + \sum_l \mu_l \pi_{W_l},$$

where $\mu_l \in (0, 1)$, and a priori, W_0 and W_l are complex subspaces of $W \oplus W$. However, since $Aj = jA$, we have that if $Av = \mu v$, then $Ajv = jAv = j\mu v = \mu jv$ since μ is real for a hermitian matrix. Therefore, W_0 and W_l are in fact quaternionic subspaces. We also have a spectral decomposition for X:

$$X = \pi_{V'} - \pi_{V''} + \sum_l (\lambda_l \pi_{V_l'} - \lambda_l \pi_{V_l''})$$

which gives us

$$X^2 = \pi_{V' \oplus V''} + \sum_l \lambda_l^2 \pi_{V_l' \oplus V_l''}$$

Since $X^2 = e^{2\pi i A}$, we get that $p_W(A) = [X]$ if and only if $\ker(A - I) = W_0 \subseteq V' \oplus V'' = \ker(X^2 - I)$, $W_l = V_l' \oplus V_l''$, and $\mu_l \in (0, 1)$ is the unique solution to $e^{2\pi i \mu_l} = \lambda_l^2$. Therefore the map makes sense. We now construct an inverse, which is given by sending $B \in BSp(\ker(X^2 - I))$ to $\pi_B + \sum_l \mu_l \pi_{V_l' \oplus V_l''} \in p_W^{-1}([X])$. $\square$

Define

$$\overline{BSp}_{V,W} = \varinjlim_{W' \leq W} BSp(V \oplus (W' - W) \oplus (W' - W))$$

where $V \subseteq W \oplus W \subset \mathcal{U} \oplus \mathcal{U}$. By the usual isometry, $V \oplus W^{\perp} \oplus W^{\perp} \cong \mathcal{U} \oplus \mathcal{U}$, so we get $\overline{BSp}_{V,W} \cong \varinjlim_W BSp(W \oplus W) = BSp \times \mathbb{Z}$. For a special representative $X \in U(W \oplus W)$, by Lemma 5.16, we get $p^{-1}([X]) = \overline{BSp}_{\ker(X^2-I),W}$.

We will now prove that $p : E \to U/Sp$ is a quasifibration. We define the following filtration of U/Sp:

$$F_n(U/Sp) = \{ [X] \mid X \text{ is a special representative, } \dim_{\mathbb{C}} \ker(X^2 - I)^{\perp} \leq 2n\}.$$

Defining $B_n(U/Sp) = F_n(U/Sp) - F_{n-1}(U/Sp)$, we will prove that $B_n(U/Sp)$ is distinguished.

Lemma 5.17. *$p^{-1}(B_n(U/Sp) \to B_n(U/Sp)$ is a Serre fibration*

Proof. This follows the same procedure as in the preceding sections. We start with the following commutative square

$$\begin{array}{ccc} \{0\} \times I^k & \xrightarrow{\alpha} & p^{-1}(B_n(U/Sp)) \\ \downarrow & & \downarrow \\ I^{k+1} & \xrightarrow[\beta]{} & B_n(U/Sp) \end{array}$$

We wish to find a lift of this diagram, i.e. a map $\gamma : I^{k+1} \to p^{-1}(B_n(U/Sp))$ making the triangles in the diagram commute.

By compactness, there exist a finite dimensional $W \subset \mathcal{U}$ such that the diagram factors as

$$\begin{array}{ccccc} \{0\} \times I^k & \xrightarrow{\alpha'} & E(W) \cap p^{-1}(B_n) & \longrightarrow & p^{-1}(B_n) \\ \downarrow & & \downarrow & & \downarrow \\ I^{k+1} & \xrightarrow[\beta']{} & U/Sp(W) \cap B_n & \longrightarrow & B_n \end{array}$$

Let $A(0, t_1, ..., t_k) = \alpha'(t_1, ..., t_k)$ and $X(t_0, t_1, ..., t_k) = \beta'(t_0, ..., t_k)$ for a special representative X of $U/Sp(W)$. For $t \in I^k, I^{k+1}$ respectively, we may write the spectral decomposition of A and X as

$$A(t) = \pi_{W_0(t)} + \sum_l \mu_l(t) \pi_{W_l(t)},$$

$$X(t) = \pi_{V(t)} - \pi_{V'(t)} + \sum_l (\lambda_l(t) \pi_{V_l(t)} - \lambda_l(t) \pi_{V_l'(t)}),$$

And in addition

$$X^2(t) = \pi_{V'(t) \oplus V''(t)} + \sum_l \lambda_l^2 \pi_{V_l'(t) \oplus V_l''(t)}$$

which means $-e^{2\pi i\mu_l(t)} = \lambda_l^2(t)$, $W_0(t) \subseteq V(t) \oplus V'(t)$, and $W_l(t) = V_l(t) \oplus V_l'(t)$ when $t \in I^k$.

Consider the following subspace of $W \oplus W$ for an m-dimensional symplectic subspace W of $\mathcal{U}$:

$$\mathrm{Perp}_{i,j}(W \oplus W) = \{(V', V'') | V', V'' \subseteq W, V' \perp V'',$$

$$\dim_{\mathbb{H}} V' = i, \dim_{\mathbb{H}} V'' = j\}$$

We may characterize this space by considering the symplectic group over $W \oplus W$. We get all possible V' and V'' by letting $Sp(W \oplus W)$ act on one pair (V', V'') by left multiplication. That means $Sp(W \oplus W)$ acts transitively on $\mathrm{Perp}_{i,j}(W \oplus W)$. Since we know the dimension of $W \oplus W$ to be $2m$, we may denote $Sp(W \oplus W)$ by Sp_{2m}. The stabilizer is given by the symplectic matrices that can be decomposed into a symplectic matrix that acts on V', a symplectic matric that acts on V'', and a symplectic matrix that acts on $(W \oplus W - (V' \oplus V''))$. Thus the stabilizer is $Sp_i \oplus Sp_j \oplus Sp_{2m-i-j} \cong Sp_i \times Sp_j \times Sp_{2m-i-j}$. We therefore have

$$\mathrm{Perp}_{i,j}(W \oplus W) \cong Sp_{2m}/(Sp_i \times Sp_j \times Sp_{2m-i-j})$$

We have a natural map $P : \mathrm{Perp}_{i,j}(W \oplus W) \to BSp_{i+j}(W \oplus W)$ given by $P(V', V'') = V' \oplus V''$. We characterize $BSp_{i+j}(W \oplus W)$ in a similar way. We have that Sp_{2m} acts transitively on $BSp_{i+j}(W \oplus W)$, with stabilizer the symplectic matrices that can be decomposed into the direct sum of a symplectic matrix that acts on $V \in BU_{i+j}(W)$, and a symplectic matrix that acts on $(W \oplus W) - V$. That is, a symplectic matrix of the form $Sp_{i+j} \oplus Sp_{2m-i-j} \cong Sp_{i+j} \times Sp_{2m-i-j}$. Thus $BU_{2n}(W) \cong U_m/(U_{2n} \times U_{m-2n})$. Since $\mathrm{Perp}_{(i+j),0} \cong BSp_{i+j}$, P is a projection, so $P : \mathrm{Perp}_{i,j}(W \oplus W) \to BSp_{i+j}(W \oplus W)$ is a fibration. That means we can find a lift ω'' to the following commutative diagram.

$$\begin{array}{ccc} \{0\} \times I^k & \xrightarrow{\alpha''} & \mathrm{Perp}_{i,j}(W \oplus W) \\ \downarrow & \overset{\omega''}{\nearrow} & \downarrow P \\ I^{k+1} & \xrightarrow[\beta'']{} & BSp_{i+j}(W \oplus W) \end{array}$$

Let $i = \dim W_0(0)$, and $j = \dim W_1(0)$ i.e the dimension of the eigenspaces of $A(0)$ corresponding to $\mu = 1$ and $\mu = 0$ respectively. Let $\alpha'' : I^k \to \mathrm{Perp}_{i,j}(W \oplus W)$ be given by $\alpha''(t) = (W_0(t), W_1(t))$ and let $\beta : I^{k+1} \to BSp_{i+j}(W \oplus W)$ be given by $\beta''(t) = V(t) \oplus V'(t)$, where we have $W_0(t) \oplus W_1(t) = V(t) \oplus V'(t)$. Here we consider $V(t) \oplus V'(t)$ as a single subspace of $W \oplus W$, "forgetting" its decomposition. We can do this since the coset is only dependent on the square of a special representative, which means any decomposition of $V(t) \oplus V'(t)$ will result in a special representative of the same coset. Define $\omega''(t) = (W'(t), (V(t) \oplus V'(t)) - W'(t))$, where $W'(t)$ is obtained from $W_0(t)$ by a homotopy. Let $\mu_l(t) \in (0,1)$ be the unique solution to $e^{2\pi i \mu_l(t)} = \lambda_l^2(t)$. We can now define $\omega' : I^{k+1} \to E(W) \cap p^{-1}(B_n)$ by

$$\omega'(t) = \pi_{W'(t)} - 0\pi_{(V(t)\oplus V'(t))-W'(t)} + \sum_l \mu_l(t)\pi_{V_l(t)}$$

and by inclusion, we obtain a lift ω to our original diagram. $\square$

We now check the last two requirements of theorem 2.2.
Define a neighborhood N_n of F_{n-1} in F_n as

$$N_n = \{\, [X] \in F_n U \mid X \text{ is a special representative, } \dim_{\mathbb{C}} \mathrm{Eig}_{e^{2\pi i[1/3,2/3]}} X^2 < 2n\},$$

where $\mathrm{Eig}_S X$ denote the direct sum of eigenspaces corresponding to eigenvalues in S.

