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Convergence of stiff, nonlinear stochastic differential equations

Stability and consistency analysis of drift-implicit methods using a one-sided Lipschitz condition

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Abstract

This work presents an overview of how to analyze numerical methods approximating stiff, nonlinear stochastic differential equations (SDEs). They appear in several problems of practical interest, which makes the convergence theory relevant too.

The use of the one-sided Lipschitz condition for SDEs is presented. The main result is a mean-square stability inequality that does not depend on the stiffness of the problem, proved for drift-implicit methods up to order 1. For non-positive one-sided Lipschitz constants it indicates no step size restriction. Consistency of drift-implicit methods up to order 1 is proved, also independent of the stiffness of the problem. Order of convergence is deduced from this stability property and the order of consistency. A similar stability inequality is proved for general Lipschitz continuous drift-implicit methods, as an alteration of the proof from the work of Winkler (2003) [28]. The result is a slightly improved step size condition.

Discussion and definition of suitable stability and consistency principles are presented. There are many different concepts, serving different purposes. This work argues why certain principles are more fitting and defines B-consistency and B-convergence for SDEs.

A significant part of the work is devoted to presenting the theory needed to analyze stiff, nonlinear SDEs and general SDEs, both numerically and analytically. The properties of the new convergence theorem are tested numerically by examples.

Sammendrag

Dette arbeidet gir en oversikt over hvordan man kan analysere numeriske metoder for å approksimere stive, ikke-lineære stokastiske differensialligninger (SDEer). Slike systemer inntrer i flere problemer i praksis, hvilket gjør konvergens-teorien relevant også.

Nytten av den énsidige Lipschitz-betingelsen for SDEer presenteres. Hovedresultatet er en stabilitetsulikhet i L_2 normen for stokastiske variabler, som ikke avhenger av stivheten til problemet, hvilket gjelder for drift-implisitte metoder opp til orden 1. For ikke-positive énsidige Lipschitz-konstanter er det ingen begrensning på steglengden. Konsistenthet av drift-implisitte metoder opp til orden 1 bevises, også uavhengig av stivheten til problemet. Kovergens-raten kan deduseres fra stabilitet-egenskapen og konsistenthet. En lignende stabilitetsulikhet bevises for generelle Lipschitz-kontinuerlige, drift-implisitte metoder, ved en modifisering av beviset i arbeidet til Winkler (2003) [28]. Resultatet gir en forbedret betingelse på steglengden.

Hensiktsmessige definisjoner av konsepter som stabilitet og konsistenthet diskuteres. De eksisterende konseptene er flerfoldige, med ulike hensikter Oppgaven argumenterer for hvordan noen kan være mer passende, og definerer B-konvergens og B-konsistenthet for SDEer.

En stor del av arbeidet presenterer teori nødvendig for å analysere stive, ikke-lineære SDEer og SDEer generelt, både numerisk og analytisk. Egenskapene til det nye konvergensteoremet testes numerisk med eksempler

Acknowledgements

Notation

$\langle y, x \rangle$	Euclidean inner product, $y^T \cdot x$
$\lvert x \rvert$	Euclidean norm of x, $\langle x, x \rangle^{1/2}$
$\lVert x \rVert$	The L_2 norm for random variables, $\left(\mathbb{E}(\lvert x \rvert^2) \right)^{1/2}$
$\lVert x \rVert_p$	The L_p norm for random variables, $\left(\mathbb{E}(\lvert x \rvert^p) \right)^{1/p}$
$\lVert A \rVert_{op}$	The operator norm for $A : \mathbb{R}^d \to \mathbb{R}^d$, $\lVert A \rVert_{op} = \sup\{ \lvert Ax \rvert : x \in \mathbb{R}^d, \lvert x \rvert = 1 \}$
Y_{t_i}	The exact solution of the SDE at time point t_i
X_n	The numerical approximation of the SDE at time point t_n
$\hat{Y}_n$	The numerical approximation going one step from $Y_{t_{n-1}}$
$\hat{l}_n$	Local error of numerical method, $\hat{l}_n = Y_{t_n} - \hat{Y}_n$
l_n	Residual error
$\mathcal{F}_t$	Filtration at time point t
$\mathcal{F}_n$	Filtration at time point t_n
J_f	Jacobian matrix of a vector-valued function f
L_g	Lipschitz constant of the function g

Contents

1   Introduction

Stiffness of a dynamical system $y'(t) = f(y(t))$ is a property which has no clear definition [7]. The solution of such a system might vary slowly, but the system damps perturbations aggressively or contain certain parts that oscillate quickly but with small amplitude. The rapid damping makes it much better to use a backward difference when approximating derivatives, thus implying implicit methods to be essential [27].

Explicit numerical methods (the standard numerical solvers) for stiff problems are only stable for very fine grids. This is computationally heavy. By stable meaning here that the numerical approximation is close to the exact solution. An implicit method can relax the grid criterion, and in addition, provide a more accurate error estimate. Stiff systems appear often in electric circuits and chemical reactions modeled as differential equations [11, 23]. In such systems, it can be natural to model with noise, due to thermal noise or diffusion, which makes this work of interest to modelling stiff noisy systems in such fields.

This work aims at investigating the convergence theory of stiff differential equations when noise is introduced. We aim to describe how error bounds are affected by noise, and investigate what can specifically be said about the stability for a noisy system.

The one-sided Lipschitz condition is well-known for its ability to describe systems with such aggressive damping. The Lipschitz and one-sided Lipschitz condition are, for a vector-valued function f, and $x, y \in \mathbb{R}^d$, respectively

$$|f(x) - f(y)| \leq L|x - y| \tag{1.1a}$$
$$\langle f(x) - f(y), x - y \rangle \leq \nu |x - y|^2 \tag{1.1b}$$

for some inner product $\langle ., . \rangle$ and corresponding norm $|.|$. The Lipschitz constant is always larger than zero, while ν can take any value.

1.1   Example: Linear ODE

The following simple example show some aspects of how a one-sided Lipschitz condition is better at describing the damping behaviour of a system. The grid criterion and improved error bound is also exemplified. The linear ordinary differential equation (ODE)

$$y'(t) = Ay(t) = \begin{pmatrix} -\mu & 1 \\ 1 & -\mu \end{pmatrix} y(t), \quad y(0) = (a, b) \tag{1.2}$$

$\mu > 1$, has an exact solution, found via the eigenvalues of A, which are $(-\mu + 1, -\mu - 1) = (\lambda_1, \lambda_2)$. The system has solution [7]

$$y(t) = (1/2(e^{\lambda_1 t} + e^{\lambda_2 t}), 1/2(e^{\lambda_1 t} - e^{\lambda_2 t}))$$

Large μ makes this system stiff. The linear system has $f(y(t)) = Ay(t)$, so from (1.1) it is clear that the two conditions result in $L = |-\mu - 1| = \mu + 1$, the steepest possible line, but

$\nu = -\mu + 1$, the largest eigenvalue. This holds for symmetric A using the Euclidean norm [7].

Let us assume for a moment that we only know that the system $y'(t) = f(y(t))$ was Lipschitz continuous. We denote $y^*(t)$ the solution with initial value $y^*(0)$, and $\tilde{y}(t)$ the solution with initial value $\tilde{y}(0)$. Now estimating the global error assuming that f is Lipschitz continuous. The difference between the two solutions can then be bounded by

$$|y^*(t) - \tilde{y}(t)| \leq |y^*(0) - \tilde{y}_0| + \int_0^t |f(y^*(s)) - f(\tilde{y}(s))|ds$$

$$\leq |y^*(0) - \tilde{y}_0| + \int_0^t L|y^*(s) - \tilde{y}(s)|ds \leq |y^*(0) - \tilde{y}_0|e^{tL}$$

by the Gronwall inequality (2.4). However, if the one-sided Lipschitz condition was assumed

$$|y^*(t) - \tilde{y}(t)|^2 \leq |y^*(0) - \tilde{y}_0|^2 + \int_0^t 2\langle y^*(s) - \tilde{y}(s), f(y(s)) - f(\tilde{y}(s)) \rangle ds$$

$$\leq |y^*(0) - \tilde{y}_0|^2 + \int_0^t 2|y^*(s) - \tilde{y}(s)|^2 ds \leq |y^*(0) - \tilde{y}_0|^2 e^{2\nu t} \qquad (1.3)$$

which can be compared to the Lipschitz case by taking the square root. In the linear example, we have positive L and negative ν, so the error bound on the exact solutions are vastly different. The Lipschitz case will generally result in large error bounds. The one-sided Lipschitz case will, however, for $-\mu + 1 = \nu < 0$, estimate an error that is exponentially damped.

Now solving the above using the backward Euler method, being

$$x_{n+1} = x_n + f(x_{n+1})h$$

and denote the solutions $x_{n+1}^*, \tilde{x}_{n+1}$ for the initial values $x_0^*, \tilde{x}_0$. When f is Lipschitz continuous the following bound can be shown [7]

$$|x_{n+1}^* - \tilde{x}_{n+1}| = \frac{1}{1 - hL}|x_0^* - \tilde{x}_0| \exp(TL)$$

which is very comparable to the exact solution. It is required that $hL < 1$. If f is one-sided Lipschitz continuous, the following bound can be shown [7]

$$|x_{n+1}^* - \tilde{x}_{n+1}|^2 \leq |x_0^* - \tilde{x}_0| \exp(TC)$$

where C only depends on ν, having the same sign. The requirement here is that $h < 1/\nu$. Again the comparability to the exact solution is evident. For negative ν there is no step size restriction and the error bound is much sharper.

This example tells us that the Lipschitz condition fails to describe *contractions*, i.e. that the different solution paths are squeezed together, it only describes the maximal growth in some direction. This is a property of dissipative systems, having $\nu < 0$. The stiffness

observed here was more specifically a contraction of the system, not an explosion. The implicit method we chose did conserve the contractive behaviour.

In general, even though a system might have a large Lipschitz constant, one-sided Lipschitz constants might turn out to be drastically smaller, independent of the stiffness, and highly useful. Assuming Lipschitz continuity is necessary to prove convergence. When a parameter is independent of stiffness we mean, in this work, that the Lipschitz constant can be arbitrarily large, while (1.1a) still holds. ν can become arbitrarily negative without ruining convergence. For highly positive ν the classic step size condition applies. Using (1.1a), we will show that convergence still holds for arbitrarily negative ν, for arbitrary step size.

Approaching stiff, nonlinear differential equations this way turned out to be just what Dahlquist needed for his foundational work on stability theory for such systems [6]. Since then, the field of B-convergence and the like has become a well-studied subject [7, 10]. The result is a rich and more general stability theory for numerical methods. The application of these ideas to SDEs form a new and important field of study, though at the time small.

Chemical reaction systems is a group of problems that can often be represented as stiff ODEs, and including randomness due to diffusion makes it a stiff SDE. In general, singular perturbation problems are another source to stiff systems, and electric circuits provide notoriously stiff problems due to elements like capacitors and resistors with parameters on vastly different scales. The Van der Pol oscillator is such a stiff system from a circuit, which is analyzed numerically in the final section.

This work will prove an alteration of the mean-square stability inequality of general drift-implicit methods with Lipschitz condition, from [28]. From this, convergence follows easily by consistency, with the same rate of convergence as the consistency order. A similar stability inequality is proved for drift with a one-sided Lipschitz condition. By combining the same approach with some ideas of Hu [15], the stability inequality is independent of stiffness. The method applies to numerical schemes up to order 1. If the one-sided Lipschitz condition is non-positive, there is no step size restriction. Such a stiffness-independent stability inequality is the first of its kind, to our knowledge. This is our main result. B-consistency and B-convergence is defined and proven for the drift-implicit Euler and Milstein methods. This gives the usual convergence rate 1/2 and 1 respectively, independent of stiffness.

The second section is devoted to presenting background theory and discusses some important concepts from numerics. A swift summary of the most important aspects of stochastic calculus is presented, being used throughout the work.

The third section contains the main results described, starting with a discussion on existing proofs and concepts. The fourth and final section shows two examples of stiff systems, one linear and one nonlinear. The order of convergence for the nonlinear system is computed based on the comparison with a higher-order Runge-Kutta method.

1.2   Literature overview

Hu's paper [15] is to our knowledge the first to apply ideas on one-sided Lipschitz continuity for SDEs, and applies this to the drift-implicit Euler. Their work is a big inspiration to the main result here.

The work by Higham and Kloeden represents a thorough theoretical foundation for systems with one-sided Lipschitz continuous drift [14]. It is based more on probability theory, proving convergence of the split-step backward Euler method (SSBE), which they show implies convergence of the backward (drift-implicit) Euler method. The error measure is more general, by use of a continuous extension, i.e. measuring error along the entire time interval. They prove convergence of all moments with optimal rate, though under stricter assumptions. This result is claimed to be improved in [25], where terms like *B-convergence* is used, known from deterministic equations. They extend the results for the SSBE method from [14] to split-step one-leg theta methods.

The work most comparable to what is done here is [2], though results are obtained differently and independently. They define stochastic C-stability, and deduce convergence from C-stability and B-consistency. The notion of B-consistency agrees with this work.

General drift-implicit methods with Lipschitz continuous drift and diffusion is essential when solving stochastic differential-algebraic equations [23, 28, 29]. Implicit multi-step methods are also available for SDEs, see [18]. Explicit methods work poorly, though can detect stiffness [27].

Assuming that the stiffness is located in the drift f of the SDE, not the diffusion g, implies that drift-implicit methods are essential. This is the most studied case, and what is considered here. There exists some work on fully implicit methods [24].

It is more common to seek contractive stability properties [14, 25], while our stability inequality is more connected to convergence and specific boundedness of perturbations. The work in [22] uses a contraction theoretic approach to describe the behaviour of the exact solution.

It has been shown that the explicit Euler fails to satisfy moment bounds for any moment when the drift only satisfies a one-sided Lipschitz condition [17], but other explicit methods show mean-square convergence, such as a projected Euler method [2], tamed Euler [16] and tamed Milstein schemes [26], and balanced methods [24]. The inclusion of Poisson driven jumps has also been considered in [13]. Most of the existing work, this included, is restricted to strong convergence.

An adapted version of the Van der Pol equation has also been applied in stochastic modeling of neuro signals, by the FitzNagumo-system with noise, which is also stiff [9]. This equation is essential in mathematical neuroscience.

2 Background theory

2.1 Stability principles

Stability is another term that lacks a clear definition [7]. All its concepts try to explain how initial errors and perturbations are propagated, in some way, but remain bounded. Stability is especially important for describing methods that work well for stiff problems. The different principles can roughly be divided into two categories, boundedness properties, and contraction properties [7]. A boundedness stability property simply states that the numerical solution does not blow up, at least for step size $h \to 0$ [27]. A contractive stability property could e.g. be what we saw in section 1.1, where for one-sided Lipschitz continuous f with $\nu < 0$, the error $x_{n+1}^* - \tilde{x}_{n+1}$ is smaller than the initial error $|x_0^* - \tilde{x}_0|$. This section considers deterministic systems only.

A-stability is probably the simplest definition of a stability property. The test equation $y'(t) = \mu y(t)$, $\mu \in \mathbb{C}^-$, is contractive, and if the numerical solution by a method also satisfies $|x_{n+1}| \leq |x_n|$ when approximating the test equation, for any $h > 0$, then it is A-stable [27]. From the example in the introduction, we see that the implicit Euler is A-stable.

Dahlquist wanted to generalize A-stability, and did so for multistep methods [6]. The mentioned application of one-sided Lipschitz conditions was initiated there. Soon the concepts of B-stability, B-convergence, and B-consistency were established too [4, 10]. These concepts originally applied to Runge-Kutta methods, but the definitions also work in more general settings.

B-stability has become an ambiguous term, where Butcher defined it as the property of a Runge-Kutta method when a problem satisfies the one-sided Lipschitz condition (1.1a) with $\nu \leq 0$, the methods solutions $|x_{n+1}^* - \tilde{x}_{n+1}| \leq |x_n^* - \tilde{x}_n|$ [4]. This is a contractivity condition, like A-stability. Others have defined it differently, as a boundedness property [7].

B-consistency and B-convergence apply for f satisfying (1.1a) with any ν. Their definitions are [7]

Definition 2.1. *A method is B-consistent of order γ if the ODE satisfies a one-sided Lipschitz condition* (1.1a)*, and the local error $\hat{l}_{n+1}$ satisfies*

$$|\hat{l}_{n+1}| \leq C\, h^{\gamma+1}, \quad \forall\, h \in (0, h_0]$$

where C and h_0 are independent of stiffness.

Definition 2.2. *A method is B-convergent of order γ if the ODE satisfies a one-sided Lipschitz condition* (1.1a)*, and the global error e_{n+1} satisfies*

$$|e_{n+1}| = |y(t_{n+1}) - x_{n+1}| = \leq C\, h^{\gamma}, \quad \forall\, h \in (0, h_0]$$

where C and h_0 are independent of stiffness.

C-stability is a stability definition that does not assume $\nu \leq 0$, but it only considers $|x_{n+1}^* - \tilde{x}_{n+1}| \leq C|x_n^* - \tilde{x}_n|$, for small enough step sizes, and C independent of stiffness. It is a boundedness principle, not a contraction principle, used for stiff systems.

A boundedness property that resembles what will be used later is BS-stability. It is related to D-stability which is often called zero-stability for multistep methods. It considers a numerical scheme, where every discrete step i is perturbed by a perturbation d_i. It can be defined as [7]

Definition 2.3. *For problems satisfying a one-sided Lipschitz condition, a method is BS-stable if*

$$|x_{n+1}^* - \tilde{x}_{n+1}| \leq C \max(|d_1|, |d_2|, ..., |d_{n+1}|) , \ \forall h \in (0, h_0]$$

for some C and h_0 independent of stiffness.

Contractivity principles always provide a boundedness property, by its nature. The boundedness principles thus form a broader class. Such principles are more often used to decide convergence and rate of convergence. An example of this is the stability inequality for stiff SDEs, resembling BS-stability, which is established in section 3.4.

