Henrik Bjørnerud Romnes

Composition operators on the Hardy space of Dirichlet series

Master's thesis in Mathematical Sciences
Supervisor: Ole Fredrik Brevig
June 2020

Master's thesis

NTNU
Norwegian University of Science and Technology
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Kunnskap for en bedre verden

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Abstract

We consider Dirichlet series with square summable coefficients, constituting the Hardy space $\mathscr{H}^2$. The purpose of this thesis is to study composition operators on this space. In particular, we prove a result by Gordon and Hedenmalm which gives a description of the analytic functions that generate bounded composition operators on the Hardy space of Dirichlet series.

Sammendrag

Vi betrakter Dirichlet-rekker med kvadratisk summerbare koeffisienter som utgjør Hardy-rommet $\mathscr{H}^2$. Formålet med denne avhandlingen er å studere komposisjonsoperatorer på dette rommet. Eksempelvis beviser vi et resultat av Gordon og Hedenmalm som beskriver de analytiske funksjonene som generer begrensede komposisjonsoperatorer på Hardy-rommet av Dirichlet-rekker.

Preface

This thesis was written as part of my master of science degree at the Norwegian University of Science and Technology during the period August 2019 to June 2020.

I would like to thank Ole Fredrik Brevig for suggesting an interesting topic for my thesis and for all the helpful advise along the way. I would also like to thank Hervé Queffélec for giving me access to the upcoming second edition of his book.

Henrik Bjørnerud Romnes
Trondheim, 2020

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Introduction

In this thesis we are concerned with Dirichlet series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad s \in \mathbb{C},$$

having square summable coefficients. We refer to the space of all such series as the Hardy space $\mathscr{H}^2$. Every element of $\mathscr{H}^2$ is analytic in $\mathbb{C}_{1/2}$, where $\mathbb{C}_{1/2} := \{s \in \mathbb{C} : \operatorname{Re} s > 1/2\}$. Our main focus is the study of composition operators $\mathscr{C}_\varphi$ on this space. For any analytic function $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ we define

$$\mathscr{C}_\varphi f := f \circ \varphi, \quad f \in \mathscr{H}^2.$$

It is of particular interest to know which additional properties that must be assigned to φ, in order to make the associated composition operator act boundedly on $\mathscr{H}^2$. A complete account for this problem was given in a paper by Gordon and Hedenmalm [9].

A similar theory has previously been developed on the Hardy space H^2 of analytic functions on the unit disk, whose power series has square summable coefficients. Most of this work is attributed to J.E. Littlewood. In particular, he showed that every analytic self-map φ of the unit disk generates a bounded composition operator $\mathscr{C}_\varphi : H^2 \to H^2$. However, if we want to ensure that a map φ generates a bounded composition operator on $\mathscr{H}^2$, then the assumption that φ is an analytic self-map of $\mathbb{C}_{1/2}$ is not sufficient. First we have to impose certain arithmetic restrictions on the map φ so that the composition $f \circ \varphi$ becomes a Dirichlet series. That is, the map φ must be of the form

$$\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s},$$

where the second term is assumed to a convergent Dirichlet series and c_0 is a non-negative integer. Then we have to make sure that φ has the right mapping properties, so that the composition $f \circ \varphi$ belongs to $\mathscr{H}^2$. In particular, it turns out that φ must have an analytic extension to the half-plane $\mathbb{C}_0$.

The norm of a composition operator $\mathscr{C}_\varphi$, either on H^2 or $\mathscr{H}^2$, is closely related to the mapping properties of the generating function φ. In H^2, the operator norm of $\mathscr{C}_\varphi$ is related to where φ maps the origin. The closer $\varphi(0)$ is to the boundary of $\mathbb{D}$, the larger the operator norm will be. In the case where $\varphi(0) = 0$ the associated composition operator is a contraction. This is called Littlewood's subordination principle. In the space $\mathscr{H}^2$ it is the point $w = \varphi(+\infty)$ that, to some extent, controls the operator norm of $\mathscr{C}_\varphi$. The operator norm is larger when the point w is closer to the boundary of $\mathbb{C}_{1/2}$. If $\varphi(+\infty) = +\infty$, then the operator $\mathscr{C}_\varphi$ is a contraction. The composition operators on H^2 and $\mathscr{H}^2$ have, in fact, lower and upper bounds that only depend on the points $\varphi(0)$ and $\varphi(+\infty)$, respectively.

We would like to know when these operators attains their upper bounds. More accurately, what characterizes those analytic maps φ that maximizes the operator norm of $\mathscr{C}_\varphi$? This question was answered in a paper by Shapiro [16], for the space H^2. He found that the operator norm $\mathscr{C}_\varphi$ is maximal if and only if the map φ satisfies a certain property, referred to as being *inner*. A map φ is called inner if the radial limit

$$\lim_{r \to 1^-} |\varphi(re^{i\theta})| = 1,$$

almost everywhere. This means that an inner function fixes the boundary points of the unit disk. An analogues result to this was provided by Brevig and Perfekt [5], for the Hardy space of Dirichlet series. They found that the operator norm of $\mathscr{C}_\varphi$ is again maximal if and only if φ in some sense maps the boundary of $\mathbb{C}_0$ to $\mathbb{C}_{1/2}$. This makes φ analogous to the inner functions on $\mathbb{D}$.

A recurring theme in the study of norms of composition operators is the existence of subordination principles. A composition operator $\mathscr{C}_\varphi$ is called subordinate to $\mathscr{C}_\psi$ if

$$||\mathscr{C}_\varphi f|| \leq ||\mathscr{C}_\psi f||$$

for every $f \in \mathscr{H}^2$. Shapiro's result (Theorem 1.20), that we mentioned above, tells us that any analytic self-map of the unit disk φ, with $\varphi(0) = w$, generates a composition operator that is subordinate to any composition operator generated by an inner function ψ, with $\psi(0) = w$. Similarly, the result by Brevig and Perfekt (Theorem 5.3) provides a subordination principle for the composition operators on $\mathscr{H}^2$. In the same paper they deduce another subordination principle for composition operators generated by a certain set of analytic functions. These are of the form

$$\varphi_{\mathbf{c}}(s) = c + \sum_{j=1}^{d} c_j p_j^{-s},$$

where $\mathbf{c} = (c_1, ..., c_d)$ and p_j is the j-th prime. The subordination principle says that a composition operator $\mathscr{C}_{\varphi_{\mathbf{b}}}$ is subordinate to $\mathscr{C}_{\varphi_{\mathbf{c}}}$, if the sequence $\mathbf{c}$ majorizes the sequence $\mathbf{b}$. We will not prove these results from [5]. However, we are going to answer a question from [5] regarding the existence of a more general subordination principle. Namely, will one of the composition operators always be subordinate to the other, even if we do not assume that one sequence majorize the other? This questioned will be answered by an example showing that such a subordination principle does not hold.

The thesis is organized as follows. The first chapter is an exposition of the more familiar Hardy space H^2. This is meant to work as a source of comparison for the forthcoming chapters. The second chapter is dedicated to the study of Dirichlet series, and we are mostly interested in those that converge to a bounded analytic function. In chapter 3 we narrow our attention to the Dirichlet series with square summable coefficients, which constitutes the

Hardy space $\mathscr{H}^2$. In chapter 4 we consider composition operators on $\mathscr{H}^2$. We show here how one can obtain a characterization of the analytic maps that generate bounded composition operators on this space. The final chapter deals with some results regarding the norm of such operators. In particular, we answer a question posed in a recent paper by Brevig and Perfekt [5].

Chapter 1

The Hardy Space H^2

The aim of this chapter is to give an account for the general theory of the Hardy space H^2 of analytic functions on the unit disk. Most of the attention is given to compositions operators generated by an analytic self-map $\varphi : \mathbb{D} \to \mathbb{D}$. A fundamental result in this regard is Littlewood's subordination theorem for which two proofs will be given. In addition, there will be a characterization of the norm of a composition operator in terms of inner functions, which are mainly based upon the results of Shapiro [16]. Other results in the present chapter can be found in [5], [7], [12] and [15].

The Hardy space H^2 is the set of analytic functions on the open unit disk, denoted by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, whose power series representation has square summable coefficients. That is, for an analytic function f on $\mathbb{D}$ with power series

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n, \tag{1.1}$$

we say that $f \in H^2$ if and only if $\sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty$. The norm of an H^2-function is defined as

$$||f||_{H^2} = \left(\sum_{n=0}^{\infty} |\hat{f}(n)|^2 \right)^{\frac{1}{2}}. \tag{1.2}$$

For two functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ and $g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n$ in H^2, we define their inner product by

$$\langle f, g \rangle_{H^2} = \sum_{n=0}^{\infty} \hat{f}(n) \overline{\hat{g}(n)}. \tag{1.3}$$

The sequence $\{\hat{f}(n)\}_{n=0}^{\infty}$ of power series coefficients belongs to the Hilbert space ℓ^2 by definition. Similarly, every sequence in ℓ^2 defines an analytic function on the open unit disk belonging to H^2 by means of the map $\{\hat{f}(n)\}_{n=0}^{\infty} \longmapsto \sum_{n=0}^{\infty} \hat{f}(n) z^n$. From the above it is clear that H^2 is isometrically isomorphic to ℓ^2. We conclude that the Hardy space H^2 is a Hilbert space.

The norm defined on H^2 has another equivalent representation in terms of integral means.

Let $M_2^2(f,r)$ denote the integral mean

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^2\,d\theta,$$

where f is assumed to be an analytic function on $\mathbb{D}$ and $0 \leq r < 1$. If we now use the series representation (1.1) of f in the integral mean formula, we get

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{n=0}^{\infty}\hat{f}(n)r^n e^{in\theta}\right|^2 d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\hat{f}(m)\overline{\hat{f}(n)}r^{n+m}e^{i(n-m)\theta}d\theta \\ &= \sum_{n=0}^{\infty}|\hat{f}(n)|^2 r^{2n}. \end{aligned} \tag{1.4}$$

The last equality follows from the fact that the exponential functions $\{e^{in\theta}\}_{n=0}^{\infty}$ defines an orthogonal set in $L^2([0,2\pi])$. Now it seems reasonable that as r approaches 1 from below, $M_2(f,r)$ converges to $||f||_{H^2}$. We will now see that this is the case.

Lemma 1.1. *Let f be an analytic function on $\mathbb{D}$. Then,*

$$||f||_{H^2} = \lim_{r\to 1^-} M_2(f,r).$$

Proof. From the equality (1.4) it is clear that $M_2(f,r)$ is an increasing function of r. We therefore have

$$M_2^2(f,r) = \sum_{n=0}^{\infty}|\hat{f}(n)|^2 r^{2n} \leq \sum_{n=0}^{\infty}|\hat{f}(n)|^2 = ||f||_{H^2}^2,$$

whenever $f \in H^2$ and $0 \leq r < 1$. So $M_2(f,r)$ is bounded by the H^2 norm. It remains to show that the whenever $\lim_{r\to 1^-} M_2(f,r) = M < \infty$, then f belongs to H^2 and $||f||_{H^2} \leq M$. If $\lim_{r\to 1^-} M_2(f,r) = M < \infty$, then the partial sums of the series (1.4) are bounded by M^2:

$$\sum_{n=0}^{N}|\hat{f}(n)|^2 r^{2n} \leq \sum_{n=0}^{\infty}|\hat{f}(n)|^2 r^{2n} \leq M^2.$$

As $r \to 1^-$, these partial sums converges to those of $||f||_{H^2}^2$, which must therefore be bounded by M^2 as well. If every partial sum of $||f||_{H^2}^2$ is bounded by M^2, then this is also true for the entire series. This completes the proof. □

We denote by H^∞ the set of bounded analytic function on the unit disk. We give it the supremum norm, that is, for $f \in H^\infty$ we have

$$||f||_{H^\infty} = \sup_{z\in\mathbb{D}}|f(z)|.$$

The next result is an immediate consequence of Lemma 1.1.

Corollary 1.2. *The space of bounded analytic functions on the unit disc H^∞ is a subset of H^2.*

Proof. Clearly,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^2\,d\theta \le \frac{1}{2\pi}\int_{-\pi}^{\pi}||f||_{H^\infty}^2\,d\theta = ||f||_{H^\infty}^2,$$

which holds true for every $0 < r < 1$. So for any $f \in H^\infty$ we get $\lim_{r\to 1^-} M_2(f,r) \le ||f||_{H^\infty}$. Hence, by Lemma 1.1, $f \in H^2$. □

Lemma 1.3. *Every norm convergent sequence in H^2 converges uniformly on compact subsets of $\mathbb{D}$.*

Proof. It will first be necessary to establish an estimate for the pointwise growth of a function f in H^2. From the triangle inequality and the Cauchy-Schwarz inequality we immediately have

$$|f(z)| = \left|\sum_{n=0}^{\infty}\hat{f}(n)z^n\right| \le \sum_{n=0}^{\infty}|\hat{f}(n)||z|^n \le \left(\sum_{n=0}^{\infty}|\hat{f}(n)|^2\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty}|z|^{2n}\right)^{\frac{1}{2}}.$$

We recognize the last two sums as the H^2 norm of f and a geometric series summing up to $(\sqrt{1-|z|^2})^{-1}$, respectively. This leaves us with the following estimate:

$$|f(z)| \le \frac{||f||_{H^2}}{\sqrt{1-|z|^2}}. \tag{1.5}$$

Now suppose that we have a sequence $\{f_j\}_{j=0}^{\infty}$ in H^2 which converges to a function f, in the sense that $||f_j - f||_{H^2} \to 0$. On every closed disk $|z| \le R$, with $0 < R < 1$, the estimate (1.5) implies that

$$\sup_{|z|\le R}|f_j(z) - f(z)| \le \frac{||f_j - f||_{H^2}}{\sqrt{1-R^2}}.$$

Hence, $\{f_j\}_{j=0}^{\infty}$ converges uniformly on the closed disk $|z| \le R$. For any compact subset A of $\mathbb{D}$, we can choose R such that A is contained in the disk $|z| \le R$. Since $\{f_j\}_{j=0}^{\infty}$ converges uniformly in this disk, it must also converge uniformly in A. We see that the sequence converges uniformly on every compact subset of $\mathbb{D}$. □

Definition 1.4. *The* ***reproducing kernel*** *for a point $z_0 \in \mathbb{D}$ is the function*

$$k_{z_0}(z) = \sum_{n=0}^{\infty}\overline{z_0}^n z^n = \frac{1}{1-\overline{z_0}z}.$$

It is obvious that the reproducing kernel for any point in the unit disk constitutes an H^2 function. An important property of a reproducing kernel k_{z_0} is that the value of a function $f \in H^2$ at z_0 is given by the inner product of f and k_{z_0}. That is, $f(z_0) = \langle f, k_{z_0}\rangle_{H^2}$. This relationship is immediate since

$$\langle f, k_{z_0}\rangle_{H^2} = \sum_{n=0}^{\infty}\hat{f}(n)z_0^n = f(z_0).$$

We can also easily determine the norm of a reproducing kernel k_{z_0}:

$$||k_{z_0}||_{H^2}^2 = \sum_{n=0}^{\infty}|\overline{z_0}|^{2n} = \frac{1}{1-|z_0|^2}.$$

Observe now that we can write the pointwise estimate (1.5) as

$$|f(z_0)| \leq ||f||_{H^2} ||k_{z_0}||_{H^2},$$

which is valid for any fixed $z_0 \in \mathbb{D}$.

An important property of functions in H^2 is the existence of non-tangential boundary values almost everywhere on $\mathbb{T}$. For $f \in H^2$ we define the function f_r by

$$f_r(e^{i\theta}) := f(re^{i\theta}) = \sum_{n=0}^{\infty} \hat{f}(n) r^n e^{in\theta},$$

where $0 < r < 1$. Suppose $g \in L^2(\mathbb{T})$ have the Fourier series representation $\sum_{n=0}^{\infty} \hat{f}(n) e^{in\theta}$. The function f_r will then converge radially to g.

Theorem 1.5. *Suppose $f \in H^2$. Then there exists a function $g \in L^2(\mathbb{T})$ such that the limit*

$$\lim_{r \to 1^-} f_r(e^{i\theta}) = g(e^{i\theta})$$

exists for almost every θ. In addition, we have $||f||_{H^2} = ||g||_{L^2}$.

Proof. It is clear that $f_r \in L^2(\mathbb{T})$ and that $||f_r||_{L^2} \leq ||f||_{H^2}$. Since f_r is bounded in L^2 for all $0 < r < 1$, there exists a sequence r_n converging to 1 so that f_{r_n} converges to some function $g \in L^2$ a.e. Denote the Fourier coefficients of g by $\hat{g}(k)$. Then

$$\hat{g}(k) = \langle g, e^{i\theta k} \rangle_{L^2} = \lim_{n \to \infty} \langle f_{r_n}, e^{i\theta k} \rangle_{L^2} = \begin{cases} \lim_{n \to \infty} \hat{f}(k) r_n^k & k \geq 0 \\ 0 & k < 0 \end{cases}.$$

We see that $\hat{g}(k) = \hat{f}(k)$, so then $g(e^{i\theta}) = \sum_{k=0}^{\infty} \hat{f}(k) e^{ik\theta}$. □

The next result provides us yet another expression for the H^2-norm, which will be particularly useful later on in the study of composition operators.

Theorem 1.6 (Littlewood-Paley Identity)**.** *For every holomorphic function $f \in H^2$ on the unit disk we have*

$$||f||_{H^2}^2 = |f(0)|^2 + 2 \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} dA(z), \tag{1.6}$$

where dA denotes the normalized Lebesgue measure on $\mathbb{D}$ ($dA = \frac{1}{\pi} dxdy$).

Proof. We start by considering the right hand side of (1.6). We write the integral in polar coordinates:

$$\begin{aligned} \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} dA(z) &= \int_{-\pi}^{\pi} \frac{1}{\pi} \int_0^1 |f'(re^{i\theta})|^2 \left(\log \frac{1}{r} \right) r dr d\theta \\ &= \int_0^1 \left(\frac{1}{\pi} \int_{-\pi}^{\pi} |f'(re^{i\theta})|^2 d\theta \right) \left(\log \frac{1}{r} \right) r dr. \end{aligned}$$

The second equality follows from Fubini's theorem. In addition, after multiplying and dividing by 2, the integral with respect to θ can be recognized as $M_2^2(f', r)$. After calculating $f'(z) = \sum_{n=1}^{\infty} n\hat{f}(n)z^{n-1}$, we get

$$\begin{aligned} 2\int_0^1 \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(re^{i\theta})|^2 d\theta\right)\left(\log\frac{1}{r}\right) rdr &= 2\int_0^1 M_2^2(f', r)\left(\log\frac{1}{r}\right) rdr \\ &= 2\int_0^1 \left(\sum_{n=1}^{\infty} n^2|\hat{f}(n)|^2 r^{2n-2}\right)\left(\log\frac{1}{r}\right) rdr \\ &= 2\sum_{n=1}^{\infty} n^2|\hat{f}(n)|^2 \int_0^1 r^{2n-2}\left(\log\frac{1}{r}\right) rdr \\ &= 2\sum_{n=1}^{\infty} n^2|\hat{f}(n)|^2 \frac{1}{4n^2} \\ &= \frac{1}{2}\sum_{n=1}^{\infty} |\hat{f}(n)|^2 \end{aligned}$$

Upon multiplying the last expression by 2 and adding $|f(0)|^2$ the result follows. □

Definition 1.7. *An analytic map $\varphi : \mathbb{D} \to \mathbb{D}$ with radial limits equal to one almost everywhere is called an inner function. That is, φ is inner if*

$$\lim_{r\to 1^-} |\varphi(re^{i\theta})| = 1,$$

for almost every $e^{i\theta} \in \mathbb{T}$.

The function $f_n(z) = z^n$, for some $n \in \mathbb{N} = \{1, 2, 3, ...\}$, provides a simple example of an inner function. It clearly defines an analytic function on the unit disk and $\lim_{r\to 1^-} |f_n(re^{i\theta})| = \lim_{r\to 1^-} |re^{i\theta n}| = 1$.

Lemma 1.8. *If φ is an inner function and $k, l \in \mathbb{N}, k \geq l$, then*

$$\langle \varphi^k, \varphi^l \rangle_{L^2(\mathbb{T})} = \varphi(0)^{k-l}.$$

Proof. By definition we have

$$\langle \varphi^k, \varphi^l \rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} \varphi^k \overline{\varphi^l}\, dm = \int_{\mathbb{T}} |\varphi|^{2l} \varphi^{k-l} dm.$$

Since φ is inner, $|\varphi|$ is equal to 1 almost everywhere on $\mathbb{T}$. Therefore,

$$\langle \varphi^k, \varphi^l \rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} \varphi^{k-l} dm = \varphi(0)^{k-l},$$

where the last equality follows from the mean value property of analytic functions and the fact that φ^{k-l} is analytic in $\mathbb{D}$ whenever $k \geq l$. □

For an element $b \in H^\infty$ we define the linear operator M_b of pointwise multiplication by

$$M_b f := bf,$$

for every $f \in H^2$. This multiplication operator satisfies the following property:

Lemma 1.9. *Let φ be an analytic self-map of the unit disc. Then for $f \in H^2$, we have*

$$||M_\varphi f||_{H^2} \leq ||f||_{H^2}.$$

That is, M_φ is a contraction on H^2.

Proof. The norm of $\varphi \in H^\infty$ is given by

$$||\varphi||_{H^\infty} = \sup_{z \in \mathbb{D}} |\varphi(z)|.$$

For any $f \in H^2$ and $0 < r < 1$ we have that

$$\begin{aligned} M_2^2(\varphi f, r) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi(re^{i\theta}) f(re^{i\theta})|^2 \, d\theta \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} ||\varphi||_{H^\infty}^2 |f(re^{i\theta})|^2 \, d\theta \\ &\leq ||\varphi||_{H^\infty}^2 ||f||_{H^2}^2. \end{aligned}$$

If φ is a self-map of $\mathbb{D}$, then $||\varphi||_{H^\infty} \leq 1$. It follows that $||M_\varphi f||_{H^2} \leq ||f||_{H^2}$, since $\lim_{r\to\infty} M_2(\varphi f, r) = ||M_\varphi f||_{H^2}$. □

Definition 1.10. *Suppose φ is a holomorphic self-map of the unit disc. We then define the composition operator $\mathscr{C}_\varphi : H^2 \to H^2$ by*

$$\mathscr{C}_\varphi f := f \circ \varphi.$$

It is now time to prove Littlewood's subordination theorem, which is a fundamental result in the study of composition operators on the Hardy space H^2.