We are going to deform X^2 such that the eigenvalues inside the range $e^{2\pi i[1/3,2/3]}$ will correspond to all non-unit eigenvalues of the deformed matrix, while the remaining eigenvalues will correspond to eigenvalue 1. That way, the resulting matrix will be a special representative of $[X] \in F_{n-1}$ as theorem 2.2 requires. Define $f : I \to I$ by

$$f(x) = \begin{cases} 1, & x \geq \frac{2}{3} \\ 3x - 1, & \frac{1}{3} \leq x \leq \frac{2}{3} \\ 0, & x \leq \frac{1}{3} \end{cases}$$

We note that $f \simeq \mathrm{Id}$ rel ∂I. This ensures that the eigenvalue 1 will not be deformed to a non-unit eigenvalue. Let H be such a homotopy, for example by $H(x,t) = t(f(x)) + (1-t)x$. It follows that there exists an $h : S^1 \times I \to S^1$ that makes the following diagram commute:

$$\begin{array}{ccc} I & \xrightarrow{H_t} & I \\ {\scriptstyle e^{2\pi i(\cdot)}}\downarrow & & \downarrow{\scriptstyle e^{2\pi i(\cdot)}} \\ S^1 & \xrightarrow[h_t]{} & S^1 \end{array}$$

For $A \in E$ of the form $A = \sum_i \mu_i \pi_{W_i}$, we define a new hermitian matrix $H_t(A)$ where $t \in I$

$$H_t(A) = \sum_i H_t(\mu_i)\pi_{W_i}.$$

similarly, we define $h_t : U \to U$ by

$$h_t(X) = \sum_i h_t(\lambda_i)\pi_{W_i} = \sum e^{2\pi i H_t(\mu_i)}\pi_{W_i}.$$

Note that $h_t : N_n \to N_n$ satisfy $h_0 = \mathrm{Id}$ and $h_1(N_n) \subseteq F_{n-1}U$. In addition, h_t is covered by $H_t : p^{-1}(N_n) \to p^{-1}(N_n)$, where $H_0 = \mathrm{Id}$. What remains, is to show that the induced map on the fibers: $H_1 : p^{-1}(X) \to p^{-1}(h_1(X))$ is a weak equivalence, and we have proven that p is a quasifibration.

By the construction of H_t, $\ker(X^2 - I) \subseteq \ker(H_1(X^2) - I)$. This means that to prove p being a quasifibration is reduced to proving the following lemma:

Lemma 5.18. *Suppose we have finite dimensional quaternionic spaces $V \subseteq V' \subseteq W \oplus W$. Let $V'' \subseteq V' - V$. Then the map $\overline{BSp}_{V,W} \to \overline{BSp}_{V',W}$ given by sending X to $X \oplus V''$ is a homotopy equivalence.*

Proof. We have that both $\overline{BSp}_{V,W}$ and $\overline{BSp}_{V',W}$ are congruent to $BSp \times \mathbb{Z}$. In ([8], 1.2), if we consider $\mathrm{Vect}^n_{\mathbb{H}}(C)$ instead of $\mathrm{Vect}^n_{\mathbb{C}}(C)$, we will get the quaternionic analogue of the formula given in lemma 4.6. We have that $\widetilde{K}_{\mathbb{H}}(C) \cong [C, BSp \times \mathbb{Z}]$, for any pointed compact space C. Where $[A, B]$ denotes the homotopy classes of maps from A to B.

We therefore get an induced map

$$\widetilde{K}_{\mathbb{H}}(C) \cong [C, \overline{BSp}_{V,W}] \to [C, \overline{BSp}_{V',W}] \cong \widetilde{K}_{\mathbb{H}}(C).$$

This means any coset representative is mapped to another representative of the same coset in $K_{\mathbb{H}}(C)$. Since $V \subseteq V'$, the map has to be addition of a trivial bundle, so the map is an isomorphism. In particular, for $C = S^i$, we get an isomorphism of homotopy groups. □

5.4 $\Omega Sp \simeq Sp/U$

Let $\mathcal{U} \cong \mathbb{H}^\infty$ be a countably infinite dimensional quaternionic inner product space. For finite dimensional $W \subset \mathcal{U}$, let $Sp(W)$ be the space of quaternionic isometries of W. We have a map $Sp(V) \to Sp(W)$ given by sending X to $X \oplus I_{W-V}$ for $V \subseteq W$. We may therefore take the direct limit: $Sp = \varinjlim_W Sp(W)$.

Define

$$E(W) = \{\, A \in H(W) \mid \mu_l \in [-1,1]\ \forall l,\ Aj = -jA\},$$

where $H(W)$ denotes the space of all complex linear hermitian transformations of W, and μ_l denotes the eigenvalues of A. Note that E(W) is contractible by the contracting homotopy, which holds since if $Aj = -jA$, and t is real, $(tA)j = -j(tA)$ Define a map $p_W : E(W) \to Sp(W)$ as $p_W(A) = -e^{\pi i A}$.

We will now find a nice representation for $Sp/U(W)$.

Lemma 5.19. *Let $W \subset \mathcal{U}$ be a finite dimensional quaternionic subspace. Then there is an isomorphism*

$$Sp/U(W) \cong \{\, V \mid V \text{ is a complex subspace of } W,\ W = V \oplus jV\}.$$

Proof. We can recognize V as a complex Lagrangian subspace of W. This is because we have that $U(2n) \cap Sp(2n, C) = Sp(n)$. Therefore we may define a symplectic form on W viewed as a $2n$ -dimensional complex space. On the complex space, j corresponds to a quaternionic structure. We choose the quaternionic structure J to represent j. We may therefore define the symplectic form to be (v, Jw) where $(\cdot,\cdot)$ is the standard complex inner product. Since $v \perp Jv$ for all v, we have that $(v_1, Jv_2) = 0$ for all $v_1, v_2 \in V$. V is therefore a complex Lagrangian subspace. We know that $U(2n)$ acts transitively on W viewed as a complex space, so the subspace of $U(2n)$ that preserves the symplectic form has to act transitively on the space in question. But such a space is precisely $Sp(n)$. To find the stabilizer, we look for the transformations og $V \oplus jV$ that induces automorphisms on V and jV. Since V is complex, such tranformations must be the group $U(W)$. The orbit-stabilizer theorem tells us that the space is congruent to $Sp/U(W)$. □

We now proceed to identify the fiber of p_W.

Lemma 5.20. *Let $W \subset \mathcal{U}$ be a finite dimensional quaternionic subspace. For $X \in Sp(W), p_W^{-1}(X) \cong Sp/U(\ker(X - I))$.*

Proof. For $A \in E(W)$, we find its spectral decomposition

$$A = \pi_{W'} - \pi_{W''} + \sum_l (\mu_l \pi_{W'_l} - \mu_l \pi_{W''_l}),$$

Where $\mu_l \in (0, 1)$. In addition, assume $Av = \mu v$. Then $Ajv = -jAv = -j\mu v = -\mu jv$. This means that $jW' = W''$ and $jW'_l = W''_l$.

Since $Sp_n \subseteq U_{2n}$, X inherits a spectral decomposition:

$$X = \pi_V - \pi_{V'} + \sum_l (\lambda_l \pi_{V'_l} + \overline{\lambda}_l \pi_{V''_l}),$$

Where $|\lambda_l| = 1$, and $\mathrm{Im}(\lambda_l) < 0$. Assume $Xv = \lambda v$. Then $Xjv = jXv = j\lambda v = \overline{\lambda} jv$. This means that $jV'_l = V''_l$, and in particular, since real numbers equal their complex conjugate, you get $jV = V$ and $jV' = V'$. This means that V and V' are quaternionic subspaces of W.

Therefore, $p_W(A) = X$ if and only if $W' \oplus W'' = V = \ker(X - I)$, $W'_l = V'_l$, $W''_l = V''_l$ and $\mu_l \in (0, 1)$ are the unique solutions in that range to $-e^{\pi i \mu_l} = \lambda_l$. We may therefore define a map $\phi : p_W^{-1}(X) \to Sp/U(\ker(X - I))$, given by $\phi(A) = W'$, since we have that $W' \oplus jW'' = \ker(X - I)$. We find an inverse $\psi : Sp/U(\ker(X - I)) \to p_W^{-1}(X)$ given by $\psi(V) = \pi_V - \pi_{(jV)} + \sum_l (\mu_l \pi_{V'_l} - \mu_l \pi_{V''_l})$. □

We would like to have a map $E(V) \to E(W)$ given by sending A to $A \oplus \pi_{W-V}$ as we have done for hermitian matrices up until now. However, $A' = A \oplus \pi_{W-V}$ will not satisfy $A'j = -jA'$. Therefore we modify the following way. For Y quaternionic space, let $Y^{\mathbb{C}} = \{v \in Y \mid iv = vi\}$. One can verify that each entry of v is of the form $a + bi$. We define a map $E(V) \to E(W)$ by sending A to $A \oplus \pi_{(W-V)^{\mathbb{C}}}$. This means that the eigenspace corresponding to $\mu = 1$ is the space $W' \oplus (W - V)^{\mathbb{C}}$. We then have that the eigenspace corresponding to $\mu = -1$ is the space $W'' \oplus j(W - V)^{\mathbb{C}}$. One can verify that $W - V =$

$(W-V)^{\mathbb{C}}+j(W-V)^{\mathbb{C}}$, which means $X \oplus \pi_{W-V} \in Sp(W)$. We now take the direct limit over all $W \subset \mathcal{U}$, which yields $p: E \to Sp$.

For $V \subseteq W \subset \mathcal{U}$, define

$$\overline{Sp/U}_{V,W} = \lim_{\to} {}_{W' \geq W} Sp/U(V \oplus (W'-W))$$

Completely analogous to previous sections, we get that $\overline{Sp/U}_{V,W} \cong Sp/U$, $p^{-1}(X) = \overline{Sp/U}_{\ker(X-I),W}$, and therefore $p^{-1}(X) \cong Sp/U$.