2.2 General tools

This section presents some mathematical tools that are extensively used in analysis of differential equations, both for analytical and approximate solutions. They will also be of use when considering stochastic differential equations, and proving different properties of solutions. The discrete versions of Gronwall's inequality will most often be used when the next error can be bounded by a series of the previous errors, to get a general error bound. The Gronwall inequalities are used when proving the error bounds of the linear ODE (1.2).

The first tool is the integral version of Gronwall's inequality.

Lemma 2.1. *When α constant w.r.t. t, the following holds*

$$x(t) \leq \alpha + \int_0^t b(s)x(s)ds \quad \Rightarrow \quad x(t) \leq \alpha \exp(\int_0^t b(s)ds) \tag{2.1}$$

The following version of the discrete Gronwall inequality is given in [5], and is practical when the elements a_i have individual coefficients b_i. We strive to keep as much detail from the original inequality, which is useful when developing sharp estimates. The analogy to (2.1) is evident.

Lemma 2.2. *For sequences $\{a_n\}, \{b_n\}$ of real numbers, where $b_n \geq 0$, and constant C, satisfying*

$$a_n \leq C + \sum_{j=n_0}^{n-1} b_j a_j \tag{2.2}$$

Then

$$a_n \le C \prod_{j=n_0}^{n-1} (1 + b_j) \le C \exp\left(\sum_{j=n_0}^{n-1} b_j\right) \tag{2.3}$$

Note. The above implies that for

$$a_n \le C_1 + C_2 \frac{1}{N} \sum_{i=1}^{n-1} a_i, \quad n = 1, ..., N, \implies \max_{n=1,...,N} a_n \le C_1 \exp(C_2) \tag{2.4}$$

which is also used later on. The inequality will still be referred to as Gronwall's inequality.

More detailed versions of the above lemma, using properties of C, are available in the literature, and can perhaps improve the estimates we present later on. The inequality will be of use when proving stability of contractive systems, and an explicit error bound is of interest.

The following lemma will be called Young's inequality, since it is deduced from the fact that $2\langle a_i, a_j \rangle \le |a_i|^2 + |a_j|^2$ for any $a_i, a_j \in \mathbb{R}^d$.

Lemma 2.3.
$$|a_1 + a_2 + ... + a_n|^2 \le n(|a_1|^2 + |a_2|^2 + ... + |a_n|^2)$$

The mean value theorem comes in many forms, but one that will prove itself useful is the following, given in [21].

Lemma 2.4. *For a differentiable vector-valued functions f,*

$$f(y) - f(x) = \int_0^1 J_f(sy + (1-s)x)ds \cdot (y - x) = M(y - x) \tag{2.5}$$

where J_f is the Jacobian of f.

The final two lemmas in this section consider functions f satisfying a one-sided Lipschitz condition, and is taken from [7, 15], where the former is a reference for a thorough analysis of this subject for deterministic equations.

Lemma 2.5. *The function f satisfies the one-sided Lipschitz condition iff the Jacobian of f, J_f satisfies*

$$\langle x, J_f(y)x \rangle \le \nu |x|^2 \quad \forall x, y \in \mathbb{R}^d. \tag{2.6}$$

Lemma 2.6. *If the matrix J satisfies the one-sided Lipschitz condition, $t\nu < 1$ and $t > 0$ we have*

$$\|(I - tJ)^{-1}\|_{op} \le (1 - \nu t)^{-1} \tag{2.7}$$

where $\|.\|_{op}$ is the operator norm, $\|A\|_{op} = \sup\{|Ax| : x \in \mathbb{R}^d, |x| = 1\}$.

2.3   Stochastic differential equations

Stochastic differential equations (SDEs) are differential equations affected by noise, which is a generalization of ODEs. Since the differential equation is affected by random variables, the solution will be a *stochastic process*.

Definition 2.4. *A stochastic process is a family of random variables, $\{Y_t\}_{t \in T}$ taking values in $\mathbb{R}^n$ defined on a probability space $(\Omega, \mathcal{F}, P)$, where Ω is the sample space, $\mathcal{F}$ the σ-algebra containing the set of events, also known as a filtration, and P is the probability measure.*

The notation for such processes is chosen Y_t. The stochastic process is thus a function $Y : \mathcal{I} \times \Omega \to \mathbb{R}^n$, where again $\mathcal{I}$ is some interval of time. $Y(t, \omega)$ is said to be adapted if $Y_t \in \mathcal{F}_t$, meaning that at time t the value of Y_t can be determined from the information available from events up to time t. We often omit writing the ω, being reminded that it is stochastic by the new notation. The numerical approximations we later discuss are also stochastic processes.

The solution of the differential equation shall be found by integration, but this demands the establishment of stochastic integrals, i.e. stochastic calculus. We will follow some of the necessary theory presented in [1, 8, 20], to solve such equations. The underlying theory is heavy with measure theoretic arguments, to define the tools rigorously, but will not be treated here.

The stochastic process at the core of SDEs is the Wiener process, also known as Brownian motion. The n-dimensional Wiener process is a vector of n independent scalar Wiener processes, where a scalar Wiener process also satisfies the following definition, given in [20]:

Definition 2.5. *The n-dimensional Wiener process is a stochastic process W_t satisfying the following criteria:*

i) *W_t is a Gaussian process with variance t in all directions, which implies that the random variable*
$Z = (W_{t_1}, ..., W_{t_k}) \in \mathbf{R}^{nk}$ *has a multinormal distribution.*

ii) *W_t has independent increments, i.e. for any $p < r \leq s < t$, $(W_r - W_p)$ is independent of $(W_t - W_s)$. The expectation of each increment is zero.*

iii) *W_t is almost surely (a.s.) continuous in t.*

Remark. To be precise in $iii)$, there exists an a.s. continuous version of W_t, by Kolmogorov's continuity theorem, and we assume that W_t is such a version.

The *standard* Brownian motion starts in zero and has expectation zero, which carries over to Z too. Thus, in the above $\mathbb{E}(W_s(W_t - W_s)) = \mathbb{E}(W_s)\mathbb{E}(W_t - W_s) = 0$. $\mathbb{E}(W_s)$ can be treated as the increment from the starting value, zero.

The auto-covariance of Z, assuming $s \leq t$, will then for each entry in its covariance matrix C look like

$$
\begin{aligned}
C(s,t) = Cov(W_s, W_t) &= \mathbb{E}(W_s W_t) - \mathbb{E}(W_s) E(W_t) \\
&= \mathbb{E}(W_s^2 + W_s(W_t - W_s)) = \mathbb{E}(W_s^2) = ns
\end{aligned}
$$

For general s, t this equals $n \min(s,t) = n(t \wedge s)$, where we have used the independent incremet property.

To get two different, correlated Wiener processes, one would generate two independent Brownian motions, and mix them with a Cholesky transformation (which is linear) to obtain the desired correlation. This we will not consider any further here.

The integration problem we consider, for X_t n-dimensional, can be formulated as

$$
Y_t = Y_0 + \int_0^t f(s, Y_s) ds + \int_0^t g(s, Y_s) dW_s, \tag{2.8}
$$

or, in short

$$
dY_t = f(t, Y_t) dt + g(t, Y_t) dW_t \tag{2.9}
$$

where $f : \mathcal{I} \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathcal{I} \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$, and W_t is an m-dimensional Wiener process. It is common to call f the drift and g the diffusion. They are together referred to as coefficient functions. The product $g(t, Y_t) dW_t$ is thus like a matrix-vector product. The latter integral is a *stochastic integral* with respect to a Brownian motion. One should note that it is the integral equation we solve.

The gateway into the world stochastic differential equations is to be able to integrate functions with respect to the Brownian motion. Based on measure theory one can show how this is done, but there is also a matter of definition playing a part. The stochastic integral is the limit in probability of the Riemann sum

$$
\lim_{n \to \infty} \sum_{\pi_n} g(\tilde{t}_i, Y_{\tilde{t}_i})(W_{t_{i+1}} - W_{t_i}) = \int_0^t g(s, Y_s) dW_s
$$

where π_n is some partition of our time interval into n points, with mesh size going to zero with increasing n. Where to evaluate the integrands is up to us to define. Two natural choices of definitions lead to the Itô and Stratonovich integrals. The Itô case chooses $\tilde{t}_i = t_i$, while Stratonovich $\tilde{t}_i = t_i + \frac{1}{2}\Delta t_i$, $\Delta t_i = t_{i+1} - t_i$. They will give different solutions and numerical schemes, but are anyhow connected. The numerical schemes have different properties, which is partly why different fields of application prefer the different definitions. The Stratonovich integral is for instance not a martingale, but it does obey the product rule of regular calculus. The Itô integral is a martingale. We will only consider the Itô case in the remaining.

The Itô integral is possible to define for functions g in (2.9) satisfying that each $\omega \to g(s, \omega)$ only depends on the history of $W_t(\omega)$ up to time s. In other words, $g(t, \omega)$ is adapted to the filtration generated by W_t. Together with these two remaining conditions, it will characterize all functions possible to integrate in the Itô sense [20]:

   i) $(t, \omega) \rightarrow g(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable. $\mathcal{B}$ denotes the Borel σ-algebra on $[0, \infty)$.

   ii) $\mathbb{E}(\int_s^t g(t, \omega)^2 dt) < \infty$.

For such a g, and f with properties

   i) $P(\int_0^t |f(s, \omega)| ds < \infty$ for all $t \geq 0) = 1$

   ii) f is adapted to the filtration generated by W_t

we call X_t from (2.9) an *Itô process*.

The Itô integral is linear and has expectation zero. The integral $\int_s^t g(r, \omega) dW_r$ is $\mathcal{F}_t$-measurable, which is important for iterated integrals. A consequence when defining the integral is the famous Itô isometry, which is used extensively in any field analyzing SDEs. For a function $g(t, \omega)$ in an Itô process

$$\mathbb{E}\Big(\sum_{i=0}^{m}(\int_s^t g_i(r, \omega) dW_r^i)^2\Big) = \mathbb{E}\Big(\sum_{i=1}^{m} \int_s^t g_i(r, \omega)^2 dt\Big) \qquad (2.10)$$

which is finite by assumption. Above g_i are vector-valued functions, columns of g, and dW_t^i the scalar noises of dW_t. The notation $g(t, Y_t) dW_t$ is practical, but in cases like this, it seems more comprehensible to consider the individual noises.

When an integrand $f(Y_s)$ is jointly measurable and integrable on $\mathcal{I} \times \Omega$, Fubini's theorem gives interchangeability of expectation and integral [18], i.e.

$$\mathbb{E}(\int_r^t f(Y_s) ds) = \int_r^t \mathbb{E}(f(Y_s)) ds \qquad (2.11)$$

This applies e.g. for $g_i(t, Y_t)$ continuous in t, Y_t, which will always be the case in this work.

Since the solution of an SDE is a stochastic process, the definition of uniqueness has to have this in mind too. *Pathwise uniqueness* is the property when X_t and X_t' are two solutions of the SDE with the same starting values and driven by the same Brownian motion, $X_t = X_t'$ with probability 1. If (2.8) holds for some filtration, it will hold for the filtration generated by X_t and W_t. A strong solution is a process X_t satisfying the SDE, where X_t is adapted to the filtration generated by the Brownian motion only, and we are fine using only this filtration. A very useful result by Itô is the following theorem regarding existence and uniqueness of solutions, given in [20] (theorem 5.2.1):

Theorem 2.1. *Let f_i and $g_{i,j}$ be globally Lipschitz continuous with respect to Y_t, for all i, j. In addition assume, $f(t, x) + g(t, x) \leq C(1 + |x|)$, for $x \in \mathbb{R}^d$. Let Y_0 have finite second moment, and is independent of the filtration generated by the Wiener process W_t. Then the SDE (2.8) has a strong solution and pathwise uniqueness holds.*

Remark. Relaxing the globally Lipschitz condition to locally is also possible [8]. The class of solvable equations is now fairly large.

From now on we will without loss of generality assume autonomous systems, as any dynamical system can be made time-invariant by augmenting it with an unknown variable z, representing time, and $dz = 1dt, z_0 = 0$. This gives slightly simpler notation.

A transformation $u(Y_t)$ of the Itô process Y_t satisfying (2.9) is also possible, using what is called the Itô formula. We assume the system to be autonomous for simplicity, and only give the general formula for multiple dimensions. The following theorem is given in [20].

Theorem 2.2. *Let* $u : \mathbf{R}^n \to \mathbf{R}^p$ *be twice continuously differentiable, and* X_t *an* n-*dimensional Itô process. The* p-*dimensional process* $Y_t = u(X_t)$ *with initial value* $Y_{t_0} = u(X_{t_0})$, *written component wise in differential form as*

$$dY_t^k = \sum_i \frac{\partial u_k}{\partial x_i}(X_t)dX_i + \sum_{i,j} \frac{\partial^2 u_k}{\partial x_i \partial x_j}(X_t)dX_i dX_j \tag{2.12}$$

is again an Itô process, where $dW_i dW_j = \delta_{i,j}dt$, $dW_i dt = dt dW_i = 0$ *and* $i, j = 1, ..., n$. *Here* $\delta_{i,j}$ *represents the Kronecker-delta.*

It is nothing but a composition of p transformations of X_t from $\mathbb{R}^n \to \mathbb{R}$ element-wise. The notation in integral form gets quite extensive, and since we want to separate the deterministic and stochastic integrals, defining the following two operators is useful.

$$\mathcal{L}_0 = \sum_{i=1}^n \Big(f_i \frac{\partial}{\partial x_i} + \frac{1}{2}g_{i,i}^2 \frac{\partial^2}{\partial x_i^2}\Big) \tag{2.13}$$

$$\mathcal{L}_1^j = \sum_{i=1}^n g_{i,j} \frac{\partial}{\partial x_i} \tag{2.14}$$

$$\mathcal{L}_1 u(x)dW_t = \sum_{j=1}^m \mathcal{L}_1^j u(x)dW_t^j \tag{2.15}$$

We stick to the compact notation (2.15) in the following. Then the Itô formula can be written, for component k

$$du_k(X_t) = \mathcal{L}_0 u_k(X_t)dt + \mathcal{L}_1 u_k(X_t)dW_t$$

In this work we will abuse this notation slightly, by using it for functions $u : \mathbb{R}^n \to \mathbb{R}^p$ too, though understanding it as using the operators $\mathcal{L}_i$ element-wise on u. We thus simply omit the subscript k above for the vector-valued function u.

We now turn to stochastic Taylor expansions, which are important when designing numerical methods for solving SDEs. Such an expansion come about by repeatedly using the Itô

formula in the integrals of (2.8), assuming sufficient differentiability of the relevant functions. The series are also known as Wagner-Platen series, described in detail in [18]. For simplicity we write $t_0 = 0$. Using the $\mathcal{L}_i$ operators defined in (2.13)-(2.14), we get that (2.8) can be written

$$
\begin{aligned}
Y_t = Y_0 &+ \int_0^t \left((f(Y_0) + \int_0^{s_1} (\mathcal{L}_0 f)(Y_{s_2}) ds_2 + \int_0^{s_1} (\mathcal{L}_1 f)(Y_{s_2}) dW_{s_2} \right) ds_1 \\
&+ \int_0^t \left((g(Y_0) + \int_0^{s_1} (\mathcal{L}_0 g)(Y_{s_2}) ds_2 + \int_0^{s_1} (\mathcal{L}_1 g)(Y_{s_2}) dW_{s_2} \right) dW_{s_1}
\end{aligned}
\tag{2.16}
$$

Repeated use of the operators would give a series evaluated at Y_0, when differentiability is sufficient. Remember that the operator $\mathcal{L}_1$ is really a composition of m operators for each noise term. Such an expansion is also applicable for a transformation $F(Y_t)$, by the Itô formula, yielding the transformations stochastic Taylor expansion. This is also known as the Wagner-Platen series. A sufficiently differentiable function $f(Y_s)$ can thus be expanded further, giving

$$
\begin{aligned}
f(Y_s) = f(Y_0) & \\
&+ \int_0^s \left[(\mathcal{L}_0 f)(Y_0) + \int_0^{s_1} \mathcal{L}_0(\mathcal{L}_0 f)(Y_{s_2}) ds_2 + \int_0^{s_1} \mathcal{L}_1(\mathcal{L}_0 f)(Y_{s_2}) dW_{s_2} \right] ds_1 \\
&+ \int_0^s \left[(\mathcal{L}_1 f)(Y_0) + \int_0^{s_1} \mathcal{L}_0(\mathcal{L}_1 f)(Y_{s_2}) ds_2 + \int_0^{s_1} \mathcal{L}_1(\mathcal{L}_1 f)(Y_{s_2}) dW_{s_2} \right] dW_{s_1} \\
&= f(Y_0) + h(\mathcal{L}_0 f)(Y_0) + (\mathcal{L}_1 f)(Y_0) W_s \\
&+ \int_0^s \left[\int_0^{s_1} \mathcal{L}_0(\mathcal{L}_0 f)(Y_{s_2}) ds_2 + \int_0^{s_1} \mathcal{L}_1(\mathcal{L}_0 f)(Y_{s_2}) dW_{s_2} \right] ds_1 \\
&+ \int_0^s \left[\int_0^{s_1} \mathcal{L}_0(\mathcal{L}_1 f)(Y_{s_2}) ds_2 + \int_0^{s_1} \mathcal{L}_1(\mathcal{L}_1 f)(Y_{s_2}) dW_{s_2} \right] dW_{s_1}
\end{aligned}
\tag{2.17}
$$

In general, the truncated series can be expressed as [18]

$$
f(Y_t) = f(Y_0) + \sum_\alpha \mathcal{L}_\alpha f_\alpha(Y_0) I_\alpha
\tag{2.18}
$$

where α is called a multi-index, of the form $(a_1, ..., a_k)$, $a_i \in [0, 1, 2, ..., m]$ generally for m-dimensional noise. The multi-index tells what operators are used in what order, and what iterated integral is involved in that term. Each a_i denotes one integrator in a multiple integral.

