Eiolf Kaspersen

Obstructions to the Surjectivity of the Thom Homomorphism

Master's thesis in Mathematical Sciences
Supervisor: Gereon Quick
June 2020

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Abstract

We present the basic properties of stable homotopy theory and generalised cohomology and construct the Thom homomorphism from complex cobordism to singular cohomology. We then study this homomorphism in detail and show that it is not in general surjective by constructing examples of singular cohomology classes which cannot be lifted to MU. Finally, we show that such cohomology classes can appear in Eilenberg-MacLane spaces, and we determine when $MU^n(K(G,n)) \to H^n(K(G,n);\mathbb{Z}_2)$ is surjective if $n \geq 3$ and G is a finitely generated abelian group.

Sammendrag

Vi presenterer de grunnlegende egenskapene til stabil homotopiteori og generalisert kohomologi og konstruerer Thom-homomorfien fra kompleks kobordisme til singulær kohomologi. Deretter studerer vi denne homomorfien mer detaljert og viser at den ikke generelt er surjektiv ved å konstruere eksempler på singulære kohomologiklasser som ikke kan løftes til MU. Til slutt viser vi at slike kohomologiklasser kan oppstå i Eilenberg-MacLane-rom, og vi fastslår når $MU^n(K(G,n)) \to H^n(K(G,n);\mathbb{Z}_2)$ er surjektiv hvis $n \geq 3$ og G er en endeliggenerert abelsk gruppe.

Contents

1 Introduction

In this master thesis, we will mainly be working with the Thom homomorphism, which is an important tool for understanding the relationship between different cohomology theories. Among the many different algebraic invariants used to study topological spaces, cohomology theories themselves come in many forms. Notable examples include singular cohomology, complex cobordism and K-theory.

We will be focusing on *multiplicative* cohomology theories, which have the elegant property that they assign a *ring* to every topological space, rather than just *groups*. If we have a topological space X and use different cohomology theories, we can of course expect this to result in different rings. It turns out that for many cohomology theories, there exist ring homomorphisms between these cohomology rings, which can be realised as maps between the cohomology theories themselves. If we can understand these maps, then we will have a much better understanding of how different cohomology theories relate to one another. For example, if such a map is injective, that would indicate that no information is lost by moving from the first cohomology theory to the second.

In particular, there exists a map from complex cobordism to any other complex oriented cohomology theory. This is known as the Thom homomorphism. As an even more specific example, we always have a ring homomorphism from the complex cobordism of a space to its singular cohomology. In general, cobordism rings are much larger than singular cohomology rings, since cobordism is a stronger cohomology theory. One might therefore expect that this map is surjective. However, this is not always the case, although constructing counterexamples can be quite difficult.

This thesis has two objectives. Firstly, we will present the methods that are necessary for the construction of the Thom homomorphism. This will include a presentation of cohomology in a general setting, as well as a closer look at complex cobordism. Some methods for making computations of generalised cohomology groups will also be needed. Once all of this has been established, we will see how it can all be used to construct the Thom homomorphism.

Secondly, we will present different methods for constructing spaces for which the Thom homomorphism is not surjective. There are several ways to detect that a space has cohomology classes that are not in the image of the Thom homomorphism, and we will show how these methods work and use them actively when constructing counterexamples. The spaces examined by Conner and Smith [3] will play an important role here.

The part of the thesis that can be considered original is a proof of the following: If G is a finitely generated abelian group and $n \geq 3$, then $MU^n(K(G,n)) \to H^n(K(G,n);\mathbb{Z}_2)$ is surjective if and only if G is of the form

$$G \cong \mathbb{Z}_{p_1^{r_1}} \oplus \cdots \oplus \mathbb{Z}_{p_s^{r_s}}, \tag{1}$$

where all the primes p_i are odd. We will see that this result is largely covered by a more extensive result due to Tamanoi [10]. However, it is worth pointing out the proof given in this thesis will show parts of Tamanoi's results using significantly less advanced methods.

We will now give a brief outline of the structure of this thesis. Section 2 will provide the basic definitions and properties of spectra, culminating in the definition of the stable homotopy category. We expand on this in Section 3 by using the Brown representability theorem to show how the study of generalised cohomology is connected to the study of spectra. This section will also deal with the construction of Eilenberg-MacLane spaces, which will play an important role in our main result.

In Section 4 we construct the Atiyah-Hirzebruch spectral sequence and use this to compute the generalised cohomology of $\mathbb{C}P^\infty$. The Milnor short exact sequence will also be needed for this computation. We then present the basic properties of complex cobordism in Section 5 by examining the spectrum MU. This section ends with the construction of the Thom homomorphism.

Finally, in Section 6, we use the Steenrod squares, Brown-Peterson cohomology and the Atiyah-Hirzebruch spectral sequence for connective K-theory to find ways to recognise that a cohomology class cannot be lifted to complex cobordism. We apply these methods to spaces constructed by Conner and Smith in [3]. Finally, we examine the properties of Eilenberg-MacLane spaces by proving our main result and comparing it to Tamanoi's work in [10].

I would like to thank my supervisor, Professor Gereon Quick, for being incredibly helpful and positive at every stage of the process of writing this thesis. Also deserving of thanks are my friends, family and fellow students, too many to name, for always being supportive. A special shout-out goes to Therese Strand, for helping out with the proofreading.

2 Stable Homotopy Theory

We start by exploring the fundamentals of stable homotopy theory. The goal of this section will be to define the stable homotopy category, where most of our work will take place. We will need several definitions in order to achieve this, and unless stated otherwise, all definitions in this section will be taken from [14]. The first we will need is the smash product of topological spaces. We will only be working with topological spaces with basepoints, and these will occasionally be referred to as "spaces".

Definition 2.1. Let X and Y be pointed topological spaces. We define the *smash product* of X and Y, denoted $X \wedge Y$, as

$$X \wedge Y = \frac{X \times Y}{X \vee Y}. \tag{2}$$

It is worth taking the time to understand why this is a meaningful definition. When we think of the wedge product of two spaces as a subspace of their Cartesian product, the wedge product is just the pairs of points where one of the points is a basepoint. In other words, writing x_0 and y_0 for the basepoints of X and Y, respectively, we can equivalently define the smash product as:

$$X \wedge Y = X \times Y \Big/ \begin{array}{ll} (x_0, y) \sim (x_0, y') & \forall y, y' \in Y \\ (x, y_0) \sim (x', y_0) & \forall x, x' \in X. \end{array} \tag{3}$$

This point of view shows us that the smash product can be thought of as a Cartesian product, modified so that there is still only one basepoint. We can now use the notion of the smash product to define the reduced suspension of a space.

Definition 2.2. The *reduced suspension* of a topological space X is defined as

$$\Sigma X = S^1 \wedge X. \tag{4}$$

It is again useful to look at an equivalent definition to get a better understanding of what this means geometrically. If we write

$$\Sigma X = X \times [0,1] \Big/ \begin{array}{ll} (x, 0) \sim (x', 0) & \forall x, x' \in X \\ (x, 0) \sim (x', 0) & \forall x, x' \in X \\ (x_0, t) \sim (x_0, t') & \forall t, t' \in [0,1], \end{array} \tag{5}$$

we can easily see that this defines the same reduced suspension. Intuitively, we can therefore think of the reduced suspension as multiplying the space by the unit interval, pinching together each end and, again, making sure the basepoint is still only one point. An advantage of this point of view is that it provides an intuitive understanding of what the suspension of spheres looks like. We can then see that $\Sigma S^n \cong S^{n+1}$.

Any map f between two spaces will induce a canonical map between the reduced suspensions of the spaces in an obvious way. We write this new map as

Σf so that

$$\begin{aligned} \Sigma f\colon \Sigma X &\longrightarrow \Sigma Y \\ (s,x) &\longmapsto (s, f(x)). \end{aligned} \tag{6}$$

We can then see that Σ is a functor from the category of pointed topological spaces to itself. A natural question to ask is whether Σ is part of an adjoint pair. This motivates our definition of the loop space functor.

Definition 2.3. Let X be a pointed topological space. The *loop space* of X, written as ΩX, is defined as the space of maps from S^1 to X. We topologise the space using the compact-open topology.

Given a map f, we get a map between loop spaces by composition with f. This means that it makes sense to think of Ω as a functor as well. We can now observe an important property of these functors. In the proof, and onward, we will let $[X, Y]$ denote the set of homotopy classes of basepoint-preserving maps from X to Y.

Lemma 2.4. *The functors (Σ, Ω) form an adjoint pair.*

Proof. To show that Σ is a left adjoint to Ω, we need to find an isomorphism between $[\Sigma X, Y]$ and $[X, \Omega Y]$. At first, we will ignore the equivalence relation on ΣX and define the isomorphism as if the suspension was just the Cartesian product $S^1 \times X$. We define maps ϕ and ψ as follows:

$$\begin{aligned} \phi\colon [\Sigma X, Y] &\longrightarrow [X, \Omega Y] \\ f &\longmapsto \phi(f)\colon \quad X \longrightarrow \Omega Y \\ &\qquad\qquad\qquad x \longmapsto \phi(f)_x\colon S^1 \longrightarrow Y \\ &\qquad\qquad\qquad\qquad\qquad\qquad s \longmapsto f(s,x) \end{aligned} \tag{7}$$

$$\begin{aligned} \psi\colon [X, \Omega Y] &\longrightarrow [\Sigma X, Y] \\ g &\longrightarrow \psi(g)\colon \quad \Sigma X \longrightarrow Y \\ &\qquad\qquad\quad (s,x) \longmapsto g(x)(s) \end{aligned} \tag{8}$$

We verify that ϕ and ψ are inverses by straightforward calculations:

$$\begin{aligned} \psi(\phi(f))(s,x) &= (\phi(f))(x)(s) = \phi(f)_x(s) = f(s,x) \\ \phi(\psi(g))(x)(s) &= \phi(\psi(g))_x(s) = \psi(g)(s,x) = g(x)(s) \end{aligned} \tag{9}$$

The reason that we take the suspension of X, rather than the Cartesian product, becomes clear when we verify that all maps are well defined and basepoint-preserving. We let x_0, y_0 and s_0 be the basepoints of X, Y and S^1, respectively. The canonical choice of basepoint in ΩY is the map which maps all of S^1 to y_0. Now, let $f \in [\Sigma X, Y]$. We want to show that the maps

$\phi(f)$ and $\phi(f)_x$ are basepoint-preserving. Since f is a basepoint-preserving map, we have $f(s_0, x) = f(s, x_0) = y_0$. Evaluating $\phi(f)_{x_0}$ on S^1, we get

$$\phi(f)_{x_0}(s) = f(s_0, x) = y_0 \quad \forall s \in S^1,$$

which means that $\phi(f)$ maps x_0 to the basepoint of ΩY. Furthermore, for any $x \in X$, we have

$$\phi(f)_x(s_0) = f(s_0, x) = y_0,$$

so $\phi(f)_x$ is a basepoint-preserving map in ΩY.

Similar calculations show us that ψ is well-defined, even though we have an equivalence relation on $S^1 \times X$. Finally, we can easily see that the construction of ψ and ϕ agrees with the homotopy equivalences on the sets of maps. $\square$

With the basic properties of Σ established, we can use this to define spectra, which is an essential structure in stable homotopy theory.

Definition 2.5. A *spectrum* is a sequence of pointed topological spaces $\{E_n\}$ with continuous maps, known as *structure maps*, $\epsilon_n \colon \Sigma E_n \longrightarrow E_{n+1}$ for all n.

$$\begin{tikzcd} \cdots \arrow[dr] & \Sigma E_{n-1} \arrow[dr, "\epsilon_{n-1}"] & \Sigma E_n \arrow[dr, "\epsilon_n"] & \Sigma E_{n+1} \arrow[dr] & \cdots \\ \cdots & E_{n-1} & E_n & E_{n+1} & \cdots \end{tikzcd} \tag{10}$$

The main idea of spectra is to take a topological space into higher and higher dimensions. As before, the spheres provide an important example. If we let each space E_n be the n-sphere, we can let our structure maps be the identity $\Sigma S^n = S^{n+1}$. In fact, this construction works for any topological space, not just spheres.

Definition 2.6. Let X be a pointed topological space. The *suspension spectrum* $\Sigma^\infty X$ of X is the spectrum whose n'th space is given by

$$E_n = \begin{cases} \Sigma^n X, & n \geq 0 \\ *, & n < 0. \end{cases} \tag{11}$$

In some cases, it will be useful to look at structure maps from a different point of view. Rather than studying maps $\epsilon_n \colon \Sigma E_n \longrightarrow E_{n+1}$, we can by Lemma 2.4 study maps $\epsilon'_n \colon E_n \longrightarrow \Omega E_{n+1}$ instead. This provides an alternative way to visualize spectra:

$$\begin{tikzcd} \cdots \arrow[dr] & E_{n-1} \arrow[dr, "\epsilon'_{n-1}"] & E_n \arrow[dr, "\epsilon'_n"] & E_{n+1} \arrow[dr] & \cdots \\ \cdots & \Omega E_{n-1} & \Omega E_n & \Omega E_{n+1} & \cdots \end{tikzcd} \tag{12}$$

There are some basic ways of creating new spectra out of old, which will be useful later on. Firstly, we may define the suspension of a spectrum. This is done in the intuitively obvious way, by taking the suspension of each space.

Definition 2.7. Let E be a spectrum. The *k'th suspension of* E is the spectrum $\Sigma^k E$ whose n'th space is $\Sigma^k E_n$.

This allows us to essentially move everything in a spectrum into higher dimensions. It is also useful to do the opposite, known as the desuspension. We do this by shifting every space to the right.

Definition 2.8. Let E be a spectrum. The *k'th desuspension of* E is the spectrum $\Sigma^{-k}E$ with spaces

$$(\Sigma^{-k}E)_n = E_{n-k}. \tag{13}$$

In the cases where we are dealing with suspension spectra, we will write $\Sigma^{\infty+k}X$ for the suspension of the suspension spectrum and $\Sigma^{\infty-k}$ for the desuspension.

As stated earlier, our goal is to define the stable homotopy category, and we are now close to having defined our objects, the spectra. However, we do not need the full generality of topological spaces, and will instead limit ourselves to spectra that consist of CW-complexes. The definition is similar, but we will be more restrictive when it comes to which structure maps we allow.

Definition 2.9. A *CW-spectrum* is a sequence of CW-complexes E_n with continuous maps $\epsilon_n \colon \Sigma E_n \longrightarrow E_{n+1}$ for all n, such that each ϵ_n is an injective map whose image is a subcomplex of E_{n+1}.

These are the objects in the stable homotopy category. We now wish to understand the morphisms in the category. Unfortunately, they have to be defined in several steps. This makes the terminology somewhat confusing, although several attempts have been made to avoid this. We will employ the convention used by Adams in [2], where he defines *functions*, then *maps* and finally *morphisms*, adding an equivalence relation in every step. We start with the functions.

Definition 2.10. Let E and F be spectra. A *function of degree* k from E to F is a sequence of continuous maps $f_n \colon E_n \longrightarrow F_{n-k}$ such that the following diagram commutes for all n:

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\epsilon_n"] \arrow[d, "\Sigma f_n"] & E_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma F_{n-k} \arrow[r, "\epsilon_{n-k}"] & F_{n-k+1} \end{tikzcd} \tag{14}$$

In other words, a function maps every space in E to a space in F, and these maps have to commute with the structure maps of both spectra. In the case where the function is of degree 0, we get the following, simplified diagram:

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\epsilon_n"] \arrow[d, "\Sigma f_n"] & E_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma F_n \arrow[r, "\epsilon_n"] & F_{n+1}. \end{tikzcd} \tag{15}$$

While we could have defined the morphisms in the stable homotopy category to be these functions, they do not give us a sufficient level of generality. To see why, we can reflect on where the word "stable" in "stable homotopy theory" comes from. The main idea is to see which properties of spaces "stabilize" as we move into higher and higher dimensions.

The most famous example is the stable homotopy groups of spheres. The homotopy groups $\pi_{n+k}(S^n)$ can take different forms for different values of n, but it has been shown that from a certain point and onward, the groups are all isomorphic. In other words, $\pi_{n+k}(S^n) \cong \pi_{m+k}(S^m)$ for all $m, n \geq N$ for some $N \in \mathbb{N}$. We can also observe this phenomenon using spectra, because all the spaces S^n can be found in the suspension spectrum of S^0, often written $\mathbb{S}$. In that case, we can see that we do not need the entire spectrum to find the stable homotopy groups of the spheres. In fact it is sufficient to understand what happens from a certain point and on to infinity. This motivates our next definition.

Definition 2.11. Let E be a CW-spectrum. We say that E' is a *subspectrum* of E if E' is a CW-spectrum where every space E'_n is a subcomplex of E_n, and every structure map ϵ'_n is the restriction of the structure map ϵ_n to the subcomplex E'_n.

We say that the subspectrum E' is a *cofinal subspectrum* of E if for every finite subcomplex $K \subset E_n$, there is a natural number r such that the composite map

$$\Sigma^r E_n \xrightarrow{\Sigma^{r-1}\epsilon_n} \Sigma^{r-1} E_{n+1} \xrightarrow{\Sigma^{r-2}\epsilon_{n+1}} \cdots \xrightarrow{\Sigma\epsilon_{n+r-2}} \Sigma E_{n+r-1} \xrightarrow{\epsilon_{n+r-1}} E_{n+r} \tag{16}$$

maps $\Sigma^r K$ into E'_{n+r}.

This means that for a subspectrum E' to be cofinal, it has to be "large enough" that any cell in the spectrum E is eventually mapped into E'. A simple way to create a cofinal subspectrum is to keep all the spaces E_n for n larger than some number k, and reduce all lower spaces to a single point. With all this in mind, we define maps of spectra.