We now prove that $p: E \to Sp$ is a quasifibration. We start by defining a filtration on Sp. Note that since $V_l' = jV_l''$, $V_l' \oplus V_l''$ can be thought of as a quaternionic subspace of W, with quaternionic dimension equal to the complex dimension of W_l'. Therefore, we may define the filtration as:

$$F_n Sp = \{ X \in Sp \mid \dim_{\mathbb{H}} \ker(X-I)^{\perp} \leq n\}.$$

Define $B_nSp = F_nSp - F_{n-1}Sp$. We start by proving that B_nSp is distinguished:

Lemma 5.21. *$p^{-1}(B_nSp) \to B_nSp$ is a Serre fibration.*

Proof. We start with the following commutative square

$$\begin{array}{ccc} \{0\} \times I^k & \xrightarrow{\alpha} & p^{-1}(B_n) \\ \downarrow & & \downarrow \\ I^{k+1} & \xrightarrow[\beta]{} & B_n \end{array}$$

We wish to find a lift of this diagram, i.e. a map $\gamma: I^{k+1} \to p^{-1}(B_n)$ making the triangles in the diagram commute.

By compactness, there exists a finite dimensional $W \subset \mathcal{U}$ such that the diagram factors as

$$\begin{array}{ccccc} \{0\} \times I^k & \xrightarrow{\alpha'} & E(W) \cap p^{-1}(B_n) & \longrightarrow & p^{-1}(B_n) \\ \downarrow & & \downarrow & & \downarrow \\ I^{k+1} & \xrightarrow[\beta']{} & Sp(W) \cap B_n & \longrightarrow & B_n \end{array}$$

Let $A(0,t_1,...,t_k) = \alpha'(t_1,...,t_k)$ and $X(t_0,t_1,...,t_k) = \beta'(t_0,...,t_k)$. For $t \in I^k, I^{k+1}$ respectively, we may write the spectral decomposition of A and X as

$$A(t) = \pi_{W'(t)} - \pi_{W''(t)} + \sum_l (\mu_l(t)\pi_{W_l'(t)} - \mu_l(t)\pi_{W_l''(t)}),$$

$$X(t) = \pi_{V(t)} - \pi_{V'(t)} + \sum_l (\lambda_l(t)\pi_{V_l'(t)} + \overline{\lambda}_l(t)\pi_{V_l''(t)}),$$

where $-e^{\pi i \mu_l(t)} = \lambda_l(t)$, $W'(t) \oplus W''(t) = V(t)$, $W_l'(t) = V_l'(t)$ and $W_l''(t) = V_l''(t)$ when $t \in I^k$. Consider the following space given an m-dimensional quaternionic subspace W of $\mathcal{U}$:

$$\begin{aligned}\mathrm{Perp}_{n,n}(W) = \{\, (V', V'') \mid\ & V', V'' \text{ are complex subspaces of } W,\ V' = jV'', \\ & \dim_{\mathbb{C}} V' = \dim_{\mathbb{C}} V'' = n\}\end{aligned}$$

We may characterize this space by considering the symplectic group over W. $Sp(W)$ acts transitively on itself, so we get all possible V' and V'' by applying all symplectic transformations on one pair of (V', V''). The stabilizer is the space of symplectic matrices that induces automorphisms on V' and V'' simultaneously. Such a symplectic matrix is of the form $U_n \oplus U_n \oplus Sp_{m-n}$. That means that

$$\mathrm{Perp}_{n,n}(W) \cong Sp_m / (U_n \times U_n \times Sp_{m-n})$$

We have a natural map

$$P : \mathrm{Perp}_{n,n}(W) \to BSp_n(W)$$

by $P(V', V'') = V' \oplus V''$. Where we found earlier that $BSp_n(W) \cong Sp_m/(Sp_n \times Sp_{m-n}$. We have that $U_n \oplus U_n \subseteq Sp_n$. To see this, decompose the quaternion $a + bi + cj + dk$ into $(a + bi) + (c + di)j$ (for the sake of illustration, we use the dot product as our inner product, and the standard norm). We let U act on $a + bi$ which gives us a new complex number $e + fi$ with the property that $e^2 + f^2 = a^2 + b^2$. We then let U' act on $c + di$, which gives us $g + hi$, such that $c^2 + d^2 = g^2 + h^2$. Combining these sends $a + bi + (c + di)j$ to $e + fi + (g + hi)j$ such that $a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + g^2 + h^2$. This is therefore a quaternionic isometry. This means that $BSp_n(W) \subseteq \mathrm{Perp}_{n,n}(W)$. We therefore have that P is a projection, which makes it a fibration. That means we can find a lift ω'' to the following commutative diagram.

$$\begin{array}{ccc} \{0\} \times I^k & \xrightarrow{\alpha''} & \mathrm{Perp}_{n,n}(W) \\ \downarrow & \overset{\omega''}{\nearrow} & \downarrow P \\ I^{k+1} & \xrightarrow[\beta'']{} & BSp_n(W) \end{array}$$

Let $n = \dim W'(0)$. Let $\alpha'' : I^k \to \mathrm{Perp}_{n,n}(W)$ be given by $\alpha''(t) = (W'(t), V'(t) - W'(t))$ and let $\beta : I^{k+1} \to BSp_n(W \oplus W)$ be given by $\beta''(t) = V'(t)$. Since $V'(t) \in BU_n\ \forall t$, we have that $V'(t)$ has constant dimension. Therefore, we may define $\omega''(t) = (W'(t), V'(t) - W'(t))$, where $W'(t)$ is obtained from $W'(t)$ by a homotopy. Let $\mu_l(t) \in (0, 1)$ be the unique solution to $-e^{\pi i \mu_l(t)} = \lambda_l(t)$. We can now define $\omega' : I^{k+1} \to E(W) \cap p^{-1}(B_n)$ by

$$\omega'(t) = \pi_{W'(t)} - \pi_{V'(t) - W'(t)} + \sum_l (\mu_l(t) \pi_{V_l'(t)} - \mu_l(t) \pi_{V_l''(t)})$$

and by inclusion, we obtain a lift ω to our original diagram. □

We define a neighborhood N_n of $F_{n-1}Sp$ in F_nSp given by

$$N_n = \{ X \in F_nSp \mid \dim_{\mathbb{H}} Eig_{-e^{\pi i[-1/2,1/2]}} X < n \}.$$

In oder to find a homotopy deforming N_n so that it lies inside $F_{n-1}Sp$, we define a function $f : [-1,1] \to [-1,1]$

$$f(x) = \begin{cases} -1, & x \le -\frac{1}{2} \\ 2x, & -\frac{1}{2} \le x \le \frac{1}{2} \\ 1, & x \ge \frac{1}{2} \end{cases}$$

Then $f \simeq \mathrm{Id}\ \mathrm{rel}\{-1,1\}$. $H(x,t) = t \cdot f + (1-t) \cdot x$ is a homotopy that satisfies this. Let $h : S^1 \times I \to S^1$ be defined so that the following square commutes for all $t \in I$:

$$\begin{array}{ccc} [-1,1] & \xrightarrow{H_t} & [-1,1] \\ \downarrow{\scriptstyle -e^{\pi i(\cdot)}} & & \downarrow{\scriptstyle -e^{\pi i(\cdot)}} \\ S^1 & \xrightarrow[h_t]{} & S^1 \end{array}$$

The induced homotopy $h : N \times I \to N$, given by sending the eigenvalues $\lambda_l = -e^{\pi i \mu_l}$ to $-e^{\pi i H_t(\mu_l)}$ deforms N_n into a subspace of $F_{n-1}Sp$ for all n, and is covered by a homotopy $H : p^{-1}(N) \times I \to p^{-1}(N)$, as was required by theorem 2.2.

It remains to check that H_1 induces weak equivalences on fibers, that is, that $H_1 : p^{-1}(X) \to p^{-1}(h_1(X))$ is a weak equivalence. Following the same argumentation leading up to lemma 4.6, this boils down to proving the following lemma.

Lemma 5.22. *Let $V \subseteq V'$ be quaternionic subspaces of a finite dimensional quaternionic subspace $W \subset \mathcal{U}$. Then the map $f : \overline{Sp/U}_{V,W} \to \overline{Sp/U}_{V',W}$ given by sending A to $A \oplus L$ for a complex matrix L such that $L \oplus jL \cong V' - V$ is a homotopy equivalence.*

Proof. since $\overline{Sp/U}_{V,W} \cong Sp/U \subseteq Sp$, we can represent an element A in $\overline{Sp/U}_{V,W}$ by an element S in Sp, while we can choose L such that its representative in Sp is $I_{V'-V}$. Consider the following commutative square:

$$\begin{array}{ccc} \varinjlim_{W' \ge W}(V \oplus (W' - W)) & \xrightarrow{\phi} & \varinjlim_{W' \ge W}(V' \oplus (W' - W)) \\ \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S \oplus I_{V'-V}} \\ \varinjlim_{W' \ge W}(V \oplus (W' - W)) & \xrightarrow[\phi]{} & \varinjlim_{W' \ge W}(V' \oplus (W' - W)) \end{array}$$

Connecting the notation to lemma 2.4, we have $f = \phi_*$ and $S \oplus I_{V'-V} = \phi S \phi^{-1} \oplus I_{W-V} = f(S) = \phi_*(S)$. From the second equality, we get that ϕ is a symplectic matrix and therefore an isometry. Since S and $I_{V'-V} \in Sp$, we can use lemma 2.4 to conclude that f is a homotopy equivalence. In particular, f is a weak homotopy equivalence. □