An example is $\alpha = (1, 0, 1)$, which corresponds to the term

$$
(\mathcal{L}_1 \mathcal{L}_0 \mathcal{L}_1 f)(Y_0) \int_0^t \int_0^{s_1} \int_0^{s_2} dW_{s_3} ds_2 dW_{s_1}
\tag{2.19}
$$

The notation is very useful when creating higher-order methods, which we give an example of in the next section. Integrating from t_n to $t_n + h$, the order of a term $\mathcal{L}_\alpha f(X_0) I_\alpha$ in h, for

$m = 1$, is simply [18]

$$O(h^n): \ n = \sum_{a_i=0} 1 + \sum_{a_i=1} \frac{1}{2} \tag{2.20}$$

In other words, each integrand representing time ds in the iterated integral, contributes with order 1, while each stochastic integrand dW_s contributes with order $1/2$. In this, numerical methods for SDEs can be of order $\frac{k}{2}$, $k \in \mathbb{N}$.

If the diffusion is independent of the solution, i.e. $g = g(t)$ generally, not a function of X_t, we say that the noise is additive. If the diffusion is dependent on the solution, $g = g(t, X_t)$ and the noise is multiplicative. The Wagner-Platen series for additive noise simplifies a lot for the diffusion, as $\mathcal{L}_1 g(s) \equiv 0$. The same simplifications will happen to the numerical methods.

2.4   Numerical methods

One way to construct numerical approximations of an SDE is to take a small step h in the integrals of (2.8), select constant integrands, and approximate the integral as

$$X_1 = X_0 + f(\tilde{X}_{0,1})h + g(\tilde{X}_{0,2})\Delta W \tag{2.21}$$

where $\tilde{X}_{0,i} \in [Y_0, Y_h]$ and $\Delta W = W_h - W_0 \sim \mathcal{N}(0, h)$, normally distributed. Y_t is the exact solution of the SDE. We get an explicit method if $\tilde{X}_{0,1} = \tilde{X}_{0,2} = X_0$, known as the Euler-Maruyama method.

A more systematic approach is to use the stochastic Taylor expansion (2.16), from which we see that the explicit Euler corresponds to the approximation of $f(X_s), g(X_s)$ of the lowest order, and exclude all higher-order terms. The explicit Euler-Maruyama method, also called the forward Euler method, is thus

$$X_{n+1} = X_n + hf(X_n) + g(X_n)\Delta W_n, \quad \Delta W_n = W_{n+1} - W_n \sim \mathcal{N}(0, h)$$

The above example is so far just a creative attempt to approximate the integrals. The two integrals can be approximated differently, meaning that we can have a method that is drift-implicit, i.e. explicitly calculated for the diffusion. Diffusion-implicit methods would require truncating the random variables used, otherwise, generally, finite absolute moments will not exist [18]. Methods with implicitness in both drift and diffusion are called fully implicit. The stochastic Taylor expansion is a more systematic approach to creating schemes and can provide approximation order.

The explicit Euler-Maruyama method is the simplest method there is, and for more complex methods one can perform the same integral approximation and include more terms from the Wagner-Platen series. The Milstein method will include the next term $\alpha = (1, 1)$,

$\mathcal{L}_1\mathcal{L}_1 g(X_0)I_{(1,1)}$, which in the scalar case is $(g_x g)(X_0)\int_0^t \int_0^{s_1} dW_{s_2} dW_{s_1} = (g_x g)(X_0)\frac{1}{2}(\Delta W^2 - h)$, where g_x denotes the derivative of g wrt. X.

A drift-implicit (one-legged) θ-method for the SDE can be written as

$$X_{n+1} = X_n + f\big((1-\theta)X_n + \theta X_{n+1}\big)h + g(X_n)\Delta W_n \qquad (2.22)$$

where $\theta \in [0,1]$. The drift-implicit Euler-Maruyama method, also called backward Euler, implicit Euler-Maruyama, is the implicit one-step method obtained when $\theta = 1$ above, i.e.

$$X_{n+1} = X_n + f(X_{n+1})h + g(X_n)\Delta W_n \qquad (2.23)$$

The method is fully drift-implicit, and is the simplest implicit scheme.

A general one-step scheme, either drift-implicit or explicit, can be denoted

$$X_{n+1} = X_n + \phi(X_n, X_{n+1}; h, t_n) + \psi(X_n, I_n; h, t_n) \qquad (2.24)$$

where the functions ψ, ϕ are increment functions, I_n is the (approximation of) the iterated integrals from the method, consisting of sampled random variables.

The concepts of convergence, stability, and consistency have to have randomness in mind, and we are applying the L_p norms for random variables.

$$\|X\|_p = (\mathbb{E}\,|X|^p))^{1/p}, \quad \|X\|_2 := \|X\| = (\mathbb{E}(|X|^2)^{1/2}$$

This usual definition of strong convergence is given in [12], [18].

Definition 2.6. *A numerical approximation X_N to the continuous solution Y_T converges strongly of order γ if*

$$\mathbb{E}(|Y_T - X_N|) \le Ch^\gamma$$

where the constant C depends on the model problem only, and the step size $h = T/N$ is sufficiently small.

The Euler-Maruyama method converges of order $1/2$, which is well-known from the literature, but will also become clear throughout this paper. For additive noise $g_x = 0$, so the strong order of convergence turns out to be 1. This also holds for the drift-implicit versions of the scheme.

The above definition does not guarantee that the higher moments of the solution converge. The concept of mean-square convergence deals with the second moment, which relates to approximating the variance of the solution accurately. The usual definition is [12]

Definition 2.7. *A numerical approximation X_N converges strongly in the mean-square sense to the continuous solution Y_T of order γ if*

$$\|Y_T - X_N\| = \mathbb{E}(|Y_T - X_N|^2)^{1/2} \le Ch^\gamma$$

where the constant C depends on the model problem only, and the step size $h = T/N$ is sufficiently small.

Since there is a clear definition of both the mean-square convergence above and the definition for B-convergence for deterministic equations, a B-convergence definition for SDE methods is also appropriate. For SDE methods it will be simply referred to as B-convergence.

Definition 2.8. *A numerical method is mean-square B-convergent of order γ when*

$$\mathbb{E}(|Y_T - X_N|^2)^{1/2} \leq Ch^\gamma$$

for sufficiently small h, and some constant C independent of the grid and the stiffness of the problem.

Mean-square convergence implies convergence in probability by the Markov inequality, i.e. for $\delta > 0$, we have

$$P(|X_n - X|^2 > \delta^2) \leq \frac{\mathbb{E}(|X_n - X|^2)}{\delta^2}$$
$$\Rightarrow \quad P(|X_n - X| > \delta) \leq \frac{\mathbb{E}(|X_n - X|^2)}{\delta^2}$$
$$\Rightarrow \quad \lim_{n\to\infty} P(|X_n - X| > \delta) \leq \lim_{n\to\infty} \frac{\mathbb{E}(|X_n - X|^2)}{\delta^2} = 0$$
$$\Rightarrow \quad \lim_{n\to\infty} P(|X_n - X| > \delta) = 0$$

which in turn implies convergence in distribution, e.g. by the Portmanteau theorem). The if the scheme is strongly convergent of order γ in the L_p norm, then it is strongly convergent of order γ too when the system is locally Lipschitz ([18], chapter 10).

Another definition of convergence might not consider the paths of the process, but how e.g. the moments of the numerical solution relate to the moments of the continuous one. This property is called weak order of convergence. It is given here for completeness. The numerical scheme has weak order of convergence γ if there is a bound

$$|E(\Psi(X_N)) - E(\Psi(Y_T))| \leq Ch^\gamma$$

that holds for fixed $\tau = nh \in \mathcal{I}$, and all $\Psi : \mathbf{R}^n \to \mathbf{R} \in \mathcal{C}$, some class of test functions $\mathcal{C}$ [18]. Often this class of test functions contains polynomials up to some order k. The Euler-Maruyama method converges of order 1 in the weak sense.

The local error $\hat{l}_n$ is defined as

$$\hat{l}_n = Y_{t_n} - \hat{Y}_n, \tag{2.25}$$
$$\hat{Y}_n = Y_{t_{n-1}} + \phi(Y_{t_{n-1}}, \hat{Y}_n; h, t_{n-1}) + \psi(Y_{t_{n-1}}, I_{n-1}; h, t_{n-1})$$

i.e. the error between the exact solution Y_{t_n} and a solution $\hat{Y}_n$ produced by the numerical method, in one step from the exact solution.

Consistency is the property that the local error converges to zero as the step size reduces. In section 3 two stability inequalities are proven, which both indicate how the local error will lead to convergence. Each inequality demands a slightly different consistency and is defined in each of the subsections.

2.4.1 Stability and one-sided Lipschitz condition for SDEs

This section shows some of the implications for SDEs when assuming a one-sided Lipschitz condition for the drift. The term dissipative is also used for SDEs whose drift satisfies a one-sided Lipschitz condition with non-positive ν.

First, we include some different stability concepts of numerical approximation of SDEs, for comparison. Linear stability analysis is based on the approximation of the following scalar test equation, known as geometric Brownian motion.

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{2.26}$$

$\mu \in \mathbb{C}^-$, with exact solution [1]

$$X_t = X_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right) \tag{2.27}$$

This SDE has a drift with negative one-sided Lipschitz constant. The following three principles are contraction principles.

Definition 2.9. *[13] If two different paths of the exact solution of an SDE* (2.9) *satisfies*

$$\|Y_t - \tilde{Y}_t\| \le \exp(\beta t)\|Y_0 - \tilde{Y}_0\|, \quad \beta \le 0$$

then it is exponentially stable in the mean-square sense.

Definition 2.10. *[12] A numerical scheme is mean-square stable if* $\mathbb{E}(|X_{n+1}|^2) \le \mathbb{E}(|X_n|^2)$ *when approximating* (2.26) *with* $(1 + h\mu)^2 + h\sigma^2 \le 1$.

Note. The condition on h, σ, μ is equivalent to that the continuous solution is exponentially mean-square stable, that is, being contractive in the L_2 norm. The analogue equivalence goes for the condition in the below definition. Mean-square stability is a much more used property than the following, which is quite demanding.

Definition 2.11. *[12] A numerical scheme is asymptotically stable if* $\lim_{j\to\infty} \mathbb{E}(|X_j|) \to 0$ *when approximating* (2.26) *with* $\mathbb{E}(\log|1 + h\mu + \sqrt{h}\sigma\mathcal{N}(0,1)|) < 0$.

Consider an SDE of the form (2.9), with a drift f satisfying a one-sided Lipschitz condition (1.1a) with parameter ν, and a diffusion g being Lipschitz continuous with parameter L_g. Comparing two exact solutions of the SDE (2.9) $Y_t, \tilde{Y}_t$ starting at $Y_0, \tilde{Y}_0$ respectively, then (from [19], lemma 1.1)

$$\begin{aligned}
\|Y_t - \tilde{Y}_t\|^2 &= \mathbb{E}(|Y_t - \tilde{Y}_t)|^2) \\
&= |Y_0 - \tilde{Y}_0|^2 + 2\,\mathbb{E}\int_0^t (f(Y_s) - f(\tilde{Y}_s))(Y_s - \tilde{Y}_s)ds \\
&+ \mathbb{E}(\int (g(Y_s)) - g(\tilde{Y}_s)dW_s)^2
\end{aligned}$$

by the Itô formula. Next, we use the Itô isometry on the last term, the stochastic integral, giving

$$\|Y_t - \tilde{Y}_t\|^2 \le |Y_0 - \tilde{Y}_0|^2 + 2\,\mathbb{E}\int_0^t (f(Y_s) - f(\tilde{Y}_s))(Y_s - \tilde{Y}_s)ds + \mathbb{E}\int (g(Y_s) - g(\tilde{Y}_s))^2 ds$$

Then using the Lipschitz and the one-sided Lipschitz condition of g and f respectively, we get

$$\|Y_t - \tilde{Y}_t\|^2 \le |Y_0 - \tilde{Y}_0|^2 + (2\nu + L_g^2) \int_0^t \mathbb{E}\, |Y_s - \tilde{Y}_s|^2 ds \le e^{(2\nu + L_g^2)t}|Y_0 - \tilde{Y}_0|^2 \quad (2.28)$$

where Gronwall's integral inequality, lemma 2.1, was used in the final step. By this, we see that the described SDE will be exponentially stable in the mean-square sense when $\nu + \frac{1}{2}L_g^2 < 0$. Large noise can thus ruin the contractivity property, but the term dissipative still applies. If the SDE has additive noise, the continuous solution is contractive like deterministic ones.

Theorem 2.1 ensures existence and uniqueness of an exact solution, though it requires that f is Lipschitz continuous, in addition to linear growth and bounded moments. For the numerical analysis to make sense, this must hold, so Lipschitz continuous f and g are required. Initial values with bounded second moments will always be assumed. Assuming that f is Lipschitz continuous does not destroy any numerical result later on, through assuming one-sided Lipschitz continuity with independence of stiffness. Whenever the one-sided Lipschitz constant gets large and positive, the classic theory of restricted step sizes apply.

2.4.2   Creation of implicit methods

This section shows the construction of the backward Euler method. In the following way, one can create other implicit methods too, of higher order, stochastic θ-methods, also ones that are implicit in the diffusion. Using the stochastic Taylor expansion (2.17) for the SDE, which is the essential tool for higher-order methods, gives

$$Y_{t+h} = Y_t + \int_t^{t+h} f(Y_s)ds + \int_t^{t+h} g(Y_s)dW_s$$

$$= Y_t + hf(Y_t) + g(Y_t)\Delta W_t + R_1$$

$$R_1 = \int_t^{t+h} \left(\int_t^s \mathcal{L}_0 f(Y_{s_1})ds_1 + \int_t^s \mathcal{L}_1 f(Y_{s_1})dW_{s_1} \right) ds$$

$$+ \int_t^{t+h} \left(\int_t^s \mathcal{L}_0 g(Y_{s_1})ds_1 + \int_t^s \mathcal{L}_1 g(Y_{s_1})dW_{s_1} \right) dW_s$$

Now replacing $f(Y_t)$ by the following, to obtain the drift-implicit Euler formulation when neglecting R_1, R_2.

$$f(Y_t) = f(Y_{t+h}) - \int_t^{t+h} \mathcal{L}_0 f(Y_s)ds - \int_t^{t+h} \mathcal{L}_1 f(Y_s)dW_s$$

$$Y_{t+h} = Y_t + hf(Y_{t+h}) + g(Y_t)\Delta W_t + R_1 + R_2$$

$$R_2 = -h\left(\int_t^{t+h} \mathcal{L}_0 f(Y_s)ds + \int_t^{t+h} \mathcal{L}_1 f(Y_s)dW_s \right)$$

The numerical scheme is thus just as previously presented in (2.23). The next section presents two drift-implicit methods of higher order.

A piece-wise linear continuous extension of the numerical solution can be made for methods of order $1/2$, and it converges to the exact solution of the same rate as the discrete does, i.e. the two continuous processes have mean-square distance of order $O(h^{1/2})$ along the entire time interval [19] (p. 20, (1.46)).

2.4.3 Drift-implicit methods of order 1 and 1.5

The following higher-order methods can be derived the same way as above, though including the higher-order terms in their order. As we saw above, the term R_2 came about by swapping $f(Y_t)$ with $f(Y_{t+1})$, and was already $O(h^3)$. If we now want to create an implicit method of higher order, it is the approximation of $\int_t^{t+h} g(Y_s)dW_s$ that shall be refined, which gives higher order for the term $\mathbb{E}(|R_1|^2)$. Skipping to the result, the scalar drift-implicit Milstein method, consistent of order 1, can be shown to be

$$X_{n+1} = X_n + hf(X_{n+1}) + g(X_n)\Delta W_n + \frac{1}{2}g'(X_n)g(X_n)((\Delta W_n)^2 - h)$$

Regarding the general drift-implicit method (2.24), clearly, Milstein and Euler have the same increment function ϕ.

The derivative term $g'(X_n)g(X_n)$ can also be calculated by difference methods, instead of the analytical derivative. This is the idea behind Runge-Kutta (RK) methods, which also is well-studied for SDEs [18]. The difference methods can either be explicitly defined or implicitly. If we want a method by even higher order than Milstein's method, the derivatives of f will appear in some way.

The following is a drift-implicit stochastic RK method of order 1.5 from [18]. Since we will only use it in a multidimensional example with scalar noise, we restrict the method to this setting, while the general form is given in the reference.

$$H_1 = X_n + f(X_n)h + g(X_n)\sqrt{h}$$
$$H_2 = X_n + f(X_n)h - g(X_n)\sqrt{h}$$
$$H_3 = H_1 + g(H_1)\sqrt{h}$$
$$H_4 = H_1 - g(H_1)\sqrt{h}$$

$$
\begin{aligned}
X_{n+1} = X_n &+ \frac{h}{2}\big(f(X_{n+1}) + f(X_n)\big) \\
&+ g(X_n)\Delta W_n \\
&+ \frac{1}{2\sqrt{h}}\big(f(H_1) - f(H_2)\big)\big(I_{(1,0)} - \tfrac{1}{2}h\Delta W\big) \\
&+ \frac{1}{2\sqrt{h}}\big(g(H_1) - g(H_2)\big)I_{(1,1)} \\
&+ \frac{1}{2}\big(g(H_1) - 2g(Y_n) + g(H_2)\big)I_{(0,1)} \\
&+ \frac{1}{2}\big(g(H_3) - g(H_4) - g(H_1) + g(H_2)\big)I_{(1,1,1)} \quad\quad (2.29)
\end{aligned}
$$

The H_i are called internal stages, which in this case are explicitly given for all i. The only implicit part of the above is the term $\frac{h}{2}f(X_{n+1})$. Therefore, solving the nonlinear equation is not particularly more difficult than for the implicit Euler method, but the method is very expensive in terms of function evaluations. The latter is a disadvantage of RK methods [27].

The multi-index I_α is very useful above. The iterated integrals are exact when the multi-index only consist of the same number. In this case, there exist fairly simple recursive relations giving the higher-order integrals, i.e. $I_{(j,j)}$. The mixed integrals, $I_{(0,1)}, I_{(1,0)}$, will have to be approximated by sampling one additional normally distributed variable. For multiple Wiener processes, the difficulty increases drastically [19], which makes higher-order schemes for SDEs with multiplicative, multidimensional noise a challenge.

$$I_0 = h \qquad I_1 = \Delta W_n \qquad I_{(1,1)} = \frac{1}{2}\Delta W_n^2 - \frac{1}{2}h$$

$$I_{(1,1,1)} = \frac{1}{3!}(I_1^3 + 3hI_1)$$

$$I_{(0,1)} = \frac{h}{2}\Big(I_1 - \frac{1}{\sqrt{3}}\zeta_n\Big)$$

$$I_{(1,0)} = \frac{h}{2}\Big(I_1 + \frac{1}{\sqrt{3}}\zeta_n\Big), \quad \zeta_n \sim \mathcal{N}(0,h) \sim \Delta W_n$$

The RK method will be used as a reference solution in a later example when the analytical solution is not known.