Definition 2.12. Let E and F be CW-spectra. A *map of degree* k from E to F is an equivalence class of functions of degree k from cofinal subspectra of E to F. The equivalence relation is as follows: Let $f_1 \colon E_1 \longrightarrow F$ and $f_2 \colon E_2 \longrightarrow F$ be functions from cofinal subspectra E_1, E_2 to F. We say that f_1 and f_2 are equivalent if there exists a cofinal subspectrum $E_3 \subset E$ such that $f_{1|E_3} = f_{2|E_3}$.

Using our construction of a cofinal subspectrum from earlier, we can see that with this definition, it no longer matters what a function does on the whole spectrum. All that is of interest, is what the function does from a certain point, and on to infinity.

The final equivalence relation we want on the functions is something that resembles the homotopy equivalence of continuous maps. In order to do this, we first need the construction of cylinder spectra.

Definition 2.13. Let E be a CW-spectrum. The *cylinder spectrum of* E, written $\mathrm{Cyl}(E)$, has spaces

$$\mathrm{Cyl}(E)_n = [0,1] \times E_n \Big/ (t, x_0) \sim (t', x_0) \quad \forall t \in [0,1] \tag{17}$$

where x_0 is the basepoint of E_n. The structure maps of $\mathrm{Cyl}(E)$ are induced by the structure maps of E in the obvious way.

We can then note that each end of the n'th space of the cylinder spectrum is homeomorphic to E_n. This gives us two canonical functions from E to $\mathrm{Cyl}(E)$. We call these $i^0 = \{i^0_n\}$ and $i^1 = \{i^1_n\}$, and define them by

$$\begin{aligned} i^0_n \colon E_n &\longrightarrow \mathrm{Cyl}(E) \\ x &\longmapsto (0, x) \\ i^1_n \colon E_n &\longrightarrow \mathrm{Cyl}(E) \\ x &\longmapsto (1, x). \end{aligned} \tag{18}$$

Then, we have everything we need for the definition of homotopy equivalence.

Definition 2.14. Let E and F be CW-spectra. Two maps f and g from E to F are said to be *homotopy equivalent* if there exists a map $h \colon \mathrm{Cyl}(E) \longrightarrow F$ such that $f = h \circ i^0$ and $g = h \circ i^1$.

By taking these homotopy classes of maps, we can finally define the stable homotopy category.

Definition 2.15. The *stable homotopy category* is the category where the objects are CW-spectra, and the morphisms are homotopy classes of maps of spectra. We write $[E, F]_k$ for the set of morphisms of degree k from E to F.

3 Generalized Cohomology

In this section, we will see how all cohomology theories can be described by the same set of axioms and how we can use these to talk about cohomology in a general setting. We then use the important Brown representability theorem to connect this to the stable homotopy category from Section 2. Finally, we will examine how singular cohomology can be described using these methods. A general source for this section is again [14], and exceptions will be pointed out.

3.1 The General Case

Since all cohomology theories have both a reduced and an unreduced form, we have to make a choice about how to proceed. We will describe the axioms for a reduced cohomology and then show how to construct an unreduced cohomology theory based on a reduced one. In the following definition **CW** denotes the category of pointed CW-complexes, where the basepoint is a 0-cell, and all maps are basepoint-preserving.

Definition 3.1. A *reduced cohomology theory* $\tilde{E}$ *on* ***CW*** is a sequence of functors $\{\widetilde{E}^n\}$ from **CW** to **Ab** such that the following four axioms hold for all n:

1. **(Suspension)** There is a natural isomorphism $\widetilde{E}^n(X) \cong \widetilde{E}^{n+1}(\Sigma X)$ for all X in **CW**.
2. **(Homotopy invariance)** If f and g are homotopic maps $X \to Y$ in **CW**, then they induce the same maps $f^* = g^* \colon \widetilde{E}^n(X) \to \widetilde{E}^n(Y)$ in cohomology.
3. **(Exactness)** If A is a subcomplex of X, then the sequence $\widetilde{E}^n({}^X\!/_A) \to \widetilde{E}^n(X) \to \widetilde{E}^n(A)$ is exact.
4. **(Additivity)** If $X = \bigvee_\alpha X_\alpha$ is a wedge sum of spaces $\{X_\alpha\}$, then the inclusions $\iota_\alpha \colon X_\alpha \hookrightarrow X$ induce an isomorphism $\prod_\alpha \iota_\alpha \colon \widetilde{E}^n(X) \cong \prod_\alpha \widetilde{E}^n(X_\alpha)$.

The group $\widetilde{E}^n(X)$ is called the *n'th reduced E-cohomology of X*.

We will not assume that the dimension axiom holds, since this gives us more freedom to define interesting cohomology theories, such as complex cobordism. Moving on, we would of course like to be able to talk about unreduced cohomology in a general setting as well. It turns out that all we have to do to go from a reduced cohomology theory to an unreduced one, is add one extra point to the spaces.

Definition 3.2. Let $\tilde{E}$ be a reduced cohomology theory on **CW** and X a cell-complex. We define *the unreduced cohomology theory* E as the functors $\{E^n\}$ from **CW** to **Ab** given by $E^n(X) = \widetilde{E}^n(X_+)$, where X_+ denotes the space we get by adding a disjoint basepoint to X. Given a subcomplex A of X, we define the relative E-cohomology $E^n(X,A) = \widetilde{E}^n({}^X/_A)$.

Although we will not prove this, it can be shown that this definition gives us all the usual properties of an unreduced cohomology theory, except, of course, the dimension axiom. We emphasise some of the more important properties that will be used later.

Proposition 3.3. *Any unreduced cohomology theory E has the following properties:*

- *If A is a subcomplex of X, there exist homomorphisms $E^n(A) \to E^{n+1}(X,A)$ for all n, such that the sequence*

$$\begin{tikzcd} \cdots \arrow[r] & E^n(X,A) \arrow[r] & E^n(X) \arrow[r] & E^n(A) \arrow[dll, rounded corners] \\ & E^{n+1}(X,A) \arrow[r] & E^{n+1}(X) \arrow[r] & E^{n+1}(A) \arrow[r] & \cdots \end{tikzcd} \tag{19}$$

  is a long exact sequence.

- *For subcomplexes $A, B \subset X$ such that $A \cup B = X$, there is a Mayer-Vietoris sequence*

$$\begin{tikzcd} \cdots \arrow[r] & E^n(X) \arrow[r] & E^n(A) \oplus E^n(B) \arrow[r] & E^n(A \cap B) \arrow[dll, rounded corners] \\ & E^{n+1}(X) \arrow[r] & E^{n+1}(A) \oplus E^n(B) \arrow[r] & E^{n+1}(A \cap B) \arrow[r] & \cdots \end{tikzcd} \tag{20}$$

3.2 Brown Representability

Having established the basics of generalised cohomology, we would like to see how this relates to the stable homotopy category. This is made possible by the Brown representability theorem.

Theorem 3.4. *(Brown representability) Let E be a reduced cohomology theory. Then there exists a CW-spectrum, also denoted E, such that $\widetilde{E}^n(X) \cong [\Sigma^\infty X, E]_{-n}$ for all X.*
Conversely, let E be a CW-spectrum. Then the functors $\widetilde{E}^n$ defined by $\widetilde{E}^n(X) = [\Sigma^\infty X, E]_{-n}$ satisfy the axioms for a reduced cohomology theory.

This very strong theorem allows us to turn questions about cohomology into questions about morphisms in the stable homotopy category. Furthermore, it allows us to extend our definition of cohomology so that it encompasses more than just topological spaces. We have defined cohomology in terms of maps in

$[\Sigma^\infty X, E]$, but there is no reason that we should have to restrict ourselves to the cases where one of the spectra is a suspension spectrum. We therefore have a natural way of defining the cohomology of a spectrum.

Definition 3.5. Let E and F be CW-spectra. We define the *n'th reduced E-cohomology of F* as $\widetilde{E}^n(F) = [F, E]_{-n}$.

There is no natural way of adding an extra basepoint to a spectrum, so we have no notion of the unreduced cohomology of a spectrum. A second observation to be made is that E-cohomology behaves especially nicely if E is in a certain class of spectra.

Definition 3.6. A CW-spectrum E is called an *Ω-spectrum* if for all n, the structure map $\epsilon'_n \colon E_n \longrightarrow \Omega E_{n+1}$ is a weak homotopy equivalence.

In the case where E is an Ω-spectrum, we can see that

$$[\Sigma^n X, E_n] \cong [\Sigma^n X, \Omega E_{n+1}] \cong [\Sigma^{n+1} X, E_{n+1}]. \tag{21}$$

This means that if we understand the homotopy classes of maps in some degree of the spectra, we understand it all. The cohomology groups can therefore be computed by $\widetilde{E}^n(X) = [X, E_n]$. Another important theorem tells us that this is the only case we need to focus on. [2]

Theorem 3.7. *In the stable homotopy category, every spectrum is isomorphic to an Ω-spectrum.*

As stated earlier, it is often necessary to use both reduced and unreduced cohomology, and we will need ways to go from one to the other. We have defined unreduced cohomology in terms of the reduced cohomology and would now like to go the other way. It turns out there is an easy way to do this, and it is again related to what happens in a single point.

Lemma 3.8. *For a cohomology theory E, we have $\widetilde{E}^n(X) \cong E^n(X, x_0)$ for all X, where x_0 is the basepoint of X.*

Proof. Assume that E is an Ω-spectrum. We then claim that $E^n(X, x_0) \cong \operatorname{Ker}\,[E^n(X) \to E^n(x_0)]$. To see this, observe the long exact sequence in cohomology

$$\begin{tikzcd} \cdots \arrow[r] & E^{n-1}(X,x_0) \arrow[r] & E^{n-1}(X) \arrow[r] & E^{n-1}(x_0) \arrow[dll, rounded corners] \\ & E^n(X,x_0) \arrow[r] & E^n(X) \arrow[r] & E^n(x_0) \arrow[r] & \cdots \end{tikzcd} \tag{22}$$

Here the map $E^{n-1}(X) \to E^{n-1}(x_0)$, or equivalently $\widetilde{E}^{n-1}(X_+) \to \widetilde{E}^{n-1}(x_{0+})$, is induced by the inclusion $x_0 \to X$. Now, any map in $[x_{0+}, E_{n-1}]$ can be extended to a map $X_+ \to E_{n-1}$ that is constant on everything but the "new" basepoint. This implies that the map $E^{n-1}(X) \to E^{n-1}(x_0)$ is surjective, and the claims follows.

Next, we see that Ker $[E^n(X) \to E^n(x_0)]$ consists of the homotopy classes of maps $X_+ \to E_n$ that go to zero in $[x_{0+}, E_n]$. These are easily seen to be precisely the maps that send the "old" basepoint of X to the basepoint of E_n, which is the definition of $\widetilde{E}^n(X)$. This concludes the proof. □

So far we have been referring to the sets $[\Sigma^\infty X, E]$ as *groups* without showing that they have a group structure. We can define the group operation in much the same way that we do with homotopy groups. We will assume that E is an Ω-spectrum. Then, by (21), we have that $\widetilde{E}^n(X) \cong [\Sigma X, E_{n+1}]$. When we are dealing with the suspension of a space, we have a notion of the "equator" of the space. By collapsing this equator, we get a pinch map $p\colon \Sigma X \to \Sigma X \vee \Sigma X$. If we let $f, g \in [\Sigma X, E_{n+1}]$, we can define the class of $f + g$ by the composition

$$\Sigma X \xrightarrow{p} \Sigma X \vee \Sigma X \xrightarrow{f \vee g} E_{n+1}. \tag{23}$$

The proof that this is a group is analogous to the proof for homotopy groups.

Many cohomology theories have a multiplicative structure as well as an additive one, and this structure can be described in terms of spectra too. In order to define this, we need to make an important assumption. We will from now on assume that there exists a smash product of spectra $E \wedge F$. This smash product is associative and commutative, and $\mathbb{S}$, the suspension spectrum of the sphere, is the identity element. The construction of this product is quite complicated, and we will not go into the details here. One possible construction can be found in [2].

Definition 3.9. A spectrum E is called a *ring spectrum* if there exists a map of spectra $\mu\colon E \wedge E \to E$, called the *multiplication map*, and a map $u\colon \mathbb{S} \to E$, called the *unit map*, such that the following diagrams commute:

$$\begin{array}{ccc} E \wedge E \wedge E & \xrightarrow{\mu \wedge 1} & E \wedge E \\ {\scriptstyle 1\wedge\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ E \wedge E & \xrightarrow{\mu} & E \end{array} \qquad \begin{array}{ccc} \mathbb{S} \wedge E & \xrightarrow{\cong} & E \\ {\scriptstyle u\wedge 1}\downarrow & & \downarrow{\scriptstyle 1} \\ E \wedge E & \xrightarrow{\mu} & E \end{array} \qquad \begin{array}{ccc} E \wedge \mathbb{S} & \xrightarrow{\cong} & E \\ {\scriptstyle 1\wedge u}\downarrow & & \downarrow{\scriptstyle 1} \\ E \wedge E & \xrightarrow{\mu} & E \end{array} \tag{24}$$

The cohomology theory corresponding to a ring spectrum is called a *multiplicative cohomology theory*.

It should be easy to see that the first diagram gives us associativity of the multiplication, while the last two correspond to having a two-sided identity. If we now let $f, g \in [\Sigma^\infty X, E]$, we can use the multiplication on E to define $f \cdot g$ as the composite map

$$\Sigma^\infty X \xrightarrow{\Delta} \Sigma^\infty X \wedge \Sigma^\infty X \xrightarrow{f \wedge g} E \wedge E \xrightarrow{\mu} E, \tag{25}$$

where Δ denotes the diagonal map of spectra.

In order to understand the spectra that represent cohomology theories, it is useful to examine the homotopy groups of these spectra. We define them in much the same way that we define homotopy groups of spaces, but rather than using maps from spheres, we will use maps from the sphere spectrum.

Definition 3.10. Let E be a CW-spectrum. The *n'th homotopy group of E*, denoted $\pi_n(E)$ is defined as $\pi_n(E) = [\mathbb{S}, E]_n$. The sum $\pi_*(E) = \bigoplus_n \pi_n(E)$ is called the *coefficient ring* of E.

We will accept without proof that $\pi_*(E)$ and $\pi_0(E)$ are rings. With this in mind, it is possible to define a certain type of cohomology theory that will be particularly interesting to use. For this definition, recall that $\mathbb{C}P^1 \cong S^2$, and observe that $\widetilde{E}^2(S^2) \cong [\Sigma^\infty S^2, E]_{-2} \cong [\mathbb{S}, E]_0 \cong \pi_0(E)$.

Definition 3.11. A multiplicative cohomology theory E is called *complex oriented* if the homomorphism $\widetilde{E}^2(\mathbb{C}P^\infty) \to \widetilde{E}^2(S^2)$, induced by the natural inclusion $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$ is surjective. An element $t \in \widetilde{E}^2(\mathbb{C}P^\infty)$ is called a *complex orientation of E* if t maps to the multiplicative identity of $\pi_*(E)$.

All the cohomology theories we will deal with from now on will be complex oriented.

3.3 Eilenberg-MacLane spectra

We will end this section by looking at a specific example of Brown representability, namely how the Eilenberg-MacLane spectrum $H\mathbb{Z}$ represents singular cohomology. This requires an understanding of Eilenberg-MacLane spaces.

Definition 3.12. Let G be a group and $n \geq 1$ an integer. The *Eilenberg-MacLane space $K(G, n)$* is a topological space which is homotopy equivalent to a CW-complex such that

$$\pi_k K(G, n) \cong \begin{cases} G, & k = n \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

Since all homotopy groups of order higher than 1 are abelian, we see that $K(G, n)$ can only exist if G is abelian or if $n = 1$. However, in those cases, $K(G, n)$ always exists, and it is unique up to homotopy. All this is proved in [9]. Rather than rewriting those proofs, we will focus on the method for constructing these spaces as cell-complexes.

Let G be an abelian group, and let $n \geq 2$. (The case $n = 1$ uses a different method which will not be presented here.) We start by making a free $\mathbb{Z}$-resolution of G:

$$0 \longrightarrow \mathbb{Z}^V \xrightarrow{\ i\ } \mathbb{Z}^W \longrightarrow G \longrightarrow 0 \tag{27}$$

where V and W are sets of generators for the groups. We then construct a wedge sum of spheres, with one n-sphere for each generator of $\mathbb{Z}^W$:

$$X = \bigvee_{w \in W} S^n \tag{28}$$

We can now observe that the $\pi_n(X) \cong \mathbb{Z}^W$. Now, for each generator $v \in V$, its image $i(v)$ is in $\pi_n(X)$. Therefore, each $i(v)$ determines an attaching map $S^n \to X$. We use these maps to attach an $(n+1)$-cell for each $v \in V$, and let X_1 denote the new complex. It can then easily be shown that the $\pi_n(X_1) \cong G$, as desired. Since we have not used any cells of lower dimension than n, all the lower homotopy groups are trivial.

The higher homotopy groups may be nontrivial, and we remove these one by one. Let $\pi_{n+1}(X_1)$ be generated by the set U. Then each $u \in U$ determines a map $S^{n+1} \to X_1$. We then attach an $(n+2)$-cell to X_1 for each generator $u \in U$, and observe that this makes π_{n+1} trivial. By continuing this process inductively, we produce the desired space $K(G,n)$.