5.5 $\Omega Sp/U \simeq U/O$

Let $\mathcal{U} \cong \mathbb{H}^\infty$ be a an infinite dimensional quaternionic space equipped with a real inner product such that multiplication with i and j are real isometries. Let $W \subset \mathcal{U}$ be a right quaternionic subspace, that is, a vector space where the scalars are multiplied on the right. Then we may view $Sp(W)$ as the real linear isometries X of W that for $v \in W$ and $\alpha \in \mathbb{H}$ satisfy $X(v\alpha) = (Xv)\alpha$. Since complex numbers are commutative, we want the complex isometries of W to be the real isometries that is right quaternion linear, as well as left complex linear. So $U(W)$ is a subspace of $Sp(W)$ that satisfies $X(iv) = i(Xv)$. Let $W^{\mathbb{R}}$ be the real subspace of W. More precisely: $W^{\mathbb{R}} = \{ v \mid vi = iv \text{ and } vj = jv\}$. This lets us describe quarternions as multiplying elements in $W^{\mathbb{R}}$ with $1, i, j$ and k. We are going to use this to find an expression of $\mathfrak{sp}(W)$, the Lie algebra of $Sp(W)$, that consists of quaternionic matrices that satisfies $A = A^*$, where $*$ denotes quaternion conjugate transpose. Such a matrix must be a skew-symmetric tranformation on the real part of W, and a symmetric transformation on each of the imaginary parts of W. Therefore:

$$\mathfrak{sp}(W) = \mathfrak{o}(W^{\mathbb{R}}) \oplus iS(W^{\mathbb{R}}) \oplus jS(W^{\mathbb{R}}) \oplus kS(W^{\mathbb{R}})$$

Where $S(X)$ denotes symmetric limear transformations of a space X. $\mathfrak{u}(W)$ is the Lie subalgebra of $\mathfrak{sp}(W)$ of the form $\mathfrak{o}(W^{\mathbb{R}}) \oplus iS(W^{\mathbb{R}})$. Since we are working with $Sp/U(W)$, we are interested in the transformations of the form $jS(W^{\mathbb{R}}) \oplus kS(W^{\mathbb{R}})$. Define

$$E(W) = \{ jB + kC \mid \mu_l^B, \mu_l^C \in [-1, 1] \ \forall l\} \subseteq jS(W^{\mathbb{R}}) \oplus kS(W^{\mathbb{R}}),$$

Where μ_l^B and μ_l^C are the eigenvalues of B and C respectively. We have a map $p_W : E(W) \to Sp/U(W)$ given by $p_W(A) = [je^{\frac{1}{2}\pi A}]$. We now find a nice representation for $U/O(W)$.

Lemma 5.23. *Let W be a finite quaternionic space with a real inner product. Then there is an isomorphism*

$$U/O(W) \cong \{ V \mid V \text{ is a right complex subspace of } W,\ W = V \oplus iV = V \oplus Vj\}.$$

Proof. We recognize this space as the (real) Lagrangian Grassmannian of W. Since we view i and j as real isometries, we should view W as a $4n$-dimensional real vector space, which makes V a $2n$-dimensional real subspace. We may now choose two real symplectic forms on W corresponding to i and j respectively, the first being $(v, J_1 w)$ and the second being (v, wJ_2), where J_1, J_2 are quaternionic structures corresponding to i and j respectively, and $(\cdot, \cdot)$ denotes the standard real inner product. Now, for any two $v_1, v_2 \in V$, $(v_1, J_1 v_2) = 0$ and $(v_1, v_2 J_2) = 0$. It follows that V is a real Lagrangian transformation, so the space is the Lagrangian Grassmannian of W. It has been proven by for example [14] that the real Lagrangian Grassmannian is isomorphic to $U/O(W)$. Therefore, so is our space. $\square$

For $V \subseteq W$ we have a map $U/O(V) \to U/O(W)$ given by $[A] \mapsto [A \oplus I_{W-V}]$, or equivalently, by the isomorphism in lemma 5.23: $V \mapsto V \oplus L$, where L is a fixed complex space such that $L \oplus iL = L \oplus Lj = W - V$, and corresponds to $[I_{W-V}]$ under the isomorphism. We now have that $U/O = \lim_{\to} {}_W U/O(W)$.

We now aim to understand the coset representatives of $Sp/U(W)$. Analogous to section 5.2 and 5.3, we have the following two lemmas:

Lemma 5.24. *Let $W \subset \mathcal{U}$ be a finite right quaternionic subspace. Suppose $A \in \mathfrak{sp}(W)$ has the property $Ai = -iA$, then $X = e^A$ has the property $Xi = iX^{-1}$*

Proof.

$$-iXi = -ie^A i = e^{-iAi} = e^{-(A)} = X^{-1}$$

□

Where i is thought of as a linear transformation with $-i$ as its inverse.

Lemma 5.25. *Let $W \subset \mathcal{U}$ be a finite right quaternionic subspace. If $Y, Z \in Sp(W)$ has the property $-iYi = Y^{-1}$ and $-iZi = Z^{-1}$, then there exists an $X \in U(W)$ such that $jY = XZ$ if and only if $-Y^2 = Z^2$.*

Proof. Suppose there is an $X \in U(W)$ such that $jY = XZ$. We have that

$$iY^{-1} = i(Z^{-1}X^{-1}j) = ZiX^{-1}j = ZX^{-1}ij = -(ZX^{-1}j)i$$

and

$$Yi = -(jXZ)i$$

Since $iY^{-1} = Yi$, we have that $-ZX^{-1}ji = -jXZi$, which means $ZX^{-1}j = jXZ$. We have that $(jX)^{-1} = -X^{-1}j$, so writing $C = jX$, where we note that $Y = -CZ$ we get $-ZC^{-1} = CZ$, which means $Z = -CZC$, so $Z^2 = -CZCZ = -(-CZ)(-CZ) = -Y^2$

Conversely, suppose $-Y^2 = Z^2$. Then $Y = -(Y^{-1}Z)Z$, so $jY = -j(Y^{-1}Z)Z$. We therefore show that $-jY^{-1}Z \in U(W)$. But $jY = -j(Y^{-1}Z)Z$ implies $jYZ^{-1} = -j(Y^{-1}Z)$, so $-jY^{-1}Zi = i(jYZ^{-1}) = i(-jY^{-1}Z)$. Therefore $-jY^{-1}Z \in U(W)$. □

This means $p_W(A) = [X]$ if and only if $X^2 = -e^{\pi A}$, so we call such an X a special representative. The following lemma shows that any coset has a special representative.

Lemma 5.26. *Every $[X] \in Sp/U(W)$ has a special representative.*

Proof. $Sp/U(W)$ is geodesically complete by the same logic as in lemma 5.9, and the geodesics γ of $Sp/U(W)$ is given by $\gamma(t) = [Ye^{tB}]$ for $Y \in Sp(W)$ and $B \in \mathfrak{u}(W)^{\perp}$, which yields the special representatives. □

We now determine the fiber of p_W.

Lemma 5.27. *Let $W \subset \mathcal{U}$ be a finite dimensional right quaternionic subspace. For a special representative X of $[X] \in Sp/U(W)$, we have $p_W^{-1}([X]) \cong U/O(\ker(X^2 - I))$.*

Proof. Let $A \in E(W)$. We find its spectral decomposition, which is given by eigenvalues in the range $[-i, i]$. As A is a right skew-hermitian matrix, we make sure to write the eigenvalues on the right.

$$A = \pi_{W'} i - \pi_{W''} i + \sum_l (\pi_{W'_l} i\mu_l - \pi_{W''_l} i\mu_l),$$

Where $\mu_l \in (0,1)$ and W', W'', W'_l and W''_l are right complex spaces. Notice that if $Av = vi\mu$, then $A(iv) = -iAv = -ivi\mu = iv(-i\mu)$, and if $A(vj) = (Av)j = vi\mu j = vj(-i\mu)$. This means that $iW' = W''$, $W'j = W''$, $iW'_l = W''_l$, and $W'_l j = W''_l$.

Similarly, write the spectral decomposition of the special representative X, being a right quaternionic transformation, as

$$X = \pi_{V'} - \pi_{V''} + \pi_{V'_0} i - \pi_{V''_0} i + \sum_l (\pi_{V'_l} \lambda_l + \pi_{V''_l} \overline{\lambda_l} - \pi_{\widetilde{V}'_l} \lambda_l - \pi_{\widetilde{V}''_l} \overline{\lambda_l}),$$

where $|\lambda_l| = 1$, and λ_l sitting in the second quadrant of the unit circle, i.e. $\mathrm{Im}(\lambda^2) < 0$ and $\mathrm{Im}(\lambda) > 0$, and a priori, $V', V'', V'_l, V''_l, \widetilde{V}'_l$ and $\widetilde{V}''_l$ are right complex vector spaces. However, we have that if $Xv = v\lambda$, then $Xiv = -iX^{-1}v = iX^*v = iv\overline{\lambda}$, and $Xvj = v\lambda j = vj\overline{\lambda}$. This means that $iV' = V' = V'j$, $iV'' = V'' = V''j$, $iV'_0 = V''_0$, $V'_0 j = V''_0$, $iV'_l = V''_l$, $V'_l j = V''_l$, $i\widetilde{V}'_l = \widetilde{V}''_l$ and $\widetilde{V}'_l j = \widetilde{V}''_l$. In particular, V' and V'' are in fact quaternionic spaces.