Theorem 1.11. *Let φ be a holomorphic self-map of $\mathbb{D}$ that fixes the origin. Then the composition operator $\mathscr{C}_\varphi$ is a contraction on H^2. That is, $||\mathscr{C}_\varphi f||_{H^2} \leq ||f||_{H^2}$, for every function $f \in H^2$. In particular, whenever a function f is in H^2, then the composition $f \circ \varphi$ is in H^2 as well.*

The following proof of this result is based on Littlewood's original ideas and can be found in [15].

Proof. The main idea of the proof is to make use of an operator known as the backward shift, denoted by B. The operator acts on elements in H^2 in the following way,

$$Bf(z) = \sum_{n=0}^{\infty} \hat{f}(n+1) z^n, \qquad f \in H^2.$$

As the operator B shifts the coefficients of f to the left, it annihilates the original constant term $\hat{f}(0) = f(0)$. Observe now that $zBf(z) = f(z) - f(0)$, giving the identity

$$f(z) = f(0) + zBf(z). \tag{1.7}$$

In addition to this there is another useful identity of the backward shift, namely

$$B^n f(0) = \hat{f}(n). \tag{1.8}$$

The way we are going to prove the result is by first proving it for polynomials and then extend the result to every holomorphic function in H^2. Therefore, we begin by letting f be a polynomial. The composition $f \circ \varphi$ is then bounded on $\mathbb{D}$, so the integral

$$M_2^2(f, r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |(f \circ \varphi)(re^{i\theta})|^2 d\theta$$

remains bounded as $r \to 1^-$. It follows that $f \circ \varphi$ lies in H^2. Now, we wish to estimate the norm of the composition $\mathscr{C}_\varphi f$ with the help of the identities (1.7) and (1.8). We start by turning (1.8) into an identity concerning our composition $\mathscr{C}_\varphi f$. This is done simply by substituting in $\varphi(z)$ for z. We now have

$$f(\varphi(z)) = f(0) + \varphi(z)(Bf)(\varphi(z)).$$

Equivalently, we can write this as

$$\mathscr{C}_\varphi f = f(0) + M_\varphi \mathscr{C}_\varphi B f. \tag{1.9}$$

It is assumed that $\varphi(0) = 0$, from which is follows that every term in the power series of φ share the factor z. Consequently, this must also true for the second term of (1.9) as this in turn fixes the origin. What we now know, in particular, is that the second term of (1.9) has a power series without a constant term. The integral of this power series around some circle of radius $r < 1$ about the origin will then vanish. Therefore, the inner product of the two terms on the right hand side of (1.9) will be zero, making them orthogonal. It follows that

$$||\mathscr{C}_\varphi f||_{H^2}^2 = |f(0)|^2 + ||M_\varphi \mathscr{C}_\varphi B f||_{H^2}^2 \leq |f(0)|^2 + ||\mathscr{C}_\varphi B f||_{H^2}^2.$$

The inequality is due to the fact that multiplication operator acts contractively on H^2. But now we also know that

$$||\mathscr{C}_\varphi B f||_{H^2}^2 \leq |Bf(0)|^2 + ||\mathscr{C}_\varphi B^2 f||_{H^2}^2.$$

Continuing in this manner eventually gives the following norm estimate for $C_\varphi f$:

$$||\mathscr{C}_\varphi f||_{H^2}^2 \leq \sum_{k=0}^{n} |B^k f(0)|^2 + ||\mathscr{C}_\varphi B^{n+1} f||_{H^2}^2 \tag{1.10}$$

This holds for every positive integer n. Therefore, since f is assumed to be a polynomial, we can choose n to be the degree of f. Hence, $B^{n+1} f(0) = 0$. So from the identity (1.8) and equation (1.10) we have

$$||\mathscr{C}_\varphi f||_{H^2}^2 \leq \sum_{k=0}^{n} |B^k f(0)|^2 = \sum_{k=0}^{n} |\hat{f}(k)|^2 = ||f||_{H^2}^2.$$

This shows that the composition operator have the desired property on the subspace of H^2 consisting of holomorphic polynomials. It remains to prove the result for functions in H^2 which are not polynomials.

From now on let f be any H^2 function. In order to take advantage of how the composistion operator acts on polynomials, we consider the j-th partial sum of the Taylor series for f, denoted f_j. Obviously, $f_j \to f$ in the H^2 norm. We know from Lemma 1.3 that f_j converges to f uniformly on compact subsets of $\mathbb{D}$. This implies that $f_j \circ \varphi \to f \circ \varphi$ uniformly on compact subsets of $\mathbb{D}$. Any circle of radius $r \in (0,1)$ form a compact subset of $\mathbb{D}$, hence

$$M_2(f \circ \varphi, r) = \lim_{j\to\infty} M_2(f_j \circ \varphi, r) \leq \limsup_{j\to\infty} ||f_j \circ \varphi||_{H^2}.$$

We have already proved that $||f_j \circ \varphi||_{H^2} \leq ||f_j||_{H^2}$, so

$$\limsup_{j\to\infty} ||f_j \circ \varphi||_{H^2} \leq \limsup_{j\to\infty} ||f_j||_{H^2}.$$

Finally, since $||f_j||_{H^2} \leq ||f||_{H^2}$, we have

$$\limsup_{j\to\infty} ||f_j||_{H^2} \leq ||f||_{H^2}.$$

This means that $M_2(f \circ \varphi, r) \leq ||f||_{H^2}$ for $0 < r < 1$, so by letting r tend to 1 we get

$$\lim_{r\to 1^-} M_2(f \circ \varphi, r) = ||\mathscr{C}_\varphi f||_{H^2} \leq ||f||_{H^2}.$$

This completes the proof. □

Littlewood's subordination theorem can also be proved through a result on subharmonic functions. Such an approach is reasonable because $|f|^\alpha$ is subharmonic whenever f is an analytic function and $\alpha > 0$. The proof can be carried out in the following way:

Proof. As before we let φ be a holomorphic self-map of $\mathbb{D}$, with $\varphi(0) = 0$. Then by Schwarz lemma, we have $|\varphi(z)| \leq |z|$ for every $z \in \mathbb{D}$. Let G be a subharmonic function on $\mathbb{D}$, and denote the composition $G \circ \varphi$ by g. We start by using the subharmonic property of G to find a function H, harmonic in $|z| < r$ and equal to G on $|z| = r$, such that $G(z) \leq H(z)$ for every $|z| \leq r$. Denote the composition $H \circ \varphi$ by h. Clearly, $g(z) \leq h(z)$ on $|z| = r$. Now we easily see that

$$\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} h(re^{i\theta})d\theta.$$

The composition of a harmonic function with a holomorphic function is harmonic, so h is harmonic. The mean value of a harmonic function over a circle of radius r is given by its value at the center of that circle. This means that

$$\frac{1}{2\pi}\int_0^{2\pi} h(re^{i\theta})d\theta = h(0).$$

Since $h(z) = H(\varphi(z))$ and φ fixes the origin, we get that $h(0) = H(0)$. We can now go the other way around and express $H(0)$ as the mean value of H around the circle of radius r:

$$H(0) = \frac{1}{2\pi}\int_0^{2\pi} H(re^{i\theta})d\theta.$$

By definition, $H(z) = G(z)$ on $|z| = r$. Hence,

$$\frac{1}{2\pi}\int_0^{2\pi} H(re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} G(re^{i\theta})d\theta.$$

To summarize, we have

$$\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} G(re^{i\theta})d\theta.$$

This implies that for any analytic function f on $\mathbb{D}$, it must be true that $M_2(\mathscr{C}_\varphi f, r) \leq M_2(f, r)$ for every $0 < r < 1$. □

Now that we have established that the composition operator $\mathscr{C}_\varphi$ is bounded whenever φ fixes the origin, it remains to prove that the operator is bounded still when $\varphi(0) = w \neq 0$. For this purpose we define, for every point $w \in \mathbb{D}$, the Möbius transformation

$$\alpha_w(z) = \frac{w - z}{1 - \overline{w}z}.$$

This function maps the unit disc to itself, while interchanging the origin with the point w.

Theorem 1.12. *Suppose φ is a holomorphic self-map of $\mathbb{D}$. Then $\mathscr{C}_\varphi$ is a bounded operator on H^2, with*

$$||\mathscr{C}_\varphi|| \leq \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}}.$$

Proof. If $\varphi(0) = w$, then the map $\psi = \alpha_w \circ \varphi$ is a holomorphic self-map of $\mathbb{D}$ fixing the origin. The function α_w is its own inverse, so $\varphi = \alpha_w \circ \psi$. The composition operator related to the function φ can now be written as $\mathscr{C}_\varphi = \mathscr{C}_\psi \mathscr{C}_{\alpha_w}$. By Littlewood's subordination theorem it follows that $\mathscr{C}_\psi$ is bounded. It remains to show that $\mathscr{C}_{\alpha_w}$ is bounded, since then $\mathscr{C}_\varphi$ becomes the product of two bounded operators and must in turn be bounded. From Theorem 1.1 we know that any analytic function f on $\mathbb{D}$ satisfies

$$||f||_{H^2}^2 = \lim_{r \to 1^-} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta.$$

Assume now that the function f is analytic in a domain $\delta\mathbb{D}$, with $\delta > 1$. Then the limit can be moved inside the integral, yielding

$$||f||_{H^2}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta.$$

If consider the composition of f with α_w, we get

$$\begin{aligned} ||f\circ\alpha_w||^2_{H^2} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\alpha_w(e^{i\theta}))|^2 d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(e^{it})|^2|\alpha_w'(e^{it})|dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(e^{it})|^2\frac{1-|w|^2}{|1-\overline{w}e^{it}|^2}dt \\ &\leq \frac{1-|w|^2}{(1-|w|)^2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(e^{it})|^2dt\right) \\ &= \frac{1+|w|}{1-|w|}||f||^2_{H^2}. \end{aligned}$$

This means that $\mathscr{C}_{\alpha_w}$ acts boundedly on analytic functions in $\delta\mathbb{D}$. This is also true, in particular, when f is a polynomial. This takes us to the same situation as in the proof of Theorem 1.11, where we extended the result from being valid for polynomials to all of H^2. The argument in this case is exactly the same, and is therefore omitted. To summarize, we have found that the operator $\mathscr{C}_{\alpha_w}$ is bounded on H^2 and

$$||\mathscr{C}_{\alpha_w}|| \leq \left(\frac{1+|w|}{1-|w|}\right)^{\frac{1}{2}}.$$

The operator $\mathscr{C}_\varphi$ is now a product of bounded operators and is therefore bounded. Since $\mathscr{C}_\psi$ is a contraction, we get the following estimate for the operator norm:

$$||\mathscr{C}_\varphi|| \leq ||\mathscr{C}_\psi||\,||\mathscr{C}_{\alpha_w}|| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{1}{2}}. \qquad \square$$

We now return to the topic of reproducing kernels, which proves to be a useful tool in the investigation of composition operators. The next result reveals a relationship between reproducing kernels and the adjoint of a composition operator.

Lemma 1.13. *Let $\mathscr{C}_\varphi$ be a composition operator on H^2 and k_{z_0} be the reproducing kernel generated by an arbitrary point z_0 on $\mathbb{D}$. Then,*

$$\mathscr{C}_\varphi^* k_{z_0} = k_{\varphi(z_0)}.$$

Proof. Recall that $f(z_0) = \langle f, k_{z_0}\rangle_{H^2}$. First,

$$\langle f, \mathscr{C}_\varphi^* k_{z_0}\rangle_{H^2} = \langle \mathscr{C}_\varphi f, k_{z_0}\rangle_{H^2} = f(\varphi(z_0)).$$

Secondly,

$$\langle f, k_{\varphi(z_0)}\rangle_{H^2} = f(\varphi(z_0)).$$

Since these equalities hold for every $f \in H^2$, we must have $\mathscr{C}_\varphi^* k_{z_0} = k_{\varphi(z_0)}$. $\square$

We can make immediate use of the previous lemma by proving that the composition operator $\mathscr{C}_{\alpha_w}$ actually attains the upper bound provided in the proof of Theorem 1.12. More precisely:

Lemma 1.14. *For $\alpha_w(z) = \frac{w-z}{1-\overline{w}z}$ we have*

$$||\mathscr{C}_{\alpha_w}|| = \left(\frac{1+|w|}{1-|w|}\right)^{\frac{1}{2}}.$$

Proof. We know that $||k_{\varphi(z)}||_{H^2} \leq ||\mathscr{C}_{\varphi(z)}||\,||k_z||_{H^2}$, which is valid for every $z \in \mathbb{D}$. It follows that

$$||\mathscr{C}_{\alpha_w}||^2 \geq \sup_{z\in\mathbb{D}} \frac{||k_{\alpha_w(z)}||^2_{H^2}}{||k_z||^2_{H^2}} = \sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\alpha_w(z)|^2}. \tag{1.11}$$

We need to choose z such that the latter fraction becomes as large as possible. The fraction happens to increase the most when z tends to the boundary in the opposite direction of w. So we set $z = -\frac{w}{|w|}r$ and find

$$\alpha_w\left(-\frac{w}{|w|}r\right) = \frac{w + r\frac{w}{|w|}}{1+|w|r} = \frac{\frac{w}{|w|}(|w|+r)}{1+|w|r}.$$

Equation (1.11) now takes the form

$$\begin{aligned}
||\mathscr{C}_{\alpha_w}||^2 \geq \sup_{z\in\mathbb{D}} \frac{1-|-\frac{w}{|w|}r|^2}{1-|\alpha_w(-\frac{w}{|w|}r)|^2} &= \lim_{r\to 1^-} \frac{1-r^2}{1-\left(\frac{|w|+r}{1+|w|r}\right)^2} \\
&= \lim_{r\to 1^-} \frac{(1-r^2)(1+|w|r)^2}{(1+|w|r)^2-(|w|+r)^2} \\
&= \lim_{r\to 1^-} \frac{(1-r^2)(1+|w|r)^2}{1+|w|^2r^2-|w|^2+r^2} \\
&= \lim_{r\to 1^-} \frac{(1-r^2)(1+|w|r)^2}{(1-r^2)(1-|w|^2)} \\
&= \frac{(1+|w|)^2}{1-|w|^2} \\
&= \frac{1+|w|}{1-|w|}.
\end{aligned}$$

Hence, the proof is complete. □

Lemma 1.13 also gives us an elegant way of establishing a lower bound for the operator norm of a composition operator. We also make use of Theorem 1.12 and a trivial inequality to provide a suitable upper bound for the same operator.

Theorem 1.15. *For every composition operator $\mathscr{C}_\varphi$ we have the following bounds for its norm:*

$$\frac{1}{\sqrt{1-|\varphi(0)|^2}} \leq ||\mathscr{C}_\varphi|| \leq \frac{2}{\sqrt{1-|\varphi(0)|^2}}.$$

Proof. Let $z_0 = 0$, so that $\mathscr{C}_\varphi^* k_0 = k_{\varphi(0)}$ by the lemma above. Earlier we showed that the norm of a reproducing kernel is given by $(1-|z_0|^2)^{-1/2}$, so now $||k_0|| = 1$ and $||k_{\varphi(0)}||_{H^2} = (1-|\varphi(0)|^2)^{-1/2}$. Further, we have

$$||k_{\varphi(0)}||_{H^2} = ||\mathscr{C}_\varphi^* k_0||_{H^2} \leq ||\mathscr{C}_\varphi^*||\,||k_0||_{H^2}.$$

The first inequality now follows from the fact that $||\mathscr{C}_\varphi^*|| = ||\mathscr{C}_\varphi||$. Now for $0 \le x < 1$ we have

$$\frac{1+x}{1-x} = \frac{(1+x)(1+x)}{(1-x)(1+x)} = \frac{(1+x)^2}{1-x^2}.$$

Therefore,

$$\sqrt{\frac{1+x}{1-x}} = \frac{1+x}{\sqrt{1-x^2}} \le \frac{2}{\sqrt{1-x^2}}.$$

From theorem (1.12) and the above, we have

$$||\mathscr{C}_\varphi|| \le \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}} \le \frac{2}{\sqrt{1-|\varphi(0)|^2}}. \qquad \square$$

We have seen that $\mathscr{C}_{\alpha_w}$ is an example of a composition operator that attains the upper bound from Theorem 1.12. Our next goal is to identify the analytic maps φ that generate such composition operators in general. We shall see that it is both a necessary and sufficient condition that φ is an inner function. We begin by giving a definition.

Definition 1.16. *Suppose φ is holomorphic on $\mathbb{D}$. The function N_φ is called the Nevanlinna counting function and is defined as*

$$N_\varphi(w) := \sum_{z \in \varphi^{-1}\{w\}} \log \frac{1}{|z|}, \qquad w \ne \varphi(0).$$

The multiplicity of the preimages is taken into account. If the preimage of a point w is empty, then we set $N_\varphi(w) = 0$.

The Nevannlinna counting function appears after a change of variable $w = \varphi(z)$ in the Littlewood-Paley identity. The formula (1.6) now takes the form

$$||\mathscr{C}_\varphi f||_{H^2}^2 = |f(\varphi(0))|^2 + 2\int_{\mathbb{D}} |f'(w)|^2 N_\varphi(w) dA(w). \tag{1.12}$$

The following Lemma was originally proved by Shapiro [16]. A slightly stronger result was given by Brevig and Perfekt in [5], which we will state and prove here. We will also provide some extra details to the proof.

Lemma 1.17. *Let φ be a holomorphic self-map of $\mathbb{D}$ that fixes the origin. For $0 \le \delta \le 1$, define the set $E_\delta := \{z \in \mathbb{T} : |\varphi(z)| < \delta\}$. Then*

$$||\mathscr{C}_\varphi f||_{H^2}^2 \le C_\delta |f(0)|^2 + (1 - C_\delta)||f||_{H^2}^2,$$

where $C_\delta = \frac{1}{2}\frac{1-\delta}{1+\delta} m(E_\delta)$.

Proof. For $w \in \mathbb{D}$ and $z \in \mathbb{T}$, we define the function

$$\varphi_w(z) := \alpha_w \circ \varphi(z) = \frac{w - \varphi(z)}{1 - \overline{w}\varphi(z)}. \tag{1.13}$$

As before, the function α_w denotes the Möbius transformation interchanging the origin and the point w. Further, we have

$$\begin{aligned}|\varphi_w(z)|^2 = \left|\frac{w-\varphi(z)}{1-\overline{w}\varphi(z)}\right|^2 &= \left(\frac{w-\varphi(z)}{1-\overline{w}\varphi(z)}\right)\overline{\left(\frac{w-\varphi(z)}{1-\overline{w}\varphi(z)}\right)} \\ &= \left(\frac{w-\varphi(z)}{1-\overline{w}\varphi(z)}\right)\left(\frac{\overline{w}-\overline{\varphi(z)}}{1-w\overline{\varphi(z)}}\right) \\ &= \frac{|w|^2 - w\overline{\varphi(z)} - \overline{w}\varphi(z) + |\varphi(z)|^2}{|1-\overline{w}\varphi(z)|^2}.\end{aligned}$$

The denominator in the last expression can be written as

$$|1-\overline{w}\varphi(z)|^2 = 1 - w\overline{\varphi(z)} - \overline{w}\varphi(z) + |w|^2 + |\varphi(z)|^2.$$

From this we can deduce the expression

$$1-|\varphi_w(z)|^2 = \frac{1-|w|^2-|\varphi(z)|^2+|w|^2|\varphi(z)|^2}{|1-\overline{w}\varphi(z)|^2} = \frac{(1-|w|^2)(1-|\varphi(z)|^2)}{|1-\overline{w}\varphi(z)|^2}.$$

Now if $z \in E_\delta$, then

$$\begin{aligned}1-|\varphi_w(z)|^2 &\geq \frac{(1-|w|^2)(1-\delta^2)}{(1+|\varphi(z)|)^2} \\ &\geq \frac{(1-|w|^2)(1-\delta^2)}{(1+\delta)^2} \\ &= \frac{1-\delta}{1+\delta}(1-|w|^2).\end{aligned}$$

We also need the inequality $1-x \leq \log\frac{1}{x}$, which remains true for $0 < x < 1$. Together, these two inequalities implies that

$$\log\frac{1}{|\varphi_w(z)|^2} \geq 1-|\varphi_w(z)|^2 \geq \frac{1-\delta}{1+\delta}(1-|w|^2),$$

or equivalently,

$$\log|\varphi_w(z)| \leq -\frac{1}{2}\frac{1-\delta}{1+\delta}(1-|w|^2). \tag{1.14}$$

Applying Jensen's formula [1] to the function $\varphi_w(z)$ gives

$$\log|\varphi_w(0)| = \sum_{k=1}^{n}\log|a_k| + \int_{\mathbb{T}}\log|\varphi_w(z)|dm(z), \tag{1.15}$$

where $a_1, ..., a_k$ denotes the zeros of $\varphi_w(z)$ in $\mathbb{D}$. Note that if $\varphi(z) = w$, then $\varphi_w(z) = 0$. This implies

$$N_\varphi(w) \leq -\sum_{k=1}^{n}\log|a_k|.$$

Using equation (1.15) and the fact that $\varphi_w(0) = w$, we get

$$N_\varphi(w) \leq \log \frac{1}{|w|} + \int_{\mathbb{T}} \log |\varphi_w(z)| dm(z).$$

Since $E_\delta \subseteq \mathbb{T}$ and $\log|\varphi_w(z)| \leq 0$ for a.e. $z \in \mathbb{T}$, it follows from the monotonicity of the Lebesgue integral that

$$N_\varphi(w) \leq \log \frac{1}{|w|} + \int_{E_\delta} \log |\varphi_w(z)| dm(z).$$

We can use the inequality (1.14) to estimate the integral over E_δ which gives

$$\int_{E_\delta} \log |\varphi_w(z)| dm(z) \leq \int_{E_\delta} -\frac{1}{2}\frac{1-\delta}{1+\delta}(1-|w|^2) dm(z) = -\frac{1}{2}\frac{1-\delta}{1+\delta}(1-|w|^2)m(E_\delta).$$

In total, we have

$$N_\varphi(w) \leq \log \frac{1}{|w|} - \frac{1}{2}\frac{1-\delta}{1+\delta}(1-|w|^2)m(E_\delta) = \log \frac{1}{|w|} - C_\delta(1-|w|^2).$$

To continue the proof we make use of the change of variable formula (1.12) and obtain

$$\begin{aligned} ||\mathscr{C}_\varphi f||_{H^2}^2 &= |f(0)|^2 + 2\int_{\mathbb{D}} |f'(w)|^2 N_\varphi(w)\, dA(w) \\ &\leq |f(0)|^2 + 2\int_{\mathbb{D}} |f'(w)|^2 \left(\log \frac{1}{|w|} - C_\delta(1-|w|^2)\right) dA(w). \end{aligned}$$

After a similar calculation as in the proof of the Littlewood-Paley identity we find

$$\begin{aligned} 2\int_{\mathbb{D}} |f'(w)|^2 \left(\log \frac{1}{|w|} - C_\delta(1-|w|^2)\right) dA(w) &= \sum_{n=1}^{\infty} |\hat{f}(n)|^2 \left(1 - 2C_\delta \frac{n}{n+1}\right) \\ &\leq (1-C_\delta) \sum_{n=1}^{\infty} |\hat{f}(n)|^2, \end{aligned}$$

which completes the proof. □

The next theorem is the main result in Shapiro's paper [16].