To assemble these spaces into a spectrum, all we need to do is make the observation that $\Omega K(G,n+1)$ is a $K(G,n)$-space. By Lemma 2.4, we have

$$\begin{aligned} \pi_k(\Omega K(G,n+1)) &= [S^k, \Omega K(G,n+1)] \\ &\cong [\Sigma S^k, K(G,n+1)] = \pi_{k+1}(K(G,n+1)), \end{aligned} \tag{29}$$

and the claims follows. Since any two $K(G,n)$-spaces are homotopy equivalent, we have a spectrum

$$\begin{tikzcd} \dots \arrow[dr] & K(G,n-1) \arrow[dr, "\sim"] & K(G,n) \arrow[dr, "\sim"] & K(G,n+1) \arrow[dr] & \dots \\ \dots & \Omega K(G,n-1) & \Omega K(G,n) & \Omega K(G,n+1) & \dots \end{tikzcd} \tag{30}$$

Definition 3.13. Let G be a group. The *Eilenberg-MacLane spectrum HG* is the spectrum where the spaces are given by $(HG)_n = K(G,n)$ and the structure maps are as in (30).

By the Brown representability theorem, the spectrum HG represents a cohomology theory, and it can be shown that this is in fact singular cohomology with G-coefficients [9]. In particular, $H\mathbb{Z}$ represents integral cohomology. Since $H\mathbb{Z}$ is, by construction, an Ω-spectrum, there is a bijection $H^n(X;\mathbb{Z}) \cong [X, K(\mathbb{Z},n)]$ for every space X. We will now construct this bijection without proving that it is in fact bijective.

Let X be a $K(\mathbb{Z},n)$-space. We must now define the *fundamental class* of $H^n(X;\mathbb{Z})$. By construction, all homotopy groups lower than n are trivial for X. Therefore, by the Hurewicz theorem, the UCT short exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X)) \longrightarrow H^n(X;n) \longrightarrow \mathrm{Hom}(H_n(X,\mathbb{Z})) \longrightarrow 0 \tag{31}$$

simplifies to a bijection $H^n(X;n) \cong \mathrm{Hom}(H_n(X),\mathbb{Z})$. Now, since $\mathbb{Z}$ is the n'th homotopy group of X, we have

$$H^n(X;\mathbb{Z}) \cong \mathrm{Hom}(H_n(X), \pi_n(X)). \tag{32}$$

In other words, every n'th cohomology class determines a map from the n'th homology of X to the n'th homotopy group of X. By the Hurewicz theorem, there is a canonical isomorphism, called the Hurewicz-homomorphism, going in the other direction. This allows us to define the fundamental class.

Definition 3.14. The cohomology class ι_n corresponding to the inverse of the Hurewicz homomorphism in

$$H^n(K(\mathbb{Z},n);\mathbb{Z}) \cong \mathrm{Hom}(H_n(X),\mathbb{Z}) \tag{33}$$

is called the *fundamental class of* $K(\mathbb{Z},n)$.

Using this cohomology class, we define the desired map by

$$\begin{aligned} [X, K(\mathbb{Z},n)] &\longrightarrow H^n(X;\mathbb{Z})) \\ [f] &\longmapsto f^*(\iota_n). \end{aligned} \tag{34}$$

As a simple example, we can compute the integral cohomology of the sphere S^n. We get

$$\widetilde{H}^k(S^n;\mathbb{Z}) \cong [S^n,(H\mathbb{Z})_k] = [S^n, K(\mathbb{Z},k)] = \pi_n(K(\mathbb{Z},k)) \cong \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n \end{cases} \tag{35}$$

as expected.

4 The Atiyah-Hirzebruch Spectral Sequence

In order to increase our understanding of generalised cohomology, we would like to compute the generalised cohomology of some spaces. In certain cases, this can be done even without specifying which cohomology theory we are dealing with. To do this, we will need a powerful tool, known as the Atiyah-Hirzebruch spectral sequence. In this section we will see that for any cohomology theory A and finite dimensional CW-complex X, there is a multiplicative cohomology spectral sequence

$$E_2^{p,q} = H^p(X; \pi_{-q}A) \Rightarrow A^{p+q}(X). \tag{36}$$

In other words, we can use the singular cohomology of a CW-complex to find its generalised A-cohomology. We are now using the notation A for cohomology theories to avoid confusion with the pages E of the spectral sequence. After presenting the spectral sequence, we will use it to compute the A-cohomology of some important spaces.

4.1 Construction of the Spectral Sequence

Our construction of the sequence will be based on [14], where only the dual case in homology is presented. The construction begins with the theory of exact couples.

Definition 4.1. An *exact couple* is a pair of abelian groups (M, N) together with group homomorphisms (i, j, k) such that the (noncommutative) diagram

$$\begin{tikzcd} N \arrow[rr, "i_0"] & & N \arrow[dl, "j_0"] \\ & M \arrow[ul, "k_0"] & \end{tikzcd} \tag{37}$$

is exact. In other words,

$$\begin{aligned} \text{Im } i_0 &= \text{Ker } j_0 \\ \text{Im } j_0 &= \text{Ker } k_0 \\ \text{Im } k_0 &= \text{Ker } i_0. \end{aligned} \tag{38}$$

We will now see that we can create a new exact couple using an old one. First, we define a differential d_0 on M by letting $d_0 = j_0 k_0$. We then observe that $d_0^2 = (j_0 k_0)(j_0 k_0) = j_0(k_0 j_0)k_0 = 0$. This implies that we have a chain complex

$$\cdots \xrightarrow{d_0} M \xrightarrow{d_0} M \xrightarrow{d_0} M \xrightarrow{d_0} \cdots \tag{39}$$

We will use the homology of this chain complex to create a new exact couple, given by the following diagram:

$$
\begin{tikzcd}
N_1 \arrow[rr, "i_1"] & & N_1 \arrow[dl, "j_1"] \\
& M_1. \arrow[ul, "k_1"] &
\end{tikzcd}
\tag{40}
$$

Here, the groups and homomorphisms are given by

$$
M_1 = \frac{\text{Ker } d_0}{\text{Im } d_0}, \quad N_1 = \text{Im } i_0, \quad i_1 = i_{0|N_1}, \quad j_1 = j_0 i_0^{-1}, \quad k_1 = \bar{k}_0. \tag{41}
$$

Some abuse of notation should be addressed here. The homomorphism $\bar{k}_0$ is the same as k_0, but defined on the homology classes of N_1 rather than the elements of N_0. We can easily see that this is well-defined, since two representatives of a class in N_1 differ by an element of Im j_0, which is mapped to 0 in N_1 by exactness.

Furthermore, the homomorphism i_0 is not in general an isomorphism. However, j_1 will still be well-defined if we simply interpret i_0^{-1} as choosing an element of the pre-image. This can be verified quickly. Let $x \in N$, and let a and b be in the pre-image $i_0^{-1}(x)$. Then $(a - b) \in \text{Ker } i_0 = \text{Im } k_0$, and consequently $j_0(a - b) \in \text{Im } j_0 k_0 = \text{Im } d_0$ which is divided out in M_1.

With the knowledge that all the homomorphisms are well-defined, we observe that diagram (40) is also an exact couple. The proof is nothing but a simple diagram chase. Now, letting $d_1 = j_1 k_1 = j_0 i_0^{-1} \bar{k}_0$, we can repeat the process, producing a whole sequence of exact couples. In the sequence $\{M_1, M_2, \ldots\}$ every group is given by the homology of the previous one's chain complex, implying that we can create a spectral sequence from an exact couple. This is the method we will use to construct the Atiyah-Hirzebruch spectral sequence.

Let A be an unreduced cohomology theory, and let X be a finite-dimensional CW-complex with skeleta X_p. By Proposition 3.3 there is a long exact sequence

$$
\begin{tikzcd}
\cdots \arrow[r] & A^{p+q}(X_p, X_{p-1}) \arrow[r, "k"] & A^{p+q}(X_p) \arrow[r, "i"] & A^{p+q}(X_{p-1}) \arrow[dll, rounded corners, "j"'] & \\
& A^{p+q+1}(X_p, X_{p-1}) \arrow[r, "k"] & A^{p+q+1}(X_p) \arrow[r, "i"] & A^{p+q+1}(X_{p-1}) \arrow[r] & \cdots
\end{tikzcd}
\tag{42}
$$

Taking the direct sum over p and q, we can assemble these groups into an exact couple

$$
\begin{tikzcd}
\bigoplus_{p,q} A^{p+q}(X_p) \arrow[rr, "i"] & & \bigoplus_{p,q} A^{p+q}(X_p) \arrow[dl, "j"] \\
& \bigoplus_{p,q} A^{p+q}(X_p, X_{p-1}) \arrow[ul, "k"] &
\end{tikzcd}
\tag{43}
$$

We let the sum $\bigoplus_{p,q} A^{p+q}(X_p, X_{p-1})$ define the first page of the spectral sequence so that $E_1^{p,q} = A^{p+q}(X_p, X_{p-1})$. Using the differential as defined for exact couples, we can generate a new exact couple and let the second page be given, once again, by the direct sum at the bottom of the diagram. Continuing this process, we get the desired spectral sequence.

We need to see that the differentials move between the right groups for this to be a cohomology spectral sequence. On the n'th page, d_n is given by $j_n k_n = j_0 i_0^{-(n-1)} \bar{k}_0$. Examining the long exact sequence (42), we see that i_0 decreases p by 1 and increases q by 1. Furthermore, j_0 increases p by 1, while k_0 leaves p and q unchanged. We can then see that

$$d_n \colon E_n^{p,q} \longrightarrow E_n^{p+n,q-n+1} \tag{44}$$

as desired. We can therefore conclude that this is in fact a spectral sequence.

The key to proving that the spectral sequence converges to the desired cohomology, lies in showing that the n'th page can be computed by

$$E_r^{p,q} \cong \frac{\text{Im } (A^{p+q}(X_{p+r-1}, X_{p-1}) \to A^{p+q}(X_p, X_{p-1}))}{\text{Im } (A^{p+q-1}(X_{p-1}, X_{p-r}) \to A^{p+q}(X_p, X_{p-1}))}. \tag{45}$$

We will omit this proof, as well as the proof that the spectral sequence is multiplicative. Both are shown in [6]. We will, however, indicate why the second page of the spectral sequence can be computed using singular cohomology. On the first page, we can see that the groups are given by

$$E_1^{p,q} = A^{p+q}(X_p, X_{p-1}) = \widetilde{A}^{p+q}(X_p/X_{p-1}) \cong \widetilde{A}^{p+q}(\bigvee_\alpha S^p), \tag{46}$$

where α counts the p-cells of X. Using the additivity axiom, we see that

$$\widetilde{A}^{p+q}(\bigvee_\alpha S^p) \cong \bigoplus_\alpha \widetilde{A}^{p+q}(S^p) \cong \bigoplus_\alpha \pi_{-q} A. \tag{47}$$

In other words, the first page of the spectral sequence is given by the cellular chains of X with coefficients in $\pi_{-q}A$. It can then be shown that the first differential is the same as the cellular boundary map, which implies that the second page is given by the cellular cohomology of X.

4.2 The E-Cohomology of Complex Projective Space

We now wish to compute the generalised A-cohomology of complex projective space $\mathbb{C}P^\infty$ for a complex-orientable cohomology theory A. Since the Atiyah-Hirzebruch spectral sequence only converges for finite-dimensional CW-complexes, we start by computing the cohomology of $\mathbb{C}P^n$. Recall that the singular cohomology of $\mathbb{C}P^n$ is given by

$$H^k(\mathbb{C}P^n; M) \cong \begin{cases} M, & 0 \leq k \leq 2n, \ k \text{ even} \\ 0, & \text{otherwise.} \end{cases} \tag{48}$$

By construction, $E_2^{p,q} = H^p(\mathbb{C}P^n; \pi_{-q}A)$, and therefore the second page of the spectral sequence looks like this:

$$
\begin{array}{c|cccccccccc}
q & & & & & & & & & & \\
 & \vdots & & & & & & & & & \\
3 & \pi_{-3}A & & & & & & & & & \\
2 & \pi_{-2}A & 0 & \pi_{-2}A & 0 & \pi_{-2}A & 0 & \cdots & \pi_{-2}A & 0 & 0 \\
1 & \pi_{-1}A & 0 & \pi_{-1}A & 0 & \pi_{-1}A & 0 & \cdots & \pi_{-1}A & 0 & 0 \\
0 & \pi_0 A & 0 & \pi_0 A & 0 & \pi_0 A & 0 & \cdots & \pi_0 A & 0 & 0 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & & 2n & & \quad p \\
-1 & \pi_1 A & 0 & \pi_1 A & 0 & \pi_1 A & 0 & \cdots & \pi_1 A & 0 & 0 \\
-2 & \pi_2 A & & & & & & & & & \\
 & \vdots & & & & & & & & &
\end{array}
\tag{49}
$$

We now examine the position $(p, q) = (2, 0)$ more closely. On the E_2-page, this is given by $\pi_0 A$. By (45), we can compute the corresponding group on the r'th page by

$$
E_r^{2,0} \cong \frac{\text{Im } (A^2((\mathbb{C}P^n)_{r+1}, (\mathbb{C}P^n)_1) \to A^2((\mathbb{C}P^n)_2, (\mathbb{C}P^n)_1))}{\text{Im } (A^1((\mathbb{C}P^n)_1, (\mathbb{C}P^n)_{2-r}) \to A^2((\mathbb{C}P^n)_2, (\mathbb{C}P^n)_1))}, \tag{50}
$$

where $(\mathbb{C}P^n)_r$ denotes the r-skeleton of $\mathbb{C}P^n$. To see what happens on the E_∞-page, we let r approach infinity and note that $(\mathbb{C}P^n)_1 = \text{pt}$ and $(\mathbb{C}P^n)_2 = \mathbb{C}P^1$. We then get

$$
E_\infty^{2,0} \cong \frac{\text{Im } (A^2(\mathbb{C}P^n, \text{pt}) \to A^2(\mathbb{C}P^1, \text{pt}))}{\text{Im } (A^1(\text{pt}) \to A^2(\mathbb{C}P^1, \text{pt}))}. \tag{51}
$$

We now claim that this is $\widetilde{A}^2(\mathbb{C}P^1)$. Firstly, we note that the lower map is obviously zero. Secondly, by the complex-orientability of A, the map $\widetilde{A}^2(\mathbb{C}P^\infty) \to \widetilde{A}^2(\mathbb{C}P^1)$ is surjective. This map factors through $\widetilde{A}^2(\mathbb{C}P^n)$, and therefore the upper map in (51) is also surjective. We can then conclude that

$$
E_\infty^{2,0} \cong \widetilde{A}^2(\mathbb{C}P^1) \cong \widetilde{A}^2(S^2) = \pi_0 A = E_2^{2,0}. \tag{52}
$$

This means that everything in $E_2^{2,0}$ survives to the E_∞-page, which implies that all differentials going into or out of $(2, 0)$ are zero. In particular, $d_2 \colon E_2^{0,1} \to E_2^{2,0}$ is trivial. A similar examination of the position $(0, 1)$ shows that $E_2^{0,1} = \pi_{-1}A$ survives to the infinite page. Since the spectral sequence is multiplicative, these two groups, together with the differential between them, determine the rest of the spectral sequence. This implies that all the differentials are zero, and the spectral sequence collapses at the second page. Examining the E_2-page, we see that the cohomology is given by

$$
A^*(\mathbb{C}P^n) \cong (\pi_* A)[\![t]\!] \Big/ (t^n), \tag{53}
$$

where t is the image of the complex orientation of A in $A^*(\mathbb{C}P^n)$.

4.3 The Milnor Short Exact Sequence

We continue by using the cohomology of $\mathbb{C}P^n$ to compute the cohomology of $\mathbb{C}P^\infty$. This can be done using a short exact sequence due to Milnor [8]. First, we need a better understanding of the first derived functor $\varprojlim^1$. We will not use its properties as a derived functor, so we will instead define it in a more direct manner. Let

$$\cdots \xrightarrow{f_4} A_3 \xrightarrow{f_3} A_2 \xrightarrow{f_2} A_1 \tag{54}$$

be a sequence of abelian groups and homomorphisms. We define a homomorphism d by

$$\begin{aligned} d\colon \prod_n A_n &\longrightarrow \prod_n A_n \\ (a_1, a_2, a_3, \dots) &\longmapsto (a_1 - f_2(a_2), a_2 - f_3(a_3), \dots). \end{aligned} \tag{55}$$

It should then be easy to see that $\varprojlim\{A_n\} \cong \operatorname{Ker} d$. By taking the cokernel instead, we can define $\varprojlim^1\{A_n\} = \operatorname{Cok} d$. This coincides with the usual construction of the first derived functor, although we will not prove this. We are now ready to state Milnor's result.

Proposition 4.2. *Let $X^1 \subset X^2 \subset \cdots$ be a sequence of CW-complexes with union X, and let E be a cohomology theory. Then there is a short exact sequence*

$$0 \longrightarrow \varprojlim_n{}^1 E^{k-1}(X^n) \longrightarrow E^k(X) \longrightarrow \varprojlim_n E^k(X^n) \longrightarrow 0 \tag{56}$$

for all k.

