Doctoral theses at NTNU, 2021:270

Mads Hustad Sandøy

Higher homological algebra and support varieties

Doctoral thesis

NTNU
Norwegian University of Science and Technology
Thesis for the Degree of
Philosophiae Doctor
Faculty of Information Technology and Electrical
Engineering
Department of Mathematical Sciences

NTNU
Norwegian University of
Science and Technology

Mads Hustad Sandøy

Higher homological algebra and support varieties

Thesis for the Degree of Philosophiae Doctor

Trondheim, August 2021

Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

NTNU
Norwegian University of
Science and Technology

NTNU
Norwegian University of Science and Technology

Thesis for the Degree of Philosophiae Doctor

Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

ISBN 978-82-326-6472-6 (printed ver.)
ISBN 978-82-326-6314-9 (electronic ver.)
ISSN 1503-8181 (printed ver.)
ISSN 2703-8084 (online ver.)

Doctoral theses at NTNU, 2021:270

Printed by NTNU Grafisk senter

Acknowledgments

First and most importantly of all, I thank my advisor Øyvind Solberg for his guidance, understanding and support. I thank him for everything from trying to teach me to appreciate small victories to encouraging me to follow my mathematical interests.

I owe thanks and a depth of gratitude to many more.

To various members of the algebra group, for answering my more or less naive and foolish questions whenever I've knocked at their office doors.

To the department, for excellent working conditions and administrative and technical support.

To the lunch-and-coffee-break crew (you know who you are), for conversations and company. After more than a year of long periods of working from home, I even miss the conversations about triathlons. And that's even though I can still remember them. Vividly!

To Johanne, for your understanding and for being an amazing colleague and collaborator.

To Louis-Philippe, for your kindness, for introducing me to many interesting mathematical topics, and for mathematical discussions.

Thanks are also due to my coauthors, to my friends, and to my family.

I thank my dear Margrethe. Thank you for your patience when I am reading, thinking, and working on mathematics even when I probably shouldn't.

Mads Hustad Sandøy
Sandøya, July 2021

Contents

Introduction

Along with this introduction, four papers together constitute this thesis:

- *On support varieties and tensor products for finite dimensional algebra*, Journal of Algebra (2020), volume 547, pages 226-237;
- *Higher Koszul duality and connections with n-hereditary algebras*;
- *Classification results for n-hereditary monomial algebras*;
- *Skew group algebras, the* ***(Fg)*** *property and self-injective radical cube zero algebras.*

All of these are concerned with or motivated by applications to a theory of support varieties defined via Hochschild cohomology, although this is not immediately obvious for the second and third paper. Moreover, all but the first paper have connections with or are concerned with (generalized) Koszul algebras and (higher) hereditary algebras.

In the following, we begin by giving some light background before we expand upon and explain some of the connections just outlined with a particular focus on showing how the second and third papers are related to and motivated by the aforementioned theory of support varieties. We end the introduction by discussing some avenues for future work.

Support varieties

The celebrated theory of cohomological support varieties for modular representations of finite groups was introduced in the early eighties by Carlson [**6**, **7**]. Analogous theories of varieties have been produced in many settings in the years since, e.g. for restricted Lie algebras [**15**] and finite dimensional cocommutative Hopf algebras, and support varieties for complete intersections have been introduced by Avramov and Buchweitz [**1**].

Solberg and Snashall [**33**] launched an investigation of cohomological support varieties of arbitrary finitely generated modules over finite dimensional algebras via the action of the Hochschild cohomology ring on the Ext-algebras of modules. In [**12**, **33**], it was shown that these varieties have many of the same elementary properties as those in the setting of group algebras, at least provided certain finite generation properties hold: e.g. modules of finite projective dimension have trivial varieties, every closed homogeneous subvariety of an appropriately chosen subring of the Hochschild cohomology ring can be realized by a module, and decomposable modules have reducible varieties.

One crucial result has nevertheless proven elusive, namely some version of a tensor product formula: in the case of group algebras, the variety of a tensor product (over the base field) of two modules is precisely the intersection of the varieties of the modules. The first paper listed above is related to this as we investigate the possibility of certain bimodule versions of such a formula, in particular showing that certain reasonable versions of such formulas cannot hold in full generality.

The (Fg) property and higher hereditary algebras

For group representations, cocommutative Hopf algebras, and restricted Lie algebras, the direct sum of all Ext-groups between any two finitely generated modules is a finitely generated module over the Noetherian (graded) commutative ring defining the support varieties. Call this property (**Fg**). This condition is of pivotal importance in all aforementioned settings. It is known that not all finite dimensional algebras satisfy (**Fg**), and one may thus ask, "When does a finite dimensional algebra satisfy (**Fg**)?"

In the framework of [**12**, **33**], one equivalent way to state this property is as follows: one says that a finite dimensional algebra Λ has (**Fg**) provided

$$\operatorname{Ext}^*_{\Lambda^e}(\Lambda, U) = \oplus_{i\geq 0} \operatorname{Ext}^i_{\Lambda^e}(\Lambda, U)$$

is a Noetherian module over the Hochschild cohomology ring of Λ

$$\operatorname{HH}^*(\Lambda) = \operatorname{Ext}^*_{\Lambda^e}(\Lambda, \Lambda)$$

for every finitely generated Λ^e-module U, where $\Lambda^e := \Lambda^{\mathrm{op}} \otimes_k \Lambda$ is the enveloping algebra of Λ. Note that it is in this more restricted sense we use the term henceforth. Also note that any finite dimensional algebra satisfying (**Fg**) must be Gorenstein by [**12**].

Since answering the question in general even in this sense is likely to be a hard problem, we narrowed our scope and looked at situations that seemed more tractable. In doing so, we believe we have found links with higher Auslander–Reiten theory and n-hereditary algebras. These areas have been much studied in recent years (see e.g. references cited in the introductions to the second and third paper listed above, i.e. respectively [**17**] and [**31**]). These areas have been shown to have connections with e.g. algebraic geometry and combinatorics [**11**, **19**], and both are "higher" generalizations of classical theories. Note that for the latter, setting $n = 1$ yields ordinary, honest hereditary algebras.

One suggestion of why pursuing such a link might be fruitful comes from the classification of the representation infinite weakly symmetric radical-cube-zero algebras satisfying (**Fg**) given by [**13**]. Reviewing that classification, one can see that, with some exceptions, the classes all seem to essentially consist of the Koszul duals of the preprojective algebras of tame hereditary algebras. Moreover, to all of them one can attach an extended Dynkin graph via the type of a self-injective radical-cube-zero algebra in the sense of [**13**, Definition 7.1].

Classical hereditary algebras also show up in connection with algebras satisfying (**Fg**) in other ways: e.g. whenever the base field is algebraically closed, it is known that the representation finite self-injective algebras all satisfy (**Fg**) by, essentially, a combination of the results in [**16**] and [**10**]. Recall that an algebra is periodic provided it has a periodic projective resolution when considered as a bimodule. Then, roughly speaking, the results in the former allows one to deduce that a periodic algebra must satisfy (**Fg**), whereas the latter yields that all representation finite self-injective algebras are periodic. Of course, from the work of Riedtmann and others (see e.g. [**4**, **28**, **29**]), we know that to each representation finite self-injective algebra we can attach a representation finite hereditary algebra, at least provided the base field is algebraically closed and of odd characteristic.

Additionally, any finite dimensional algebra derived equivalent to a tame hereditary algebra of an extended Dynkin type with bipartite orientation has a trivial extension that can easily be seen to be (**Fg**) by combining the main results in [**14**] and [**23**]: by the former, any trivial extension (see [**17**, Section 2.3] for a definition)

$$\Delta A = A \oplus DA$$

of such a hereditary algebra A has (**Fg**), it is well known that trivial extensions of derived equivalent algebras are derived equivalent [**27**], and the (**Fg**) property is preserved by derived equivalences by the latter reference.

Since n-hereditary algebras for $n > 1$ also come in two flavours of a similar kind, i.e. n-representation-finite [**21**, Definition 2.2] and n-representation infinite tame [**20**, Definition 6.10] (henceforth, respectively n-RF and n-RI tame, and we note that one can also see [**17**, Section 5] in the second paper for background and definitions for n-RF and n-RI algebras.) This suggests that one might – perhaps a bit naively – expect to be able to find new classes of self-injective algebras satisfying (**Fg**) near classes of n-hereditary algebras of those flavours. Additionally, in the same perhaps naive vein, one might hope to develop useful methods for verifying that an algebra has (**Fg**) using techniques and results involving such n-hereditary algebras.

In fact, it is not too hard to find examples of this happening: [**20**, Section 5] introduces the class of n-RI algebras of type $\widetilde{A}$ and [**20**, Example 6.11] shows that these are n-RI tame. Any n-hereditary algebra has an associated higher preprojective algebra and by the same example those associated to n-RI algebras of type $\widetilde{A}$ are of the following form: if S is a polynomial ring in $n+1$ variables over an algebraically closed field k of characteristic zero and G is a finite abelian subgroup of $\mathrm{SL}_{n+1}(k)$, then the associated higher preprojective algebra is of the form SG, the skew group algebra of S and G as in, say, [**9**, **26**] and which one can recall has underlying vector space given by $S \otimes_k kG$ and multiplication given by

$$sg \cdot th = sg(t)gh$$

with $s, t \in S$ and $g, h \in kG$.

If one lets S^G be the invariant subring of S, then by [**20**, Example 6.11] SG is finitely generated as a module over S^G, which is itself Noetherian. Now, SG is a Koszul algebra with Koszul dual EG if we let E be the exterior algebra over k in the same number of variables. See e.g. [**2**] for definitions and background on Koszul algebras and Koszul duals. Alternatively, one can use the definitions [**17**, Definition 3.4, Definition 3.6] in the second paper listed above. According to [**14**, Theorem 1.3], to check whether a Koszul algebra has (**Fg**), it suffices to check whether its Koszul dual is finitely generated as a module over a Noetherian central subalgebra. Consequently, EG must thus satisfy (**Fg**).

One can also note that this possible connection with n-RI tame algebras is utilised in the fourth paper listed above, i.e. [**32**]. In [**32**], we almost finish the classification of radical-cube-zero selfinjective algebras satisfying (**Fg**) begun in [**13**, **30**], leaving open only the case of the algebras of type $\widetilde{A}_n$. After using the n-quasi-Veronese construction as in [**25**] to reduce to a normal form that is a twisted trivial extension of a bipartite tame hereditary algebra, we are able to employ results about the latter class in a crucial simplifying step for the main result of that paper. See also [**32**] for definitions.

The (Fg) property and higher Koszul algebras

There are also other reasons to investigate such a link, as we now explain: In the general setting in which Solberg and Snashall introduced support varieties [**12**, **33**], the best understood case is perhaps that of Koszul algebras. In [**5**], one finds work of Briggs and Gelinas suggesting why this should have been so: [**5**] shows that the Hochschild cohomology of Λ, i.e. $\mathrm{HH}^*(\Lambda)$, surjects along a well-known canonical map onto the A_∞-centre of $\mathrm{Ext}^*(\Lambda_0, \Lambda_0)$. See e.g. the surveys [**22**, **24**] on A_∞-algebras and related notions for definitions. In particular, Koszul algebras are characterized as having $\mathrm{Ext}^*(\Lambda_0, \Lambda_0)$ for $\Lambda_0 = \Lambda/\operatorname{rad}\Lambda$ with trivial A_∞-structure, allowing one to work with the graded centre instead. Thus, verifying **(Fg)** becomes far easier than what would otherwise be the case.

However, as A_∞-techniques are subtle and little is known even in many well-studied settings – say in group representation theory [**34**] – this is somehow unfortunate, and working around this obstruction was partly the motivation for the second article listed above, i.e. [**17**]: higher Koszul algebras Λ replace $\Lambda/\operatorname{rad}\Lambda$ with a Λ_0-tilting module T having properties that force $\mathrm{Ext}^*(T, T)$ to have trivial A_∞-structure. This suggests that it is a natural and perhaps tractable class of algebras to investigate with an eye towards future applications involving the **(Fg)** property, and in [**17**] we do this by characterising "well-graded" Frobenius higher Koszul algebras in terms of certain associated algebras being n-RI. Recall that a basic self-injective algebra is necessarily Frobenius, where the latter simply means that the algebra and the k-dual of the algebra are isomorphic as right modules. Alternatively, see [**17**, Section 2.3] for a definition.

In particular, using the results in [**17**, Section 6], one can deduce that $\mathrm{Ext}^*_{\Delta A}(A, A)$ has trivial A_∞-structure for A some n-RI algebra, where the same is not necessarily true for the Ext-algebra of the simples of ΔA, e.g. whenever A is basic tame hereditary with an orientation of its quiver that is not bipartite.

The (Fg) property, periodic and higher almost Koszul algebras

Another reasonably well-understood and perhaps tractable class of **(Fg)** algebras are the periodic algebras. As stated before, these are defined by the algebra considered as a bimodule having a periodic projective resolution. Also as mentioned before, a periodic algebra must satisfy (**Fg**) as a consequence of [**16**].

Recent work by Chan et al. [**8**] has shown that the trivial extension of an algebra being periodic is closely connected to the fractionally Calabi–Yau property of that algebra. Recall that if A is a finite dimensional algebra of finite global dimension and $D^b(\operatorname{mod} A)$ is its bounded derived category, then the latter has a Serre functor (see [**17**, Definition 4.4]) given by the derived Nakayama functor ν. One calls A fractionally Calabi–Yau provided there are integers $\ell > 0$ and m such that ν^ℓ is naturally isomorphic to $[m]$ as functors on $D^b(\operatorname{mod} A)$, where $[m]$ is the mth power of the shift functor on $D^b(\operatorname{mod} A)$.

Examples of the fractionally Calabi–Yau algebras are in particular given by some n-representation finite algebras as in [**21**]. Moreover, there is also a weaker notion called a twisted fractionally Calabi-Yau algebra, where the defining natural isomorphism is taken only up to a twist by an algebra automorphism. Herschend and Iyama show in [**18**] that all n-RF algebras are twisted fractionally Calabi–Yau, and they ask whether all n-RF algebras are actually fractionally Calabi–Yau. Similarly, there is a notion of a twisted periodic algebra, wherein the algebra considered as a bimodule has a projective resolution that is periodic up to a twist by an algebra automorphism. However, one can note that these do not necessarily satisfy (**Fg**).

While the connection between tame and representation finite hereditary algebas is well-understood, the same cannot be said for n-RI tame algebras and n-RF algebras for $n > 1$. Nevertheless, [**20**, Theorem 5.10] shows that the higher type A n-RF algebras introduced in [**21**] are quotients of n-RI algebras of type $\widetilde{A}$ by ideals generated by some idempotent, and one might suspect that similar things can be said more generally. Hence, studying n-RF algebras potentially provides several possible avenues for finding classes of **(Fg)** algebras via trivial extensions and related constructions, and this might also lead to results of independent interest.

This was thus partially the motivation for the third paper, i.e. [**31**], in which we classify the quadratic monomial 2-hereditary algebras with higher preprojective algebra given by a planar quiver with potential, showing that there are essentially only two, both being 2-RF. See e.g. [**31**] for definitions.

Moreover, without the planarity assumption, we show that for each $n \geq 3$ there is exactly one quadratic monomial n-RF algebra. Roughly speaking, the strategy was to try to find new classes of n-RF algebras by looking for intersections with homologically "well-behaved" classes of algebras such as monomial algebras.

Moreover, this is also connected to the work in section 7 of the second article, i.e. [**17**], in which we introduce a generalization of the almost Koszul algebras of [**3**] and characterize these in terms of associated algebras being n-RF. It is easy to show that these higher almost Koszul algebras are twisted periodic, but we do not know whether they are periodic.

Future work

Unpublished work of the author using dg-homological algebra shows that trivial extension algebras that are higher Koszul have resolutions similar to those one obtains in the classical Koszul case, say as in [**2**]. Note that when Λ is higher Koszul in the sense of [**17**, Definition 3.4], this is defined relative to a Λ_0 tilting module T; see also [**17**, Section 2-3] for definitions. Using this, we believe we are able to show that the canonical map from $\mathrm{HH}^*(\Lambda)$ to $\mathrm{Ext}^*_\Lambda(T,T)$ surjects onto the latter's graded centre, hence establishing **(Fg)** for these reduces to verifying that $\mathrm{Ext}^*_\Lambda(T,T)$ is finitely generated as a module over its graded centre.

We also hope to generalize the results in the preceding paragraph to more general higher Koszul algebras by explicitly constructing resolutions or by other means. Furthermore, we would investigate more closely the connection between **(Fg)** algebras and tame n-hereditary algebras. Using the aforementioned unpublished work, it should already be possible to show that an n-hereditary algebra is tame if and only if its trivial extension is higher Koszul and satisfies **(Fg)**, but we believe more can be said.

Bibliography

[1] Luchezar L. Avramov and Ragnar-Olaf Buchweitz, *Support varieties and cohomology over complete intersections*, Invent. Math. **142** (2000), no. 2, 285–318. MR1794064

[2] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527. MR1322847 (96k:17010)

[3] Sheila Brenner, Michael C. R. Butler, and Alastair D. King, *Periodic algebras which are almost Koszul*, Algebr. Represent. Theory **5** (2002), no. 4, 331–367. MR1930968

[4] Otto Bretscher, Christian Läser, and Christine Riedtmann, *Self-injective and simply connected algebras*, Manuscripta Math. **36** (1981/82), no. 3, 253–307. MR641977

[5] Benjamin Briggs and Vincent Gelinas, *The A_∞-centre of the Yoneda algebra and the characteristic action of Hochschild cohomology on the derived category*, 2017.

[6] Jon F. Carlson, *The complexity and varieties of modules*, Integral representations and applications (Oberwolfach, 1980), 1981, pp. 415–422. MR646116

[7] ———, *The varieties and the cohomology ring of a module*, J. Algebra **85** (1983), no. 1, 104–143. MR723070

[8] Aaron Chan, Erik Darpö, Osamu Iyama, and René Marczinzik, *Periodic trivial extension algebras and fractionally Calabi-Yau algebras*, 2020.

[9] Miriam Cohen and Susan Montgomery, *Group-graded rings, smash products, and group actions*, Trans. Amer. Math. Soc. **282** (1984), no. 1, 237–258. MR728711

[10] Alex S. Dugas, *Periodic resolutions and self-injective algebras of finite type*, J. Pure Appl. Algebra **214** (2010), no. 6, 990–1000. MR2580674

[11] Tobias Dyckerhoff, Gustavo Jasso, and Tashi Walde, *Simplicial structures in higher Auslander-Reiten theory*, Adv. Math. **355** (2019), 106762, 73. MR3994443

[12] Karin Erdmann, Miles Holloway, Rachel Taillefer, Nicole Snashall, and Øyvind Solberg, *Support varieties for selfinjective algebras*, K-Theory **33** (2004), no. 1, 67–87. MR2199789

[13] Karin Erdmann and Øyvind Solberg, *Radical cube zero selfinjective algebras of finite complexity*, J. Pure Appl. Algebra **215** (2011), no. 7, 1747–1768. MR2771643

[14] ———, *Radical cube zero weakly symmetric algebras and support varieties*, J. Pure Appl. Algebra **215** (2011), no. 2, 185–200. MR2720683

[15] Eric M. Friedlander and Brian J. Parshall, *Geometry of p-unipotent Lie algebras*, J. Algebra **109** (1987), no. 1, 25–45. MR898334

[16] Edward L. Green, Nicole Snashall, and Øyvind Solberg, *The Hochschild cohomology ring of a selfinjective algebra of finite representation type*, Proc. Amer. Math. Soc. **131** (2003), no. 11, 3387–3393. MR1990627

[17] Johanne Haugland and Mads Hustad Sandøy, *Higher Koszul duality and connections with n-hereditary algebras* (2021).

[18] Martin Herschend and Osamu Iyama, *n-representation-finite algebras and twisted fractionally Calabi-Yau algebras*, Bull. Lond. Math. Soc. **43** (2011), no. 3, 449–466. MR2820136

[19] Martin Herschend, Osamu Iyama, Hiroyuki Minamoto, and Steffen Oppermann, *Representation theory of Geigle-Lenzing complete intersections*, 2014. arXiv:1409.0668.

[20] Martin Herschend, Osamu Iyama, and Steffen Oppermann, *n-representation infinite algebras*, Adv. Math. **252** (2014), 292–342. MR3144232

[21] Osamu Iyama and Steffen Oppermann, *n-representation-finite algebras and n-APR tilting*, Trans. Amer. Math. Soc. **363** (2011), no. 12, 6575–6614. MR2833569

[22] Bernhard Keller, *A-infinity algebras, modules and functor categories*, Trends in representation theory of algebras and related topics, 2006, pp. 67–93. MR2258042

[23] Julian Külshammer, Chrysostomos Psaroudakis, and Øystein Skartsæterhagen, *Derived invariance of support varieties*, Proc. Amer. Math. Soc. **147** (2019), no. 1, 1–14. MR3876726

[24] Di M. Lu, John H. Palmieri, Quan S. Wu, and James J. Zhang, *A_∞-algebras for ring theorists*, Proceedings of the International Conference on Algebra, 2004, pp. 91–128. MR2058967

[25] Hiroyuki Minamoto and Izuru Mori, *The structure of AS-Gorenstein algebras*, Adv. Math. **226** (2011), no. 5, 4061–4095. MR2770441

[26] Idun Reiten and Christine Riedtmann, *Skew group algebras in the representation theory of Artin algebras*, J. Algebra **92** (1985), no. 1, 224–282. MR772481

[27] Jeremy Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra **61** (1989), no. 3, 303–317. MR1027750

[28] Christine Riedtmann, *Representation-finite self-injective algebras of class A_n*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), 1980, pp. 449–520. MR607169

[29] ______, *Representation-finite self-injective algebras of class D_n*, Compositio Math. **49** (1983), no. 2, 231–282. MR704393

[30] Shumbana Said, *Radical cube zero self injective algebras and support varieties.*, Ph.D. Thesis, 2015.

[31] Mads Sandøy and Louis-Philippe Thibault, *Classification results for n-hereditary monomial algebras* (2021).

[32] Mads Hustad Sandøy, *Skew group algebras, the (**Fg**) property and self-injective radical cube zero algebras* (2021).

[33] Nicole Snashall and Øyvind Solberg, *Support varieties and Hochschild cohomology rings*, Proc. London Math. Soc. (3) **88** (2004), no. 3, 705–732. MR2044054

[34] Mikael Vejdemo-Johansson, *Algorithms in A_∞-algebras*, Georgian Mathematical Journal **25** (2018Dec), no. 4, 629–635.

Paper 1

ON SUPPORT VARIETIES AND TENSOR PRODUCTS FOR FINITE DIMENSIONAL ALGEBRA

Petter Andreas Bergh Mads Hustad Sandøy Øyvind Solberg

Published in Journal of Algebra (2020), volume 547, pages 226-237,

Original publication at: https://doi.org/10.1016/j.jalgebra.2019.10.059.

Journal of Algebra 547 (2020) 226–237

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On support varieties and tensor products for finite dimensional algebras

Petter Andreas Bergh, Mads Hustad Sandøy *, Øyvind Solberg

Institutt for matematiske fag, NTNU, N-7491 Trondheim, Norway

A R T I C L E   I N F O

Article history:
Received 18 June 2019
Available online 2 December 2019
Communicated by Markus Linckelmann

MSC:
16D20
16E40
16S80
16T05
18D10
18E30
81R50

Keywords:
Support varieties
Tensor products
Quantum complete intersections

A B S T R A C T

It has been asked whether there is a version of the tensor product property for support varieties over finite dimensional algebras defined in terms of Hochschild cohomology. We show that in general no such version can exist. In particular, we show that for certain quantum complete intersections, there are modules and bimodules for which the variety of the tensor product is not even contained in the variety of the one-sided module.

1. Introduction

In [11,12], Carlson introduced cohomological support varieties for modules over group algebras of finite groups, using the maximal ideal spectrum of the group cohomology ring.

* Corresponding author.
E-mail addresses: petter.bergh@ntnu.no (P.A. Bergh), mads.sandoy@ntnu.no (M. Hustad Sandøy), oyvind.solberg@ntnu.no (Ø. Solberg).

https://doi.org/10.1016/j.jalgebra.2019.10.059

These varieties behave well with respect to the typical operations such as directs sums and syzygies. Moreover, they encode important homological information. For example, the dimension of the support variety of a module equals the complexity of the module. In particular, the variety of a module is trivial if and only if the module is projective.

Shortly after these cohomological support varieties were introduced, it was shown in [1] that the variety of a tensor product of modules equals the intersection of the varieties of the modules. This property is commonly referred to as the *tensor product property*. As shown in [14], it holds also for modules over finite dimensional cocommutative Hopf algebras; for such algebras, there is a theory of support varieties generalizing that for groups. In fact, one can define support varieties over any finite dimensional Hopf algebra, cocommutative or not, using the Hopf algebra cohomology ring. However, it is not known if this cohomology ring is finitely generated in general. What *is* known is that the tensor product property may or may not hold for non-cocommutative Hopf algebras having finitely generated cohomology rings. Namely, as shown in [6,18,19], there are examples of such algebras where the tensor product property holds, and examples where it does not.

Why do we care about the tensor product property? There are several reasons. Not only does it look good; it indicates that the homological behavior of a tensor product is closely related to each of the factors. When the property does not hold, some peculiar things can happen; examples in [6] show that the tensor product of two modules in one order can be projective, but non-projective in the other order. Another reason why the tensor product property is of interest is that in many cases, it is connected with the classification of thick subcategories. It is an ingredient in Balmer's classification of thick tensor ideals of tensor triangulated categories (cf. [2]), and a necessary consequence of Benson, Iyengar and Krause's stratification approach in [4,5], as shown in [4, Theorem 7.3]. In general, one is often in a situation where some triangulated tensor category (where the tensor product is not necessarily symmetric) acts on a triangulated category, and where the latter comes with a theory of support varieties relative to some cohomology ring; this is studied in detail in [10]. If the appropriate tensor product property holds, then it is sometimes the case that the thick subcategories are actually tensor ideals.

In [13,20,21], a theory of support varieties for arbitrary finite dimensional algebras was developed, using Hochschild cohomology rings. For such an algebra A, there is in general no natural tensor product between one-sided modules, as is the case for Hopf algebras. However, one can tensor any left A-module with a bimodule, and obtain a new left A-module. It has therefore been asked whether some version of the tensor product property holds in this setting. In other words, given a bimodule B and a left A-module M, is there an equality

$$\mathrm{V}(B \otimes_A M) = \mathrm{V}(B) \cap \mathrm{V}(M)$$

of support varieties? This does not immediately make sense: how should we define the support variety of a bimodule? If we just use the same definition as for one-sided modules, then the support variety of any bimodule which is one-sided projective is trivial. In this case, the variety of the tensor product $A \otimes_A M$ would be $\mathrm{V}(M)$, whereas $\mathrm{V}(A) \cap \mathrm{V}(M)$ would always be trivial. However, as we explain at the end of Section 2, there are actually several possible meaningful ways of defining a support variety theory for bimodules, using Hochschild cohomology. On the other hand, we show that the tensor product property can *never* hold in general, regardless of which bimodule version of support variety theory we use. In fact, we show in Theorem 2.2 that when A is a quantum complete intersection of a certain type, then there exists a left A-module M and a bimodule B for which

$$\mathrm{V}(B \otimes_A M) \nsubseteq \mathrm{V}(M)$$

One consequence of the failure of such an inclusion is that in the stable module category and the bounded derived category of A-modules, there are thick subcategories that are not tensor ideals.

2. Support varieties and tensor products

Let us first recall the basics on the theory of support varieties for finite dimensional algebras, using Hochschild cohomology. We only give a very brief overview; for details, we refer the reader to [13,20,21].

Let k be a field and A a finite dimensional k-algebra with radical $\mathfrak{r}$. All modules considered will be finitely generated left modules, and we denote the category of such A-modules by $\operatorname{mod} A$. A bimodule over A is the same thing as a left module over the enveloping algebra $A^{\mathrm{e}} = A \otimes_k A^{\mathrm{op}}$, and the Hochschild cohomology ring of A is the graded ring

$$\mathrm{HH}^*(A) = \bigoplus_{n=0}^{\infty} \operatorname{Ext}^n_{A^{\mathrm{e}}}(A, A)$$

with the Yoneda product. This ring is graded-commutative, and so its even part $\mathrm{HH}^{2*}(A)$ is commutative in the ordinary sense. Now let M and N be A-modules, and consider the graded vector space

$$\operatorname{Ext}^*_A(M, N) = \bigoplus_{n=0}^{\infty} \operatorname{Ext}^n_A(M, N)$$

The Yoneda product makes this into a graded left module over $\operatorname{Ext}^*_A(N, N)$, and a graded right module over $\operatorname{Ext}^*_A(M, M)$. Since for every $L \in \operatorname{mod} A$ the tensor product $- \otimes_A L$ induces a homomorphism

$$\varphi_L \colon \mathrm{HH}^*(A) \to \operatorname{Ext}^*_A(L, L)$$

of graded rings, we see that $\operatorname{Ext}_A^*(M,N)$ becomes a module over $\operatorname{HH}^*(A)$ in two ways: via the ring homomorphisms φ_N and φ_M. However, the scalar multiplication via these two ring homomorphisms coincide up to a sign.

Now suppose that H is a graded subalgebra of $\operatorname{HH}^{2*}(A)$. Then for every pair (M,N) of A-modules, we can define the support variety $\operatorname{V}_H(M,N)$ using the maximal ideal spectrum of H:

$$\operatorname{V}_H(M,N) = \{\mathfrak{m} \in \operatorname{MaxSpec} H \mid \operatorname{Ann}_H\left(\operatorname{Ext}_A^*(M,N)\right) \subseteq \mathfrak{m}\}$$

There are equalities

$$\operatorname{V}_H(M,M) = \operatorname{V}_H(M,A/\mathfrak{r}) = \operatorname{V}_H(A/\mathfrak{r},M)$$

and we define this to be the support variety $\operatorname{V}_H(M)$ of the single module M. These support varieties share many of the properties enjoyed by the cohomological support varieties for modules over group rings, in particular when H is noetherian and $\operatorname{Ext}_A^*(M,N)$ is a finitely generated H-module for all $M,N \in \operatorname{mod} A$. If this is the case, we say that the algebra A satisfies **Fg** with respect to H. Note that by [21, Proposition 5.7], the (even part of the) Hochschild cohomology ring is universal with this property, in the following sense: the algebra A satisfies **Fg** with respect to some $H \subseteq \operatorname{HH}^*(A)$ if and only if $\operatorname{HH}^*(A)$ is noetherian and $\operatorname{Ext}_A^*(A/\mathfrak{r},A/\mathfrak{r})$ is a finitely generated $\operatorname{HH}^*(A)$-module.

The finite dimensional algebras we shall study are of a very special form, namely quantum complete intersections. These are quantum commutative analogues of truncated polynomial rings. Let us therefore fix some notation that we shall use throughout.

Setup. (1) Fix an algebraically closed field k, together with two integers $c \geq 2$ and $a \geq 2$.

(2) Define an integer $\bar{a}$ by

$$\bar{a} = \begin{cases} a & \text{if } \operatorname{char} k = 0 \\ a/\gcd(a, \operatorname{char} k) & \text{if } \operatorname{char} k > 0 \end{cases}$$

and fix a primitive $\bar{a}$th root of unity $q \in k$.