Theorem 1.18. *Suppose that φ is a holomorphic self-map of $\mathbb{D}$ with $\varphi(0) = 0$. Then the following are equivalent:*

1. *φ is inner.*
2. *$\mathscr{C}_\varphi : H^2 \to H^2$ is an isometry.*
3. *$||\mathscr{C}_\varphi|_{H_0^2}|| = 1$.*

Proof. ($1 \implies 2$) Assume that φ is inner. Since $\varphi(0)=0$, it follows from Lemma 1.8 that $\langle \varphi^m, \varphi^n \rangle_{H^2} = \delta_{m,n}$, so

$$\begin{aligned} ||\mathscr{C}_\varphi f||_{H^2}^2 &= \langle \mathscr{C}_\varphi f, \mathscr{C}_\varphi f \rangle_{H^2} \\ &= \sum_{m=0}^\infty \sum_{n=0}^\infty \hat{f}(m)\overline{\hat{f}(n)} \langle \varphi^m, \varphi^n \rangle_{H^2} \\ &= \sum_{n=0}^\infty |\hat{f}(n)|^2 \\ &= ||f||_{H^2}^2. \end{aligned}$$

This is true for every $f \in H^2$. Hence, $\mathscr{C}_\varphi$ is an isometry.

($2 \implies 3$) If $||\mathscr{C}_\varphi f||_{H^2} = ||f||_{H^2}$ for every $f \in H^2$, then certainly this must also be true for every f in the subspace H_0^2. Hence, the operator norm of $\mathscr{C}_\varphi$ is still 1 when restricted to this subspace.

($3 \implies 1$) We prove this using a contrapositive argument. That is, we want to show that whenever φ is not inner, then the operator norm of $\mathscr{C}_\varphi$ is less than 1 when restriced to H_0^2. If φ is not inner, then it is possible to find a $\delta \in (0,1)$ such that the set E_δ defined in Lemma 1.17 has positive measure. When we restrict $\mathscr{C}_\varphi$ to the subspace H_0^2, then the lemma states that

$$||\mathscr{C}_\varphi f||_{H^2}^2 \leq (1 - C_\delta)||f||_{H^2}^2.$$

Since $0 < C_\delta < 1$, it follows that $||\mathscr{C}_\varphi|| < 1$. □

We also want to prove an analogous result for when φ does not fix the origin. For this result we will need a lemma:

Lemma 1.19. *Suppose φ and $\tilde{\varphi}$ are inner functions. Then the composition $\varphi \circ \tilde{\varphi}$ is also inner.*

Proof. Consider the integral

$$\int_{\mathbb{T}} |\varphi(\tilde{\varphi}(z))| \, dm.$$

The map $\tilde{\varphi}$ is inner, so there exists a subset $E \subseteq \mathbb{T}$ where $|\tilde{\varphi}(e^{i\theta})| = 1$ for all $e^{i\theta} \in E$ and $m(\mathbb{T} \backslash E) = 0$. We therefore have

$$\int_{\mathbb{T}} |\varphi(\tilde{\varphi}(z))| \, dm = \int_E |\varphi(\tilde{\varphi}(z))| \, dm = \int_E |\varphi(e^{i\tilde{\theta}})| \, dm = 1,$$

since φ is inner. Hence, $\varphi \circ \tilde{\varphi}$ is inner. □

Theorem 1.20. *Suppose that φ is a holomorphic self-map of $\mathbb{D}$ with $\varphi(0) = w \neq 0$. Then the following are equivalent:*

1. *φ is inner.*
2. *$||\mathscr{C}_\varphi f||_{H^2} = ||\mathscr{C}_{\alpha_w} f||_{H^2}$ for every $f \in H^2$.*

3. $||\mathscr{C}_\varphi|| = \sqrt{\frac{1+|w|}{1-|w|}}$.

Proof. (1 $\implies$ 2) Let $f_w = f \circ \alpha_w$ and $\varphi_w = \alpha_w \circ \varphi$. Since α_w is its own inverse, we see that $f_w \circ \varphi_w = f \circ \varphi$. From Lemma 1.19 we know that if both φ and α_w are inner function, then that is also the case for φ_w. Observe that

$$|\alpha_w(e^{i\theta})| = \left|\frac{w - e^{i\theta}}{1 - \overline{w}e^{i\theta}}\right| = 1,$$

because

$$(w - e^{i\theta})(\overline{w} - e^{-i\theta}) = (1 - \overline{w}e^{i\theta})(1 - we^{-i\theta}).$$

We can therefore conclude that φ_w is inner. Now

$$||\mathscr{C}_\varphi f||_{H^2} = ||f \circ \varphi||_{H^2} = ||f_w \circ \varphi_w||_{H^2}.$$

We know now that φ_w is an inner function which fixes the origin, so by Theorem 1.18 we have

$$||f_w \circ \varphi_w||_{H^2} = ||f_w||_{H^2} = ||\mathscr{C}_{\alpha_w} f||_{H^2}.$$

(2 $\implies$ 3) Lemma 1.14 tells us that $||\mathscr{C}_{\alpha_w}|| = \sqrt{\frac{1+|w|}{1-|w|}}$, so this is trivial.

(3 $\implies$ 1) As before, we prove the contrapositive. So assume that φ is not inner. Define φ_w as in (1.13). Then $\varphi = \alpha_w \circ \varphi_w$, since α_w is its own inverse. For $f \in H^2$, we have

$$f \circ \varphi = f \circ \alpha_w \circ \varphi_w = f \circ \alpha_w \circ \varphi_w + f(w) - f(w) = \mathscr{C}_{\varphi_w}(f \circ \alpha_w - f(w)) + f(w).$$

After writing $g = f \circ \alpha_w - f(w)$, the previous equality simplifies to

$$\mathscr{C}_\varphi f = \mathscr{C}_{\varphi_w} g + f(w).$$

Observe that $g(0) = \varphi_w(0) = 0$, which implies $\mathscr{C}_{\varphi_w} g(0) = 0$. This makes $\mathscr{C}_{\varphi_w} g$ orthogonal to constant functions in H^2. Because $\varphi_w(0) = 0$ we know from before that $||\mathscr{C}_{\varphi_w}|_{H_0^2}|| = \sqrt{\epsilon} < 1$. This yields

$$||\mathscr{C}_\varphi f||^2_{H^2} = ||\mathscr{C}_{\varphi_w} g||^2_{H^2} + |f(w)|^2 \leq \epsilon ||g||^2 + |f(w)|^2 = \epsilon ||\mathscr{C}_{\alpha_w} f - f(w)||^2 + |f(w)|^2.$$

We now need to do another observation, namely that $f(0) = \langle f, 1\rangle_{H^2}$. From this we get

$$\langle \mathscr{C}_{\alpha_w} f, f(w)\rangle_{H^2} = \overline{f(w)}\mathscr{C}_{\alpha_w} f(0) = \overline{f(w)} f(w) = |f(w)|^2.$$

Further,

$$\begin{aligned} ||\mathscr{C}_{\alpha_w} f - f(w)||^2_{H^2} &= ||\mathscr{C}_{\alpha_w} f||^2_{H^2} - 2\mathrm{Re}\langle \mathscr{C}_{\alpha_w} f, f(w)\rangle_{H^2} + |f(w)|^2 \\ &= ||\mathscr{C}_{\alpha_w} f||^2_{H^2} - 2|f(w)|^2 + |f(w)|^2 \\ &= ||\mathscr{C}_{\alpha_w} f||^2_{H^2} - |f(w)|^2. \end{aligned}$$

So now we have

$$\begin{aligned} ||\mathscr{C}_\varphi f||^2_{H^2} \leq \epsilon ||\mathscr{C}_{\alpha_w} f - f(w)||^2_{H^2} + |f(w)|^2 &\leq \epsilon ||\mathscr{C}_{\alpha_w} f||^2_{H^2} - \epsilon |f(w)|^2 - |f(w)|^2 \\ &= \epsilon ||\mathscr{C}_{\alpha_w} f||^2_{H^2} + (1-\epsilon)|f(w)|^2. \end{aligned}$$

We continue by estimating $|f(w)|$ using (1.5):

$$|f(w)| \leq \frac{||f||_{H^2}}{\sqrt{1-|w|^2}}.$$

From Theorem 1.12 it now follows that

$$\begin{aligned}||\mathscr{C}_\varphi f||_{H^2}^2 &\leq \epsilon\left(\frac{1+|w|}{1-|w|}\right)||f||_{H^2}^2 + (1-\epsilon)\frac{||f||_{H^2}^2}{1-|w|^2}\\ &= \left(\epsilon + \frac{1-\epsilon}{1+|w|^2}\right)\left(\frac{1+|w|}{1-|w|}\right)||f||_{H^2}^2.\end{aligned}$$

We assume that $w \neq 0$, so

$$\epsilon + \frac{1-\epsilon}{1+|w|^2} < 1,$$

and in turn

$$||\mathscr{C}_\varphi|| < \sqrt{\frac{1+|w|}{1-|w|}}.$$

□

Chapter 2

Bounded Dirichlet series

In this chapter we establish some properties of Dirichlet series that will be useful later on. We consider the space $\mathscr{H}^\infty$ of bounded Dirichlet series on $\mathbb{C}_0$ and, in particular, we give a proof of Bohr's theorem. Also, we introduce the notion of a vertical limit function and prove certain results on the topic. Most of the theory in this chapter can be found in [13].

A Dirichlet series is a series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \tag{2.1}$$

where $s = \sigma + it \in \mathbb{C}$ and a_n is a sequence of complex numbers. Denote by $\mathbb{C}_\theta$ the set

$$\mathbb{C}_\theta = \{s \in \mathbb{C} : \operatorname{Re} s > \theta\}.$$

To every convergent Dirichlet series f we associate a number σ_c defined by

$$\sigma_c(f) = \inf\{\theta \in \mathbb{R} : f \text{ is convergent in } \mathbb{C}_\theta\},$$

which we refer to as the abscissa of convergence. A classical and important example of a convergent Dirichlet series is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

By the standard theory of convergent series we find that the abscissa of convergence for the zeta function is $\sigma_c(\zeta) = 1$.

Definition 2.1. *We denote the space of convergent Dirichlet series by $\mathcal{D}$. That is*

$$\mathcal{D} := \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \,\middle|\, \sigma_c(f) < \infty \right\}.$$

We also consider the abscissa of uniform convergence and absolute convergence, denoted by σ_u and σ_a, respectively. These numbers are defined analogously to the abscissa of convergence σ_c. Note that if σ_u is the abscissa of uniform convergence of a Dirichlet series f, then it is understood that f converges uniformly on $\overline{\mathbb{C}_{\sigma_u+\delta}}$, for any $\delta > 0$. Consequently, for every fixed $\delta > 0$ there exists a constant M such that $|f(s)| \leq M$ on $\overline{\mathbb{C}_{\sigma_u+\delta}}$. For a function $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{D}$ we always have $\sigma_a(f) - \sigma_c(f) \leq 1$. Indeed, for any $\varepsilon > 0$ we must have $|a_n| \leq C n^{\sigma_c + \varepsilon/2}$, where the constant C is adjusted depending on the choice of ε. It follows that

$$\sum_{n=1}^{\infty} |a_n| n^{-\sigma_c - 1 - \varepsilon} \leq C \sum_{n=1}^{\infty} n^{-1-\varepsilon/2},$$

which means that f certainly converges absolutely in $\mathbb{C}_{\sigma_c+1+\varepsilon}$. Since ε was arbitrary we get $\sigma_a(f) \leq \sigma_c + 1$. Consider now the alternating zeta function defined by

$$\zeta^*(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}.$$

It is well known that $\sigma_c(\zeta^*) = 0$. Observe that the alternating zeta function converges absolutely in the domain where the zeta function converges regularly. Therefore, $\sigma_a(\zeta^*) = 1$ and we get $\sigma_a(\zeta^*) - \sigma_c(\zeta^*) = 1$. Hence, the relation $\sigma_a - \sigma_c \leq 1$ is in fact optimal. There also exists another relation of this kind, namely $\sigma_a - \sigma_u \leq 1/2$. This result is due to Bohr ([13], Theorem 4.4.2) and is much more involved. The constant $1/2$ is optimal as well, which is a result from [2].

We now introduce an important set of Dirichlet series denoted by $\mathscr{H}^\infty$, which is the set of convergent Dirichlet series that can be analytically continued to a bounded function on $\mathbb{C}_0$. If we let $H^\infty(\mathbb{C}_0)$ denote the set of bounded analytic functions on $\mathbb{C}_0$, then we can write

$$\mathscr{H}^\infty = H^\infty(\mathbb{C}_0) \cap \mathcal{D}.$$

The norm on $\mathscr{H}^\infty$ is defined as

$$||f||_{\mathscr{H}^\infty} = \sup_{s \in \mathbb{C}_0} |f(s)|.$$

2.1 Bohr's theorem

The next goal is to prove an important theorem regarding the convergence of a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ in the space $\mathscr{H}^\infty$, named after Bohr. To that end, we follow two ideas from [13]. We start out with a lemma that gives us a rough estimate of the coefficients a_n.

Lemma 2.2. *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathscr{H}^\infty$. Then $|a_n| \leq ||f||_{\mathscr{H}^\infty}$ for every positive integer n.*

Proof. Let $\rho > 0$ and consider the integral

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(\rho + it) m^{\rho+it}\, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\sum_{n=1}^{\infty} a_n n^{-\rho-it} \right) m^{\rho+it}\, dt.$$

Since $f \in \mathscr{H}^\infty$ it is a convergent Dirichlet series in some half-plane. We can then choose $\rho > \sigma_a(f)$ so that $\sum_{n=1}^\infty |a_n| n^{-\rho} < \infty$. For sufficiently large ρ we then have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\sum_{n=1}^{\infty} a_n n^{-\rho-it} \right) m^{\rho+it}\, dt = a_m + \sum_{\substack{n=1\\ n\neq m}}^{\infty} a_n (nm)^{-\rho} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} n^{-it} m^{it}\, dt = a_m.$$

After a change of variables $s = \rho + it$ we can write

$$a_m = \lim_{T\to\infty} \frac{1}{2iT} \int_{\rho-iT}^{\rho+iT} f(s) m^s ds.$$

Denote by Γ_ε, for some $0 < \varepsilon < \rho$, the rectangle with corners at $\varepsilon - iT$, $\varepsilon + iT$, $\rho + iT$ and $\rho - iT$. Then the Cauchy integral formula gives

$$\begin{aligned} 0 &= \lim_{T\to\infty} \frac{1}{2iT} \int_{\Gamma_\varepsilon} f(s) n^s\, ds \\ &= \lim_{T\to\infty} \frac{1}{2iT} \left(\int_{\rho-iT}^{\rho+iT} \int_{\rho+iT}^{\varepsilon+iT} \int_{\varepsilon+iT}^{\varepsilon-iT} \int_{\varepsilon-iT}^{\rho-iT} \right) f(s) n^s\, ds, \end{aligned}$$

or

$$a_n = \lim_{T\to\infty} \frac{1}{2iT} \left(\int_{\varepsilon+iT}^{\rho+iT} f(s) n^s\, ds + \int_{\varepsilon-iT}^{\varepsilon+iT} f(s) n^s\, ds - \int_{\varepsilon-iT}^{\rho-iT} f(s) n^s\, ds \right).$$

Estimation of the first and third integral yields

$$\left| \frac{1}{2iT} \int_{\varepsilon+iT}^{\rho+iT} f(s) n^s\, ds \right| \leq \frac{\rho n^\rho ||f||_{\mathscr{H}^\infty}}{2T},$$

and

$$\left| \frac{1}{2iT} \int_{\varepsilon-iT}^{\rho-iT} f(s) n^s\, ds \right| \leq \frac{\rho n^\rho ||f||_{\mathscr{H}^\infty}}{2T}.$$

Clearly, the integrals goes to zero as $T \to \infty$. For the remaining integral we have

$$\left| \frac{1}{2iT} \int_{\varepsilon-iT}^{\varepsilon+iT} f(s) n^s\, ds \right| \leq n^\varepsilon ||f||_{\mathscr{H}^\infty}.$$

Now, when we let ε approach zero we get $|a_n| \leq ||f||_{\mathscr{H}^\infty}$. □

From this result we can infer a useful property of the functions in $\mathscr{H}^\infty$, namely that $\sigma_a(f) \leq 1$ for every $f \in \mathscr{H}^\infty$. Indeed, for $f(s) = \sum_{n=1}^\infty a_n n^{-s} \in \mathscr{H}^\infty$, we have $\sum_{n=1}^\infty |a_n| n^{-\sigma} \leq ||f||_{\mathscr{H}^\infty} \sum_{n=1}^\infty n^{-\sigma}$. This property will be helpful in proving our next result.

Lemma 2.3. *Let $f(s) = \sum_{n=1}^\infty a_n n^{-s} \in \mathscr{H}^\infty$ and $S_N f(s) = \sum_{n=1}^N a_n n^{-s}$ be its partial sum. Then there exists a constant C such that*

$$||S_N f||_{\mathscr{H}^\infty} \leq C \log N ||f||_{\mathscr{H}^\infty}.$$

Proof. For the proof we will need Perron's formula ([13], Theorem 4.2.3):

$$A(x) := \sum_{n\leq x} a_n = \frac{1}{2i\pi}\int_{\rho-iT}^{\rho+iT} f(s)\frac{x^s}{s}\,ds + O\left[\frac{x^\rho}{T}\sum_{n\geq 1}\frac{|a_n|}{n^\rho|\log(x/n)|}\right].$$

Here, $\rho > \max(0, \sigma_a(f))$, $x, T \geq 1$ and x is assumed not to be an integer. Since $f \in \mathscr{H}^\infty$ we know that $\sigma_a(f) \leq 1$, so we can choose $\rho = 2$. We let $x = N + 1/2$ for some positive integer N. We need to estimate the error term. First we find

$$|\log(x/n)| \geq |1 - n/x| = \left|\frac{n-x}{x}\right| = \left|\frac{n - N - 1/2}{N + 1/2}\right| \leq \frac{1}{2(N+1/2)}.$$

We see that $|\log(x/n)|^{-1} = O(x)$. This gives

$$\frac{x^2}{T}\sum_{n\geq 1}\frac{|a_n|}{n^2|\log(x/n)|} \leq \frac{x^3}{T}\sum_{n\geq 1}\frac{|a_n|}{n^2} \geq \frac{x^3}{T}||f||_{\mathscr{H}^\infty}$$

If we now choose $T = x^3$, then we find that the error term of $A(x)$ is dominated by $C||f||_{\mathscr{H}^\infty}$, for some constant C. It remains to estimate the integral. Let $0 < \varepsilon < 2$ and consider the rectangle determined by the points $2 - iT$, $2 + iT$, $\varepsilon + iT$ and $\varepsilon - iT$. Cauchy's integral formula gives

$$\begin{aligned}\int_{2-iT}^{2+iT} f(s)\frac{x^s}{s}\,ds &= \int_{\varepsilon-iT}^{\varepsilon+iT} f(s)\frac{x^s}{s}\,ds + \int_{\varepsilon+iT}^{2+iT} f(s)\frac{x^s}{s}\,ds + \int_{2-iT}^{\varepsilon-iT} f(s)\frac{x^s}{s}\,ds\\ &= \int_{\varepsilon-iT}^{\varepsilon+iT} f(s)\frac{x^s}{s}\,ds + \int_{\varepsilon}^{2} f(u+iT)\frac{x^{u+iT}}{u+iT}\,du - \int_{\varepsilon}^{2} f(u-iT)\frac{x^{u-iT}}{u-iT}\,du.\end{aligned}$$

We have

$$\left|\int_{\varepsilon}^{2} f(u+iT)\frac{x^{u+iT}}{u+iT}\,du\right| \leq \frac{x^2||f||_{\mathscr{H}^\infty}}{T} = \frac{||f||_{\mathscr{H}^\infty}}{x},$$

because $T = x^3$. We also get

$$\left|\int_{\varepsilon}^{2} f(u-iT)\frac{x^{u-iT}}{u-iT}\,du\right| \leq \frac{||f||_{\mathscr{H}^\infty}}{x}.$$

For the remaining integral we do a change of variables $s = \varepsilon + it$. This gives

$$\int_{\varepsilon-iT}^{\varepsilon+iT} f(s)\frac{x^s}{s}\,ds = \int_{-T}^{T} f(\varepsilon+it)\frac{x^{\varepsilon+it}}{\varepsilon+it}i\,dt.$$

We now estimate the integral in the following way.

$$\left|\int_{-T}^{T} f(\varepsilon+it)\frac{x^{\varepsilon+it}}{\varepsilon+it}i\,dt\right| \leq x^\varepsilon||f||_{\mathscr{H}^\infty}\int_{-T}^{T}\frac{1}{\sqrt{\varepsilon^2+t^2}}\,dt.$$

Let $u = t/\varepsilon$ in order to get

$$\begin{aligned}
x^{\varepsilon}||f||_{\mathscr{H}^\infty}\int_{-T}^{T}\frac{1}{\sqrt{\varepsilon^2+t^2}}\,dt &= 2x^{\varepsilon}||f||_{\mathscr{H}^\infty}\int_{0}^{T/\varepsilon}\frac{1}{\sqrt{u^2+1}}\,du \\
&= 2x^{\varepsilon}||f||_{\mathscr{H}^\infty}\left(\int_0^1\frac{1}{\sqrt{u^2+1}}\,du+\int_1^{T/\varepsilon}\frac{1}{\sqrt{u^2+1}}\,du\right) \\
&\leq 2x^{\varepsilon}||f||_{\mathscr{H}^\infty}\left(1+\int_1^{T/\varepsilon}\frac{du}{u}\right) \\
&\leq 4x^{\varepsilon}||f||_{\mathscr{H}^\infty}\log(T/\varepsilon).
\end{aligned}$$

We observe that this integral is the largest contributor to the size of $|A(x)|$. If we now conveniently pick $\varepsilon = \frac{1}{\log(x)}$, we get

$$|A(x)| \leq C||f||_{\mathscr{H}^\infty}\log(x^3\log(x))$$

for some constant C. Clearly, $\log(x^3\log(x)) = O(\log(x))$. Hence,

$$|A(x)| \leq C\log(x)||f||_{\mathscr{H}^\infty}.$$

We readily see that the $\mathscr{H}^\infty$-norm is invariant under vertical translations. Also, if $g(s) = f(s+\sigma)$, then it is clear that $||g||_{\mathscr{H}^\infty} \leq ||f||_{\mathscr{H}^\infty}$ for $\sigma > 0$. For any $s_0 \in \mathbb{C}_0$ we get

$$|S_N f(s_0)| = \left|\sum_{n=1}^{N} a_n n^{-s_0}\right| \leq C\log N||f||_{\mathscr{H}^\infty},$$

which ends the proof. □

We are now ready to state and prove Bohr's Theorem [13].