We also compute the spectral decomposition for X^2:

$$X^2 = \pi_{V' \oplus V''} - \pi_{V'_0 \oplus V''_0} + \sum_l (\pi_{V'_l \oplus \widetilde{V}'_l} \lambda_l^2 + \pi_{V''_l \oplus \widetilde{V}''_l} \overline{\lambda_l}^2).$$

Now, if $A \in p_W^{-1}([X])$, then we must have $-e^{\pi i A} = X^2$, which means $\mu_l \in (0,1)$ is the unique solution to $-e^{\pi i \mu_l} = \lambda_l^2$, $W' \oplus W'' = V' \oplus V'' = \ker(X^2 - I)$, $W'_l = V'_l \oplus \widetilde{V}'_l$, and $W''_l = V''_l \oplus \widetilde{V}''_l$.

We therefore have a map $\phi : p_W^{-1}([X]) \to U/O(\ker(X^2 - I))$, with $\phi(A) = W'$, with an inverse $\psi(V) = \pi_{W'} i - \pi_{iW'} i + \sum_l (\pi_{V'_l \oplus \widetilde{V}'_l} i\mu_l - \pi_{V''_l \oplus \widetilde{V}''_l} i\mu_l)$ □

Similar to the previous instances, for $V \subseteq W$, we want a map $i_{V,W} : E(V) \to E(W)$ such that $W - V$ will be part of the eigenspace of X^2 corresponding to eigenvalue $\lambda^2 = 1$, i.e to the eigenspace of A corresponding to the eigenvalues $\mu = i$ and $\mu = -i$. But we also want that $iW' = W'' = W'j$. $W' = (k+1)(W - V)^{\mathbb{R}}$ and $W'' = (i-j)(W-V)^{\mathbb{R}}$ satisfies this, since for $V \in (W-V)^{\mathbb{R}}$ $i(k+1)v = (-j+i)v \in W''$ and $(k+1)vj = (k+1)jv = (-i+j)v = (i-j)(-v) \in W''$. Therefore we define

$$i_{V,W}(A) = A \oplus (\pi_{(k+1)(W-V)^{\mathbb{R}}} i - \pi_{(i-j)((W-V)^{\mathbb{R}}} i).$$

Taking direct limits over all $W \subset \mathcal{U}$, we get a map $p : E \to Sp/U$. We show this this is a quasifibration with fiber U/O.

First we define the following space for $V \subseteq W$:

$$\overline{U/O}_{V,W} = \lim_{\rightarrow} {}_{W' \geq W} U/O(V \oplus (W' - W))$$

Completely analogous to previous sections, we get that $\overline{U/O}_{V,W} \cong U/O$, as well as $\overline{U/O}_{\ker(X^2-I),W} \cong p^{-1}(W)$ which establishes U/O as the fiber of p.

We define the following filtration on Sp/U:

$$F_n(Sp/U) = \{\, [X] \mid X \text{ is a special representative, } \dim_{\mathbb{C}} \ker(X^2 - I)^{\perp} \leq 2n\},$$

and we let $B_n(Sp/U) = F_n(Sp/U) - F_{n-1}(Sp/U)$.

We prove distinguishness the same way as the previous sections:

Lemma 5.28. *$p^{-1}(B_n(Sp/U)) \to B_n(Sp/U)$ is a Serre fibration.*

Proof. This follow the same procedure as in the preceding sections. We start with the following commutative square

$$\begin{array}{ccc}
\{0\} \times I^k & \xrightarrow{\alpha} & p^{-1}(B_n(Sp/U)) \\
\downarrow & & \downarrow \\
I^{k+1} & \xrightarrow[\beta]{} & B_n(Sp/U)
\end{array}$$

We wish to find a lift of this diagram, i.e. a map $\gamma : I^{k+1} \to p^{-1}(B_n(Sp/U))$ making the triangles in the diagram commute.

By compactness, there exist a finite dimensional $W \subset \mathcal{U}$ such that the diagram factors as

$$\begin{array}{ccccc}
\{0\} \times I^k & \xrightarrow{\alpha'} & E(W) \cap p^{-1}(B_n) & \longrightarrow & p^{-1}(B_n) \\
\downarrow & & \downarrow & & \downarrow \\
I^{k+1} & \xrightarrow[\beta']{} & Sp/U(W) \cap B_n & \longrightarrow & B_n
\end{array}$$

Let $A(0, t_1, ..., t_k) = \alpha'(t_1, ..., t_k)$ and $X(t_0, t_1, ..., t_k) = \beta'(t_0, ..., t_k)$ for a special representative X of $Sp/U(W)$. For $t \in I^k, I^{k+1}$ respectively, we may write the spectral decomposition of A(t) and X(t) as

$$A(t) = \pi_{W'(t)} i - \pi_{W''(t)} i + \sum_l \pi_{W'_l(t)} i\mu_l(t) - \pi_{W''_l(t)} i\mu_l(t),$$

$$X(t) = \pi_{V'(t)} - \pi_{V''(t)} + \pi_{V'_0(t)} i - \pi_{V''_0(t)} i$$
$$+ \sum_l (\pi_{V'_l(t)} \lambda_l(t) + \pi_{V''_l(t)} \overline{\lambda_l}(t) - \pi_{\widetilde{V}'_l(t)} \lambda_l(t) - \pi_{\widetilde{V}''_l(t)} \overline{\lambda_l}(t)),$$

where $-e^{\pi i \mu_l(t)} = \lambda_l(t)^2$, $W'(t) \oplus W''(t) = V'(t) \oplus V''(t), W'_l(t) = V'_l(t) \oplus \widetilde{V}'_l(t)$, and $W''_l(t) = V''_l(t) \oplus \widetilde{V}''_l(t)$

Consider the following subspace of an 2m-dimensional complex subspace W of $\mathcal{U}$:

$$\begin{aligned} \mathrm{Perp}_{d,d}(W) = \{\, (V', V'') \mid\ & V', V'' \text{ are complex subspaces of } W, \\ & iV' = V'' = V'j,\ \dim_{\mathbb{C}} V' = \dim_{\mathbb{C}} V'' = d\}. \end{aligned}$$

We may characterize this space by considering the unitary group over W. We get all possible V' and V'' by letting $U(W)$ act on W, and observing the induced tranformations on a given V' and V''. That means $U(W)$ acts transitively on $\mathrm{Perp}_{d,d}(W)$. The stabilizer is given by the unitary matrices that can be decomposed into an orthogonal matrix that acts on V', an orthogonal matrix that acts on V'', and a unitary matrix that acts on $(W - (V' \times V''))$. Thus the stabilizer is $O_{2d} \times O_{2d} \times U_{2m-2d}$. We therefore have

$$\mathrm{Perp}_n(W) \cong U_{2m}/(O_{2d} \times O_{2d} \times U_{2m-2d})$$

Define $BU_n(Y) = \{\, V \mid V \subseteq Y,\ \dim_{\mathbb{C}} V = n\}$

We have a natural map $P : \mathrm{Perp}_{d,d}(W) \to BU_{2d}(W)$ given by $P(V', V'') = V' \oplus V''$. We characterize $BU_{2d}(W)$ in a similar way. We have that U_{2m} acts transitively on $BU_{2d}(W)$, with stabilizer the unitary matrices that can be decomposed into the direct sum of a unitary matrix that acts on $V \in BU_{2d}(W)$, and a unitary matrix that acts on $W - V$. That is, a unitary matrix of the form $U_{2d} \oplus U_{2m-2d} \cong U_{2d} \times U_{2m-2d}$. Thus $BU_{2d}(W) \cong U_{2m}/(U_{2d} \times U_{2m-2d})$. The argument used in the previous section for determining that $U_n \times U_n \subseteq Sp_n$ can be used for determining that $O_{2n} \times O_{2n} \subseteq U_{2n}$. This means that $U_{2m}/(U_{2d} \times U_{2m-2d}) \subseteq U_{2m}/(U_{2d} \times U_{2m-2d})$. It follows that P is a projection, so $P : \mathrm{Perp}_n(W) \to BU_{2n}(W)$ is a fibration. That means we can find a lift ω'' to the following commutative diagram.

$$\begin{array}{ccc} \{0\} \times I^k & \xrightarrow{\alpha''} & \mathrm{Perp}_n(W) \\ \downarrow & \overset{\omega''}{\nearrow} & \downarrow P \\ I^{k+1} & \xrightarrow[\beta'']{} & BU_{2n}(W) \end{array}$$

Let $d = \dim W'(0)$, i.e the dimension of the eigenspaces of $A(0)$ corresponding to $\mu = i$. Let $\alpha'' : I^k \to \mathrm{Perp}_{d,d}(W)$ be given by $\alpha''(t) = (W'(t), W''(t))$ and let $\beta : I^{k+1} \to BU_{2d}(W \oplus W)$ be given by $\beta''(t) = V'(t) \oplus V''(t)$, where we have $W'(t) \oplus W''(t) = V'(t) \oplus V''(t)$. Like in section 5.5, we consider $V'(t) \oplus V''(t)$ as a single subspace of W, "forgetting" its decomposition. We can do this since the coset is only dependent on the square of a special representative, which means any decomposition of $V_0(t) \oplus V'_0(t)$ will result in a special representative of the same coset. Define $\omega''(t) = (\widetilde{W}'(t), (V'(t) \oplus V''(t)) - \widetilde{W}'(t))$, where $\widetilde{W}'(t)$ is obtained from $W'(t)$ by a homotopy. Let $\mu_l(t) \in (0,1)$ be the unique solution to $-e^{\pi\mu_l(t)} = \lambda_l^2(t)$. We can now define $\omega' : I^{k+1} \to E(W) \cap p^{-1}(B_n)$ by

$$\omega'(t) = \pi_{\widetilde{W}'(t)} i - \pi_{(V'(t)\oplus V''(t)) - \widetilde{W}'(t)} i + \sum_l (\pi_{V_l'(t)} \mu_l(t) i - \pi_{V_l''(t)} \mu_l(t) i)$$

and by inclusion, we obtain a lift ω to our original diagram. □

We define a neighborhood N_n of $F_{n-1}(Sp/U)$ in $F_n(Sp/U)$ as

$$N_n = \{ [X] \mid X \text{ is a special representative, } \dim_{\mathbb{C}} \mathrm{Eig}_{-e^{\pi i[1/3,2/3]}} X^2 < 2n \}.$$