(3) Denote by A_q^c the quantum complete intersection

$$k\langle x_1,\ldots,x_c\rangle / \left(x_1^a,\ldots,x_c^a,\{x_ix_j - qx_jx_i\}_{i<j}\right)$$

This is a local selfinjective algebra of dimension a^c, and by [8, Theorem 5.5] it satisfies **Fg** with respect to $\operatorname{HH}^{2*}(A_q^c)$. In [3], it was shown that one can actually define rank varieties over this algebra, and that these varieties behave very much like the rank varieties for group algebras. It was then shown in [7] that these rank varieties are isomorphic to the support varieties one obtains by using a suitable polynomial subalgebra of the Hochschild cohomology ring. We now point out some facts about this algebra and its support varieties.

Fact 2.1. (1) By [8, Theorem 5.3], the Ext-algebra $\operatorname{Ext}^*_{A_q^c}(k,k)$ of the simple module k admits a presentation

$$k\langle z_1,\dots,z_c,y_1,\dots,y_c\rangle/\mathfrak{a}$$

where $\mathfrak{a}$ is the ideal generated by the relations

$$\begin{pmatrix} z_iz_j - z_jz_i & \text{for all } i,j \\ z_iy_j - y_jz_i & \text{for all } i,j \\ y_iy_j + qy_jy_i & \text{for all } i>j \\ y_i^2 & \text{for all } i \text{ if } a>2 \\ y_i^2 - z_i & \text{for all } i \text{ if } a=2 \end{pmatrix}$$

Here, the homological degree of each y_i is one, whereas that of each z_i is two. In particular, the z_i generate a polynomial subalgebra $k[z_1,\dots,z_c]$ over which $\operatorname{Ext}^*_{A_q^c}(k,k)$ is finitely generated as a module.

(2) As explained in [7, Section 2], it follows from [17, Corollary 3.5] that the image of the ring homomorphism

$$\varphi_k\colon \operatorname{HH}^{2*}(A_q^c) \to \operatorname{Ext}^*_{A_q^c}(k,k)$$

is the whole polynomial subalgebra $k[z_1,\dots,z_c]$. Consequently, there exists a polynomial subalgebra $k[\eta_1,\dots,\eta_c]$ of $\operatorname{HH}^{2*}(A_q^c)$ with the following properties: each η_i is a homogeneous element in $\operatorname{HH}^{2*}(A_q^c)$ of degree two with $\varphi_k(\eta_i)=z_i$, and A_q^c satisfies **Fg** with respect to $k[\eta_1,\dots,\eta_c]$.

We now prove our main result. It shows that there exists an A_q^c-module M and a bimodule B for which the support variety of the tensor product $B\otimes_{A_q^c} M$ is not contained in the support variety of M.

Theorem 2.2. *Let $k[\eta_1,\dots,\eta_c]$ be a polynomial subalgebra of $\operatorname{HH}^{2*}(A_q^c)$ as in* Fact 2.1. *Then for every graded subalgebra H of $\operatorname{HH}^*(A_q^c)$ with*

$$k[\eta_1,\dots,\eta_c] \subseteq H \subseteq \operatorname{HH}^{2*}(A_q^c)$$

the following hold:

(1) the algebra H is noetherian, and A_q^c satisfies **Fg** *with respect to H;*

(2) there exists an A_q^c-module M and a bimodule B with $\operatorname{V}_H(B\otimes_{A_q^c} M) \not\subseteq \operatorname{V}_H(M)$.

Proof. Let us simplify notation a bit and write A for our algebra A_q^c. Since it satisfies **Fg** with respect to $k[\eta_1,\dots,\eta_c]$, it follows from [13, Proposition 2.4] that the Hochschild cohomology ring $\operatorname{HH}^*(A)$ is finitely generated as a module over $k[\eta_1,\dots,\eta_c]$. Note that the assumption in [13, Proposition 2.4] is that **Fg** holds with respect to a graded subalgebra of $\operatorname{HH}^*(A)$ whose degree zero part coincides with $\operatorname{HH}^0(A)$, which is the center of

A. This is not the case for the polynomial subalgebra $k[\eta_1, \dots, \eta_c]$, since the center of A is not of dimension one. However, this assumption is not needed in the result.

Since $\mathrm{HH}^*(A)$ is finitely generated as a module over the noetherian ring $k[\eta_1, \dots, \eta_c]$, the same is true for H, since this is a $k[\eta_1, \dots, \eta_c]$-submodule of $\mathrm{HH}^*(A)$. Then H is noetherian as a ring, since it contains $k[\eta_1, \dots, \eta_c]$ as a subring. Moreover, since $\mathrm{Ext}_A^*(k,k)$ is finitely generated over $k[\eta_1, \dots, \eta_c]$, it must also be finitely generated over the bigger algebra H. This proves (1).

To prove (2), we first show that we may without loss of generality assume that $H = k[\eta_1, \dots, \eta_c]$. To do this, consider the ring homomorphism

$$\varphi_k \colon \mathrm{HH}^*(A) \to \mathrm{Ext}_A^*(k,k)$$

By Fact 2.1, the image of $\mathrm{HH}^{2*}(A)$ is the polynomial subalgebra $k[z_1, \dots, z_c]$ of $\mathrm{Ext}_A^*(k,k)$, and this is also the image of $k[\eta_1, \dots, \eta_c]$; after all, that is how we constructed $k[\eta_1, \dots, \eta_c]$ in the first place. Therefore, since $k[\eta_1, \dots, \eta_c] \subseteq H \subseteq \mathrm{HH}^{2*}(A)$, we see that the image of $k[\eta_1, \dots, \eta_c]$ is the same as that of H, namely $k[z_1, \dots, z_c]$. Now take any A-module X, and consider its support variety $\mathrm{V}_H(X)$, which by definition is the set

$$\{\mathfrak{m} \in \operatorname{MaxSpec} H \mid \operatorname{Ann}_H (\mathrm{Ext}_A^*(X,X)) \subseteq \mathfrak{m}\}$$

By [20, Theorem 3.2], there is an equality

$$\mathrm{V}_H(X) = \{\mathfrak{m} \in \operatorname{MaxSpec} H \mid \operatorname{Ann}_H (\mathrm{Ext}_A^*(X,k)) \subseteq \mathfrak{m}\}$$

and so by [9, Proposition 3.6] the variety $\mathrm{V}_H(X)$ is isomorphic to the set of maximal ideals of $k[z_1, \dots, z_c]$ containing the annihilator of $\mathrm{Ext}_A^*(X,k)$. Here we view $\mathrm{Ext}_A^*(X,k)$ as a left module over $\mathrm{Ext}_A^*(k,k)$, and in this way it becomes a module over the subalgebra $k[z_1, \dots, z_c]$. The isomorphism respects inclusions of varieties, and this proves the claim.

In light of the above, we now take $H = k[\eta_1, \dots, \eta_c]$. Since k is algebraically closed, we may identify the maximal ideal spectrum of H with the affine space k^c. For a point $\lambda = (\lambda_1, \dots, \lambda_c)$ in k^c, we denote the corresponding maximal ideal $(\eta_1 - \lambda_1, \dots, \eta_c - \lambda_c)$ in H by $\mathfrak{m}_\lambda$, and when λ is nonzero we denote the corresponding line

$$\{(\gamma\lambda_1, \dots, \gamma\lambda_c) \mid \gamma \in k\}$$

in k^c by ℓ_λ. Moreover, we denote the element $\sum_{i=1}^c \lambda_i x_i$ in A by u_λ, and by $F(\lambda)$ the point $(\lambda_1^a, \dots, \lambda_c^a)$ in k^c. By [7, Proposition 3.5], the support variety $\mathrm{V}_H(Au_\lambda)$ of the cyclic A-module Au_λ equals $\ell_{F(\lambda)}$, that is, there is an equality

$$\mathrm{V}_H(Au_\lambda) = \left\{\mathfrak{m}_{\gamma F(\lambda)} \mid \gamma \in k\right\} = \{(\eta_1 - \gamma\lambda_1^a, \dots, \eta_c - \gamma\lambda_c^a) \mid \gamma \in k\}$$

Note that $F(\lambda) = 0$ if and only if $\lambda = 0$.

Now take any point $\mu = (\mu_1, \dots, \mu_c)$ in k^c with $\mu_i \neq 0$ for all i, and consider the automorphism $\psi_\mu \colon A \to A$ given by $x_i \mapsto \mu_i x_i$. What happens to the cyclic A-module Au_λ when we twist it by this automorphism? In general, for an A-module X and an automorphism ψ of A, the twisted module ${}_\psi X$ is the same as X as a vector space, but for $w \in A$ and $x \in X$ the scalar multiplication is $w \cdot x = \psi(w)x$. Now denote the point $(\mu_1^{-1}\lambda_1, \dots, \mu_c^{-1}\lambda_c)$ in k^c by $\mu^{-1}\lambda$, and consider the map

$$\begin{aligned} Au_{\mu^{-1}\lambda} &\to {}_{\psi_\mu}(Au_\lambda) \\ wu_{\mu^{-1}\lambda} &\mapsto \psi_\mu(w)u_\lambda \end{aligned}$$

Note that since $u_{\mu^{-1}\lambda} = \psi_\mu^{-1}(u_\lambda)$, this map is obtained by simply applying ψ_μ to the elements in $Au_{\mu^{-1}\lambda}$. It is k-linear, and for every element $v \in A$ and $wu_{\mu^{-1}\lambda} \in Au_{\mu^{-1}\lambda}$ there are equalities

$$\begin{aligned} \psi_\mu\left(v \cdot (wu_{\mu^{-1}\lambda})\right) &= \psi_\mu\left(vwu_{\mu^{-1}\lambda}\right) \\ &= \psi_\mu(u)\psi_\mu(w)u_\lambda \\ &= u \cdot (\psi_\mu(w)u_\lambda) \end{aligned}$$

Thus the map is an A-homomorphism. Similarly, the inverse automorphism ψ_μ^{-1} induces an A-homomorphism in the other direction, hence $Au_{\mu^{-1}\lambda}$ and ${}_{\psi_\mu}(Au_\lambda)$ are isomorphic A-modules. Using [7, Proposition 3.5] again, we now see that $\mathrm{V}_H\left({}_{\psi_\mu}(Au_\lambda)\right)$ equals the line $\ell_{F(\mu^{-1}\lambda)}$.

Twisting an A-module X by an automorphism ψ is the same as tensoring with the bimodule ${}_\psi A_1$, i.e. ${}_\psi X \simeq {}_\psi A_1 \otimes_A X$. Therefore, with λ and μ as above, the support variety $\mathrm{V}_H\left({}_{\psi_\mu}A_1 \otimes_A Au_\lambda\right)$ is the line $\ell_{F(\mu^{-1}\lambda)}$. On the other hand, the support variety $\mathrm{V}_H(Au_\lambda)$ is the line $\ell_{F(\lambda)}$, which generically differs from $\ell_{F(\mu^{-1}\lambda)}$. For example, with $\lambda = (1, \dots, 1)$, any μ whose components are not all the same when raised to the ath power will do. Consequently, for this λ and such a μ, we see that $\mathrm{V}_H\left({}_{\psi_\mu}A_1 \otimes_A Au_\lambda\right) \nsubseteq \mathrm{V}_H(Au_\lambda)$. $\square$

As a consequence of the theorem, there cannot exist a bimodule version of the tensor product property for support varieties over the algebra A_q^c.

Corollary 2.3. *Let H, M and B be as in* Theorem 2.2*, and suppose that V_H^b is some support variety theory on the category of A_q^c-bimodules, defined in terms of the maximal ideal spectrum of H. Then $\mathrm{V}_H(B \otimes_{A_q^c} M) \neq \mathrm{V}_H^b(B) \cap \mathrm{V}_H(M)$.*

For a finite dimensional algebra A, there are actually several possible ways of defining support varieties for bimodules. Namely, take any commutative graded subalgebra H of $\mathrm{HH}^*(A)$. For a bimodule B, we can view $\mathrm{Ext}_{A^e}^*(B, A)$ as a left module over $\mathrm{HH}^*(A)$, and in this way it becomes an H-module. We can then define

$$\mathrm{V}_H^b(B) = \{\mathfrak{m} \in \operatorname{MaxSpec} H \mid \operatorname{Ann}_H\left(\mathrm{Ext}_{A^e}^*(B, A)\right) \subseteq \mathfrak{m}\}$$

Similarly, we can use the fact that $\mathrm{Ext}^*_{A^e}(A, B)$ is a right module over $\mathrm{HH}^*(A)$ and obtain another support variety. These types of one-sided support varieties were studied in [9], where it was shown that they satisfy many of the properties one expects for a meaningful theory of support.

Now suppose that we take a bimodule B which is projective as a left A-module. Then if we take any exact sequence η of bimodules, the sequence $\eta \otimes_A B$ remains exact. Thus we obtain a ring homomorphism

$$\begin{gathered} \mathrm{HH}^*(A) \to \mathrm{Ext}^*_{A^e}(B, B) \\ \eta \mapsto \eta \otimes_A B \end{gathered}$$

of graded rings, and we can define

$$\mathrm{V}^b_H(B) = \{\mathfrak{m} \in \operatorname{MaxSpec} H \mid \operatorname{Ann}_H\left(\mathrm{Ext}^*_{A^e}(B, B)\right) \subseteq \mathfrak{m}\}$$

Similarly, if B is projective as a right A-module, we obtain a version by tensoring with B on the left. Consequently, for bimodules which are projective as both left and right A-modules, there are totally at least four ways of defining support varieties using H, and there is in general no reason to expect them to be equivalent.

Suppose now that A is a finite dimensional selfinjective algebra satisfying **Fg** with respect to some subalgebra H of its Hochschild cohomology ring. We then ask: what are the consequences of having a tensor product formula for bimodules acting on left modules? In order to investigate this, assume that

$$\mathrm{V}_H(B \otimes_A M) = \mathrm{V}^b_H(B) \cap \mathrm{V}_H(M)$$

for all B in a tensor closed subcategory $\mathscr{X}$ of bimodules and all left A-modules M, where V_H is the usual support variety theory on left modules and V^b_H is some support variety theory for bimodules in $\mathscr{X}$ (defined in terms of the same geometric space as V_H, namely the maximal ideal spectrum of H). Then

$$\begin{aligned} \mathrm{V}^b_H(B_1 \otimes_A B_2) \cap \mathrm{V}_H(M) &= \mathrm{V}_H((B_1 \otimes_A B_2) \otimes_A M) \\ &= \mathrm{V}_H(B_1 \otimes_A (B_2 \otimes_A M)) \\ &= \mathrm{V}^b_H(B_1) \cap \mathrm{V}_H(B_2 \otimes_A M) \\ &= \mathrm{V}^b_H(B_1) \cap \mathrm{V}^b_H(B_2) \cap \mathrm{V}_H(M) \\ &= \mathrm{V}^b_H(B_2) \cap \mathrm{V}^b_H(B_1) \cap \mathrm{V}_H(M) \\ &= \mathrm{V}_H(B_2 \otimes_A (B_1 \otimes_A M)) \\ &= \mathrm{V}_H((B_2 \otimes_A B_1) \otimes_A M) \\ &= \mathrm{V}^b_H(B_2 \otimes_A B_1) \cap \mathrm{V}_H(M) \end{aligned}$$

for all B_1 and B_2 in $\mathscr{X}$ og all left A-modules M. Then we claim that the equality

$$\mathrm{V}_H^b(B_1 \otimes_A B_2) = \mathrm{V}_H^b(B_2 \otimes_A B_1)$$

holds for all bimodules B_1 and B_2 in $\mathscr{X}$. To see this, choose $M = A/\mathfrak{r}$, where $\mathfrak{r}$ is the radical of A. Then $\mathrm{V}_H(M)$ is the whole defining maximal ideal spectrum of H, so that $\mathrm{V}_H^b(B_1 \otimes_A B_2) = \mathrm{V}_H^b(B_2 \otimes_A B_1)$. Hence, one consequence is that the bimodule support variety V_H^b must be independent of the order of the terms in a tensor product of bimodules, and therefore forcing some type of symmetry on the tensor products of bimodules in $\mathscr{X}$.

Let $\eta\colon \Omega^n_{A^\mathrm{e}}(A) \to A$ represent a homogeneous element in H, where $\Omega^n_{A^\mathrm{e}}(A)$ is the nth syzygy in a minimal projective resolution of A over A^e. Taking the pushout along this homomorphism and the minimal projective resolution of A over A^e gives rise to a short exact sequence

$$0 \to A \to M_\eta \to \Omega^{n-1}_{A^\mathrm{e}}(A) \to 0$$

as defined in [13]. The bimodules M_η for homogeneous elements η in H have the following property

$$\mathrm{V}_H(M_{\eta_1} \otimes_A \cdots \otimes_A M_{\eta_t} \otimes_A M) = \mathrm{V}_H(\langle \eta_1, \ldots, \eta_t \rangle) \cap \mathrm{V}_H(M).$$

If there is a support variety V_H^b of bimodules such that

$$\mathrm{V}_H^b(M_{\eta_1} \otimes_A \cdots \otimes_A M_{\eta_t}) = \mathrm{V}(\langle \eta_1, \ldots, \eta_t \rangle),$$

then V_H^b must in particular satisfy

$$\mathrm{V}_H^b(M_{\eta_1} \otimes_A M_{\eta_2}) = \mathrm{V}_H^b(M_{\eta_2} \otimes_A M_{\eta_1}).$$

For example, let $\mathrm{V}_H^b(B) = \mathrm{V}_H(B \otimes_A A/\mathfrak{r})$ for a bimodule B. Then it follows that

$$\mathrm{V}_H^b(M_{\eta_1} \otimes_A \cdots \otimes_A M_{\eta_t}) = \mathrm{V}_H(\langle \eta_1, \ldots, \eta_t \rangle)$$

for all homogeneous elements η_i in H, and V_H^b satisfies the above symmetry condition. Since

$$\begin{aligned} \mathrm{Ext}^*_A(B \otimes_A A/\mathfrak{r}, A/\mathfrak{r}) &\simeq \mathrm{Ext}^*_{A^\mathrm{e}}(B, \mathrm{Hom}_A(A/\mathfrak{r}, A/\mathfrak{r})) \\ &\simeq \mathrm{Ext}^*_{A^\mathrm{e}}(B, A/\mathfrak{r} \otimes_k A/\mathfrak{r}) \\ &\simeq \mathrm{Ext}^*_{A^\mathrm{e}}(B, A^\mathrm{e}/\operatorname{rad} A^\mathrm{e}) \end{aligned}$$

as H-modules, and $A/\mathfrak{r} \otimes_k A/\mathfrak{r} \simeq A^\mathrm{e}/\operatorname{rad} A^\mathrm{e}$ when $A/\mathfrak{r}$ is separable over the field k, then applying similar arguments as in [20] we obtain that

$$\begin{aligned}\mathrm{V}^b_H(B) &= \mathrm{V}(\mathrm{Ann}_H \mathrm{Ext}^*_{A^{\mathrm{e}}}(B, A^{\mathrm{e}}/\operatorname{rad} A^{\mathrm{e}})) \\ &= \mathrm{V}(\mathrm{Ann}_H \mathrm{Ext}^*_{A^{\mathrm{e}}}(B, B)) \\ &= \mathrm{V}(\mathrm{Ann}_H \mathrm{Ext}^*_{A^{\mathrm{e}}}(A^{\mathrm{e}}/\operatorname{rad} A^{\mathrm{e}}, B)).\end{aligned}$$

In other words, adapting the notion from [20],

$$\mathrm{V}^b_H(B) = \mathrm{V}^b_H(B, A^{\mathrm{e}}/\operatorname{rad} A^{\mathrm{e}}) = \mathrm{V}^b_H(B, B) = \mathrm{V}^b_H(A^{\mathrm{e}}/\operatorname{rad} A^{\mathrm{e}}, B).$$

Then it is natural to ask how we can/should choose $\mathscr{X}$. If we are thinking in terms of subcategories of the stable category of bimodules, can we choose $\mathscr{X}$ to be the tensor closed subcategory generated by the bimodules M_η for all homogeneous elements η in H? If all M_η's are in $\mathscr{X}$, we do not know how $M_{\eta_1} \otimes_A M_{\eta_2}$ and $M_{\eta_2} \otimes_A M_{\eta_1}$ are related as bimodules in general.

Let us now return to our quantum complete intersection A^c_q. Corollary 2.3, which is a direct consequence of Theorem 2.2, shows that the tensor product property for support varieties over this algebra cannot hold in general, now matter how one defines support varieties for bimodules. Another consequence of Theorem 2.2 is that not all the thick subcategories of the derived category and the stable module category of A^c_q are tensor ideals. In order to explain this, let us first briefly describe a general framework where one typically is interested in such questions; for details, we refer to [10]. Let $\mathscr{C}$ be a triangulated tensor category, that is, a triangulated category which is at the same time a (possibly non-symmetric) tensor category, and where the two structures are compatible. Furthermore, suppose that $\mathscr{C}$ acts on a triangulated category $\mathscr{D}$. This means that there exists an additive bifunctor

$$\begin{aligned}\mathscr{C} \times \mathscr{D} &\to \mathscr{D} \\ (C, D) &\mapsto C * D\end{aligned}$$

which is compatible in a natural way with the structures of both $\mathscr{C}$ and $\mathscr{D}$. Finally, suppose that H is a commutative graded subalgebra of the graded endomorphism ring $\mathrm{End}^*_{\mathscr{C}}(I)$ of the unit object I in $\mathscr{C}$, or, more generally, that there exists a ring homomorphism $H \to \mathrm{End}^*_{\mathscr{C}}(I)$. Then for all objects $D_1, D_2 \in \mathscr{D}$, the graded homomorphism group $\mathrm{Hom}^*_{\mathscr{D}}(D_1, D_2)$ becomes a left and a right H-module, and left and right scalar multiplication coincide up to a sign. One can then define the support variety $\mathrm{V}_H(D_1, D_2)$ as usual, in terms of the variety of the annihilator ideal $\mathrm{Ann}_H(\mathrm{Hom}^*_{\mathscr{D}}(D_1, D_2))$. For a single object $D \in \mathscr{D}$, one defines the support variety by $\mathrm{V}_H(D) = \mathrm{V}_H(D, D)$. If H is Noetherian and the graded H-modules $\mathrm{Hom}^*_{\mathscr{D}}(D_1, D_2)$ are finitely generated for all objects D_1 and D_2 in $\mathscr{D}$, then one obtains a meaningful theory of support varieties.

Given any triangulated category, it is of great interest to classify its thick subcategories. The first example of such a classification was the celebrated result of Hopkins-Neeman, for the category of perfect complexes over a commutative noetherian ring (cf.

[15,16]). That particular classification result showed for free that all the thick subcategories are actually thick tensor ideals. Now given $\mathscr{C}$ and $\mathscr{D}$ as above, one may ask for a similar classification of thick subcategories of $\mathscr{D}$, and whether these are all tensor ideals. Here, the notion of tensor ideals in $\mathscr{D}$ refers to the action of $\mathscr{C}$ on $\mathscr{D}$: a thick subcategory $\mathscr{A} \subseteq \mathscr{D}$ is a tensor ideal if $C * A \in \mathscr{A}$ for all $C \in \mathscr{C}$ and $A \in \mathscr{A}$.

Suppose that V is a closed homogeneous subvariety of $\operatorname{MaxSpec} H$, and define a full subcategory $\mathscr{A}_V$ of $\mathscr{D}$ by

$$\mathscr{A}_V = \{D \in \mathscr{D} \mid \mathrm{V}_H(D) \subseteq V\}$$

This is a thick subcategory of $\mathscr{D}$, and there are several classes of examples of triangulated categories where *all* the thick subcategories are of this form. For example, this is the case for the category of perfect complexes over a commutative noetherian ring. The crucial point now is that whenever $\mathrm{V}_H(C * D) \subseteq \mathrm{V}_H(D)$ for all objects $C \in \mathscr{C}$ and $D \in \mathscr{D}$, then $\mathscr{A}_V$ is automatically a thick tensor ideal for all V. This indicates the importance of the inclusion property

$$\mathrm{V}_H(C * D) \subseteq \mathrm{V}_H(D)$$

for support varieties in the setting of a triangulated tensor category acting on a triangulated category.

Now consider our quantum complete intersection $A = A_q^c$ again. This is a selfinjective algebra, and so the stable module category $\underline{\operatorname{mod}}\, A$ is triangulated. The enveloping algebra A^{e} is also selfinjective, and its stable module category $\underline{\operatorname{mod}}\, A^{\mathrm{e}}$, that is, the stable module category of A-bimodules, is a triangulated tensor category. It acts on $\underline{\operatorname{mod}}\, A$ by tensor products over A, and so we are in a setting where all of the above applies. However, let H, M and B be as in Theorem 2.2. Since $\mathrm{V}_H(B \otimes_A M) \nsubseteq \mathrm{V}_H(M)$, not all thick subcategories of $\underline{\operatorname{mod}}\, A$ can be tensor ideals. Namely, take $V = \mathrm{V}_H(M)$ and define $\mathscr{A}_V$ as above. This is a thick subcategory of $\underline{\operatorname{mod}}\, A$, but it is not a tensor ideal since $M \in \mathscr{A}_V$ but $B \otimes_A M \notin \mathscr{A}_V$. Finally, note that the bimodule B we used in the proof of Theorem 2.2 is actually projective as a left and as a right A-module. The bounded derived category of such bimodules is also a triangulated tensor category, and it acts on the bounded derived category $\mathbf{D}^{\mathbf{b}}(\operatorname{mod} A)$ of A-modules. Thus also in $\mathbf{D}^{\mathbf{b}}(\operatorname{mod} A)$ there are thick subcategories that are not tensor ideals.

References

[1] G.S. Avrunin, L.L. Scott, Quillen stratification for modules, Invent. Math. 66 (2) (1982) 277–286.

[2] P. Balmer, The spectrum of prime ideals in tensor triangulated categories, J. Reine Angew. Math. 588 (2005) 149–168.

[3] D. Benson, K. Erdmann, M. Holloway, Rank varieties for a class of finite-dimensional local algebras, J. Pure Appl. Algebra 211 (2) (2007) 497–510.

[4] D. Benson, S.B. Iyengar, H. Krause, Stratifying triangulated categories, J. Topol. 4 (3) (2011) 641–666.

[5] D. Benson, S.B. Iyengar, H. Krause, Stratifying modular representations of finite groups, Ann. Math. (2) 174 (3) (2011) 1643–1684.

[6] D. Benson, S. Witherspoon, Examples of support varieties for Hopf algebras with noncommutative tensor products, Arch. Math. (Basel) 102 (6) (2014) 513–520.

[7] P.A. Bergh, K. Erdmann, The Avrunin-Scott theorem for quantum complete intersections, J. Algebra 322 (2) (2009) 479–488.

[8] P.A. Bergh, S. Oppermann, Cohomology of twisted tensor products, J. Algebra 320 (8) (2008) 3327–3338.

[9] P.A. Bergh, Ø. Solberg, Relative support varieties, Q. J. Math. 61 (2) (2010) 171–182.

[10] A.B. Buan, H. Krause, N. Snashall, Ø. Solberg, Support varieties – an axiomatic approach, preprint, arXiv:1710.08685.

[11] J.F. Carlson, The complexity and varieties of modules, in: Integral Representations and Applications, Oberwolfach, 1980, in: Lecture Notes in Math., vol. 882, Springer, Berlin-New York, 1981, pp. 415–422.

[12] J.F. Carlson, The varieties and the cohomology ring of a module, J. Algebra 85 (1) (1983) 104–143.

[13] K. Erdmann, M. Holloway, N. Snashall, Ø. Solberg, R. Taillefer, Support varieties for selfinjective algebras, K-Theory 33 (1) (2004) 67–87.

[14] E.M. Friedlander, J. Pevtsova, π-supports for modules for finite group schemes, Duke Math. J. 139 (2) (2007) 317–368.

[15] M.J. Hopkins, Global methods in homotopy theory, in: Homotopy Theory, Durham, 1985, in: London Math. Soc. Lecture Note Ser., vol. 117, Cambridge Univ. Press, Cambridge, 1987, pp. 73–96.

[16] A. Neeman, The chromatic tower for $D(R)$, with an appendix by Marcel Bökstedt, Topology 31 (3) (1992) 519–532.

[17] S. Oppermann, Hochschild cohomology and homology of quantum complete intersections, Algebra Number Theory 4 (7) (2010) 821–838.

[18] J. Pevtsova, S. Witherspoon, Tensor ideals and varieties for modules of quantum elementary Abelian groups, Proc. Am. Math. Soc. 143 (9) (2015) 3727–3741.

[19] J.Y. Plavnik, S. Witherspoon, Tensor products and support varieties for some noncocommutative Hopf algebras, Algebr. Represent. Theory 21 (2) (2018) 259–276.

[20] N. Snashall, Ø. Solberg, Support varieties and Hochschild cohomology rings, Proc. Lond. Math. Soc. (3) 88 (3) (2004) 705–732.

[21] Ø. Solberg, Support varieties for modules and complexes, in: Trends in Representation Theory of Algebras and Related Topics, in: Contemp. Math., vol. 406, Amer. Math. Soc., Providence, RI, 2006, pp. 239–270.

Paper 2

HIGHER KOSZUL DUALITY AND CONNECTIONS WITH n-HEREDITARY ALGEBRAS

Johanne Haugland Mads Hustad Sandøy

HIGHER KOSZUL DUALITY AND CONNECTIONS WITH n-HEREDITARY ALGEBRAS

JOHANNE HAUGLAND AND MADS HUSTAD SANDØY

Abstract. We establish a connection between two areas of independent interest in representation theory, namely Koszul duality and higher homological algebra. This is done through a generalization of the notion of T-Koszul algebras, for which we prove that an analogue of classical Koszul duality still holds. Our approach is motivated by and has applications for n-hereditary algebras. In particular, we characterize an important class of n-T-Koszul algebras of highest degree a in terms of $(na-1)$-representation infinite algebras. As a consequence, we see that an algebra is n-representation infinite if and only if its trivial extension is $(n+1)$-Koszul with respect to its degree 0 part. Furthermore, we show that when an n-representation infinite algebra is n-representation tame, then the bounded derived categories of graded modules over the trivial extension and over the associated $(n+1)$-preprojective algebra are equivalent. In the n-representation finite case, we introduce the notion of almost n-T-Koszul algebras and obtain similar results.

Contents

2010 *Mathematics Subject Classification.* 16G20, 16S37, 16W50, 18E30, 18G20.
Key words and phrases. Generalized Koszul algebra, Auslander–Reiten theory, preprojective algebra, n-representation finite algebra, n-representation infinite algebra, graded Frobenius algebra.

1. Introduction

Global dimension is a useful measure for the objects one studies in representation theory of finite dimensional algebras. However, while algebras of global dimension 0 and 1 are exceptionally well understood, it seems quite difficult to develop a general theory for algebras of higher global dimension. This is a background for studying the class of n-hereditary algebras [6,9,12,13,15,16,19–21]. These algebras play an important role in higher Auslander–Reiten theory [17,18,24], which has been shown to have connections to commutative algebra, both commutative and non-commutative algebraic geometry, combinatorics, and conformal field theory [1,7,14,22,32]. An n-hereditary algebra has global dimension less than or equal to n and is either *n-representation finite* or *n-representation infinite*. As one might expect, these notions coincide with the classical definitions of representation finite and infinite hereditary algebras in the case $n = 1$.