Theorem 2.4 (Bohr's theorem)**.** *Suppose we have a function $f \in \mathscr{H}^\infty$, with Dirichlet series representation $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ in some half-plane $\mathbb{C}_\theta$. Then this Dirichlet series converges uniformly in the half-plane $\overline{\mathbb{C}_\varepsilon}$ for all $\varepsilon > 0$.*

Proof. The series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges uniformly in some half-plane $\overline{\mathbb{C}_\varepsilon}$ if and only if the related series $\sum_{n=1}^{\infty} a_n n^{-s-\varepsilon}$ converges uniformly on every closed half-plane contained in $\mathbb{C}_0$. So we shall show that the latter series converges uniformly on every closed half-plane contained $\mathbb{C}_0$ for all $\varepsilon > 0$. We consider the partial sum $S_N f(s) = \sum_{n=1}^{N} a_n n^{-s}$ and write

$$\sum_{n=1}^{N} a_n n^{-s-\varepsilon} = \sum_{n=1}^{N}(S_n f(s) - S_{n-1}f(s))n^{-\varepsilon}.$$

An application of the standard partial summation formula gives

$$\sum_{n=1}^{N}(S_n f(s) - S_{n-1}(s))n^{-\varepsilon} = S_N f(s)N^{-\varepsilon} - \sum_{n=1}^{N-1} S_n f(s)((n+1)^{-\varepsilon} - n^{-\varepsilon}).$$

Using Lemma 2.3 we find that the term $S_N f(s)N^{-\varepsilon}$ is bounded by $C \log N ||f||_{\mathscr{H}^\infty}/N^{\varepsilon}$. Let $g(x) = x^{-\varepsilon}$. Then $g(n+1) - g(n) = g'(c)$ for some $c \in [n, n+1]$, by the mean value theorem. Since g is decreasing monotonically in this interval we must have $g'(c) \leq g'(n)$. Hence, the terms $S_n f(s)((n+1)^{-\varepsilon} - n^{-\varepsilon})$ are bounded by $C' \log n ||f||_{\mathscr{H}^\infty}/(n^{\varepsilon+1})$, again by Lemma 2.3 and the mean value theorem applied to $n^{-\varepsilon}$. The bounds are independent of s, so we get uniform convergence. □

We are now going to prove yet another result regarding uniform convergence of functions that can be represented as Dirichlet series. The next theorem is, in fact, a stronger version of Bohr's Theorem and of great utility.

Theorem 2.5. *Let $\varphi : \mathbb{C}_\theta \to \mathbb{C}_\nu$ be an analytic function, with $\varphi \in \mathcal{D}$. Then $\sigma_u(\varphi) \leq \theta$.*

Proof. We define $\psi(s) := \varphi(s + \theta) - \nu$. Now, ψ is an analytic function on $\mathbb{C}_0$ which can represented as a Dirichlet series in some half-plane. So we need to prove that $\sigma_u(\psi) \leq 0$. Herglotz representation theorem ([8], Theorem 3.5) tells us that every harmonic non-negative function h on $\mathbb{C}_0$ can be expressed as

$$h(\sigma + it) = c\sigma + \int_{\mathbb{R}} P_\sigma(t - \tau)\, d\mu(\tau),$$

where $c \geq 0$ is a constant and P_σ is the poisson kernel, that is

$$P_\sigma(v) = \frac{1}{\pi} \frac{\sigma}{\sigma^2 + v^2}.$$

If $0 < \sigma \leq \theta$, then we must have $\sigma P_\sigma(v) \leq \theta P_\sigma(v)$ or $P_\sigma(v) \leq \frac{\theta}{\sigma} P_\sigma(v)$. Of course, this implies that $h(\sigma + it) \leq \frac{\theta}{\sigma} h(\theta + it)$. We know that ψ can represented as a Dirichlet series for a large enough choice of $\operatorname{Re} s$. Hence, there exists a half-plane $\mathbb{C}_\vartheta$ on which ψ is bounded by a constant, say M. Now, for any $0 < \alpha < 1$, the function $\operatorname{Re} \psi^\alpha$ constitute a non-negative harmonic function, and we denote it by h. If we write $\psi(s) = |\psi(s)| e^{i \arg(\psi)}$, then we see that

$$\begin{aligned} h = \operatorname{Re}(\psi^\alpha) &= \cos(\alpha \arg(\psi)) |\psi|^\alpha \\ &\geq \cos(\alpha\pi/2)) |\psi|^\alpha, \end{aligned}$$

since $-\pi/2 \leq \arg(\psi) \leq \pi/2$. We can write this inequality as $|\psi|^\alpha \leq K(\alpha) h$, where $K(\alpha)$ is a constant depending on α. Combining this with the inequalities above gives

$$|\psi(\sigma + it)|^\alpha \leq K(\alpha) h(\sigma + it) \leq K(\alpha) \frac{\theta}{\sigma} h(\theta + it) \leq K(\alpha) \frac{\theta}{\sigma} |\psi(\theta + it)|^\alpha \leq K(\alpha) \frac{\theta}{\sigma} M^\alpha$$

We observe that ψ remains bounded in the half-plane $\mathbb{C}_\sigma$ for every $\sigma > 0$. Now we just apply Bohr's theorem (Theorem 2.4) to finish the proof. □

Quite remarkably, a Dirichlet series converging to an analytic function in some possibly distant half-plane, actually converges to that function in every half-plane where it is analytic.

2.2 Vertical limit functions

We will now introduce the concept of a vertical limit function, which will be of great importance in the study of composition operators on $\mathscr{H}^2$. In order to do so, we need the notion of a multiplicative character. We say that $\chi : \mathbb{N} \to \mathbb{T}$ is a multiplicative character if it satisfies $\chi(mn) = \chi(m)\chi(n)$ for all positive integers m and n. We denote the set of all such characters by $\mathcal{M}$. It is convenient to identify the set $\mathcal{M}$ with $\mathbb{T}^\infty$, the infinite dimensional Cartesian product of $\mathbb{T}$. This identification is done as follows. Take a point $z = (z_1, z_2, ...) \in \mathbb{T}^\infty$ and let $\chi(p_j) = z_j$, where p_j denotes the j-th prime. The character χ is now defined for every prime number and we extend it to all of $\mathbb{N}$ by letting $\chi(n) = \chi(p_1^{r_1}) \cdots \chi(p_m^{r_m})$, where $n = p_1^{r_1} \cdots p_m^{r_m}$. Note that this definition of χ forces it to be multiplicative. Moreover, $\mathbb{T}^\infty$ is a compact group under point-wise multiplication. It therefore exists a unique Haar measure on $\mathbb{T}^\infty$, which happens to coincide with the infinite product measure generated by the normalized Lebesgue measure on $\mathbb{T}$.

Consider now the function $f(s) = \sum_{n=1}^\infty a_n n^{-s} \in \mathcal{D}$. For any character $\chi \in \mathcal{M}$ we define the function f_χ by

$$f_\chi(s) = \sum_{n=1}^\infty a_n \chi(n) n^{-s}. \tag{2.2}$$

We call this a vertical limit function. To understand the reasoning behind this name we look at the character $\chi(n) = n^{-it}$. This obviously defines a multiplicative character. Also, for $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ we have $f_\chi(s) = f(s + i\tau) =: f_\tau(s)$. That is, f_χ is a vertical translation of f. The interesting thing is, as we shall see shortly, that every vertical limit function f_χ corresponds to the limit of a sequence of vertical translations of f. We start out by proving a famous theorem by Kronecker, and we will give an analytic proof due to Bohr [3].

Definition 2.6. *Let $a_1, ..., a_n$ be a set of real numbers. We say that the numbers are $\mathbb{Q}$-linearly independent if*

$$\sum_{j=1}^n c_j a_j = 0, \quad \text{with } c_1, ..., c_n \in \mathbb{Z},$$

only when $c_j = 0$ for $1 \leq j \leq n$.

One typical example of such a set is the following. If $p_1, ..., p_j$ is a set of j unique primes, then the real numbers $\log p_1, ..., \log p_j$ are $\mathbb{Q}$-linearly independent. This follows from the relation

$$\sum_{i=1}^j c_i \log p_i = \log \prod_{1 \leq i \leq j} p_i^{c_i},$$

and the fundamental theorem of arithmetic.

Theorem 2.7 (Kronecker's Theorem)**.** *Suppose $\xi_1, ..., \xi_k$ are $\mathbb{Q}$-linearly independent real numbers. Let $\alpha_1, ..., \alpha_k$ be arbitrary real numbers and let $\epsilon > 0$. Then there exists integers $n_1, ..., n_k$ and a real number t satisfying*

$$|t\xi_j - \alpha_j - n_j| < \epsilon, \quad j = 1, 2, ..., k.$$

Proof. We want to carry out an analytic proof of this result and we will rely on the following fact. The exponential function $e^{2\pi ix}$ is equal to 1 if and only if x is an integer. Consider the function

$$f(t) = 1 + \sum_{m=1}^{k} e^{2\pi i(t\xi_m - \alpha_m)}.$$

Clearly, $|f(t)| \leq k+1$. Observe that $|f(t)|$ is close to $k+1$ if and only if $(t\xi_m - \alpha_m)$ is close to an integer for all m. In this language, Kronecker's Theorem states that we can get $|f(t)|$ arbitrarily close to $k+1$ by choosing t sufficiently large.

Let F denote the function

$$F(x_1, .., x_k) = 1 + x_1 + \cdots + x_k.$$

If we now raise the functions f and F to their p-th power, for some integer p, and use the multinomial theorem, they take the form

$$(f(t))^p = \sum b_\nu e^{i\beta_\nu t},$$
$$(F(x_1, ..., x_k))^p = \sum b_{\lambda_1,\ldots,\lambda_k} x_1^{\lambda_1} \cdots x_k^{\lambda_k}.$$

Note that the β_ν must all be different due to the linear independence of the ξ_i. The p-th power of the functions f and F must therefore have the same number of terms, and the absolute value of the coefficients b_ν and $b_{\lambda_1,\ldots,\lambda_k}$ coincide. This yields

$$\sum |b_\nu| = \sum b_{\lambda_1,\ldots,\lambda_k} = (F(1, ..., 1))^p = (k+1)^p.$$

For any fixed $\nu = \nu'$ we have the following identity,

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{T} \int_0^T (f(t))^p e^{-i\beta_{\nu'} t}\, dt &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\sum b_\nu e^{i\beta_\nu t}\right) e^{-i\beta_{\nu'} t}\, dt \\
&= b_{\nu'} + \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\sum_{\nu\neq\nu'} b_\nu e^{i\beta_\nu t}\right) e^{-i\beta_{\nu'} t}\, dt \\
&= b_{\nu'} + \sum_{\nu\neq\nu'} b_\nu \left(\lim_{T\to\infty} \frac{1}{T} \int_0^T e^{i(\beta_\nu - \beta_{\nu'})t}\, dt\right) \\
&= b_{\nu'} + \sum_{\nu\neq\nu'} b_\nu \left(\lim_{T\to\infty} \frac{e^{i(\beta_\nu - \beta_{\nu'})T} - 1}{(\beta_\nu - \beta_{\nu'})iT}\right) \\
&= b_{\nu'}
\end{aligned}$$

If we now assume that there exists a constant C such that $|f(t)| \leq C < k+1$ for all $t \in \mathbb{R}$, then every coefficient b_ν of $(f(t))^p$ satisfies

$$|b_\nu| = \left|\lim_{T\to\infty} \frac{1}{T} \int_0^T (f(t))^p e^{-i\beta_\nu t}\, dt\right| \leq C^p.$$

As k increases by 1, the number of terms in the polynomial development of $(1+x_1+...+x_k)^p$ is multiplied with at most $p+1$. This means that the total number of terms must be less than $(p+1)^k$. We therefore have

$$\sum |b_\nu| < (p+1)^k C^p,$$

which implies that

$$\frac{(p+1)^k C^p}{\sum |b_\nu|} = \frac{(p+1)^k C^p}{(k+1)^p} > 1$$

for all p. However, since $C < k+1$ we have that

$$\frac{(p+1)^k C^p}{(k+1)^p} \to 0,$$

as $p \to \infty$. Therefore, our assumption that $|f(t)|$ is less than C for every $t \in \mathbb{R}$ leads to a contradiction. Consequently, the proof is complete. □

The result below was originally presented in [10], and we will follow a proof from [13].

Theorem 2.8. *Suppose that $f(s) = \sum_{n=1} a_n n^{-s}$, with $\sigma_u(f) = \theta$. Then the vertical limit functions f_χ are precisely the limits of some sequence $\{f_{\tau_N}\}$ of vertical translations of f, in the half-plane $\mathbb{C}_\theta$. The sequence $\{f_{\tau_N}\}$ converges uniformly on $\overline{\mathbb{C}_{\theta+\delta}}$, for every $\delta > 0$. Moreover, we have $\sigma_u(f_\chi) = \theta$.*

Proof. Assume that we have a sequence of vertical translations $\{f_{\tau_N}\}_{N\geq 1}$ converging to an analytic function on $\mathbb{C}_\theta$. It is clear that the function defined as $\xi_{\tau_N}(n) := n^{-i\tau_N}$ must converge to some limit function $\xi(n)$, as $N \to \infty$. We readily see that $|\xi(n)| = 1$ and that ξ is completely multiplicative. Hence, there exists a character $\chi \in \mathcal{M}$ such that $\chi(n) = \xi(n)$ for all n. Let $\vartheta > \theta$. We know that f converges uniformly on $\overline{\mathbb{C}_\vartheta}$. We can then take the limit as N approaches infinity of the translations f_{τ_N}, and pass the limit inside the summation. This will, in conjunction with the argument above, ensure that f_{τ_N} converges to f_χ on $\mathbb{C}_\vartheta$. Since ϑ was arbitrary, we actually have convergence on all of $\mathbb{C}_\theta$ and also $\sigma_u(f_\chi) = \theta$.

On the other hand, we want to show that for any character $\chi \in \mathcal{M}$ there exists a sequence of real numbers $\{\tau_N\}_{N\geq 1}$ such that $\{f_{\tau_N}\}_{N\geq 1}$ converges to f_χ. For this purpose, we recall Kronecker's Theorem (Theorem 2.7). In the language of vertical limit functions the theorem says that there exists $\tau \in \mathbb{R}$ such that

$$|\chi(n) - n^{-i\tau}| < \varepsilon, \quad n = 1, ..., N,$$

for any $\varepsilon > 0$ and positive integer N. Then for all N we should be able to find τ_N such that

$$|\chi(n) - n^{-i\tau_N}| < 1/N \quad n = 1, ..., N.$$

As N approach infinity we see that $n^{-i\tau_N} \to \chi(n)$, which is what we wanted to prove. □

Since all characters χ has modulus 1 it is also clear that $\sigma_a(f_\chi) = \sigma_a(f)$. We will now establish a product formula for vertical limit functions. The lemma is a slight improvement of a result from [13].

Lemma 2.9. *Assume we have a character $\chi \in \mathcal{M}$ and two convergent Dirichlet series f and g. Then the product formula $(fg)_\chi = f_\chi g_\chi$ holds in $\mathbb{C}_\theta$ whenever $\sigma_u(f) \leq \theta$ and $\sigma_u(g) \leq \theta$.*

Proof. In a remote half-plane where both f and g converges absolutely we can write the product $f(s)g(s) = \left(\sum_{n=1}^{\infty} a_n n^{-s}\right)\left(\sum_{m=1}^{\infty} b_m m^{-s}\right)$ as

$$f(s)g(s) = \sum_{n=1}^{\infty} c_n n^{-s},$$

where

$$c_n = \sum_{ij=n} a_i b_j.$$

Then

$$(fg)_\chi = \sum_{n=1}^{\infty} c_n \chi(n) n^{-s}.$$

On the other hand we have

$$f_\chi g_\chi = \sum_{n=1}^{\infty} d_n n^{-s},$$

where

$$d_n = \sum_{ij=n} a_i b_j \chi(i)\chi(j).$$

But χ is a completely multiplicative character, so

$$d_n = \chi(n) \sum_{ij=n} a_i b_j = \chi(n) c_n.$$

We see that $(fg)_\chi = f_\chi g_\chi$ in a half-plane where f and g converges absolutely. Since $\sigma_u(f) \leq \theta$ and $\sigma_u(g) \leq \theta$, the two functions are bounded on $\mathbb{C}_\theta$. The product of two bounded functions are again bounded, so by Bohr's theorem it follows that $\sigma_u(fg) \leq \theta$. The product formula therefore holds in all of $\mathbb{C}_\theta$. □

The next result tells us that the image of a function is invariant under vertical translations. We follow a proof from [5].

Lemma 2.10. *Let $\psi : \mathbb{C}_\theta \to \mathbb{C}_\nu$ be analytic, with $\psi(s) = \sum_{n=1}^{\infty} c_n n^{-s}$. Then $\psi(\mathbb{C}_\theta) = \psi_\chi(\mathbb{C}_\theta)$.*

Proof. We observe that $\psi(+\infty) = \psi_\chi(\infty) = c_1$, so the result is clear for constant ψ. Assume now that ψ is non-constant. For a point $w \in \mathbb{C}_\theta$ we find a closed disc K which contains w in its interior and satisfies

$$M = \inf_{s \in \partial K} |\psi_\chi(s) - \psi_\chi(w)| > 0.$$

By Theorem 2.8 there exists a sequence of vertical translation of ψ converging to ψ_χ on K. Let τ_k be a sequence such that $\psi(s + i\tau_k) \to \psi_\chi(s)$. By choosing k large enough we can ensure that

$$|\psi(s + i\tau_k) - \psi_\chi| < M.$$

We add and subtract $\psi_\chi(w)$ on the left-hand side, so that

$$|\psi(s+i\tau_k) - \psi_\chi(w) - (\psi_\chi(s) - \psi_\chi(w))| < M,$$

valid for $s \in K$. Denote by f and g the functions

$$\begin{aligned} f(s) &= \psi_\chi(s) - \psi_\chi(w) \\ g(s) &= \psi(s+i\tau_k) - \psi_\chi(w) - (\psi_\chi(s) - \psi_\chi(w)). \end{aligned}$$

The function f is clearly zero at $s = w$. Rouché's theorem tells us that f and $f+g$ has the same number so zeros in K. We have

$$(f+g)(s) = \psi(s+i\tau_k) - \psi_\chi(w),$$

so there must be a value of s in K making $\psi(s+i\tau_k) = \psi_\chi(w)$, and the proof is done. □

Chapter 3

The Hardy space $\mathscr{H}^2$

We consider here the Hardy space of Dirichlet series $\mathscr{H}^2$. We prove several results concerning the $\mathscr{H}^2$-norm. In that regard, we define the space $H^2_{\mathrm{i}}(\mathbb{C}_\theta, \alpha)$ of analytic functions on $\mathbb{C}_\theta$ which, after a Möbius transformation, belongs to Hardy space H^2. In the end, we consider the almost sure behaviour of vertical limit functions.

We now introduce the Hardy space $\mathscr{H}^2$ of Dirichlet series, which possess many similar properties to the space H^2 and is defined analogously. For any Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad s = \sigma + it \in \mathbb{C}, \tag{3.1}$$

we say that f belongs to $\mathscr{H}^2$ if and only if $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. The norm and inner product on $\mathscr{H}^2$ are defined as

$$||f||_{\mathscr{H}^2} = \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} \quad \text{and} \quad \langle f, g \rangle_{\mathscr{H}^2} = \sum_{n=1}^{\infty} a_n \overline{b_n},$$

respectively. A simple calculation involving the Cauchy-Schwarz inequality gives

$$|f(s)|^2 \leq \left(\sum_{n=1}^{\infty} |a_n n^{-s}| \right)^2 \leq \sum_{n=1}^{\infty} |a_n|^2 \sum_{n=1}^{\infty} n^{-2\sigma}, \tag{3.2}$$

for any $f \in \mathscr{H}^2$. Observe now that f certainly converges absolutely in the half-plane $\mathbb{C}_{\frac{1}{2}}$, and we therefore have $\sigma_a(f) \leq \frac{1}{2}$. This also provides us with a basic convergence result.