Let $f : I \to I$ be given by

$$f(x) = \begin{cases} 1, & x \geq \frac{2}{3} \\ 3x - 1, & \frac{1}{3} \leq x \leq \frac{2}{3} \\ 0, & x \leq \frac{1}{3} \end{cases}$$

Then $f \simeq \mathrm{Id}$ rel ∂I, and $H(x,t) = tf(x) + (1-t)x$ is a homotopy that satisfies this. There exists a unique homotopy $h : S^1 \times I \to S^1$ such that the following square commutes

$$\begin{array}{ccc} I & \xrightarrow{H_t} & I \\ \downarrow{\scriptstyle -e^{\pi i(\cdot)}} & & \downarrow{\scriptstyle -e^{\pi i(\cdot)}} \\ S^1 & \xrightarrow[h_t]{} & S^1 \end{array}$$

The induced homotopy $h : N \times I \to N$, given by sending the eigenvalues $\lambda_l = -e^{\pi i \mu_l}$ to $-e^{\pi i H_t(\mu_l)}$ deforms N_n into a subspace of $F_{n-1}Sp$ for all n, and is covered by a homotopy $H : p^{-1}(N) \times I \to p^{-1}(N)$, as was required by theorem 2.2.

It remains to check that H_1 induces weak equivalences on fibers, that is, that $H_1 : p^{-1}(X) \to p^{-1}(h_1(X))$ is a weak equivalence. Following the same argumentation leading up to lemma 4.6, this boils down to proving the following lemma.

Lemma 5.29. *Let $V \subseteq V'$ be quaternionic subspaces of a finite dimensional quaternionic subspace $W \subset \mathcal{U}$. Then the map $f : \overline{U/O}_{V,W} \to \overline{U/O}_{V',W}$ given by sending A to $A \oplus L$ for a complex matrix L such that $L \oplus jL \cong L \oplus iL \cong V' - V$ is a homotopy equivalence.*

Proof. since $\overline{U/O}_{V,W} \cong U/O \subseteq U$, we can represent a representative of an element A in $\overline{U/O}_{V,W}$ by an element S in U, while we choose L such that its representative in U is $I_{V'-V}$. Consider the following commutative square:

$$\begin{array}{ccc} \varinjlim_{W' \geq W}(V \oplus (W' - W)) & \xrightarrow{\phi} & \varinjlim_{W' \geq W}(V' \oplus (W' - W)) \\ \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S \oplus I_{V'-V}} \\ \varinjlim_{W' \geq W}(V \oplus (W' - W)) & \xrightarrow[\phi]{} & \varinjlim_{W' \geq W}(V' \oplus (W' - W)) \end{array}$$

Connecting the notation to lemma 2.4, we have $f = \phi_*$ and $S \oplus I_{V'-V} = \phi S \phi^{-1} \oplus I_{V'-V} = f(S) = \phi_*(S)$. From the second equality, we get that ϕ is a unitary matrix and therefore an isometry. Since S and $I_{V'-V} \in U$, we can use lemma 2.4 to conclude that f is a homotopy equivalence. In particular, f is a weak homotopy equivalence. □

5.6 $\Omega(U/O) \simeq BO \times \mathbb{Z}$

Let $\mathcal{U} \cong \mathbb{C}^\infty$. Let $c : \mathcal{U} \to \mathcal{U}$ be a fixed complex conjugation. In this section, we assume that all finite dimensional complex subspaces $W \subset \mathcal{U}$ are closed under the conjugation map c. Define the real subspace of W to be $W^{\mathbb{R}} = \{ v \in W \mid v = \overline{v}\}$. Let $U(W \oplus W)$ be the set of all complex isometries of $W \oplus W$, and let $O(W \oplus W)$ be the set of all $X \in U(W \oplus W)$ such that $X = \overline{X}$.

Define

$$E(W) = \{ A \in H(W \oplus W) \mid A = \overline{A},\ \mu_l \in [0,1]\ \forall l\},$$

Where μ_l denotes the eigenvalues of A. Define $p_W : E(W) \to U/O(W \oplus W)$ by $p_W(A) = [e^{\pi i A}]$.

In order to tell us more about the representatives of U/O, we present two lemmas similar to previous sections:

Lemma 5.30. *Let W be a finite dimensional complex space. If $A \in \mathfrak{u}(W \oplus W)$ has the property $\overline{A} = -A$, then $X = e^A$ has the property $\overline{X} = X^{-1}$.*

Proof.

$$\overline{X} = \overline{e^A} = e^{\overline{A}} = e^{-A} = X^{-1}.$$

□

Lemma 5.31. *Let W be a finite dimensional complex space. If Y, $Z \in U(W \oplus W)$ has the property that $Y^{-1} = \overline{Y}$ and $Z^{-1} = \overline{Z}$, then there exists an $X \in O(W \oplus W)$ such that $Y = XZ$ if and only if $Y^2 = Z^2$.*

Proof. Assume $Y = XZ$. Then

$$Y^{-1} = Z^{-1}X^{-1}$$

$$\overline{Y} = \overline{XZ} = XZ^{-1}$$

Where we have used that $\overline{X} = X$ and $Z^{-1} = \overline{Z}$. Since $Y^{-1} = \overline{Y}$, we get that $Z^{-1}X^{-1} = XZ^{-1}$ which gives $Z = XZX$. Therefore $Y^2 = XZXZ = Z^2$

Assume $Y^2 = Z^2$. We get that $Y = (Y^{-1}Z)Z$ as well as $YZ^{-1} = Y^{-1}Z$. We need to show that $Y^{-1}Z \in O(W \oplus W)$, i.e that $\overline{Y^{-1}Z} = Y^{-1}Z$. But $\overline{Y^{-1}Z} = \overline{Y^{-1}}\ \overline{Z} = YZ^{-1} = Y^{-1}Z$, so $Y^{-1}Z \in O(W \oplus W)$ □

If $X \in U(W \oplus W)$ and $X = e^A$ for $A \in \mathfrak{u}(W \oplus W)$ such that $\overline{A} = A$, we call X a special representative for $[X] \in U/O(W \oplus W)$. From the two lemmas, two special representatives are in the same coset if and only if their squares are equal. This means that a special representative X has the property that $X = e^{\pi i A}$ for an $A \in H(W \oplus W)$. And from the following lemma, every coset in $U/O(W \oplus W)$ has a special representative.

Lemma 5.32. *There exists a special representative X for every coset $[X] \in U/O(W \oplus W)$*

Proof. $U/O(W \oplus W)$ is geodesically complete, The geodesics are given by $\gamma_t = [Ye^{tB}]$ for $Y \in U(W \oplus W)$ and $B \in \mathfrak{o}(W \oplus W)^{\perp}$, which determine the special representatives. □

We now proceed to determine the fiber of p_W.

Lemma 5.33. *Let $W \subset \mathcal{U}$ be a finite dimensional complex space. If X is a special representative for $[X] \in U/O(W \oplus W)$, then $p_W^{-1}([X]) \cong BO(\ker(X^2 - I)^{\mathbb{R}})$.*

Proof. If $A \in E(W)$, then A has a spectral decomposition

$$A = \pi_{W_0} + \sum_l \mu_l \pi_{W_l}$$

where $\mu_l \in (0,1)$. In order for these eigenspaces to be regarded as subspaces, they need to be closed under conjugation. This is indeed the case since if $Av = \mu v$, $A\overline{v} = \overline{Av} = \overline{\mu v} = \mu \overline{v}$.

The special representative X has a spectral decomposition

$$X = \pi_{V_0'} - \pi_{V_0''} + \sum_l (\lambda_l \pi_{V_l'} - \lambda_l \pi_{V_l''})$$

where $\mathrm{Im}(\lambda_l) > 0$). These eigenspaces are also closed uder conjugation, since if $Xv = \lambda v$, $X\overline{v} = \overline{X^{-1}v} = \overline{\overline{\lambda}}v = \lambda\overline{v}$. Now, if $e^{2\pi i A} = X^2$, then the eigenvalues $\mu_l \in (0,1)$ must be the unique solutions to $e^{2\pi i \mu_l} = \lambda_l^2$. In addition, we must have $W_l = V_l' \oplus V_l''$ and $W_0 \subseteq V_0' \oplus V_0''$. Since W_0, V_0' and V_0'' are closed under conjugation, we can decompose them into their real and imaginary part and we get the following: $W_0^{\mathbb{R}} \oplus iW_0^{\mathbb{R}} \subseteq V_0'^{\mathbb{R}} \oplus V_0''^{\mathbb{R}} + iV_0'^{\mathbb{R}} \oplus iV_0''^{\mathbb{R}}$, and in particular, $\ker(A - I)^{\mathbb{R}} = W_0^{\mathbb{R}} \subseteq V_0'^{\mathbb{R}} \oplus V_0''^{\mathbb{R}} = \ker(X^2 - I)^{\mathbb{R}}$. Therefore, $\ker(A - I)^{\mathbb{R}} \in BO(\ker(X^2 - I)^{\mathbb{R}})$.