Like in the classical theory, n-hereditary algebras have a notion of (higher) preprojective algebras. If A is n-representation infinite and the $(n+1)$-preprojective $\Pi_{n+1}A$ is graded coherent, there is an equivalence $\mathcal{D}^b(\operatorname{mod} A) \simeq \mathcal{D}^b(\operatorname{qgr} \Pi_{n+1}A)$, where $\operatorname{qgr} \Pi_{n+1}A$ denotes the category of finitely presented graded modules modulo finite dimensional modules [29,30].

On the other hand, the bounded derived category of a finite dimensional algebra of finite global dimension is always equivalent to the stable category of finitely generated graded modules over its trivial extension [11]. Combining these two equivalences, and using the notation ΔA for the trivial extension of A, one obtains

$$\underline{\operatorname{gr}}(\Delta A) \simeq \mathcal{D}^b(\operatorname{qgr} \Pi_{n+1}A). \tag{1.1}$$

The equivalence above brings to mind the acclaimed Bernšteĭn–Gel′fand–Gel′fand-correspondence, which can be formulated as $\underline{\operatorname{gr}}\,\Lambda \simeq \mathcal{D}^b(\operatorname{qgr} \Lambda^!)$ for a finite dimensional Frobenius Koszul algebra Λ and its graded coherent Artin–Schelter regular Koszul dual $\Lambda^!$ [4]. The BGG-correspondence is known to descend from the Koszul duality equivalence between bounded derived categories of graded modules over the two algebras, as indicated in the following diagram

$$\begin{tikzcd}
\mathcal{D}^b(\operatorname{gr} \Lambda) \arrow[r, "\simeq"] \arrow[d] & \mathcal{D}^b(\operatorname{gr} \Lambda^!) \arrow[d] \\
\underline{\operatorname{gr}}\,\Lambda \arrow[r, dashed, "\simeq"] & \mathcal{D}^b(\operatorname{qgr} \Lambda^!).
\end{tikzcd}$$

It is natural to ask whether something similar is true in the n-representation infinite case. i.e. if the equivalence (1.1) is a consequence of some higher Koszul duality pattern. This is a motivating question for this paper.

Motivating question. Is the equivalence (1.1) a consequence of some higher Koszul duality pattern?

One reasonable approach to this question is to study generalizations of the notion of Koszulity. A positively graded algebra Λ generated in degrees 0 and 1 with semisimple degree 0 part is known as a Koszul algebra if Λ_0 is a graded self-orthogonal module over Λ [3,33]. This means that $\operatorname{Ext}^i_{\operatorname{gr}\Lambda}(\Lambda_0, \Lambda_0\langle j\rangle) = 0$ whenever $i \neq j$, where $\langle - \rangle$ denotes the graded shift. Using basic facts about Serre functors and triangulated equivalences, one can show that a similar statement holds for ΔA with respect to its degree 0 part $(\Delta A)_0 = A$ in the case where A is n-representation infinite. Here, the algebra A is clearly not necessarily semisimple, but it is of finite global dimension.

In [10] Green, Reiten and Solberg present a notion of Koszulity for more general graded algebras, where the degree 0 part is allowed to be an arbitrary finite dimensional algebra. Their work provides a unified approach to Koszul duality and tilting equivalence. Koszulity in this framework is defined with respect to a module T, and thus the algebras are called T-Koszul. Madsen [28] gives a simplified definition of T-Koszul algebras, which he shows to be a generalization of the original one whenever the degree 0 part is of finite global dimension.

We generalize Madsen's definition to obtain the notion of *n-T-Koszul algebras*, where n is a positive integer and $n = 1$ returns Madsen's theory. In Theorem 3.9 we prove that an analogue of classical Koszul duality holds in this generality, and we recover a version of the BGG-correspondence in Proposition 3.11. Moreover, Theorem 6.4 provides a characterization of an important class of n-T-Koszul algebras of highest degree a in terms of $(na-1)$-representation infinite algebras. More precisely, we show that a finite dimensional graded Frobenius algebra of highest degree $a \geq 1$ is n-T-Koszul if and only if $\widetilde{T} = \oplus_{i=0}^{a-1}\Omega^{-ni}T\langle i\rangle$ is a tilting object in the associated stable category and the endomorphism algebra of this object is $(na-1)$-representation infinite. As a consequence, we see in Corollary 6.6 that an algebra is n-representation infinite if and only if its trivial extension is $(n+1)$-Koszul with respect to its degree 0 part. Furthermore, we show in Corollary 6.9 that when A is n-representation infinite, then the higher Koszul dual of its trivial extension is given by the associated $(n+1)$-preprojective algebra. Combining this with our version of the BGG-correspondence, Corollary 6.10 gives an affirmative answer to our motivating question. In particular, we see that when an n-representation infinite algebra A is n-representation tame, then the bounded derived categories of graded modules over ΔA and over $\Pi_{n+1}A$ are equivalent, and that this descends to give an equivalence $\underline{\operatorname{gr}}(\Delta A) \simeq \mathcal{D}^b(\operatorname{qgr}\Pi_{n+1}A)$. Notice that in some sense, the theory we develop is a generalized Koszul dual version of parts of [30].

Having developed our theory for one part of the higher hereditary dichotomy, we ask and provide an answer to whether something similar holds in the higher representation finite case. Inspired by and seeking to generalize the notion of almost Koszul algebras as developed by Brenner, Butler and King [5], we arrive at the definition of *almost n-T-Koszul algebras*. This enables us to show a similar characterization result as in the n-T-Koszul case, namely Theorem 7.17.

This paper is organized as follows. In Section 2 we highlight relevant facts about graded algebras, before the definition and general theory of n-T-Koszul algebras is presented in Section 3. In Section 4 we give an overview of the notions of tilting objects and Serre functors, and construct an equivalence which will be heavily used later on. As a foundation for the rest of the paper, Section 5 is devoted to recalling definitions and known facts about n-hereditary algebras. Note that this section does not contain new results. In Section 6 we state and prove our results on the connections between n-T-Koszul algebras and higher representation infinite algebras. Finally, almost n-T-Koszul algebras are introduced in Section 7, and we develop their theory along the same lines as was done in Section 6.

1.1. **Conventions and notation.** Throughout this paper, let k be an algebraically closed field and n a positive integer. All algebras are algebras over k. We denote by D the duality $D(-) = \mathrm{Hom}_k(-, k)$.

Notice that A and B always denote ungraded algebras, while the notation Λ and Γ is used for graded algebras. We work with right modules, homomorphisms act on the left of elements, and we write the composition of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ as $g \circ f$. We denote by $\mathrm{Mod}\, A$ the category of A-modules and by $\mathrm{mod}\, A$ the category of finitely presented A-modules.

We write the composition of arrows $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ in a quiver as $\alpha\beta$. In our examples, we use diagrams to represent indecomposable modules. This convention is explained in more detail in Example 6.5.

Given a set of objects $\mathcal{U}$ in an additive category $\mathcal{A}$, we denote by $\mathrm{add}\,\mathcal{U}$ the full subcategory of $\mathcal{A}$ consisting of direct summands of finite direct sums of objects in $\mathcal{U}$. If $\mathcal{A}$ is triangulated, we use the notation $\mathrm{Thick}_{\mathcal{A}}(\mathcal{U})$ for the smallest thick subcategory of $\mathcal{A}$ which contains $\mathcal{U}$. When it is clear in which category our thick subcategory is generated, we often omit the subscript $\mathcal{A}$.

Moreover, note that we have certain standing assumptions given at the beginning of Section 3 and Section 6.

2. Preliminaries

In this section we recall some facts about graded algebras which will be used later in the paper. In particular, we observe how a graded algebra can be considered as a dg-category concentrated in degree 0. This plays an important role in our proofs in Section 3. We also provide an introduction to a class of algebras which will be studied in Section 6 and Section 7, namely the graded Frobenius algebras.

2.1. **Graded algebras, modules and extensions.** Consider a graded k-algebra $\Lambda = \oplus_{i \in \mathbb{Z}} \Lambda_i$. The category of graded Λ-modules and degree 0 morphisms is denoted by $\mathrm{Gr}\,\Lambda$ and the subcategory of finitely presented graded Λ-modules by $\mathrm{gr}\,\Lambda$. Recall that $\mathrm{gr}\,\Lambda$ is abelian if and only if Λ is graded right coherent, i.e. if every finitely generated homogeneous right ideal is finitely presented.

Given a graded module $M = \oplus_{i\in\mathbb{Z}} M_i$, we define the *$j$-th graded shift* of M to be the graded module $M\langle j\rangle$ with $M\langle j\rangle_i = M_{i-j}$. The following basic result relates ungraded extensions to graded ones.

Lemma 2.1. *(See* [31, Corollary 2.4.7]*.) Let M and N be graded Λ-modules. If M is finitely generated and there is a projective resolution of M such that all syzygies are finitely generated, then*

$$\operatorname{Ext}^i_\Lambda(M,N) \simeq \bigoplus_{j\in\mathbb{Z}} \operatorname{Ext}^i_{\operatorname{Gr}\Lambda}(M, N\langle j\rangle)$$

for all $i \geq 0$.

A non-zero graded module $M = \oplus_{i\in\mathbb{Z}} M_i$ is said to be *concentrated in degree m* if $M_i = 0$ for $i \neq m$. When Λ is finite dimensional and M finitely generated, there is an integer h such that $M_h \neq 0$ and $M_i = 0$ for every $i > h$. We call h the *highest degree* of M. In the same way, the *lowest degree* of M is the integer l such that $M_l \neq 0$ and $M_i = 0$ for every $i < l$.

2.2. **Graded algebras as dg-categories.** Recall that a dg-category is a k-linear category in which the morphism spaces are complexes over k and the composition is given by chain maps. We refer to [25] for general background on dg-categories.

In [27, Section 4] it is explained how one can encode the information of a graded algebra as a dg-category concentrated in degree 0. This is useful, as it enables us to apply known techniques developed for dg-categories to get information about the derived category of graded modules. Let us briefly recall this construction, emphasizing the part which will be useful in Section 3.

Given a graded algebra $\Lambda = \oplus_{i\in\mathbb{Z}}\Lambda_i$, we associate the category $\mathcal{A}$, in which $\operatorname{Ob}(\mathcal{A}) = \mathbb{Z}$ and the morphisms are given by $\operatorname{Hom}_\mathcal{A}(i,j) = \Lambda_{i-j}$. Multiplication in Λ yields composition in $\mathcal{A}$ in the natural way. Observe that the Hom-sets of $\mathcal{A}$ behaves well with respect to addition in $\mathbb{Z}$, namely that for any integers i and j, we have

$$\operatorname{Hom}_\mathcal{A}(i,0) \simeq \operatorname{Hom}_\mathcal{A}(i+j,j). \tag{2.1}$$

The category of right modules over $\mathcal{A}$, meaning k-linear functors from $\mathcal{A}^{\mathrm{op}}$ into $\operatorname{Mod} k$, is equivalent to $\operatorname{Gr}\Lambda$. Similarly, as $\mathcal{A}$ is a dg-category concentrated in degree 0, dg-modules over $\mathcal{A}$ correspond to complexes of graded Λ-modules. Consequently, one obtain $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\operatorname{Gr}\Lambda)$, i.e. that the derived category of the dg-category $\mathcal{A}$ is equivalent to the usual derived category of $\operatorname{Gr}\Lambda$.

Instead of starting with a graded algebra, one can use this construction the other way around. Given a dg-category $\mathcal{A}$ concentrated in degree 0, for which the objects are in bijection with the integers and the condition (2.1) is satisfied, we can identify the category with the graded algebra

$$\Lambda = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_\mathcal{A}(i,0),$$

in the sense that $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\operatorname{Gr}\Lambda)$. Notice that the fact that certain Hom-sets coincide is necessary in order to be able to use composition in our category to define multiplication in Λ.

2.3. **Graded Frobenius algebras.** Recall that *twisting* by a graded algebra automorphism ϕ of a graded algebra Λ yields an autoequivalence $(-)_\phi$ on $\operatorname{gr}\Lambda$. Given M in $\operatorname{gr}\Lambda$, the module M_ϕ is defined to be equal to M as a vector space with right Λ-action $m \cdot \lambda = m\phi(\lambda)$, while $(-)_\phi$ acts trivially on morphisms.

A finite dimensional positively graded algebra Λ will be called *graded Frobenius* if $D\Lambda \simeq \Lambda\langle -a\rangle$ as both graded left and graded right Λ-modules for some integer a. Notice that if Λ is concentrated in degree 0, we recover the usual notion of a Frobenius algebra. Observe also that the integer a in our definition must be equal to the highest degree of Λ, as $(D\Lambda)_i = D(\Lambda_{-i})$. We will usually assume $a \geq 1$.

Being graded Frobenius is equivalent to being Frobenius as an ungraded algebra and having a grading such that the socle is contained in the highest degree.

Lemma 2.2. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a finite dimensional algebra of highest degree a. The following are equivalent:*

(1) Λ is graded Frobenius.
(2) There exists a graded automorphism μ of Λ such that ${}_1\Lambda_\mu\langle -a\rangle \simeq D\Lambda$ as graded Λ-bimodules.
(3) Λ is Frobenius as ungraded algebra and has a grading satisfying $\operatorname{Soc}\Lambda \subseteq \Lambda_a$.

Proof. If Λ is graded Frobenius, [30, Lemma 2.9] implies that there exists a graded automorphism μ of Λ such that

$$D\Lambda \simeq {}_1\Lambda_\mu\langle -a\rangle \simeq {}_{\mu^{-1}}\Lambda_1\langle -a\rangle$$

as graded Λ-bimodules. It is hence clear that *(1)* is equivalent to *(2)*.

To see that *(1)* is equivalent to *(3)*, use that graded lifts of finite dimensional modules are unique up to isomorphism and graded shift [3, Lemma 2.5.3] together with the fact that $\operatorname{Soc} D\Lambda \subseteq (D\Lambda)_0$. □

The automorphism μ of a Frobenius algebra Λ as in the lemma above, is unique up to composition with an inner automorphism and is known as the *graded Nakayama automorphism* of Λ. We call Λ *graded symmetric* if μ can be chosen to be trivial, and note that this notion also descends to the ungraded case.

One class of examples which will be important for us, is that of trivial extension algebras. Recall that given a finite dimensional algebra A, the *trivial extension of A* is $\Delta A \coloneqq A \oplus DA$ as a vector space. The trivial extension is an algebra with multiplication $(a, f) \cdot (b, g) = (ab, ag + fb)$ for $a, b \in A$ and $f, g \in DA$. We consider ΔA as a graded algebra by letting A be in degree 0 and DA be in degree 1. Observe that ΔA is graded symmetric as it is symmetric as an ungraded algebra and satisfies $\operatorname{Soc}\Delta A \subseteq (\Delta A)_1$.

The stable category of finitely presented graded modules over a graded algebra Λ is denoted by $\underline{\operatorname{gr}}\,\Lambda$. If Λ is self-injective, the category $\operatorname{gr}\Lambda$ is a Frobenius category, and $\underline{\mathrm{gr}}\,\Lambda$ is triangulated with shift functor $\Omega^{-1}(-)$. Notice that every Frobenius algebra is self-injective. Observe that twisting by a graded automorphism ϕ of Λ descends to an autoequivalence $(-)_\phi$ on $\underline{\mathrm{gr}}\,\Lambda$. This functor commutes with taking syzygies and cosyzygies, as well as with graded shift.

We will often consider syzygies and cosyzygies of modules over self-injective algebras even when we do not work in a stable category. Whenever we do so, we assume having chosen a minimal projective or injective resolution, so that our syzygies and cosyzygies do not have any non-zero projective summands. Because of our convention with respect to (representatives of) syzygies and cosyzygies, the notions of highest and lowest degree make sense for these too.

Throughout the paper, we often need to consider basic degree arguments, as summarized in the following lemma. We include a short proof for the convenience of the reader.

Lemma 2.3. *Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a finite dimensional self-injective graded algebra of highest degree a and $\operatorname{Soc}\Lambda \subseteq \Lambda_a$. The following statements hold:*

(1) Given any non-zero element $x \in \Lambda$, there exists $\lambda \in \Lambda$ such that $x\lambda \in \Lambda_a$ is non-zero.

(2) Let P be an indecomposable projective graded Λ-module of highest degree h. Then, given any non-zero element $x \in P$, there exists $\lambda \in \Lambda$ such that $x\lambda \in P_h$ is non-zero.

(3) Let M and P be finitely generated graded Λ-modules with P indecomposable projective. Denote the highest degree of P by h. Then, for every non-zero morphism $f \in \operatorname{Hom}_{\mathrm{gr}\,\Lambda}(M,P)$, there exists an element $x \in M$ such that $f(x) \in P_h$ is non-zero.

(4) Let M be an non-projective finitely generated graded Λ-module of highest degree h and lowest degree l. Then the highest degree of $\Omega^i M$ is less than or equal to h in the case $i \leq 0$ and greater than or equal to $l+a$ in the case $i > 0$.

(5) Assume $a \geq 1$, and let M and N be modules concentrated in degree 0. Then

$$\operatorname{Hom}_{\underline{\mathrm{gr}}\,\Lambda}(M,N) \simeq \operatorname{Hom}_{\mathrm{gr}\,\Lambda}(M,N).$$

(6) Let M be a module concentrated in degree 0. Then

$$\operatorname{Hom}_{\mathrm{gr}\,\Lambda}(M,\Omega^i M\langle j\rangle) = 0$$

for $i,j < 0$.

(7) Let M be a module concentrated in degree 0. Then

$$\operatorname{Hom}_{\mathrm{gr}\,\Lambda}(M,\Omega^i M\langle j\rangle) = 0$$

for $i > 0$ and $j \geq 1-a$.

Proof. Combining the assumption $\operatorname{Soc}\Lambda \subseteq \Lambda_a$ with the facts that $\operatorname{Rad}\Lambda$ is nilpotent and $\operatorname{Soc}\Lambda = \{y \in \Lambda \mid y\operatorname{Rad}\Lambda = 0\}$, one obtains *(1)*.

Part *(2)* follows from *(1)*, as projectives are direct summands of free modules.

For *(3)*, let $y \in M$ such that $f(y) \neq 0$. By *(2)*, there exists an element $\lambda \in \Lambda$ such that $f(y)\lambda \in P_h$ is non-zero. Consequently, the element $x = y\lambda$ yields our desired conclusion.

In order to prove *(4)*, let us first consider the case $i \leq 0$. The statement clearly holds if $i = 0$. Observe next that $\operatorname{Soc} M$ has highest degree h. Hence, the injective envelope of M also has highest degree h. Since M is non-projective, the cosyzygy $\Omega^{-1}M$ is a non-zero quotient of this injective envelope, and consequently has highest degree at most h. We are thus done by induction.

For the case $i > 0$, note that each summand in the projective cover of M has highest degree greater than or equal to $l + a$. As ΩM is a submodule of this projective cover, it follows from *(3)* that ΩM also has highest degree greater than or equal to $l + a$. Moreover, the syzygy is itself non-projective of lowest degree greater than or equal to l, so the claim follows by induction.

To verify *(5)*, notice that there can be no non-zero homomorphism $M \to N$ factoring through a Λ-projective. Otherwise, one would have non-zero homomorphisms $M \to \Lambda\langle i\rangle$ and $\Lambda\langle i\rangle \to N$ for some integer i. The former is possible only if $i = -a$ by *(3)*. However, if $i = -a$, the latter is impossible as $\Lambda\langle -a\rangle$ is generated in degree $-a$.

Observe that *(6)* is immediate in the case where M is projective. Otherwise, note that the highest degree of $\Omega^i M$ is at most 0 by *(4)*. Hence, the highest degree of $\Omega^i M\langle j\rangle$ is less than or equal to j. As $j < 0$, this yields our desired conclusion.

For *(7)*, it again suffices to consider the case where M is non-projective. Applying *(4)*, our assumptions yield that the highest degree of $\Omega^i M\langle j\rangle$ is greater than or equal to 1. By *(3)*, this gives $\operatorname{Hom}_{\operatorname{gr}\Lambda}(M, \Omega^i M\langle j\rangle) = 0$, as syzygies are submodules of projectives. $\square$

3. Higher Koszul duality

Throughout the rest of this paper, let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a positively graded algebra, where Λ_0 is a finite dimensional algebra augmented over $k^{\times r}$ for some $r > 0$. We assume that Λ is locally finite dimensional, i.e. that Λ_i is finite dimensional as a vector space over k for each $i \geq 0$.

In this section we define more flexible notions of what it means for a module T to be graded self-orthogonal and an algebra to be T-Koszul than the ones Madsen introduces in [28, Definition 3.1.1 and 4.1.1]. This enables us to talk about higher T-Koszul duality for a more general class of algebras. In particular, we obtain a higher Koszul duality equivalence in Theorem 3.9 and we recover a version of the BGG-correspondence in Proposition 3.11. Note that the ideas in this section are similar to the ones in [28]. For the convenience of the reader, we nevertheless give concise proofs of this section's main results, to show that the arguments work also in our generality.

It should be noted that it is also possible to derive Theorem 3.9 by using [28, Theorem 4.3.4]. This strategy involves regrading the algebras so that they satisfy

Madsen's definition of graded self-orthogonality and tracking our original (derived) categories of graded modules through his equivalence. We spell this out in greater detail after our proof of Theorem 3.9. Proceeding in this way, one can recover generalized analogues of many of the results in [28]. We make no essential use of these results, but this approach could be relevant for future related work.

We remark that we believe it to be undesirable to work with the regraded algebras throughout, since – as will become clear – the resulting graded module categories are in some sense too big. Moreover, we consider endomorphism algebras of tilting objects, and it is less convenient to study regraded versions of these. In particular, as we want to relate our results to existing ones involving graded modules over trivial extensions or preprojective algebras, we cannot always work directly with the regraded algebras.

In order to state our main definitions, let us first recall the notion of a tilting module.

Definition 3.1. Let A be a finite dimensional algebra. A finitely generated A-module T is called a *tilting module* if the following conditions hold:

(1) $\operatorname{proj.dim}_A T < \infty$;
(2) $\operatorname{Ext}^i_A(T,T) = 0$ for $i > 0$;
(3) There is an exact sequence
$$0 \to A \to T^0 \to T^1 \to \cdots \to T^l \to 0$$
with $T^i \in \operatorname{add} T$ for $i = 0, \ldots, l$.

We now define what it means for a module to be graded n-self-orthogonal.

Definition 3.2. Let T be a finitely generated basic graded Λ-module concentrated in degree 0. We say that T is *graded n-self-orthogonal* if
$$\operatorname{Ext}^i_{\operatorname{gr}\Lambda}(T, T\langle j\rangle) = 0$$
for $i \neq nj$.

Usually, it will be clear from context what the parameter n is, so we often simply say that a module satisfying the description above is graded self-orthogonal.

Notice that this definition of graded self-orthogonality is more general than the one given in [28]. More precisely, the two definitions coincide exactly when n is equal to 1. In this case, examples of graded self-orthogonal modules are given by Λ_0 in the classical Koszul situation or tilting modules if $\Lambda = \Lambda_0$. Moreover, we see in Section 6 that n-representation infinite algebras provide examples of modules which are graded n-self-orthogonal for any choice of n.

In general, a graded self-orthogonal module T might have syzygies which are not finitely generated, so Lemma 2.1 does not apply. However, the following proposition gives a similar result for graded self-orthogonal modules. This is an analogue of [28, Proposition 3.1.2]. The proof is exactly the same, except that we use our more general version of what it means for T to be graded self-orthogonal.

Proposition 3.3. *Let T be a graded n-self-orthogonal Λ-module. Then*

$$\operatorname{Ext}^{ni}_{\Lambda}(T,T) \simeq \operatorname{Ext}^{ni}_{\operatorname{gr}\Lambda}(T,T\langle i\rangle)$$

for all $i \geq 0$.

Using our definition of a graded self-orthogonal module T, we also get a more general notion of what it means for an algebra to be Koszul with respect to T.

Definition 3.4. Assume $\operatorname{gl.dim}\Lambda_0 < \infty$ and let T be a graded Λ-module concentrated in degree 0. We say that Λ is *n-T-Koszul* or *n-Koszul with respect to T* if the following conditions hold:

(1) T is a tilting Λ_0-module.
(2) T is graded n-self-orthogonal as a Λ-module.

Remark 3.5. In Definition 3.2 we require a graded n-self-orthogonal module to be basic for consistency with [28]. As a consequence of this choice, we later assume that certain algebras are basic, for instance in Corollary 6.6. Note that this assumption does not usually play an important role in our proofs, and could be omitted if one is willing to consider n-Koszul algebras with respect to a possibly non-basic module T.

Like in the classical theory, we want a notion of a Koszul dual of a given n-T-Koszul algebra.

Definition 3.6. Let Λ be an n-T-Koszul algebra. The *n-T-Koszul dual of* Λ is given by $\Lambda^! = \oplus_{i\geq 0}\operatorname{Ext}^{ni}_{\operatorname{gr}\Lambda}(T,T\langle i\rangle)$.

Note that while the notation for the n-T-Koszul dual is potentially ambiguous, it will in this paper always be clear from context which n-T-Koszul structure the dual is computed with respect to.

By Proposition 3.3, we get the following equivalent description of the n-T-Koszul dual.

Corollary 3.7. *Let Λ be an n-T-Koszul algebra. Then there is an isomorphism of graded algebras $\Lambda^! \simeq \oplus_{i\geq 0}\operatorname{Ext}^{ni}_{\Lambda}(T,T)$.*

Given a set of objects $\mathcal{U} \subseteq \mathcal{D}^b(\operatorname{gr}\Lambda)$, let $\operatorname{Thick}^{\langle - \rangle}(\mathcal{U})$ denote the smallest thick subcategory of $\mathcal{D}^b(\operatorname{gr}\Lambda)$ which contains $\mathcal{U}$ and is closed under graded shift. Using that Λ_0 has finite global dimension and that T is a tilting Λ_0-module, one obtains that T generates the entire bounded derived category of $\operatorname{gr}\Lambda$ whenever Λ is an n-T-Koszul algebra.

Lemma 3.8. *Let Λ be a finite dimensional n-T-Koszul algebra. Then* $\operatorname{Thick}^{\langle - \rangle}(T) = \mathcal{D}^b(\operatorname{gr}\Lambda)$.

The proof of Theorem 3.9 uses notions and techniques of dg-homological algebra. Since this is the only section where these are used, we refer the reader to [25] for an introduction. Notice that we have more or less adopted the notation of that source for the reader's convenience. In particular, recall from [25] that given a dg-category $\mathcal{B}$, we define the category $\mathrm{H}^0\,\mathcal{B}$ to have the same objects as $\mathcal{B}$ and morphisms given by taking the 0-th cohomology of the morphism spaces in $\mathcal{B}$. Similarly, also the category $\tau_{\leq 0}\,\mathcal{B}$ has the same objects as $\mathcal{B}$, and morphisms given by taking subtle truncation.

We are now ready to state and prove the main result of this section, namely to show that we obtain a higher Koszul duality equivalence. This recovers [28, Theorem 4.3.4] in the case where $n = 1$ and is a version of [3, Theorem 2.12.6] in the classical Koszul case.

Theorem 3.9. *Let Λ be a finite dimensional n-T-Koszul algebra and assume that $\Lambda^!$ is graded right coherent and has finite global dimension. Then there is an equivalence $\mathcal{D}^b(\operatorname{gr}\Lambda) \simeq \mathcal{D}^b(\operatorname{gr}\Lambda^!)$ of triangulated categories.*

Proof. Consider the full subcategory $\mathcal{U} = \{T\langle i\rangle[ni] \mid i \in \mathbb{Z}\}$ of $\mathcal{D}^b(\operatorname{gr}\Lambda)$. Using a standard lift [25, Section 7.3], we replace $\mathcal{U}$ by a dg-category $\mathcal{B}$ which has objects $\{P\langle i\rangle[ni]\}$, where P is some graded projective resolution of T, and

$$\operatorname{Hom}_{\mathcal{B}}(P\langle i\rangle[ni], P\langle j\rangle[nj])^k = \prod_{m\in\mathbb{Z}} \operatorname{Hom}_{\operatorname{gr}\Lambda}(P^{m+ni}\langle i\rangle, P^{m+nj+k}\langle j\rangle).$$

In other words, morphism spaces are given by all homogeneous maps of complexes that are also homogeneous of degree 0 with respect to the grading of Λ. The morphism spaces are complexes with the standard super commutator differential defined by

$$d(f) = d_{P\langle j\rangle[nj]} \circ f - (-1)^k f \circ d_{P\langle i\rangle[ni]}$$

for f in $\operatorname{Hom}_{\mathcal{B}}(P\langle i\rangle[ni], P\langle j\rangle[nj])^k$.

Notice that $\operatorname{Thick}(\mathcal{U}) = \operatorname{Thick}^{\langle - \rangle}(T) = \mathcal{D}^b(\operatorname{gr}\Lambda)$. Since we have used a standard lift and idempotents split in $\mathcal{D}^b(\operatorname{gr}\Lambda)$, we get that $\operatorname{Thick}(\mathcal{U}) = \mathcal{D}^b(\operatorname{gr}\Lambda)$ is equivalent to $\mathcal{D}^{\mathrm{perf}}(\mathcal{B})$, i.e. the subcategory of perfect objects.

As T is graded n-self-orthogonal, the cohomology of each morphism space in $\mathcal{B}$ is concentrated in cohomological degree 0. Hence, we get a zigzag of dg-categories

$$\mathrm{H}^0\,\mathcal{B} \twoheadleftarrow \tau_{\leq 0}\mathcal{B} \hookrightarrow \mathcal{B}$$

in which the dg-functors induce quasi-equivalences. Thus, we also get an equivalence $\mathcal{D}(\mathrm{H}^0\,\mathcal{B}) \simeq \mathcal{D}(\mathcal{B})$ [25, Sec. 7.1-7.2 and 9.1]. This equivalence descends to one on the compact or perfect objects, and so we get $\mathcal{D}^{\mathrm{perf}}(\mathrm{H}^0\,\mathcal{B}) \simeq \mathcal{D}^{\mathrm{perf}}(\mathcal{B})$.