Lemma 3.1. *A sequence of Dirichlet series converging in the $\mathscr{H}^2$-norm converges uniformly, to the same limit, on closed half-planes in $\mathbb{C}_{1/2}$.*

Proof. Suppose $\{f_j\}_{j \geq 1}$ is a sequence in $\mathscr{H}^2$ converging to some limit f. Let $\theta > 1/2$, so that the closed half-plane $\overline{\mathbb{C}_\theta}$ belongs to $\mathbb{C}_{1/2}$. Since $\zeta(\operatorname{Re} s)$ decreases as $\operatorname{Re} s$ increases, we get

$$\sup_{s \in \overline{\mathbb{C}_\theta}} |f_j(s) - f(s)| \leq ||f_j - f||_{\mathscr{H}^2} \zeta(2\theta),$$

by the pointwise estimate (3.2). The zeta function remains bounded for every $\theta > 1/2$, so the right-hand side of the inequality approaches zero when $j \to \infty$ and the result follows. □

Let e_n denote the function $e_n(s) = n^{-s}$ for $n \geq 1$. We define, for a point $a \in \mathbb{C}_{1/2}$, the reproducing kernel K_a by

$$K_a(s) = \sum_{n=1}^{\infty} e_n(s)\overline{e_n(a)} = \zeta(s + \overline{a}),$$

so that the relation $f(a) = \langle f, K_a \rangle_{\mathscr{H}^2}$ holds for every $f \in \mathscr{H}^2$.

We will now look in to an alternative method for computing the $\mathscr{H}^2$-norm, as we did with the norm on H^2. However, the new expression will not be applicable to every Dirichlet series $f \in \mathscr{H}^2$. In particular, we will require $\sigma_u(f) \leq 0$. The result is due to Carlson [6].

Theorem 3.2. *Whenever the series* (2.1) *converges uniformly on* $\overline{\mathbb{C}_\varepsilon}$, *for any* $\varepsilon > 0$, *then*

$$||f||^2_{\mathscr{H}^2} = \lim_{\sigma \to 0^+} \left(\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^2 \, dt \right). \tag{3.3}$$

Proof. We start by rewriting the integrand:

$$|f(\sigma + it)|^2 = f(\sigma + it)\overline{f(\sigma + it)}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n \overline{a_m} n^{-\sigma} m^{-\sigma} n^{-it} m^{it}.$$

Now consider the integral

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left(\frac{m}{n}\right)^{it} dt.$$

We write $\left(\frac{m}{n}\right)^{it} = e^{it \log \frac{m}{n}}$, and get

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left(\frac{m}{n}\right)^{it} dt = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \cos\left(t \left|\log \frac{m}{n}\right|\right) dt = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

In total, we have for $\sigma > 0$

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^2 \, dt = \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma},$$

since f converges uniformly on $\overline{\mathbb{C}_\sigma}$. Upon letting σ go to zero from above the result follows. □

Corollary 3.3. *The space of bounded Dirichlet series* $\mathscr{H}^\infty$ *is a subset of the Hardy space* $\mathscr{H}^2$.

Proof. If $f \in \mathscr{H}^\infty$, then Bohr's theorem tells us that f converges uniformly in $\overline{\mathbb{C}_\varepsilon}$ for every $\varepsilon > 0$. Now it follows from Theorem 3.2 that the $\mathscr{H}^2$-norm of f is bounded by the $\mathscr{H}^\infty$-norm of f, which is finite. □

For the non-negative real numbers α and θ we define the Möbius transformations

$$\mathcal{T}_\alpha(z) = \alpha\frac{1-z}{1+z},$$

and

$$\mathcal{S}_\theta(s) = s + \theta.$$

The map $\mathcal{T}_\alpha$ sends the unit disc to the half-plane $\mathbb{C}_0$ and $\mathcal{S}_\theta$ sends $\mathbb{C}_0$ to $\mathbb{C}_\theta$. Now let f be an analytic function in $\mathbb{C}_\theta$ so that the composition $f \circ \mathcal{S}_\theta \circ \mathcal{T}_\alpha$ belongs to H^2. We denote the space of all such functions by $H^2_{\mathrm{i}}(\mathbb{C}_\theta, \alpha)$. Recall that the functions in H^2 has boundary values almost everywhere on the unit circle (Theorem 1.5). This property now transfers to the function $f \in H^2_{\mathrm{i}}(\mathbb{C}_\theta, \alpha)$ in the sense that the limit

$$\lim_{\sigma\to\theta^+} f(\sigma + it)$$

exists for almost every $t \in \mathbb{R}$. We define the norm on $H^2_{\mathrm{i}}(\mathbb{C}_\theta, \alpha)$ by

$$||f||^2_{H^2_{\mathrm{i}}(\mathbb{C}_\theta,\alpha)} := ||f \circ \mathcal{S}_\theta \circ \mathcal{T}_\alpha||^2_{H^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f \circ \mathcal{S}_\theta \circ \mathcal{T}_\alpha(e^{i\theta})|^2\, d\theta.$$

If we write $\mathcal{T}_\alpha(e^{i\theta}) = -i\alpha\tan(\theta/2)$ and use the substitution $t = \alpha\tan(\theta/2)$, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |f \circ \mathcal{S}_\theta \circ \mathcal{T}_\alpha(e^{i\theta})|^2\, d\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(\theta - it)|^2 \frac{\cos(\theta)+1}{\alpha}\, dt.$$

Now, with $\theta = 2\arctan(t/\alpha)$, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(\theta - it)|^2 \frac{\cos(\theta)+1}{\alpha}\, dt = \frac{1}{\pi}\int_{-\infty}^{\infty} |f(\theta + it)|^2 \frac{\alpha}{\alpha^2+t^2}\, dt.$$

We therefore have

$$||f||_{H^2_{\mathrm{i}}(\mathbb{C}_\theta,\alpha)} = \left(\frac{1}{\pi}\int_{-\infty}^{\infty} |f(\theta + it)|^2 \frac{\alpha}{\alpha^2+t^2}\, dt\right)^{\frac{1}{2}}. \tag{3.4}$$

We now prove a lemma form [4].

Lemma 3.4. *Suppose that a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges uniformly in $\overline{\mathbb{C}_\theta}$. Then*

$$||f||^2_{H^2_{\mathrm{i}}(\mathbb{C}_\theta,\beta)} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n\overline{a_m}\frac{(nm)^{\beta-\theta}}{[\max(n,m)]^{2\beta}}. \tag{3.5}$$

Proof. First we calculate the integral

$$\int_{-\infty}^{\infty} \frac{\cos\alpha t}{\beta^2+t^2}\, dt.$$

Consider the related contour integral

$$\int_C \frac{e^{i\alpha z}}{\beta^2+z^2} dz,$$

where $C = [-R, R] \cup \Gamma_R$ and Γ_R is the half-circle on the upper half plane of radius R. The integrand has one pole in C at $z = i\beta$. We get

$$\mathrm{Res}_{z=i\beta} \ \frac{e^{i\alpha z}}{\beta + z^2} = \lim_{z \to i\beta} (z - i\beta) \frac{e^{i\alpha z}}{\beta^2 + z^2} = \frac{1}{\beta e^{\alpha\beta}},$$

so

$$\int_C \frac{e^{i\alpha z}}{\beta^2 + z^2} dz = \frac{\pi}{\beta} e^{-\alpha\beta}.$$

If we now let R go to infinity and use the ML-inequality, we see that the integral over Γ_R goes to zero. This implies that

$$\int_C \frac{e^{i\alpha z}}{\beta^2 + z^2} dz = \int_{-\infty}^{\infty} \frac{\cos \alpha t}{\beta^2 + t^2} \, dt = \frac{\pi}{\beta} e^{-\alpha\beta}. \tag{3.6}$$

Consider now the integral

$$I(x) = \frac{\beta}{\pi} \int_{-\infty}^{\infty} x^{it} \frac{1}{\beta^2 + t^2} \, dt.$$

We write $x^{it} = e^{it \log x}$ so that the integral becomes

$$I(x) = \frac{\beta}{\pi} \int_{-\infty}^{\infty} \cos(|\log x| t) \frac{1}{\beta^2 + t^2} \, dt.$$

By (3.6) we have

$$I(x) = e^{-|\log x|\beta} = \frac{1}{[\max(x, 1/x)]^\beta}.$$

Finally, by uniform convergence we get

$$\begin{aligned} ||f||^2_{H^2_{\mathrm{i}}(\mathbb{C}_\theta, \beta)} &= \frac{1}{\pi} \int_{-\infty}^{\infty} |f(\theta + it)|^2 \frac{\beta}{\beta^2 + t^2} \, dt \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n \overline{a_m} (mn)^{-\theta} \left(\frac{m}{n}\right)^{it} \frac{\beta}{\beta^2 + t^2} \, dt \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n \overline{a_m} (mn)^{-\theta} I(m/n) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n \overline{a_m} \frac{(nm)^{\beta - \theta}}{[\max(n, m)]^{2\beta}}. \end{aligned}$$

□

Corollary 3.5. *If $f \in \mathscr{H}^2$ converges uniformly in $\overline{\mathbb{C}_\varepsilon}$ for all $\varepsilon > 0$, then*

$$||f||_{\mathscr{H}^2} = \lim_{\sigma \to 0} \lim_{\beta \to \infty} ||f||_{H^2_{\mathrm{i}}(\mathbb{C}_\varepsilon, \beta)}.$$

Proof. Indeed, as β approaches infinity, we have that

$$\frac{(nm)^\beta}{[\max(n, m)]^{2\beta}} = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases}$$

□

We have already seen that any function in $\mathscr{H}^2$ certainly converges uniformly in the half-plane $\mathbb{C}_{1/2+\delta}$, for $\delta \geq 0$. So the formula (3.5) holds for every $f \in \mathscr{H}^2$ when $\theta = \frac{1}{2}$. We denote the matrix on the right hand side of (3.5), in this case, by M_α. That is

$$M_\alpha := \left(\frac{(nm)^{\alpha-1/2}}{[\max(n,m)]^{2\alpha}} \right)_{n,m\geq 1}.$$

We define the norm of M_α by

$$||M_\alpha|| := \sup_{a,b\in\ell^2} \frac{|\langle M_\alpha a, b\rangle_{\ell^2}|}{||a||_{\ell^2}||b||_{\ell^2}}. \tag{3.7}$$

We will refer to the inner product on the right-hand side of (3.7) more conveniently as $B_\alpha(a,b)$, namely

$$B_\alpha(a,b) := \langle M_\alpha a, b\rangle_{\ell^2} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n b_m \frac{(nm)^{\alpha-1/2}}{[\max(n,m)]^{2\alpha}}.$$

The following result is due to Brevig [4].

Lemma 3.6. *For $0 < \alpha < \infty$ we have the sharp estimate*

$$||f||^2_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\alpha)} \leq ||M_\alpha||\, ||f||^2_{\mathscr{H}^2}.$$

Proof. We have

$$\begin{aligned} ||f||^2_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\alpha)} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n \overline{a_m} \frac{(nm)^{\alpha-1/2}}{[\max(n,m)]^{2\alpha}} \\ &\leq ||M_\alpha||\, ||f||^2_{\mathscr{H}^2}, \end{aligned}$$

by (3.5) and (3.7). The matrix M_α is real and symmetric, and is therefore self-adjoint. So the norm is attained for some $b = \overline{a}$, which is exactly what we have in the estimate above. The estimate is therefore sharp. □

The two next results, also from [4], will provide a lower and upper bound for $||M_\alpha||$, respectively.

Lemma 3.7. *For every $0 < \alpha < \infty$, we have $||M_\alpha|| \geq 2/\alpha$.*

Proof. By definition we have

$$||M_\alpha|| \geq \frac{1}{||a||_{\ell^2}||b||_{\ell^2}} \left| B_\alpha(a,b) \right|,$$

for any $a_n, b_m \in \ell^2$. So for any $0 < \varepsilon < \alpha$ we can define $a_n = n^{-1/2-\varepsilon}$ and $b_m = m^{-1/2-\varepsilon}$ so that

$$||M_\alpha|| \geq \frac{1}{\zeta(1+2\varepsilon)} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(nm)^{\alpha-1-\varepsilon}}{[\max(n,m)]^{2\alpha}}.$$

We rewrite the sums as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(nm)^{\alpha-1-\varepsilon}}{[\max(n,m)]^{2\alpha}}=\sum_{n=1}^{\infty}\left[n^{-\alpha-1-\varepsilon}\sum_{m=1}^{n}m^{\alpha-1-\varepsilon}+n^{\alpha-1-\varepsilon}\sum_{m=n+1}^{\infty}m^{-\alpha-1-\varepsilon}\right].$$

Consider first the sum $\sum_{m=1}^{n} m^{\alpha-1-\varepsilon}$. An application of Euler's summation formula gives

$$\begin{aligned}\sum_{m=1}^{n}m^{\alpha-1-\varepsilon}&=\int_1^n t^{\alpha-1-\varepsilon}\,dt+(\alpha-1-\varepsilon)\int_1^n t^{\alpha-2-\varepsilon}(t-[t])\,dt\\&=\frac{n^{\alpha-\varepsilon}}{\alpha-\varepsilon}-\frac{1}{\alpha-\varepsilon}+O\left(\alpha\int_1^n t^{\alpha-2-\varepsilon}\right)\\&=\frac{n^{\alpha-\varepsilon}}{\alpha-\varepsilon}+O(n^{\alpha-1-\varepsilon}).\end{aligned}$$

The sum $\sum_{m=n+1}^{\infty} m^{-\alpha-1-\varepsilon}$ can be written as

$$\sum_{m=n+1}^{\infty}m^{-\alpha-1-\varepsilon}=\zeta(\alpha+1+\varepsilon)-\sum_{m=1}^{n}m^{-\alpha-1-\varepsilon}.$$

We now apply the Euler summation formula to the sum $\sum_{m=1}^{n} m^{-\alpha-1-\varepsilon}$. This gives

$$\begin{aligned}\sum_{m=1}^{n}m^{-\alpha-1-\varepsilon}&=\int_1^n t^{-\alpha-1-\varepsilon}\,dt-(\alpha+1+\varepsilon)\int_1^n t^{-\alpha-2-\varepsilon}(t-[t])\,dt\\&=\frac{n^{-\alpha-\varepsilon}}{-\alpha-\varepsilon}-\frac{1}{-\alpha-\varepsilon}-(\alpha+1+\varepsilon)\int_1^{\infty} t^{-\alpha-2-\varepsilon}(t-[t])\,dt+O(n^{-\alpha-1-\varepsilon}).\end{aligned}$$

If we let n go to infinity in the equation above, then the left-hand side approaches $\zeta(\alpha+1+\varepsilon)$. The right-hand side becomes

$$-\frac{1}{-\alpha-\varepsilon}-(\alpha+1+\varepsilon)\int_1^{\infty}t^{-\alpha-2-\varepsilon}(t-[t])\,dt.$$

So we must have

$$\zeta(\alpha+1+\varepsilon)=-\frac{1}{-\alpha-\varepsilon}-(\alpha+1+\varepsilon)\int_1^{\infty}t^{-\alpha-2-\varepsilon}(t-[t])\,dt.$$

It follows that

$$\sum_{m=n+1}^{\infty}m^{-\alpha-1-\varepsilon}=\frac{n^{-\alpha-\varepsilon}}{\alpha+\varepsilon}+O(n^{-\alpha-1-\varepsilon}).$$

We now compute the following:

$$\begin{aligned}n^{-\alpha-1-\varepsilon}\sum_{m=1}^{n}m^{\alpha-1-\varepsilon}&=n^{-\alpha-1-\varepsilon}\left(\frac{n^{\alpha-\varepsilon}}{\alpha-\varepsilon}+O(n^{\alpha-1-\varepsilon})\right)=\frac{n^{-1-2\varepsilon}}{\alpha-\varepsilon}+O(n^{-2-2\varepsilon}),\\n^{\alpha-1-\varepsilon}\sum_{m=n+1}^{\infty}m^{-\alpha-1-\varepsilon}&=n^{\alpha-1-\varepsilon}\left(\frac{n^{-\alpha-\varepsilon}}{\alpha+\varepsilon}+O(n^{-\alpha-1-\varepsilon})\right)=\frac{n^{-1-2\varepsilon}}{\alpha+\varepsilon}+O(n^{-2-2\varepsilon})\end{aligned}$$

In total, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(nm)^{\alpha-1-\varepsilon}}{[\max(n,m)]^{2\alpha}} = \sum_{n=1}^{\infty}\left[\frac{n^{-1-2\varepsilon}}{\alpha-\varepsilon}+\frac{n^{-1-2\varepsilon}}{\alpha+\varepsilon}+O(n^{-2-2\varepsilon})\right]$$
$$= \zeta(1+2\varepsilon)\left(\frac{1}{\alpha-\varepsilon}+\frac{1}{\alpha+\varepsilon}\right)+O\big(\zeta(2+2\varepsilon)\big).$$

Dividing this by $\zeta(1+2\varepsilon)$ and letting $\varepsilon \to 0^+$ gives $||M_\alpha|| \geq 2/\alpha$. □

Lemma 3.8. *For every* $0 < \alpha < \infty$, *we have* $||M_\alpha|| \leq 1/\alpha + \max(1/\alpha, 1)$.

Proof. We use the Cauchy-Schwarz inequality with weights $\sqrt{m/n}$ and $\sqrt{n/m}$ to obtain

$$|B_\alpha(a,b)| \leq \left(\sum_{n=1}^{\infty}|a_n|^2\sqrt{n}\sum_{m=1}^{\infty}\frac{(nm)^{\alpha-1/2}}{[\max(n,m)]^{2\alpha}}\sqrt{\frac{1}{m}}\right)^{1/2}$$
$$\times\left(\sum_{m=1}^{\infty}|b_m|^2\sqrt{m}\sum_{n=1}^{\infty}\frac{(nm)^{\alpha-1/2}}{[\max(n,m)]^{2\alpha}}\sqrt{\frac{1}{n}}\right)^{1/2}.$$

Define

$$S_\alpha(n) = \sqrt{n}\sum_{m=1}^{\infty}\frac{(nm)^{\alpha-1/2}}{[\max(n,m)]^{2\alpha}}\sqrt{\frac{1}{m}},$$

so that

$$|B_\alpha(a,b)| \leq \left(\sum_{n=1}^{\infty}|a_n|^2 S_\alpha(n)\right)^{1/2}\left(\sum_{m=1}^{\infty}|b_m|^2 S_\alpha(m)\right)^{1/2}$$
$$\leq \left(\sup_n S_\alpha(n)\left(\sum_{n=1}^{\infty}|a_n|^2\right)\right)^{1/2}\left(\sup_n S_\alpha(m)\left(\sum_{m=1}^{\infty}|b_m|^2\right)\right)^{1/2}$$

Now for any $a, b \in \ell^2$ with $||a||_{\ell^2} = ||b||_{\ell^2} = 1$, we get

$$||M_\alpha|| \leq \sup_n S_\alpha(n),$$

since $S_\alpha(n) = S_\alpha(m)$. To continue, write $S_\alpha(n)$ as

$$S_\alpha(n) = n^{-\alpha}\sum_{m=1}^{n} m^{\alpha-1} + n^{\alpha}\sum_{m=n+1}^{\infty} m^{-\alpha-1}.$$

Consider first the sum $\sum_{m=n+1}^{\infty} m^{-\alpha-1}$. We have the estimate

$$\sum_{m=n+1}^{\infty} m^{-\alpha-1} \leq \int_n^{\infty} t^{-\alpha-1}\,dt = \frac{n^{-\alpha}}{\alpha}.$$

For the $n^{-\alpha}\sum_{m=1}^{n} m^{\alpha-1}$ we consider the cases $0 < \alpha \leq 1$ and $\alpha > 1$ separately. First, for $0 < \alpha \leq 1$, we have

$$\sum_{m=1}^{n} m^{\alpha-1} \leq \int_0^n t^{\alpha-1}\,dt = \frac{n^\alpha}{\alpha}.$$

For $\alpha > 1$ we can easily find

$$n^{-\alpha}\sum_{m=1}^{n} m^{\alpha-1} \leq n^{-\alpha}(nn^{\alpha-1}) = 1.$$

Hence,

$$S_\alpha(n) \leq \begin{cases} 2/\alpha, & 0 < \alpha \leq 1, \\ 1 + 1/\alpha, & \alpha > 1, \end{cases}$$

which completes the proof. □

In the previous chapter we introduced the notion of a vertical limit function. In [10], Hedenmalm, Lindquist and Seip prove the following result regarding the vertical limit functions corresponding to a Dirichlet series in $\mathscr{H}^2$.

Theorem 3.9. *Suppose $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ belongs to $\mathscr{H}^2$. Then the function*

$$f_\chi(s) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}$$

almost surely (with respect to the Haar measure on $\mathbb{T}^\infty$) extends to an analytic function on $\mathbb{C}_0$ belonging to the space $H^2_{\mathrm{i}}(\mathbb{C}_0)$. In addition, the non-tangential value

$$f^*(\chi) = \lim_{\sigma\to 0^+} f_\chi(\sigma)$$

exists for almost every $\chi \in \mathbb{T}^\infty$

A function $f \in \mathscr{H}^2$ will, in the worst case, only converge in the half-plane $\mathbb{C}_{1/2}$. However, the theorem above tells us that a vertical limit function f_χ of f most likely converges in the larger half-plane $\mathbb{C}_0$. In other words, the vertical limit functions are better behaved than their original functions. The following result from [9], proved by Wintner [17] and Kahane [11], gives us an example of a vertical limit function that will prove to have useful properties.

Lemma 3.10. *Let*

$$g_\chi(s) = \sum_p \chi(p) p^{-s-1/2}.$$

Then the line $\operatorname{Re} s = \frac{1}{2}$ is the abscissa of convergence for a dense set characters χ. Moreover, the line $\operatorname{Re} s = 0$ is the abscissa of convergence for almost all characters χ. In both cases, the abscissa of convergence is a natural boundary.