Proof. We start by constructing a new CW-complex, given by

$$L = X^1 \times [0,1] \cup X^2 \times [1,2] \cup \dots, \tag{57}$$

with the edges identified with each other in the obvious way. Furthermore, we define subcomplexes

$$\begin{aligned} A &= X^1 \times [0,1] \cup X^3 \times [2,3] \cup \dots \\ B &= X^2 \times [1,2] \cup X^4 \times [3,4] \cup \dots \end{aligned} \tag{58}$$

By Proposition 3.3, this leads to a Mayer-Vietoris sequence

$$\begin{tikzcd} \cdots \arrow[r] & E^{k-1}(A) \oplus E^{k-1}(B) \arrow[r] & E^{k-1}(A \cap B) \arrow[dll, rounded corners] \\ E^k(L) \arrow[r] & E^k(A) \oplus E^k(B) \arrow[r] & E^k(A \cap B) \arrow[r] & \cdots \end{tikzcd} \tag{59}$$

Technically, we need to increase the intervals $[n, n+1]$ slightly to make sure that A and B cover L. However, this complicates the notation, so we leave it out. We now observe that we have the homotopy equivalences

$$\begin{aligned} L &\sim X \\ A &\sim X^1 \sqcup X^3 \sqcup \dots \\ B &\sim X^2 \sqcup X^4 \sqcup \dots \\ A \cap B &\sim X^1 \sqcup X^2 \sqcup \dots \end{aligned} \tag{60}$$

This means that sequence (59) is isomorphic to

$$\begin{tikzcd} \cdots \arrow[r] & \prod_n E^{k-1}(X^n) \arrow[r, "i^*_{k-1}"] & \prod_n E^{k-1}(X^n) \arrow[dll, rounded corners] & \\ E^k(X) \arrow[r] & \prod_n E^k(X^n) \arrow[r, "i^*_k"] & \prod_n E^k(X^n) \arrow[r] & \cdots \end{tikzcd} \tag{61}$$

Taking the cokernel of i^*_{k-1} and the kernel of i^*_k, we can turn this into a short exact sequence

$$0 \longrightarrow \text{Cok } i^*_{k-1} \longrightarrow E^k(X) \longrightarrow \text{Ker } i^*_k \longrightarrow 0. \tag{62}$$

It can be shown that the maps i^* are compatible with the map d that we used to define $\varprojlim$ and $\varprojlim^1$. Applying this to the sequence

$$\cdots \longrightarrow E^*(X^n) \longrightarrow E^*(X^{n-1}) \longrightarrow \cdots, \tag{63}$$

we see that Cok $i^*_{k-1} = \varprojlim^1 E^{k-1}(X^n)$ and Ker $i^*_k = \varprojlim E^k(X^n)$. This gives us the desired short exact sequence

$$0 \longrightarrow \varprojlim_n{}^1 E^{k-1}(X^n) \longrightarrow E^k(X) \longrightarrow \varprojlim_n E^k(X^n) \longrightarrow 0. \tag{64}$$

□

We are especially interested in the cases where the $\varprojlim^1$-term disappears, since this makes the calculations much easier. Fortunately, these cases are easily recognisable.

Definition 4.3. Let the sequence $\{A_n\}_{n\geq 1}$ be as in (54). The sequence is said to satisfy the *Mittag-Leffler condition* if for each n, there exists an $N \geq n$ such that for all $m, m' \geq N$, we have Im $(A_m \to A_n) =$ Im $(A_{m'} \to A_n)$ under the canonical composition maps.

It is proved in [6] that if a sequence satisfies the Mittag-Leffler condition, then its $\varprojlim^1$-term is zero. We now have everything we need to compute the cohomology of $\mathbb{C}P^\infty$.

Since $\mathbb{C}P^\infty$ can be realised as the union of its skeletal filtration $\mathbb{C}P^1 \subset \mathbb{C}P^2 \subset \ldots$, we have, by Proposition 4.2, a short exact sequence

$$0 \longrightarrow \varprojlim_n{}^1 E^{k-1}(\mathbb{C}P^n) \longrightarrow E^k(\mathbb{C}P^\infty) \longrightarrow \varprojlim_n E^k(\mathbb{C}P^n) \longrightarrow 0. \tag{65}$$

We now wish to determine the $\varprojlim^1$-term. We have already computed all the cohomology groups in the sequence

$$\cdots \longrightarrow E^*(\mathbb{C}P^n) \longrightarrow E^*(\mathbb{C}P^{n-1}) \longrightarrow \cdots . \tag{66}$$

Furthermore, we see that the maps

$$(\pi_* E)[\![t]\!]/(t^n) \longrightarrow (\pi_* E)[\![t]\!]/(t^{n-1}) \tag{67}$$

are all given by "cutting off" the highest power of t. In other words, all the maps are surjective, and our sequence satisfies the Mittag-Leffler condition. This implies that the $\varprojlim^1$-term is 0. In fact, we can see this without using the Mittag-Leffler condition. The homomorphism d used to define $\varprojlim^1$ is clearly surjective in this case, and it follows that its cokernel is trivial. The short exact sequence (65) therefore simplifies to an isomorphism, and we get

$$E^*(\mathbb{C}P^\infty) \cong \varprojlim_n E^*(\mathbb{C}P^n) \cong \varprojlim_n (\pi_* E)[\![t]\!]/(t^n) \cong \pi_* E[\![t]\!]. \tag{68}$$

To summarise our results so far, we have the following proposition.

Proposition 4.4. *Let E be a cohomology theory with complex orientation t. Then*

(i) $E^*(\mathbb{C}P^n) \cong (\pi_* E)[\![t]\!]\Big/(t^n)$

(ii) $E^*(\mathbb{C}P^\infty) \cong (\pi_* E)[\![t]\!]$

For our purposes in the next section, we need to generalise this result. Let $BU(n)$ denote the n-dimensional infinite complex Grassman-manifold, often written as $G_n(\mathbb{C}^\infty)$. The cohomology of $BU(n)$ can be computed using similar methods to the ones we just used. We omit these calculations and simply state the result, having already seen how the computations work for a special case. A computation using the Atiyah-Hirzebruch spectral sequence can be found in [6], while a more direct approach is taken in [7].

Proposition 4.5. *Let E be a complex-oriented cohomology theory. The E-cohomology of $BU(n)$ is given by $E^*(BU(n)) \cong (\pi_* E)[\![x_1, \ldots, x_n]\!]$.*

It is then easy to see that this specialises to what we have seen for the case $BU(1) = \mathbb{C}P^\infty$. The generators $x_1, \ldots, x_n$ can be interpreted as generalised Chern-classes.

5 Complex Cobordism

We will now present a specific cohomology theory known as complex cobordism. This is a strong cohomology theory, which means that it gives a lot of information about the topological spaces. It is therefore generally speaking harder to compute than weaker cohomology theories. While complex cobordism has a nice geometrical interpretation, we will instead define it in terms of its spectrum, called MU. Rather than defining the spectrum directly, we will instead define a sequence of spectra $MU(n)$, and construct MU as the colimit of these. We will then see that this particular cohomology theory has an important universal property. This section is mainly based on [7]. First, we need a way to create new spaces using vector bundles.

Definition 5.1. Let $\xi = (E, \pi, B)$ be a complex vector bundle equipped with a Hermitian metric. We define the *disc bundle* $D(\xi)$ to be the fibre bundle with total space $D(E) = \{x \in E \colon |x| \leq 1\}$. We define the *sphere bundle* $S(\xi)$ to be the fibre bundle with total space $S(E) = \{x \in E \colon |x| = 1\}$.

These fibre bundles have the obvious projection maps to the original base spaces. Using these two definitions, we can define the spaces we will use to construct the spectrum MU.

Definition 5.2. Let $\xi = (E, \pi, B)$ be a complex vector bundle equipped with a Hermitian metric. The *Thom space of* ξ is the space $\mathrm{Th}(\xi) = D(E)/S(E)$.

We now have everything we need to construct $MU(n)$. We will write γ^n for the tautological n-plane bundle over $BU(n)$.

Definition 5.3. Let $n \geq 0$. The spectrum $MU(n)$ is defined by $MU(n) = \Sigma^{\infty-2n}\mathrm{Th}(\gamma^n)$.

We will use the convention that $\mathrm{Th}(\gamma^0) \cong S^0$. This may seem counterintuitive, but it will lead to some useful properties. Most importantly, we see that $MU(0)$ is isomorphic to the sphere spectrum $\mathbb{S}$.

It will also be useful to understand the spectrum $MU(1)$. In the disc bundle $D(\gamma^1)$, it is easy to see that every fibre is contractible. This implies that $D(\gamma^1)$ is homotopy equivalent to the base space $BU(1) \cong \mathbb{C}P^\infty$. Moreover, the sphere bundle $S(\gamma^1)$ is the infinite dimensional sphere S^∞, which is known to be contractible. We can then see that

$$MU(1) = \Sigma^{\infty-2}\mathrm{Th}(\gamma^1) \cong \Sigma^{\infty-2}D(\gamma^1)/S(\gamma^1) \cong \Sigma^{\infty-2}\mathbb{C}P^\infty. \tag{69}$$

In order to construct the colimit of the sequence of $\{MU(n)\}_{n\geq 0}$, we need to see that there exist maps $MU(n) \to MU(n+1)$ for all n. The key to this construction lies in the following lemma [14].

Lemma 5.4. *Let $\mathbb{1}$ be the trivial complex line bundle over $BU(n)$. Then $Th(\gamma^n \oplus \mathbb{1}) \cong \Sigma^2\, Th(\gamma^n)$.*

Proof. Firstly, observe that $D(\gamma^n \oplus \mathbb{1}) \cong D^2 \times D(\gamma^n)$, where D^2 is thought of as the unit disc in the complex plane. For the sphere bundle, we simply use the fact that $S(\xi) = \partial D(\xi)$ for any vector bundle ξ. We then have

$$\begin{aligned} \mathrm{Th}(\gamma^n \oplus \mathbb{1}) &= \frac{D(\gamma^n \oplus \mathbb{1})}{\partial D(\gamma^n \oplus \mathbb{1})} \qquad (70) \\ &= \frac{D^2 \times D(\gamma^n)}{\partial(D^2 \times D(\gamma^n))} \\ &= \frac{D^2 \times D(\gamma^n)}{S^1 \times D(\gamma^n) \cup D^2 \times S(\gamma^n)}. \end{aligned}$$

By collapsing the subspace $S^1 \times S(\gamma^n)$, we get

$$\begin{aligned} \cdots &= \frac{S^2 \times \mathrm{Th}(\gamma^n)}{\mathrm{pt} \times \mathrm{Th}(\gamma^n) \cup S^2 \times \mathrm{pt}} \qquad (71) \\ &= \frac{S^2 \times \mathrm{Th}(\gamma^n)}{S^2 \vee \mathrm{Th}(\gamma^n)} \\ &= S^2 \wedge \mathrm{Th}(\gamma^n) \\ &= \Sigma^2 \mathrm{Th}(\gamma^n). \end{aligned}$$

This concludes the proof. □

We can now observe that $\gamma^n \oplus \mathbb{1}$ is an $(n+1)$-plane bundle, and is therefore characterised by a bundle map $\gamma^n \oplus \mathbb{1} \to \gamma^{n+1}$. This induces a map on the Thom spaces as well. Using this, as well as Lemma 5.4 we can construct the map

$$\begin{aligned} MU(n) = \Sigma^{\infty-2n}\mathrm{Th}(\gamma^n) &\cong \Sigma^{\infty-2(n+1)}\Sigma^2\mathrm{Th}(\gamma^n) \qquad (72) \\ &\cong \Sigma^{\infty-2(n+1)}\mathrm{Th}(\gamma^{n+1} \oplus \mathbb{1}) \to \Sigma^{\infty-2(n+2)}\mathrm{Th}(\gamma^{n+1}) = MU(n+1). \end{aligned}$$

This finally allows us to define the spectrum MU.

Definition 5.5. The *complex cobordism spectrum* MU is the colimit of the diagram

$$\cdots \longrightarrow MU(n-1) \longrightarrow MU(n) \longrightarrow MU(n+1) \longrightarrow \cdots$$

where the maps are as in (72).

We will assume that this colimit exists and refer to [7] for a proof. Our next goal is to see that complex cobordism, the cohomology theory represented by MU, is multiplicative and complex oriented. A rigorous construction of the multiplication map $MU \wedge MU \to MU$ would require a more detailed examination of the smash product of spectra. We will instead only indicate where the map comes from.

Let γ^m and γ^n be the tautological vector bundles over $BU(m)$ and $BU(n)$, respectively. Using the projection maps

$$\begin{aligned}\pi_m \colon BU(m) \times BU(n) &\longrightarrow BU(m) \\ \pi_n \colon BU(n) \times BU(n) &\longrightarrow BU(n),\end{aligned} \tag{73}$$

we get the induced vector bundles $\pi_m^*(\gamma^m)$ and $\pi_n^*(\gamma^n)$, both over the space $BU(m) \times BU(n)$. The Whitney sum of these two vector bundles is an $(m+n)$-dimensional vector bundle, and hence it is classified by a map into $BU(m+n)$, as seen in the diagram below.

$$\begin{array}{ccc} \pi_m^*(\gamma^m) \oplus \pi_n^*(\gamma^n) & \longrightarrow & \gamma^{m+n} \\ \downarrow & & \downarrow \\ BU(m) \times BU(n) & \xrightarrow{f} & BU(m+n) \end{array} \tag{74}$$

The map f induces a map on the Thom spaces, and so it can be shown that there is a map of spectra $MU(m) \wedge MU(n) \to MU(m+n)$. This map, in turn, can be used to create a map on the colimits, which gives us the desired map $MU \wedge MU \to MU$. It can also be shown that the canonical map $\mathbb{S} \cong MU(0) \to MU$ is the unit map for this multiplication.

Next, we would like to see that MU has a canonical complex orientation. In fact, this complex orientation will be given by the map $MU(1) \to MU$, which we will call ϕ. From (69), we have that $MU(1) \cong \Sigma^{\infty-2}\mathbb{C}P^\infty$, and hence we see that ϕ is in

$$[MU(1), MU] \cong [\Sigma^{\infty-2}\mathbb{C}P^\infty, MU] \cong [\Sigma^\infty \mathbb{C}P^\infty, MU]_{-2} = \widetilde{MU}^2(\mathbb{C}P^\infty), \tag{75}$$

which is where we need a complex orientation of MU to be. Furthermore, we have

$$[MU(0), MU] \cong [\mathbb{S}, MU] \cong [\Sigma^\infty S^0, MU] \cong [\Sigma^\infty S^2, MU]_{-2} = \widetilde{MU}^2(S^2). \tag{76}$$

It is then evident that the map $\widetilde{MU}^2(\mathbb{C}P^\infty) \to \widetilde{MU}^2(S^2)$ defining the complex orientation is given by composition with the canonical map $MU(0) \to MU(1)$. Given MU's properties as a colimit, we know that the diagram

$$\begin{tikzcd} MU(0) \arrow[rr] \arrow[dr] & & MU(1) \arrow[dl] \\ & MU & \end{tikzcd} \tag{77}$$

is commutative. It follows that $\widetilde{MU}^2(\mathbb{C}P^\infty) \to \widetilde{MU}^2(S^2)$ maps ϕ to the unit map $\mathbb{S} \to MU$. This corresponds to the multiplicative identity in $\pi_*(MU)$, which proves that ϕ is a complex orientation of MU.

Before seeing how complex cobordism relates to other cohomology theories, there is one more aspect of MU that should be mentioned. The coefficient ring of a spectrum carries a lot of information about the corresponding cohomology theory, and we should therefore understand the structure of π_*MU.

Theorem 5.6. *The coefficient ring of MU is*

$$\pi_* MU \cong \mathbb{Z}[x_1, x_2, \ldots], \tag{78}$$

where the generators have degree $|x_i| = 2i$.

This ring is also known as *the Lazard ring*. It has an important universal property in the study of formal group laws, but we will not go into that here. Having established the basic properties of complex cobordism, our next objective is to construct a morphism $MU \to E$ for every complex oriented cohomology theory E. To do this, we first need an alternative way of understanding the Thom space of the universal bundle γ^n.

Lemma 5.7. *The Thom space $Th(\gamma^n)$ of the universal n-plane bundle is homotopy equivalent to $BU(n)/BU(n-1)$ for all $n \geq 1$.*

Proof. In the disc bundle $D(\gamma^n)$ we can contract every fibre in the obvious way to see that $D(E) \sim BU(n)$. It remains to show that $S(E) \sim BU(n-1)$.