The dg-category $\mathrm{H}^0\,\mathcal{B}$ is concentrated in degree 0, its objects are in natural bijection with the integers and we can identify it with a graded algebra as described in Section 2.2. As we wish this algebra to be positively graded, we let the object $P\langle i\rangle[ni]$ in $\mathrm{H}^0\,\mathcal{B}$ correspond to the integer $-i$. This yields the algebra

$$\bigoplus_{i\geq 0} \operatorname{Hom}_{\mathrm{H}^0\,\mathcal{B}}(P, P\langle i\rangle[ni]) \simeq \bigoplus_{i\geq 0} \operatorname{Ext}^{ni}_{\operatorname{gr}\Lambda}(T, T\langle i\rangle) = \Lambda^!.$$

It now follows that $\mathcal{D}(\mathrm{H}^0\,\mathcal{B}) \simeq \mathcal{D}(\operatorname{Gr}\Lambda^!)$, which again yields an equivalence $\mathcal{D}^{\mathrm{perf}}(\mathrm{H}^0\,\mathcal{B}) \simeq \mathcal{D}^{\mathrm{perf}}(\operatorname{Gr}\Lambda^!)$. As in the ungraded case, compact objects of $\mathcal{D}(\operatorname{Gr}\Lambda^!)$ coincides with perfect complexes, i.e. bounded complexes of finitely generated graded projective modules [25, Theorem 5.3]. Hence, as $\Lambda^!$ is graded right coherent of finite global dimension, we also have the equivalence $\mathcal{D}^{\mathrm{perf}}(\operatorname{Gr}\Lambda^!) \simeq \mathcal{D}^b(\operatorname{gr}\Lambda^!)$, which completes our proof. □

Let us now provide more details on how to obtain the above theorem and generalized analogues of other results in [28] using the equivalence constructed there. Observe first that given $\Lambda = \oplus_{i\geq 0}\Lambda_i$ satisfying the assumptions in Theorem 3.9, one can rescale the grading so that the regraded algebra Λ^ρ is T-Koszul in the sense of [28, Definition 4.1.1]. To be precise, let $\Lambda^\rho_i = \Lambda_j$ if $i = nj$ for some integer j and $\Lambda^\rho_i = 0$ otherwise. The category $\operatorname{gr}\Lambda$ embeds into $\operatorname{gr}\Lambda^\rho$ as the full subcategory consisting of modules which are non-zero only in degrees multiples of n. As the embedding is exact, it induces a triangulated functor between the corresponding derived categories. By [37, Lemma 13.17.4], this functor yields an equivalence $\mathcal{D}^b(\operatorname{gr}\Lambda) \xrightarrow{\simeq} \mathcal{D}^b_{\operatorname{gr}\Lambda}(\operatorname{gr}\Lambda^\rho)$, where $\mathcal{D}^b_{\operatorname{gr}\Lambda}(\operatorname{gr}\Lambda^\rho)$ denotes the full subcategory of $\mathcal{D}^b(\operatorname{gr}\Lambda^\rho)$ consisting of objects with cohomology in $\operatorname{gr}\Lambda$.

Using that Λ^ρ is T-Koszul and noticing that $(\Lambda^!)^\rho \simeq (\Lambda^\rho)^!$, we get by [28, Theorem 4.3.4] the equivalence in the upper row of the diagram

$$\begin{tikzcd}
 & \mathcal{D}^b(\operatorname{gr}\Lambda^\rho) \arrow[r, "\simeq"] & \mathcal{D}^b(\operatorname{gr}(\Lambda^!)^\rho) & \\
\mathcal{D}^b(\operatorname{gr}\Lambda) \arrow[r, "\simeq"] & \mathcal{D}^b_{\operatorname{gr}\Lambda}(\operatorname{gr}\Lambda^\rho) \arrow[u, hook] \arrow[r, dashed, "\simeq"] & \mathcal{D}^b_{\operatorname{gr}\Lambda^!}(\operatorname{gr}(\Lambda^!)^\rho) \arrow[u, hook] & \mathcal{D}^b(\operatorname{gr}\Lambda^!). \arrow[l, "\simeq"']
\end{tikzcd}$$

In order to deduce Theorem 3.9 from this, we need to show that the equivalence restricts as indicated by the dashed arrow. It is sufficient to show that objects which are non-zero only in degrees multiples of n are sent to objects satisfying the same property. Examining the construction of the equivalence, we see that it is essentially the same as the one given in the proof of Theorem 3.9 in the case $n = 1$. Consequently, we are done if the equivalences in the zig-zag and the equivalence from $\operatorname{Thick}(\mathcal{U})$ to $\mathcal{D}^{\mathrm{perf}}(\mathcal{B})$ satisfy the desired condition.

For the former equivalences, this is easily verified and is left to the reader, whereas for the latter, we begin by first recalling some necessary notions. Let $\mathcal{A}$ be the dg-category obtained by regarding the graded algebra Λ^ρ as a category as outlined in Section 2.2, and recall that $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\operatorname{Gr}\Lambda^\rho)$. Moreover, see [25, Section 1.2] for the definition of the dg-category $\operatorname{Dif}\mathcal{A}$, and [25, Section 6.2] for the definition of the triangulated functor $\mathbf{R}\,\mathrm{H}_X$ for X an $\mathcal{A}$-$\mathcal{B}$-dg-bimodule. If $\mathcal{A}$ is an ordinary algebra concentrated in cohomological degree 0, the objects of the category $\operatorname{Dif}\mathcal{A}$ are complexes of modules over $\mathcal{A}$ and the morphisms are given by homogeneous maps which do not necessarily respect the differentials. In this case, the functor $\mathbf{R}\,\mathrm{H}_X$ would be quasi-isomorphic to regular $\mathbf{R}\,\mathrm{Homs}$. The theory of standard lifts [25, Section 7.3] implies that the equivalence $\operatorname{Thick}(\mathcal{U}) \to \mathcal{D}^{\mathrm{perf}}(\mathcal{B})$ is the restriction of the functor $\mathbf{R}\,\mathrm{H}_X\colon \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$, where X is the $\mathcal{A}$-$\mathcal{B}$-dg-bimodule given by $X(j,k)^l = P^{l-k}_{j+k}$, which has property (P) as defined in [25, Section 3.1]. Hence, we get

$$\begin{aligned}\mathbf{R}\,\mathrm{H}_X(M)^l_k &= \operatorname{Hom}_{\operatorname{Dif}\mathcal{A}}(X(?,k),M)^l \\ &= \prod_{m\in\mathbb{Z}} \operatorname{Hom}_{\operatorname{Gr}\Lambda^\rho}(P^{m-k}\langle -k\rangle, M^{m+l}) \\ &\simeq \mathbf{R}\operatorname{Hom}_{\operatorname{Gr}\Lambda^\rho}(P\langle -k\rangle[-k], M)^l.\end{aligned}$$

If n does not divide k, this is zero whenever M is non-zero only in degrees that are multiples of n. Hence, one obtains that Madsen's equivalence between $\mathcal{D}^b(\operatorname{gr}\Lambda^\rho)$ and $\mathcal{D}^b(\operatorname{gr}(\Lambda^!)^\rho)$ restricts to yield an equivalence between $D^b(\operatorname{gr}\Lambda)$ and $D^b(\operatorname{gr}\Lambda^!)$ as claimed.

In our following two propositions, we denote by $K\colon \mathcal{D}^b(\operatorname{gr}\Lambda) \to \mathcal{D}^b(\operatorname{gr}\Lambda^!)$ the equivalence from Theorem 3.9. Since shifting by 1 in $\operatorname{gr}\Lambda$ corresponds to shifting by n in $\operatorname{gr}\Lambda^\rho$, the argument above together with [28, Proposition 3.2.1] yield the following.

Proposition 3.10. *Let Λ be a finite dimensional n-T-Koszul algebra and assume that $\Lambda^!$ is graded right coherent and has finite global dimension. We then have $K(M\langle i\rangle) = K(M)\langle -i\rangle[-ni]$ for $M \in \mathcal{D}^b(\operatorname{gr}\Lambda)$.*

We finish this section by showing that an analogue of the BGG-correspondence holds in our generality. Recall that $\operatorname{qgr}\Lambda^!$ is defined as the localization of $\operatorname{gr}\Lambda^!$ at the full subcategory of finite dimensional graded $\Lambda^!$-modules.

We hence have a natural functor $\mathcal{D}^b(\operatorname{gr}\Lambda^!) \to \mathcal{D}^b(\operatorname{qgr}\Lambda^!)$. In the case where Λ is graded Frobenius, there is a well-known equivalence $\mathcal{D}^b(\operatorname{gr}\Lambda)/\mathcal{D}^{\mathrm{perf}}(\operatorname{gr}\Lambda) \simeq \underline{\operatorname{gr}}\,\Lambda$ [36, Theorem 2.1]. Note that we recall this result as Theorem 4.2 in our next section. One consequently obtains a functor

$$\mathcal{D}^b(\operatorname{gr}\Lambda) \to \mathcal{D}^b(\operatorname{gr}\Lambda)/\mathcal{D}^{\mathrm{perf}}(\operatorname{gr}\Lambda) \xrightarrow{\simeq} \underline{\operatorname{gr}}\,\Lambda.$$

These two functors give the vertical arrows in the diagram in our proposition below.

Proposition 3.11. *Let Λ be a finite dimensional n-T-Koszul algebra and assume that $\Lambda^!$ is graded right coherent and has finite global dimension. If Λ is graded Frobenius, then the equivalence K descends to yield $\underline{\operatorname{gr}}\,\Lambda \simeq \mathcal{D}^b(\operatorname{qgr}\Lambda^!)$, as indicated in the following diagram*

$$\begin{tikzcd} \mathcal{D}^b(\operatorname{gr}\Lambda) \arrow[r, "K"] \arrow[d] & \mathcal{D}^b(\operatorname{gr}\Lambda^!) \arrow[d] \\ \underline{\operatorname{gr}}\,\Lambda \arrow[r, dashed, "\simeq"] & \mathcal{D}^b(\operatorname{qgr}\Lambda^!). \end{tikzcd}$$

Proof. Since $D\Lambda$ is injective, we get that the k-th cohomology of $\mathbf{R}\,\mathrm{H}_X(D\Lambda\langle i\rangle)_j$ is zero unless $k = ni = -nj$, in which case it is isomorphic to

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}^b(\operatorname{gr}\Lambda)}(T, D\Lambda) &\simeq \operatorname{Hom}_{\operatorname{gr}\Lambda}(T, D\Lambda)\\ &\simeq \operatorname{Hom}_{\operatorname{gr}\Lambda^{\mathrm{op}}}(\Lambda, DT)\\ &\simeq DT.\end{aligned}$$

Chasing this through the equivalences in the zig-zag in the proof of Theorem 3.9, we notice that this stalk complex has the $\Lambda^!$-action one expects, i.e. the action induced by letting $\Lambda^!_0 \simeq \operatorname{End}_{\operatorname{gr}\Lambda}(T) \simeq \operatorname{End}_{\Lambda_0}(T)$ act on T on the left by endomorphisms. Our argument above hence yields that K restricts to an equivalence $\operatorname{Thick}^{\langle -\rangle}(D\Lambda) \xrightarrow{\simeq} \operatorname{Thick}^{\langle -\rangle}(DT)$.

Since tilting theory implies that DT is a tilting module over $\operatorname{End}_{\Lambda_0}(T)$, one deduces that $\operatorname{Thick}^{\langle -\rangle}(DT)$ is the full subcategory of $\mathcal{D}^b(\operatorname{gr}\Lambda^!)$ of all objects with finite dimensional cohomology. As $\operatorname{qgr}\Lambda^!$ is the localization of $\operatorname{gr}\Lambda^!$ at the Serre subcategory of finite dimensional $\Lambda^!$-modules and the quotient functor in this case is known to have a left adjoint, we get that

$$\mathcal{D}^b(\operatorname{gr}\Lambda^!)/\operatorname{Thick}^{\langle -\rangle}(DT) \xrightarrow{\simeq} \mathcal{D}^b(\operatorname{qgr}\Lambda^!)$$

is an equivalence by [37, Lemma 13.17.2-3].

The triangulated quotient functor $Q\colon \mathcal{D}^b(\operatorname{gr}\Lambda^!) \to \mathcal{D}^b(\operatorname{gr}\Lambda^!)/\operatorname{Thick}^{\langle -\rangle}(DT)$ has kernel $\operatorname{Thick}^{\langle -\rangle}(DT) \simeq K\operatorname{Thick}^{\langle -\rangle}(D\Lambda)$, and hence composing it with K induces a triangulated functor

$$\overline{K}\colon \mathcal{D}^b(\operatorname{gr}\Lambda)/\operatorname{Thick}^{\langle -\rangle}(D\Lambda) \to \mathcal{D}^b(\operatorname{qgr}\Lambda^!)$$

satisfying $\overline{K}\circ P = Q\circ K$ by the universal property of quotient categories, in which P is the quotient functor

$$P\colon \mathcal{D}^b(\operatorname{gr}\Lambda) \to \mathcal{D}^b(\operatorname{gr}\Lambda)/\operatorname{Thick}^{\langle -\rangle}(D\Lambda).$$

As $\underline{\operatorname{gr}}\,\Lambda \simeq \mathcal{D}^b(\operatorname{gr}\Lambda)/\operatorname{Thick}^{\langle -\rangle}(D\Lambda)$ by [36, Theorem 2.1] and it is straightforward to check that $\overline{K}$ is an equivalence, we are hence done. $\square$

4. Tilting objects, equivalences and Serre functors

Tilting objects and the equivalences they provide play a crucial role throughout the rest of this paper. In this section we recall relevant notions and apply one of Yamaura's ideas to give an explicit construction of an equivalence which will be heavily used in Section 6 and Section 7. We also describe the correspondence of Serre functors induced by this equivalence.

Definition 4.1. Let $\mathcal{T}$ be a triangulated category. An object T in $\mathcal{T}$ is a *tilting object* if the following conditions hold:

(1) $\operatorname{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for $i \neq 0$;
(2) $\operatorname{Thick}_{\mathcal{T}}(T) = \mathcal{T}$.

The first condition in the definition above is often referred to as *rigidity*.

A triangulated category is called *algebraic* if it is triangle equivalent to the stable category of a Frobenius category. Recall that when Λ is a self-injective graded algebra, the category $\operatorname{gr}\Lambda$ is Frobenius, and consequently the stable category $\underline{\operatorname{gr}}\,\Lambda$ is an algebraic triangulated category. By Keller's tilting theorem [25, Theorem 4.3], we hence know that if T is a tilting object in $\underline{\operatorname{gr}}\,\Lambda$ and $B = \operatorname{End}_{\underline{\operatorname{gr}}\,\Lambda}(T)$ has finite global dimension, then there is a triangle equivalence $\underline{\operatorname{gr}}\,\Lambda \simeq \mathcal{D}^b(\operatorname{mod} B)$. While Keller's result is proved by applying general techniques from dg-homological algebra, we need a more explicit description of this equivalence. Recall first that $\underline{\operatorname{gr}}\,\Lambda$ can be realized as the quotient category $\mathcal{D}^b(\operatorname{gr}\Lambda)/\,\mathcal{D}^{\mathrm{perf}}(\operatorname{gr}\Lambda)$.

Theorem 4.2. *(See* [36, Theorem 2.1]*.)* *Let Λ be finite dimensional and self-injective. Then the canonical embedding $\operatorname{gr}\Lambda \to \mathcal{D}^b(\operatorname{gr}\Lambda)$ induces an equivalence $\underline{\operatorname{gr}}\,\Lambda \xrightarrow{\simeq} \mathcal{D}^b(\operatorname{gr}\Lambda)/\,\mathcal{D}^{\mathrm{perf}}(\operatorname{gr}\Lambda)$ of triangulated categories.*

Denote by G the quasi-inverse to the equivalence described in Theorem 4.2 and by P the projection functor $\mathcal{D}^b(\operatorname{gr}\Lambda) \to \mathcal{D}^b(\operatorname{gr}\Lambda)/\,\mathcal{D}^{\mathrm{perf}}(\operatorname{gr}\Lambda)$. As T has a natural structure as a left B-module, we can consider the left derived tensor functor

$$\mathcal{D}^b(\operatorname{mod} B) \xrightarrow{-\otimes^{\mathbf{L}}_B T} \mathcal{D}^b(\operatorname{gr}\Lambda).$$

Note that when we think of the tilting object T in $\underline{\operatorname{gr}}\,\Lambda$ as a graded Λ-module, we choose a representative without projective summands.

We now give an explicit description of the equivalence $\underline{\operatorname{gr}}\,\Lambda \simeq \mathcal{D}^b(\operatorname{mod} B)$. This construction and proof is essentially the same as [38, Proposition 3.14], but we show that it also works in our more general setup.

Proposition 4.3. *Let Λ be finite dimensional and self-injective and assume that $\operatorname{gl.dim}\Lambda_0 < \infty$. Consider a tilting object T in $\underline{\operatorname{gr}}\,\Lambda$ and denote its endomorphism algebra by $B = \operatorname{End}_{\underline{\operatorname{gr}}\,\Lambda}(T)$. Then the composition*

$$F\colon\ \mathcal{D}^b(\operatorname{mod} B) \xrightarrow{-\otimes^{\mathbf{L}}_B T} \mathcal{D}^b(\operatorname{gr}\Lambda) \xrightarrow{P} \mathcal{D}^b(\operatorname{gr}\Lambda)/\,\mathcal{D}^{\mathrm{perf}}(\operatorname{gr}\Lambda) \xrightarrow{G} \underline{\operatorname{gr}}\,\Lambda$$

is an equivalence of triangulated categories.

Proof. Observe first that rigidity of T yields

$$\operatorname{Hom}_{\mathcal{D}^b(B)}(B, B[i]) \simeq \operatorname{Hom}_{\underline{\operatorname{gr}}\,\Lambda}(T, \Omega^{-i}T)$$

for every $i \in \mathbb{Z}$. As $F(B)$ is isomorphic to T in $\underline{\operatorname{gr}}\,\Lambda$, this means that the restriction of F to the subcategory $\mathcal{X} = \{B[i] \mid i \in \mathbb{Z}\}$ is fully faithful. As Λ_0 has finite global dimension, so has B by [38, Corollary 3.12]. Consequently, one obtain

$$\operatorname{Thick}(B) = \operatorname{Thick}(\mathcal{X}) = \mathcal{D}^b(\operatorname{mod} B).$$

Using that $\mathcal{X}$ is closed under translation, this implies that F is fully faithful. Since $\operatorname{Thick}(T) = \underline{\operatorname{gr}}\,\Lambda$ and idempotents split in $\mathcal{D}^b(\operatorname{mod} B)$, the functor F is also essentially surjective, and hence an equivalence. $\square$

In the same way as B is the preimage of T under our equivalence above, we can also describe projective B-modules in terms of summands of T. Given a decomposition $T \simeq \oplus_{i=0}^{t} T^i$ of T, let $e_i \colon T \twoheadrightarrow T^i \hookrightarrow T$ denote the i-th projection followed by the i-th inclusion. This yields a decomposition $B \simeq \oplus_{i=0}^{t} P^i$ of B into projectives $P^i = e_i B$. Notice that the projective P^i is the preimage of the summand T^i under the equivalence F, as $e_i B \otimes_B^{\mathbf{L}} T \simeq e_i T = T^i$.

From Section 6 and on, the following notion will be crucial.

Definition 4.4. Let $\mathcal{T}$ be a k-linear Hom-finite triangulated category. An additive autoequivalence $\mathcal{S}$ on $\mathcal{T}$ is called a *Serre functor* provided there exists a bifunctorial isomorphism

$$\operatorname{Hom}_{\mathcal{T}}(X, Y) \simeq D \operatorname{Hom}_{\mathcal{T}}(Y, \mathcal{S}X)$$

for all objects X and Y in $\mathcal{T}$.

We want to compare the Serre functor on $\mathcal{D}^b(\operatorname{mod} B)$ to that of $\underline{\operatorname{gr}}\,\Lambda$ when Λ is a graded Frobenius algebra of highest degree a with Nakayama automorphism μ. In this case, it follows from Auslander–Reiten duality, see [2] and [35, Proposition I.2.3], combined with the characterization in Lemma 2.2 that $\Omega(-)_\mu\langle -a\rangle$ is a Serre functor on $\underline{\operatorname{gr}}\,\Lambda$. As B is a finite dimensional algebra of finite global dimension, the derived Nakayama functor $\nu(-) = - \otimes_B^{\mathbf{L}} DB$ is a Serre functor on $\mathcal{D}^b(\operatorname{mod} B)$. By uniqueness of the Serre functor, the equivalence F from Proposition 4.3 yields a commutative diagram

$$\begin{tikzcd}
\mathcal{D}^b(\operatorname{mod} B) \arrow[r, "F"] \arrow[d, "\nu"] & \underline{\operatorname{gr}}\,\Lambda \arrow[d, "\Omega(-)_\mu\langle -a\rangle"] \\
\mathcal{D}^b(\operatorname{mod} B) \arrow[r, "F"] & \underline{\operatorname{gr}}\,\Lambda.
\end{tikzcd}$$

Note that throughout the rest of this paper, we will often use the equivalence from Proposition 4.3 and the correspondence of the Serre functors described in the diagram above without making the reference explicitly.

5. On n-hereditary algebras

The class of *n-hereditary algebras* was introduced in [15] and consists of the disjoint union of n-representation finite and n-representation infinite algebras. In this section we recall some definitions and basic results from [15, 20, 21]. This forms a necessary background for exploring connections between the notion of n-T-Koszulity and higher hereditary algebras, which is the topic our next two sections. Note that Section 5 does not contain any new results. Throughout this section, let A be a finite dimensional algebra. Recall that if A has finite global dimension, then the derived Nakayama functor $\nu(-) = - \otimes_A^{\mathbf{L}} DA$ is a Serre functor on $\mathcal{D}^b(\operatorname{mod} A)$. We use the notation $\nu_n = \nu(-)[-n]$. The algebra A is called *n-representation* *finite* if $\operatorname{gl.dim} A \leq n$ and $\operatorname{mod} A$ contains an n-cluster tilting object. We have the following criterion for n-representation finiteness in terms of the subcategory

$$\mathcal{U} = \operatorname{add}\{\nu_n^i A \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}^b(\operatorname{mod} A).$$

Theorem 5.1. *(See* [21, Theorem 3.1]*.)* *Assume* $\operatorname{gl.dim} A \leq n$*. The following are equivalent:*

(1) A *is* n*-representation finite;*
(2) $DA \in \mathcal{U}$*;*
(3) $\nu\mathcal{U} = \mathcal{U}$*.*

In particular, an algebra A with $\operatorname{gl.dim} A \leq n$ is n-representation finite if and only if there for any indecomposable projective A-module P_i, is an integer $m_i \geq 0$ such that $\nu_n^{-m_i}(P_i)$ is indecomposable injective. We will need the following well-known property of n-representation finite algebras.

Lemma 5.2. *(See* [15, Proposition 2.3]*.)* *Let* A *be* n*-representation finite. For each indecomposable projective* A*-module* P_i*, we then have* $\mathrm{H}^l(\nu_n^{-m}(P_i)) = 0$ *for* $l \neq 0$ *and* $0 \leq m \leq m_i$*, where* m_i *is given as above.*

Moving on to the second part of the n-hereditary dichotomy, recall that A is called *n-representation infinite* if $\operatorname{gl.dim} A \leq n$ and $\mathrm{H}^i(\nu_n^{-j}(A)) = 0$ for $i \neq 0$ and $j \geq 0$.

The following basic lemma will be needed in our next two sections. This fact should be well-known, but we include a proof as we lack an explicit reference. In the proof we abuse notation by letting ν denote both the derived Nakayama functor and the ordinary Nakayama functor, as context allows one to determine which one is intended.

Lemma 5.3. *Let* $\operatorname{gl.dim} A < \infty$ *and assume that for each indecomposable projective* A*-module* P*, we have* $\mathrm{H}^i(\nu_n^{-1}(P)) = 0$ *for* $i \notin \{0, -n\}$*. Then* $\operatorname{gl.dim} A \leq n$*. If there is at least one non-injective projective* A*-module, then* $\operatorname{gl.dim} A = n$*.*

Proof. To show $\operatorname{gl.dim} A \leq n$, it is sufficient to check that $\operatorname{inj.dim} A \leq n$, as A has finite global dimension.

Let P be an indecomposable projective A-module. Assume that in computing $\nu_n^{-1}(P)$ we use a minimal injective resolution $I^\bullet$ of P. As $\operatorname{gl.dim} A < \infty$, this resolution is finite. If $\operatorname{inj.dim} P = m \notin \{0, n\}$, our assumption yields

$$\mathrm{H}^m(\nu^{-1}(P)) \simeq \mathrm{H}^{m-n}(\nu_n^{-1}(P)) = 0.$$

However, if there is no cohomology in degree m, this implies that the morphism $\nu^{-1}(I^{m-1} \to I^m)$ is an epimorphism. As $\nu^{-1}(I^m)$ is projective, this morphism must split. Since ν^{-1} is an equivalence when restricted to $\operatorname{add} DA$, this contradicts the minimality of the resolution $I^\bullet$, and we can conclude that $\operatorname{inj.dim} P = 0$ or n. In particular, one obtains $\operatorname{inj.dim} A \leq n$. If there exists P non-injective, we clearly get the second claim. $\square$

Like in the classical theory of hereditary algebras, the class of n-hereditary algebras also has an appropriate version of (higher) preprojective algebras which is nicely behaved. Given an n-hereditary algebra A, we denote the $(n+1)$-*preprojective algebra of* A by $\Pi_{n+1}A$. Recall from [21, Lemma 2.13] that

$$\Pi_{n+1}A \simeq \bigoplus_{i\geq 0} \operatorname{Hom}_{D^b(A)}(A, \nu_n^{-i}(A)).$$

If A is n-representation finite, the associated $(n+1)$-preprojective is finite dimensional and self-injective, whereas in the n-representation infinite case, the $(n+1)$-preprojective is infinite dimensional graded bimodule $(n+1)$-Calabi–Yau of Gorenstein parameter 1.

Remark 5.4. Note that other authors refer to the classes of algebra we discuss here using different terms. For instance, an n-representation finite algebra is called 'n-representation-finite n-hereditary' in [23]. This terminology is very reasonable, but as we need to mention n-representation finite algebras frequently, we stick to the notion from [20] for brevity.

6. Higher Koszul duality and n-representation infinite algebras

In this section we investigate connections between n-representation infinite algebras and the notion of higher Koszulity. Let us first present our standing assumptions.

Setup. Throughout the rest of this section, let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a finite dimensional graded Frobenius algebra of highest degree $a \geq 1$ with $\operatorname{gl.dim}\Lambda_0 < \infty$. Let T denote a basic graded Λ-module which is concentrated in degree 0 and a tilting module over Λ_0. Consider a decomposition $T \simeq \oplus_{i=0}^{t} T^i$ into indecomposable summands and assume that twisting by the Nakayama automorphism μ of Λ only permutes these summands. This means that we have a permutation, for simplicity also denoted by μ, on the set $\{1,\dots,t\}$ such that $T^i_\mu \simeq T^{\mu(i)}$. For our fixed positive integer n, we consider the module

$$\widetilde{T} = \bigoplus_{i=0}^{a-1} \Omega^{-ni} T\langle i\rangle.$$

We denote the endomorphism algebra $\operatorname{End}_{\underline{\operatorname{gr}}\,\Lambda}(\widetilde{T})$ by B.

One should note that in the classical case, the Nakayama automorphism induces a permutation of the simples, i.e. the module corresponding to our T. This justifies the assumption that twisting by the Nakayama automorphism of Λ only permutes the indecomposable summands of T. Note that using this, we immediately obtain $T_\mu \simeq T$, and hence $\Omega T_\mu\langle -a\rangle \simeq \Omega T\langle -a\rangle$.

Our first aim in this section is to describe the endomorphism algebra B as an upper triangular matrix algebra of finite global dimension. We start by recalling the following lemma.

Lemma 6.1. *(See* [8, Corollary 4.21 (4)]*.) Let A and A' be finite dimensional algebras and M an $A^{\mathrm{op}} \otimes_k A'$-module. Then the algebra*

$$\begin{bmatrix} A & M \\ 0 & A' \end{bmatrix}$$

has finite global dimension if and only if both A and A' have finite global dimension.

In Lemma 6.2 we describe B as an upper triangular matrix algebra associated to the graded algebra $\Gamma = \oplus_{i\geq 0} \operatorname{Ext}^{ni}_{\operatorname{gr}\Lambda}(T, T\langle i\rangle)$. Notice that in the case where Λ is n-T-Koszul, the algebra Γ coincides with the n-T-Koszul dual $\Lambda^!$.

Lemma 6.2. *The algebra $B = \operatorname{End}_{\underline{\operatorname{gr}}\Lambda}(\widetilde{T})$ is isomorphic to the upper triangular matrix algebra*

$$B \simeq \begin{pmatrix} \Gamma_0 & \Gamma_1 & \cdots & \Gamma_{a-1} \\ 0 & \Gamma_0 & \cdots & \Gamma_{a-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma_0 \end{pmatrix},$$

where $\Gamma = \oplus_{i\geq 0} \operatorname{Ext}^{ni}_{\operatorname{gr}\Lambda}(T, T\langle i\rangle)$. In particular, the global dimension of B is finite.

Proof. For $0 \leq i, j \leq a-1$, we consider

$$\operatorname{Hom}_{\underline{\operatorname{gr}}\Lambda}(\Omega^{-nj}T\langle j\rangle, \Omega^{-ni}T\langle i\rangle) \simeq \operatorname{Hom}_{\underline{\operatorname{gr}}\Lambda}(T, \Omega^{-n(i-j)}T\langle i-j\rangle).$$

In the case $i < j$, we note that $|i-j| \leq a-1$ and so Lemma 2.3 *(7)* applies. Consequently,

$$\operatorname{Hom}_{\underline{\operatorname{gr}}\Lambda}(T, \Omega^{-n(i-j)}T\langle i-j\rangle) \simeq \operatorname{Hom}_{\operatorname{gr}\Lambda}(T, \Omega^{-n(i-j)}T\langle i-j\rangle) = 0.$$

If $i = j$, one obtains $\operatorname{End}_{\underline{\operatorname{gr}}\Lambda}(T)$, which is isomorphic to $\operatorname{End}_{\operatorname{gr}\Lambda}(T) = \Gamma_0$ by Lemma 2.3 *(5)*. For $i > j$, we get

$$\operatorname{Hom}_{\underline{\operatorname{gr}}\Lambda}(T, \Omega^{-n(i-j)}T\langle i-j\rangle) \simeq \operatorname{Ext}^{n(i-j)}_{\operatorname{gr}\Lambda}(T, T\langle i-j\rangle) = \Gamma_{i-j}.$$

Computing our matrix with respect to the decomposition

$$\widetilde{T} = \Omega^{-n(a-1)}T\langle a-1\rangle \oplus \cdots \oplus \Omega^{-n}T\langle 1\rangle \oplus T,$$

this yields our desired description.

To see that B is of finite global dimension, notice that $\Gamma_0 \simeq \operatorname{End}_{\Lambda_0}(T)$. As $\operatorname{End}_{\Lambda_0}(T)$ is derived equivalent to Λ_0, which is of finite global dimension, Lemma 6.1 applies and the claim follows. $\square$

Note that we could also have deduced that B is of finite global dimension from [38, Corollary 3.12]. In the main result of this section, Theorem 6.4, we characterize when our algebra Λ is n-T-Koszul in terms of B being $(na-1)$-representation infinite. Our next lemma provides an important step in the proof of this result.

Recall that given a graded Λ-module $M = \oplus_{i\in\mathbb{Z}} M_i$, each graded part M_i is also a module over Λ_0. On the other hand, every Λ_0-module is trivially a graded Λ-module concentrated in degree 0. In the proof of Lemma 6.3, we repeatedly vary between thinking of graded Λ-modules concentrated in one degree and modules over the degree 0 part.

We use the notation $M_{\geq i}$ for the submodule of M with

$$(M_{\geq i})_j = \begin{cases} M_j & j \geq i \\ 0 & j < i, \end{cases}$$

while the quotient module ${}^{M}\!/_{M_{\geq i+1}}$ is denoted by $M_{\leq i}$. Note that M_i is isomorphic to ${}^{M_{\geq i}}\!/_{M_{\geq i+1}}$.