Clearly, g_χ does not actually belong to $\mathscr{H}^2$. However, if we let

$$f(s) = \sum_p \frac{p^{-s}}{\sqrt{p}\log p}$$

so that

$$f_\chi(s) = \sum_p \frac{p^{-s}}{\sqrt{p}\log p}\chi(p),$$

then we observe $f_\chi \in \mathscr{H}^2$ and $f'_\chi = -g_\chi$. Therefore, f_χ must have the same properties as g_χ. We already know from Theorem 3.9 that f_χ can be extended analytically to $\mathbb{C}_0$, almost surely. But Lemma 3.10 provides us with additional information about this particular f_χ, namely that there does not exist an analytic extension beyond the imaginary line. Also, for a dense set of characters, f_χ can not be extended analytically beyond the line $\operatorname{Re} s = \frac{1}{2}$.

Chapter 4

Composition operators on $\mathscr{H}^2$

In this chapter we use the results we have established concerning vertical limit functions and the Hardy space $\mathscr{H}^2$ to prove a result from [9].

In chapter 1 we found that analytic self-maps of the unit disc generates bounded composition operators on the Hardy space H^2. The goal of this chapter is to give a characterization of the analytic functions $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ that generates bounded composition operators on the Hardy space of Dirichlet series $\mathscr{H}^2$. Consider the following definition.

Definition 4.1 (The Gordon–Hedenmalm class)**.** *Let $\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s} = c_0 s + \psi(s)$, with $c_0 \in \mathbb{N} \cup \{0\}$. We say that φ belongs to the Gordon-Hedenmalm class, denoted by $\mathscr{G}$, if it satisfies the following properties.*

1. $\sigma_u(\psi) \leq 0$
2. *If $c_0 = 0$, then $\psi(\mathbb{C}_0) \subset \mathbb{C}_{1/2}$.*
3. *If $c_0 \geq 1$, then $\psi \equiv 0$ or $\psi(\mathbb{C}_0) \subset \mathbb{C}_0$.*

In [9], Gordon and Hedenmalm proved the following theorem.

Theorem 4.2. *A function $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ generates a bounded composition operator $\mathscr{C}_\varphi : \mathscr{H}^2 \to \mathscr{H}^2$ if and only if φ belongs to the Gordon-Hedenmalm class.*

The fact that the $\mathscr{G}$ contains every analytic function associated with a bounded composition operator on $\mathscr{H}^2$ is the main result of this chapter. For the proof of this result we will follow [9], but also the work of Queffélec and Queffélec from their unpublished second edition of [13]. The proof is rather complicated, so it is convenient to divide the proof into several parts. We will treat the proof for necessity and sufficiency of the condition $\varphi \in \mathscr{G}$ separately, and we will also distinguish between the case $c_0 = 0$ and $c_0 \geq 1$. The first step towards proving the result consists of determining when the composition $f \circ \varphi$ is a Dirichlet series, for $f \in \mathscr{H}^2$. We have the following result.

Theorem 4.3. *Suppose we have an analytic function $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$. Then φ generates a composition operator $\mathscr{C}_\varphi : \mathscr{H}^2 \to \mathcal{D}$ if and only if φ is of the form*

$$\varphi(s) = c_0 s + \psi(s), \tag{4.1}$$

where c_0 is a non-negative integer and $\psi \in \mathcal{D}$.

The proof of the theorem requires the two following lemmas:

Lemma 4.4. *Suppose we have a function $g(s) = \sum_{n=N}^{\infty} b_n n^{-s} \in \mathcal{D}$. Then*

$$\lim_{\operatorname{Re} s \to \infty} N^s g(s) = b_N.$$

Proof. First write $N^s g(s) = b_N + \sum_{n>N}^{\infty} b_n \left(\frac{N}{n}\right)^s$. This gives

$$\lim_{\operatorname{Re} s \to \infty} N^s g(s) = b_N + \lim_{\operatorname{Re} s \to \infty} \sum_{n>N}^{\infty} b_n \left(\frac{N}{n}\right)^s. \tag{4.2}$$

For $\varepsilon > 0$ and $\operatorname{Re} s \geq \sigma_a(g) + \varepsilon =: \sigma_\varepsilon$ we have

$$\left| b_n \left(\frac{N}{n}\right)^s \right| \leq \left| b_n \left(\frac{N}{n}\right)^{\sigma_\varepsilon} \right|,$$

whenever $n > N$. Clearly,

$$\sum_{n>N}^{\infty} \left| b_n \left(\frac{N}{n}\right)^{\sigma_\varepsilon} \right| < \infty.$$

Weierstrass' M-test now says that the series $\sum_{n>N}^{\infty} b_n \left(\frac{N}{n}\right)^s$ converges uniformly, which allows us to move the limit in (4.2) inside the series. Since $\lim_{\operatorname{Re} s \to \infty} b_n \left(\frac{N}{n}\right)^s = 0$ as long as $N < n$, the result follows. □

Lemma 4.5. *Suppose that c is a real number such that n^c is an integer for all $n \in \mathbb{N}$. Then, $c \in \mathbb{N} \cup \{0\}$.*

Proof. For a smooth function f we define the k-iterated difference $(\Delta^k f)(n)$, where $n \in \mathbb{N}$, as

$$(\Delta^k f)(n) = (\Delta^{k-1} g)(n), \text{ with } g(n) = f(n+1) - f(n).$$

For $k = 0$ we define $(\Delta^0 f)(n) = f(n)$. We can also write the k-iterated difference as

$$(\Delta^k f)(n) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(n+i).$$

Denote by $f^{(k)}$ the k-th derivate of f. The proof now relies on the following identity.

$$\begin{aligned}
&\int_{[0,1]^k} f^{(k)}(n + t_1 + \cdots + t_k)\, dt_1 \cdots dt_k = \\
&\int_{[0,1]^{k-1}} f^{(k-1)}(n + 1 + t_2 + \cdots + t_k) - f^{(k-1)}(n + t_2 + \cdots + t_k)\, dt_2 \cdots dt_k = \\
&\vdots \\
&= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(n+i).
\end{aligned}$$

Consider now the function $f(t) = t^c$, where c is a real number, but not an integer. Assume first that $c > 0$ and that $f(t)$ is an integer whenever t is a natural number, making the k-iterated difference of f an integer as well. We have

$$f^{(k)}(t) = c(c-1)\cdots(c-k+1)t^{c-k}.$$

If $k > c$, then $\lim_{n\to\infty}(\Delta^k f)(n) = 0$. On the other hand, since c is not an integer, it follows that $f^{(k)}$ is always positive or always negative. This means that $(\Delta^k f)(n) \neq 0$, which can be seen by inspecting its integral expression. But $(\Delta^k f)(n)$ is an integer by assumption, so then we must have $|(\Delta^k f)(n)| \geq 1$. This is a contradiction, and in turn, c must be an integer to ensure that $f(t)$ is an integer whenever t is a natural number.

Now, if $c < 0$, then $f(t)$ tends to zero as $t \to \infty$. But if $f(t)$ is an integer for all $t \in \mathbb{N}$, then $f(t)$ would have to be equal to zero for a sufficiently large choice of t. Of course, this can never happen. We can conclude that c is a non-negative integer, i.e. $c \in \mathbb{N} \cup \{0\}$. □

Now we are ready to prove Theorem 4.3.

Proof. Assume that $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ is an analytic function such that the composition $f \circ \varphi$, with $f \in \mathscr{H}^2$, is again a Dirichlet series. If $f_k(s) = k^{-s}$, then we can write

$$(f_k \circ \varphi)(s) = k^{-\varphi(s)} = \sum_{n=N(k)}^{\infty} b_n^{(k)} n^{-s}. \tag{4.3}$$

Here, $N(k)$ represents the smallest natural number n such that $b_{N(k)}^{(k)}$ is non-zero. With Lemma 4.4 in mind, it is clear that

$$(N(k))^s k^{-\varphi(s)} \to b_{N(k)}^{(k)}$$

when $\operatorname{Re} s \to \infty$, or equivalently,

$$\lim_{\operatorname{Re} s\to\infty} e^{s \log N(k) - \varphi(s)\log k} = b_{N(k)}^{(k)}. \tag{4.4}$$

Let $g(s) = s\log N(k) - \varphi(s)\log k$. Clearly, g is holomorphic in $\mathbb{C}_{1/2}$ and consequently maps this half-plane to a connected domain. Let U be an arbitrarily small open neighborhood of $\log b_{N(k)}^{(k)}$. Thanks to (4.4) and the connectivity of $g(\mathbb{C}_{1/2})$, all the values of $g(s)$, for s with sufficiently large real part, are contained in the set $U + 2i\pi l$, for some $l \in \mathbb{Z}$. This implies that

$$\lim_{\operatorname{Re} s\to\infty} g(s) = \lim_{\operatorname{Re} s\to\infty} s\log N(k) - \varphi(s)\log k = \log b_{N(k)}^{(k)} + 2i\pi l, \tag{4.5}$$

for some integer q. Upon dividing by $s \log k$ we obtain

$$\lim_{\operatorname{Re} s\to\infty} \frac{\varphi(s)}{s} = \frac{\log N(k)}{\log k}.$$

Lets define $c_0 := \frac{\log N(k)}{\log k}$ and observe that $k^{c_0} = N(k)$. The number $N(k)$ is an integer for every $k \in \mathbb{N}$, so we deduce from Lemma 4.5 that c_0 is a non-negative integer. Notice that c_0 only depends on the function φ, and not on the particular choice of k.

We now claim that $\psi(s) = \varphi(s) - c_0 s \in \mathcal{D}$. If we can show this, then the first part of the proof will be complete. As before, we multiply the Dirichlet series (4.3) by $(N(k))^s = k^{c_0 s}$ and obtain

$$k^{c_0 s} k^{-\varphi(s)} = k^{-\psi(s)} = \sum_{n=k^{c_0}}^{\infty} b_n^{(k)} \left(\frac{n}{k^{c_0}}\right)^{-s}. \tag{4.6}$$

For simplicity let $\beta_j = b_{k^{c_0}+j}^{(k)}$ and write (4.6) as

$$k^{-\psi(s)} = \beta_0 + \beta_1 \left(1 + \frac{1}{k^{c_0}}\right)^{-s} + \beta_2 \left(1 + \frac{2}{k^{c_0}}\right)^{-s} + \cdots = \beta_0 + h(s).$$

We note that $\beta_0 \neq 0$, because $\beta_0 = b_{k^{c_0}}^{(k)} = b_{N(k)}^{(k)}$ which is non-zero by definition. If we now apply the logarithm to the previous equation we get

$$-\psi(s) \log k = \log(\beta_0 + h(s)) + 2i\pi l, \tag{4.7}$$

similarly as in (4.5). According to the proof of Lemma 4.4 the function $h(s) = \sum_{n>k^{c_0}} b_n^{(k)} n^{-s}$ converges uniformly to 0 as $\operatorname{Re} s \to \infty$. This allows us to choose $\operatorname{Re} s$ large enough so that $|h(s)| < |\beta_0|$, and in turn

$$\log(\beta_0 + h(s)) = \log\left(1 + \frac{h(s)}{\beta_0}\right) + \log \beta_0 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \beta_0^{-n} h(s)^n + \log \beta_0.$$

Inserting this in to (4.7) yields

$$-\psi(s) \log k = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \beta_0^{-n} h(s)^n + \log \beta_0 + 2i\pi l.$$

We have assumed $\operatorname{Re} s$ to be large enough for $h(s)$ to converge absolutely. We can therefore expand the expression of $h(s)^n$ for every n and rearrange the terms to obtain an expression of the form

$$\psi(s) = \sum_{r=0}^{\infty} \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \gamma_{n_1,\ldots,n_r} \prod_{j=1}^{r} \left(1 + \frac{n_r}{k^{c_0}}\right)^{-s}.$$

This expression for ψ is valid for every k and converges in some half-plane. Suppose $k = k_1$ and let $\prod_{j=1}^{r} \left(1 + \frac{n_j}{k_1^{c_0}}\right)$ be any of the products in the formula above. Since ψ is independent of k, we can choose another $k = k_2$ so that

$$\prod_{j=1}^{r} \left(1 + \frac{n_j}{k_1^{c_0}}\right) = \prod_{j=1}^{r'} \left(1 + \frac{m_j}{k_2^{c_0}}\right),$$

for some positive integers r' and $m_j, 1 \leq j \leq r'$. We can write this equality as

$$\frac{p}{k_1^{c_0 q}} = \frac{p'}{k_2^{c_0 q'}}, \tag{4.8}$$

for some positive integers p, p', q and q'. In particular, (4.8) must hold for $k_1 = 2$ and $k_2 = 3$. So we must have that $p3^{c_0 q'} = p' 2^{c_0 q}$, or

$$p = \frac{p' 2^{c_0 q}}{3^{c_0 q'}}.$$

We deduce from this that $3^{c_0 q'}$ divides p', and similarly, $2^{c_0 q}$ divides p. Hence, the elements $\prod_{j=1}^{r} \left(1 + \frac{n_r}{k^{c_0}}\right)$ are all positive integers. We conclude that ψ is a convergent Dirichlet series, i.e. $\psi \in \mathcal{D}$.

It remains to prove the second part of the proof. That is, we want to show that if φ satisfies the condition (4.1), then it generates a composition operator $\mathscr{C}_\varphi : \mathscr{H}^2 \to \mathcal{D}$. So let φ be given by

$$\varphi(s) = c_0 s + \psi(s),$$

where c_0 is a non-negative integer and $\psi(s) = \sum_{n=1}^{\infty} c_n n^{-s} \in \mathcal{D}$. For any natural number k we can write

$$k^{-\varphi(s)} = k^{-c_0 s} k^{-\psi(s)} = k^{-c_0 s - c_1} \prod_{n \geq 2} k^{-c_n n^{-s}} = k^{-c_0 s - c_1} \prod_{n \geq 2} e^{-c_n n^{-s} \log k}.$$

Expanding the exponential term in its Taylor series gives

$$k^{-\varphi(s)} = k^{-c_0 s - c_1} \prod_{n \geq 2} \left(1 + \sum_{j=1}^{\infty} \frac{(-c_n \log k)^j}{j!} n^{-js}\right).$$

Let $g(s) = \sum_{k=1}^{\infty} b_k k^{-s} \in \mathscr{H}^2$. The composition $g \circ \varphi$ can now be expressed as

$$(g \circ \varphi)(s) = \sum_{k=1}^{\infty} b_k k^{-\varphi(s)} = \sum_{k=1}^{\infty} b_k k^{-c_0 s - c_1} \prod_{n \geq 2} \left(1 + \sum_{j=1}^{\infty} \frac{(-c_n \log k)^j}{j!} n^{-js}\right). \tag{4.9}$$

We want to show that the expression (4.9) constitute a convergent Dirichlet series in some half-plane. This can be done by rearrangement of the terms, but this requires the series to be absolutely convergent. In order to see that the series converges absolutely for some large $\operatorname{Re} s$ we write (4.9) as

$$\sum_{k=1}^{\infty} b_k k^{-c_0 s - c_1} \exp\left(\left(-\sum_{n=2}^{\infty} c_n n^{-s}\right) \log k\right).$$

This series converges absolutely if the series

$$\sum_{k=1}^{\infty} |b_k| k^{-c_0 \operatorname{Re} s - \operatorname{Re} c_1} \exp\left(\left(\sum_{n=2}^{\infty} |c_n| n^{-\operatorname{Re} s}\right) \log k\right). \tag{4.10}$$

converges. By assumption, $\psi(s)$ is a convergent Dirichlet series in some half-plane $\mathbb{C}_\theta$. So for some s with $\operatorname{Re} s \geq \theta$, the series $\psi(s)$ converges absolutely. Therefore, the series $\sum_{n=2}^{\infty} |c_n| n^{-\operatorname{Re} s}$ converges. Also, by Lemma 4.4, the series tends to zero when $\operatorname{Re} s$ goes to infinity. Then for $c_0 \neq 0$ the series (4.10) converges. For $c_0 = 0$ the same series certainly converges if $\operatorname{Re} c_1 > 1/2$. We will now see that this is the case. If $c_0 = 0$, then $\varphi(s) = \psi(s)$ and $\lim_{\operatorname{Re} s \to \infty} \varphi(s) = c_1$, again by Lemma 4.4. Since the image of φ is contained in $\mathbb{C}_{1/2}$, we must must have $\operatorname{Re} c_1 \geq 1/2$. If φ is a constant function, then we trivially have $\operatorname{Re} c_1 > 1/2$. Otherwise, there must exist a number $n \geq 2$ such that c_n is non-zero. Denote the first such number by N. Then $\varphi(s) = c_1 + c_N N^{-s} + O((N+1)^{-\operatorname{Re} s})$). So for $\operatorname{Re} s$ sufficiently large, the image of φ takes the form of a small punctured disc around the point c_1. This disc must be contained in $\mathbb{C}_{1/2}$, which implies that $\operatorname{Re} c_1 > 1/2$. □

We now know which restrictions we have to impose on an analytic function φ so that it generates a bounded composition operator $\mathscr{C}_\varphi : \mathscr{H}^2 \to \mathcal{D}$. Before we can give a proof of the main result of this chapter, we will need to establish some properties of such functions.

Theorem 4.6. *Suppose $\varphi : \mathbb{C}_\theta \to \mathbb{C}_\nu$ is an analytic function of the form* (4.1)*. We then have the following mapping properties.*

1. *If ψ is constant, i.e. $\psi \equiv c_1$, then $c_1 \in \overline{\mathbb{C}_{\nu - c_0\theta}}$.*
2. *If ψ is non-constant, then it sends $\mathbb{C}_\theta$ to the open half-plane $\mathbb{C}_{\nu - c_o\theta}$.*
3. *Assume again that ψ is non-constant. Then for every $\vartheta > \theta$, we have $\psi(\mathbb{C}_\vartheta) \subset \mathbb{C}_{\nu+\varepsilon-c_0\theta}$. Here, $\varepsilon > 0$ depends on the choice of ϑ. In addition,* $\operatorname{Re}\psi$ *is bounded from above on $\mathbb{C}_\vartheta$.*

Proof. (1) First note that $\operatorname{Re}\varphi(s) = c_0 \operatorname{Re} s + \operatorname{Re}\psi(s) > \nu$, or equivalently, $\operatorname{Re}\psi(s) > \nu - c_0 \operatorname{Re}(s)$. For a fixed $s \in \mathbb{C}_\theta$ we denote its real part by ϑ, so that $\operatorname{Re}\psi(s) > \nu - c_0\vartheta$. Since ψ is analytic in the half-plane $\mathbb{C}_\vartheta$, we know from Theorem 2.5 that its Dirichlet series converges uniformly in the same half-plane. So ψ must therefore be bounded in $\mathbb{C}_\vartheta$. Now consider the function $2^{-\psi}$ which, by the above, is bounded on $\mathbb{C}_\theta$. The maximum modulus principle tells us that $|2^{-\psi}| \leq 2^{c_0\vartheta-\nu}$, since the maximum must occur on the boundary of the domain. We choose an arbitrary s with real part greater than θ, so that we actually have $|2^{-\psi}| \leq 2^{c_0\theta-\nu}$. This means that Re $\psi(s) \geq \nu - c_0\theta$ on $\mathbb{C}_\theta$.

(2) Applying the open mapping theorem to the result above completes the proof.

(3) Let $F(s) = 2^{-\psi(s)}$. We want to show that $\sup|F(s)| < 2^{c_0\theta-\nu}$ for $s \in \mathbb{C}_\theta$. To that end, we will consider the function

$$M_F(x) := \sup\{|F(s)| : \operatorname{Re} s > x\},$$

for $x \geq \theta$. If $\psi(s) = \sum_{n=1}^{\infty} c_n n^{-s}$, then $F(s) \to 2^{-c_1}$ as $\operatorname{Re} s$ goes to infinity. The function ψ is nonconstant, so as $\operatorname{Re} s \to \infty$ we can write

$$\psi(s) = c_1 + c_N N^{-s} + O((N+1)^{-\operatorname{Re} s}),$$

where $N = \min\{n \in \mathbb{N} : n \geq 2, c_n \neq 0\}$. We observe that the image under ψ of some distant half-plane contains a punctured disc centered at c_1. This shows that there exists values of s such that $|F(s)| > 2^{-\operatorname{Re} c_1}$. Hence, $M_F(x)$ is nonconstant. We have seen that $M(\theta) \leq 2^{c_0\theta-\nu}$, and since $M_F(x)$ is nonconstant there must be a value of x so that $M_F(x)$ is strictly less than $2^{c_0\theta-\nu}$.

We can conclude by Hadamard's three lines theorem ([14], Theorem 12.8) that M_F is logarithmically convex. Fix x large enough so that $M_F(x) < 2^{c_0\theta-\nu}$. For $x > \vartheta > \theta$, we can write $\vartheta = (1-\lambda)\theta + \lambda x$, with $\lambda \in (0,1)$, and get

$$M_F(\vartheta) \leq M_F(\theta)^{1-\lambda} M_F(x)^{\lambda} < 2^{(c_0\theta-\nu)(1-\lambda)} 2^{(c_0\theta-\nu)\lambda} = 2^{c_0\theta-\nu},$$

which is what we wanted to show.

Now we will see that $\operatorname{Re}\psi$ is bounded above on $\mathbb{C}_\vartheta$. According to Theorem 4.3 we have $2^{-\psi} \in \mathcal{D}$. We have also just seen that $2^{-\psi}$ is bounded above on $\mathbb{C}_\vartheta$. So the Dirichlet series representation of $2^{-\psi}$ converges uniformly on $\mathbb{C}_\vartheta$, by Bohr's theorem. Assume now that $\operatorname{Re}\psi$ is not bounded above $\mathbb{C}_\vartheta$. Then there have to exist a sequence of points $\{s_n\}_{n\geq 1}$ such that

$$\lim_{n\to\infty} 2^{-\psi(s_n)} = 0.$$

On the other hand, we know that ψ tends to the constant term c_1 as $\operatorname{Re} s$ approaches infinity. Then it is obvious that the sequence $\{s_n\}_{n\geq 1}$ must remain bounded as $n \to \infty$. This implies, however, that some vertical translation of $2^{-\psi(s_n)}$ should have a zero somewhere on the real line. But then $2^{-\psi(s_n)}$ would have to be identically zero, which is not the case as it is nonconstant. We have reached a contradiction and the result follows. □

In the result above it is actually possible to replace the function ψ with a vertical limit function ψ_χ and everything would still hold true. Lemma 2.10 shows that their image must be the same. To see that $\operatorname{Re}\psi_\chi$ is bounded above one could repeat the argument used to show that $\operatorname{Re}\psi$ is bounded above.