3 Proving convergence by stability and consistency

This section contains the main results, starting with a discussion on existing theory, both for one-sided Lipschitz continuous drift, and Lipschitz continuous drift. Understanding the latter, which is the classic case, is important in order to develop results for one-sided Lipschitz continuity.

3.1 Convergence principles assuming regular Lipschitz condition

The classic way of proving numerical convergence is to assume Lipschitz continuity of our functions f and g, which hold in what is discussed here. Implicit methods are under consideration.

Two proofs will be discussed here, one by Milstein [19], and one by Winkler [28]. The proof by Milstein is a classic reference, which states that a scheme is mean-square convergent of order $p_2 - 1/2$ if the local error of a scheme satisfies

$$|\mathbb{E}(\hat{l}_n | \mathcal{F}_{n-1})| \leq K(1 + |Y_{t_{n-1}}|^2)^{1/2} h^{p_1} \tag{3.1}$$

$$\|\hat{l}_n\| \leq K(1 + |Y_{t_{n-1}}|^2)^{1/2} h^{p_2} \tag{3.2}$$

$$p_2 \geq \frac{1}{2}, \quad p_1 \geq p_2 + \frac{1}{2}$$

for some constant K, grid-independent. Thus, the scheme is convergent of order $p_2 - 1/2$, by the implications previously discussed. The local error condition can be interpreted as a consistency condition of the scheme. The definition above points out that there is a possibility that p_1 is even larger than $\gamma + 1$, though it will not help to have an even larger p_1, p_2 decides the convergence rate. The proof of Milstein is based more on results from probability theory and SDEs, compared to Winkler's proof.

Winkler structures the proof quite differently. By posing some reasonable Lipschitz conditions on the increment functions ϕ and ψ from the general one-step method (2.24), they establish a stability inequality, showing how perturbations propagate with the solution. The stability inequality resembles BS-stability, definition 2.3. The difference from BS-stability is however that the perturbations in this stability inequality have a specific order of h. In the end, convergence follows from stability and a suitable consistency condition. The consistency condition is almost indistinguishable from the one implied in Milstein, where p_2 corresponds to $\gamma + 1/2$, p_1 to $\gamma + 1$.

Winkler is able to separate the problem into several pieces. They also end up with specific error bounds, partly because it is divided into distinct parts. Since convergence is split up into stability and consistency, the effect real perturbations have on the convergence is also estimated.

3.2   Convergence principles with one-sided Lipschitz condition

The two seemingly first publications proving results for SDEs using the one-sided Lipschitz condition were first Hu [15], then Higham et al.[14]. To relate the works to the previously mentioned works, Higham has several similarities with Milstein, relying more on results from probability theory. Hu proves convergence based to a large degree on calculus results, seemingly more like Winkler. The main result of this work is inspired by the two latter, due to their compatibility.

The stiffness is by assumption in the drift f, which also is assumed to satisfy a one-sided Lipschitz condition, in both cases, denoting its parameter ν. The diffusion g a global Lipschitz condition with parameter L_g. To prove optimal rate of convergence some kind of monotonicity condition is assumed for both f and g.

Higham shows mean-square convergence for any $t \in \mathcal{I}$ for a continuous-time extension of the discrete solution of the kind

$$\bar{X}(t) = X_0 + \int_0^t f(X(s))ds + \int_0^t g(X(s))dW_s, \quad X(t) = X_n, \quad t \in [t_n, t_{n+1})$$

$X(t)$ is thus a stepping process, continuous on the right, limit on the left, and coincides with the discrete solution in the points t_i. This measure of convergence is stronger than the one from definition 2.6.

The continuous-time extension Hu uses is the linear piece-wise continuous one described in section 2.4.2, and its convergence is simply provided by regular mean-square convergence, definition 2.6. Hu proves mean-square convergence for the drift-implicit Euler method of order 1/2. The convergence is proved for step sizes $h\nu < 1$, thus for negative one-sided Lipschitz constants, this has no step size restriction. For positive ν, the error constant is very large, though grid-independent. This also occurs in the stability proof of section 3.4, which is greatly inspired by the work of Hu.

Higham proves convergence of the SSBE method very elegantly by showing its relation to an explicit method, then showing convergence for the resulting explicit method. Subsequently, they prove convergence of the backward Euler method, by showing its relation to the SSBE, then employing the aforementioned results. However, all results rely on a condition that appears when showing the first equivalence, where the step size is restricted to $h < \frac{1}{2\beta}$, for $\beta = \max(\nu + 1/2, 2L_g^2)$. The work in [25] extends the result to split-step one-leg theta methods and still has a step size restriction as strict.

Hu and Higham do not use any terms like C-stability or B-consistency. It makes sense since their works do not attempt splitting up the convergence results into such parts. More recent works have started using such terminology [2, 25]. B-convergence from definition 2.8 is clear. The works [2, 3] compare to our work, where they establish B-consistency and C-stability, and thereby deduce B-convergence. They consider drift-implicit Euler and Milstein projection-type schemes, and the SSBE method. The stability condition established here is different, but the notion of B-consistency is similar. This will be pointed out in section 3.4. They use a general, positive $L = \max(\nu, L_g)$ for the sum of (1.1a) and (1.1b), which results in the step size restriction $h < 1/L$.

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3.3 Stability and convergence of the general implicit method

This section proves that the general drift-implicit one-step scheme satisfies a stability inequality, which allows deducing convergence from consistency. The order of convergence is the same as the order of consistency. The consistency definition required is raised by the stability inequality, which will become clear. The convergence proof becomes quite simple and is stated at the end. The proof is an alteration of the one in [28], and we obtain a more sensible step size criterion. Most of the proof remains the same, and it reveals how a similar stability inequality can be established for stiff systems, which will be treated in the next section.

For the general drift-implicit method (2.24), we assume that the step size is constant for simplicity. We have $n \in \{1, ..., N\}$, $h = T/N$. The parameter I_n represent potential multiple stochastic integrals when higher-order methods are considered. The implicit increment function ϕ corresponds to the estimation of the drift, while ψ can correspond to both the diffusion integral but also higher-order elementary differentials of f due to the Itô formula and the stochastic Taylor expansion. The term is approximated explicitly. Including perturbations d_n^*, the general scheme we consider is

$$X_{n+1}^* = X_n^* + \phi(X_n^*, X_{n+1}^*; h, t_n) + \psi(X_n^*, I_n; h, t_n) + d_n^* \tag{3.3}$$

The following set of assumptions will be used later on:

Assumption A. For $t, t+h \in \mathcal{I}$, positive constants $L_{\phi,1}, L_{\phi,2}, L_\psi$, and $\mathcal{F}_t$-measurable random variables $y_t, \tilde{y}_t \in \mathbb{R}^d$:

$$\mathbb{E}(\psi(y_t; t, h, I_{t,h}) - \psi(\tilde{y}_t; t, h, I_{t,h})|\mathcal{F}_t) = 0 \tag{A1}$$

$$\mathbb{E}(|\psi(y_t; t, h, I_{t,h}) - \psi(\tilde{y}_t; t, h, I_{t,h})|^2|\mathcal{F}_t) \leq hL_\psi^2|y_t - \tilde{y}_t|^2 \tag{A2}$$

$$\mathbb{E}(|\psi(0; t, h, I_{t,h})|^2) \leq \infty \tag{A3}$$

Assumption B. For all $z, \tilde{z}, x, \tilde{x} \in \mathbb{R}^d$,

$$|\phi(x, z; h, t_n) - \phi(\tilde{x}, \tilde{z}; h, t_n)| \leq hL_{\phi,1}|x - \tilde{x}| + hL_{\phi,2}|z - \tilde{z}| \tag{B1}$$

The SDE (2.9) has a unique strong solution for Lipschitz continuous f, g, with maximal growth $C(1 + |x|)$ and initial variable with finite second moment, by theorem 2.1. For the drift-implicit Euler-Maruyama method, these conditions imply that assumptions *A* and *B* hold.

The theorem on numerical stability now follows.

Theorem 3.1. *Assuming that the general drift-implicit scheme* (3.3)*, used to approximate the SDE* (2.9)*, ψ satisfies assumption A and ϕ satisfies assumption B. Then there exists positive $h_0 < \frac{1}{\sqrt{10L_{\phi,2}}}$ s.t. for any $h \in (0, h_0]$ the following holds:*

Given $\mathcal{F}_{t_0}$-measurable, square-integrable initial values X_0^*, $\tilde{X}_0$ and all $\mathcal{F}_{t_n}$-measurable perturbations $\tilde{d}_n, d_n^*$ with finite second moments, there exists unique solutions of the perturbed drift-implicit scheme (3.3), and the global error $e_n = X_n^* - \tilde{X}_n$ satisfies

$$\max_{n=1,...,N} \|e_n\| \leq S\Big\{\|e_0\| + \max_{n=1,...,N}(\|d_n\|h^{-1/2} + \|\bar{d}_n\|h^{-1})\Big\}$$

where $d_n = d_n^* - \tilde{d}_n$, $\bar{d}_n = \mathbb{E}(d_n|\mathcal{F}_{t_{n-1}})$ and

$$S = \sqrt{2\hat{S}} \max(1, T) \tag{3.4a}$$

$$\hat{S} = \frac{5}{(1 - 10h^2 L_{\phi,2}^2)} \exp\Big(\frac{5}{(1 - 10h^2 L_{\phi,2}^2)}(L_\psi^2 T + 2T^2(L_{\phi,1}^2 + L_{\phi,2}^2))\Big) \tag{3.4b}$$

Proof. The proof can be divided into three natural parts, being firstly proving existence and uniqueness of a solution of the nonlinear algebraic equation (3.3), then proving that this solution has finite second moments, through (A3), and lastly showing the stability inequality above. Only the last part of the following proof will differ from the reference, thus we refer to [28] for details.

First we introduce some notations.

$$\Delta\phi_{n+1} = \phi(X_n^*, X_{n+1}^*; h, t_n) - \phi(\tilde{X}_n, \tilde{X}_{n+1}; h, t_n)$$
$$\Delta\psi_n = \psi(X_n^*; h, t_n, I_n) - \psi(\tilde{X}_n; h, t_n, I_n)$$

The global error e_{n+1} can thus be expressed as

$$e_{n+1} = e_n + \Delta\phi_{n+1} + \Delta\psi_n + d_n \tag{3.5}$$

$$= e_0 + \Delta\phi_{n+1} + \sum_{i=1}^{n}\Delta\phi_i + \sum_{i=0}^{n-1}\Delta\psi_i + \sum_{i=0}^{n}d_i \tag{3.6}$$

Young's inequality, lemma 2.3, gives

$$|e_{n+1}|^2 \leq 5\Big\{|e_0|^2 + |\Delta\phi_{n+1}|^2 + |\sum_{i=1}^{n}\Delta\phi_i|^2 + |\sum_{i=0}^{n}\Delta\psi_i|^2 + |\sum_{i=0}^{n}d_i|^2\Big\}$$

$$\leq 5\Big\{|e_0|^2 + |\Delta\phi_{n+1}|^2 + (n+1)\sum_{i=1}^{n}|\Delta\phi_i|^2 + |\sum_{i=0}^{n}\Delta\psi_i|^2 + |\sum_{i=0}^{n}d_i|^2\Big\}$$

$$\leq 5\Big\{|e_0|^2 + |\Delta\phi_{n+1}|^2 + N\sum_{i=1}^{n}|\Delta\phi_i|^2 + |\sum_{i=0}^{n}\Delta\psi_i|^2 + |\sum_{i=0}^{n}d_i|^2\Big\} \tag{3.7}$$

Looking at the global error in the L_2 norm, some immediate simplifications can be made using assumption (A1). Using the tower rule and $\mathcal{F}_{t_i}$-measurability of $\Delta\psi_i$, assuming $i > j$,

we see that $\mathbb{E}(\Delta\psi_i^T \Delta\psi_j) = \mathbb{E}(\mathbb{E}(\Delta\psi_i^T | \mathcal{F}_{t_i})\Delta\psi_j) = 0$, which will be the case for any $i \neq j$, by (A1). Then

$$\mathbb{E}(|\sum_{i=0}^{n} \Delta\psi_i|^2) = \mathbb{E}(|\sum_{i,j=0}^{n} \Delta\psi_i^T \Delta\psi_j|^2) \leq \sum_{i,j=0}^{n} \mathbb{E}(|\Delta\psi_i^T \Delta\psi_j|^2) = \sum_{i=0}^{n} \mathbb{E}(|\Delta\psi_i|^2)$$

$$= \sum_{i=0}^{n} \mathbb{E}\,\mathbb{E}(|\Delta\psi_i|^2\,|\mathcal{F}_{i-1}) \underbrace{\leq}_{(A2)} \sum_{i=0}^{n} h L_\psi^2 \|e_i\|^2 \tag{3.8}$$

The Lipschitz condition (B1) gives

$$|\Delta\phi_{n+1}|^2 + N\sum_{i=0}^{n} |\Delta\phi_i|^2 \leq 2h^2 L_{\phi,2}^2 |e_{n+1}|^2 + 2h^2 L_{\phi,1}^2 |e_n|^2$$

$$+ 2h^2 N \sum_{i=1}^{n} L_{\phi,2}^2 |e_i|^2 + L_{\phi,1}^2 |e_{i-1}|^2$$

$$\leq 2h^2 L_{\phi,2}^2 |e_{n+1}|^2 + 2N^2 h^2 \frac{1}{N}(L_{\phi,1}^2 + L_{\phi,2}^2)\sum_{i=0}^{n} |e_i|^2 \tag{3.9}$$

For later use we define $\hat{L} = 2(L_{\phi,1}^2 + L_{\phi,2}^2)$. Equation (3.7) is thus

$$\|e_{n+1}\|^2 \leq 10h^2 L_{\phi,2}^2 \|e_{n+1}\|^2$$

$$+ 5\Big\{\mathbb{E}(|e_0|^2) + L_\psi^2 \frac{T}{N}\sum_{i=0}^{n}\mathbb{E}(|e_i|^2) + \hat{L}\frac{T^2}{N}\sum_{i=0}^{n}\mathbb{E}(|e_i|^2) + \mathbb{E}(|\sum_{i=0}^{n} d_i|^2)\Big\}$$

The resulting implicit term on the right-hand side is $10h^2 L_{\phi,2}^2 \|e_{n+1}\|^2$, which can shortly be moved to the left-hand side. For the next inequality to hold, we will require that $1 - 10h^2 L_{\phi,2}^2 > 0$. Still, this expression can not approach zero, and the following ensures that this issue is avoided.

$$\exists h_0 < \frac{1}{\sqrt{10}L_{\phi,2}} \text{ s.t. } \forall h \in (0, h_0] : 1 - 10h^2 L_{\phi,2}^2 > 0 \tag{3.10}$$

Under the step size condition (3.10), the error can be estimated by

$$\|e_{n+1}\|^2(1 - 10h^2 L_{\phi,2}^2) \leq 5\Big\{\mathbb{E}(|e_0|^2) + L_\psi^2 \frac{T}{N}\sum_{i=0}^{n}\mathbb{E}(|e_i|^2) + \hat{L}\frac{T^2}{N}\sum_{i=0}^{n}\mathbb{E}(|e_i|^2) + \mathbb{E}(|\sum_{i=0}^{n} d_i|^2)\Big\}$$
$$\tag{3.11}$$

$$= 5\Big\{\|e_0\|^2 + \mathbf{L}\frac{1}{N}\sum_{i=0}^{n}\mathbb{E}(|e_i|^2) + \mathbb{E}(|\sum_{0=1}^{n} d_i|^2)\Big\}$$

$$\|e_{n+1}\|^2 \leq \frac{5}{(1 - 10h^2 L_{\phi,2}^2)}\Big\{\|e_0\|^2 + \mathbf{L}\frac{1}{N}\sum_{i=0}^{n}\mathbb{E}(|e_i|^2) + \mathbb{E}(|\sum_{i=0}^{n} d_i|^2)\Big\}$$

$$\mathbf{L} := (L_\psi^2 T + \hat{L}T^2)$$

Next applying Gronwall's discrete inequality (2.4) on the equation above, giving

$$\|e_{n+1}\|^2 \leq \hat{S}\Big\{\|e_0\|^2 + \mathbb{E}\,|\sum_{i=0}^{n} d_i|^2\Big\}, \quad \hat{S} = \frac{5}{(1 - 10h^2 L_{\phi,2}^2)} \exp(\frac{5}{(1 - 10h^2 L_{\phi,2}^2)} \mathbf{L}))$$

(3.12)

Lastly we treat the mean-square norm of perturbations by considering the splitting of each perturbation into

$$d_i = r_i + s_i, \quad s_i = d_i - \mathbb{E}(d_i|\mathcal{F}_{n-1}), \quad \mathbb{E}(s_i|\mathcal{F}_{n-1}) = 0, \quad r_i = \mathbb{E}(d_i|\mathcal{F}_{n-1}) \quad (3.13)$$

Then $\mathbb{E}(s_i^T s_j) = 0$, $i \neq j$ by the same tower property argument as above. The sum regarding s_i's will then be

$$\mathbb{E}(|\sum_{i=0}^{n} s_i|^2) = \sum_{i=0}^{n} \mathbb{E}(|s_i|^2)$$

$$\mathbb{E}(|\sum_{i=0}^{n} d_i|^2) \leq 2\,\mathbb{E}(|\sum_{i=0}^{n} r_i|^2) + 2\sum_{i=0}^{n} \mathbb{E}(|s_i|^2) \leq 2(n+1)\sum_{i=0}^{n} \mathbb{E}(|r_i|^2) + 2\sum_{i=0}^{n} \mathbb{E}(|s_i|^2)$$

$$\leq 2\sum_{i=0}^{n} (\mathbb{E}(|s_i|^2) + \mathbb{E}\,|r_i|^2 \frac{Nh}{h} \leq 2 \max_{i=0,\ldots,N} (\mathbb{E}\,|s_i|^2 + \mathbb{E}\,|r_i|^2 \frac{Nh}{h})N$$