Lemma 6.3. *The module $\widetilde{T}$ generates $\underline{\mathrm{gr}}\,\Lambda$ as a thick subcategory, i.e. we have* $\mathrm{Thick}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T}) = \underline{\mathrm{gr}}\,\Lambda$.

Proof. We divide the proof into two steps. In the first part, we show that the set of objects $\{\Lambda_0\langle i\rangle\}_{i\in\mathbb{Z}}$ generates $\underline{\mathrm{gr}}\,\Lambda$ as a thick subcategory. In the second part, we prove that this set is contained in $\mathrm{Thick}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T})$, which yields our desired conclusion.

Part 1:

Notice first that every graded Λ-module which is concentrated in degree i is necessarily contained in the thick subcategory generated by $\Lambda_0\langle i\rangle$. To see this, apply $\langle i\rangle$ to a finite Λ_0-projective resolution of the module, split up into short exact sequences and use that thick subcategories have the 2/3-property on distinguished triangles.

Let M be an object in $\underline{\mathrm{gr}}\,\Lambda$. Denote the highest and lowest degree of M by h and l, respectively. Observe that $M_{\geq h} = M_h$. By the argument above, we know that M_j is in $\mathrm{Thick}_{\underline{\mathrm{gr}}\,\Lambda}(\{\Lambda_0\langle i\rangle\}_{i\in\mathbb{Z}})$ for every j. Considering the short exact sequences

$$0 \longrightarrow M_{\geq j+1} \longrightarrow M_{\geq j} \longrightarrow M_j \longrightarrow 0 \tag{6.1}$$

for $j = l, \ldots, h-1$, we can hence conclude that also $M_{\geq l} = M$ is in our subcategory. This proves that $\mathrm{Thick}_{\underline{\mathrm{gr}}\,\Lambda}(\{\Lambda_0\langle i\rangle\}_{i\in\mathbb{Z}}) = \underline{\mathrm{gr}}\,\Lambda$.

Part 2:

As thick subcategories are closed under direct summands and translation, we immediately observe that $T\langle i\rangle$ is in $\mathrm{Thick}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T})$ for $i = 0, \ldots, a-1$. Since T is a tilting module over Λ_0, and $\Lambda_0\langle i\rangle$ thus has a finite coresolution in $\mathrm{add}\,T\langle i\rangle$, this implies that $\Lambda_0\langle i\rangle$ is in $\mathrm{Thick}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T})$ for $i = 0, \ldots, a-1$. Note that by our argument in *Part 1*, we hence know that every module which is concentrated in degree i for some $i = 0, \ldots, a-1$, is contained in our subcategory.

Consider the short exact sequences (6.1) for $M = \Lambda$, and recall that the module $\Lambda_{\geq 0} = \Lambda$ is projective and hence zero in $\underline{\mathrm{gr}}\,\Lambda$. By a similar argument as before, this yields that Λ_a is contained in $\mathrm{Thick}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T})$. We next explain why this entails that also $\Lambda_0\langle a\rangle$ is in our subcategory.

Since Λ is graded Frobenius, we have $\Lambda\langle -a\rangle \simeq D\Lambda$ as graded right Λ-modules, and thus $D\Lambda_0 \simeq \Lambda_a$ as Λ_0-modules. As Λ_0 has finite global dimension, this implies that Λ_0 is contained in $\mathrm{Thick}_{\mathcal{D}^b(\Lambda_0)}(\Lambda_a\langle -a\rangle)$. Composing the equivalence from Theorem 4.2 with the associated quotient functor, one obtains a triangulated functor $Q\colon \mathcal{D}^b(\mathrm{gr}\,\Lambda) \to \underline{\mathrm{gr}}\,\Lambda$. From the chain of subcategories

$$\mathrm{Thick}_{\mathcal{D}^b(\Lambda_0)}\,\Lambda_a\langle -a\rangle \subseteq \mathrm{Thick}_{\mathcal{D}^b(\mathrm{gr}\,\Lambda)}\,\Lambda_a\langle -a\rangle \subseteq Q^{-1}(\mathrm{Thick}_{\underline{\mathrm{gr}}\,\Lambda}\,\Lambda_a\langle -a\rangle),$$

we see that $\Lambda_0\langle a\rangle$ is in $\mathrm{Thick}_{\underline{\mathrm{gr}}\,\Lambda}(\Lambda_a)$, which again is contained in $\mathrm{Thick}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T})$.

Shifting the short exact sequences involved by positive integers and using the same argument as above, one obtains that $\Lambda_0\langle i\rangle$ is in $\mathrm{Thick}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T})$ for all $i \geq 0$. That $\Lambda_0\langle i\rangle$ is in $\mathrm{Thick}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T})$ for all $i < 0$ is shown similarly using the short exact sequences

$$0 \longrightarrow \Lambda_j \longrightarrow \Lambda_{\leq j} \longrightarrow \Lambda_{\leq j-1} \longrightarrow 0$$

for $j = 1, \ldots, a$. We can hence conclude that $\Lambda_0\langle i\rangle$ is in $\mathrm{Thick}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T})$ for every integer i, which finishes our proof. $\square$

We are now ready to state and prove the main result of this section.

Theorem 6.4. *The following statements are equivalent:*

(1) Λ *is n-T-Koszul.*

(2) $\widetilde{T}$ *is a tilting object in* $\underline{\mathrm{gr}}\,\Lambda$ *and* $B = \mathrm{End}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T})$ *is $(na-1)$-representation infinite.*

Proof. We begin by proving *(1)* implies *(2)*. To see that $\widetilde{T}$ is a tilting object, notice first that it generates $\underline{\mathrm{gr}}\,\Lambda$ by Lemma 6.3. Thus, we need only check rigidity, i.e. that $\mathrm{Hom}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T}, \Omega^{-l}\widetilde{T}) = 0$ whenever $l \neq 0$. Splitting up on summands of $\widetilde{T} = \oplus_{i=0}^{a-1}\Omega^{-ni}T\langle i\rangle$ and reindexing appropriately, we see that it is enough to show

$$\mathrm{Hom}_{\underline{\mathrm{gr}}\,\Lambda}(T, \Omega^{-(nk+l)}T\langle k\rangle) = 0 \quad \text{for} \quad l \neq 0 \tag{6.2}$$

for any integer k with $|k| \leq a-1$.

Assume $nk + l = 0$. Now $l \neq 0$ implies $k \neq 0$, so the condition above is satisfied as our morphisms are homogeneous of degree 0.

Let $nk + l > 0$. Now,

$$\mathrm{Hom}_{\underline{\mathrm{gr}}\,\Lambda}(T, \Omega^{-(nk+l)}T\langle k\rangle) \simeq \mathrm{Ext}^{nk+l}_{\mathrm{gr}\,\Lambda}(T, T\langle k\rangle),$$

which is zero for $l \neq 0$ as Λ is n-T-Koszul.

It remains to verify (6.2) in the case where $nk + l < 0$. As $|k| \leq a-1$, part *(7)* of Lemma 2.3 applies. We hence see that (6.2) is satisfied also in this case, which means that $\widetilde{T}$ is a tilting object in $\underline{\mathrm{gr}}\,\Lambda$.

Recall from Lemma 6.2 that B has finite global dimension. To see that B is $(na-1)$-representation infinite, we use that $\widetilde{T}$ is a tilting object in $\underline{\mathrm{gr}}\,\Lambda$. Hence, the equivalence and correspondence of Serre functors described in Section 4 yields

$$\begin{aligned}\operatorname{Hom}_{\underline{\operatorname{gr}}\,\Lambda}(\widetilde{T}, \Omega^{-(nai+l)}\widetilde{T}\langle ai\rangle) &\simeq \operatorname{Hom}_{\mathcal{D}^b(B)}(B, \nu^{-i}(B)[nai-i+l]) \\ &\simeq \operatorname{Hom}_{\mathcal{D}^b(B)}(B, \nu_{na-1}^{-i}(B)[l]) \\ &\simeq \mathrm{H}^l(\nu_{na-1}^{-i}(B)),\end{aligned} \tag{6.3}$$

where we have implicitly used that $T_\mu \simeq T$ and that the functors $\Omega^{\pm 1}(-)$, $\langle \pm 1\rangle$ and $(-)_\mu$ commute.

Splitting up on summands of $\widetilde{T}$ and reindexing appropriately, we notice that $\operatorname{Hom}_{\underline{\operatorname{gr}}\,\Lambda}(\widetilde{T}, \Omega^{-(nai+l)}\widetilde{T}\langle ai\rangle) = 0$ for $l \neq 0$ and $i > 0$ if and only if (6.2) is satisfied for $k > 0$. The latter follows by the same argument as in our proof of rigidity above, so we can conclude that $\mathrm{H}^l(\nu_{na-1}^{-i}(B)) = 0$ for $i > 0$ and $l \neq 0$. Note that when $i = 0$ and $l \neq 0$, we have $\mathrm{H}^l(\nu_{na-1}^{-i}(B)) = \mathrm{H}^l(B) = 0$. Consequently, our algebra B is $(na-1)$-representation infinite by Lemma 5.3.

To show that *(2)* implies *(1)*, we verify that given any integer k, one obtains $\operatorname{Ext}^{nk+l}_{\operatorname{gr}\Lambda}(T, T\langle k\rangle) = 0$ for $l \neq 0$. If $nk + l \leq 0$, this is immediately satisfied, so assume $nk + l > 0$. As before, we now have

$$\operatorname{Ext}^{nk+l}_{\operatorname{gr}\Lambda}(T, T\langle k\rangle) \simeq \operatorname{Hom}_{\underline{\operatorname{gr}}\,\Lambda}(T, \Omega^{-(nk+l)}T\langle k\rangle).$$

If $k < 0$, this is zero by Lemma 2.3 *(6)*, so it remains to check the case where k is non-negative.

Observe that the isomorphism

$$\operatorname{Hom}_{\underline{\operatorname{gr}}\,\Lambda}(\widetilde{T}, \Omega^{-(nai+l)}\widetilde{T}\langle ai\rangle) \simeq \mathrm{H}^l(\nu_{na-1}^{-i}(B))$$

from (6.3) still holds, as $\widetilde{T}$ is assumed to be a tilting object in $\underline{\operatorname{gr}}\,\Lambda$. As B is $(na-1)$-representation infinite, we know that $\mathrm{H}^l(\nu_{na-1}^{-i}(B)) = 0$ for $i \geq 0$ and $l \neq 0$. The isomorphism above hence yields that (6.2) is satisfied for $k \geq 0$.

This allows us to conclude that T is graded n-self-orthogonal. As T is a tilting module over Λ_0 by our standing assumptions, we have hence shown that Λ is n-T-Koszul. □

To illustrate our characterization result, we consider an example. As can be seen below, we use diagrams to represent indecomposable modules. The reader should note that in general one cannot expect modules to be represented uniquely by such diagrams, but in the cases we look at, they determine indecomposable modules up to isomorphism.

Example 6.5. Let A denote the path algebra of the quiver

$$\begin{tikzcd}
& 2 \arrow[dr, "\alpha_3"] & \\
1 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"'] & & 4 \\
& 3 \arrow[ur, "\alpha_4"'] &
\end{tikzcd}$$

modulo the ideal generated by paths of length two. The trivial extension ΔA is given by the quiver

$$\begin{tikzcd}
& 2 \arrow[dl, bend left, "\alpha_1'"] \arrow[dr, bend left, "\alpha_3"] & \\
1 \arrow[ur, bend left, "\alpha_1"] \arrow[dr, bend left, "\alpha_2"] & & 4 \arrow[ul, bend left, "\alpha_3'"] \arrow[dl, bend left, "\alpha_4'"] \\
& 3 \arrow[ul, bend left, "\alpha_2'"] \arrow[ur, bend left, "\alpha_4"] &
\end{tikzcd}$$

with the trivial extension relations, i.e. all length two zero relations with the exception of $\alpha_i\alpha_i'$ and $\alpha_i'\alpha_i$. Instead, these latter paths satisfy all length two commutativity relations, i.e. $\alpha_1\alpha_1' - \alpha_2\alpha_2'$, $\alpha_3\alpha_3' - \alpha_1'\alpha_1$, $\alpha_4'\alpha_4 - \alpha_3'\alpha_3$, and $\alpha_2'\alpha_2 - \alpha_4\alpha_4'$. Moreover, we let ΔA be graded with the trivial extension grading.

The indecomposable projective injectives for ΔA can be given as the diagrams

$$\begin{array}{ccc} & 1_0 & \\ 3_0 & & 2_0 \\ & 1_1 & \end{array} \qquad \begin{array}{ccc} & 2_0 & \\ 1_1 & & 4_0 \\ & 2_1 & \end{array} \qquad \begin{array}{ccc} & 3_0 & \\ 1_1 & & 4_0 \\ & 3_1 & \end{array} \qquad \begin{array}{ccc} & 4_0 & \\ 2_1 & & 3_1, \\ & 4_1 & \end{array}$$

where the (non-subscript) numbers represent elements of a basis for the module, each of which is annihilated by all the idempotents except for e_i with i equal to the number. The subscript numbers represent the degree of the basis element.

Let T be the tilting A-module given by the direct sum of the following modules

$$\begin{array}{ccc} & 1_0 & \\ 3_0 & & 2_0 \end{array} \qquad 2_0 \qquad 3_0 \qquad \begin{array}{ccc} 2_0 & & 3_0. \\ & 4_0 & \end{array}$$

The initial two terms of the minimal injective ΔA-resolution of the first summand of T as well as the first two cosyzygies can be given as

$$\begin{tikzcd}[column sep=small]
\begin{array}{ccc} & 3_{-1} & \\ 4_{-1} & & 1_0 \\ & 3_0 & \end{array} \oplus \begin{array}{ccc} & 2_{-1} & \\ 1_0 & & 4_{-1} \\ & 2_0 & \end{array} \arrow[dr] & & \begin{array}{ccc} & 4_{-2} & \\ 2_{-1} & & 3_{-1} \\ & 4_{-1} & \end{array} \oplus \begin{array}{ccc} & 1_{-1} & \\ 3_{-1} & & 2_{-1} \\ & 1_0 & \end{array} \oplus \begin{array}{ccc} & 4_{-2} & \\ 2_{-1} & & 3_{-1} \\ & 4_{-1} & \end{array} \arrow[dr] & \\
& \begin{array}{ccccc} & 3_{-1} & & 2_{-1} & \\ 4_{-1} & & 1_0 & & 4_{-1} \end{array} \arrow[ur] & & \begin{array}{ccccccc} & 4_{-2} & & 1_{-1} & & 4_{-2} & \\ 2_{-1} & & 3_{-1} & & 2_{-1} & & 3_{-1}. \end{array}
\end{tikzcd}$$

Looking at this part of the resolution, it is not so obvious that T is graded 2-self-orthogonal as a ΔA module, whereas by using the equivalence $\mathcal{D}^b(\operatorname{mod} A) \simeq \underline{\operatorname{gr}}\,\Delta A$ or by degree arguments as we have done before, it is immediate that $\widetilde{T} \simeq T$ is a tilting object in $\underline{\operatorname{gr}}\,\Delta A$. It is also easy to check that $\operatorname{End}_{\underline{\operatorname{gr}}\,\Delta A}(T)$ is isomorphic to the hereditary algebra given by the path algebra of the quiver of A, which is representation infinite. Using Theorem 6.4, we can hence conclude that the algebra ΔA is 2-T-Koszul.

Note that this example also illustrates that, as has been remarked on in the literature before, one cannot always expect nice minimal resolutions of T for (generalized) T-Koszul algebras.

As a consequence of Theorem 6.4, our next corollary shows that an algebra is n-representation infinite if and only if its trivial extension is $(n+1)$-Koszul with respect to its degree 0 part. This result is inspired by connections between n-representation infinite algebras and graded bimodule $(n+1)$-Calabi–Yau algebras of Gorenstein parameter 1, as studied in [1, 15, 26, 30]. In some sense, the corollary below is a T-Koszul dual version of [15, Theorem 4.36].

Note that in the first part of Corollary 6.6, we set $T = \Lambda_0$ and hence assume that the Nakayama automorphism of Λ only permutes the summands of Λ_0. This is trivially satisfied whenever our algebra is graded symmetric.

Corollary 6.6. *If $a = 1$, our algebra Λ is $(n+1)$-Koszul with respect to $T = \Lambda_0$ if and only if Λ_0 is n-representation infinite. In particular, we obtain a bijective correspondence*

$$\left\{\begin{array}{l}\textit{isomorphism classes}\\ \textit{of basic } n\textit{-representation}\\ \textit{infinite algebras}\end{array}\right\} \rightleftarrows \left\{\begin{array}{l}\textit{isomorphism classes of graded symmetric finite}\\ \textit{dimensional algebras of highest degree } 1 \textit{ which are}\\ (n+1)\textit{-Koszul with respect to their degree } 0 \textit{ part}\end{array}\right\},$$

where the maps are given by $A \longmapsto \Delta A$ and $\Lambda_0 \longleftarrow\!\shortmid \Lambda$.

Proof. Notice that $\operatorname{End}_{\underline{\operatorname{gr}}\Lambda}(\Lambda_0) \simeq \operatorname{End}_{\operatorname{gr}\Lambda}(\Lambda_0) \simeq \Lambda_0$ by Lemma 2.3 *(5)*. Observe that $\operatorname{Hom}_{\underline{\operatorname{gr}}\Lambda}(\Lambda_0, \Omega^{-i}\Lambda_0) \simeq \operatorname{Hom}_{\underline{\operatorname{gr}}\Lambda}(\Omega^{i}\Lambda_0, \Lambda_0) = 0$ for all $i \neq 0$. This follows by degree considerations similar to those used in the proof of Lemma 2.3 and using the fact that syzygies of Λ_0 are generated in degrees greater or equal to 1. Combining this with Lemma 6.3, one obtains that Λ_0 is a tilting object in $\underline{\operatorname{gr}}\,\Lambda$, and consequently our first statement follows from Theorem 6.4.

We get the bijection as a special case of this, as ΔA is a graded symmetric finite dimensional algebra of highest degree 1 and $\Lambda \simeq \Delta\Lambda_0$ as graded algebras in the case where Λ is symmetric. □

Our aim for the rest of this section is to use the theory we have developed to provide an affirmative answer to our motivating question from the introduction. As in the case of the generalized AS-regular algebras studied by Minamoto and Mori in [30], the notion of quasi-Veronese algebras is relevant.

Definition 6.7. Let $\Gamma = \oplus_{i\in\mathbb{Z}}\Gamma_i$ be a $\mathbb{Z}$-graded algebra and r a positive integer. The *r-th quasi-Veronese algebra* of Γ is a $\mathbb{Z}$-graded algebra defined by

$$\Gamma^{[r]} = \bigoplus_{i\in\mathbb{Z}} \begin{pmatrix} \Gamma_{ri} & \Gamma_{ri+1} & \cdots & \Gamma_{ri+r-1} \\ \Gamma_{ri-1} & \Gamma_{ri} & \cdots & \Gamma_{ri+r-2} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{ri-r+1} & \Gamma_{ri-r+2} & \cdots & \Gamma_{ri} \end{pmatrix}.$$

In Proposition 6.8 we show that if Λ is n-T-Koszul, then the na-th preprojective algebra of $B = \operatorname{End}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T})$ is isomorphic to a twist of the a-th quasi-Veronese of $\Lambda^!$. In order to make this precise, notice first that a graded algebra automorphism ϕ of a graded algebra Γ induces a graded algebra automorphism $\phi^{[r]}$ of $\Gamma^{[r]}$ by letting $\phi^{[r]}((\gamma_{j,k})) = (\phi(\gamma_{j,k}))$. Here we use the notation $(\gamma_{j,k})$ for the matrix with $\gamma_{j,k}$ in position (j,k). Recall also that we can define a possibly different graded algebra ${}_{\langle\phi\rangle}\Gamma$ with the same underlying vector space structure as Γ, but with multiplication $\gamma\cdot\gamma' = \phi^i(\gamma)\gamma'$ for γ' in Γ_i.

Recall that μ is the Nakayama automorphism of Λ, and denote our chosen isomorphism $T_\mu \simeq T$ from before by τ. Note that twisting by μ might non-trivially permute the summands of T. In the case where Λ is n-T-Koszul, let $\overline{\mu}$ be the graded algebra automorphism of $\Lambda^!$ defined on the i-th component

$$\Lambda^!_i = \operatorname{Ext}^{ni}_{\mathrm{gr}\,\Lambda}(T, T\langle i\rangle) \simeq \operatorname{Hom}_{\underline{\mathrm{gr}}\,\Lambda}(T, \Omega^{-ni}T\langle i\rangle)$$

by the composition

$$\operatorname{Hom}_{\underline{\mathrm{gr}}\,\Lambda}(T, \Omega^{-ni}T\langle i\rangle) \xrightarrow{(-)_\mu} \operatorname{Hom}_{\underline{\mathrm{gr}}\,\Lambda}(T_\mu, \Omega^{-ni}T_\mu\langle i\rangle) \xrightarrow{(-)^\tau} \operatorname{Hom}_{\underline{\mathrm{gr}}\,\Lambda}(T, \Omega^{-ni}T\langle i\rangle),$$

where

$$(\gamma)^\tau = \Omega^{-ni}(\tau)\langle i\rangle \circ \gamma \circ \tau^{-1}$$

for γ in $\operatorname{Hom}_{\underline{\mathrm{gr}}\,\Lambda}(T_\mu, \Omega^{-ni}T_\mu\langle i\rangle)$.

Before showing Proposition 6.8, recall that a decomposition of $\widetilde{T}$ yields a decomposition of $B = \operatorname{End}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T})$. In the proof below, we denote the summands of $\widetilde{T}$ by $X^i = \Omega^{-ni}T\langle i\rangle$, while P^i is the projective B-module which is the preimage of X^i under the equivalence $\mathcal{D}^b(\operatorname{mod} B) \xrightarrow{\simeq} \underline{\mathrm{gr}}\,\Lambda$ from Proposition 4.3.

Proposition 6.8. *Let Λ be n-T-Koszul. Then $\Pi_{na}B \simeq {}_{\langle(\overline{\mu}^{-1})^{[a]}\rangle}(\Lambda^!)^{[a]}$ as graded algebras. In particular, we have $\Pi_{na}B \simeq (\Lambda^!)^{[a]}$ in the case where Λ is graded symmetric.*

Proof. As Λ is n-T-Koszul, we know from Theorem 6.4 that $\widetilde{T}$ is a tilting object in $\underline{\mathrm{gr}}\,\Lambda$ and that B is $(na-1)$-representation infinite. The i-th component of the na-th preprojective algebra of B is given by $(\Pi_{na}B)_i = \operatorname{Hom}_{\mathcal{D}^b(B)}(B, \nu_{na-1}^{-i}B)$. For

$0 \leq j, k \leq a-1$, we hence consider

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}^b(B)}(P^k, \nu_{na-1}^{-i} P^j) &\simeq \operatorname{Hom}_{\underline{\operatorname{gr}}\,\Lambda}(X^k, \Omega^{-(na-1)i-i} X^j_{\mu^{-i}}\langle ai\rangle) \\ &\simeq \operatorname{Hom}_{\underline{\operatorname{gr}}\,\Lambda}(T, \Omega^{-n(ai+j-k)} T_{\mu^{-i}}\langle ai+j-k\rangle) \\ &\overset{(*)}{\simeq} \operatorname{Ext}^{n(ai+j-k)}_{\operatorname{gr}\Lambda}(T, T_{\mu^{-i}}\langle ai+j-k\rangle) \simeq \Lambda^!_{ai+j-k}.\end{aligned}$$

Notice that the first isomorphism is a consequence of the equivalence and correspondence of Serre functors described in Section 4, while $(*)$ is obtained by applying Lemma 2.3 *(5)* and *(7)*. The last isomorphism follows from the assumption $T_\mu \simeq T$.

Computing our matrix with respect to the decomposition

$$B \simeq P^{a-1} \oplus \cdots \oplus P^1 \oplus P^0,$$

this yields

$$(\Pi_{na} B)_i \simeq \begin{pmatrix} \Lambda^!_{ai} & \Lambda^!_{ai+1} & \cdots & \Lambda^!_{ai+a-1} \\ \Lambda^!_{ai-1} & \Lambda^!_{ai} & \cdots & \Lambda^!_{ai+a-2} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda^!_{ai-a+1} & \Lambda^!_{ai-a+2} & \cdots & \Lambda^!_{ai} \end{pmatrix},$$

which shows that our two algebras are isomorphic as graded vector spaces.

In order to see that the multiplications agree, consider the diagram

$$\begin{tikzcd}
(P^j, \nu_{na-1}^{-i'} P^{j'}) \otimes (P^k, \nu_{na-1}^{-i} P^j) \arrow[r] \arrow[d] & (P^k, \nu_{na-1}^{-(i+i')} P^{j'}) \arrow[d] \\
(\nu_{na-1}^{-i} P^j, \nu_{na-1}^{-(i+i')} P^{j'}) \otimes (P^k, \nu_{na-1}^{-i} P^j) \arrow[r] \arrow[d] & (P^k, \nu_{na-1}^{-(i+i')} P^{j'}) \arrow[d] \\
(X^j_{\mu^{-i}}(ai), X^{j'}_{\mu^{-(i+i')}}(a(i+i'))) \otimes (X^k, X^j_{\mu^{-i}}(ai)) \arrow[r] \arrow[d] & (X^k, X^{j'}_{\mu^{-(i+i')}}(a(i+i'))) \arrow[d] \\
\Lambda^!_{ai'+j'-j} \otimes \Lambda^!_{ai+j-k} \arrow[r] & \Lambda^!_{a(i+i')+j'-k}.
\end{tikzcd}$$

For simplicity, we have here suppressed the Hom-notation and denoted $\Omega^{-ni}(-)\langle i\rangle$ by $(-)(i)$. The horizontal maps are given by multiplication or composition, and the vertical maps give our isomorphism of graded algebras. In particular, the middle two horizontal maps are merely composition, whereas the top and bottom horizontal maps are the multiplication of $\Pi_{na} B$ and ${}_{\langle(\overline{\mu}^{-1})^{[a]}\rangle}(\Lambda^!)^{[a]}$, respectively. Moreover, the bottom vertical maps are given by

$$f \otimes g \mapsto \prod_{l=0}^{i'-1} \tau^{-1}_{\mu^{l-i'}}(ai'+j'-j) \circ f_{\mu^i}(-ai-j) \otimes \prod_{l=0}^{i-1} \tau^{-1}_{\mu^{l-i}}(ai+j-k) \circ g(-k)$$

and

$$f \circ g \mapsto \prod_{l=0}^{i+i'-1} \tau^{-1}_{\mu^{l-i-i'}}(a(i+i')+j'-k) \circ (f \circ g)(-k).$$

As the diagram commutes, we can conclude that $\Pi_{na}B \simeq {}_{\langle(\overline{\mu}^{-1})^{[a]}\rangle}(\Lambda^!)^{[a]}$ as graded algebras. If Λ is assumed to be graded symmetric, the Nakayama automorphism μ can be chosen to be trivial, so one obtains $\Pi_{na}B \simeq (\Lambda^!)^{[a]}$. $\square$

In the corollary below, we show that the $(n+1)$-th preprojective of an n-representation infinite algebra is isomorphic to the n-T-Koszul dual of its trivial extension. This is a T-Koszul dual version of [30, Proposition 4.20].

Corollary 6.9. *If A is basic n-representation infinite, then $\Pi_{n+1}A \simeq (\Delta A)^!$ as graded algebras.*

Proof. Let A be a basic n-representation infinite algebra. It then follows from Corollary 6.6 that ΔA is $(n+1)$-Koszul with respect to A. By Lemma 2.3 part *(5)*, one obtains $\mathrm{End}_{\underline{\mathrm{gr}}\,\Delta A}(A) \simeq \mathrm{End}_{\mathrm{gr}\,\Delta A}(A) \simeq A$. Recall that ΔA is graded symmetric of highest degree 1. Applying Proposition 6.8 to ΔA hence yields our desired conclusion. $\square$

We are now ready to give an answer to our motivating question from the introduction, namely to see that we obtain an equivalence $\underline{\mathrm{gr}}(\Delta A) \simeq \mathcal{D}^b(\mathrm{qgr}\,\Pi_{n+1}A)$ which descends from higher Koszul duality in the case where A is n-representation infinite and $\Pi_{n+1}A$ is graded right coherent.

Recall that an n-representation infinite algebra A is called *n-representation tame* if the associated $(n+1)$-preprojective $\Pi_{n+1}A$ is a noetherian algebra over its center [15, Definition 6.10]. Notice that a noetherian algebra is graded right coherent, so our result holds in this case.

Corollary 6.10. *Let A be a basic n-representation infinite algebra with $\Pi_{n+1}A$ graded right coherent. Then there is an equivalence $\mathcal{D}^b(\mathrm{gr}\,\Delta A) \simeq \mathcal{D}^b(\mathrm{gr}\,\Pi_{n+1}A)$ of triangulated categories which descends to an equivalence $\underline{\mathrm{gr}}(\Delta A) \simeq \mathcal{D}^b(\mathrm{qgr}\,\Pi_{n+1}A)$. In particular, this holds if A is n-representation tame.*

Proof. It is well-known that $\Pi_{n+1}A$ is of finite global dimension [30, Theorem 4.2]. Hence, we get the equivalence $\mathcal{D}^b(\mathrm{gr}\,\Delta A) \simeq \mathcal{D}^b(\mathrm{gr}\,\Pi_{n+1}A)$ by Theorem 3.9 combined with Corollary 6.6 and Corollary 6.9. By Proposition 3.11, this equivalence descends to yield $\underline{\mathrm{gr}}(\Delta A) \simeq \mathcal{D}^b(\mathrm{qgr}\,\Pi_{n+1}A)$. $\square$

7. Higher almost Koszulity and n-representation finite algebras

In our previous section, we gave connections between higher Koszul duality and n-representation infinite algebras. Having developed our theory for one part of the higher hereditary dichotomy, it is natural to ask whether something similar holds in the n-representation finite case. To provide an answer to this question, we introduce the notion of higher almost Koszulity. As before, this should be formulated relative to a tilting module over the degree 0 part of the algebra, which is itself assumed to be finite dimensional and of finite global dimension. Notice that after having presented the definitions and basic examples, we prove our results given the same standing assumptions as in Section 6.

Our definition of what it means for an algebra to be almost n-T-Koszul is inspired by and generalizes the notion of almost Koszulity, as introduced in [5]. Let us hence first recall the definition of an almost Koszul algebra.

Definition 7.1. (See [5, Definition 3.1].) Assume that Λ_0 is semisimple. We say that Λ is *(right) almost Koszul* if there exist integers $p, q \geq 1$ such that

(1) $\Lambda_i = 0$ for all $i > p$;
(2) There is a graded complex

$$0 \to P^{-q} \to \cdots \to P^{-1} \to P^0 \to 0$$

of projective right Λ-modules such that each P^{-i} is generated by its component P_i^{-i} and the only non-zero cohomology is Λ_0 in internal degree 0 and $P_l^{-q} \otimes_{\Lambda_0} \Lambda_p$ in internal degree $p+q$.