Define a map $\phi : p_w^{-1}([X]) \to BO(\ker(X^2 - I)^{\mathbb{R}})$ given by $\phi(A) = \ker(A - I)^{\mathbb{R}}$. This map has an inverse ψ, namely, for a real subspace $V \subseteq \ker(X^2 - I)^{\mathbb{R}}$, let $W_0 = V \oplus iV$. Then define

$$\psi(V) = \pi_{W_0} + \sum_l \mu_l \pi_{V_l' \oplus V_l''}.$$

□

For $V \subseteq W \subset \mathcal{U}$ closed under conjugation, there is a map $U/O(V \oplus V) \to U/O(W \oplus W)$ defined by sending $[X]$ to $[X \oplus I_{(W-V)\oplus(W-V)}]$, and a map $E(V) \to E(W)$ defined by sending A to $A \oplus \pi_{(W-V)\oplus 0}$. By taking direct limits, we get a map $p : E \to U/O$. We first show that this map has fiber $BO \times \mathbb{Z}$. First, for $V \subseteq W \subset \mathcal{U}$ closed under conjugation, define a map $BO(V^{\mathbb{R}} \oplus V^{\mathbb{R}}) \to BO(W^{\mathbb{R}} \oplus W^{\mathbb{R}})$ given by sending V' to $V' \oplus (W - V)^{\mathbb{R}} \oplus 0$. In addition, define

$BO(Y) = \coprod_n BO_n(Y)$. We may therefore use the real analogue of lemma 4.2 to conclude that $\varinjlim_W BO(W^{\mathbb{R}} \oplus W^{\mathbb{R}}) \cong BO \times \mathbb{Z}$.

For $V \subseteq W \oplus W$ closed under conjugation, define

$$\overline{BO}_{V^{\mathbb{R}},W^{\mathbb{R}}} = \varinjlim_{W'^{\mathbb{R}} \geq W^{\mathbb{R}}} BO(V^{\mathbb{R}} \oplus (W'^{\mathbb{R}} - W^{\mathbb{R}}) \oplus (W'^{\mathbb{R}} - W^{\mathbb{R}}))$$

for W' closed under conjugation. By a choice of isometry $V^{\mathbb{R}} \oplus (W^{\perp})^{\mathbb{R}} \oplus (W^{\perp})^{\mathbb{R}} \cong \mathcal{U}^{\mathbb{R}} \oplus \mathcal{U}^{\mathbb{R}}$, we get that $\overline{BO}_{V,W} \cong \varinjlim_W BO(W^{\mathbb{R}} \oplus W^{\mathbb{R}}) \cong BO \times \mathbb{Z}$, and in particular, setting $V = \ker(X^2 - I)$, we have that $p^{-1}([X]) \cong \overline{BO}_{\ker(X^2-I),W} \cong BO \times \mathbb{Z}$.

We now show that $p : E \to U/O$ is a quasifibration. We start by defining a filtration on U/O:

$$F_n(U/O) = \{\, [X] \in U/O \mid X \text{ is a special representative, } \dim_{\mathbb{C}} \ker(X^2 - I)^{\perp} \leq n\}.$$

Let $B_n(U/O) = F_n(U/O) - F_{n-1}(U/O)$. We now prove that B_n is distinguished

Lemma 5.34. *$p^{-1}(B_n(U/O)) \to B_n(U/O)$ is a Serre fibration*

Proof. We start with the following commutative square

$$\begin{array}{ccc} \{0\} \times I^k & \xrightarrow{\alpha} & p^{-1}(B_n(U/O)) \\ \downarrow & & \downarrow \\ I^{k+1} & \xrightarrow[\beta]{} & B_n(U/O) \end{array}$$

We wish to find a lift of this diagram, i.e. a map $\gamma : I^{k+1} \to p^{-1}(B_n(U/O))$ making the triangles in the diagram commute.

By compactness, there exist a finite dimensional $W \subset \mathcal{U}$ such that the diagram factors as

$$\begin{array}{ccccc} \{0\} \times I^k & \xrightarrow{\alpha'} & E(W) \cap p^{-1}(B_n) & \longrightarrow & p^{-1}(B_n) \\ \downarrow & & \downarrow & & \downarrow \\ I^{k+1} & \xrightarrow[\beta']{} & U/O(W \oplus W) \cap B_n & \longrightarrow & B_n \end{array}$$

Let $A(0, t_1, ..., t_k) = \alpha'(t_1, ..., t_k)$ and $X(t_0, t_1, ..., t_k) = \beta'(t_0, ..., t_k)$ for a special representative X of $U/O(W \oplus W)$. For $t \in I^k, I^{k+1}$ respectively, we may write the spectral decomposition of A and X as

$$A(t) = \pi_{W_0(t)} + \sum_l \mu_l(t) \pi_{W_l(t)},$$

$$X(t) = \pi_{V_0'(t)} - \pi_{V_0''(t)} + \sum_l (\lambda_l(t) \pi_{V_l'(t)} - \lambda_l(t) \pi_{V_l''(t)}),$$

And in addition

$$X^2(t) = \pi_{V_0'(t)\oplus V_0''(t)} + \sum_l \lambda_l^2 \pi_{V_l'(t)\oplus V_l''(t)}$$

which means $-e^{2\pi i \mu_l(t)} = \lambda_l^2(t)$, $W_0(t) \subseteq V_0(t) \oplus V_0'(t)$, and $W_l(t) = V_l(t) \oplus V_l'(t)$ when $t \in I^k$.

Consider the following subspace of $W^{\mathbb{R}} \oplus W^{\mathbb{R}}$ for an m-dimensional complex subspace W of $\mathcal{U}$ closed under conjugation:

$$\mathrm{Perp}_{i,j}(W^{\mathbb{R}} \oplus W^{\mathbb{R}}) = \{\, (V'^{\mathbb{R}}, V''^{\mathbb{R}}) \mid V'^{\mathbb{R}}, V''^{\mathbb{R}} \subseteq W^{\mathbb{R}} \oplus W^{\mathbb{R}},\ V'^{\mathbb{R}} \perp V''^{\mathbb{R}},$$

$$\dim_{\mathbb{R}} V'^{\mathbb{R}} = i, \dim_{\mathbb{R}} V''^{\mathbb{R}} = j\}.$$

We characterize this space by considering the orthogonal group over $W^{\mathbb{R}} \oplus W^{\mathbb{R}}$. We get all possible $V'^{\mathbb{R}}$ and $V''^{\mathbb{R}}$ by letting $O(W^{\mathbb{R}} \oplus W^{\mathbb{R}})$ act on one pair $(V'^{\mathbb{R}}, V''^{\mathbb{R}})$ by left multiplication. That means $O(W^{\mathbb{R}} \oplus W^{\mathbb{R}})$ acts transitively on our space. Since we know the real dimension of $W^{\mathbb{R}} \oplus W^{\mathbb{R}}$ to be $2m$, we may denote $O(W^{\mathbb{R}} \oplus W^{\mathbb{R}})$ by O_{2m}. The stabilizer is given by the orthogonal matrices that can be decomposed into an orthogonal matrix that acts on $V'^{\mathbb{R}}$, an orthogonal matrix that acts on $V''^{\mathbb{R}}$, and an orthogonal matrix that acts on $(W^{\mathbb{R}} \oplus W^{\mathbb{R}} - (V'^{\mathbb{R}} \oplus V''^{\mathbb{R}}))$. Thus the stabilizer is $O_i \oplus O_j \oplus O_{2m-i-j} \cong O_i \times O_j \times O_{2m-i-j}$. So by the orbit-stabilizer theorem tells us that

$$\mathrm{Perp}_{i,j}(W^{\mathbb{R}} \oplus W^{\mathbb{R}}) \cong O_{2m}/(O_i \times O_j \times O_{2m-i-j})$$

We have a natural map $P : \mathrm{Perp}_{i,j}(W^{\mathbb{R}} \oplus W^{\mathbb{R}}) \to BO_{i+j}(W^{\mathbb{R}} \oplus W^{\mathbb{R}})$ given by $P(V'^{\mathbb{R}}, V'\prime\mathbb{R}) = V'^{\mathbb{R}} \oplus V''^{\mathbb{R}}$. We characterize $BO_{i+j}(W^{\mathbb{R}} \oplus W^{\mathbb{R}})$ in a similar way. We have that O_{2m} acts transitively on $BO_{i+j}(W^{\mathbb{R}} \oplus W^{\mathbb{R}})$, with stabilizer the orthogonal matrices that can be decomposed into the direct sum of an orthogonal matrix that acts on $V^{\mathbb{R}} \in BO_{i+j}(W^{\mathbb{R}} \oplus W^{\mathbb{R}})$, and an orthogonal matrix that acts on $(W^{\mathbb{R}} \oplus W^{\mathbb{R}}) - V^{\mathbb{R}}$. That is, an orthogonal matrix of the form $O_{i+j} \oplus O_{2m-i-j} \cong O_{i+j} \times O_{2m-i-j}$. Thus $BO_{i+j}(W^{\mathbb{R}} \oplus W^{\mathbb{R}}) \cong O_{2m}/(O_{i+j} \times O_{2m-i-j})$. Since $\mathrm{Perp}_{(i+j),0} \cong BO_{i+j}$, P is a projection, so $P : \mathrm{Perp}_{i,j}(W^{\mathbb{R}} \oplus W^{\mathbb{R}}) \to BO_{i+j}(W^{\mathbb{R}} \oplus W^{\mathbb{R}})$ is a fibration. That means we can find a lift ω'' to the following commutative diagram.

$$\begin{array}{ccc} \{0\} \times I^k & \xrightarrow{\alpha''} & \mathrm{Perp}_{i,j}(W \oplus W) \\ \downarrow & \overset{\omega''}{\nearrow} & \downarrow P \\ I^{k+1} & \xrightarrow[\beta'']{} & BO_{i+j}(W \oplus W) \end{array}$$