Corollary 4.7. *Suppose $\varphi \in \mathscr{G}$ and that φ is not a vertical translation. That is, $\varphi(s) \neq s+i\tau$. Then φ sends $\mathbb{C}_{1/2}$ into a slightly smaller half-plane.*

Proof. The proof differs depending on the value of c_0. Consider first the case $c_0 = 0$. By assumption, φ maps $\mathbb{C}_0$ to $\mathbb{C}_{1/2}$. If φ is a constant c_1, that is, its Dirichlet series part ψ is constant, then $\operatorname{Re} c_1 > 1/2$ and the result follows by Theorem 4.6.

Now, assume that $c_0 = 1$. If $\psi = c_1$, a constant, then $\operatorname{Re} c_1 \geq 0$, since $\varphi \in \mathscr{G}$. But we have assumed that φ is not a vertical translation, and in particular, not the identity map. This implies that $\operatorname{Re} c_1 \neq 0$, which gives the result again by Theorem 4.6.

Finally, suppose that $c_0 \geq 2$. We know that $\psi(\mathbb{C}_0) \subset \mathbb{C}_0$, so $\varphi(\mathbb{C}_{1/2}) \subset \varphi(\mathbb{C}_1)$ and we are done. □

Corollary 4.8. *Let $\varphi : \mathbb{C}_\theta \to \mathbb{C}_\nu$ be an analytic function of the form* (4.1)*. Then we have the following properties:*

1. $(n^{-c_0 s})_\chi = \chi(n)^{c_0} n^{-c_0 s}$.
2. $(n^{-\varphi})_\chi = \chi(n)^{c_0} n^{-\varphi_\chi}$.

Proof. (1) Clearly, $(n^{-c_0 s})_\chi = ((n^{c_0})^{-s})_\chi = \chi(n^{c_0})(n^{c_0})^{-s} = \chi(n)^{c_0} n^{-c_0 s}$.

(2) First write $n^{-\varphi(s)} = n^{-c_0 s} n^{-\psi(s)}$. If $\chi = n^{-i\tau}$, then it easily follows that

$$(n^{-\psi})_\chi = n^{-\psi_\chi}.$$

By Theorem 2.8 this also holds for any $\chi \in \mathcal{M}$. Now, using property 1. and Lemma 2.9, we get

$$\begin{aligned}(n^{-\varphi(s)})_\chi &= (n^{-c_0 s} n^{-\psi(s)})_\chi = (n^{c_0 s})_\chi (n^{-\psi(s)})_\chi = \chi(n)^{c_0} n^{-c_0 s} n^{-\psi_\chi(s)} \\ &= (n^{-\varphi(s)})_\chi = \chi(n)^{c_0} n^{-\varphi_\chi(s)}.\end{aligned}$$

□

For a function φ of the form $\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s}$ and a character $\chi \in \mathcal{M}$, we define

$$\varphi_\chi(s) := c_0 s + \sum_{n=1}^{\infty} c_n \chi(n) n^{-s},$$

and we refer to this as a vertical limit function as well. If $f \in \mathscr{H}^2$, then the composition $f \circ \varphi$ is another Dirichlet series and we are interested in knowing what the vertical limit functions of this composition looks like. The theorem below provides us with a describtion of these vertical limit functions.

Theorem 4.9. *Let $\varphi : \mathbb{C}_\theta \to \mathbb{C}_{1/2}$ be an analytic function, where $\varphi(s) = c_0 s + \psi(s)$, with $c_0 \in \mathbb{N} \cup \{0\}$, $\psi \in \mathcal{D}$ and $\theta \geq 0$. If φ is non-constant, then for every $f \in \mathscr{H}^2$ and $\chi \in \mathcal{M}$ we have*

$$(f \circ \varphi)_\chi(s) = f_{\chi^{c_0}} \circ \varphi_\chi(s) \quad \text{for all} \quad s \in \mathbb{C}_\theta.$$

Proof. Consider the partial sum $f_N(s) = \sum_{n=1}^{N} a_n n^{-s}$. By Theorem 2.9 it follows that

$$(f_N \circ \varphi)_\chi(s) = \sum_{n=1}^{N} a_n \chi(n)^{c_0} n^{-\varphi_\chi(s)} = (f_N)_{\chi^{c_0}} \circ \varphi_\chi(s).$$

We need to show that the identity above remains valid as $N \to \infty$. The function f converges absolutely in $\mathbb{C}_{1/2}$, since $f \in \mathscr{H}^2$. We can then write, for $s \in \mathbb{C}_\theta$,

$$(f \circ \varphi)(s) = \sum_{n=1}^{\infty} a_n n^{-\varphi(s)}.$$

To continue, we will need the following mapping property of φ. Let $\operatorname{Re} s > \vartheta > \theta$. Then we have $\operatorname{Re} \varphi(s) \geq 1/2 + \varepsilon$, for some $\varepsilon > 0$. If $c_0 = 0$, then this follows immediately from Theorem 4.6, since φ is nonconstant. If $c_0 \geq 1$, then we use Theorem 4.6 again and find

$$\operatorname{Re} \varphi(s) = c_0 \operatorname{Re} s + \operatorname{Re} \psi(s) \geq c_0 \operatorname{Re} s + 1/2 - c_0 \theta \geq c_0(\vartheta - \theta) + 1/2 > 1/2.$$

This mapping property of φ ensures that $f_N \circ \varphi$ converges uniformly to $f \circ \varphi$ on $\mathbb{C}_\vartheta$, since we have

$$|(f \circ \varphi)(s)|^2 \leq \left(\sum_{n=1}^{\infty} |a_n n^{-\varphi(s)}| \right)^2 \leq \sum_{n=1}^{\infty} |a_n|^2 \sum_{n=1}^{\infty} n^{-2 \operatorname{Re} s},$$

by the Cauchy–Schwarz inequality. □

4.1 Necessity

We are now going to prove the necessity part of Theorem 4.2, and for that we need a lemma.

Lemma 4.10. *Suppose we have an analytic function φ on $\mathbb{C}_{1/2}$. If $n^{-\varphi}$ can be extended analytically to a function f_n on an open simply connected superset Ω of $\mathbb{C}_{1/2}$, for $n = 2, 3$, then φ can be extended analytically to Ω as well.*

Proof. If $f_n = n^{-\varphi(s)}$ on $\mathbb{C}_{1/2}$, then

$$\frac{f_n'(s)}{f_n(s)} = -\varphi'(s)\log n,$$

or

$$\varphi'(s) = -\frac{f_n'(s)}{f_n(s)\log n}.$$

This expression for the derivative of φ is valid for $s \in \mathbb{C}_{1/2}$. Also, it constitutes a meromorphic function in some superset Ω and we have therefore obtained an extension of φ'. We wish to show that this extension has no poles in Ω. We know that f_n is analytic everywhere in its domain. Hence, if f_n'/f_n has a pole at some point $s_0 \in \Omega$, then s_0 is a zero of f_n. So fix $s_0 \in \Omega$ and assume $f_n(s_0) = 0$. Denote the order of s_0 by h_n. We now have

$$\operatorname{Res}_{s=s_0}\{\varphi'(s)\} = \operatorname{Res}_{s=s_0}\left\{-\frac{f_n'(s)}{f_n(s)\log n}\right\} = \frac{-h_n}{\log n}.$$

This equality is supposed to hold for $n = 2, 3$. If we divide the equation for $n = 2$ and $n = 3$ on each other we we obtain

$$\frac{\log 2}{\log 3} = \frac{h_2}{h_3} \in \mathbb{Q}.$$

Since $\log 2/\log 3$ is not rational, we have arrived at a contradiction. As a consequence, we have shown that the extension of φ' is analytic in Ω. Finally, Ω is a simply connected domain, so φ can be extended to Ω as well. □

Proof. (Necessity) Our starting point is a bounded composition operator $\mathscr{C}_\varphi$ on $\mathscr{H}^2$, and we want to show that $\varphi(s) = c_0 s + \psi(s) \in \mathscr{G}$. We have already seen that φ must be of the form $\varphi(s) = c_0 s + \psi(s)$, with $c_0 \in \mathbb{N} \cup \{0\}$ and $\psi \in \mathcal{D}$. We consider first the case $c_0 \geq 1$. Let $f_n = n^{-\varphi_\chi} = (n^{-\varphi})_\chi$. By assumption, we have $n^{-\varphi} \in \mathscr{H}^2$. Theorem 3.9 says that f_n almost surely has an extension to $\mathbb{C}_0$. If we now apply Lemma 4.10 to f_n, with $\Omega = \mathbb{C}_0$, we find that φ_χ can be almost surely extended to $\mathbb{C}_0$. Now consider a function $f \in \mathscr{H}^2$. Then the composition $f \circ \varphi$ is in $\mathscr{H}^2$ as well. So, again by Theorem 3.9, we know that $(f \circ \varphi)_\chi$ can be almost surely extended to $\mathbb{C}_0$. Since the map $z \to z^{c_0}$ preserves the measure on $\mathbb{T}$, we get that the map $\chi \to \chi^{c_0}$ is measure-preserving on $\mathbb{T}$ as well. This implies that $f_{\chi^{c_0}}$ can be almost surely extended to $\mathbb{C}_0$. Denote by $M \subset \mathcal{M}$ the subset of characters for which these functions can be extended to $\mathbb{C}_0$. We know from Theorem 4.9 that we can write

$$(f \circ \varphi)_\chi(s) = f_{\chi^{c_0}} \circ \varphi_\chi(s), \tag{4.11}$$

for every $s \in \mathbb{C}_{1/2}$. What we now need to show is that there exists a character $\chi \in M$ such that $\varphi_\chi(\mathbb{C}_0) \subset \mathbb{C}_0$. We can describe this equivalently as the set

$$\Gamma = \{s \in \mathbb{C}_0 : \operatorname{Re} \varphi_\chi(s) > 0\}$$

begin equal to $\mathbb{C}_0$ for some χ. The image $\varphi_\chi(\mathbb{C}_0)$ must be a connected open set, by continuity and the open mapping theorem. The function φ sends $\mathbb{C}_{1/2}$ to $\mathbb{C}_{1/2}$, and so does φ_χ. From this it is clear that Γ contains $\mathbb{C}_{1/2}$. Moreover, Γ is an open set. Now we denote by Γ_c the connected part of Γ that contains $\mathbb{C}_{1/2}$. If we assume that $\Gamma \neq \mathbb{C}_0$, then we are certainly able to find a point s_0 on the boundary of Γ_c which is also in $\mathbb{C}_0$. Note that Γ_c have to be open in $\mathbb{C}_0$. If not, there would have to be points immediately to the left of that boundary which satisfies $\operatorname{Re} \varphi_\chi(s) \leq 0$, while the points on the boundary satisfies $\operatorname{Re} \varphi_\chi(s) > 0$. This would contradict the connectedness of $\varphi_\chi(\mathbb{C}_0)$. Since φ_χ maps points in Γ_c to $\mathbb{C}_0$ and points in $\mathbb{C}_0/\Gamma$ to $\mathbb{C}/\mathbb{C}_0$, we get by connectedness that φ_χ maps the boundary $\partial\Gamma_c$ to the imaginary axis. Now we want to show that we can find a point $s_0 \in \partial\Gamma_c \cap \mathbb{C}_0$ such that $\varphi'_\chi(s_0) \neq 0$. We know that φ_χ is nonconstant, so φ'_χ can not be identically zero. And since φ'_χ is analytic, it must have isolated zeros. Denote by Z the zeros of φ'_χ in $\mathbb{C}_0$. Let $B \in \mathbb{C}_0$ be a ball containing s_0. The real part of an analytic function is harmonic, so we have

$$\int_B h(x + iy)\, dxdy = 0,$$

for $h = \operatorname{Re} \varphi_\chi$. Clearly, there exists points $u, v \in B \backslash Z$ satisfying $h(u) < 0$ and $h(v) > 0$. One can now find a path from u to v in B which does not intersect with Z, and somewhere along this path, say at w, we must have $h(w) = 0$. This means that $\varphi'_\chi(w) \neq 0$. We have shown that there are points $s_0 \in \partial\Gamma_c \cap \mathbb{C}_0$ with $\varphi'_\chi(w) \neq 0$. This implies that φ_χ is conformal close to the point s_0. Remember that the formula (4.11) can be extended analytically to Γ_c, in particular. Since φ_χ is conformal near s_0 we see that the formula

$$f_{\chi^{c_0}} = (f \circ \varphi)_\chi \circ \varphi_\chi^{-1} \tag{4.12}$$

is an analytic extension of $f_{\chi^{c_0}}$ to some small part of the imaginary axis close to the point $\varphi_\chi(s_0) \in \partial\mathbb{C}_0$. However, Lemma 3.10 shows that there exists an $f \in \mathscr{H}^2$ such that $f_{\chi^{c_0}}$ almost surely has a natural boundary at $\partial\mathbb{C}_0$. This implies that the formula (4.12) does not hold in general and there must be some $\chi \in M$ for which $\varphi_\chi(\mathbb{C}_0) \subset \mathbb{C}_0$. It then follows by Theorem 4.6 that $\psi_\chi(\mathbb{C}_0) \subset \mathbb{C}_0$. Recall that the action of twisting does not change the mapping properties of a function (Lemma 2.10). Hence, $\psi(\mathbb{C}_0) \subset \mathbb{C}_0$.

We consider now the case $c_0 = 0$. We first note that $f_{\chi^{c_0}} = f$, so that

$$(f \circ \varphi)_\chi(s) = f \circ \varphi_\chi(s).$$

This time we want to show that $\varphi_\chi(\mathbb{C}_0) \subset \mathbb{C}_{1/2}$. The idea is the same as before. Let Γ be the set

$$\Gamma = \{s \in \mathbb{C}_0 : \operatorname{Re} \varphi_\chi(s) > 1/2\},$$

and Γ_c the connected part containing $\mathbb{C}_{1/2}$. Again, if $\Gamma \neq \mathbb{C}_0$, then there is a point s_0 on the boundary of Γ_c which is also in $\mathbb{C}_0$. It is safe to assume that this point satisfies $\varphi'_\chi(s_0) \neq 0$, making φ_χ conformal near s_0. The point $\varphi_\chi(s_0)$ lies on the boundary $\partial\mathbb{C}_{1/2}$, so that the formula

$$f = (f\circ\varphi)_\chi \circ \varphi_\chi^{-1},$$

which is valid in Γ_c, is also an analytic continuation of f to some small part of $\partial\mathbb{C}_{1/2}$. We now also need to find an example of a function $f \in \mathscr{H}^2$ that can not be extended past $\mathbb{C}_{1/2}$. This is again given by Lemma 3.10. Indeed, since there exists a dense set of characters χ such that

$$f_\chi(s) = \sum_p \frac{p^{-s}}{\sqrt{p}\log p}\chi(p)$$

has $\operatorname{Re} s = \frac{1}{2}$ as a natural boundary, we can choose one of these to be our example. This completes the proof of the necessity part. □

4.2 Sufficiency

In this section we prove the sufficiency part of Theorem 4.2. We start out with a lemma from [4] which provides an upper bound for the norm of a composition operator when $c_0 = 0$.

Lemma 4.11. *Assume that* $\varphi \in \mathscr{G}$, *with* $c_0 = 0$, *and* $f \in \mathscr{H}^2$. *If* $\alpha = \operatorname{Re} c_1 - 1/2$, *where* $c_1 = \varphi(+\infty)$, *then*

$$||\mathscr{C}_\varphi f||_{\mathscr{H}^2} \leq ||f||_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\alpha)}.$$

Proof. We can assume that c_1 is real-valued. This is justified by the fact that we can always consider a vertical translation of φ for which the constant term is real-valued and the $\mathscr{H}^2$-norm is not affected by a vertical translation. Suppose now that $f(s) = \sum_{n=1}^N a_n n^{-s}$ is a Dirichlet polynomial. It follows from Corollary 3.5 that

$$||\mathscr{C}_\varphi f||_{\mathscr{H}^2} = \lim_{\beta\to\infty} ||f\circ\varphi||_{H^2_{\mathrm{i}}(\mathbb{C}_0,\beta)} = \lim_{\beta\to\infty} ||f\circ\varphi\circ\mathcal{T}_\beta||_{H^2},$$

for $\varphi \in \mathscr{G}$ with $c_0 = 0$. Define $F := f\circ\mathcal{S}_{1/2}\circ\mathcal{T}_\alpha$ and $\phi := \mathcal{T}_\alpha^{-1}\circ\mathcal{S}_{1/2}^{-1}\circ\varphi\circ\mathcal{T}_\beta$. Observe that $F \in H^2$ and that ϕ is an analytic self-map of the unit disc. We also have $f\circ\varphi\circ\mathcal{T}_\beta = F\circ\phi$. By Theorem 1.12 it follows that

$$||f\circ\varphi\circ\mathcal{T}_\beta||_{H^2} = ||F\circ\phi||_{H^2} \leq \sqrt{\frac{1+|\phi(0)|}{1-|\phi(0)|}}||F||_{H^2}.$$

Further, we have

$$\lim_{\beta\to\infty}\phi(0) = \lim_{\beta\to\infty}\mathcal{T}_\alpha^{-1}(\varphi(\beta)-1/2) = \mathcal{T}_\alpha^{-1}(c_1 - 1/2).$$

For $\alpha = c_1 - 1/2$ we see that $\lim_{\beta\to\infty}\phi(0) = 0$. Hence,

$$||\mathscr{C}_\varphi f||_{\mathscr{H}^2} \leq ||F||_{H^2} = ||f\circ\mathcal{S}_{1/2}\circ\mathcal{T}_\alpha||_{H^2} = ||f||_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\alpha)}. \tag{4.13}$$

For an arbitrary $f \in \mathscr{H}^2$ there exists a sequence $\{f_j\}_{j\geq 1}$ of Dirichlet polynomials converging to f in the $\mathscr{H}^2$-norm. By (4.13) and Lemma 3.6 we have

$$||\mathscr{C}_\varphi f_j||_{\mathscr{H}^2} \leq ||f_j||_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\alpha)} \leq \sqrt{||M_\alpha||}\,||f_j||_{\mathscr{H}^2}.$$

By assumption, the right hand side of the inequality above remains bounded when j approaches infinity. This shows that the result must hold for every $f \in \mathscr{H}^2$. □

It is worth mentioning here that the upper bound provided in the previous lemma is actually attained for some φ. In particular, we have the following lemma:

Lemma 4.12. *For every* $\operatorname{Re} c_1 > 1/2$ *there exists* $\varphi \in \mathscr{G}$*, with* $c_0 = 0$*, such that* $\varphi(+\infty) = \operatorname{Re} c_1$ *and*

$$||\mathscr{C}_\varphi f||_{\mathscr{H}^2} = ||f||_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\alpha)},$$

for all $f \in \mathscr{H}^2$.

Proof. We define

$$\varphi_\alpha(s) := \mathcal{S}_{1/2} \circ \mathcal{T}_\alpha(2^{-s}) = \frac{1}{2} + \alpha\frac{1-2^{-s}}{1+2^{-s}}.$$

Observe that φ_α is a Dirichlet series sending $\mathbb{C}_0$ to $\mathbb{C}_{1/2}$. Also, $\varphi_\alpha(+\infty) = c_1 = 1/2 + \alpha$. Consider the subspace $\mathscr{X}$ of $\mathscr{H}^2$, defined by

$$\mathscr{X} := \left\{ f(s) = \sum_{k=0}^{\infty} a_{2^k} 2^{-ks} \right\}.$$

We can map $\mathscr{X}$ onto H^2 by sending 2^{-s} to z, making the two spaces isometrically isomorphic. Finally, since the composition $f \circ \varphi_\alpha$ belongs to $\mathscr{X}$ for all $f \in \mathscr{H}^2$, we have

$$||\mathscr{C}_\varphi f||_{\mathscr{H}^2} = ||F||_{H^2} = ||f \circ \mathcal{S}_{1/2} \circ \mathcal{T}_\alpha||_{H^2} = ||f||_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\alpha)}.$$ □

Proof. (Sufficiency) We want to show that $\varphi \in \mathscr{G}$ is a sufficient condition for $\mathscr{C}_\varphi$ to be a bounded operator on $\mathscr{H}^2$. Assume first that $c_0 = 0$. Then the result follows immediately by Lemma 4.11. For $c_0 \geq 1$ we can follow a similar approach as the proof of Lemma 4.11. The qualitative difference in this case is that $\varphi(\mathbb{C}_0) \subset \mathbb{C}_0$, which means that we can leave out the horizontal shift $\mathcal{S}_{1/2}$. We define instead $F := f \circ \mathcal{T}_\alpha$ and $\phi := \mathcal{T}_\alpha^{-1} \circ \varphi \circ \mathcal{T}_\beta$, for some Dirichlet polynomial f. As before, we get

$$||f \circ \varphi \circ \mathcal{T}_\beta||_{H^2} = ||F \circ \phi||_{H^2} \leq \sqrt{\frac{1+|\phi(0)|}{1-|\phi(0)|}}||F||_{H^2}.$$

Again, we check what happens to $\phi(0)$ as we let $\beta \to \infty$:

$$\lim_{\beta\to\infty} \phi(0) = \lim_{\beta\to\infty} \mathcal{T}_\alpha^{-1}(\varphi(\beta)) = \lim_{\beta\to\infty} \frac{\varphi(\beta)-\alpha}{\varphi(\beta)+\alpha} = \lim_{\beta\to\infty} \frac{1-\alpha/\varphi(\beta)}{1+\alpha/\varphi(\beta)}.$$

This time we choose $\alpha = c_0\beta$, such that $\varphi(\beta) = \alpha + O(1)$ as $\beta, \alpha \to \infty$. Then

$$\lim_{\beta\to\infty} \phi(0) = 0.$$

The composition $f \circ \varphi$ is a convergent Dirichlet series in some half-plane by Theorem 4.3. Moreover, since f is a polynomial, the composition $f \circ \varphi$ is bounded on $\mathbb{C}_0$. So the Dirichlet series associated to this composition converges uniformly on closed half-planes in $\mathbb{C}_0$. Now we can use Corollary 3.5 to get

$$\lim_{\beta \to \infty} ||F||_{H^2} = \lim_{\beta \to \infty} ||f||_{H^2_{\mathrm{i}}(\mathbb{C}_0, \alpha)} = ||f||_{\mathscr{H}^2},$$

so that

$$||\mathscr{C}_\varphi f||_{\mathscr{H}^2} \leq ||f||_{\mathscr{H}^2}.$$

This completes the sufficiency part of the proof. □

Chapter 5

Norms of composition operators on $\mathscr{H}^2$

In this chapter we study the norms of composition operators on the Hardy space $\mathscr{H}^2$. We primarily consider results from a paper by Brevig and Perfekt [5].