Inserting the above into (3.12), and taking the maximum,

$$\max_{n=1,\ldots,N} \|e_n\|^2 \leq \hat{S}\Big\{\|e_0\|^2 + 2 \max_{i=1,\ldots,N} (\mathbb{E}\,|s_i|^2 T/h + \mathbb{E}\,|r_i|^2 \frac{T^2}{h^2})\Big\}$$

$$\max_{n=1,\ldots,N} \|e_n\| \leq \tilde{S}\Big\{\|e_0\| + 2 \max_{i=1,\ldots,N} (\|s_i\|\sqrt{T/h} + \|r_i\|T/h)\Big\}$$

$$\max_{n=1,\ldots,N} \|e_n\| \leq S\Big\{\|e_0\| + \max_{i=1,\ldots,N} (\|s_i\|h^{-1/2} + \|r_i\|h^{-1})\Big\}$$

$$S = \sqrt{2\hat{S}}\,\max(1, T)$$

The final transition back to the original perturbations is made by replacing $r_i = \bar{d}_i$, and the fact that

$$\|s_i\|^2 = \|d_i - \bar{d}_i\| = \mathbb{E}\,|d_i - \bar{d}_i|^2 \leq \mathbb{E}\,|d_i - \bar{d}_i|^2 + \mathbb{E}\,|\bar{d}_i|^2 = \mathbb{E}\,|d_i|^2 - 2\,\mathbb{E}(d_i\bar{d}_i) + 2\,\mathbb{E}\,|\bar{d}_i|^2$$
$$= \mathbb{E}\,|d_i|^2 - 2\,\mathbb{E}(\mathbb{E}(d_i\bar{d}_i|\mathcal{F}_{t_{i-1}})) + 2\,\mathbb{E}\,|\bar{d}_i|^2 = \mathbb{E}\,|d_i|^2 = \|d_i\|^2$$

(3.14)

$\square$

Remark. Again note that most of the ideas applied above are identical to the ones by Winkler [28]. The small difference from the reference, of writing (3.6) in five terms as opposed to four, results in a different step size requirement. The main point is that the implicit term $\Delta\phi_{n+1}$ is excluded from the sum, since it must be treated differently. Once we use Young's

inequality on the sum, the implicit term avoid having a coefficient N. This differs from the reference, which instead of (3.10) ends up conditioning that

$$8TL^2_{\phi,2}h_0 < \frac{1}{2} \quad \Rightarrow \quad h_0 < \frac{1}{16TL^2_{\phi,2}}$$

since in (3.9), their coefficient of $|e_n|^2$ is $8Nh^2L^2_{\phi,2} = 8ThL^2_{\phi,2}$, and the criterion $8Th_0L^2_{\phi,2} < 1$ must hold.

The factor 16 above comes from repeated use of Young's inequality and is not necessarily observable in a numerical experiment. The same goes for the factor $\sqrt{10}$ the relation $h_0 < 1/(\sqrt{10}L_{\phi,2})$. This step size criterion is essentially identical to the one for deterministic equations.

The stability constant $\hat{S}$ in (3.4b) is huge in several problems, comparable to what we saw in the introduction for Lipschitz continuous functions f, though even larger now. This is partly due to noise, which contributes exponentially like for the backward Euler method in the introduction. $\hat{S}$ is dependent on h in (3.4b). This formulation thus preserves the details of the proof. The dependency on h can be avoided by e.g. assuming that $h_0^2 < (2(10L^2_{\phi,2}))^{-1}$, then

$$\frac{5}{1 - 10h^2L^2_{\phi,2}} < 10$$

If $L_{\phi,2}$ becomes large, the theorem only admits very small step sizes.

For a system with additive noise, any term with L_ψ will vanish, meaning that e.g. $\mathbf{L} = \hat{L}T^2$. If the implicit increment function ϕ only depends on the implicit part, as for the drift-implicit Euler-Maruyama method, the Lipschitz constant $L_{\phi,1}$ also vanishes in the same way. This results in

$$\hat{S} = 8\exp(4T^2L^2_{\phi,2}) \quad \text{when}$$
$$4h^2L^2_{\phi,2} \leq 1/2$$

with this approach. Some coefficients are reduced, due to fewer terms when applying Young's inequality. The result is a much smaller stability constant.

It also coincides with local error assumptions made by Milstein [19].

Definition 3.1. *A numerical method is consistent of order γ when*

$$\|\hat{l}_n\| \leq C\,h^{\gamma+1/2}$$
$$\|\mathbb{E}(\hat{l}_n|\mathcal{F}_{n-1})\| \leq \bar{C}\,h^{\gamma+1}$$

for sufficiently small h, and some grid-independent constants $C, \bar{C}$.

The theorem on convergence now follows easily, like in [28].

Theorem 3.2. *A general drift-implicit one-step method* (2.24) *is convergent of order γ if it is stable in the above sense, $Y_0 = X_0$, and consistent of order γ.*

Proof. Using the definition of the local error (2.25), and $\hat{Y}_{n+1}$,

$$
\begin{aligned}
Y_{t_{n+1}} &= \hat{l}_{n+1} + \hat{Y}_{n+1} \\
&= \hat{l}_{n+1} + Y_{t_n} + \phi(Y_{t_n}, \hat{Y}_{n+1}; h, t_n) + \psi(Y_{t_n}; h, t_n, I_n)
\end{aligned}
$$

$$
\begin{aligned}
Y_{t_1} &= \hat{l}_1 + \hat{Y}_{t_1} \\
&= \hat{l}_1 + Y_{t_0} + \phi(Y_{t_0}, \hat{Y}_1; h, t_n) + \psi(Y_{t_0}; h, t_0, I_0)
\end{aligned}
$$

The above formulation shows that the exact solution $Y_{t_{n+1}}$ can be formulated as the solution using the numerical method (3.3) from Y_{t_0} with perturbation $d_i^* = \hat{l}_i$ in each step. The approximate solution X_n, starting from $X_0 = Y_{t_0}$, is the solution obtained by using the same method with no perturbations.

The next step is inserting the exact and approximate solution in the stability inequality above, theorem 3.1, which holds by assumption. As pointed out, the two solutions are both solved with method (3.3), where the perturbations of Y_{t_i} are $\hat{l}_i$, and perturbations of X_i are zero. Then, the difference of perturbations is $d_i = d_i^* - \tilde{d}_i = \hat{l}_i - 0$. Since the method is assumed stable, we have from theorem 3.1

$$
\max_{n=1,\ldots,N} \|Y_{t_n} - X_n\| \le \max_{n=1,\ldots,N} S\{\|\hat{l}_n\| h^{-1/2} + \|\mathbb{E}(\hat{l}_n \,|\, \mathcal{F}_{n-1})\| h^{-1}\} \le O(h^\gamma) + O(h^\gamma)
$$

By assumption and definition 3.1, $\|\hat{l}_n\| = O(h^{\gamma+1/2})$ and $\|\mathbb{E}(\hat{l}_n \,|\, \mathcal{F}_{n-1})\| = O(h^{\gamma+1})$.

Inserted into the above, this yields $\max_{n=1,\ldots,N} \|Y_{t_n} - X_n\| = O(h^\gamma)$. Then also $\|Y_{t_N} - X_N\| = O(h^\gamma)$, which is the definition of convergence of order γ. $\square$

For later reference, it would also be possible to create a recursion from (3.5) in the proof, but it would cause a far worse error constant in the end. This will be illustrated at the end of the next section when considering drifts satisfying a one-sided Lipschitz condition. In this case, we have not found a way to complete the proof by using (3.6), one has to use (3.5).

3.4   Convergence and stability of drift-implicit methods with one-sided Lipschitz condition

This section proves that certain implicit methods converge with order γ, provided consistency of order γ independent of stiffness, and stability independent of stiffness. The stability inequality here is similar to the previous one, though independent of stiffness under certain conditions. The general ideas are very similar to the previous section, inspired by Winkler [28], combined with the ideas of Hu [15]. First, a description of the admissible methods is given, as it is much more restricted than in the previous section.

The next theorem regards one-step approximations of SDE systems (2.9) for which the increment function ϕ satisfies a one-sided Lipschitz condition (1.1a). We now have to restrict the methods under consideration. The increment functions ϕ will here only be fully implicit, i.e. we are considering

$$X^*_{n+1} = X^*_n + \phi(X^*_{n+1}; h, t_n) + \psi(X^*_n; h, t_n, I_n) + d^*_n \tag{3.15}$$

To avoid any differential terms of f, which is assumed stiff, Taylor based methods only up to order 1 are applicable. For any higher-order method, e.g. the Runge-Kutta method of order 1.5 (2.29), the term $I_{(1,0)}(\mathcal{L}^1 f)$ has to be included. In the following, assumption B is replaced by assumption C, ensuring that we do not rely on the Lipschitz constant of f.

Assumption C. For all $x, \tilde{x} \in \mathbb{R}^d$, $t, t + h \in \mathcal{I}$, some constant ν, ϕ satisfies

$$\langle \phi(x; h, t) - \phi(\tilde{x}, h, t), x - \tilde{x} \rangle \leq h\nu|x - \tilde{x}|^2 \tag{C1}$$

There exists a unique solution of the SDE (2.9), with Lipschitz continuous f, g, with growth maximally $C(1 + |x|)$, and bounded second moment of the initial variable, by theorem 2.1. When assuming that f is one-sided Lipschitz continuous, assumption C follows for the fully implicit Milstein method and the drift-implicit Euler, since then $f = \phi$. Assumption A follows for both methods.

The theorem on numerical stability independent of stiffness now follows.

Theorem 3.3. *Assuming that assumption A and C hold for ψ, ϕ respectively, from (3.15). Given $\mathcal{F}_{t_0}$-measurable, square-integrable initial values $X^*_0, \tilde{X}_0$ and all $\mathcal{F}_{t_n}$-measurable perturbations $\tilde{d}_n, d^*_n$ with finite second moments, there exists unique solutions to (3.15), being $\{\tilde{X}_i\}^N_{i=0}$ and $\{X^*_i\}^N_{i=0}$ respectively.*

In addition, there exists positive $h_0 < 1/\nu$ s.t. for any $h \in (0, h_0]$, $e_n := X^*_n - \tilde{X}_n$, e_n satisfies

$$\max_{n=1,\dots,N} \|e_n\| \leq S\left\{ \|e_0\| + \max_{i=1,\dots,N}(\|d_i\|h^{-1/2} + \|\bar{d}_i\|h^{-1}) \right\}$$

$$S = \sqrt{2\hat{S}} \max(1, T)$$

where $d_n = d^*_n - \tilde{d}_n$, $\bar{d}_n = \mathbb{E}(d_n|\mathcal{F}_{t_{n-1}})$ and $\hat{S}$ is independent of stiffness.

If $\nu < 0$, then for any $h > 0$

$$\hat{S} = 3\exp(3L_\psi^2(-\frac{1}{2\nu}))$$

If $\nu = 0$, then for any $h > 0$

$$\hat{S} = 3\exp(3L_\psi^2 T)$$

Proof. We assume that the solution of (3.15) has an existing and unique solution for each step, and that it is bounded in the L_2 norm by (A3).

We start off considering general ν, though eventually encounter a point where the sign matters, and the cases are treated differently.

Similarly to the previous section, we denote

$$e_{n+1} = X_{n+1}^* - \tilde{X}_{n+1}$$
$$\Delta\phi_{n+1} = \phi(X_{n+1}^*; h, t_n) - \phi(\tilde{X}_{n+1}; h, t_n)$$
$$\Delta\psi_n = \psi(X_n^*; h, t_n, I_n) - \psi(\tilde{X}_n; h, t_n, I_n)$$

Then, with Jacobian of ϕ denoted J_ϕ

$$e_{n+1} = e_n + \Delta\phi_{n+1} + \Delta\psi_n + d_n$$
$$e_{n+1} - \Delta\phi_{n+1} = e_n + \Delta\psi_n + d_n$$
$$e_{n+1} - h\{\int_0^1 J_\phi(\tilde{X}_{n+1}\theta + (1-\theta)X_{n+1}^*)d\theta\}e_{n+1} = e_n + \Delta\psi_n + d_n$$

Where the last transition is made using the mean-value theorem, lemma 2.4.

Now defining $A_n = \left(I - h\int_0^1 J_\phi(\tilde{X}_n\theta + (1-\theta)X_n^*)d\theta\right)^{-1}$. Since $h > 0$, by assuming that $h\nu < 1$, lemma 2.6 is applicable. It is worth pointing out that $h\nu$ can not approach 1 either.

There exists $h_0 > 0$ dependent on ν s.t. for all $h \in (0, h_0]$, $h\nu < 1$ is satisfied. When $\nu \leq 0$ no restrictions on h occur here. From this we see that $\|A_i\|_{op} \leq (1 - h\nu)^{-1} := \eta$, some bounded η, dependent on h.

$$e_{n+1} = A_{n+1}(e_n + \Delta\psi_n + d_n)$$
$$= \{\prod_{i=1}^{n+1} A_i\}e_0 + \sum_{i=0}^{n}\{\prod_{j=i+1}^{n+1} A_j\}(\Delta\psi_i + d_i) \tag{3.16}$$

Using Young's inequality (lemma 2.3) and (A1) the mean-square global error can now be bounded by

$$\mathbb{E}\,|e_{n+1}|^2 \leq 3\,\mathbb{E}\,|\{\prod_{i=1}^{n+1} A_i\}e_0|^2 + 3\,\mathbb{E}\,|\sum_{i=0}^{n}\{\prod_{j=i+1}^{n+1} A_j\}d_i|^2 + 3\sum_{i=0}^{n}\mathbb{E}\,|\{\prod_{j=i+1}^{n+1} A_j\}\Delta\psi_i|^2$$

Above the tower property and assumption (A1) allows for writing the sum of $\Delta\psi_i$ this way, just like in (3.8). Inserting the operator norm of A_j, with bound $\frac{1}{1-h\nu} = \eta$, and apply assumption (A2), results in

$$\mathbb{E}\,|e_{n+1}|^2 \le 3\eta^{2(n+1)}\|e_0\|^2 + 3\,\mathbb{E}\,|\sum_{i=0}^{n}\eta^{n-i+1}d_i|^2 + 3\sum_{i=0}^{n}\eta^{2(n-i+1)}\,\mathbb{E}\,|\Delta\psi_i|^2 \qquad (3.17)$$

Applying the Lipschitz condition for the diffusion (A2) gives the following result, to which we subsequently apply the discrete Gronwall inequality (lemma 2.2), with $b_i = 3hL_\psi^2\eta^{2(n-i+1)}$.

$$\mathbb{E}\,|e_{n+1}|^2 \le 3\eta^{2(n+1)}\|e_0\|^2 + 3\,\mathbb{E}\,|\sum_{i=0}^{n}\eta^{n-i+1}d_i|^2 + 3\sum_{i=0}^{n}\eta^{2(n-i+1)}\,\mathbb{E}\,|e_i|^2 hL_\psi^2$$

$$\le 3\Big\{\eta^{2(n+1)}\|e_0\|^2 + \mathbb{E}\,|\sum_{i=0}^{n}\eta^{n-i+1}d_i|^2\Big\}\exp(\sum_{i=0}^{n}b_i) \qquad (3.18)$$

First looking at the case of positive ν, which implies $\eta > 1$, implying that $b_{i+1} > b_i$, but it can nonetheless be shown to be bounded by some grid-independent constant. Conditioning that $h\nu < 1$ gives η convex in this region, thus $\eta \le (1 + Ch)$, where C depends on ν only.

$$\sum_{i=0}^{n}3hL_\psi\eta^{2(n-i+1)} \le nh3L_\psi^2\eta^{2(n+1)}$$

$$\le T3L_\psi^2(1+Ch)^{2(n+1)}$$

$$(1+Ch)^{2n} \le \exp(2CT) \qquad (3.19)$$

An estimate like (3.19) can be used to bound $\eta^{2(n+1)}, \eta^{n-i+1}$ and $\exp(\sum b_i)$. The estimate is clearly very large, but still grid- and stiffness-independent.

Firstly, if $\nu = 0$ then $\eta = 1$, and from (3.18) we see that

$$\exp(\sum_{i=0}^{n}3hL_\psi^2) \le \exp(3TL_\psi^2)$$

When $\nu < 0$, $\eta < 1$, so

$$\exp(\sum_{i=0}^{n}b_i) = \exp(3hL_\psi^2\sum_{i=0}^{n}\eta^{2(n-i+1)})$$

$$\sum_{i=0}^{n}\eta^{2(n-i+1)} \le \eta^{2(n+1)}\frac{1-\eta^{-2(n+1)}}{1-\eta^{-2}} = \frac{1-\eta^{2(n+1)}}{\eta^{-2}-1} \le \frac{1}{\eta^{-2}-1}$$

$$\frac{h}{\eta^{-2}-1} = \frac{h}{h^2\nu^2-2h\nu} = \frac{1}{h\nu^2-2\nu} \le \frac{-1}{2\nu}$$

$$\exp(\sum_{i=0}^{n}b_i) \le \exp(\frac{-3L_\psi^2}{2\nu})$$

There is big damping on $\|e_0\|$ and the perturbations when $\eta \in (0,1)$, but we simply estimate η^k as 1. For $\eta \in (0,1)$, i.e. $\nu < 0$ the entire bound will look like

$$\mathbb{E}\,|e_{n+1}|^2 \le 3\Big\{\|e_0\|^2 + \mathbb{E}\,|\sum_{i=0}^{n} d_i|^2\Big\}\exp(\frac{-3L_\psi^2}{2\nu})$$

It is stiffness independent since it only decreases when ν large in magnitude.