Let $i = \dim W_0(0)$, and $j = \dim W_1(0)$ i.e the dimension of the eigenspaces of $A(0)$ corresponding to $\mu = 1$ and $\mu = 0$ respectively. Let $\alpha'' : I^k \to \mathrm{Perp}_{i,j}(W^{\mathbb{R}} \oplus W^{\mathbb{R}})$ be given by $\alpha''(t) = (W_0^{\mathbb{R}}(t), W_1^{\mathbb{R}}(t))$ and let $\beta : I^{k+1} \to BO_{i+j}(W \oplus W)$ be given by $\beta''(t) = V_0^{\mathbb{R}}(t) \oplus V_0'^{\mathbb{R}}(t)$, where we have $W_0^{\mathbb{R}}(t) \oplus$

$W_1^{\mathbb{R}}(t) = V_0^{\mathbb{R}}(t) \oplus V_0'^{\mathbb{R}}(t)$. Here we consider $V_0^{\mathbb{R}}(t) \oplus V_0'^{\mathbb{R}}(t)$ as a single subspace of $W^{\mathbb{R}} \oplus W^{\mathbb{R}}$, "forgetting" its decomposition. We can do this since the coset is only dependent on the square of a special representative, which means any decomposition of $V_0^{\mathbb{R}}(t) \oplus V_0'^{\mathbb{R}}(t)$ will result in a special representative of the same coset. Define $\omega''(t) = (W_0'^{\mathbb{R}}(t), (V_0^{\mathbb{R}}(t) \oplus V_0'^{\mathbb{R}}(t)) - W_0'^{\mathbb{R}}(t))$, where $W_0'^{\mathbb{R}}(t)$ is obtained from $W_0^{\mathbb{R}}(t)$ by a homotopy. Let $\mu_l(t) \in (0,1)$ be the unique solution to $e^{2\pi i \mu_l(t)} = \lambda_l^2(t)$. Let $W_0'(t) = W_0'^{\mathbb{R}}(t) \oplus iW_0'^{\mathbb{R}}(t)$, and let $V_0(t) \oplus V_0'(t) = V_0^{\mathbb{R}}(t) \oplus V_0'^{\mathbb{R}}(t) \oplus i(V_0^{\mathbb{R}}(t) \oplus V_0'^{\mathbb{R}}(t))$ We can now define $\omega' : I^{k+1} \to E(W) \cap p^{-1}(B_n)$ by

$$\omega'(t) = \pi_{W_0'(t)} - 0\pi_{(V_0(t) \oplus V_0'(t)) - W_0'(t)} + \sum_l \mu_l(t)\pi_{V_l(t)}$$

and by inclusion, we obtain a lift ω to our original diagram. □

Define a neighborhood N_n of $F_{n-1}(U/O)$ in $F_n(U/O)$ as

$$N_n = \{\, [X] \in F_n(U/O) \mid X \text{ is a special representative, } \dim_{\mathbb{C}} \operatorname{Eig}_{e^{2\pi i[1/3,2/3]}} X^2 < n\}.$$

Define $f : I \to I$ by

In the same way as before, we are going to deform N_n into a subspace of $F_{n-1}(U/O)$. To do that, we define the following function

$$f(x) = \begin{cases} 1, & x \geq \frac{2}{3} \\ 3x - 1, & \frac{1}{3} \leq x \leq \frac{2}{3} \\ 0, & x \leq \frac{1}{3} \end{cases}$$

We note that $f \simeq \mathrm{Id}$ rel ∂I. Let $H(x,t) = t(f(x)) + (1-t)x$ be such a homotopy. It follows that there exists an $h : S^1 \times I \to S^1$ that makes the following diagram commute:

$$\begin{array}{ccc} I & \xrightarrow{H_t} & I \\ {\scriptstyle e^{2\pi i(\cdot)}}\downarrow & & \downarrow{\scriptstyle e^{2\pi i(\cdot)}} \\ S^1 & \xrightarrow[h_t]{} & S^1 \end{array}$$

For $A \in E$ of the form $A = \sum_i \mu_i \pi_{W_i}$, we define a new hermitian matrix $H_t(A)$ where $t \in I$

$$H_t(A) = \sum_i H_t(\mu_i)\pi_{W_i}.$$

similarly, we define $h_t : U \to U$ by

$$h_t(X) = \sum_i h_t(\lambda_i)\pi_{W_i} = \sum e^{2\pi i H_t(\mu_i)}\pi_{W_i}.$$

Note that $h_t : N_n \to N_n$ satisfy $h_0 = \mathrm{Id}$ and $h_1(N_n) \subseteq F_{n-1}U$. In addition, h_t is covered by $H_t : p^{-1}(N_n) \to p^{-1}(N_n)$, where $H_0 = \mathrm{Id}$. What remains, is to show that the induced map on the fibers: $H_1 : p^{-1}(X) \to p^{-1}(h_1(X))$ is a weak equivalence, and we have proven that p is a quasifibration. This boils down to the following lemma:

Lemma 5.35. *Suppose* $V^{\mathbb{R}} \subseteq V'^{\mathbb{R}} \subseteq W^{\mathbb{R}} \oplus W^{\mathbb{R}}$*, and* $V''^{\mathbb{R}} \subseteq V'^{\mathbb{R}} - V^{\mathbb{R}}$*. Then the map* $\overline{BO}_{V^{\mathbb{R}},W^{\mathbb{R}}} \to \overline{BO}_{V'^{\mathbb{R}},W^{\mathbb{R}}}$ *given by sending* $Y to Y \oplus V''^{\mathbb{R}}$ *is a weak equivalence.*

Proof. We have that both $\overline{BO}_{V^{\mathbb{R}},W^{\mathbb{R}}}$ and $\overline{BO}_{V'^{\mathbb{R}},W^{\mathbb{R}}}$ are congruent to $BO \times \mathbb{Z}$.

We know that $\widetilde{K}_{\mathbb{C}}(C) \cong [C, BU \times \mathbb{Z}]$ (See [8], 1.2), for any pointed compact space C. Where $[A, B]$ denotes the homotopy classes of maps from A to B. Adjusting the arguments in ([8], 1.2) by considering $\text{Vect}_{\mathbb{R}}$ instead of $\text{Vect}_{\mathbb{C}}$, we get that $\widetilde{K}_{\mathbb{R}}(C) \cong [C, BO \times \mathbb{Z}]$

We therefore get an induced map

$$\widetilde{K}_{\mathbb{R}}(C) \cong [C, \overline{BO}_{V^{\mathbb{R}},W^{\mathbb{R}}}] \to [C, \overline{BO}_{V'^{\mathbb{R}},W^{\mathbb{R}}}] \cong \widetilde{K}_{\mathbb{C}}(C).$$

This means any coset representative is mapped to another representative of the same coset in $K_{\mathbb{C}}(C)$. Since $V^{\mathbb{R}} \subseteq V'^{\mathbb{R}}$, the map has to be addition of a trivial bundle, so the map is an isomorphism. In particular, for $C = S^i$, we get an isomorphism of homotopy groups. □

Thus we have proved the real Bott periodicity theorem.

6 Analysis

As stated in the introduction, this proof can give us some insight into why complex Bott periodicity is simpler than real Bott periodicity. In this section, a short analysis of the proof is given, to shed some light on this phenomenon. Let us recall the iterated loop space for BU and BO, where each arrow means applying Ω.
BU:

$$BU \to U \to BU \times \mathbb{Z}$$

BO:

$$BO \to O \to O/U \to U/Sp \to BSp \times \mathbb{Z} \to Sp \to Sp/U \to U/O \to BO \times \mathbb{Z}.$$

For the real case to exhibit a two-periodic pattern, we would have had to have $\Omega O \simeq BO \times \mathbb{Z}$. In order to see what stops us from arriving at that conclusion, let us take a closer look at the proof of lemma 5.3, and compare it to lemma 4.3. Recall that in lemma 4.3

$$A = \pi_{V'} + \sum_i \mu_i \pi_{W_i},$$

$$X = \pi_V + \sum_i \lambda_i \pi_{V_i}.$$

The reason we were able to find an isomorphism between $BU(\ker(X - I))$ and $p_W^{-1}(X)$, was because we found that $V' \subseteq \ker(X - I) = V$ could be any complex subspace of $\ker(X - I)$. However, this was not the case in lemma 5.3. Recall that in lemma 5.3

$$A = i\pi_{V'} - i\pi_{V''} + \sum_j \mu_j \pi_{W_j},$$

$$X = \pi_V + \sum_j \lambda_j \pi_{V_j},$$

In order to get an isomorphism between $BO(\ker(X-I))$ and $p_W^{-1}(X)$, we would need that $V' \subseteq \ker(X-I)) = V$ could be any real subspace of $\ker(X-I)$. However, we had the added constraint that μ and $-\mu$ had eigenspaces of equal dimension. In particular, $\dim(V') = \dim(V'')$, and since $V' \oplus V'' = V$, we had to have that $\dim(V') = \frac{1}{2}\dim(V)$. This means that V' could not be an arbitrary real subspace of $\ker(X-I)$. Therefore $p_W^{-1}(X)$ could not be isomorphic to $BO(\ker(X-I))$, so we could not have $\Omega O \simeq BO \times \mathbb{Z}$.

The fact that $-\mu$ and μ form a pair of eigenvalues with eigenspaces of equal dimension for $A \in \mathfrak{o}(W)$ implies that the eigenvalues $\lambda = -e^{\pi\mu}$ and $\overline{\lambda} = -e^{\pi(-\mu)}$ of $X \in SO(W)$ also form a pair of eigenvalues that has eigenspace of equal dimension. This puts a constraint on the elements of the orthogonal group that is not present in the unitary group, and can be seen as the cause of the added complexity in the real case of Bott periodicity theorem in this particular proof.

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