Recall that $BU(n)$ can be constructed as $EU(n)/U(n)$, where $EU(n)$ is the space of orthonormal n-frames in $\mathbb{C}^\infty$, and $U(n)$ is the unitary group. We identify $U(n-1)$ with its image in $U(n)$ under the inclusion map. Then $U(n-1)$ also acts on $EU(n)$, and we see that

$$\frac{EU(n)}{U(n-1)} \cong BU(n-1) \times S^\infty \sim BU(n-1), \tag{79}$$

since S^∞ is contractible. We use this to create a fibre bundle over $BU(n)$. There is a fibration $EU(n)/U(n-1) \to EU(n)/U(n-1)$ defined by letting the larger group $U(n)$ act on $EU(n)$. The fibre is given by $U(n)/U(n-1)$, which is known to be homeomorphic to S^{2n-1}. In other words, we have a fibre bundle

$$\frac{U(n)}{U(n-1)} \longrightarrow \frac{EU(n)}{U(n-1)} \longrightarrow \frac{EU(n)}{U(n)} \tag{80}$$

which is isomorphic to

$$S^{2n-1} \longrightarrow BU(n-1) \longrightarrow BU(n). \tag{81}$$

However, this fibre bundle is the same as the sphere bundle $S(\gamma^n)$. This shows that the total space $S(E)$ is homotopy equivalent to $BU(n-1)$. The proof is concluded by seeing that

$$\mathrm{Th}(\gamma^n) = \frac{D(E)}{S(E)} \sim \frac{BU(n)}{BU(n-1)}. \tag{82}$$

□

We now turn our attention to generalised E-cohomology again. In order to construct the maps $MU \to E$, where E is assumed to be complex-oriented, we need a particular cohomology class in $E^{2n}(BU(n), BU(n-1))$. As seen in

Proposition 4.5, we have the cohomology rings $E^*(BU(n)) \cong \pi_* E[\![x_1, \dots, x_n]\!]$ and, naturally, $E^*(BU(n-1)) \cong \pi_* E[\![x_1, \dots, x_{n-1}]\!]$. The homomorphism $E^*(BU(n)) \to E^*(BU(n-1))$ is given by removing the last generator x_n. The homomorphism is therefore surjective. This implies that in the the long exact sequence

$$\cdots \longrightarrow E^{2n}(BU(n), BU(n-1)) \longrightarrow E^{2n}(BU(n)) \longrightarrow E^{2n}(BU(n-1)) \longrightarrow \cdots, \tag{83}$$

the group $E^{2n}(BU(n), BU(n-1))$ is given by the kernel of $E^{2n}(BU(n)) \to E^{2n}(BU(n-1))$. It is easy to see that x_n generates this kernel. In other words,

$$\text{Ker } (\pi_* E[\![x_1, \dots x_n]\!] \to \pi_* E[\![x_1, \dots x_{n-1}]\!]) \cong x_n(\pi_* E)[\![x_1, \dots x_n]\!]. \tag{84}$$

This x_n will now be our choice of distinguished cohomology class. By construction, x_n is in $E^{2n}(BU(n), BU(n-1))$. However, using the Brown representability theorem and Lemma 5.7, we see that

$$\begin{aligned} x_n &\in E^{2n}(BU(n), BU(n-1)) \\ &= \widetilde{E}^{2n}(BU(n)/BU(n-1)) \\ &\cong [\Sigma^\infty BU(n)/BU(n-1), E]_{-2n} \\ &\cong [\Sigma^{\infty-2n}\text{Th}(\gamma^n), E] \\ &= [MU(n), E]. \end{aligned} \tag{85}$$

This means that our cohomology class x_n determines a morphism from $MU(n)$ to E, which we will denote by ϕ_n. Clearly, this construction works for all n. The following theorem shows that these morphisms work particularly nicely in relation to our maps between the spectra of the form $MU(n)$.

Theorem 5.8. *Let E be a complex oriented cohomology theory, and let the morphisms $\{\phi_n\}$ be as defined above. Then the following diagram commutes for all $n \geq 0$.*

$$\begin{tikzcd} MU(n) \arrow[rr] \arrow[dr, "\phi_n"'] & & MU(n+1) \arrow[dl, "\phi_{n+1}"] \\ & E & \end{tikzcd} \tag{86}$$

A proof of this theorem would, again, require a closer look at the multiplicative structure of MU, so we will move on without proving the theorem. A sketch of the proof can be found in [7].

The important consequence of Theorem 5.8 is that, since MU is a colimit, the maps ϕ_n factor through MU. As the following diagram shows, we therefore get a morphism $MU \to E$.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{MU}(n) \arrow[rr] \arrow[dr] \arrow[ddr, bend right] & & \mathrm{MU}(n+1) \arrow[r] \arrow[dl] \arrow[ddl, bend left] & \cdots \\
& & \mathrm{MU} \arrow[d, dashed] & & \\
& & \mathrm{E} & &
\end{tikzcd} \tag{87}$$

Definition 5.9. Let E be a complex oriented cohomology theory. The morphism $MU \to E$, defined as in (87), is called the *Thom map for E*. The induced map on the cohomology rings $MU^*(X) \to E^*(X)$ is called the *Thom homomorphism for E*.

The rest of this thesis will be dedicated to the study of this homomorphism.

6 The Thom Homomorphism

We are now interested in examining the Thom homomorphism from complex cobordism to singular cohomology with coefficients in $\mathbb{Z}$ or $\mathbb{Z}_p$. Since the cohomology groups obtained from complex cobordism are, generally speaking, larger that singular cohomology groups, one might expect that the maps $MU^*(X) \to H^*(X;\mathbb{Z})$ and $MU^*(X) \to H^*(X;\mathbb{Z}_p)$ will always be surjective. However, this is not the case. In this final section we will present several counterexamples. We will use different methods to construct CW-complexes and find cohomology classes which cannot be lifted to complex cobordism. Several tools will be needed to do this, and we start with the Steenrod squares.

6.1 The Steenrod Squares

The Steenrod squares are operations on singular cohomology with $\mathbb{Z}_2$-coefficients. Similar operations exist for mod-p cohomology with p odd, but we will not use those here. The basic idea of these operations is to generalise the cup-product. Firstly, it is evident that by turning the ordinary cup-product

$$H^p(X;\mathbb{Z}_2) \times H^q(X;\mathbb{Z}_2) \to H^{p+q}(X;\mathbb{Z}_2) \tag{88}$$

into a squaring operation, we get a product $H^p(X,\mathbb{Z}_2) \to H^{2p}(X;\mathbb{Z}_2)$. Now, we wish to generalise this so that we do not necessarily end up in degree $2p$.

A detailed construction of these cohomology operations can be found in [9]. Therefore, we leave out the construction and simply present the Steenrod squares in terms of their properties. Before we can state these properties, we need to define the *Bockstein homomorphism.* The short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ 2\ } \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0 \tag{89}$$

induces a long exact sequence of cohomology groups with connecting homomorphism

$$H^n(X;\mathbb{Z}_2) \xrightarrow{\ \beta\ } H^{n+1}(X;\mathbb{Z}). \tag{90}$$

By composing with the homomorphism induced by the reduction map $\mathbb{Z} \to \mathbb{Z}_2$, we get the sequence

$$H^n(X;\mathbb{Z}_2) \xrightarrow{\ \beta\ } H^{n+1}(X;\mathbb{Z}) \xrightarrow{\ r\ } H^{n+1}(X;\mathbb{Z}_2). \tag{91}$$

It is a confusing fact that the term "Bockstein homomorphism" is often used both for the homomorphism (90) and the composition (91), as is the notation β. Here, β will always denote the map $H^n(X;\mathbb{Z}_2) \to H^{n+1}(X;\mathbb{Z})$, while the composition will be denoted by $r \circ \beta$ (or Sq^1, for reasons that will become clear later).

We are now ready to state the properties of the Steenrod squares.

Theorem 6.1. *For all $i \geq 0$ there exist group homomorphisms $H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2)$ for all X such that*

1. *$Sq^i = 0$ if $i > n$*

2. *$Sq^0 = \mathrm{id}$*

3. *Sq^1 is the Bockstein homomorphism $r \circ \beta \colon H^n(X;\mathbb{Z}_2) \to H^{n+1}(X;\mathbb{Z}_2)$*

4. *Sq^n is the cup product square $H^n(X;\mathbb{Z}_2) \to H^{2n}(X;\mathbb{Z}_2)$*

5. *For any continuous map $f \colon X \to Y$, the following diagram commutes:*

$$\begin{tikzcd} H^n(Y;\mathbb{Z}_2) \arrow[r, "Sq^i"] \arrow[d, "f^*"] & H^{n+i}(Y;\mathbb{Z}_2) \arrow[d, "f^*"] \\ H^n(X;\mathbb{Z}_2) \arrow[r, "Sq^i"] & H^{n+i}(X;\mathbb{Z}_2) \end{tikzcd} \tag{92}$$

6. *The Steenrod squares commute with the reduced suspension. In other words, the following diagram commutes.*

$$\begin{tikzcd} H^n(X;\mathbb{Z}_2) \arrow[r, "Sq^i"] \arrow[d, "\Sigma"] & H^{n+i}(X;\mathbb{Z}_2) \arrow[d, "\Sigma"] \\ H^{n+1}(\Sigma X;\mathbb{Z}_2) \arrow[r, "Sq^i"] & H^{n+i+1}(\Sigma X;\mathbb{Z}_2) \end{tikzcd} \tag{93}$$

7. ***(Cartan formula)*** *$Sq^n(xy) = \sum_{i+j=n} Sq^i(x)Sq^j(y)$*

8. ***Adem relations*** *If $a < 2b$, then $Sq^a Sq^b = \sum_c \binom{b-c-1}{a-2c}_{mod\ 2} Sq^{a+b-c}Sq^c$, where we remove the terms where the binomial coefficient is not defined.*

In many cases, we would like to deal with sums and compositions of these operations. This motivates the definition of the Milnor operations.

Definition 6.2. The Milnor operations Q_i for $i \geq 0$ are defined recursively by

$$\begin{aligned} Q_0 &= \mathrm{Sq}^1 \\ Q_n &= \mathrm{Sq}^{2^n} Q_{n-1} + Q_{n-1}\mathrm{Sq}^{2^n}. \end{aligned} \tag{94}$$

Since these will be needed later, we write out the first few Q_i:

$$\begin{aligned} Q_0 &= \mathrm{Sq}^1 \\ Q_1 &= \mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^1\mathrm{Sq}^2 \\ Q_2 &= \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^4\mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^4 + \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^4 \end{aligned} \tag{95}$$

These operations will allow us to understand the properties of certain cohomology classes better. The key lies in another useful cohomology theory.

6.2 Brown-Peterson Cohomology

We now introduce a cohomology theory which will give us a method for constructing nonliftable cohomology classes. One important advantage of Brown-Peterson cohomology is that it retains much of the information from complex cobordism, although it is easier to compute. Recall that the coefficient ring of complex cobordism is given by $\pi_* MU = \mathbb{Z}[x_2, x_4, \ldots]$. We will see that Brown-Peterson cohomology has a similar coefficient ring, but with much fewer generators. Instead of constructing the cohomology theory like we did with complex cobordism, we will simply state the relevant properties of Brown-Peterson cohomology, and refer to [11] for more details.

Brown-Peterson cohomology is of course represented by a spectrum, which is denoted by BP and called the *Brown-Peterson spectrum.* Although there is in fact a separate version of BP-cohomology for every prime p, this is usually omitted from the notation. We will therefore always write BP and make sure to state elsewhere which prime we are dealing with. Recall that $\mathbb{Z}_{(p)}$ denotes the integers localised at the prime p. The coefficient ring of BP is then given by

$$\pi_* BP = \mathbb{Z}_{(p)}[v_1, v_2, \ldots], \tag{96}$$

where the generators are in degree $|v_i| = 2(p^i - 1)$. Although this is already a significant simplification of complex cobordism, we can make the coefficient ring even smaller. By removing all generators higher than v_n, it is clear that there is a quotient map

$$\mathbb{Z}_{(p)}[v_1, v_2, \ldots] \longrightarrow \mathbb{Z}_{(p)}[v_1, \ldots, v_n]. \tag{97}$$

It turns out that this ring homomorphism can be realised as a map of spectra. In that way, we can define cohomology theories $BP\langle n\rangle$ for all natural numbers n with coefficient rings $\pi_* BP\langle n\rangle = \mathbb{Z}_{(p)}[v_1, \ldots, v_n]$. We can then see that it makes sense to think of $BP\langle 0\rangle$ as $H\mathbb{Z}_{(p)}$, singular cohomology with coefficients in $\mathbb{Z}_{(p)}$. Finally, we define $BP\langle -1\rangle$ to be $H\mathbb{Z}_p$. With the obvious maps between these spectra, we have what is called a *tower* of cohomology theories

$$BP \longrightarrow \ldots \longrightarrow BP\langle n\rangle \longrightarrow BP\langle n-1\rangle \longrightarrow \ldots \longrightarrow BP\langle -1\rangle. \tag{98}$$

One reason why this is interesting is that there is a Thom map $BP \to H\mathbb{Z}_p$ similar to the one for MU. This map is the same as the one we get by going through any number of steps in the tower (98). This means that if a cohomology class in $H^*(X; \mathbb{Z}_p)$ cannot be lifted to $BP\langle n\rangle^*(X)$, it cannot be lifted to $BP^*(X)$ either. Furthermore, it is known that the images of MU and BP in $H\mathbb{Z}_p$ coincide, which implies that such a cohomology class cannot be in the image of the Thom map from MU.

Turning our attention back to Brown-Peterson cohomology, there exists a fibre sequence of spectra

$$\Sigma^{|v_n|} BP\langle n\rangle \longrightarrow BP\langle n\rangle \longrightarrow BP\langle n-1\rangle. \tag{99}$$

This sequence induces a long exact sequence in BP-cohomology

$$\begin{tikzcd} \cdots \arrow[r] & BP\langle n\rangle^{k+|v_n|}(X) \arrow[r] & BP\langle n\rangle^{k}(X) \arrow[r] & BP\langle n-1\rangle^{k}(X) \arrow[dll, rounded corners, "\partial"] \\ & BP\langle n\rangle^{k+|v_n|+1}(X) \arrow[r] & BP\langle n\rangle^{k+1}(X) \arrow[r] & \cdots \end{tikzcd} \tag{100}$$

We now take a closer look at the connecting homomorphism ∂. Let $p = 2$. Since there are maps from $BP\langle n\rangle$ and $BP\langle n-1\rangle$ to singular cohomology, we can make the diagram

$$\begin{tikzcd} BP\langle n-1\rangle^{k}(X) \arrow[r, "\partial"] \arrow[d] & BP\langle n\rangle^{k+|v_n|+1}(X) \arrow[d] \\ H^{k}(X;\mathbb{Z}_2) & H^{k+|v_n|+1}(X;\mathbb{Z}_2). \end{tikzcd} \tag{101}$$

A natural question to ask is whether there exists a map between the cohomology groups at the bottom that makes this diagram commute. It turns out this is the n'th Milnor operation Q_n. We add this to the diagram, as well as the previous term of the long exact sequence in BP-cohomology. We also note that v_n is in degree $2(2^n - 1) = 2^{n+1} - 2$. This results in the diagram

$$\begin{tikzcd} BP\langle n\rangle^{k}(X) \arrow[r, "q"] \arrow[dr, dashed] & BP\langle n-1\rangle^{k}(X) \arrow[r, "\partial"] \arrow[d, "\theta_{n-1}"] & BP\langle n\rangle^{k+2^{n+1}-1}(X) \arrow[d, "\theta_n"] \\ & H^{k}(X;\mathbb{Z}_2) \arrow[r, "Q_n"] & H^{k+2^{n+1}-1}(X;\mathbb{Z}_2). \end{tikzcd} \tag{102}$$

We now have a way to recognise cohomology classes that cannot be lifted to Brown-Peterson cohomology.

Lemma 6.3. *Let α be a cohomology class in $H^k(X;\mathbb{Z}_2)$ such that $Q_n(\alpha) \neq 0$. Then α is not in the image of the Thom homomorphism from $BP\langle n\rangle^k(X)$.*

Proof. Assume that α is in the image of the Thom homomorphism, meaning there exists some $\beta \in BP\langle n\rangle^k(X)$ which maps to α. Then, by observing diagram (102), we see that $(Q_n \circ \theta_{n-1} \circ q)(\beta) \neq 0$. However, by commutativity of the square, $(\theta_n \circ \partial \circ q)(\beta) \neq 0$, which is impossible since the top row of the diagram is exact. We therefore have a contradiction, and the result follows. $\square$

As stated earlier, such cohomology classes cannot be in the image of the Thom homomorphism from MU either. This means that the Milnor operations provide an obstruction to lifting cohomology classes to complex cobordism. We now have a useful tool for constructing specific examples.

6.3 Atiyah-Hirzebruch for Connective K-theory

There is one more method which we will use to recognise cases of nonsurjectivity. Rather than looking at the Thom map from MU to $H\mathbb{Z}$ directly, we can instead look at a similar Thom map from connective K-theory to singular cohomology. In fact, the homomorphism from complex cobordism factors through K-theory. This has the fortunate implication that if a cohomology class cannot be lifted to K-theory, it cannot be lifted to cobordism either.

We now need to understand the Thom map from K-theory better. It turns out we can construct this using the Atiyah-Hirzebruch spectral sequence. Let ku denote connective K-theory, which has coefficient ring $\pi_* ku \cong \mathbb{Z}[x]$, with x in degree 2. The reason we use *connective* K-theory is that it has no generators in negative degrees. Consequently, its Atiyah-Hirzebruch spectral sequence has nontrivial terms only in the fourth quadrant. For a CW-complex X, the E_2-page looks like this:

$$
\begin{array}{c|ccccc}
 & 0 & 1 & 2 & 3 & 4 \quad \rightarrow p \\
\hline
0 & H^0(X;\mathbb{Z}) & H^1(X;\mathbb{Z}) & H^2(X;\mathbb{Z}) & H^3(X;\mathbb{Z}) & \cdots \\
\text{-1} & 0 & 0 & 0 & 0 & \cdots \\
\text{-2} & H^0(X;\mathbb{Z}) & H^1(X;\mathbb{Z}) & H^2(X;\mathbb{Z}) & H^3(X;\mathbb{Z}) & \cdots \\
\text{-3} & 0 & \cdots & & & \\
q \downarrow & \vdots & & & &
\end{array}
\tag{103}
$$

In particular, the top row is the singular cohomology of X. We can see that no nontrivial differentials go into these groups, on any page. Therefore, there are well-defined maps

$$E_\infty^{p,0} \longrightarrow E_2^{p,0} \tag{104}$$

known as *edge maps*. Since the E_2-page is singular cohomology and the E_∞-page is K-theory, this can be interpreted as a map $ku^*(X) \to H^*(X;\mathbb{Z})$. This is the Thom map from connective K-theory to singular cohomology. More details can be found in [13].

The key observation to make is that if an element of $H^*(X;\mathbb{Z})$ is such that one of the differentials does not map it to zero, then it does not survive to the E_∞-page. It follows that such an element is not in the image of the Thom homomorphism.