The rest of the proof concerns the sum of perturbations. This goes exactly like in the previous section, and is skipped. The desired stability inequality for $\nu \le 0$ is obtained, as before, with

$$\hat{S} = 3\exp(3L_\psi^2(-\frac{1}{2\nu}))\ ,\ \nu < 0$$
$$\hat{S} = 3\exp(3L_\psi^2 T)\ ,\ \nu = 0$$
$$S = \sqrt{2\hat{S}}\max(1,T)$$

$\square$

Remark. The inequalities used for positive ν are particularly rough since $\eta > 1$, C potentially large. The proof also uses that $1 + b_i \le \exp(b_i)$, which is also very rough for large b_i. We expect something that is growing much faster than in the case of non-positive ν, which is true. With $\nu > 0$ the SDE is no longer dissipative, so one can expect initial errors and perturbations to grow large.

Now picking up the thread mentioned at the end of the previous section. A partial explanation of this very large error bound can be found in the fact that the proof was only able to express the recursion on e_n as in (3.16), as opposed to (3.11). For the one-sided Lipschitz condition we have only been able to express this as (3.16), and it works fine for non-positive ν. This approach would give poor results for a Lipschitz continuous drift function too, since we could estimate from (3.5)

$$|e_{n+1}| = |e_n + h\Delta f_{n+1} + \Delta g_n \Delta W_n + d_n|$$
$$|e_{n+1}|(1 - 4hL_f) \le 4(|e_n|\Delta g_n \Delta W_n + d_n)$$
$$|e_{n+1}| \le \frac{4}{1 - 4hL_f}(|e_n|\Delta g_n \Delta W_n + d) = B_n(|e_n| + \Delta g_n \Delta W_n + d_n)$$

which corresponds to (3.16). After this, one could apply (3.19). This approach will give a similar result as the one above for $\nu > 0$, and will again give a very large error bound. The stability constant $\hat{S}$ is nonetheless stiffness-independent for one-sided Lipschitz continuous ν.

The resulting dependency of $\hat{S}$ on ν for $\nu < 0$ resembles the bound of the exact solution – smaller error bound for ν more negative. For $\nu = 0$, the error bound is highly similar to (2.28), the bound for the exact solution. Here it will not be $\exp((\nu + 1/2L_g^2)t)$ that decides the bound of the error, but $\max(1,T)\exp(-3L_g^2/4\nu)$, for negative ν. For $\nu = 0$ the discrete error bound is proportional to $\max(1,T)\exp(3L_g^2 T/2)$, which is in a sense close. The final error bound is however also dependent on the local order, as will be shown shortly.

Again the stability inequality implies a certain consistency definition. The following defines B-consistency.

Definition 3.2. *A numerical method is B-consistent of order γ when*

$$\|\hat{l}_n\| \leq C\, h^{\gamma+1/2}$$

$$\|\,\mathbb{E}(\hat{l}_n|\mathcal{F}_{n-1})\| \leq \bar{C}\, h^{\gamma+1}$$

for sufficiently small h independent of stiffness, and some grid-independent constants $C, \bar{C}$, independent of stiffness.

The stiffness-independent counterparts of the definitions in section 3.3 are now established. The stability inequality of theorem 3.3 also allows for deducing the rate of B-convergence, when assuming B-consistency. This constitutes the following theorem.

Theorem 3.4. *A drift-implicit one-step method up to order 1 with one-sided Lipschitz continuous drift f is B-convergent of order γ if it is stable in the sense of theorem 3.3, $Y_0 = X_0$, and B-consistent of order γ.*

Proof. This is proven exactly like theorem 3.2, though with stiffness-independent stability and consistency principles.

The exact solution can be viewed as the solution with the method (3.15) from Y_0, with perturbations $\hat{l}_i$ in each step. The numerical solution is the unperturbed solution from $X_0 = Y_0$ using the method (3.15). By assuming stiffness-independent stability and B-consistency, convergence rate γ follows, independent of stiffness. $\qquad\square$

Note that definition 3.2 of B-consistency is highly similar to the definition 3.1 of consistency, only adding a stiffness criterion. In addition, the definition coincides with B-consistency defined in [2]. They define B-consistency of order γ to be when the local error satisfies

$$\|\hat{l}_n - \mathbb{E}(\hat{l}_n|\mathcal{F}_{n-1})\| \leq C\, h^{\gamma+1/2}$$

$$\|\,\mathbb{E}(\hat{l}_n|\mathcal{F}_{n-1})\| \leq \bar{C}\, h^{\gamma+1}$$

$C, \bar{C}$ independent of grid and stiffness. The equivalence to definition 3.2 can be seen by studying the splitting of the perturbation in (3.13) and (3.14). From the convergence proof, it was shown that the exact solution is the method solution from Y_0, perturbed by the local errors. The numerical solution is just the solution from Y_0 with no perturbations, thus

$$d_i = \hat{l}_i - 0 = \hat{l}_i\,, \qquad\qquad \text{and, using (3.13), this is}$$

$$s_i = d_i - \mathbb{E}(d_i|\mathcal{F}_{n-1}) = \hat{l}_i - \mathbb{E}(\hat{l}_i|\mathcal{F}_{n-1})\,, \quad \text{then applying (3.14), giving}$$

$$\|\hat{l}_i - \mathbb{E}(\hat{l}_i|\mathcal{F}_{n-1})\| \leq \|d_i\| = \|\hat{l}_i\|$$

The results above are highly comparable to [2, 3], though obtained differently and independently. The methods they consider are related to ours, but different. They do not however consider non-positive ν, which for the results here gives no step size restriction. Their results require step size restrictions for all one-sided Lipschitz constants.

3.5   B-consistency of methods up to order 1

The consistency defined above requires that the local error $\hat{l}_{n+1}$ has order $\gamma + 1/2$ in the L_2-norm, and $\gamma + 1$ for $\| \mathbb{E}(\hat{l}_{n+1}|\mathcal{F}_n)\|$. Using the assumptions from theorem 3.3 and a monotonicity assumption on f, g we are able to prove stiffness-independent consistency for drift-implicit Euler and Milstein type methods. This proof is inspired by ideas from [2, 7, 19]. Again, the methods of higher order will have to include derivatives of f, which for stiff systems will be large. They also and up in the increment function ψ, thus changes its properties.

$$X_{n+1} = X_n + \phi(X_{n+1}; h, t_n) + \psi(X_n; h, t_n, I_n) \tag{3.20}$$

The choices are not many when we only consider these two methods. In both methods $\phi(X_n; h, t_n) = hf(t_n, X_n)$, while ψ is not equal in the two cases. For the drift-implicit Euler method $\psi(X_n; h, t_n, I_n) = g(t_n, X_n)I_{(1)}$ The Milstein method simply adds the term $\mathcal{L}_1 g(t_n, X_n)I_{(1,1)}$, which in the case of scalar noise is $\frac{1}{2}J_g(t_n, X_n)g(t_n, X_n)((\Delta W_n)^2 - h)$, J_g being the Jacobian of g. When $g \in C^1(\mathbb{R})$ the increment function ψ is Lipschitz continuous for the drift-implicit Milstein and Euler method.

This is a smoothness and a monotonicity assumption.

Assumption D. Assuming that both $\{f, g\} \ni \theta \in C^1$ and :

$$|\theta(t, x)| \le K(1 + |x|^q) \tag{D1}$$

$$|\theta(t, x_2) - \theta(t, x_1)| \le K(1 + |x_1|^{q-1} + |x_2|^{q-1})|x_2 - x_1| \tag{D2}$$

$$|\theta(t_1, x) - \theta(t_2, x)| \le K(1 + |x|^q)|t_1 - t_2|^{1/2} \tag{D3}$$

for some constant K possibly different in each estimate, $q \in (1, \infty)$, $t \in \mathcal{I}$, $x, x_1, x_2 \in \mathbb{R}^d$.

Lemma 3.1. *The drift-implicit Euler method is B-consistent of order $1/2$, independent of stiffness, and the drift-implicit Milstein method is B-consistent of order 1, also independent of stiffness, under the assumption A for ψ from (3.20), assumption C for f and assumption D for f, g.*

Proof. We assume scalar noise, $m = 1$, for simplicity. Then $g(X_i) \in \mathbb{R}^{d \times 1}$, $\Delta W_n \in \mathbb{R}$. If there were multidimensional noise, one would use the Itô isometry (2.10) and Fubini's theorem (2.11), to bound

$$\mathbb{E}(|\sum_{i=1}^m \int_{t_n}^{t_{n+1}} g_i(s, Y_s)dW_s^i|^2) = \sum_{i=1}^m \int_{t_n}^{t_{n+1}} \mathbb{E}(|g_i(s, Y_s)|^2)ds$$

then using the condition (D1), e.g. for g, bounding the above by [2]

$$\mathbb{E}(|\sum_{i=1}^m \int_{t_n}^{t_{n+1}} g_i(s, Y_s)dW_s^i|^2) \le \sum_{i=1}^m \int_{t_n}^{t_{n+1}} \sup_{t \in \mathcal{I}} K(1 + |Y_t|^q)ds$$

$$= m \int_{t_n}^{t_{n+1}} \sup_{t \in \mathcal{I}} K(1 + \mathbb{E}(|Y_t|^q))ds$$

$$\|\sum_{i=1}^m \int_{t_n}^{t_{n+1}} g_i(s, Y_s)dW_s^i\| \le C(Km)^{1/2} \sup_{t \in \mathcal{I}}(1 + \mathbb{E}(|Y_t|^q))|t_{n+1} - t_n|^{1/2}$$

This would be applied below when appropriate, simply scaling by a factor $m^{1/2}$ in relevant terms.

The next step shows the relation between the local error $\hat{l}_i$ and the residual error l_i, which is very useful. The residual error is the difference between the analytical solution at t_{n+1}, and the solution using the method one step from t_n, though inserted the analytical solution, i.e.

$$l_{n+1} = Y_{t_{n+1}} - (Y_{t_n} + hf(t_{n+1}, Y_{t_{n+1}}) + \psi(Y_{t_n}; h, t_n, I_n)) \tag{3.21}$$

$$\hat{l}_{n+1} = Y_{t_{n+1}} - \hat{Y}_{n+1} = Y_{t_{n+1}} - (Y_{t_n} + hf(t_{n+1}, \hat{Y}_{n+1}) + \psi(Y_{t_n}; h, t_n, I_n))$$

$$\hat{l}_{n+1} = l_{n+1} + h(f(t_{n+1}, Y_{t_{n+1}}) - f(t_{n+1}, \hat{Y}_{n+1}))$$

$$l_{n+1} = \hat{l}_{n+1} - h(f(t_{n+1}, Y_{t_{n+1}})) \tag{3.22}$$

In (3.22) it is very efficient to make use of the mean-value theorem (lemma 2.4), and the implications for the Jacobian of f with a one-sided Lipschitz condition, assumption C [7]. Then for $h\nu < 1$ we get

$$\underbrace{\left(I - h \int_0^1 J_f(uY_{t_{n+1}} + (1-u)\hat{Y}_{n+1})du\right)}_{A_n^{-1}} \hat{l}_{n+1} = A_n^{-1}\hat{l}_{n+1} = l_{n+1}$$

$$\|\hat{l}_{n+1}\| \le (1 - h\nu)^{-1}\|l_{n+1}\|$$

From (3.21) in integral form

$$l_{n+1} = Y_{t_n} + \int_{t_n}^{t_{n+1}} f(s, Y_s)ds + \int_{t_n}^{t_{n+1}} g(s, Y_s)dW_s$$

$$- (Y_{t_n} + hf(t_{n+1}, Y_{t_{n+1}}) + \psi(Y_{t_n}; h, t_n, I_n))$$

$$l_{n+1} = \int_{t_n}^{t_{n+1}} (f(s, Y_s) - f(t_{n+1}, Y_{t_{n+1}}))ds + \int_{t_n}^{t_{n+1}} g(s, Y_s)dW_s - \psi(Y_{t_n}; h, t_n, I_n)$$

$$\tag{3.23}$$

First showing the order of the implicit Euler method, where $\psi(Y_{t_n}) = g(Y_{t_n})\Delta W_n$. This reveals how it can be done for the Milstein case. For (3.23) we apply the triangle inequality, Itô isometry, Fubini's theorem and Jensen's inequality to get

$$\|l_{n+1}\| \le \|\int_{t_n}^{t_{n+1}} (f(s, Y_s) - f(t_{n+1}, Y_{t_{n+1}}))ds\| + \|\int_{t_n}^{t_{n+1}} (g(s, Y_s) - g(t_n, Y_{t_n}))dW_s\|$$

$$\le \|\int_{t_n}^{t_{n+1}} (f(s, Y_s) - f(t_{n+1}, Y_{t_{n+1}}))ds\| + \left(\mathbb{E}\int_{t_n}^{t_{n+1}} |g(s, Y_s) - g(t_n, Y_{t_n})|^2 ds\right)^{1/2}$$

$$\le \|\int_{t_n}^{t_{n+1}} (f(s, Y_s) - f(t_{n+1}, Y_{t_{n+1}}))ds\| + \left(\int_{t_n}^{t_{n+1}} \mathbb{E}(|g(s, Y_s) - g(t_n, Y_{t_n})|^2)ds\right)^{1/2}$$

$$\le \int_{t_n}^{t_{n+1}} \|(f(s, Y_s) - f(t_{n+1}, Y_{t_{n+1}}))\|ds + \left(\int_{t_n}^{t_{n+1}} \|(g(s, Y_s) - g(t_n, Y_{t_n}))\|^2 ds\right)^{1/2}$$

$$\tag{3.24}$$

Assumption D implies for $x_1, x_2 \in \mathbb{R}^d$, and $t_1, t_2 \in \mathcal{I}$, that

$$
\begin{aligned}
|\theta(t_1, x_1) - \theta(t_2, x_2)| &\leq |\theta(t_1, x_1) - \theta(t_1, x_2)| + |\theta(t_1, x_2) - \theta(t_2, x_2)| \\
&\leq K(1 + |x_1|^{q-1} + |x_2|^{q-1})|x_2 - x_1| + K(1 + |x_2|^q)|t_1 - t_2|^{1/2}
\end{aligned}
\tag{3.25}
$$

By these implications, the first term of (3.24) can be estimated by

$$
\begin{aligned}
\|(f(s, Y_s) - f(t_{n+1}, Y_{t_{n+1}}))\| &\leq K\|(1 + |Y_s|^{q-1} + |Y_{t_{n+1}}|^{q-1})|Y_s - Y_{t_{n+1}}|\| \\
&\quad + K\|(1 + |Y_{t_{n+1}}|^q)|s - t_{n+1}|^{1/2}\| \\
&\leq K(1 + 2\sup_{t \in \mathcal{I}} \mathbb{E}(|Y_t|^{q-1}))\|Y_s - Y_{t_{n+1}}\| \\
&\quad + K(1 + \sup_{t \in \mathcal{I}} \mathbb{E}(|Y_t|^q))|h|^{1/2}
\end{aligned}
\tag{3.26}
$$

The benefit from expressing the local error by the residual error is evident now, since we are using the exact solution at every point, and can estimate $\|Y_s - Y_{t_{n+1}}\|$ using assumption (D3). Knowing that $t_{n+1} > s$, we write this as

$$
Y_{t_{n+1}} - Y_s = \int_s^{t_{n+1}} f(s_1, Y_{s_1}) ds_1 + \int_s^{t_{n+1}} g(s_1, Y_{s_1}) dW_{s_1}
$$

By the same arguments as in (3.24), the above can be bounded by

$$
\|Y_{t_{n+1}} - Y_s\| \leq \int_s^{t_{n+1}} \|f(s_1, Y_{s_1})\| ds_1 + \left(\int_s^{t_{n+1}} \|g(s_1, Y_{s_1})\|^2 ds_1 \right)^{1/2}
$$

$$
\underbrace{\leq}_{(D3)} K(1 + \sup_{t \in \mathcal{I}} \mathbb{E}(|Y_t|^q)) \int_s^{t_{n+1}} ds_1
$$

$$
+ K(1 + \sup_{t \in \mathcal{I}} \mathbb{E}(|Y_t|^q)) \left(\int_s^{t_{n+1}} ds_1 \right)^{1/2} = O(h^{1/2})
$$

This is the order of (3.26). This gives for the first term of (3.24), by integration

$$
\int_{t_n}^{t_{n+1}} \|(f(s, Y_s) - f(t_{n+1}, Y_{t_{n+1}}))\| ds \leq K' h^{3/2}
$$

By an identical approximation, we also get that $\|g(s, Y_s) - g(t_n, Y_{t_n}))\| = O(h^{1/2})$, since they satisfy the same conditions. The integral and final order is then

$$
\left(\int_{t_n}^{t_{n+1}} \|(g(s, Y_s) - g(t_n, Y_{t_n}))\|^2 ds \right)^{1/2} \leq \left(K^2(1 + \sup_{t \in \mathcal{I}} \|Y_t\|^q)h \int_{t_n}^{t_{n+1}} ds \right)^{1/2}
\tag{3.27}
$$

$$
= hK(1 + \sup_{t \in \mathcal{I}} \|Y_t\|^q) = O(h)
\tag{3.28}
$$

This term is thus the lowest in order of h, and determines the order of the residual error. It is also the term that differs from the Milstein method, which we treat next. We only have to prove that ψ approximates the stochastic integral of g of order $O(h^{3/2})$. The

first term of (3.24) occurs there too, whose order is already proven. Milstein's method has $\psi(Y_{t_n}; h, t_n, I_n) = g(t_n, Y_{t_n})\Delta W_n + \mathcal{L}_1 g(t_n, Y_n) I_{(1,1)}$. This is inserted into (3.23), and also substitute $g(s, Y_s)$ by

$$g(s, Y_s) = g(t_n, Y_{t_n}) + \int_{t_n}^s \mathcal{L}_0 g(s_1, Y_{s_1}) ds_1 + \int_{t_n}^s \mathcal{L}_1 g(s_1, Y_{s_1}) dW_{s_1}$$

When inserted, the term $g(t_n, Y_{t_n})$ cancels with the one-step method. The residual error is thus

$$l_{n+1} = \int_{t_n}^{t_{n+1}} (f(s, Y_s) - f(t_{n+1}, Y_{t_{n+1}})) ds$$

$$+ \int_{t_n}^{t_{n+1}} \left(\int_{t_n}^s \mathcal{L}_0 g(s_1, Y_{s_1}) ds_1 + \int_{t_n}^s \mathcal{L}_1 g(s_1, Y_{s_1}) dW_{s_1} \right) dW_s - \mathcal{L}_1 g(t_n, Y_n) I_{(1,1)}$$

By order of the iterated integrals, and the fact that we already showed that the first term is $O(h^{3/2})$, we only need to consider the $I_{(1,1)}$ integral part.