If Λ is almost Koszul for integers p and q, one also says that Λ is (p, q)*-Koszul*.

Roughly speaking, by iteratively taking tensor products over the degree 0 part, we see that if Λ is almost Koszul, then Λ_0 has a somewhat periodic projective resolution which is properly piecewise linear for $p > 1$. This may remind one of the behaviour of the inverse Serre functor of an n-representation finite algebra on indecomposable projectives. However, note that for the latter the periods may be different for different indecomposable projectives. This highlights one additional area in which we must generalize the notion of almost Koszulity, namely that the length of the period of graded n-self-orthogonality can vary for different summands of our tilting module.

Motivated by our observations above, let us now define what it means for a module to be almost graded n-self-orthogonal. Recall that we consider a fixed decomposition $T \simeq \oplus_{i=1}^t T^i$ into indecomposable summands.

Definition 7.2. Let $T \simeq \oplus_{i=1}^t T^i$ be a finitely generated basic graded Λ-module concentrated in degree 0. We say that T is *almost graded n-self-orthogonal* if for each $i \in \{1, \dots, t\}$, there exists an object $T' \in \operatorname{add} T$ and positive integers l_i and g_i such that the following conditions hold:

(1) $\Omega^{-l_i} T^i \simeq T'\langle -g_i \rangle$;
(2) $\operatorname{Ext}^j_{\operatorname{gr}\Lambda}(T, T^i\langle k \rangle) = 0$ for $j \neq nk$ and $j < l_i$.

This leads to our definition of what it means for an algebra to be almost n-T-Koszul.

Definition 7.3. Assume $\operatorname{gl.dim}\Lambda_0 < \infty$ and let T be a graded Λ-module concentrated in degree 0. We say that Λ is *almost n-T-Koszul* or *almost n-Koszul with respect to T* if the following conditions hold:

(1) T is a tilting Λ_0-module.
(2) T is almost graded n-self-orthogonal as a Λ-module.

Whenever we work with an almost n-T-Koszul algebra, we use the notation l_i and g_i for integers given as in Definition 7.2.

As a first class of examples, we verify that Definition 7.3 is indeed a generalization of Definition 7.1.

Example 7.4. Let Λ be a (p,q)-Koszul algebra. We show that Λ is almost 1-Koszul with respect to Λ_0. It is immediate that $\operatorname{gl.dim}\Lambda_0 < \infty$ and that Λ_0 is a tilting module over itself. To see that Λ is almost 1-Koszul with respect to Λ_0, we must hence check that Λ_0 is almost graded 1-self-orthogonal as a Λ-module. Note that by letting $l_i = q+1$ and $g_i = p+q$ for every $i \in \{1,\dots,t\}$, we get that condition (2) of Definition 7.1 implies conditions (1) and (2) of Definition 7.2. To see this, we use the fact that an algebra is left (p,q)-Koszul if and only if it is right (p,q)-Koszul, i.e. [5, Proposition 3.4]. Hence, we get a left projective resolution of Λ_0, which can be dualized to yield a right injective resolution of Λ_0.

Trivial extensions of n-representation finite algebras provide another important class of examples of algebras satisfying Definition 7.3, as can be seen through the theory we develop in the rest of this section. Our main result is Theorem 7.17, which is an almost n-T-Koszul analogue of the characterization result in Section 6, i.e. Theorem 6.4. We divide the proof of Theorem 7.17 into a series of smaller steps. In order to state our precise result, we need information about the relation between the integers l_i and g_i of an almost n-T-Koszul algebra. As will become clear from the proof of our characterization result, the notion given in the definition below is sufficient. Recall that we consider a fixed decomposition $T \simeq \oplus_{i=1}^{t} T^i$ into indecomposable summands.

Definition 7.5. An almost n-T-Koszul algebra Λ of highest degree a is called *(n, m_i, σ_i)-T-Koszul* or *(n, m_i, σ_i)-Koszul with respect to T* if for each $i \in \{1,\dots,t\}$, there exists non-negative integers m_i and σ_i with $\sigma_i \leq a-1$ such that

(1) $l_i = nam_i - n\sigma_i + 1$;
(2) $g_i = a(m_i+1) - \sigma_i$;
(3) There is no integer k satisfying $0 < nk < l_i$ and $\Omega^{-nk}T^i \simeq T'\langle -k\rangle$ with $T' \in \operatorname{add} T$.

We say that an algebra is *(n,m,σ)-T-Koszul* if it is (n, m_i, σ_i)-T-Koszul with $m_i = m$ and $\sigma_i = \sigma$ for all i.

One can think of part (3) in the definition above as a minimality condition for each l_i, as explained in the following remark.

Remark 7.6. When T is almost graded n-self-orthogonal, part (3) in the definition above is equivalent to that there exist no integers l_i' and g_i' with $l_i' < l_i$ satisfying Definition 7.2. Note in particular that given such integers, one must have $l_i' = ng_i'$ as T is almost graded n-self-orthogonal. This contradicts the third requirement in Definition 7.5.

Similarly as in Example 7.4, we see that almost Koszul algebras give rise to natural examples of algebras which are (n, m_i, σ_i)-T-Koszul.

Example 7.7. Let Λ be a (p, q)-Koszul algebra in the sense of Definition 7.1 and assume that Λ is graded Frobenius of highest degree $a \geq 2$. Then Λ is $(1, m, \sigma)$-Koszul with respect to Λ_0, where m and σ are the unique integers such that $q = am - \sigma$ with $0 \leq \sigma \leq a - 1$. Note that as $p = a$, it follows from Example 7.4 that part (1) and (2) of Definition 7.5 are satisfied. As the integers l_i and g_i do not depend on the parameter i, we simply denote them by l and g.

It remains to check minimality, i.e. that part (3) of Definition 7.5 holds. Assume to the contrary that there exist integers l' and g' as described in Remark 7.6. As $n = 1$, this in particular means that

$$\Omega^{-l'}\Lambda_0 \simeq \Lambda_0\langle -l' \rangle. \tag{7.1}$$

By the existence of the almost Koszul resolution from Definition 7.1, we have an epimorphism $I^{l'-1}\langle 1 - l' \rangle \twoheadrightarrow \Omega^{-l'}\Lambda_0$, where $I^{l'-1}$ is a summand of $D\Lambda$ as graded modules. Since Λ is graded Frobenius and hence $D\Lambda \simeq \Lambda\langle -a \rangle$, the module $I^{l'-1}\langle 1 - l' \rangle$ is a direct summand of $\Lambda\langle 1 - l' - a \rangle$. Consequently, the top of $I^{l'-1}\langle 1 - l' \rangle$ is concentrated in degree $1 - l' - a$. However, by the isomorphism (7.1), the projective module $I^{l'-1}\langle 1 - l' \rangle$ projects onto a semisimple module concentrated in degree $-l'$. This yields that $\operatorname{Top} I^{l'-1}\langle 1 - l' \rangle$ is concentrated in degree $-l'$, which is a contradiction as $a \geq 2$.

Recall that a Dynkin quiver is said to have *bipartite orientation* if every vertex is either a sink or a source. Just as in the study of almost Koszul algebras in [5], trivial extensions of bipartite Dynkin quivers provide an important class of algebras which are (n, m_i, σ_i)-T-Koszul.

Example 7.8. Let Λ be given by the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_0"] & 2 \arrow[l, bend left, "\alpha_0'"] \arrow[r, bend right, "\alpha_1'"'] & 3 \arrow[l, bend right, "\alpha_1"'] \end{tikzcd}$$

with relations $\alpha_0\alpha_1'$, $\alpha_1\alpha_0'$, and $\alpha_0'\alpha_0 - \alpha_1'\alpha_1$. This algebra is graded symmetric of highest degree 2 with grading induced by letting the arrows be in degree 1. The indecomposable projective injectives can be represented by the diagrams

$$\begin{smallmatrix} 1_0 \\ 2_1 \\ 1_2 \end{smallmatrix} \qquad \begin{smallmatrix} & 2_0 & \\ 1_1 & & 3_1 \\ & 2_2 & \end{smallmatrix} \qquad \begin{smallmatrix} 3_0 \\ 2_1 \\ 3_2 \end{smallmatrix},$$

where the subscripts indicate the degrees of the basis elements. Computing injective resolutions of the simples, one can check directly that Λ is $(1,1,0)$-Koszul with respect to $\Lambda_0 = \Lambda/\operatorname{Rad}\Lambda$, i.e. it is $(1, m_i, \sigma_i)$-Λ_0-Koszul with $(m_i)_{i=1}^3 = (1,1,1)$ and $(\sigma_i)_{i=1}^3 = (0,0,0)$. Moreover, one can verify that $\widetilde{\Lambda_0}$ is a tilting object in $\underline{\operatorname{gr}}\,\Lambda$ with 1-representation finite endomorphism algebra. Note that this is a specific case of what we prove more generally in our characterization result for (n, m_i, σ_i)-T-Koszul algebras given in Theorem 7.17. In particular, we see that the endomorphism algebra of $\widetilde{\Lambda_0}$ in $\underline{\operatorname{gr}}\,\Lambda$ decomposes as the direct sum of the endomorphism algebras of

$$1_0 \qquad\qquad \begin{smallmatrix} & 2_{-1} & \\ 1_0 & & 3_0 \end{smallmatrix} \qquad\qquad 3_0$$

and

$$\begin{smallmatrix} 1_{-1} \\ 2_0 \end{smallmatrix} \qquad\qquad 2_0 \qquad\qquad \begin{smallmatrix} 3_{-1} \\ 2_0 \end{smallmatrix},$$

which are respectively isomorphic to the path algebras of the quivers

$$1 \longleftarrow 2 \longrightarrow 3$$

and

$$1 \longrightarrow 2 \longleftarrow 3.$$

Note that Λ in the example above is the trivial extension of a bipartite Dynkin quiver of type A_3 endowed with the grading given by putting arrows in degree 1. The behaviour exhibited in the example is typical of the general case, and we summarize this in the following proposition. See for instance [12, Section 3.1] for an overview of the Coxeter numbers of different Dynkin quivers.

Proposition 7.9. *Let Q be a bipartite Dynkin quiver with Coxeter number $h \geq 4$. Consider $\Lambda = \Delta kQ$ with grading given by putting arrows in degree* 1. *Then Λ is $(1, \frac{h-2}{2}, 0)$-Λ_0-Koszul if h is even and $(1, \frac{h-1}{2}, 1)$-Λ_0-Koszul otherwise.*

Proof. As Q is a bipartite Dynkin quiver and $h \geq 4$, it follows from [5, Proposition 3.11, Corollary 4.3] that Λ is $(2, h-2)$-Koszul in the sense of Definition 7.1. Our conclusion now follows by the argument in Example 7.7. $\square$

From now on, we make the same standing assumptions as we did in order to develop our theory in Section 6.

Setup. Throughout the rest of this paper, we use the standing assumptions described at the beginning of Section 6.

Given these assumptions, let us first show that the data of an (n, m_i, σ_i)-T-Koszul algebra determines a permutation on the set $\{1, \dots, t\}$ in a natural way.

Lemma 7.10. *Let Λ be (n, m_i, σ_i)-T-Koszul. There is then a permutation π on the set $\{1, \dots, t\}$ such that*

$$\Omega^{-l_i} T^i \simeq T^{\pi(i)}\langle -g_i \rangle$$

for each $i \in \{1, \dots, t\}$.

Proof. Let $i \in \{1, \dots, t\}$. As T is almost graded n-self-orthogonal, there exists an object $T' \in \operatorname{add} T$ such that

$$\Omega^{-l_i} T^i \simeq T'\langle -g_i \rangle.$$

Recall that T is concentrated in degree 0 and that $a \geq 1$. Since it follows from Lemma 2.2 that $\operatorname{Soc}\Lambda \subseteq \Lambda_a$, this implies that T^i is not projective as a Λ-module by Lemma 2.3 *(3)*. As $\Omega^{-1}(-)$ is an equivalence on the stable category, the object T' is indecomposable, and consequently $T' \simeq T^{i'}$ for some $i' \in \{1, \dots, t\}$. This allows us to define the map

$$\pi \colon \{1, \dots, t\} \to \{1, \dots, t\}$$

by setting $\pi(i) = i'$.

We next show that π is injective and hence a permutation. Let $\pi(i) = \pi(j)$ and assume $l_i \neq l_j$. Without loss of generality, we consider the case $l_i > l_j$. Our assumption yields

$$\Omega^{-(l_i - l_j)} T^i \simeq T^j\langle -(g_i - g_j) \rangle.$$

Observe that the integers $l_i' = l_i - l_j$ and $g_i' = g_i - g_j$ hence satisfy Definition 7.2. Note in particular that $0 < l_i' < l_i$ and that positivity of l_i' combined with T being almost graded n-self-orthogonal implies positivity of g_i'. This contradicts part (3) of Definition 7.5 by Remark 7.6, so we must have $l_i = l_j$, which implies $T^i \simeq T^j$. As T is basic, this means that $i = j$, which finishes our proof. $\square$

Using our fixed decomposition $T \simeq \oplus_{i=1}^t T^i$ together with the definition of $\widetilde{T}$, we see that the algebra $B = \operatorname{End}_{\underline{\operatorname{gr}}\Lambda}(\widetilde{T})$ decomposes as

$$B \simeq \bigoplus_{i=1}^{t} \bigoplus_{j=0}^{a-1} \operatorname{Hom}_{\underline{\operatorname{gr}}\Lambda}(\widetilde{T}, X^{i,j}),$$

where $X^{i,j} = \Omega^{-nj} T^i \langle j \rangle$. Hence, the indecomposable projective B-modules

$$P^{i,j} = \operatorname{Hom}_{\underline{\operatorname{gr}}\Lambda}(\widetilde{T}, X^{i,j})$$

are indexed by the set

$$J = \{(i,j) \mid 1 \leq i \leq t \text{ and } 0 \leq j \leq a-1\}.$$

Notice that if $\widetilde{T}$ is a tilting object in $\underline{\operatorname{gr}}\Lambda$, then $X^{i,j}$ is the image of $P^{i,j}$ under the equivalence $\mathcal{D}^b(\operatorname{mod} B) \simeq \underline{\operatorname{gr}}\Lambda$, which was explicitly constructed in Proposition 4.3.

Given a permutation σ on the index set J, we let σ_j^L and σ_i^R be defined by

$$\sigma(i,j) = (\sigma_j^L(i), \sigma_i^R(j)).$$

We are now ready to state and prove the first part of our characterization result. Note that this direction in the proof of Theorem 7.17 explains and justifies the somewhat technical definition of an (n, m_i, σ_i)-T-Koszul algebra.

Theorem 7.11. *If $\widetilde{T}$ is a tilting object in $\underline{\mathrm{gr}}\,\Lambda$ and $B = \mathrm{End}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T})$ is $(na-1)$-representation finite, then there exist integers m_i and σ_i such that Λ is (n, m_i, σ_i)-T-Koszul.*

Proof. By [12, Proposition 0.2], there is a permutation σ on J such that for every pair (i, j) in J there is an integer $m_{i,j} \geq 0$ with

$$\nu_{na-1}^{-m_{i,j}} P^{i,j} \simeq I^{\sigma(i,j)},$$

as B is $(na-1)$-representation finite. Applying ν_{na-1}^{-1} on both sides, we get

$$\nu_{na-1}^{-m_{i,j}-1} P^{i,j} \simeq P^{\sigma(i,j)}[na-1].$$

Since $\widetilde{T}$ is a tilting object in $\underline{\mathrm{gr}}\,\Lambda$, we have an equivalence $\mathcal{D}^b(\mathrm{mod}\, B) \simeq \underline{\mathrm{gr}}\,\Lambda$ as described in Proposition 4.3. Using that $X^{i,j} = \Omega^{-nj} T^i\langle j \rangle$ is the image of $P^{i,j}$ under this equivalence, combined with the correspondence of Serre functors, one obtains

$$\Omega^{-(na-1)(m_{i,j}+1)-(m_{i,j}+1)} X^{i,j}_{\mu^{-m_{i,j}-1}} \langle a(m_{i,j}+1) \rangle \simeq \Omega^{-(na-1)} X^{\sigma(i,j)}.$$

This again yields

$$\Omega^{-nam_{i,j}-1} X^{\mu^{-m_{i,j}-1}(i),j} \simeq X^{\sigma(i,j)} \langle -a(m_{i,j}+1) \rangle, \tag{7.2}$$

as $(-)_\mu$ commutes with cosyzygies and graded shifts and permutes the summands of T. It follows that for each pair (i, j) in J, we get

$$\Omega^{-nam_{i,j}-1-n(j-\sigma_i^R(j))} T^{\mu^{-m_{i,j}-1}(i)} \simeq T^{\sigma_j^L(i)} \langle -a(m_{i,j}+1) + \sigma_i^R(j) - j \rangle. \tag{7.3}$$

Twisting by $\mu^{m_{i,j}+1}$ and setting $j = 0$, one obtains

$$\Omega^{-(nam_{i,0}-n\sigma_i^R(0)+1)} T^i \simeq T^{\mu^{m_{i,0}+1}(\sigma_0^L(i))} \langle -a(m_{i,0}+1) + \sigma_i^R(0) \rangle. \tag{7.4}$$

Letting $m_i := m_{i,0}$ and $\sigma_i := \sigma_i^R(0)$, we hence see that l_i and g_i can be chosen so that part (1) of the definition for being almost graded n-self-orthogonal is satisfied for T, and that parts (1) and (2) of being (n, m_i, σ_i)-T-Koszul is satisfied for Λ. Note that since g_i of this form is always positive, so is l_i, as can be seen by applying Lemma 2.3 *(6)*.

In order to show part (3) of Definition 7.5, consider an integer k satisfying $0 < nk < l_i$. Note that we can write $k = qa - r$ with $q \geq 1$ and $0 \leq r \leq a-1$. Aiming for a contradiction, assume that there is an integer $j \in \{1, \ldots, t\}$ with

$$\Omega^{-n(qa-r)} T^i \simeq T^j \langle -(qa-r) \rangle.$$

Twisting by $(-)_{\mu^{-q}}$ and using the equivalence $\mathcal{D}^b(\mathrm{mod}\, B) \simeq \underline{\mathrm{gr}}\,\Lambda$ in a similar way as in the beginning of this proof, we obtain

$$\nu_{na-1}^{-q} P^{i,0} \simeq P^{\mu^{-q}(j),r}.$$

Applying ν_{na-1} on both sides yields

$$\nu_{na-1}^{-(q-1)} P^{i,0} \simeq I^{\mu^{-q}(j),r}[-na+1]. \tag{7.5}$$

From the assumption $nk < l_i$ along with the description of l_i, we deduce that $0 \leq q-1 \leq m_i$. As long as $na > 1$, the expression (7.5) hence contradicts Lemma 5.2, so we can conclude that the third condition of Definition 7.5 is satisfied. If $na = 1$, the algebra B is semisimple. In particular, this implies that $l_i = 1$, so the condition is trivially satisfied in this case.

It remains to prove that T satisfies part (2) of Definition 7.2, i.e. that for each $i \in \{1, \dots, t\}$, we have $\operatorname{Ext}^{nk+l}_{\operatorname{gr}\Lambda}(T, T^i\langle k\rangle) = 0$ for $l \neq 0$ and $nk+l < l_i$. If $nk+l \leq 0$, this is immediately clear, so we can assume $nk+l > 0$. This yields

$$\operatorname{Ext}^{nk+l}_{\operatorname{gr}\Lambda}(T, T^i\langle k\rangle) \simeq \operatorname{Hom}_{\underline{\operatorname{gr}}\Lambda}(T, \Omega^{-(nk+l)} T^i\langle k\rangle).$$

In the case $k < 0$, this is zero by Lemma 2.3 *(6)*, and we can thus assume $k \geq 0$.

As $\widetilde{T}$ is a tilting object in $\underline{\operatorname{gr}}\Lambda$, a similar argument as in the proof of Theorem 6.4 yields an isomorphism

$$\operatorname{Hom}_{\underline{\operatorname{gr}}\Lambda}(\widetilde{T}, \Omega^{-(nam+l)} X^{\mu^{-m}(i),j}\langle am\rangle) \simeq \operatorname{H}^l(\nu_{na-1}^{-m}(P^{i,j})) \tag{7.6}$$

for every pair (i,j) in J. By Lemma 5.2, we know that $\operatorname{H}^l(\nu_{na-1}^{-m}(P^{i,j})) = 0$ for $l \neq 0$ and $0 \leq m \leq m_{i,j}$ as B is $(na-1)$-representation finite. Using that $(-)_\mu$ is an equivalence on $\underline{\operatorname{gr}}\Lambda$, that $\widetilde{T}_\mu \simeq \widetilde{T}$ and splitting up on summands of $\widetilde{T} = \oplus_{s=0}^{a-1} \Omega^{-ns} T\langle s\rangle$, this yields

$$\operatorname{Hom}_{\underline{\operatorname{gr}}\Lambda}(T, \Omega^{-(n(am-s+j)+l)} T^i\langle am-s+j\rangle) = 0 \tag{7.7}$$

for $l \neq 0$ and $0 \leq m \leq m_{i,j}$. We simplify this by letting $j = 0$. Hence, we have $m_{i,j} = m_i$. In the case $k \leq am_i$, we can write $k = am - s$ for appropriate values of m and s, so (7.7) implies our desired conclusion in this case. If $k > am_i$, we use the isomorphism $T^i \simeq \Omega^{l_i} T^{\pi(i)}\langle -g_i\rangle$ to rewrite

$$\operatorname{Hom}_{\underline{\operatorname{gr}}\Lambda}(T, \Omega^{-(nk+l)} T^i\langle k\rangle) \simeq \operatorname{Hom}_{\underline{\operatorname{gr}}\Lambda}(T, \Omega^{l_i-(nk+l)} T^{\pi(i)}\langle k-g_i\rangle).$$

When $nk+l < l_i$, this is 0 by Lemma 2.3 *(7)*. To see this, notice that the assumption $k > am_i$ combined with the definition of g_i yields $k - g_i \geq 1-a$. This finishes our proof. $\square$

Before giving a result which explains why our choices of m_i and σ_i are reasonable, we need the following lemma.

Lemma 7.12. *If $\widetilde{T}$ is a tilting object in $\underline{\operatorname{gr}}\Lambda$, then the algebra $B = \operatorname{End}_{\underline{\operatorname{gr}}\Lambda}(\widetilde{T})$ is basic.*

Proof. As $\widetilde{T}$ is a tilting object in $\underline{\operatorname{gr}}\Lambda$, it suffices to show that $\widetilde{T}$ is basic. Note that the indecomposable summands of $\widetilde{T}$ are of the form $\Omega^{-nj} T^i\langle j\rangle$ with $0 \leq i \leq t$ and $0 \leq j \leq a-1$. Assume that we have isomorphic summands

$$\Omega^{-nj} T^i\langle j\rangle \simeq \Omega^{-nl} T^k\langle l\rangle.$$

If $j = l$, it follows that $i = k$ as T is basic. Without loss of generality, we hence assume $j > l$. Consider now

$$\mathrm{Hom}_{\mathrm{gr}\,\Lambda}(T^i, T^i) \simeq \mathrm{Hom}_{\mathrm{gr}\,\Lambda}(T^i, \Omega^{-n(l-j)}T^k\langle l-j\rangle),$$

which is non-zero as $T^i \neq 0$. This contradicts Lemma 2.3 *(7)*, as $l - j \geq 1 - a$ and $-n(l-j) > 0$, so we can conclude that $(i,j) = (k,l)$. □

Recall from [12, Proposition 0.2] and the proof of Theorem 7.11 that when B is $(na-1)$-representation finite, there is a permutation σ on J such that for every pair (i,j) in J there is an integer $m_{i,j} \geq 0$ with

$$\nu_{na-1}^{-m_{i,j}} P^{i,j} \simeq I^{\sigma(i,j)}.$$

As before, we use the notation

$$\sigma(i,j) = (\sigma_j^L(i), \sigma_i^R(j)).$$

The proposition below provides more information about how the permutation σ and the integers $m_{i,j}$ associated to B being $(na-1)$-representation finite are related to the parameters m_i and σ_i.

Proposition 7.13. *If $\widetilde{T}$ is a tilting object in $\underline{\mathrm{gr}}\,\Lambda$ and $B = \mathrm{End}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T})$ is $(na-1)$-representation finite, then Λ is (n, m_i, σ_i)-T-Koszul with $m_i = m_{i,0}$ and $\sigma_i = \sigma_i^R(0)$ and we have*

$$\sigma_i^R(j) = \begin{cases} \sigma_i + j & \text{if } \sigma_i + j \leq a-1 \\ \sigma_i + j - a & \text{if } \sigma_i + j > a-1 \end{cases}$$

and

$$m_{i,j} = \begin{cases} m_i & \text{if } j \leq \sigma_i^R(j) \\ m_i - 1 & \text{if } j > \sigma_i^R(j). \end{cases}$$

Additionally, if π is the permutation on $\{1,\dots,t\}$ induced by Λ being (n, m_i, σ_i)-T-Koszul, we have

$$\sigma_j^L(i) = \mu^{-m_{i,j}-1}(\pi(i)).$$

Proof. Recall first that Λ is (n, m_i, σ_i)-T-Koszul with $m_i = m_{i,0}$ and $\sigma_i = \sigma_i^R(0)$ by Theorem 7.11 and its proof. From now, consider a fixed integer $i \in \{1,\dots,t\}$ and let $0 \leq j \leq a-1$.

Our next aim is to verify the first two equations in the formulation of the proposition. Note that to get the desired expression for $\sigma_i^R(j)$, it is enough to show that

$$\sigma_i^R(j) = \begin{cases} \sigma_i^R(0) + j & \text{if } j \leq \sigma_i^R(j) \\ \sigma_i^R(0) + j - a & \text{if } j > \sigma_i^R(j). \end{cases}$$

To see that this is sufficient, observe that given the expression above, one has $j \leq \sigma_i^R(j)$ if and only if $\sigma_i + j \leq a-1$. Indeed, if $j \leq \sigma_i^R(j)$, our formula gives

$$\sigma_i^R(j) = \sigma_i^R(0) + j = \sigma_i + j,$$

so $\sigma_i + j \le a-1$. On the other hand, the assumption $j > \sigma_i^R(j)$ yields

$$\sigma_i^R(j) = \sigma_i^R(0) + j - a = \sigma_i + j - a,$$

which implies $\sigma_i + j > a-1$.

Assume $j \le \sigma_i^R(j)$. Observe that one obtains

$$\Omega^{-nam_{i,j}-1} X^{\mu^{-m_{i,j}-1}(i),0} \simeq X^{\sigma(i,j)-(0,j)} \langle -a(m_{i,j}+1) \rangle$$

by applying $\Omega^{nj}(-)\langle -j \rangle$ to (7.2). Our assumption yields $0 \le \sigma_i^R(j) - j \le a-1$, so we can run the argument at the beginning of the proof of Theorem 7.11 in reverse to get

$$\nu_{na-1}^{-m_{i,j}} P^{i,0} \simeq I^{\sigma(i,j)-(0,j)}.$$

Recall that $\mathrm{H}^0(\nu_{na-1}^{-1}-) \simeq \tau_{na-1}^{-1}$ as endofunctors on $\operatorname{mod} B$, where τ_{na-1}^{-1} denotes the $(na-1)$-Auslander–Reiten translation. Note that the τ_{na-1}^{-1}-orbit of a projective B-module contains precisely one injective [19, Proposition 1.3]. Compare our expression above with

$$\nu_{na-1}^{-m_{i,0}} P^{i,0} \simeq I^{\sigma(i,0)}.$$

If $na > 1$, we deduce that $m_{i,j} = m_{i,0}$ and $I^{\sigma(i,j)-(0,j)} \simeq I^{\sigma(i,0)}$. If $na = 1$, then B is semisimple. This implies $m_{i,j} = m_{i,0} = 0$, and the same conclusion thus follows. In particular, this yields

$$\sigma(i,j) - (0,j) = \sigma(i,0)$$

as B is basic. Consequently, we obtain our desired expressions for $\sigma_i^R(j)$ and $m_{i,j}$ once we have made the substitutions $m_i = m_{i,0}$ and $\sigma_i = \sigma_i^R(0)$.

For the second case, assume $j > \sigma_i^R(j)$. Note that we now necessarily have $na > 1$ as $m_i = 0$ implies $\sigma_i = 0$. Apply $\Omega^{-n(a-j)}(-)\langle a-j \rangle$ to (7.2) to get

$$\Omega^{-na(m_{i,j}+1)-1} X^{\mu^{-(m_{i,j}+1)}(i),0} \simeq X^{\sigma(i,j)+(0,a-j)} \langle -a((m_{i,j}+1)+1) \rangle.$$

Our assumption yields $0 < \sigma_i^R(j) + a - j \le a-1$. Twisting by $(-)_{\mu^{-1}}$ and again reversing the argument at the beginning of the proof of Theorem 7.11, we hence obtain

$$\nu_{na-1}^{-(m_{i,j}+1)} P^{i,0} \simeq I^{\mu^{-1}(\sigma_j^L(i)),\sigma_i^R(j)+a-j}.$$

Similarly as above, this leads to our desired expressions for $\sigma_i^R(j)$ and $m_{i,j}$.

It remains to check that $\sigma_j^L(i) = \mu^{-m_{i,j}-1}(\pi(i))$. This follows by applying what we have shown so far to (7.3). $\square$

Our next aim is to prove the other direction of this section's main result. Let us first give an overview of some useful observations.

Lemma 7.14. *Let Λ be (n, m_i, σ_i)-T-Koszul. The following statements hold for $1 \le i \le t$:*

(1) We have $\pi \circ \mu = \mu \circ \pi$, where π is the permutation on $\{1, \dots, t\}$ induced by Λ being (n, m_i, σ_i)-T-Koszul.

(2) The constants l_i and g_i satisfy $l_i = l_{\mu(i)}$ and $g_i = g_{\mu(i)}$.

(3) *The constants m_i and σ_i satisfy $m_i = m_{\mu(i)}$ and $\sigma_i = \sigma_{\mu(i)}$.*
(4) *We have $g_i \geq a$. Moreover, if $m_i = 0$, then $\sigma_i = 0$.*

Proof. For part *(1)* and *(2)*, recall that $\Omega^{\pm 1}(-)$ and $\langle \pm 1 \rangle$ both commute with $(-)_\mu$. This implies that $\Omega^{-l_i} T^{\mu(i)} \langle g_i \rangle \simeq T^{\mu(\pi(i))}$ and $\Omega^{-l_{\mu(i)}} T^{\mu(i)} \langle g_{\mu(i)} \rangle \simeq T^{\pi(\mu(i))}$, and hence arguments similar to those in Remark 7.6 and Lemma 7.10 are sufficient.

Comparing the expressions for g_i and $g_{\mu(i)}$, we see that part *(3)* follows from *(2)* by a number theoretical argument.

Part *(4)* is a consequence of the definition of l_i and g_i. To be precise, it is clear that $m_i = 0$ implies $\sigma_i = 0$ as l_i is positive. Using this, the assumption $\sigma_i \leq a - 1$ yields our first statement. □

Compared to what was the case for n-T-Koszul algebras, it is somewhat more involved to show that $\widetilde{T}$ is a tilting object in $\underline{\mathrm{gr}}\,\Lambda$ whenever Λ is (n, m_i, σ_i)-T-Koszul. We hence prove this as a separate result.