We begin this chapter by establishing a lower and upper bound for the operator norm of a composition operator $\mathscr{C}_\varphi : \mathscr{H}^2 \to \mathscr{H}^2$, similarly as we did for the composition operators on H^2 in Theorem 1.15. For the lower bound we will, as before, use a reproducing kernel argument. We need the following lemma.

Lemma 5.1. *For a composition operator $\mathscr{C}_\varphi : \mathscr{H}^2 \to \mathscr{H}^2$ and a reproducing kernel K_a we have*

$$\mathscr{C}_\varphi^* K_a = K_{\varphi(a)}.$$

Proof. The proof is identical to that of Lemma 1.13. □

Theorem 5.2. *Suppose $\varphi \in \mathscr{G}$ and let $\alpha = \operatorname{Re}\varphi(+\infty) - 1/2$. Then*

$$\sqrt{\zeta(2\operatorname{Re}\varphi(+\infty))} \leq ||\mathscr{C}_\varphi|| \leq 1/\alpha + \max(1/\alpha, 1).$$

Proof. By lemma 5.1 it follows that

$$||K_{\varphi(a)}||_{\mathscr{H}^2} = ||\mathscr{C}_\varphi^* K_a||_{\mathscr{H}^2} \leq ||\mathscr{C}_\varphi^*||\, ||K_a||_{\mathscr{H}^2} = ||\mathscr{C}_\varphi||\, ||K_a||_{\mathscr{H}^2}.$$

The norm of the reproducing kernel is given by

$$||K_a||_{\mathscr{H}^2}^2 = \sum_{n=1}^{\infty} \left|\overline{n^{-a}}\right|^2 = \zeta(2\operatorname{Re} a).$$

Letting a go to infinity yields $||K_a||_{\mathscr{H}^2} = 1$. Now we have

$$||\mathscr{C}_\varphi|| \geq ||K_{\varphi(a)}||_{\mathscr{H}^2} = \sqrt{\zeta(2\operatorname{Re}\varphi(+\infty))}.$$ □

For the upper bound we combine Lemma 3.6 and Lemma 4.11, which gives

$$||\mathscr{C}_\varphi f||_{\mathscr{H}^2} \leq ||f||_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\alpha)} \leq ||M_\alpha||\, ||f||_{\mathscr{H}^2},$$

valid for every $f \in \mathscr{H}^2$. Lemma 3.8 tells us that $||M_\alpha|| \leq 1/\alpha + \max(1/\alpha, 1)$, so the proof is complete.

The next result we are going to consider provides an interesting analogy to the result by Shapiro that we proved in chapter 1 (Theorem 1.20). Recall that the result was concerned with composition operators generated by inner functions. We therefore need to clarify what it means for a Dirichlet series f in $\mathscr{H}^2$ to be inner. Recall that the non-tangential boundary value

$$f^*(\chi) = \lim_{\sigma\to 0^+} f_\chi(\sigma),$$

exists for almost every $\chi \in \mathbb{T}^\infty$. If $|f^*(\chi)| = 1$ for almost every $\chi \in \mathbb{T}^\infty$, then we say that f is inner. We can now state the result ([5], Theorem 21).

Theorem 5.3. *Suppose that $\varphi \in \mathscr{G}$, with $c_0 = 0$, maps $\mathbb{C}_0^*$ into $\mathbb{C}_{1/2}$ and that $\varphi(+\infty) = \omega$. Let Θ be a Riemann map from $\mathbb{D}$ to $\mathbb{C}_{1/2}$ with $\Theta(0) = \omega$ and set $\psi(s) = \Theta(2^{-s})$. Then*

$$||\mathscr{C}_\varphi f||_{\mathscr{H}^2} \le ||\mathscr{C}_\psi f||_{\mathscr{H}^2}, \quad f \in \mathscr{H}^2, \tag{5.1}$$

and the following are equivalent.

1. *$\Theta^{-1} \circ \varphi$ is inner.*
2. *$||\mathscr{C}_\varphi f||_{\mathscr{H}^2} = ||\mathscr{C}_\psi f||_{\mathscr{H}^2}$.*
3. *$||\mathscr{C}_\varphi|| = ||\mathscr{C}_\psi||$.*

The Riemann map Θ is a special case of the function

$$\varphi_\alpha(s) = \mathcal{S}_{1/2} \circ \mathcal{T}_\alpha(2^{-s}) = \frac{1}{2} + \alpha\frac{1-2^{-s}}{1+2^{-s}}$$

that we saw earlier. We only need to ensure that $\varphi_\alpha(0) = \omega$, which is done by letting $\alpha = \omega - 1/2$. Shapiro's result (Theorem 1.20) tells us that the operator norm $||\mathscr{C}_\varphi||$ is maximal if and only if the analytic self-map φ of the unit disc maps the boundary to the boundary. Since Möbius transformations sends boundaries to boundaries, we see that condition 1 in the result above is equivalent to $\varphi^*(\chi) = 1/2$ for almost every $\chi \in \mathbb{T}^\infty$. The operator norm of $\mathscr{C}_{\varphi_\alpha}$ is maximal by Lemma 4.12. This means that the maps $\varphi \in \mathscr{G}$, with $c_0 = 0$, that fixes the boundary in the described sense, generate composition operators that attains its upper bound.

From here on out, we will study the norms of composition operators generated by affine symbols, mainly based on a paper by Brevig and Perfekt [5]. An affine symbol is a Dirichlet series of the form

$$\varphi_{\mathbf{c}}(s) = c + \sum_{j=1}^{d} c_j p_j^{-s}, \tag{5.2}$$

where $\mathbf{c} = (c_1, ..., c_d)$ and p_j is the j-th prime. Since we are going to consider composition operators generated by such symbols it will be necessary to determine when they belong to the Gordon-Hedenmalm class. We see immediately that we must have $\operatorname{Re} c > 1/2$. In addition, we must have $\sum_{j\ge 1} |c_j| \le \operatorname{Re} c - 1/2$. Let $\mathbb{C}_0^*$ denote the extended half-plane $\mathbb{C}_0 \cup \{\infty\}$. The image of this half-plane under an affine symbol is given by the following result from [5].

Lemma 5.4. *Let φ be an affine symbol of the form* (5.2) *belonging to $\mathscr{G}$. Then $\varphi(\mathbb{C}_0^*) = \mathbb{D}(c, r)$, where*

$$r = \sum_{j=1}^{d} |c_j| \leq \operatorname{Re} c - 1/2.$$

For the remainder of this chapter we will consider a special family of sequences that we denote by $\mathscr{L}(d, r)$. This is the set of sequences $\mathbf{c} = (c_1, ..., c_d)$, with $c_j \geq 0$, and the sum of the sequence being equal to r. To each sequence $\mathbf{c} \in \mathscr{L}(d, r)$ we associate a function

$$L_{\mathbf{c}}(s) := \sum_{j=1}^{d} c_j p_j^{-s}.$$

Further, we write $\mathbf{c}^{\downarrow}$ to denote the decreasing rearrangement of a sequence $\mathbf{c}$. If $\mathbf{b}$ and $\mathbf{c}$ are two sequences in $\mathscr{L}(d, r)$ satisfying

$$\sum_{j=1}^{k} b_j^{\downarrow} \leq \sum_{j=1}^{k} c_j^{\downarrow}$$

for $k = 1, 2, ..., d-1$, we say that $\mathbf{b}$ majorizes $\mathbf{c}$ and write $\mathbf{b} \prec \mathbf{c}$. In [5] the following result was proved.

Lemma 5.5. *Let $1 \leq q \leq$. If $\boldsymbol{b}$, $\boldsymbol{c} \in \mathscr{L}(d, r)$ and $\boldsymbol{b} \prec \boldsymbol{c}$, then $||L_{\boldsymbol{b}}||_{\mathscr{H}^q} \leq ||L_{\boldsymbol{c}}||_{\mathscr{H}^q}$. The inequality is strict if $\boldsymbol{b}$ is not a permutation of $\boldsymbol{c}$ and $1 < q < \infty$.*

Let f be a Dirichlet series in $\mathscr{H}^2$ and

$$\varphi_{\mathbf{c}}(s) = c + \sum_{j=1}^{d} c_j p_j^{-s} = c + L_{\mathbf{c}}(s),$$

with $\mathbf{c} \in \mathscr{L}(d, r)$. Taylor expanding f about the point $s = c$ yields

$$f(s) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(s - c)^k.$$

The composition of f with $\varphi_{\mathbf{c}}$ can now be expressed as

$$\mathscr{C}_{\varphi_{\mathbf{c}}} f(s) = f(c + L_{\mathbf{c}}(s)) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(L_{\mathbf{c}}(s))^k.$$

We can expand the k-th power of $L_{\mathbf{c}}(s)$ using the multinomial formula and obtain an expression of the form

$$(L_{\mathbf{c}}(s))^k = \sum_{|\alpha|=k} \binom{k}{\alpha} \prod_{j=1}^{d} c_j^{\alpha_j} p_j^{-\alpha_j s}.$$

Clearly, $\langle m^{-s}, n^{-s}\rangle_{\mathscr{H}^2} = 0$ whenever $m \neq n$. Hence, by the fundamental theorem of arithmetic, we find that $\langle (L_{\mathbf{c}}(s))^k, (L_{\mathbf{c}}(s))^l\rangle_{\mathscr{H}^2} = 0$ whenever $k \neq l$. This gives

$$||\mathscr{C}_{\varphi_{\mathbf{c}}} f||^2_{\mathscr{H}^2} = \sum_{k=0}^{\infty} \frac{|f^{(k)}(c)|^2}{(k!)^2} ||L^k_{\mathbf{c}}||^2_{\mathscr{H}^2} \tag{5.3}$$

We are led to the following subordination principle. If $\mathbf{b} \prec \mathbf{c}$, then

$$||\mathscr{C}_{\varphi_{\mathbf{b}}} f||^2_{\mathscr{H}^2} \leq ||\mathscr{C}_{\varphi_{\mathbf{c}}} f||^2_{\mathscr{H}^2}.$$

This is stated more precisely in the following theorem from [5].

Theorem 5.6. *Fix and* r *such that* $\operatorname{Re} c - 1/2 \geq r > 0$ *and let* d *be a positive integer. For* $\boldsymbol{c} \in \mathscr{L}(d, r)$*, let*

$$\varphi_{\boldsymbol{c}} := c + \sum_{n=1}^{d} c_j p_j^{-s}.$$

Suppose that $\boldsymbol{b}, \boldsymbol{c} \in \mathscr{L}(d, r)$ *and* $\boldsymbol{b} \prec \boldsymbol{c}$*. Then*

$$||\mathscr{C}_{\varphi_{\mathbf{b}}} f||^2_{\mathscr{H}^2} \leq ||\mathscr{C}_{\varphi_{\mathbf{c}}} f||^2_{\mathscr{H}^2}, \quad f \in \mathscr{H}^2.$$

In addition, we have the following equivalent statements.

1. $\mathbf{b}$ *is a permutation of* $\mathbf{c}$.
2. $||\mathscr{C}_{\varphi_{\mathbf{b}}} f||_{\mathscr{H}^2} = ||\mathscr{C}_{\varphi_{\mathbf{c}}} f||_{\mathscr{H}^2}$ *for every* $f \in \mathscr{H}^2$.
3. $||\mathscr{C}_{\varphi_{\mathbf{b}}}|| = ||\mathscr{C}_{\varphi_{\mathbf{c}}}||$.

The first fart of the result result follows from (5.3) and Lemma 5.5.

In ([5], Section 7.2), the following example of two affine symbols was given:

$$\varphi_a = c + \frac{r}{2}(2^{-s} + 3^{-s}),$$
$$\varphi_b = c + \frac{r}{6}(4 \cdot 2^{-s} + 3^{-s} + 5^{-s}).$$

They found that neither of them majorizes the other, yet one of the associated composition operators is subordinate to the other. This made them pose the following question.

If $\varphi_{\mathbf{b}}$ and $\varphi_{\mathbf{c}}$ are two affine symbols with the same mapping properties, is it true that either $\mathscr{C}_{\varphi_{\mathbf{b}}}$ is subordinate to $\mathscr{C}_{\varphi_{\mathbf{b}}}$, or $\mathscr{C}_{\varphi_{\mathbf{b}}}$ is subordinate to $\mathscr{C}_{\varphi_{\mathbf{b}}}$?

We are now going to answer this question, and see that a general subordination principle of this kind does not hold. We want to construct a counterexample and begin with a basic observation. If $\varphi_{\mathbf{b}}$ and $\varphi_{\mathbf{c}}$ are two affine symbols, then we can not have $||L^k_{\mathbf{b}}||^2_{\mathscr{H}^2} \leq ||L^k_{\mathbf{c}}||^2_{\mathscr{H}^2}$ or $||L^k_{\mathbf{c}}||^2_{\mathscr{H}^2} \leq ||L^k_{\mathbf{b}}||^2_{\mathscr{H}^2}$ for all $k \in \mathbb{N}$, since this would lead to one operator being subordinate to the other by the arguments above. We start by finding an example such that the above inequalities changes direction at least once for different values of k. For simplicity, we will work with the space $\mathscr{L}(3,1)$. Consider the example:

$$\varphi_1(s) = c + \frac{2}{3}\cdot 2^{-s} + \frac{1}{3}\cdot 3^{-s},$$
$$\varphi_2(s) = c + \frac{3}{4}\cdot 2^{-s} + \frac{1}{8}\cdot 3^{-s} + \frac{1}{8}\cdot 5^{-s}.$$

The value of c must satisfy $\sum_{j\geq 1}|c_j| \leq \operatorname{Re} c - 1/2$, so that φ_1 and φ_2 belongs to $\mathscr{G}$. We can fix $c = 2$. We compute the relevant norms:

$$\begin{aligned} ||\varphi_1 - c||^2_{\mathscr{H}^2} &= \frac{5}{9} = 0.55555... \\ ||\varphi_2 - c||^2_{\mathscr{H}^2} &= \frac{19}{32} = 0.59375... \\ ||(\varphi_1 - c)^2||^2_{\mathscr{H}^2} &= \frac{11}{27} = 0.40740... \\ ||(\varphi_2 - c)^2||^2_{\mathscr{H}^2} &= \frac{795}{2048} = 0.38818... \end{aligned}$$

Observe that

$$||\varphi_1 - c||^2_{\mathscr{H}^2} < ||\varphi_2 - c||^2_{\mathscr{H}^2}$$

and

$$||(\varphi_1 - c)^2||^2_{\mathscr{H}^2} > ||(\varphi_2 - c)^2||^2_{\mathscr{H}^2}.$$

The inequalities changes direction already for $k = 2$. It remains to find two Dirichlet series f and g so that

$$||\mathscr{C}_{\varphi_1} f|| < ||\mathscr{C}_{\varphi_2} f|| \tag{5.4}$$

and

$$||\mathscr{C}_{\varphi_1} g|| > ||\mathscr{C}_{\varphi_2} g||. \tag{5.5}$$

The easiest choice here would be $f(s) = m^{c-s}$ and $g(s) = n^{c-s}$ for some $m, n \geq 1$. In order to determine what m and n should be, we have to examine the formula (5.3). We see that $|f^{(k)}(c)| = (\log m)^k$ and $g^{(k)}(c) = (\log n)^k$. In order to make f satisfy (5.4), its derivative have to grow slowly with respect to k. A natural choice is then $m = 2$. In this case, the value of $|f^{(k)}(c)|$ decreases as k increases. We compute the first 10 terms in the formula (5.3) for the composition operators $||\mathscr{C}_{\varphi_1} f||$ and $||\mathscr{C}_{\varphi_2} f||$, with the help of a computer. It is worth mentioning that the computation could be checked by hand, if we approximate the logarithmic term by rational numbers. We omit the first term since this is the same for both composition operators. This gives

$$\sum_{k=1}^{10} \frac{|f^{(k)}(c)|^2}{(k!)^2} ||(\varphi_1 - c)^k||^2_{\mathscr{H}^2} = 0.02005...,$$
$$\sum_{k=1}^{10} \frac{|f^{(k)}(c)|^2}{(k!)^2} ||(\varphi_2 - c)^k||^2_{\mathscr{H}^2} = 0.02098....$$

The second sum is larger then the first, as expected. But we need to know that this inequality holds when we include all the terms in (5.3). We immediately see that $\varphi_1 \in \mathscr{H}^\infty$ and $||(\varphi_1 - c)^k||_{\mathscr{H}^\infty} \leq 1$. By Theorem 3.2 it is clear that $||(\varphi_1 - c)^k||^2_{\mathscr{H}^2} \leq ||(\varphi_1 - c)^k||_{\mathscr{H}^\infty} \leq 1$, so we get the following estimate.

$$\begin{aligned}\sum_{k=J}^{\infty} \frac{|f^{(k)}(c)|^2}{(k!)^2} ||(\varphi_1 - c)^k||^2_{\mathscr{H}^2} &\leq \sum_{k=J}^{\infty} \frac{(\log 2)^{2k}}{(k!)^2} \\ &\leq \frac{1}{(J!)^2} \sum_{k=J}^{\infty} (\log 2)^{2k}.\end{aligned}$$

We observe that the right-hand side is geometric series. We therefore have

$$\begin{aligned}\frac{1}{(J!)^2} \sum_{k=J}^{\infty} (\log 2)^{2k} &= \frac{1}{(J!)^2} \frac{(\log 2)^{2J}}{1 - (\log 2)^2} \\ &\leq \frac{1}{(J!)^2} \frac{(\log 2)^{2}}{1 - (\log 2)^2} \\ &\leq \frac{1}{(J!)^2}.\end{aligned}$$

The last inequality follows from the fact that the map $x \mapsto \frac{x}{1-x}$ is increasing on the open unit interval and $(\log 2)^2 < 1/2$. Now, for $J = 11$, we find

$$\sum_{k=11}^{\infty} \frac{|f^{(k)}(c)|^2}{(k!)^2} ||(\varphi_1 - c)^k||^2_{\mathscr{H}^2} \leq \frac{1}{(11!)^2} < 6.3 \cdot 10^{-16}.$$

Finally, we have

$$\begin{aligned}||\mathscr{C}_{\varphi_1} f||^2_{\mathscr{H}^2} &= \sum_{k=0}^{10} \frac{|f^{(k)}(c)|^2}{(k!)^2} ||(\varphi_1 - c)^k||^2_{\mathscr{H}^2} + \sum_{k=11}^{\infty} \frac{|f^{(k)}(c)|^2}{(k!)^2} ||(\varphi_1 - c)^k||^2_{\mathscr{H}^2} \\ &\leq 0.02006... \\ &\leq \sum_{k=0}^{10} \frac{|f^{(k)}(c)|^2}{(k!)^2} ||(\varphi_2 - c)^k||^2_{\mathscr{H}^2} \\ &\leq ||\mathscr{C}_{\varphi_2} f||^2_{\mathscr{H}^2},\end{aligned}$$

which proves that $f(s) = 2^{c-s}$ satisfies (5.4). We are left with the problem of choosing n so that $g(s) = n^{c-s}$ satisfies (5.5). We will see that $n = 4$ is sufficient and we repeat the argument we used for f. First,

$$\begin{aligned}\sum_{k=1}^{10} \frac{|g^{(k)}(c)|^2}{(k!)^2} ||(\varphi_1 - c)^k||^2_{\mathscr{H}^2} &= 0.00962... , \\ \sum_{k=1}^{10} \frac{|g^{(k)}(c)|^2}{(k!)^2} ||(\varphi_2 - c)^k||^2_{\mathscr{H}^2} &= 0.00917... .\end{aligned}$$

This time, the first sum is larger than the second. Again, we estimate the remaining terms:

$$\begin{aligned}\sum_{k=J}^{\infty} \frac{|g^{(k)}(c)|^2}{(k!)^2} ||(\varphi_2 - c)^k||^2_{\mathscr{H}^2} &\leq \sum_{k=J}^{\infty} \frac{(\log 4)^{2k}}{(k!)^2} \\ &= \sum_{k=J}^{\infty} \frac{4^k (\log 2)^{2k}}{(k!)^2}.\end{aligned}$$

Since we are going to consider $J > 10$ it is safe to assume that $4^k < k!$. Hence,

$$\begin{aligned}\sum_{k=J}^{\infty} \frac{4^k (\log 2)^{2k}}{(k!)^2} &\leq \sum_{k=J}^{\infty} \frac{(\log 2)^{2k}}{k!} \\ &\leq \frac{1}{J!} \sum_{k=J}^{\infty} (\log 2)^{2k} \\ &\leq \frac{1}{J!}.\end{aligned}$$

In the latter inequality we have just repeated the argument above. So we get

$$\sum_{k=11}^{\infty} \frac{|g^{(k)}(c)|^2}{(k!)^2} ||(\varphi_2 - c)^k||^2_{\mathscr{H}^2} \leq \frac{1}{11!} < 2.51 \cdot 10^{-8}.$$

The function g now satisfies (5.5) since

$$\begin{aligned}||\mathscr{C}_{\varphi_2} g||^2_{\mathscr{H}^2} &= \sum_{k=0}^{10} \frac{|g^{(k)}(c)|^2}{(k!)^2} ||(\varphi_2 - c)^k||^2_{\mathscr{H}^2} + \sum_{k=11}^{\infty} \frac{|g^{(k)}(c)|^2}{(k!)^2} ||(\varphi_2 - c)^k||^2_{\mathscr{H}^2} \\ &\leq 0.00918... \\ &\leq \sum_{k=0}^{10} \frac{|g^{(k)}(c)|^2}{(k!)^2} ||(\varphi_1 - c)^k||^2_{\mathscr{H}^2} \\ &\leq ||\mathscr{C}_{\varphi_1} g||^2_{\mathscr{H}^2}.\end{aligned}$$

This answers the question and we can conclude that such a subordination principle does not hold.

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