Since g is differentiable and Lipschitz continuous, its Jacobian J_g satisfies $\|J_g(x)\|_{op} \le L_g, \forall x \in \mathbb{R}^d$. The term of interest can be written

$$\mathcal{L}_1 g(s_1, Y_{s_1}) = J_g(s_1, Y_{s_1}) g(s_1, Y_{s_1})$$

$$\int_{t_n}^{t_{n+1}} \int_{t_n}^s \left(\mathcal{L}_1 g(s_1, Y_{s_1}) - \mathcal{L}_1 g(t_n, Y_n) \right) dW_{s_1} dW_s$$

$$\le \int_{t_n}^{t_{n+1}} \int_{t_n}^s L_g \left(g(s_1, Y_{s_1}) - g(t_n, Y_n) \right) dW_{s_1} dW_s \qquad (3.29)$$

Next we consider the norm of the term, and use the Itô isometry twice. The final relation we insert is the fact proven above, that $\|g(s, Y_s) - g(t_n, Y_{t_n}))\|^2 = O(h^1)$.

$$\mathbb{E}\left(\left| \int_{t_n}^{t_{n+1}} \int_{t_n}^s L_g \left(g(s_1, Y_{s_1}) - g(t_n, Y_n) \right) dW_{s_1} dW_s \right|^2 \right)$$

$$\le \int_{t_n}^{t_{n+1}} \mathbb{E}\left(\left| \int_{t_n}^s L_g \left(g(s_1, Y_{s_1}) - g(t_n, Y_n) \right) dW_{s_1} \right|^2 \right) ds$$

$$\le \int_{t_n}^{t_{n+1}} \int_{t_n}^s L_g^2 \mathbb{E}\left(\left| g(s_1, Y_{s_1}) - g(t_n, Y_n) \right|^2 \right) ds_1 ds \le \int_{t_n}^{t_{n+1}} \int_{t_n}^s O(h) \, ds_1 ds = O(h^3)$$

By taking the square root, the desired order is evident.

Finally, the definition asks that $\| \mathbb{E}(\hat{l}_{n+1} | \mathcal{F}_n)\| = O(h^{\gamma+1})$. This will be $O(h^2)$ for both methods. The proof is much simpler than the above, since expectation is linear, and the Itô integral is a martingale. By the same reasons as above, we rather consider the residual error l_n. The martingale property gives that the stochastic integral and ψ cancels, for both the Euler and Milstein method. Then we are left with

$$\| \mathbb{E}(\int_{t_n}^{t_{n+1}} (f(s, Y_s) - f(t_{n+1}, Y_{t_{n+1}})) ds \, | \mathcal{F}_n)\| = O(h^2)$$

(Details in [19], proof of theorem 1.1)

$\square$

4 Examples

4.1 Linear constant coefficient SDE

We now revisit the linear example from the introduction, though with scalar, multiplicative noise. In the scalar case, the solution is known as geometric Brownian motion.

$$Y_t = \begin{pmatrix} -\mu & 1 \\ 1 & -\mu \end{pmatrix} Y_t dt + \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} Y_t dW_t = AY_t dt + BY_t dW_t, \quad Y_0 = (a, b)$$

for some constants $\mu > 1, \sigma$. This has an exact solution, which can be written for commuting matrices A, B [18]

$$Y_t = Y_0 \exp((A - \frac{1}{2}B^2)t + BW_t)$$

The drift has Lipschitz constant $L_A = \mu + 1$ and one-sided Lipschitz constant $\nu = -\mu + 1$. The diffusion is Lipschitz continuous with constant $L_B = \sigma$. Since the drift satisfies a one-sided Lipschitz condition, with negative ν for $\mu > 1$, the stability theorem

A drift-implicit scheme up to order 1 will look like

$$X_{n+1} = X_n + AX_{n+1}h + \psi(X_n; h, t_n, I_n), \quad \text{solving for } X_{n+1} \text{ gives}$$

$$X_{n+1} = (I - hA)^{-1}(X_n + \psi(X_n; h, t_n, I_n))$$

where for constant step size and matrix A, the inverse of the matrix $(I - hA)$ only has to be computed once, and is clearly equal for any path. The simplicity of the system opens up for fast simulations.

For the Milstein method, $\psi(X_n; h, t_n, I_n) = BX_n \Delta W_n + \frac{1}{2}(B^2 X_n)((\Delta W_n)^2 - h)$. For the backward Euler method $\psi(X_n; h, t_n, I_n) = BX_n \Delta W_n$

For different μ, T, and σ the order of convergence is found by calculating paths with increasing step size. The same Wiener process is used across the different step sizes, The global error is calculated from step size $T/2048$ up to $T/4$ in each figure, thus 10 different step sizes. We use the same initial value for all paths, $X_0 = (2, 0)$.
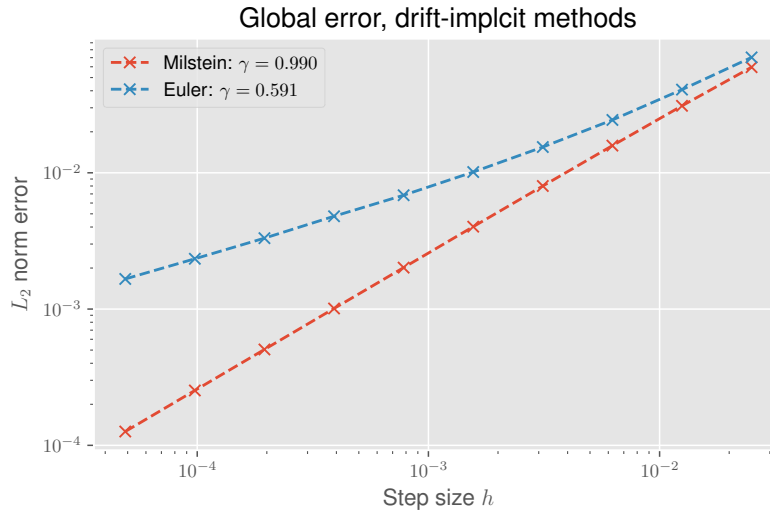
Figure 1: 10000 paths, $\mu = 7$, $\sigma = 1$, $T = 0.1$. Measuring the expected order of convergence.

Figure 1 shows that the order of convergence of the two drift-implicit methods is observable, and coincides with the theory. The lines cross for the larger step sizes. It also seems that the implicit Euler method has faster convergence in this region. This is natural, as the error of the method consists of two terms, that is, the drift is approximated of order 1, while diffusion is approximated of order 1/2. There might however be regions where the error constant for the diffusion in practice is very small. Then something closer to rate 1 can be observed. This is even more clear in the next section.
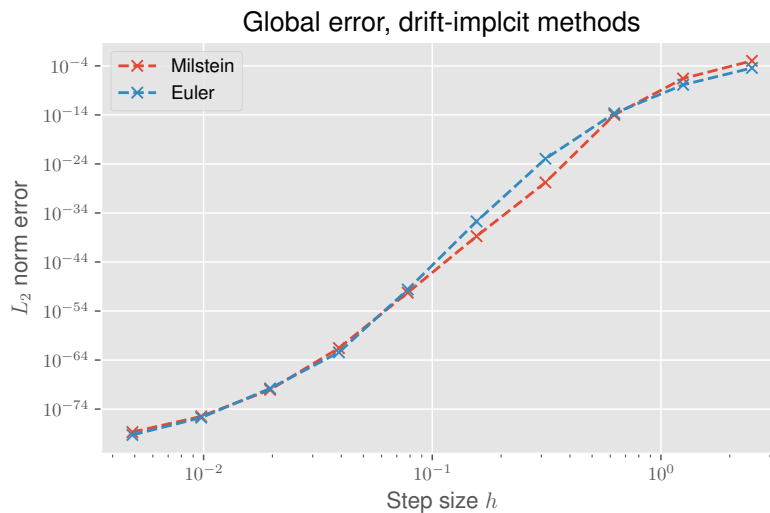

Figure 2: Violating the Lipschitz step size criterion. 10000 paths, $\mu = 20$, $\sigma = 2$, $T = 10$.

Figure 2 shows that we have convergence for all step sizes. The convergence is extremely rapid, since the system is very contractive, with $\nu = -19$. The five points of the largest step size are not mean-square stable in the sense of definition 2.10. The six points of the largest step size violate the classic condition of $h < 1/L$. The eight points of the largest step size violate the step-size restriction of $h < 1/\sqrt{10}L$, from the stability inequality in theorem

3.1. The stability constant from theorem 3.1 is extremely large. However, we know that the system satisfies a one-sided Lipschitz condition with $\nu = -\mu + 1$, thus allowing the use of theorem 3.3. This tells us that there should be no step size restriction, which seems to hold experimentally.

The system for $\nu = -19, L_\psi = 2, T = 10$ has stability constant $S = \sqrt{2\hat{S}T} = 28.7$. Assuming that the global error is largest in the last point, which is plotted in figure 2, and that the local error is constant for all steps, we can calculate what the convergence bound from theorem 3.4. The local error in the first step can thus be applied. For the linear system, using the implicit Euler, $N = 4$, $h = T/N = 2.5$, we have $\|\hat{l}_1\| = \|Y_h - X_1\| = 0.13$, calculated from 10000 paths. We also compute $\|\mathbb{E}(\hat{l}_1|\mathcal{F}_0)\| = 0.04$. This results in an error bound $28.7(0.04 + 0.13) = 4.88$. It is obviously larger than the error seen from figure 2, though such error constants will always be very much larger than the actual errors.

4.2   Van der Pol oscillator

This section looks at a classic example, namely the noisy Van der Pol oscillator. This example is frequently used when it comes to stiff nonlinear ODEs. The deterministic version of the following problem is analyzed in [27]. In [18] they look briefly at the noisy Duffing-Van der Pol oscillator, which is related to this problem. The previous example was linear and stiff, and not entirely

Denote $Y_t = (Y_1, Y_2)$. In this example, we only consider $\mu > 0$.

$$dY_t = \begin{pmatrix} Y_2 \\ \mu((1 - Y_1^2)Y_2 - Y_1) \end{pmatrix} dt + \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} Y_t dW_t, \ Y_0 = (2,0)^T \qquad (4.1)$$

With no noise, the path from $t_0 = 0$ to $T = 2.75$ is plotted in figure 3, for one period. For each period there are two sharp transients, making the problem stiff. On the interval before the first transient and between the transients, the solution changes slowly, and larger step sizes should be admissible. The Jacobian of f is

$$J_f = \begin{pmatrix} 0 & 1 \\ \mu(-2Y_1Y_2 - 1) & \mu(1 - Y_1^2) \end{pmatrix}$$

We now want to analyze the properties of the drift function, which is stiff for large μ.

$$\langle Y_t, f(Y_t) \rangle = Y_1 Y_2(1 - \mu) + \mu Y_2^2(1 - Y_1^2)$$

The second term is negative for $|Y_1| > 1$, which is expected to hold in the early evolution of the equation. In this part Y_2 is negative, but in magnitude, we expect the second term to be bigger than the first on this initial interval, for $\mu \geq 2$. These two conditions are reasonable, we can assume contractivity of the system in this region, but all we can say is that the system satisfies the one-sided Lipschitz condition with $\nu \leq 0$. Our system is random. The noise can at any time point bring us into the transient since it is a random variable in $\mathbb{R}$.
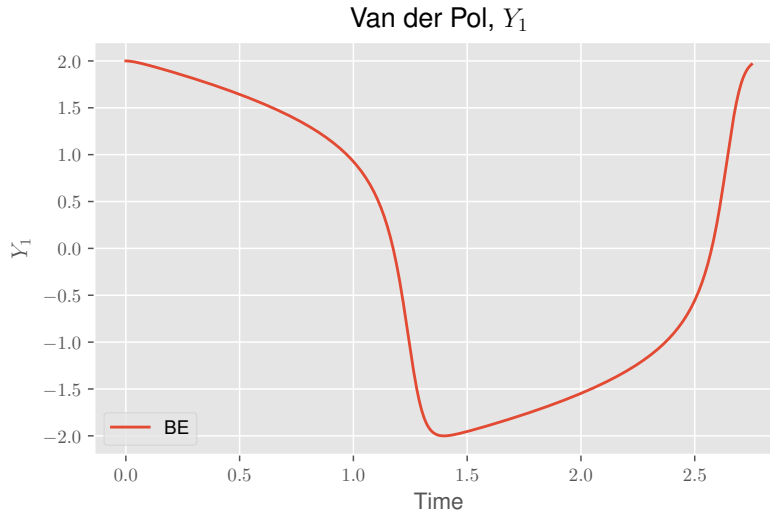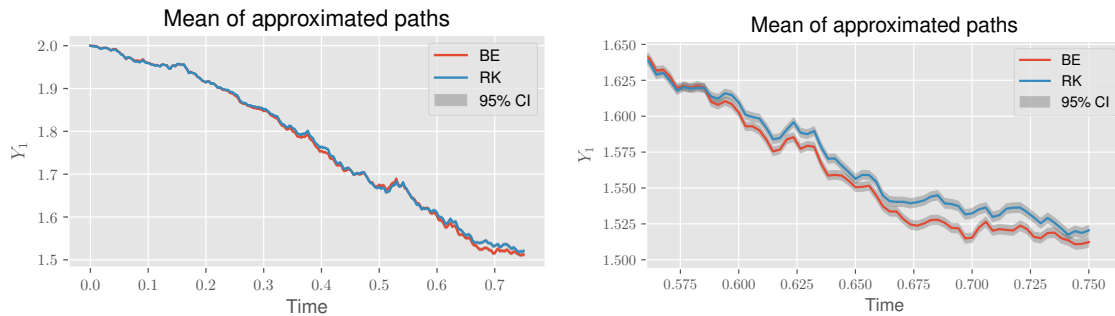
Figure 3: The deterministic Van der Pol equation for $\mu = 10$, $T = 2.75$, $N = 1024$. BE is the backward Euler method.

In figure 4b the mean of 1000 noisy paths until $T = 0.75$ is plotted. 256 uniform steps have been used. Accordingly, we assume that for such levels of noise, $\sigma \leq 2$, $Y_1 \in (1, 2]$ on this interval. Then we have a contraction with $\nu = 0$.

(a) Means of simulations on the non-transient phase.

(b) Zooming in on the mean of paths.

Figure 4: The mean of 1000 simulations of the Van der Pol oscillator, using $\mu = 10$, $\sigma = 2$, 256 steps. BE is the backward Euler method, and RK the Runge-Kutta method. The 95 percent confidence intervals (CI) are for the estimator of the mean in each point.

4.2.1   Implementation

A short description of how to solve the nonlinear equation using the drift-implicit Euler method is given here, which also reveals how this is done for the 1.5 order stochastic Runge-Kutta scheme (2.29). The analytical solution is not known for this problem, so the comparison is made with the solution with the RK method on a very fine grid.

Moving everything in our numerical scheme to one side, and denoting the resulting function

as

$$F(Y_n) = Y_n - hf(Y_n) - C \qquad (4.2)$$
$$C = Y_{n-1} + g(Y_{n-1})\Delta W_n \qquad (4.3)$$

for which we must solve $F(Y_n) = 0$. The parameter C is in each time step constant since it comprises everything that is either explicitly given or sampled in our scheme. For the RK scheme, the only change that matters for solving F is that the implicit part f has coefficient $h/2$, and a different C.

The Newton iteration can now be formulated, with Jacobian of F denoted dF, as

$$dF\Delta Y_n^{(m+1)} = -F(Y_n^{(m)}), \quad Y_n^{(m+1)} = \Delta Y_n^{(m+1)} + Y_n^{(m)}, \qquad (4.4)$$

$Y_n^{(m)}$ is the approximation of Y_n at iteration m of the algorithm. The algorithm needs a starting point, and one can choose from many, here the previous point $Y_{n-1} = Y_n^{(0)}$. This calculation uses the Jacobian at the previous point, e.g. $dF = dF(Y_{n-1}) = I - h\nabla f(Y_{n-1})$, though other approximations are also possible. In practice the Jacobian can be evaluated even less frequently, to save function evaluations and speed up computations. We will here use the exact Jacobian of the function F, though it can also be approximated numerically, by difference methods. The latter point would be useful in automatic analysis of systems, e.g. auto-generated dynamical systems from circuits.

All solvers have been implemented by the author, in Python. The newton iterations run up to 15 iterations but breaks off if the updating increment $|\Delta Y_i^{(j)}| < 10^{-12}$.

Simulating 1000 paths results in a relatively slow program. Since we are solving a non-linear equation at each discrete time point, one has to treat each path individually. Using parallelization, we can use all cores of the computer/server simultaneously, through pythons multiprocessing library.

Each convergence plot is made with step sizes from $h = T/2048$ up to $h = T/32$. The RK solution is calculated with $h = T/4096$.

It is useful to set the seed of the random number generator to a known value so that the simulations can be reproduced exactly. Finally, it should be pointed out that the same Wiener process should be used across all solutions when measuring convergence. One can simply adjust the step in a slicing procedure of the original sampled Wiener process, according to the step size change.

4.2.2   Results

In figure 4 and 5 the confidence interval in grey shows the 95 % quantiles for the mean. This indicates that we are able to estimate the mean with large accuracy, even for relatively large step sizes and few simulations.