Proposition 7.15. *If Λ is (n, m_i, σ_i)-T-Koszul, then $\widetilde{T}$ is a tilting object in $\underline{\mathrm{gr}}\,\Lambda$.*

Proof. Since Lemma 6.3 yields $\mathrm{Thick}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T}) = \underline{\mathrm{gr}}\,\Lambda$, we only need to check rigidity. As in the proof of Theorem 6.4, it is enough to verify that

$$\mathrm{Hom}_{\underline{\mathrm{gr}}\,\Lambda}(T, \Omega^{-(nk+l)} T\langle k \rangle) = 0 \quad \text{for} \quad l \neq 0$$

for any integer k with $|k| \leq a - 1$. In the cases $nk + l = 0$ and $nk + l < 0$, the argument is exactly the same as in the proof of Theorem 6.4, so assume $nk + l > 0$. For each summand T^i of T, one now obtains

$$\mathrm{Hom}_{\underline{\mathrm{gr}}\,\Lambda}(T, \Omega^{-(nk+l)} T^i\langle k \rangle) \simeq \mathrm{Ext}^{nk+l}_{\mathrm{gr}\,\Lambda}(T, T^i\langle k \rangle).$$

In the case $nk + l < l_i$, this is zero for $l \neq 0$ as T is almost graded n-self-orthogonal. Otherwise, we use the isomorphism $T^i \simeq \Omega^{l_i} T^{\pi(i)} \langle -g_i \rangle$ to rewrite the expression above. In the case $nk + l = l_i$, we get

$$\mathrm{Hom}_{\underline{\mathrm{gr}}\,\Lambda}(T, \Omega^{-(nk+l-l_i)} T^{\pi(i)}\langle k - g_i \rangle) = \mathrm{Hom}_{\underline{\mathrm{gr}}\,\Lambda}(T, T^{\pi(i)}\langle k - g_i \rangle).$$

This is zero as $|k| \leq a - 1$ together with Lemma 7.14 *(4)* yields $k - g_i < 0$. If $nk + l > l_i$, one obtains

$$\mathrm{Hom}_{\underline{\mathrm{gr}}\,\Lambda}(T, \Omega^{-(nk+l-l_i)} T^{\pi(i)}\langle k - g_i \rangle) \simeq \mathrm{Ext}^{nk+l-l_i}_{\mathrm{gr}\,\Lambda}(T, T^{\pi(i)}\langle k - g_i \rangle).$$

As $nk + l - l_i > 0$ and $k - g_i < 0$, the first expression can not be written as an n-multiple of the second. If $nk + l - l_i < l_{\pi(i)}$, we are hence done. Otherwise, we iterate the argument until we reach our desired conclusion. □

We are now ready to show the other direction of Theorem 7.17.

Theorem 7.16. *If Λ is (n, m_i, σ_i)-T-Koszul, then $\widetilde{T}$ is a tilting object in $\underline{\mathrm{gr}}\,\Lambda$ and $B = \mathrm{End}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T})$ is $(na - 1)$-representation finite.*

Proof. Since $\widetilde{T}$ is a tilting object in $\underline{\mathrm{gr}}\,\Lambda$ by Proposition 7.15, we only need to show that $B = \mathrm{End}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T})$ is $(na-1)$-representation finite. Let us first use the integers m_i and σ_i to define $\sigma_i^R(j)$, $m_{i,j}$ and $\sigma_j^L(i)$ for (i,j) in J by the formulas in the formulation of Proposition 7.13. Note that this yields $0 \leq \sigma_i^R(j) \leq a-1$, as well as $1 \leq \sigma_j^L(i) \leq t$ and $m_{i,j} \geq 0$. The latter is a consequence of Lemma 7.14 *(4)*.

Using that Λ is assumed to be (n, m_i, σ_i)-T-Koszul, we see that (7.4) is satisfied. Furthermore, we can run the argument at the beginning of the proof of Theorem 7.11 in reverse, using that $\widetilde{T}$ is a tilting object in $\underline{\mathrm{gr}}\,\Lambda$. Consequently, one obtains

$$\nu_{na-1}^{-m_{i,j}} P^{i,j} \simeq I^{\sigma(i,j)}$$

for every indecomposable projective B-module $P^{i,j}$, where

$$\sigma(i,j) := (\sigma_j^L(i), \sigma_i^R(j)).$$

Our next aim is to show that σ is a permutation on J. As J is a finite set, it is enough to check injectivity. Recall that μ and π are permutations, and hence injective. Combining this with Lemma 7.14 *(1)* and *(3)*, notice that also σ_0^L is injective.

Assume that $\sigma(i,j) = \sigma(k,l)$ for (i,j) and (k,l) in J. If $j \leq \sigma_i^R(j)$ and $l \leq \sigma_k^R(l)$, we see that

$$\sigma_0^L(i) = \sigma_j^L(i) = \sigma_l^L(k) = \sigma_0^L(k),$$

so $i = k$ by injectivity of σ_0^L. As we in this case also have

$$\sigma_i^R(0) + j = \sigma_i^R(j) = \sigma_k^R(l) = \sigma_k^R(0) + l,$$

it follows that $j = l$, so σ is injective. The argument in the case $j > \sigma_i^R(j)$ and $l > \sigma_k^R(l)$ is similar.

By symmetry, it remains to consider the case where $j \leq \sigma_i^R(j)$ and $l > \sigma_k^R(l)$. Here, the assumption $\sigma(i,j) = \sigma(k,l)$ yields

$$\sigma_0^L(i) = \sigma_j^L(i) = \sigma_l^L(k) = \mu(\sigma_0^L(k)).$$

Consequently, Lemma 7.14 *(1)* and *(3)* imply that $i = \mu(k)$ and $\sigma_i^R(0) = \sigma_k^R(0)$. As we in this case also have

$$\sigma_i^R(0) + j = \sigma_i^R(j) = \sigma_k^R(l) = \sigma_k^R(0) + l - a,$$

this means that $j = l - a$, contradicting the assumption $0 \leq j, l \leq a-1$. Hence, this case is impossible, and we can conclude that σ is a permutation.

It now follows that every indecomposable injective, and hence also DB, is contained in the subcategory

$$\mathcal{U} = \mathrm{add}\{\nu_{na-1}^l B \mid l \in \mathbb{Z}\} \subseteq \mathcal{D}^b(\mathrm{mod}\, B).$$

By Theorem 5.1, it thus remains to prove that $\mathrm{gl.dim}\, B \leq na-1$. To show this, observe first that B has finite global dimension by Lemma 6.2. As $\widetilde{T}$ is a tilting object in $\underline{\mathrm{gr}}\,\Lambda$, it follows from (7.6) in the proof of Theorem 7.11 that we have

$$\begin{aligned}\mathrm{H}^l(\nu_{na-1}^{-1}(P^{i,j})) &\simeq \mathrm{Hom}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T}, \Omega^{-(na+l)} X^{\mu^{-1}(i),j}\langle a\rangle) \\ &\simeq \bigoplus_{s=0}^{a-1} \mathrm{Hom}_{\underline{\mathrm{gr}}\,\Lambda}(T, \Omega^{-(n(a+j-s)+l)} T^i \langle a+j-s\rangle)\end{aligned}$$

for every pair (i,j) in J. We want to show that this is zero whenever $l \notin \{1-na, 0\}$. Note that the argument for this is similar to the proof of Proposition 7.15. In particular, it is enough to consider the case $n(a+j-s)+l \geq l_i$ for each i, since the remaining cases are covered by our previous proof. Using that $\Omega^{-l_i}T^i \simeq T^{\pi(i)}\langle -g_i\rangle$, the summands in our expression above can be rewritten as

$$\mathrm{Hom}_{\underline{\mathrm{gr}}\,\Lambda}(T, \Omega^{-n(\sigma_i+j-s-am_i)-(na-1+l)} T^{\pi(i)} \langle \sigma_i + j - s - am_i\rangle).$$

If $n(\sigma_i + j - s - am_i) + na - 1 + l < l_{\pi(i)}$, this is non-zero only when l is as claimed. Otherwise, Lemma 7.14 *(4)* implies that we get a negative graded shift in the next step of the iteration, and we are done by the same argument as in the proof of Proposition 7.15. From this, one can see that the assumptions in Lemma 5.3 are satisfied, and hence $\mathrm{gl.dim}\, B \leq na-1$. Applying Theorem 5.1, we conclude that B is $(na-1)$-representation finite, which finishes our proof. □

Altogether, combining Theorem 7.11 and Theorem 7.16, we have now proved this section's main result. Recall that we use the standing assumptions described at the beginning of Section 6.

Theorem 7.17. *The following statements are equivalent:*

(1) There exist integers m_i and σ_i such that Λ is (n, m_i, σ_i)-T-Koszul.
(2) $\widetilde{T}$ is a tilting object in $\underline{\mathrm{gr}}\,\Lambda$ and $B = \mathrm{End}_{\underline{\mathrm{gr}}\,\Lambda}(\widetilde{T})$ is $(na-1)$-representation finite.

Moreover, the parameters m_i, σ_i and the permutation π obtained from Λ being (n, m_i, σ_i)-T-Koszul correspond to the parameter $m_{i,j}$ and the permutation σ obtained from B being $(na-1)$-representation finite as described in Proposition 7.13.

We now present some consequences of our characterization theorem similar to the ones in Section 6. Notice that unlike the corresponding result for n-representation infinite algebras, the following corollary is not – as far as we know – an analogue of anything existing in the literature. Mutatis mutandis, the proof is the same as that of Corollary 6.6 and is hence omitted. The parameters of Λ and Λ_0 in the statement correspond as described in Theorem 7.17.

Note that in the first part of the corollary below, we set $T = \Lambda_0$ and hence assume that the Nakayama automorphism of Λ only permutes the summands of Λ_0. This is trivially satisfied whenever our algebra is graded symmetric.

Corollary 7.18. *If $a = 1$, our algebra Λ is $(n+1, m_i, \sigma_i)$-Koszul with respect to $T = \Lambda_0$ if and only if Λ_0 is n-representation finite. In particular, we obtain a* *bijective correspondence*

$$\left\{\begin{array}{l}\textit{isomorphism classes of}\\ \textit{basic } n\textit{-representation finite}\\ \textit{algebras } A\end{array}\right\} \rightleftarrows \left\{\begin{array}{l}\textit{isomorphism classes of graded symmetric finite}\\ \textit{dimensional algebras of highest degree } 1 \textit{ which}\\ \textit{are } (n+1, m_i, \sigma_i)\textit{-Koszul with respect to their}\\ \textit{degree } 0 \textit{ parts}\end{array}\right\},$$

where the maps are given by $A \longmapsto \Delta A$ *and* $\Lambda_0 \longleftarrow \Lambda$.

Just like in Section 6, it is natural to consider the notion of an almost n-T-Koszul dual of a given almost n-T-Koszul algebra.

Definition 7.19. Let Λ be an almost n-T-Koszul algebra. The *almost n-T-Koszul dual of* Λ is given by $\Lambda^! = \oplus_{i\geq 0} \operatorname{Ext}^{ni}_{\operatorname{gr}\Lambda}(T, T\langle i\rangle)$.

As before, note that while the notation $\Lambda^!$ is potentially ambiguous, it is for us always clear from context which structure the dual is computed with respect to.

Our next proposition shows that if Λ is (n, m_i, σ_i)-T-Koszul, then the na-th preprojective algebra of $B = \operatorname{End}_{\underline{\operatorname{gr}}\Lambda}(\widetilde{T})$ is isomorphic to a twist of the a-th quasi-Veronese of $\Lambda^!$. The proof is exactly the same as that of the corresponding result in Section 6, namely Proposition 6.8.

Proposition 7.20. *Let* Λ *be* (n, m_i, σ_i)*-*T*-Koszul. Then* $\Pi_{na}B \simeq {}_{\langle(\overline{\mu}^{-1})^{[a]}\rangle}(\Lambda^!)^{[a]}$ *as graded algebras. In particular, we have* $\Pi_{na}B \simeq (\Lambda^!)^{[a]}$ *in the case where* Λ *is graded symmetric.*

The proof of our final corollary is similar to that of Corollary 6.9 and is hence omitted.

Corollary 7.21. *If* A *is basic* n*-representation finite, then* $\Pi_{n+1}A \simeq (\Delta A)^!$ *as graded algebras.*

Acknowledgements. The authors would like to thank Steffen Oppermann for helpful discussions and Louis-Philippe Thibault and Øyvind Solberg for careful reading and helpful suggestions on a previous version of this paper. Parts of this work was carried out while the first author participated in the Junior Trimester Program "New Trends in Representation Theory" at the Hausdorff Research Institute for Mathematics in Bonn. She would like to thank the Institute for excellent working conditions. The authors profited from use of the software QPA [34] to compute examples which motivated parts of the paper.

References

[1] Claire Amiot, Osamu Iyama, and Idun Reiten, *Stable categories of Cohen-Macaulay modules and cluster categories*, Amer. J. Math. **137** (2015), no. 3, 813–857.

[2] Maurice Auslander, Idun Reiten, and Sverre O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1995.

[3] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527.

[4] Joseph N. Bernšteĭn, Izrail' M. Gel′fand, and Sergei I. Gel′fand, *Algebraic vector bundles on* $\mathbf{P}^n$ *and problems of linear algebra*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 66–67.

[5] Sheila Brenner, Michael C. R. Butler, and Alastair D. King, *Periodic algebras which are almost Koszul*, Algebr. Represent. Theory **5** (2002), no. 4, 331–367.

[6] Erik Darpö and Osamu Iyama, *d-representation-finite self-injective algebras*, Adv. Math. **362** (2020), 106932, 50.

[7] David E. Evans and Mathew Pugh, *The Nakayama automorphism of the almost Calabi-Yau algebras associated to* $SU(3)$ *modular invariants*, Comm. Math. Phys. **312** (2012), no. 1, 179–222.

[8] Robert M. Fossum, Phillip A. Griffith, and Idun Reiten, *Trivial extensions of abelian categories*, Lecture Notes in Mathematics, Vol. 456, Springer-Verlag, Berlin-New York, 1975. Homological algebra of trivial extensions of abelian categories with applications to ring theory.

[9] Joseph Grant and Osamu Iyama, *Higher preprojective algebras, Koszul algebras, and superpotentials*, Compos. Math. **156** (2020), no. 12, 2588–2627.

[10] Edward L. Green, Idun Reiten, and Øyvind Solberg, *Dualities on generalized Koszul algebras*, Mem. Amer. Math. Soc. **159** (2002), no. 754, xvi+67.

[11] Dieter Happel, *On the derived category of a finite-dimensional algebra*, Comment. Math. Helv. **62** (1987), no. 3, 339–389.

[12] Martin Herschend and Osamu Iyama, *n-representation-finite algebras and twisted fractionally Calabi–Yau algebras*, Bull. Lond. Math. Soc. **43** (2011), no. 3, 449–466.

[13] ———, *Selfinjective quivers with potential and 2-representation-finite algebras*, Compos. Math. **147** (2011), no. 6, 1885–1920.

[14] Martin Herschend, Osamu Iyama, Hiroyuki Minamoto, and Steffen Oppermann, *Representation theory of Geigle-Lenzing complete intersections*, arXiv:1409.0668 (2020).

[15] Martin Herschend, Osamu Iyama, and Steffen Oppermann, *n-representation infinite algebras*, Adv. Math. **252** (2014), 292–342.

[16] Osamu Iyama, *Auslander correspondence*, Adv. Math. **210** (2007), no. 1, 51–82.

[17] ———, *Higher-dimensional Auslander–Reiten theory on maximal orthogonal subcategories*, Adv. Math. **210** (2007), no. 1, 22–50.

[18] ———, *Auslander–Reiten theory revisited*, Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 349–397.

[19] ———, *Cluster tilting for higher Auslander algebras*, Adv. Math. **226** (2011), no. 1, 1–61.

[20] Osamu Iyama and Steffen Oppermann, *n-representation-finite algebras and n-APR tilting*, Trans. Amer. Math. Soc. **363** (2011), no. 12, 6575–6614.

[21] ———, *Stable categories of higher preprojective algebras*, Adv. Math. **244** (2013), 23–68.

[22] Osamu Iyama and Michael Wemyss, *Maximal modifications and Auslander-Reiten duality for non-isolated singularities*, Invent. Math. **197** (2014), no. 3, 521–586.

[23] Gustavo Jasso and Julian Külshammer, *Higher Nakayama algebras I: Construction*, Adv. Math. **351** (2019), 1139–1200. With an appendix by Külshammer and Chrysostomos Psaroudakis and an appendix by Sondre Kvamme.

[24] Gustavo Jasso and Sondre Kvamme, *An introduction to higher Auslander–Reiten theory*, Bull. Lond. Math. Soc. **51** (2019), no. 1, 1–24.

[25] Bernhard Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 1.

[26] ———, *Deformed Calabi–Yau completions*, J. Reine Angew. Math. **654** (2011), 125–180. With an appendix by Michel Van den Bergh.

[27] Dag Oskar Madsen, *Ext-algebras and derived equivalences*, Colloq. Math. **104** (2006), no. 1, 113–140.

[28] ———, *On a common generalization of Koszul duality and tilting equivalence*, Adv. Math. **227** (2011), no. 6, 2327–2348.

[29] Hiroyuki Minamoto, *Ampleness of two-sided tilting complexes*, Int. Math. Res. Not. IMRN **1** (2012), 67–101.

[30] Hiroyuki Minamoto and Izuru Mori, *The structure of AS-Gorenstein algebras*, Adv. Math. **226** (2011), no. 5, 4061–4095.

[31] Constantin Năstăsescu and Freddy Van Oystaeyen, *Methods of graded rings*, Lecture Notes in Mathematics, vol. 1836, Springer-Verlag, Berlin, 2004.

[32] Steffen Oppermann and Hugh Thomas, *Higher-dimensional cluster combinatorics and representation theory*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 6, 1679–1737.

[33] Stewart B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39–60.

[34] The QPA-team, *QPA - Quivers, path algebras and representations* (Version 1.31; 2018), `https://folk.ntnu.no/oyvinso/QPA/`.

[35] Idun Reiten and Michel Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. **15** (2002), no. 2, 295–366.

[36] Jeremy Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra **61** (1989), no. 3, 303–317.

[37] The Stacks Project Authors, *Stacks Project* (2018), `https://stacks.math.columbia.edu`.

[38] Kota Yamaura, *Realizing stable categories as derived categories*, Adv. Math. **248** (2013), 784–819.

Department of mathematical sciences, NTNU, NO-7491 Trondheim, Norway

Email address: `johanne.haugland@ntnu.no`

Email address: `mads.sandoy@ntnu.no`

Paper 3

CLASSIFICATION RESULTS FOR n-HEREDITARY MONOMIAL ALGEBRAS

Mads Hustad Sandøy   Louis-Philippe Thibault

CLASSIFICATION RESULTS FOR n-HEREDITARY MONOMIAL ALGEBRAS

MADS HUSTAD SANDØY AND LOUIS-PHILIPPE THIBAULT

Abstract. We classify n-hereditary monomial algebras in three natural contexts: First, we give a classification of the n-hereditary truncated path algebras. We show that they are exactly the n-representation-finite Nakayama algebras classified by Vaso. Next, we classify partially the n-hereditary quadratic monomial algebras. In the case $n = 2$, we prove that there are only two examples, provided that the preprojective algebra is a planar quiver with potential. The first one is a Nakayama algebra and the second one is obtained by mutating $\mathbb{A}_3 \otimes_k \mathbb{A}_3$, where $\mathbb{A}_3$ is the Dynkin quiver of type A with bipartite orientation. In the case $n \geq 3$, we show that the only n-representation finite algebras are the n-representation-finite Nakayama algebras with quadratic relations.

Contents

1. Introduction

Auslander–Reiten theory has proven to be a central tool in the study of the representation theory of Artin algebras [ARS97]. In 2004, Iyama introduced a generalisation of some of the key concepts to a 'higher-dimensional' paradigm [Iya07a, Iya07b]. To put it in his own words, "in these Auslander–Reiten theories, the number '2' is quite symbolic". For example, the Auslander correspondence establishes a bijection between finite-dimensional representation-finite algebras and finite-dimensional algebras of global dimension at most 2 and dominant dimension at least

2020 *Mathematics Subject Classification.* 16G20 (Primary), 16E30.

Key words and phrases. n-hereditary algebra, n-representation-finite algebra, n-representation-infinite algebra, preprojective algebra, Jacobian algebra, selfinjective algebra, Calabi–Yau algebra, Auslander–Reiten theory.

2 [Aus71]. This realisation was the starting point of very fruitful research which has had applications in representation theory, commutative algebra, as well as commutative and categorical algebraic geometry (e.g. [Iya11, IO11, MM11, IO13, HIO14, HIMO14, IW14, AIR15, IJ17, DJW19, JK19, BHon]).

Auslander–Reiten theory is particularly nice over finite-dimensional hereditary algebras Λ. For example, there is a trichotomy in the representation theory of these algebras into preprojective, regular and preinjective modules. Moreover, their preprojective algebra $\Pi = T_\Lambda \operatorname{Ext}^1_\Lambda(D\Lambda, \Lambda)$ provides very useful information [BGL87]. This motivated the study of the so-called n-hereditary algebras, which consist of the n-representation-finite (henceforth abbreviated as n-RF) [Iya07b, HI11a, HI11b, Iya11, IO11, IO13] and n-representation-infinite (henceforth n-RI) [HIO14] algebras. These are finite-dimensional algebras of global dimension n which enjoy properties analogous to hereditary algebras in the classical theory. There is also a natural generalisation of the preprojective algebra over these algebras.

Many instances of n-hereditary algebras were discovered over the years (e.g. [HI11b, IO13, AIR15, Pet19, Pas20, BHon]). For example, algebras of higher type A and type $\tilde{A}$ are n-RF and n-RI, respectively [IO11, HIO14]. The defining properties of n-hereditary algebras are rather strong, so classes of examples should be expected to be somewhat special. However, it seems that we are still in an early stage, and that many more classes of examples and classification results have yet to be discovered. Such results would allow an even better understanding of the role of these algebras.

The aim of this paper is to study characteristics of certain n-hereditary monomial algebras. On many occasions, we use the fact that n-hereditary algebras Λ enjoy the property that $\operatorname{Ext}^j_{\Lambda^e}(\Lambda, \Lambda^e) = 0$ for all $0 < j < n$ [IO13], which we refer to as the vanishing-of-Ext condition. Since monomial algebras have a nice bimodule resolution, provided by Bardzell [Bar97], we have a good control over these extension groups. Using that fact and a classification of the n-representation-finite Nakayama algebras by Vaso [Vas19], we obtain the following result for truncated path algebras.

Theorem A (Proposition 3.6, Theorem 3.7)**.** *Let $\Lambda = kQ/\mathcal{J}^\ell$ be a truncated path algebra, where $\ell \geq 2$, Q is a finite quiver and $\mathcal{J}$ is the arrow ideal. Let $\mathbb{A}_m$ be the linearly oriented Dynkin quiver of type A with m vertices.*

(1) If Q is acyclic and $\operatorname{Ext}^j_{\Lambda^e}(\Lambda, \Lambda^e) = 0$ for all $0 < j < \operatorname{gl.dim} \Lambda$, then $Q = \mathbb{A}_m$, for some m.

(2) The following are equivalent:
  - *(a) Λ is n-hereditary;*
  - *(b) $\Lambda \cong k\mathbb{A}_m/\mathcal{J}^\ell$, for some m, and $\ell \mid m-1$ or $\ell = 2$.*

  In this case, $n = 2\frac{m-1}{\ell}$ and Λ is an n-representation-finite Nakayama algebra.

We note that the vanishing-of-Ext condition already allows us to reduce the number of cases by a lot.

Next, we move to the study of quadratic monomial algebras. Our main results are given as follows.

Theorem B (Theorem 4.1, Corollary 4.20, Theorem 4.26)**.** *Let $\Lambda = kQ/I$ be a quadratic monomial algebra of global dimension n.*

(1) Suppose that $n = 2$.
  - *(a) If $\operatorname{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) = 0$, then Q is an (r, s)-star quiver (Definition 4.19).*

(b) *If Λ is n-hereditary and the preprojective algebra $\Pi(\Lambda)$ is a planar quiver with potential, then Λ is given by one of the following two 2-RF algebras:*

(1.1)

where the dotted arcs denote relations. Note that the first algebra is the Nakayama algebra $k\mathbb{A}_3/\mathcal{J}^2$.

(2) *Suppose that $n \geq 3$ and Λ is n-RF. Then $\Lambda \cong k\mathbb{A}_{n+1}/\mathcal{J}^2$.*

Perhaps surprisingly, we see that the class of 2-RF quadratic monomial algebras is richer than those in higher global dimension. In the $n = 2$ case, we assumed that the preprojective algebra was a planar QP. There are examples of other 2-RF quadratic monomial algebras where this property is not satisfied, see Example 4.24. This assumption appears often, at least implicitly, in different results aimed at understanding some selfinjective Jacobian algebras and 2-RF algebras (e.g. [HI11b, Pet19, Pas20]). Note that all examples covered in the previous theorem were already known to be n-RF. The algebra corresponding to the $(4,4)$-star above is a cut of $\Pi(\mathbb{A}_3^{\mathrm{bip}} \otimes_k \mathbb{A}_3^{\mathrm{bip}})$, where $\mathbb{A}_3^{\mathrm{bip}}$ is the Dynkin quiver of type A with bipartite orientation and Π denotes the higher preprojective algebra.

Acknowledgements. We thank Martin Herschend for pointing out a mistake in the statement of Proposition 3.6 when the results were first announced at the fd-seminar in June 2020, and we thank Steffen Oppermann and Øyvind Solberg for carefully reading an earlier version of the manuscript.

Setup. Let k be an algebraically closed field. The k-dual $\mathrm{Hom}_k(-,k)$ is denoted by D. Unless specified otherwise, all modules are left modules. The idempotent associated to a vertex i is denoted by e_i. If a and b are arrows in a quiver, then ab denotes the path b followed by a. The head of an arrow $a\colon i \to j$ is denoted by $h(a)$ and equals j, and the tail is denoted by $t(a)$ and equals i. These extend to paths $p = p_\ell p_{\ell-1} \cdots p_1$ by letting $h(p) = h(p_\ell)$ and $t(p) = t(p_1)$. Moreover, the length of a path $p = p_\ell p_{\ell-1} \cdots p_1$ is ℓ and this is denoted by $L(p)$. The syzygy of a module N is the kernel of the projective cover of N and this is denoted by ΩN. If Λ is a k-algebra, then $\mathsf{mod}\,\Lambda$ denotes the category of finitely generated left modules and $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$ the bounded derived category. When $\Lambda = kQ/I$ is a basic algebra, we always assume that Q is a connected quiver.

2. Preliminaries

2.1. **n-hereditary algebras.** Let Λ be a finite-dimensional algebra of global dimension n. Let

$$\mathbb{S} := D\Lambda \overset{\mathbf{L}}{\otimes}_\Lambda - : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$$

be the Serre functor with inverse

$$\mathbb{S}^{-1} = \mathbf{R}\mathrm{Hom}_\Lambda(D\Lambda, -) : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda).$$

Denote by $\mathbb{S}_n$ the composition $\mathbb{S}_n := \mathbb{S} \circ [-n]$.

Definition 2.1. We say that Λ is

- *n-representation-finite (n-RF)* if for any indecomposable projective $P \in \mathsf{proj}\,\Lambda$, there exists $i \geq 0$ such that $\mathbb{S}_n^{-i}(P) \in \mathsf{inj}\,\Lambda$, the category of finitely generated injective modules.
- *n-representation-infinite (n-RI)* if $\mathbb{S}_n^{-i}(\Lambda) \in \mathsf{mod}\,\Lambda$ for any $i \geq 0$.
- *n-hereditary* if $\mathrm{H}^j(\mathbb{S}_n^i(\Lambda)) = 0$ for all $i, j \in \mathbb{Z}$ such that $j \notin n\mathbb{Z}$.

These definitions, as written, were given in [HIO14], but the concept of n-RF algebras was studied before in [Iya07b, HI11a, HI11b, Iya11, IO11, IO13].

We have the following dichotomy.

Theorem 2.2 ([HIO14, Theorem 3.4])**.** *Let Λ be a ring-indecomposable k-algebra. Then Λ is n-hereditary if and only if it is either n-RF or n-RI.*

Recall that hereditary algebras Λ are *formal*, that is, for any $X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$, there is an isomorphism

$$X \cong \bigoplus_{i\in\mathbb{Z}} \mathrm{H}^j(X)[-j].$$

An important feature of n-hereditary algebras is that a certain generalisation of this property holds. This follows from [Iya11, Lemma 5.2].

Proposition 2.3. *Let Λ be an n-hereditary algebra. Then for any $i \in \mathbb{Z}$ and an indecomposable projective module $P \in \mathsf{proj}\,\Lambda$, there exists $j \in \mathbb{Z}$ such that*

$$\mathbb{S}_n^i(P) \cong \mathrm{H}^{nj}(\mathbb{S}_n^i(P))[-nj].$$

As a consequence, n-hereditary algebras satisfy a condition which is closely related to the *vosnex ("vanishing of small negative extensions")* property (see [IO13, Notation 3.5]).

Corollary 2.4. *Let Λ be an n-hereditary algebra. Then*

$$\mathrm{Ext}^{\ell}_{\Lambda}(D\Lambda, \Lambda) = 0 \tag{2.1}$$

for all $0 < \ell < n$.

We refer to this property as the *vanishing-of-*Ext condition.

As for classical hereditary algebras, preprojective algebras play an important role.

Definition 2.5. Let Λ be a finite-dimensional algebra of global dimension n. The $(n+1)$-*preprojective algebra* $\Pi(\Lambda)$ is defined as

$$\Pi(\Lambda) := T_\Lambda \operatorname{Ext}^n_\Lambda(D\Lambda, \Lambda) \cong \bigoplus_{\ell \geq 0} \mathrm{H}^0(\mathbb{S}_n^{-\ell}(\Lambda)).$$

Note that $\mathrm{Ext}^{\ell}_{\Lambda}(D\Lambda, \Lambda) \cong \mathrm{Ext}^{\ell}_{\Lambda^e}(\Lambda, \Lambda^e)$ [GI19, Lemma 2.9], a fact that we use often.

Preprojective algebras and n-hereditary algebras are connected in the following way.

Theorem 2.6. *Let Λ be a finite-dimensional algebra.*

(1) If Λ is an n-representation-finite algebra. Then $\Pi(\Lambda)$ is a selfinjective algebra. The converse holds if Λ has global dimension 2.

(2) The following are equivalent.

  a) Λ is n-representation-infinite;

  b) $\Pi(\Lambda)$ is a bimodule Calabi–Yau algebra of Gorenstein parameter 1.

Here, *(1)* is due to [IO13, Corollary 3.4 & Corollary 3.8], whereas *(2)* is an amalgam of results from [Kel11, Theorem 4.8], [MM11, Corollary 4.13], [HIO14, Theorem 4.36], and [AIR15, Theorem 3.4]. We refer to the papers for definitions.

In the case where Λ is Koszul, we have a good understanding of the construction of preprojective algebras. To present the construction we need certain notions of derivatives which we define below, and we note that they are used extensively in this paper, and not just in the context of Koszul algebras.