We now need to see how the differentials act on the singular cohomology. From diagram (103), we can see that all the differentials on the second page are trivial. The third page is therefore identical, but with possibly nontrivial differentials. The following lemma gives us a formula for these differentials [4].

Lemma 6.4. *The differentials $d_3 \colon H^n(X;\mathbb{Z}) \to H^{n+3}(X;\mathbb{Z})$ on the E_3-page of the Atiyah-Hirzebruch spectral sequence for connected K-theory are given by $d_3 = \beta \circ Sq^2 \circ r$.*

This means that we have a second method we can use, which is to find integral cohomology classes which are not mapped to zero by the differential d_3.

6.4 Basic Examples

Now that we have established ways to recognise cohomology classes that are not in the image of the Thom homomorphism, we can look at some simple examples. We start with real projective space, $\mathbb{R}P^\infty$. This example is particularly interesting because we can compute the action of the Steenrod squares on the cohomology classes in their entirety.

Recall that the $\mathbb{Z}_2$-cohomology of $\mathbb{R}P^\infty$ is given by

$$H^*(\mathbb{R}P^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha], \tag{105}$$

with the generator α in degree 1. Since we can use the Cartan formula to find out what happens in the higher degrees, it is enough to determine how the Steenrod squares act on α. Using the basic properties of the Steenrod squares, we see that

$$\begin{aligned} \mathrm{Sq}^0(\alpha) &= \alpha \\ \mathrm{Sq}^1(\alpha) &= \alpha^2 \\ \mathrm{Sq}^n(\alpha) &= 0, \quad n \geq 2. \end{aligned} \tag{106}$$

To simplify the calculation, we let Sq denote the sum of all the Steenrod squares $\mathrm{Sq}^0 + \mathrm{Sq}^1 + \ldots$. It is then easy to see that the Cartan formula is equivalent to the fact that Sq is a ring homomorphism. This makes it a simple task to determine what happens to the powers α^n. Since $\mathrm{Sq}(\alpha) = \mathrm{Sq}^0(\alpha) + \mathrm{Sq}^1(\alpha) = \alpha + \alpha^2$, we get

$$\mathrm{Sq}(\alpha^n) = (\mathrm{Sq}(\alpha))^n = (\alpha + \alpha^2)^n = \sum_{i=0}^{n} \binom{n}{i} \alpha^{n-i}(\alpha^2)^i = \sum_{i=0}^{n} \binom{n}{i} \alpha^{n+i}. \tag{107}$$

It clearly follows that

$$\mathrm{Sq}^i(\alpha^n) = \sum_{i=0}^{n} \binom{n}{i} \alpha^{n+i}. \tag{108}$$

This formula does of course not make sense for $i \geq n$. However, by Theorem 6.1, those operations are all 0, so we do not need a formula for those anyway. It is now an easy task to see that

$$\begin{aligned} Q_0(\alpha) &= \mathrm{Sq}^1(\alpha) = \alpha^2 \\ Q_1(\alpha) &= \mathrm{Sq}^2\mathrm{Sq}^1(\alpha) + \mathrm{Sq}^1\mathrm{Sq}^2(\alpha) = \alpha^4 \\ Q_2(\alpha) &= \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1(\alpha) = \alpha^8 \end{aligned} \tag{109}$$

and so on. In general, we get that $Q_i(\alpha) = \alpha^{2^{i+1}}$. Since all the Milnor operations are nontrivial on α, it follows from Lemma 6.3 that α cannot be lifted to any level of the BP-tower, and therefore not to MU either. Note that we could have determined the actions of the Milnor operations on α without using formula (108) since it would have been enough to recognise the cases where the Steenrod squares are equal to the cup product square.

This result may not be surprising. While the mod-2-cohomology of $\mathbb{R}P^\infty$ is nontrivial in all positive degrees, recall that the integral cohomology is given by

$$H^n(\mathbb{R}P^\infty;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & n=0 \\ \mathbb{Z}_2, & n=2,4,\dots \\ 0, & \text{otherwise.} \end{cases} \tag{110}$$

Unlike the mod-2-cohomology, the integral cohomology is trivial in odd degrees, which indicates that $H\mathbb{Z}_2$ contains information about $\mathbb{R}P^\infty$ which cannot be found in "higher" cohomology theories.

We can of course not expect all examples to be as familiar as real projective space. To obtain a higher degree of generality, we wish to find ways to construct CW-complexes with our desired properties. This becomes a question of which attaching maps we can use. We therefore need a better understanding of how maps between spheres affect the Steenrod squares.

Let $f\colon S^{n+i-1} \to S^n$. We can then form the complex

$$K_f = S^n \cup_f e^{n+i}. \tag{111}$$

This complex can only have nontrivial cohomology in degree n and $n+i$. Our question is therefore: When is the Steenrod square $\mathrm{Sq}^i\colon H^n(K_f;\mathbb{Z}_2) \to H^{n+i}(K_f;\mathbb{Z}_2)$ nontrivial? This question was answered by Adams in [1]. Before we state his result, we prove a special case to illustrate the methods used.

Lemma 6.5. *Let K_f be a CW-complex of the form $S^n \cup_f e^{n+i}$, where $i \geq 3$ is an odd number. Then the Steenrod square $Sq^i\colon H^n(K_f;\mathbb{Z}_2) \to H^{n+i}(K_f;\mathbb{Z}_2)$ is trivial.*

Proof. We write $i = 2k+1$. Then there is an Adem relation

$$\mathrm{Sq}^1\mathrm{Sq}^{2k} = \sum_c \binom{2k-c-1}{1-2c} \mathrm{Sq}^{2k+1-c}\mathrm{Sq}^c. \tag{112}$$

This is clearly only defined for $c = 0$, implying

$$\mathrm{Sq}^1\mathrm{Sq}^{2k} = \binom{2k-1}{1} \mathrm{Sq}^{2k+1}\mathrm{Sq}^0 = \mathrm{Sq}^{2k+1}. \tag{113}$$

The Steenrod square in question can therefore be expressed as the composition

$$H^n(K_f;\mathbb{Z}_2) \xrightarrow{\mathrm{Sq}^{i-1}} H^{n+i-1}(K_f;\mathbb{Z}_2) \xrightarrow{\mathrm{Sq}^1} H^{n+i}(K_f;\mathbb{Z}_2). \tag{114}$$

However, K_f does not have any cells in dimension $n+i-1$, which implies that $H^{n+i-1}(K_f;\mathbb{Z}_2) = 0$. It follows that Sq^i is trivial. $\square$

Decompositions similar to (114) exist for all other numbers, except the ones of the form 2^k. In fact, the full result narrows down the possible values of i to a handful of numbers.

Theorem 6.6. *Let K_f be as above. Then the Steenrod square $Sq^i\colon H^n(K_f;\mathbb{Z}_2) \to H^{n+i}(K_f;\mathbb{Z}_2)$ is nontrivial only if i is $1, 2, 4$ or 8.*

For these numbers, it is of course not guaranteed that the Steenrod square will be nonzero; it depends on the attaching map too. Our next goal is to see which maps we can use. We start with the case $i = 1$.

Proposition 6.7. *Let the complex $K_r = S^n \cup_r e^{n+1}$ be given by attaching an $(n+1)$-cell to S^n by a map of degree r. Then $Sq^1\colon H^n(K_r;\mathbb{Z}_2) \to H^{n+1}(K_r;\mathbb{Z}_2)$ is nontrivial if and only if $r \equiv 2 \pmod 4$.*

Proof. First, we note that if r is odd, then we have the cellular cochain complex

$$\begin{array}{ccccccc} & & (n) & & (n+1) & & \\ 0 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow{r} & \mathbb{Z}_2 & \longrightarrow & 0. \end{array} \tag{115}$$

It is easy to see that the map r is an isomorphism. It follows that $H^n(K_r;\mathbb{Z}_2)$ is zero, which makes Sq^1 trivial. Now, let r be even, and write $r = 2k$. In mod-2-cohomology we have the cellular cochain complex

$$\begin{array}{ccccccc} & & (n) & & (n+1) & & \\ 0 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow{2k} & \mathbb{Z}_2 & \longrightarrow & 0, \end{array} \tag{116}$$

and in integral cohomology the complex

$$\begin{array}{ccccccc} & & (n) & & (n+1) & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2k} & \mathbb{Z} & \longrightarrow & 0. \end{array} \tag{117}$$

This gives us the cohomology groups

$$\begin{aligned} H^n(K_{2k};\mathbb{Z}) &\cong 0 & \qquad H^n(K_{2k};\mathbb{Z}_2) &\cong \mathbb{Z}_2 \\ H^{n+1}(K_{2k};\mathbb{Z}) &\cong \mathbb{Z}_{2k} & \qquad H^{n+1}(K_{2k};\mathbb{Z}_2) &\cong \mathbb{Z}_2. \end{aligned}$$

Now, recall that Sq^1 is given by the connecting homomorphism in the long exact sequence induced by

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}_2. \tag{118}$$

In the long exact sequence in cohomology, we now have

$$\begin{array}{ccccc} H^n(K_{2k};\mathbb{Z}) & \longrightarrow & H^n(K_{2k};\mathbb{Z}_2) & \xrightarrow{\beta} & H^{n+1}(K_{2k};\mathbb{Z}) \\ \| & & \| & & \| \\ 0 & & \mathbb{Z}_2 & & \mathbb{Z}_{2k}. \end{array} \tag{119}$$

It is clear that β is an injection which sends the generator of $H^n(K_{2k};\mathbb{Z}_2)$ to the element k in $H^{n+1}(K_{2k};\mathbb{Z})$. Finally, the reduction map

$$H^{n+1}(K_{2k};\mathbb{Z}) \longrightarrow H^{n+1}(K_{2k};\mathbb{Z}_2) \tag{120}$$

maps k to zero if and only if k is even. Since Sq^1 is given by the composition $r \circ \beta$, the proof is finished. □

When constructing CW-complexes, we now have a way to tell if Sq^1 will be trivial or not.

We may now move on to the cases $i = 2, 4, 8$. If we recognise these numbers, it should be no surprise that the relevant maps are the Hopf maps η, ν and σ. We illustrate the situation for η. This is the Hopf invariant one map $S^3 \to S^2$, but by abuse of notation we also let it denote the suspensions $S^{n+1} \to S^n$. Now, recall that complex projective space is constructed using the Hopf map η and its suspensions. In particular, we have $\mathbb{C}P^2 = S^2 \cup_\eta e^4$. By similar computations as the ones for $\mathbb{R}P^\infty$ earlier, or by observing that Sq^2 is the cup product square in degree 2, we see that $\mathrm{Sq}^2 \colon H^2(\mathbb{C}P^2;\mathbb{Z}_2) \to H^4(\mathbb{C}P^2;\mathbb{Z}_2)$ is an isomorphism.

The complex $K_\eta = S^n \cup_\eta e^{n+2}$ is nothing but a repeated suspension of $\mathbb{C}P^2$. Since the Steenrod squares commute with suspensions, we have a commutative diagram

$$\begin{tikzcd} H^2(\mathbb{C}P^2;\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^2"] \arrow[d, "\Sigma^{n-2}"] & H^4(\mathbb{C}P^2;\mathbb{Z}_2) \arrow[d, "\Sigma^{n-2}"] \\ H^n(K_\eta;\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^2"] & H^{n+2}(K_\eta;\mathbb{Z}_2), \end{tikzcd} \tag{121}$$

which proves that Sq^2 is an isomorphism on the cohomology of K_η. Using the quaternions and octonions, it can similarly be shown that constructions using ν and σ also lead to the relevant Steenrod squares being isomorphisms. While these are not strictly speaking the only maps we can use, all the possible maps have some relation to the Hopf maps. The full result, from [12], is as follows.

Proposition 6.8. *Let $i = 2$ (or $4, 8$). Let $f \colon S^{n+i-1} \to S^n$ and $K_f = S^n \cup_f e^{n+i}$ where $n > i$. Then $Sq^i \colon H^n(K_f;\mathbb{Z}_2) \to H^{n+i}(K_f;\mathbb{Z}_2)$ is an isomorphism if and only if $f \equiv \eta$ (or ν, σ) mod $2\pi_{n+i-1}S^n$.*

We are now close to being able to create CW-complexes where certain Milnor operations are nonzero. The problem is that given, for example, a complex of the form $X = S^n \cup_2 e^{n+1}$, we have not yet seen a method for using a Hopf map to attach a cell to the $(n+1)$-cell in X. This motivates the next type of maps we will study.

6.5 Extensions and Coextensions

The two types of maps we will now present were defined by Toda in [12]. They arise from diagrams of the form

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[rr, dashed, bend right, "0"'] & Y \arrow[r, "\alpha"] & Z \end{tikzcd} \tag{122}$$

where the composition $\alpha \circ \beta$ is null-homotopic. There are two maps which can be created from this diagram, and we start with the extension.

Definition 6.9. Let X, Y, Z be topological spaces and α, β continuous maps as in diagram (122). An *extension of* α is a continuous map $\bar{\alpha}\colon Y \cup_\beta CX \to Z$ such that $\bar{\alpha}_{|Y} = \alpha$.

In other words, we are extending the map α so that it is defined on the mapping cone $Y \cup_\beta CX$ rather than just the space Y. It is not difficult to see that such a construction is only possible if $\alpha \circ \beta = 0$. First, to clarify the notation we use, we will describe the cone CX by

$$CX = X \times [0,1] \Big/ X \times \{1\} \cup \{x_0\} \times [0,1], \tag{123}$$

and construct the mapping cone $Y \cup_\beta CX$ by identifying the points $(x, 0)$ with their image in Y. Now, let h be a homotopy between $\alpha \circ \beta$ and the constant map. In other words,

$$h\colon X \times [0,1] \longrightarrow Z, \tag{124}$$

where $h_0 = \alpha \circ \beta$ and h_1 is the constant map. We then construct the extension by

$$\begin{aligned} \bar{\alpha}\colon Y \cup_\beta CX &\longrightarrow Z \\ y &\longmapsto \alpha(y), \qquad && y \in Y \\ (x,t) &\longmapsto h(x,t), \qquad && (x,t) \in X \times [0,1]. \end{aligned} \tag{125}$$

We are essentially using the null-homotopy to create the extension. In fact, such an extension can only exist if the composition $\alpha \circ \beta$ is null-homotopic, which should be easy to see from this construction. We may now define the somewhat similar coextensions.

Definition 6.10. Let (X, Y, Z, α, β) be as in diagram (122). A *coextension of* β is a continuous map $\widetilde{\beta}\colon \Sigma X \to Z \cup_\alpha Y$ such that

$$\begin{aligned} \widetilde{\beta}(x,t) &\in Z && \text{if } t \in [0, 1/2] \\ \widetilde{\beta}(x,t) &= (\beta(x), 2t-1) && \text{if } t \in [1/2, 1]. \end{aligned} \tag{126}$$

This means that we want the bottom half of ΣX to be mapped into Z, and the top half to be mapped into CX by the extension of β to the cones. We can construct this in a similar way to how we constructed the extension. Let h be a null-homotopy of $\alpha \circ \beta$ so that

$$h\colon X \times [0,1] \longrightarrow Z, \tag{127}$$

with $h_0 = \alpha \circ \beta$ and h_1 is constant. We may then give an explicit description of the coextension.

$$\begin{aligned} \widetilde{\beta}\colon \Sigma X &\longrightarrow Z \cup_\alpha Y \\ (x,t) &\longmapsto \begin{cases} h(x, 1-2t), & t \in [0, 1/2] \\ (\beta(x), 2t-1), & t \in [1/2, 1] \end{cases} \end{aligned} \tag{128}$$

This shows that a coextension exists if $(\alpha \circ \beta) \sim 0$. As with the extensions, the converse is also true: If a coextension exists, then $(\alpha \circ \beta) \sim 0$.

Our next goal is to see how these maps interact with the Steenrod squares. Assume that $\alpha \circ \beta$ is null-homotopic in

$$S^{n+i-2} \xrightarrow{\beta} S^{n-1} \xrightarrow{\alpha} X. \tag{129}$$

We may now form a coextension

$$S^{n+i-1} \xrightarrow{\widetilde{\beta}} X \cup_\alpha e^n, \tag{130}$$

Here it is of course important to observe that e^n is the cone of S^{n-1}. By using the coextension to attach an $(n+i+1)$-cell, we get the complex

$$Y = X \cup_\alpha e^n \cup_{\widetilde{\beta}} e^{n+i}. \tag{131}$$

The next lemma [12] reveals that this type of complex will be useful for our purposes.

Lemma 6.11. *Let the CW-complex Y be defined as above. Assume also that β is such that Sq^i acts nontrivially on the CW-complex $S^{n-1} \cup_\beta e^{n+i-1}$. Then $Sq^i\colon H^n(Y;\mathbb{Z}_2) \to H^{n+i}(Y;\mathbb{Z}_2)$ is nonzero. In fact, it maps the cohomology from the cell e^n isomorphically to the cohomology from e^{n+i}.*

It should be noted that our formulation is somewhat informal here. By "the cohomology from e^n" we mean the image of the homomorphism

$$q^*\colon H^n(S^n \cup e^{n+i};\mathbb{Z}_2) \longrightarrow H^n(Y;\mathbb{Z}_2) \tag{132}$$

induced by the map which collapses the subcomplex X.