The mean $\hat{\mu}$ is estimated from all observations of the path at that time point. For 1000

paths this interval is rather narrow, as seen from figure 4b. For a coarser estimation, we also present figure 5. The approximation of the mean consists of 100 paths of 64 points each.
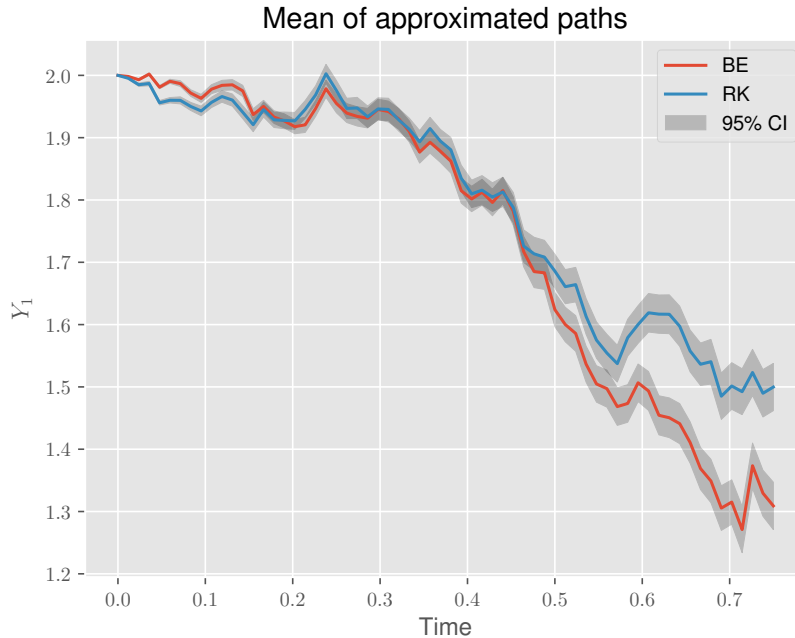

Figure 5: Paths with 95 percent confidence intervals for each point estimated. This interval is for the estimator of the mean.

In figure 6 the mean-square global error of the system is plotted. The backward Euler is compared to the RK method to tell the error. Using a linear regression on the resulting errors, the slope of the line is estimated to 1.031, i.e. measuring order $\gamma \approx 1$ of convergence. This is better than what one can prove since we do have multiplicative noise. Nonetheless, the noise might be so small in this case that it practically does not affect the system behaviour – the drift rules the equation effectively entirely. This is the same phenomenon as described for the linear example. The error constant for estimating the diffusion becomes so small that we only observe the order 1 component.

The error is measured with the same reference solution, of 4096 steps, only the backward Euler solution varies, so this is not the cause of a better convergence rate. The number of paths seems sufficiently large, as the line occurring is quite straight for each of the convergence plots here. For fewer paths, the line would appear less straight but still contain the same trend.
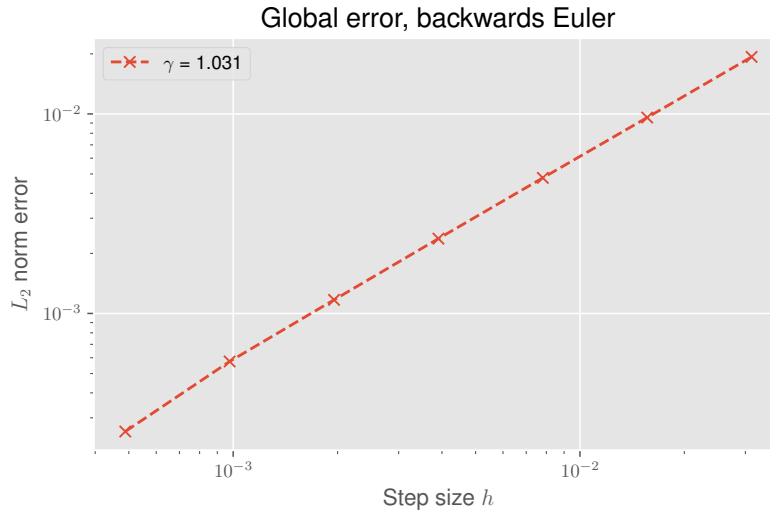
Figure 6: Global error for different step sizes of the Van der Pol system, measured in the mean-square sense. Parameters used are $\mu = 2$, $\sigma = 0.01$, 1000 paths. reference solution with $N = 4096$, and the smallest step size of backward Euler at $T/N = 0.75/2048$.

For a larger diffusion constant σ, same μ, the order of convergence seems more like we would expect, from figure 7. It is still a systematically faster convergence rate than expected, which implies that the observed order is now a mix of the order 1 and order 1/2 component. This phenomenon is known from deterministic equations too [27]. Notice on the y-axis that the global error is now larger, which is as expected since we allow larger volatility in our solutions.
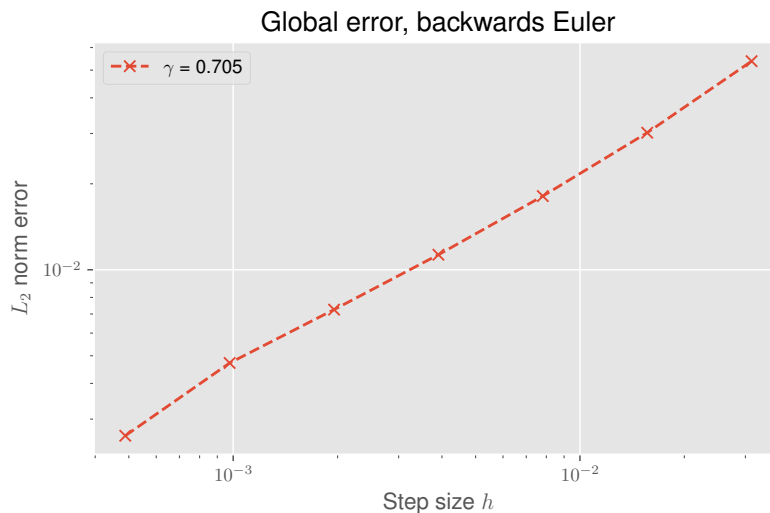

Figure 7: Global error for different step sizes of the Van der Pol system. Parameters used are $\mu = 2$, $\sigma = 0.1$, 1000 paths. Reference solution with $N = 4096$, and smallest step size of backward Euler at $T/N = 0.75/2048$.

The difference between figure 7 and figure 8 is that the latter has a larger drift constant, thus it is more stiff and contractive. Even though we estimated ν to be maximally zero, and could
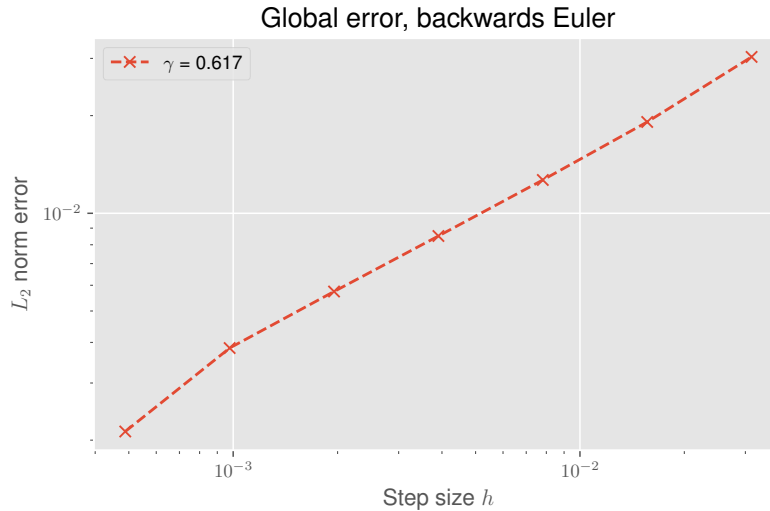
Figure 8: Global error for different step sizes of the Van der Pol system. Parameters used are $\mu = 10$, $\sigma = 0.1$, 1000 paths. Reference solution with $N = 4096$, and smallest step size of backward Euler at $T/N = 0.75/2048$.

not be more specific, the unobservable one-sided Lipschitz constant of the system might be more negative. The stiffer case has a slightly lower convergence rate, though the order is still as expected. The error constants are however very similar, slightly smaller for the stiffer case. This coincides with the error bound of theorem 3.4 on convergence for stiff systems. The stability constant S is proportional to $\exp(-1/\nu)$, which in both cases is close to zero, giving very similar error bounds.

Theoretically, it was seen that $\nu \leq 0$ in a certain region that we assume the solution to be in. This is vague, but the reality for many nonlinear systems. For a more contractive problem, S should decrease slightly, which is what we observe numerically.

The final plot, figure 9, measures convergence for even larger diffusion. For certain paths, the numerical method diverges for $N = 32$ steps. The error is therefore not measured for this step size. This tells us that paths exit the contractive region, and the large step size causes divergence. This was something we anticipated initially since the one-sided Lipschitz continuity of the drift depends on the solution. This represents a more general behaviour of nonlinear systems. The noise can take on any numerical value, thus there is a possibility of exiting the slowly changing phase at any time. This is a big difference from deterministic equations. The convergence is as we know measured in the *mean-square sense*. For the transient phase that the solution shortly would enter, several of the step sizes used in all these plots would cause divergence. On a transient a step size in line with the classic theory is appropriate.
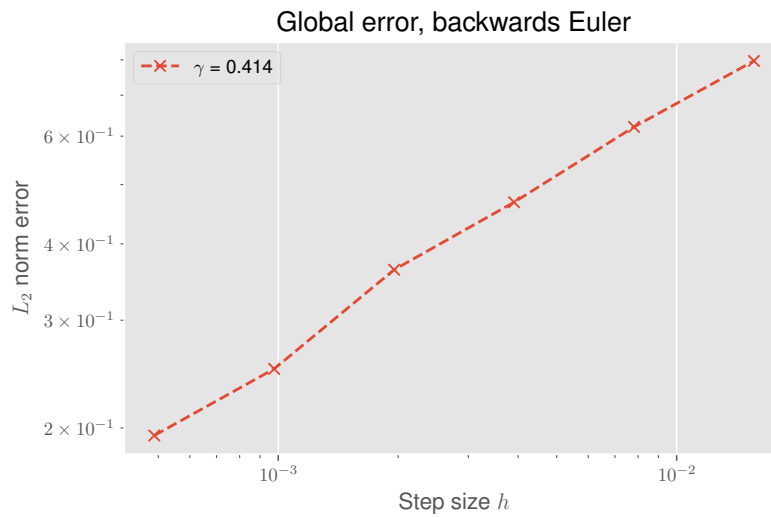
Figure 9: Convergence for $\mu = 10$, $\sigma = 1$. The approximation for $N = 32$ diverges, and is not plotted. Rate of convergence p is measured from the points plotted.

5 Remarks and summary

Sections 3.4 and 3.5 have presented three fundamental principles when proving convergence for stiff stochastic differential equations, obtaining rate of convergence from stability and consistency order. Our work is the first to our knowledge that proves this type of stability property independent of stiffness. The proof was possible due to the ideas in [28]. It has the property that non-positive one-sided Lipschitz constants do not cause step size restrictions. Other literature mostly treats other methods and stability properties, for which step size restrictions may be realistic. However, the existence and uniqueness of the algebraic equation we have to solve in each step simply has to be assumed for now. One might encounter step size restrictions proving this. This issue would be appropriate to resolve.

The assumptions needed to prove stability and consistency coincide with other literature on the subject. We have discussed how the stochastic theory relates to the deterministic and defined the concepts B-consistency and B-convergence. These concepts are supported by the current literature available both on SDEs and ODEs. Our focus is on convergence in the mean-square sense, though in the literature bounds of all moments have been proved under stronger assumptions. Also, stronger continuous extensions are obtained in other works.

We have seen some use of the one-sided Lipschitz condition for SDEs, both using regular calculus and techniques from probability theory. The theory for both SDEs and ODEs is much larger than what was presented here. The background theory for deterministic systems displays some powerful properties of the one-sided Lipschitz condition, such as expressing the local error as a scaled residual error. Hopefully one could include more of this well-studied field into the field of stiff SDEs.

Methods of higher order are so far not possible to include in the stability inequality of theorem 3.3 since it involves stochastic integrals and the estimation of elementary differentials of f, which are large due to stiffness. The same issue would disturb proving B-consistency. Other conditions might advance the theory. Clarifying this would be a natural next step. Work on methods of higher order for stiff SDEs is not known to us but is abundant for deterministic equations [27].

The improved step size condition obtained for the stability in the general Lipschitz setting [28] is more of a correction, but nonetheless useful. The theorem applies to implicit methods for non-stiff systems, which has many applications. It now coincides with step size criteria for ODEs.

Finding explicit error bounds is not common in the literature, though it was possible here. This allows for comparison between the Lipschitz and one-sided Lipschitz case, and with the deterministic bounds. There is a slight difference in how we are able to approach the stability inequality in the two settings, where the Lipschitz case allowed for a seemingly more greedy recursion. If one can find a way to treat the one-sided Lipschitz proof more like the global Lipschitz proof, the error constant might become sharper.

Experimentally, the order is as expected, the error constant is on a much smaller scale, and convergence is obtained for large step sizes when the one-sided Lipschitz constant is non-

positive. The Van der Pol example shows a typical behaviour of nonlinear equation, where the one-sided Lipschitz continuity depends on the solution. We are not able to say much about *how* negative the one-sided Lipschitz constant is at a given point, but typically one can anticipate that the system should behave contractively. Since stiff, nonlinear equations is a *very* broad and diverse class of problems, further work should carry out more numerical tests.

The numerical examples in section 4 confirm that the order of convergence is as expected, and that large step sizes are admitted for non-positive one-sided Lipschitz constants. The nonlinear system reacts quite as anticipated to changes in stiffness and noise, by the convergence and stability theorems.

The stability principle sought in this work is not of the contractive kind, so this theory will not reveal contractions of the numerical solution. Introducing non-stiff noise in a stiff system will lead to larger error bounds than for no noise, both for the exact and numerical solution. We have proven relations that bound this increase, and the exact and numerical bounds are similar to a certain degree. The error bounds are explicitly given, and non-positive one-sided Lipschitz constants indicate no step size restrictions. Negative one-sided Lipschitz constants yield particularly improved error bounds.

References

[1] L. Arnold. *Stochastic differential equations*. Wiley-Interscience, 1974.

[2] W.-J. Beyn, E. Isaak, and R. Kruse. Stochastic c-stability and b-consistency of explicit and implicit euler-type schemes. *Journal of Scientific Computing*, 67(3):955–987, 2016.

[3] W.-J. Beyn, E. Isaak, and R. Kruse. Stochastic c-stability and b-consistency of explicit and implicit milstein-type schemes. *Journal of Scientific Computing*, 70(3):1042–1077, 2017.

[4] J. C. Butcher. A stability property of implicit runge-kutta methods. *BIT Numerical Mathematics*, 15(4):358–361, 1975.

[5] D. S. Clark. Short proof of a discrete gronwall inequality. *Discrete applied mathematics*, 16(3):279–281, 1987.

[6] G. Dahlquist. Error analysis for a class of methods for stiff non-linear initial value problems. In *Numerical analysis*, pages 60–72. Springer, 1976.

[7] K. Dekker and J. G. Verwer. *Stability of Runge-Kutta Methods for Stiff Nonlinear Differential Equations*. Elsevier Science Ltd., 1984.

[8] R. Durrett. *Stochastic Calculus: A Practical Introduction*. CRC Press, 1996.

[9] G. B. Ermentrout and D. H. Terman. *Mathematical foundations of neuroscience*. Springer Science & Business Media, 2010.

[10] R. Frank, J. Schneid, and C. W. Ueberhuber. The concept of b-convergence. *SIAM Journal on Numerical Analysis*, 18(5):753–780, 1981.

[11] D. T. Gillespie. The chemical langevin equation. *The Journal of Chemical Physics*, 113(1):297–306, 2000.

[12] D. J. Higham. An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM review*, 43(3):525–546, 2001.

[13] D. J. Higham and P. E. Kloeden. Numerical methods for nonlinear stochastic differential equations with jumps. *Numerische Mathematik*, 101(1):101–119, 2005.

[14] D. J. Higham, X. Mao, and A. M. Stuart. Strong convergence of euler-type methods for nonlinear stochastic differential equations. *SIAM Journal on Numerical Analysis*, 40(3):1041–1063, 2002.

[15] Y. Hu. Semi-implicit euler-maruyama scheme for stiff stochastic equations. In *Stochastic Analysis and Related Topics V*, pages 183–202. Springer, 1996.

[16] M. Hutzenthaler and A. Jentzen. Numerical approximations of stochastic differential equations with non-globally lipschitz continuous coefficients. *Memoirs Of The American Mathematical Society*, 236, 2015.

[17] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden. Strong and weak divergence in finite time of euler's method for stochastic differential equations with non-globally lipschitz continuous coefficients. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 467(2130):1563–1576, 2011.

[18] P. E. Kloeden and E. Platen. *Numerical solution of stochastic differential equations*. Springer Science & Business Media, 2013.

[19] G. N. Milstein. *Numerical integration of stochastic differential equations*. Springer Science & Business Media, 1994.

[20] B. Øksendal. *Stochastic differential equations*. Springer, 2003.

[21] J. M. Ortega and W. C. Rheinboldt. *Iterative solution of nonlinear equations in several variables*. Siam, 1970.

[22] Q.-C. Pham, N. Tabareau, and J.-J. Slotine. A contraction theory approach to stochastic incremental stability. *IEEE Transactions on Automatic Control*, 54(4):816–820, 2009.

[23] O. Schein and G. Denk. Numerical solution of stochastic differential-algebraic equations with applications to transient noise simulation of microelectronic circuits. *Journal of Computational and Applied Mathematics*, 100(1):77–92, 1998.

[24] M. V. Tretyakov and Z. Zhang. A fundamental mean-square convergence theorem for sdes with locally lipschitz coefficients and its applications. *SIAM Journal on Numerical Analysis*, 51(6):3135–3162, 2013.

[25] X. Wang and S. Gan. B-convergence of split-step one-leg theta methods for stochastic differential equations. *Journal of Applied Mathematics and Computing*, 38(1-2):489–503, 2012.

[26] X. Wang and S. Gan. The tamed milstein method for commutative stochastic differential equations with non-globally lipschitz continuous coefficients. *Journal of Difference Equations and Applications*, 19(3):466–490, 2013.

[27] G. Wanner and E. Hairer. *Solving ordinary differential equations II*. Springer Berlin Heidelberg, 1996.

[28] R. Winkler. Stochastic differential algebraic equations of index 1 and applications in circuit simulation. *Journal of computational and applied mathematics*, 157(2):477–505, 2003.

[29] R. Winkler. Stochastic differential algebraic equations in transient noise analysis. In *Scientific computing in electrical engineering*, pages 151–156. Springer, 2006.