Proof. By Theorem 6.1, the Steenrod squares commute with homomorphisms induced by continuous maps. In particular, Sq^i commutes with the map q^* as defined above. This leads to the following commutative diagram

$$\begin{array}{ccc} H^n(S^n \cup_{q\circ\widetilde{\beta}} e^{n+i};\mathbb{Z}_2) & \xrightarrow{\mathrm{Sq}^i} & H^{n+i}(S^n \cup_{q\circ\widetilde{\beta}} e^{n+i};\mathbb{Z}_2) \\ \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} \\ H^n(Y;\mathbb{Z}_2) & \xrightarrow{\mathrm{Sq}^i} & H^{n+i}(Y;\mathbb{Z}_2). \end{array} \tag{133}$$

We can see that in the space $S^n \cup_{q\circ\widetilde{\beta}} e^{n+i}$, the $(n{+}i)$-cell is attached by the map $q \circ \widetilde{\beta}$, which is by definition in the same homotopy class as $\Sigma\beta$. Since the Steenrod squares commute with suspensions, this implies that the squaring operation in the top row of the diagram is an isomorphism. Furthermore, the q^* to the right in the diagram is clearly an injection. A close look at the diagram then proves the claim. □

Alternatively, we can do the same using extensions. The map α extends to

$$S^n \cup_\beta e^{n+i} \xrightarrow{\bar{\alpha}} S^{n-k}. \tag{134}$$

Again, this can be used as an attaching map, resulting in the complex

$$X = S^{n-k} \cup_{\bar{\alpha}} e^{n+1} \cup_\beta e^{n+i+1}. \tag{135}$$

It should be easy to see that this is precisely the same complex as the one made using coextensions. This means that we can also use extensions to construct complexes where the Steenrod squares are nonzero. In [3], Conner and Smith combine both methods to construct examples, and we will now present one of these complexes.

We start with the maps

$$S^{n+4} \xrightarrow{\nu} S^{n+1} \xrightarrow{\eta} S^n \xrightarrow{2} S^n \tag{136}$$

where η and ν are Hopf maps, and 2 denotes a map of degree two. We will also assume that n is sufficiently large that all the homotopy groups are stable. The composition $\eta \circ \nu$ is in the fourth stable homotopy group, which is known to be trivial [12]. In other words, $\eta \circ \nu$ is null-homotopic. Furthermore, η generates the first stable homotopy group which is isomorphic to $\mathbb{Z}_2$. Therefore, the composition $2 \circ \eta$ is also null-homotopic. We then have a coextension of ν:

$$S^{n+5} \xrightarrow{\widetilde{\nu}} S^n \cup_\eta e^{n+2}. \tag{137}$$

We can also make an extension of 2:

$$S^n \cup_\eta e^{n+2} \xrightarrow{\bar{2}} S^n \tag{138}$$

Combining these two, we get the composition

$$S^{n+5} \xrightarrow{\widetilde{\nu}} S^n \cup_\eta e^{n+2} \xrightarrow{\bar{2}} S^n. \tag{139}$$

This composition is in the fifth stable homotopy group of spheres, which is also trivial. The map $\bar{2}$ therefore extends to

$$S^n \cup_\eta e^{n+2} \cup_{\widetilde{\nu}} e^{n+6} \xrightarrow{\bar{2}} S^n. \tag{140}$$

Finally, we use this map to assemble the complex

$$X = S^n \cup_2 C(S^n \cup_\eta e^{n+2} \cup_{\widetilde{\nu}} e^{n+6}) = S^n \cup_2 e^{n+1} \cup_\eta e^{n+3} \cup_{\widetilde{\nu}} e^{n+7}. \tag{141}$$

We will now see that this space has the properties we were looking for. Firstly, it is easily seen that the cohomology is given by

$$\widetilde{H}^k(X;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & k = n, n+1, n+3, n+7 \\ 0, & \text{otherwise.} \end{cases} \tag{142}$$

By Proposition 6.7, Sq^1 maps the generator of the n'th cohomology to the generator of $H^{n+1}(X;\mathbb{Z}_2)$. Furthermore, by Theorem 6.6 and Lemma 6.11, we know that

$$\begin{aligned} \mathrm{Sq}^2 &\colon H^{n+1}(X;\mathbb{Z}_2) \longrightarrow H^{n+3}(X;\mathbb{Z}_2), \\ \mathrm{Sq}^4 &\colon H^{n+3}(X;\mathbb{Z}_2) \longrightarrow H^{n+7}(X;\mathbb{Z}_2) \end{aligned} \tag{143}$$

are isomorphisms, so the composition $\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1 \colon H^n(X;\mathbb{Z}_2) \to H^{n+7}(X;\mathbb{Z}_2)$ is also an isomorphism. All the other terms of the second Milnor operation (see (95)) factor through trivial cohomology groups. We can therefore conclude that Q_2 is nonzero on $H^n(X;\mathbb{Z}_2)$. Then, by Lemma 6.3, the Thom homomorphism is not surjective.

Although Conner and Smith do not mention this in their article, it is possible to reverse their process to obtain a slightly different CW-complex. We start with the diagram

$$S^{n+4} \xrightarrow{2} S^{n+4} \xrightarrow{\eta} S^{n+3} \xrightarrow{\nu} S^n. \tag{144}$$

Using the same method as before, we obtain the complex

$$X = S^n \cup_{\widetilde{\nu}} e^{n+4} \cup_\eta e^{n+6} \cup_{\widetilde{2}} e^{n+7}. \tag{145}$$

Here, $\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^4$ acts nontrivially on the n'th cohomology of X, again showing that the Thom homomorphism is not surjective. However, in this case we can also use the Atiyah-Hirzebruch spectral sequence for K-theory. We look at the group $H^{n+4}(X;\mathbb{Z})$ and want to determine how the third-page differential acts on it. As stated earlier, this is given by $d_3 = \beta \circ \mathrm{Sq}^2 \circ r$. We get the diagram

$$\begin{tikzcd} H^{n+4}(X;\mathbb{Z}) \arrow[r, "r"] & H^{n+4}(X;\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^2"] & H^{n+6}(X;\mathbb{Z}_2) \arrow[r, "\beta"] \arrow[dr, "\mathrm{Sq}^1"'] & H^{n+7}(X;\mathbb{Z}) \arrow[d, "r"] \\ & & & H^{n+7}(X;\mathbb{Z}_2) \end{tikzcd} \tag{146}$$

Here, all the maps are clearly nonzero, and the generator of $H^{n+4}(X;\mathbb{Z})$ is carried through to $H^{n+7}(X;\mathbb{Z})$. Therefore, this element does not make it to the fourth page of the spectral sequence and cannot be in the image of the Thom homomorphism from K-theory.

It is also shown in [3] that this example can be taken one step further, so that Q_3 is nonzero. To do this, we need to compute a specific homotopy group. We will therefore briefly state some basic properties which we will use. Firstly, recall that a pair of spaces (X, A) is called *n-connected* if all the relative homotopy groups $\pi_i(X, A)$ are trivial for $i \leq n$. We then have the following two lemmas.

Lemma 6.12. *Let X be a pointed space, and let $A \subset X$ be a subspace such that the basepoint x_0 is in A. Then there is a long exact sequence*

$$\begin{tikzcd} \cdots \arrow[r] & \pi_n(A) \arrow[r] & \pi_n(X) \arrow[r] & \pi_n(X,A) \arrow[dll, rounded corners] & \\ & \pi_{n-1}(A) \arrow[r] & \pi_{n-1}(X) \arrow[r] & \pi_{n-1}(X,A) \arrow[r] & \cdots . \end{tikzcd} \tag{147}$$

Lemma 6.13. *Let $A \subset X$ be spaces such that A is m-connected and (X, A) is n-connected. Then there is an isomorphism $\pi_i(X, A) \cong \pi_i(X/A)$ for all $i \leq m + n$.*

Combining these, we see that if our spaces are sufficiently highly connected, we have a long exact sequence where we can replace the relative homotopy groups with the homotopy groups of the quotient space X/A. More details about the homomorphisms that make up the long exact sequence can be found in [5].

We now return to Conner and Smith's examples. We start out in a similar fashion as before, but this time we also use the third Hopf map σ. This gives us the diagram

$$S^{n+11} \xrightarrow{\sigma} S^{n+4} \xrightarrow{\nu} S^{n+1} \xrightarrow{\eta} S^n \xrightarrow{2} S^n. \tag{148}$$

As before, the compositions $\eta \circ \nu$ and $2 \circ \nu$ are zero. Forming an extension of 2 and a coextension of ν, we get

$$S^{n+12} \xrightarrow{\sigma} S^{n+5} \xrightarrow{\widetilde{\nu}} S^n \cup_\eta e^{n+2} \xrightarrow{\bar{2}} S^n. \tag{149}$$

By the same argument as before, $\bar{2} \circ \widetilde{\nu}$ is null-homotopic. In order to proceed, we also need the composition $\widetilde{\nu} \circ \sigma$ to be null-homotopic, but this requires more work to show. The possible homotopy classes this map could belong to are given by $\pi_{n+12}(S^n \cup_\eta e^{n+2})$. We use the methods mentioned above to compute this homotopy group. Since we are working in the stable range, the conditions of Lemma 6.13 are fulfilled. We therefore have

$$\pi_k(S^n \cup_\eta e^{n+12}, S^n) \cong \pi_k(S^{n+2}) \quad \forall k, \tag{150}$$

and we get a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{n+12}(S^n) \arrow[r] & \pi_{n+12}(S^n \cup_\eta e^{n+2}) \arrow[r] & \pi_{n+12}(S^{n+2}) \arrow[dll, "\eta", rounded corners] \\
& \pi_{n+11}(S^n) \arrow[r] & \cdots .
\end{tikzcd} \tag{151}$$

Here we can see that the connecting homomorphism is η since that is the map which attaches the $(n+2)$-cell to the rest of the complex. Most of these groups are just stable homotopy groups of spheres, which have already been computed in low degrees. From [12] we get that

$$\begin{aligned}
\pi_{n+12}(S^n) &\cong 0 \\
\pi_{n+12}(S^{n+2}) &\cong \mathbb{Z}_2(\eta \circ \mu) + \mathbb{Z}_3\beta_1 \\
\pi_{n+11}(S^n) &\cong \mathbb{Z}_8\zeta + \mathbb{Z}_9\alpha_3' + \mathbb{Z}_7\alpha_{1,7}.
\end{aligned} \tag{152}$$

We do not need to understand all of these generators, but we will note that we have the relations

$$\begin{aligned}
\eta^2 \circ \mu &= \zeta \\
\eta \circ \beta_1 &= 0.
\end{aligned} \tag{153}$$

From the long exact sequence we can then see that

$$\pi_{n+12}(S^n \cup_\eta e^{n+2}) \cong \mathrm{Ker}\ (\pi_{n+12}(S^{n+2}) \longrightarrow \pi_{n+11}(S^n)) \cong \mathbb{Z}_3. \tag{154}$$

This shows that the composition $\widetilde{\nu} \circ \sigma$ is in a homotopy group isomorphic to $\mathbb{Z}_3$. However, ν and σ are generators of groups of order 8 and 16, respectively, and therefore $\widetilde{\nu} \circ \sigma$ cannot possibly be a generator of a group of order 3. In other words, $\widetilde{\nu} \circ \sigma$ must be homotopic to the constant map. Returning to diagram (149), we form a coextension of σ and an extension of $\bar{2}$ to get

$$S^{n+13} \xrightarrow{\widetilde{\sigma}} S^n \cup_\eta e^{n+2} \cup_{\widetilde{\nu}} e^{n+6} \xrightarrow{\bar{2}} S^n. \tag{155}$$

At this point, our notation gets difficult if we are to write all the bars and tildes to signify extensions of extensions, etc. We will therefore omit these when necessary and trust the reader to understand the maps based on their context. Finally, we see that the composition $\bar{2} \circ \widetilde{\sigma}$ is in the 13th stable homotopy group of spheres, which by [12] is isomorphic to $\mathbb{Z}_3$. However, our composition is clearly divisible by 2 and must therefore be 0. We may then form the coextension

$$S^{n+14} \xrightarrow{\widetilde{\sigma}} S^n \cup_{\bar{2}} e^{n+1} \cup_\eta e^{n+3} \cup_{\widetilde{\nu}} e^{n+7} \tag{156}$$

and form the complex

$$Y = S^n \cup_{\bar{2}} e^{n+1} \cup_\eta e^{n+3} \cup_{\widetilde{\nu}} e^{n+7} \cup_{\widetilde{\sigma}} e^{n+15}. \tag{157}$$

By the same argument as before, we can see that $\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1 \colon H^n(Y;\mathbb{Z}_2) \to H^{n+15}(Y;\mathbb{Z}_2)$ is an isomorphism, while all the other terms of Q_3 factor through trivial cohomology groups. This shows that Q_3 is in fact nonzero on the cohomology of Y.

6.6 Eilenberg-MacLane Spaces

In this final section we will examine how the Thom homomorphism acts on the cohomology of Eilenberg-MacLane spaces. We will use the technique outlined in section 3.3 to construct these spaces, and we will see how this relates to our methods for proving nonsurjectivity. As it turns out, these methods will be enough to find a sufficient criterion for certain Eilenberg-MacLane spaces to have nonliftable cohomology classes. To our knowledge, this is an original proof. We will then contextualise this result by presenting a stronger theorem due to Tamanoi [10], and seeing how our result relates to this.

We begin with an examination of the spaces $K(\mathbb{Z}, n)$. It is to be expected that these spaces will have different properties depending on the number n. For $n = 1$, we can easily see that S^1 is a $K(\mathbb{Z}, 1)$-space. Since this is a one-dimensional complex, it cannot have any nontrivial Steenrod squares. Moving on to the next case, it is known that $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$-space. From Proposition 4.4, we know that $\mathbb{C}P^\infty$ has cohomology only in even degrees. It follows that Sq^1 can never be nonzero, and our methods can take us no further.

The case where $n \geq 3$ is more interesting. To construct this, we simply start with S^n, since this clearly has the right n'th homotopy group. Then, we must determine the next homotopy group and attach $(n+2)$-cells accordingly. We know that for $n \geq 3$ we have $\pi_{n+1}(S^n) \cong \mathbb{Z}_2$, generated by η. We attach an $(n+2)$-cell using this map and get the CW-complex

$$X = S^n \cup_\eta e^{n+2}. \tag{158}$$

The next step is to determine the homotopy groups of this complex. To compute these, we must first know how connected the pair (X, S^n) is. We have a long exact sequence:

$$\begin{tikzcd} \cdots \arrow[r] & \pi_{n+1}(S^n) \arrow[r, "\cong"] & \pi_{n+1}(X) \arrow[r] & \pi_{n+1}(X, S^n) \arrow[dll, rounded corners] & \\ & \pi_n(S^n) \arrow[r, "\cong"] & \pi_n(X) \arrow[r] & \pi_n(X, S^n) \arrow[r] & 0. \end{tikzcd} \tag{159}$$

Here, the maps $\pi_n(S^n) \to \pi_n(X)$ and $\pi_{n+1}(S^n) \to \pi_{n+1}(X)$ are clearly isomorphisms, since the $(n+2)$-cell of X does not affect the homotopy groups lower than $n+2$. This is because by cellular approximation, we may assume that all maps are cellular. Therefore, it is enough to look at the n-skeleton or $(n+1)$-skeleton, respectively. It follows that the groups $\pi_n(X, S^n)$ and $\pi_{n+1}(X, S^n)$ are trivial. The same is obviously true for all lower homotopy groups. We can conclude that (X, S^n) is $(n+1)$-connected.

Furthermore, S^n is $(n-1)$-connected. By Lemma 6.13, we then have that $\pi_i(X, S^n) \cong \pi_i(X/S^n)$ for all $i \leq 2n$. Observing that $X/S^n \cong S^{n+2}$, we get a long exact sequence involving S^n, X and S^{n+2}. To find $\pi_{n+2}(X)$, we look at the relevant part of the sequence and fill in the homotopy groups that are known, as well as their generators. This yields the sequence

$$\begin{array}{ccccccccc}
\pi_{n+3}(S^{n+2}) & \xrightarrow{\eta} & \pi_{n+2}(S^n) & \longrightarrow & \pi_{n+2}(X) & \longrightarrow & \pi_{n+2}(S^{n+2}) & \xrightarrow{\eta} & \pi_{n+1}(S^n) \\
\| & & \| & & & & \| & & \| \\
\mathbb{Z}_2 & & \mathbb{Z}_2 & & & & \mathbb{Z} & & \mathbb{Z}_2 \\
 & & & & & & & & \\
\langle\eta\rangle & & \langle\eta^2\rangle & & & & \langle 1\rangle & & \langle\eta\rangle
\end{array} \tag{160}$$

Since we know that the connecting homomorphism is given by η, it is easy to see that

$$\pi_{n+2}(X) \cong \mathrm{Ker}(\pi_{n+2}(S^{n+2}) \longrightarrow \pi_{n+1}(S^n)) \cong \mathbb{Z}. \tag{161}$$

This kernel is clearly generated by a map of degree 2 on S^{n+2}. By pulling this map back to π_{n+2} we get the map which results from taking a coextension of 2 in the diagram

$$S^{n+1} \xrightarrow{2} S^{n+1} \xrightarrow{\eta} S^n. \tag{162}$$

To remove the $(n+2)$nd homotopy group of our complex, we must attach an $(n+3)$-cell using this map, which results in the complex

$$Y = S^n \cup_\eta e^{n+2} \cup_{\tilde{2}} e^{n+3}. \tag{163}$$

We have of course seen spaces of this type before. By the same arguments as before, we know that

$$\begin{aligned} Q_1 &\colon H^n(Y;\mathbb{Z}_2) \longrightarrow H^{n+3}(Y;\mathbb{Z}_2) \quad \text{and} \\ d_3 &\colon H^n(Y;\mathbb{Z}) \longrightarrow H^{n+3}(Y;\mathbb{Z}) \end{aligned} \tag{164}$$

are both isomorphisms, implying that the generators of $H^n(Y;\mathbb{Z}_2)$ and $H^n(Y;\mathbb{Z})$ cannot be lifted to $BP\langle 1\rangle$ and ku, respectively. Although it may seem like we are repeating ourselves by constructing more complexes of the type examined by Conner and Smith, it is interesting to see how similar complexes occur more "naturally" as Eilenberg-MacLane spaces.

Of course, we would like to understand the whole space $K(\mathbb{Z}, n)$, rather than just the $(n+3)$-skeleton. However, it turns out these higher cells are irrelevant. Through a similar, but long-winded, calculation, we can see that the $(n+4)$-cells will be attached directly to the n-cell of Y. It follows that these cells cannot affect the $(n+3)$rd cohomology. The same results are therefore valid for $K(\mathbb{Z}, n)$.

We now turn our attention to spaces of the form $K(\mathbb{Z}_2, n)$. Once again, we let $n \geq 3$. To construct such a space, we start with a free resolution of $\mathbb{Z}_2$:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0. \tag{165}$$

This leads to the complex

$$X = S^n \cup_2 e^{n+1}. \tag{166}$$

We can easily observe that S^n is $(n-1)$-connected, while (X, S^n) is n-connected. By Lemma 6.13, we know that $\pi_i(X, S^n) \cong \pi_i(X/S^n)$ for all $i \leq 2n-1 \leq n+2$, since $n \geq 3$. We will now find $\pi_{n+1}(X)$ by looking at another long exact sequence of homotopy groups:

$$\begin{tikzcd}
\pi_{n+2}(S^{n+1}) \arrow[r, "2"] \arrow[d, equal] & \pi_{n+1}(S^n) \arrow[r] \arrow[d, equal] & \pi_{n+1}(X) \arrow[r] & \pi_{n+1}(S^{n+1}) \arrow[r, "2"] \arrow[d, equal] & \pi_n(S^n) \arrow[d, equal] \\
\mathbb{Z}_2 & \mathbb{Z}_2 & & \mathbb{Z} & \mathbb{Z} \\
\langle \eta \rangle & \langle \eta \rangle & & \langle 1 \rangle & \langle 1 \rangle
\end{tikzcd} \tag{167}$$

It is easy to see that the desired homotopy group is $\mathbb{Z}_2$, generated by η. This gives us the complex

$$Y = S^n \cup_2 e^{n+1} \cup_\eta e^{n+2}. \tag{168}$$

Before moving on to the homotopy groups of Y, we will also need to find $\pi_{n+2}(X)$. The relevant part of the long exact sequence is

$$\begin{tikzcd}
\pi_{n+2}(S^n) \arrow[r] \arrow[d, equal] & \pi_{n+2}(X) \arrow[r, two heads] & \pi_{n+2}(S^{n+1}) \arrow[r, "0"] \arrow[d, equal] \arrow[l, dashed, bend right] & \cdots \\
\mathbb{Z}_2 & & \mathbb{Z}_2 & \\
\langle \eta^2 \rangle & & \langle \eta \rangle &
\end{tikzcd} \tag{169}$$

An important question to ask now is whether there exists a splitting as indicated with the dashed arrow above. Such a splitting would have to be induced by a map $S^{n+1} \to X$. However, we know which possible maps this could be, since we have already computed $\pi_{n+1}(X) \cong \mathbb{Z}_2$. The generator of this group is the homotopy class of η, which maps all of S^{n+1} into the n-cell of X. Therefore, the composition

$$S^{n+1} \xrightarrow{\eta} X \longrightarrow X/S^n \tag{170}$$

is zero. Therefore, there is no splitting that composes to the identity on $\pi_{n+2}(S^{n+1})$. We can then see that the map $\pi_{n+2}(S^n) \to \pi_{n+2}(X)$ cannot be zero, as this would force the isomorphism $\pi_{n+2}(X) \cong \pi_{n+2}(S^{n+1})$. Since a map from $\mathbb{Z}_2$ must either be zero or an injection, it must be injective. A consequence of this is that $\pi_{n+2}(X)$ fits into the following short exact sequence:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \pi_{n+2}(X) \longrightarrow \mathbb{Z}_2 \longrightarrow 0. \tag{171}$$

We know that this sequence does not split, and it follows that $\pi_{n+2}(X) \cong \mathbb{Z}_4$. We also want to find a generator. The coextension $\widetilde{\eta}\colon S^{n+2} \to S^n \cup_2 e^{n+1}$ clearly belongs to this group. It is not null-homotopic and not the image of the nontrivial class in $\pi_{n+2}(S^n)$. The group is therefore generated by $\widetilde{\eta}$.

We now turn our attention back to $Y = S^n \cup_2 e^{n+1} \cup_\eta e^{n+2}$, since we need to find its $(n{+}3)$rd homotopy group. Observing that $Y/X \cong S^{n+2}$, we get a long exact sequence

$$\begin{array}{ccccccccc}
\pi_{n+3}(S^{n+2}) & \xrightarrow{\eta} & \pi_{n+2}(X) & \longrightarrow & \pi_{n+2}(Y) & \longrightarrow & \pi_{n+2}(S^{n+2}) & \xrightarrow{\eta} & \pi_{n+1}(X) \\
\| & & \| & & & & \| & & \| \\
\mathbb{Z} & & \mathbb{Z}_4 & & & & \mathbb{Z} & & \mathbb{Z}_2 \\
 & & & & & & & & \\
\langle \eta \rangle & & \langle \widetilde{\eta} \rangle & & & & \langle 1 \rangle & & \langle \eta \rangle
\end{array} \tag{172}$$

It follows that $\pi_{n+2}(Y) \cong \mathbb{Z}_2 \times \mathbb{Z}$, and that it is generated by the pair $(\widetilde{\eta}, \widetilde{2})$. We must therefore attach two $(n{+}3)$-cells, using each of the generators. The map $\widetilde{\eta}$ attaches a cell to the subcomplex $S^n \cup_2 e^{n+1}$, while $\widetilde{2}$ attaches a cell to $S^n \cup_\eta e^{n+2}$. The resulting complex Z is given by taking

$$S^n \cup_2 e^{n+1} \cup_{\widetilde{\eta}} e^{n+2} \coprod S^n \cup_\eta e^{n+2} \cup_{\widetilde{2}} e^{n+3} \tag{173}$$

and identifying the two n-cells with each other in the obvious way. The interesting thing to note about this complex, and in fact $K(\mathbb{Z}_2, n)$, is that both of the terms in the first Milnor operation $Q_1 = \mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^1\mathrm{Sq}^2$ are nonzero on its cohomology. We can then see that the nonzero class in $H^n(Z; \mathbb{Z}_2)$ does not lift to Brown-Peterson cohomology. The integral cohomology $H^n(Z; \mathbb{Z})$ is trivial, so we cannot use our method for connected K-theory.

We are also interested in the spaces $K(\mathbb{Z}_k, n)$, where k is any integer, not just 2. First, let k be an even number and $n \geq 3$. Most of the work here is already done, since these spaces look very similar to $K(\mathbb{Z}_2, n)$, at least in low dimensional cells. If we go through the construction of $K(\mathbb{Z}_2, n)$ again, we can see that nothing changes, except for the degree of the initial attaching map. The $(n{+}3)$-skeleton of $K(\mathbb{Z}_k, n)$ is therefore given by

$$S^n \cup_k e^{n+1} \cup_{\widetilde{\eta}} e^{n+2} \coprod S^n \cup_\eta e^{n+2} \cup_{\widetilde{2}} e^{n+3} \tag{174}$$

where we as before identify the n-cells with each other. Here, this composition $\mathrm{Sq}^1\mathrm{Sq}^2$ is clearly nonzero, while $\mathrm{Sq}^2\mathrm{Sq}^1$ will depend on the value k. Recalling Lemma 6.7, we see that $\mathrm{Sq}^2\mathrm{Sq}^1$ is also nonzero when $k \equiv 2 \pmod 4$ and zero for $k \equiv 0 \pmod 4$. Either way, Q_1 is nonzero, and we have an obstruction to lifting the n'th mod-2 cohomology to Brown-Peterson cohomology.

The case where k is odd is quickly dealt with, at least as long as we only focus on the cohomology in degree n. The construction begins with the complex

$$S^n \cup_k e^{n+1}, \tag{175}$$

and all the other cells will have dimension $n{+}2$ or greater. It is then easy to see that $H^n(K(\mathbb{Z}_k, n); \mathbb{Z}_2) = 0$ when k is odd, so we have no nontrivial cohomology classes to lift.

We summarise what we have done so far in a proposition.

Proposition 6.14. *Let n, k be integers such that $n \geq 3$ and k is even. We then have the following:*

1. *The fundamental class of $H^n(K(\mathbb{Z}, n); \mathbb{Z})$ does not lift to ku.*
2. *The nontrivial class in $H^n(K(\mathbb{Z}, n); \mathbb{Z}_2)$ does not lift to BP.*
3. *The nontrivial class in $H^n(K(\mathbb{Z}_k, n); \mathbb{Z}_2)$ does not lift to BP.*

It turns out that we can say a lot more about Eilenberg-MacLane spaces using only what we know about the spaces above. The important tool that allows us to generalise these results to more general cases, is the following lemma, from [5].

Lemma 6.15. *Let G and H be abelian groups, and let $n \geq 1$ be an integer. Then $K(G, n) \times K(H, n)$ is a $K(G \times H, n)$-space.*

This is a very useful tool since it allows us to break down many Eilenberg-MacLane spaces into smaller parts. We now have everything we need to state our main result.

Theorem 6.16. *Let G be an abelian group, and let n, k be integers such that $n \geq 3$ and k is even. Let $X = K(G \times \mathbb{Z}, n)$ and $Y = K(G \times \mathbb{Z}_k, n)$. Then the following Thom homomorphisms are not surjective:*

$$\begin{aligned} &\text{1. } ku^n(X) \longrightarrow H^n(X; \mathbb{Z}) \\ &\text{2. } BP^n(X) \longrightarrow H^n(X; \mathbb{Z}_2) \\ &\text{3. } BP^n(Y) \longrightarrow H^n(Y; \mathbb{Z}_2) \end{aligned} \tag{176}$$

Proof. We will only prove the first map is nonsurjective, since the other cases use the exact same method. By Lemma 6.15, we can construct the space X as $K(G, n) \times K(\mathbb{Z}, n)$. Since we know by Proposition 6.14 that the Thom map for $K(\mathbb{Z}, n)$ is nonsurjective, we just need to show that taking the product with $K(G, n)$ does not affect this.

To do this, we take a closer look at the cell-structure of X. We can think of $K(\mathbb{Z}, n)$ as having one 0-cell, one n-cell, one $(n + 2)$-cell, etc. The space $K(G, n)$ has one 0-cell, and an unknown number of cells in dimension n and higher. Recall now that the structure of a product of CW-complexes works as follows: For every pair of an r-cell in $K(G, n)$ and s-cell in $K(\mathbb{Z}, n)$, we get an $(r + s)$-cell in the product, with characteristic map given by the product of the characteristic maps of each cell. Since we only have one 0-cell in each space, it follows that the wedge sum

$$W = K(G, n) \vee K(\mathbb{Z}, n) \tag{177}$$

is a subcomplex of X. Using Proposition 6.14, it is now easy to see that in W, the fundamental class of $K(\mathbb{Z}, n)$ is not in the image of the Thom map from ku. It remains to show that the rest of X does not change this.

The cells in X which are not in W all have dimension $2n$ or higher. It is easy to see that if n is sufficiently large, they will not change the relevant cohomology groups. We will therefore focus on the case $n = 3$. To show nonsurjectivity using the previous methods, we only use the cohomology up to degree $n + 3 = 6$. It follows that we only need to understand the 7-skeleton of X. Here we can see that the 6-cells of $X \backslash W$ are only attached to the 3-cells of X. They will therefore not affect the relevant 6-cell. Likewise, the 7-cells are attached to cells of dimension 3 and 4, and it follows that they cannot change the 6'th cohomology of X. This concludes the proof for the first Thom homomorphism, and the other ones can be proved by the same method. □

We now have a good understanding of how the Thom map acts on the cohomology of Eilenberg-MacLane spaces of groups of the form $\mathbb{Z}$ or $\mathbb{Z}_k$. In addition, we have seen what happens if we take the product of one of these groups with some group G. This means that we have a complete understanding of what happens if we only use the groups $\mathbb{Z}$ and $\mathbb{Z}_k$. These are of course the finitely generated abelian groups.

Corollary 6.17. *Let G be a finitely generated abelian group, and let $n \geq 3$. Then the Thom map*

$$BP^n(K(G,n)) \longrightarrow H^n(K(G,n);\mathbb{Z}_2) \tag{178}$$

is surjective if and only if G is of the form

$$G \cong \mathbb{Z}_{p_1^{r_1}} \oplus \cdots \oplus \mathbb{Z}_{p_s^{r_s}}, \tag{179}$$

where all the primes p_i are odd.

Proof. If G is of the form (179), then its n'th cohomology with coefficients in $\mathbb{Z}_2$ is 0. This follows from Lemma 6.15 and the observation that $H^n(K(\mathbb{Z}_k,n);\mathbb{Z}_2) = 0$ when k is odd. Then the Thom homomorphism is trivially surjective. The case where G has a summand $\mathbb{Z}$ or $\mathbb{Z}_{2^k}$ follows from Theorem 6.16. □

When we are dealing with finitely generated abelian groups, we have therefore got a complete result about the n'th (mod 2)-cohomology of $K(G,n)$. When it comes to the higher cohomology groups, our methods could not prove that the Thom homomorphism is nonsurjective. That does not mean that the Thom homomorphism is necessarily surjective, only that our methods are insufficient to prove otherwise.

A more comprehensive statement about the image of the Thom homomorphism from BP for Eilenberg-MacLane spaces can be found in [10]. In this article, Tamanoi proves that there is an expression for the image for spaces $K(G,n)$, where G is a finitely generated $\mathbb{Z}_{(p)}$-module. To state his result, we must first define the fundamental class of $H^n(K(\mathbb{Z}_{(p)},n);\mathbb{Z}_p)$. As in the case for $K(\mathbb{Z},n)$, we can use the universal coefficient theorem. Letting $X = K(\mathbb{Z}_{(p)},n)$, we get the short exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X),\mathbb{Z}_p) \longrightarrow H^n(X;\mathbb{Z}_p) \longrightarrow \mathrm{Hom}(H_n(X),\mathbb{Z}_p) \longrightarrow 0 \tag{180}$$

Since $H_{n-1}(X) = 0$, we get an isomorphism

$$H^n(X; \mathbb{Z}_p) \cong \operatorname{Hom}(H_n(X), \mathbb{Z}_p) \cong \operatorname{Hom}(\mathbb{Z}_{(p)}, \mathbb{Z}_p). \tag{181}$$

Now, there is a canonical map from $\mathbb{Z}_{(p)}$ to $\mathbb{Z}_p$ given by reduction modulo p. This leads to our definition.

Definition 6.18. The *fundamental class of* $H^n(K(\mathbb{Z}_{(p)}, n); \mathbb{Z}_p)$ is the cohomology class corresponding to the canonical map $\mathbb{Z}_{(p)} \to \mathbb{Z}_p$ under the isomorphism (181) above.

Although Tamanoi provides an answer for all $\mathbb{Z}_{(p)}$-modules, we will not repeat the entire result here. We will however take a closer look at the case that "overlaps" with Theorem 6.16 and Corollary 6.17.

Theorem 6.19. *Let $n \geq 2$ and $j \geq 1$ be integers, and let p be prime. Let BP be the Brown-Peterson cohomology for the prime p, and let $\iota \in H^n(K(\mathbb{Z}/p^j, n); \mathbb{Z}_p)$ be the fundamental class. Then the image of the Thom homomorphism*

$$BP^*(K(\mathbb{Z}/p^j, n)) \longrightarrow H^n(K(\mathbb{Z}/p^j, n); \mathbb{Z}_p) \tag{182}$$

is given by

$$\mathbb{Z}_p[Q_{s_{n-1}} Q_{s_{n-2}} \cdots Q_{s_1} r \partial \iota \mid 0 < s_1 < \cdots < s_{n-1}], \tag{183}$$

where ∂ is the Bockstein homomorphism induced by

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}_p \longrightarrow 0 \tag{184}$$

and r is reduction to $\mathbb{Z}_p$-cohomology.

We can now see how this agrees with our earlier result for the case $p = 2$. The image of the Thom homomorphism is a polynomial ring where the generator with the lowest degree is

$$x_1 = Q_{n-1} \cdots Q_1 \partial \iota. \tag{185}$$

Recalling that the Milnor operations have degrees $|Q_i| = 2^i - 1$ and the Bockstein has degree 1 we see that x_1 has degree

$$|x_i| = \left(\sum_{i=1}^{n-1} 2^i - 1 \right) + 1 = 2 + 2^2 + \ldots 2^{n-1} - (n-1) + 1 = 2^n - n + 1. \tag{186}$$

This is certainly greater than n, which agrees with our claim that the n'th cohomology is not in the image. However, Tamanoi's result clearly shows that we have to go into even higher degrees than $n+1$ to find nontrivial cohomology classes which are in the image of the Thom homomorphism. We also note that Tamanoi deals with the case $n = 2$ as well, which our result does not cover. It can therefore safely be said that [10] deals with the image of the Thom homomorphism for these spaces more thoroughly than Theorem 6.16. However, it is interesting to note that while Tamanoi used an extensive look into the Steenrod algebra to prove Theorem 6.19, we have seen that it is possible to prove a weaker result using much simpler methods.

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