Notation for derivatives. Let S be a semisimple k-algebra and V be an S-bimodule. Let $p = v_\ell \otimes \cdots \otimes v_1 \in V^{\otimes_S \ell}$. We define the linear morphisms

$$\delta^{\mathcal{L}}_m(p) := v_{\ell-m} \otimes \cdots \otimes v_1 \quad \text{and} \quad \delta^{\mathcal{R}}_m(p) := v_\ell \otimes \cdots \otimes v_{m+1}.$$

for $m < \ell$ and we let both equal 0 when $\ell = m$.

Moreover, we define

$$\mathcal{L}_m(p) := v_\ell \otimes \cdots \otimes v_{\ell-m+1} = \delta^{\mathcal{R}}_{\ell-m}(p) \quad \text{and} \quad \mathcal{R}_m(p) = v_m \otimes \cdots \otimes v_1 = \delta^{\mathcal{L}}_{\ell-m}(p).$$

The subscript is dropped if $m = 1$.

We also define linear morphisms associated to elements $q \in V^{\otimes m}$, for $m \leq \ell$:

$$\delta^{\mathcal{L}}_q(p) := \begin{cases} a & \text{if } p = q \otimes a \\ 0 & \text{else} \end{cases} \quad \text{and} \quad \delta^{\mathcal{R}}_q(p) := \begin{cases} b & \text{if } p = b \otimes q \\ 0 & \text{else.} \end{cases}$$

Similarly, we define

$$\mathcal{L}_q(p) := \begin{cases} b & \text{if } p = b \otimes q \otimes a \\ 0 & \text{else} \end{cases} \quad \text{and} \quad \mathcal{R}_q(p) := \begin{cases} a & \text{if } p = b \otimes q \otimes a \\ 0 & \text{else.} \end{cases}$$

When $p = b \otimes q \otimes a$ for some paths a and b, we say that q *divides* p and denote this by $q|p$.

Description of the n-preprojective algebra of a Koszul n-hereditary algebra. Recall that if Λ is Koszul, it can be given as a tensor algebra $T_S V/\langle M\rangle$ where, as in the previous section, S is some semisimple k-algebra, V is an S-bimodule, and $M \subset V \otimes_S V$ is a subbimodule [BGS96]. Let then

$$K_\ell := \bigcap_{\mu=0}^{\ell-2} (V^{\otimes \mu} \otimes M \otimes V^{\otimes \ell-\mu-2})$$

be the terms appearing in the minimal Koszul resolution of Λ according to [BGS96]. Moreover, given a vector space V, let $\mathcal{B}(V)$ be a basis.

Proposition 2.7 ([GI19, Proposition 3.12], [Thi20, Corollary 3.3])**.** *Let $\Lambda = T_S V/\langle M\rangle$ be a finite-dimensional Koszul algebra of global dimension n. Let $\{e_i \,|\, 1 \leq i \leq m\}$ be a complete set of primitive orthogonal idempotents in Λ. Let $\overline{V}$ be the vector space obtained from V by adding a basis element $e_i a_q e_j$ for each element $q \in \mathcal{B}(e_j K_n e_i)$. Let $\overline{M}$ be the union of M with the set $\widetilde{M}$ of quadratic relations given by*

$$\widetilde{M} := \left\{ \sum_{q \in \mathcal{B}(K_n)} a_q \delta^{\mathcal{R}}_p(q) + (-1)^n \sum_{q \in \mathcal{B}(K_n)} \delta^{\mathcal{L}}_p(q) a_q \quad \middle| \quad p \in \mathcal{B}(K_{n-1}) \right\}.$$

There is an isomorphism of algebras

$$\Pi \cong T_S \overline{V}/\langle \overline{M}\rangle.$$

2.2. **Monomial algebras.** In this subsection, we define monomial algebras and describe certain minimal projective resolutions.

Definition 2.8. Let $\Lambda = kQ/I$, where Q is a finite quiver and I an admissible ideal. We say that Λ is a *monomial algebra* if I can be generated by a finite number of paths.

There is a nice description of the minimal projective Λ-bimodule resolution of Λ, due to Bardzell [Bar97]. Let M be a minimal set of paths of minimal length which generates I. Given a path p, define the *support* to be the set of all vertices dividing p. For every directed path T in Q, there is a natural order $<$ on the support of T. Let $M(T)$ be the set of relations which divide T.

Definition 2.9. Let $p \in M(T)$. We define the *left construction associated to p along T* by induction. Let $r_2 \in M(T)$ be the path (if it exists) in $M(T)$ which is minimal with respect to $t(p) < h(r_2) < h(p)$. Now assume we have constructed $r_1 = p, r_2, \ldots, r_j$. Let

$$L_{j+1} = \{r \in M(T) \mid h(r_{j-1}) \leq t(r) < h(r_j)\}.$$

If $L_{j+1} \neq \varnothing$, let r_{j+1} be such that $t(r_{j+1})$ is minimal in L_{j+1}.

Definition 2.10. Let $p \in M$ and $\ell \geq 2$ be an integer. We define

$$\mathrm{AS}_p(\ell) := \{(r_1 = p, r_2, \ldots, r_{\ell-1}) \mid (r_1, r_2, \ldots, r_{\ell-1}) \text{ is a sequence of paths associated to } p \text{ in the left construction}\}.$$

For each element $(r_1, \ldots, r_{\ell-1}) \in \mathrm{AS}_p(\ell)$, define p^ℓ to be the path from $t(p)$ to $h(r_{\ell-1})$ and let $\mathrm{AP}_p(\ell)$ be the set of all p^ℓ. Finally, we define

$$\mathrm{AP}(\ell) := \bigcup_{p \in M} \mathrm{AP}_p(\ell),$$

if $\ell \geq 2$ and $\mathrm{AP}(0) := Q_0$, $\mathrm{AP}(1) := Q_1$.

The vector spaces $k\,\mathrm{AP}(\ell)$ are the kQ_0-bimodules which appear in the minimal resolution we want to construct. Note that $\mathrm{AP}(2) = M$. If $p \in \mathrm{AP}(\ell)$, define

$$\mathrm{Sub}(p) := \{q \in \mathrm{AP}(\ell - 1) \mid q \text{ divides } p\}.$$

Lemma 2.11 ([Bar97, Lemma 3.3])**.** *The set* $\mathrm{Sub}(p)$ *contains two paths* p_0 *and* p_1 *such that* $t(p_0) = t(p)$ *and* $h(p_1) = h(p)$*. Moreover, if* ℓ *is odd, then* $\mathrm{Sub}(p) = \{p_0, p_1\}$*.*

We are now ready to define morphisms

$$d_\ell : \bigoplus_{p \in \mathrm{AP}(\ell)} \Lambda e_{h(p)} \otimes_k e_{t(p)}\Lambda \to \bigoplus_{p \in \mathrm{AP}(\ell-1)} \Lambda e_{h(p)} \otimes_k e_{t(p)}\Lambda,$$

noting that we give our conventions with respect to idempotents, and heads and tails of arrows and paths in the setup immediately following the introduction. Recall that if $p \in \mathrm{AP}(\ell)$ and $q \in \mathrm{Sub}(p)$, we write $p = \mathcal{L}_q(p) q \mathcal{R}_q(p)$. By the previous lemma, we have that $\mathrm{Sub}(p) = \{p_0, p_1\}$ if ℓ is odd, in which case $p = \mathcal{L}_{p_0}(p)p_0$ and $p = p_1\mathcal{R}_{p_1}(p)$. Then we define

$$d_\ell((e_{h(p)} \otimes e_{t(p)})_p) := \begin{cases} (\mathcal{L}_{p_0}(p)e_{h(p_0)} \otimes e_{t(p_0)})_{p_0} - (e_{h(p_1)} \otimes e_{t(p_1)}\mathcal{R}_{p_1}(p))_{p_1} & \text{if } \ell \text{ is odd} \\ \sum_{q \in \mathrm{Sub}(p)} (\mathcal{L}_q(p)e_h \otimes e_t \mathcal{R}_q(p))_q & \text{if } \ell \text{ is even.} \end{cases}$$

Here, we use the notation $(- \otimes -)_p$ to denote an element in the p-th component in $\bigoplus_p \Lambda e_i \otimes_k e_j \Lambda$.

Theorem 2.12 ([Bar97, Theorem 4.1])**.** *The complex*

$$\cdots \xrightarrow{d_{n+1}} \bigoplus_{p\in \mathrm{AP}(n)} \Lambda e_{h(p)} \otimes_k e_{t(p)}\Lambda \xrightarrow{d_n} \cdots \xrightarrow{d_1} \bigoplus_{e_i\in \mathrm{AP}(0)} \Lambda e_i \otimes_k e_i\Lambda \xrightarrow{\mu} \Lambda \to 0, \qquad (2.2)$$

where $\mu((e_i\otimes e_i)_{e_i}) = e_i$, is a minimal projective resolution of Λ as a Λ-bimodule.

2.3. **Computing $\mathrm{Ext}^{\ell}_{\Lambda^e}(\Lambda,\Lambda^e)$.** In the next sections, we use on many occasions Corollary 2.4 as an obstruction for certain algebras to be n-hereditary. We therefore explain here how to compute $\mathrm{Ext}^{\ell}_{\Lambda^e}(\Lambda,\Lambda^e)$ for $1 \le \ell \le n$.

Let Λ be a basic finite-dimensional algebra. By [Hap89, Section 1.5], Λ has a minimal projective bimodule resolution of the form

$$P_\bullet : \cdots \xrightarrow{d_{n+1}} \bigoplus_{p\in \mathcal{B}(E^n(i,j))} \Lambda e_{h(p)} \otimes_k e_{t(p)}\Lambda \xrightarrow{d_n} \cdots \xrightarrow{d_1} \bigoplus_{e_i\in \mathcal{B}(E^0(i,j))} \Lambda e_i \otimes_k e_i\Lambda \to 0,$$

where $E^{\ell}(i,j) := \mathrm{Ext}^{\ell}_{\Lambda}(S_i,S_j)$ and S_i denotes the simple module at vertex i. In the case where Λ is monomial, we have $E^{\ell}(i,j) \cong e_j k\,\mathrm{AP}(\ell)e_i$. Note that, in general, it is hard to determine the differentials d_ℓ.

In order to compute $\mathrm{Ext}^{\ell}_{\Lambda^e}(\Lambda,\Lambda^e)$, we apply $\mathrm{Hom}_{\Lambda^e}(-,\Lambda^e)$ to $P_\bullet$ and use the isomorphisms

$$\begin{array}{rcccc} \Psi : \mathrm{Hom}_{\Lambda^e}(\Lambda e_j \otimes_k e_i\Lambda, \Lambda^e) & \cong & e_j\Lambda \otimes_k \Lambda e_i & \cong & \Lambda e_i \otimes_k e_j\Lambda \\ \phi & \mapsto & \phi(e_j\otimes e_i) & & \\ & & e_j\otimes e_i & \mapsto & e_i \otimes e_j \end{array}$$

to obtain a complex

$$\mathrm{Hom}_{\Lambda^e}(P_\bullet,\Lambda^e) : 0 \to \bigoplus_{e_i\in \mathcal{B}(E^0(i,j))} \Lambda e_i \otimes_k e_i \Lambda \xrightarrow{\tilde{d}_1} \cdots \xrightarrow{\tilde{d}_n} \bigoplus_{p\in \mathcal{B}(E^n(i,j))} \Lambda e_{t(p)} \otimes_k e_{h(p)}\Lambda \to \cdots, \quad (2.3)$$

where $\tilde{d}_\ell(e_i\otimes e_j) = \Psi(\Psi^{-1}(e_i\otimes e_j)\circ d_\ell)$.

Computing $\mathrm{Ext}^{\ell}_{\Lambda^e}(\Lambda,\Lambda^e)$ requires the understanding of the morphisms $\tilde{d}_\ell$, which we do have in the case where Λ is monomial. In fact, we have

$$\tilde{d}_\ell((e_{t(p)}\otimes e_{h(p)})_p) = \begin{cases} \sum_{q\in \mathrm{AP}(\ell)} (e_{t(q)}\otimes e_{h(q)}\delta^{\mathcal{R}}_p(q))_q - \sum_{q\in\mathrm{AP}(\ell)} (\delta^{\mathcal{L}}_p(q)e_{t(q)}\otimes e_{h(q)})_q & \text{if } \ell \text{ is odd} \\ \sum_{q\in \mathrm{AP}(\ell)\,|\, p\in \mathrm{Sub}(q)} (\mathcal{R}_p(q)e_{t(q)}\otimes e_{h(q)}\mathcal{L}_p(q))_q & \text{if } \ell \text{ is even.} \end{cases}$$

In further sections, we use these to describe cocycles and coboundaries, allowing us to show that some $\mathrm{Ext}^{\ell}_{\Lambda^e}(\Lambda,\Lambda^e)$ does not vanish for some algebra, thus preventing them from being n-hereditary. Using Corollary 2.4, we can already give a necessary condition for a monomial algebra to be n-hereditary. This is analogous to results established in [GI19, Proof of Theorem 3.14] and [Thi20, Proof of Theorem 3.6] in the case where Λ is Koszul.

Lemma 2.13. *Let Λ be a monomial algebra and define*

$$\delta(E(i,j)^{\ell}) := \{w\in E(i,j)^{\ell-1} \mid w \in \mathrm{Sub}(w') \text{ for some } w'\in E(i,j)^{\ell}\}.$$

Then $E(i,j)^{\ell-1} = \delta(E(i,j)^{\ell})$ for all $2\le \ell\le n$.

Proof. Suppose by contradiction that there exists $w \in E(i,j)^{\ell-1}$ which does not divide any element of $E(i,j)^{\ell}$, for some ℓ. Then

$$\tilde{d}_\ell((e_{t(w)}\otimes e_{h(w)})_w) = 0,$$

which means that $(e_{t(w)}\otimes e_{h(w)})_w$ is an $(\ell-1)$-cocycle in $\mathrm{Hom}_{\Lambda^e}(P_\bullet,\Lambda^e)$. However, it is not a coboundary, implying that $\mathrm{Ext}^{\ell-1}_{\Lambda^e}(\Lambda,\Lambda^e)\neq 0$. This contradicts Corollary 2.4. $\square$

Corollary 2.14. *Let $\Lambda = kQ/\langle M\rangle$ be a basic n-hereditary monomial algebra where M is a set of relations given by paths in kQ. Then every arrow in Q_1 is part of a relation in M.*

Proof. By Lemma 2.13, we have that $\delta(M) = V$. □

3. Classification of n-hereditary truncated path algebras

In this section, we assume that $\Lambda = kQ/I$ is a monomial finite-dimensional algebra, that is, the ideal I in a presentation of Λ can be chosen to be generated by paths. Moreover, for the rest of the text, whenever Λ is assumed to be monomial, we also assume $I = \langle M\rangle$ with M a minimal set of paths of minimal length.

Recall that, by the vanishing-of-Ext condition, we have that $\mathrm{Ext}^i_{\Lambda^e}(\Lambda, \Lambda^e) = 0$ for all $0 < i < n$ for any n-hereditary algebra Λ. We thus seek to understand what knowledge one can obtain from this property. As an application, we use this information to classify the truncated path algebras $\Lambda = kQ/\mathcal{J}^\ell$, where $\mathcal{J}$ is the arrow ideal, which are n-hereditary in the second subsection.

3.1. **Vanishing-of-Ext condition for monomial algebras.** In this subsection, we find necessary conditions on the quiver and relations of monomial path algebras in order to satisfy the vanishing-of-Ext condition. To be more precise, we only look into the vanishing of the first Ext. Recall that, by Lemma 2.13, every arrow has to be part of at least one relation, otherwise $\mathrm{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) \neq 0$. This is a first obstruction, which does not require the monomial hypothesis. We therefore assume this property for the class of algebras we consider in this subsection. Throughout, we let Λ be a monomial algebra in which every arrow divides at least one relation.

The main strategy is to construct cocycle elements which are not coboundaries in the complex (2.3), defined as $\mathrm{Hom}_{\Lambda^e}(P_\bullet, \Lambda^e)$, where $P_\bullet$ is the minimal projective Λ-bimodule resolution of Λ, described in the preliminaries. We refer to Sections 2.2 and 2.3 for more details and the notation.

Proposition 3.1. *Suppose that there exists an arrow a which is the start (resp. the end) of every relation it divides and such that $t(a)$ (resp. $h(a)$) is not a source (resp. a sink). Then $\mathrm{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) \neq 0$.*

Proof. Assume that there is an arrow a which is the end of every relation r_i it divides and such that $h(a)$ is not a sink. The other case is dual. We consider the element

$$(ae_{t(a)} \otimes_k e_{h(a)})_a \in \bigoplus_{v \in Q_1} \Lambda e_{t(v)} \otimes_k e_{h(v)}\Lambda$$

from complex (2.3). Then

$$\tilde{d}_2((ae_{t(a)} \otimes_k e_{h(a)})_a) = \sum_i (a\mathcal{R}_1(r_i)e_{t(r_i)} \otimes_k e_{h(a)})_{r_i} = 0,$$

so it is a cocycle in complex (2.3). However, since $h(a)$ is not a sink, $(ae_{t(a)} \otimes_k e_{h(a)})_a$ cannot be a coboundary. In fact, let b be an arrow such that $h(a) = t(b)$. Then,

$$\tilde{d}_1((e_{h(a)} \otimes_k e_{h(a)})_{e_{h(a)}} = (ae_{t(a)} \otimes_k e_{h(a)})_a + (e_{h(a)} \otimes_k e_{h(b)}b)_b + \dots$$

This is the only place where $(ae_{t(a)} \otimes_k e_{h(a)})_a$ appears as a summand of an element in the image of $\tilde{d}_1$. The same is true for $(e_{h(a)} \otimes_k e_{h(b)}b)_b$, which means that this term cannot be cancelled by other elements in the image of $\tilde{d}_1$. Therefore, $(ae_{t(a)} \otimes_k e_{h(a)})_a$ is not a coboundary and $\mathrm{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) \neq 0$. □

We say that two relations *intersect* with each other if there is at least one arrow which divides both of them. We have the following corollary.

Corollary 3.2. *Assume that there is a relation r which does not intersect with any other relation and such that $t(r)$ and $h(r)$ are not both a source and a sink. Then* $\operatorname{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) \neq 0$.

Continuing on the same ideas, we explore what happens at sinks and sources. We show that the vanishing-of-Ext conditions implies that sinks and sources divide only one arrow.

Proposition 3.3. *Assume that there is a vertex i in Q which is a sink (resp. a source), such that there is at least two arrows having i as head (resp. as tail). Then* $\operatorname{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) \neq 0$.

Proof. We suppose that i is a sink. The other case is dual. Let a and b be two arrows such that $h(a) = h(b) = i$. We claim that the element $(ae_{t(a)} \otimes_k e_i)_a \in \bigoplus_{v \in Q_1} \Lambda e_{t(v)} \otimes_k e_{h(v)} \Lambda$ is a cocycle in degree 1. In fact, since $h(a)$ is a sink, every relation r containing a is of the form $r = aR_r(a)$ for some path $R_r(a)$. Therefore,

$$\tilde{d}_2((ae_{t(a)} \otimes_k e_i)_a) = \sum_{a|r} (aR_r(a)e_{t(r)} \otimes e_i)_r = 0.$$

This is however not a coboundary. In fact, since i is also the head of another arrow, we have that

$$\tilde{d}_1((e_i \otimes e_i)_{e_i}) = (ae_{t(a)} \otimes_k e_i)_a + (be_{t(b)} \otimes_k e_i)_b + \dots$$

By the same reasoning as in Proposition 3.1, we conclude that $\operatorname{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) \neq 0$. □

3.2. **n-hereditary truncated path algebras.** We now consider the case of truncated path algebras $\Lambda = kQ/\mathcal{J}^\ell$ for some $\ell \geq 2$, where Q is a finite quiver and $\mathcal{J}$ is the arrow ideal. In this case, the terms in the Bardzell resolution (2.2) are particularly easy to describe. Indeed, the vector space $k\,\mathrm{AP}(i)$ is generated by all paths of length $\frac{i}{2} \cdot \ell$ if i is even and those of length $\left(\frac{i-1}{2} \cdot \ell + 1\right)$ if i is odd. Let $L(p)$ denote the length of a path p. We use the following results.

Theorem 3.4. [DHZL08, Theorem 2] *Let Λ be a truncated path algebra. If N is a non-zero Λ-module with skeleton σ, then the syzygy of N*

$$\Omega N \cong \bigoplus_{q\ \sigma\text{-critical}} \Lambda q.$$

We refer to the paper for the definitions of skeletons σ and of σ-critical paths.

We also need the following result regarding extensions of certain kinds of indecomposable modules.

Proposition 3.5 ([Vas19, Proposition 3.1])**.** *Let Λ be a finite-dimensional algebra. Let $N \in \operatorname{mod} \Lambda$ be a non-projective indecomposable module. If ΩN is decomposable, then* $\operatorname{Ext}^1_\Lambda(N, \Lambda) \neq 0$.

Let now $\mathbb{A}_m$ be the linearly oriented Dynkin quiver of type A

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \text{- - - - -} \bullet \longrightarrow \bullet \longrightarrow \bullet$$

with m vertices.

Proposition 3.6. *Let Q be a finite acyclic quiver and assume that $Q \neq \mathbb{A}_m$. Let $\Lambda := kQ/\mathcal{J}^\ell$ for some $\ell \geq 2$ be a truncated path algebra. Then there exists $0 < j < \operatorname{gl.dim} \Lambda$ such that* $\operatorname{Ext}^j_{\Lambda^e}(\Lambda, \Lambda^e) \neq 0$.

Proof. By Proposition 3.3, if Q is a Dynkin quiver of type A with a non-linear orientation, then $\operatorname{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) \neq 0$. Since $Q \neq \mathbb{A}_m$, there exists a vertex i which divides at least 3 arrows. If i is either a source or a sink, then $\operatorname{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) \neq 0$ by Proposition 3.3 as well.

Now suppose that i is the head of at least two arrows and the tail of at least one arrow. The opposite case is treated similarly. Among the arrows with head i we pick two, say, a_r and b_s

satisfying that $a_r \neq b_s$ and that there exist paths $T_1 := a_r \cdots a_1$ and $T_2 := b_s \cdots b_1$ which are maximal in the following sense: without loss of generality, we let T_1 be the longest path in kQ ending at i and T_2 the maximal path in kQ ending at i not divided by a_r. Note that this uses that Q is acyclic. In particular, we assume $L(T_2) \leq L(T_1)$. Moreover, we let $T_3 := c_t \cdots c_1$ be the longest path in kQ beginning in i.

We may also assume that $h(T_3)$ is only a sink to the arrow c_t and $t(T_i)$ is a source to only one arrow for $i = 1, 2$, since otherwise $\operatorname{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) \neq 0$ and we are done.

We split the proof into the following cases:

- **C1**: $L(T_3T_2) \leq \ell - 1$
- **C2**: $L(T_3T_2) \geq \ell$
  - **a)** $L(T_2) \leq \ell - 1$
  - **b)** $L(T_2) \geq \ell$
    - **i)** $L(T_3) \geq \ell - 1$
    - **ii)** $L(T_3) \leq \ell - 2$

C1: If $L(T_3T_2) \leq \ell - 1$, then b_s does not divide any relation, by the maximality assumption on the length of T_3T_2. As a consequence of Lemma 2.13, $\operatorname{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) \neq 0$ and we are done.

C2: Now suppose that $L(T_3T_2) \geq \ell$.

a) If $L(T_2) \leq \ell - 1$, then for any relation path p (of length ℓ) such that b_s divides p, we have that

$$L(\mathcal{L}_{b_s}(p)) \geq \ell - s \geq \max(1, \ell - r),$$

where $r := L(T_1)$ and $s := L(T_2)$. The first inequality is explained by the maximality assumption on $L(T_3T_2)$. For the second inequality, recall that we have assumed without loss of generality that $r \geq s$. This means that any path of maximal length starting at i is of length at least $\max(1, \ell - r)$. Therefore, the element

$$(e_{t(b_s)} \otimes_k e_i a_r \cdots a_{\max(1, r-\ell+2)})_{b_s} \in \bigoplus_{v \in Q_1} \Lambda e_{t(v)} \otimes_k e_{h(v)} \Lambda$$

is a non-trivial cocycle. It is not a coboundary since the only two Λ-bimodule generators in $\bigoplus_{i \in Q_0} \Lambda e_i \otimes_k e_i \Lambda$ which map non-trivially via $\tilde{d}_1$ to an element in $\Lambda e_{t(b_s)} \otimes_k e_i \Lambda$ are

$$(e_{t(b_s)} \otimes_k e_{t(b_s)})_{e_{t(b_s)}} \mapsto (e_{t(b_s)} \otimes_k e_i b_s)_{b_s} + \ldots$$

and

$$(e_i \otimes_k e_i)_{e_i} \mapsto (b_s e_{t(b_s)} \otimes_k e_i)_{b_s} + \ldots$$

and they cannot be linearly combined to obtain our cocycle. Thus, $\operatorname{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) \neq 0$.

b.i) We now consider the case where $L(T_2) \geq \ell$. Let $j \in \mathbb{N}_{\geq 1}$ be such that the length of the paths in $k\,\mathrm{AP}(j)$ is less than or equal to $L(T_2)$, but the length of the paths in $k\,\mathrm{AP}(j+1)$ is strictly bigger than $L(T_2)$. If $L(T_3) \geq \ell - 1$, then $0 < j < \operatorname{gl.dim} \Lambda$, since $k\,\mathrm{AP}(j+1)$ is non empty, as it contains a path dividing T_3T_2. Let $T := b_s \cdots b_x$ be the path in $k\,\mathrm{AP}(j)$ ending at i and dividing T_2. Then the element

$$(e_{t(T)} \otimes_k e_i a_r \cdots a_{r-\ell+2})_T \in \bigoplus_{p \in \mathrm{AP}(j)} \Lambda e_{t(p)} \otimes_k e_{h(p)} \Lambda$$

is a cocycle. Indeed, for any $T' \in \mathrm{AP}(j+1)$ which is divided by T, we have $L(\mathcal{L}_T(T')) \geq 1$. This is explained by the fact that $L(T') > L(T_2)$ and the maximality assumption on the length of T_2. In fact, if $L(\mathcal{L}_T(T')) = 0$, then $h(T') = i$ and $L(T_3T') > L(T_3T_2)$, contradicting our hypothesis.

The element is not a coboundary for a similar reason as above.

b.ii) Now, if $L(T_3) \leq \ell - 2$, then we consider the indecomposable injective module I associated to the vertex $h := h(T_3)$. We show that either there are more than one σ-critical paths or there is only one σ-critical path q and Λq is projective. In the former case, we conclude by Theorem 3.4 and Proposition 3.5 that $\mathrm{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) \cong \mathrm{Ext}^1_\Lambda(D\Lambda, \Lambda) \neq 0$. In the latter case, we obtain the same conclusion since $\mathrm{proj.dim}\, I = 1$.

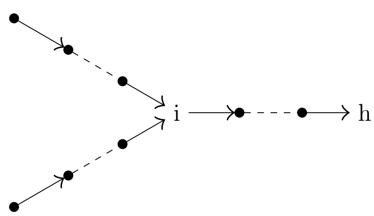
We call *branching points* the vertices which divide at least 3 arrows. Let $\mathcal{S}_h$ be the support of paths of length $\ell - 1$ in kQ which end in h. Let $\mathcal{B}_h$ be the set of branching points which are in $\mathcal{S}_h$. Since $L(T_3) \leq \ell - 2$, we have that $i \in \mathcal{B}_h$.

Let $\mathcal{S}'_h \subset \mathcal{S}_h$ be the set of vertices which start the paths of length $\ell - 1$ that end in h. Note that $P := \bigoplus_{\iota \in \mathcal{S}'_h} \Lambda e_\iota^{m(\iota)}$ is the projective cover of I, where $m(\iota)$ is the number of paths of length ℓ which ends in h and starts in ι. Because $L(T_3) < \ell - 1$, we have $|\mathcal{S}'_h| \geq 2$, since it contains a vertex in T_1 and T_2. Thus, I is not a projective module.

Let $x \in B_h$ be such that there exist arrows α and β ending in x. Then either paths of the form αp or of the form βq are in the skeleton σ, for $p, q \in \sigma$. The paths not in σ must then be σ-critical. In fact, they get identified via $P \twoheadrightarrow I$. Thus, every such branching point gives rise to σ-critical paths.

Now suppose that there exists a vertex $x \in \mathcal{B}_h$ which is the start of an arrow α not in a path of length $\ell - 1$ ending in h. Then for any skeleton σ, we have that any path of the form αp, for $p \in \sigma$, is σ-critical, since it goes to 0 via $P \twoheadrightarrow I$.

Therefore, in order to have only one σ-critical path, it is necessary that the full subquiver $\bar{Q}$ containing all the directed paths connected to the branching points in $\mathcal{B}_h$ is given by



In this case, we have that $\Omega I \cong \Lambda e_i$ is projective. □

The n-representation-finite Nakayama algebras were classified by Vaso in [Vas19]. Using his classification, we obtain as a corollary of the previous proposition a classification of all n-hereditary algebras of the form $\Lambda = kQ/\mathcal{J}^\ell$.

Theorem 3.7. *Let $\Lambda = kQ/\mathcal{J}^\ell$ for some $\ell \geq 2$ and finite quiver Q. The following are equivalent.*

(1) Λ is n-hereditary;
(2) $\Lambda \cong k\mathbb{A}_m/\mathcal{J}^\ell$, for some m, and $\ell \mid m - 1$ or $\ell = 2$.

In this case, $n = 2\frac{m-1}{\ell}$ and Λ is a Nakayama n-representation-finite algebra.

Proof. By [DHZL08, Theorem 5], any truncated path algebra of finite global dimension must have an acyclic quiver. By Proposition 3.6, if Λ is n-hereditary, then its quiver must be $\mathbb{A}_m$, since n-hereditary algebras satisfy the property that $\mathrm{Ext}^i_{\Lambda^e}(\Lambda, \Lambda^e) = 0$ for all $0 < i < n$. Therefore, Λ

is an n-representation-finite Nakayama algebra. The result thus follows from [Vas19, Theorem 3]. □

4. Classification of n-hereditary quadratic monomial algebras

In this section, we give a partial classification of the n-hereditary quadratic monomial algebras. Let $\Lambda = kQ/I$ be such an algebra. In the first subsection, we tackle the case $n = 2$. With the additional assumption that the preprojective algebra can be given by a planar selfinjective quiver with potential, we show that there are only two examples. Then, in the next subsection, we show that provided $n \geq 3$, the only n-hereditary quadratic monomial algebras are the Nakayama ones given in the previous section.

4.1. **The case $n = 2$.** The goal of this section is to prove the following theorem.

Theorem 4.1. *Let $\Lambda = kQ/I$ be a 2-hereditary quadratic monomial algebra. Assume that $\Pi(\Lambda)$ is given by a planar quiver with potential. Then Λ is one of the two bounded quiver algebras given in (1.1). These algebras are 2-representation-finite.*

These two algebras were already known to be 2-representation-finite. In fact, the first one appears already in [IO13, Theorem 3.12]. The second one is a cut, a notion defined below, of $\Pi(\mathbb{A}_3^{\mathrm{bip}} \otimes_k \mathbb{A}_3^{\mathrm{bip}})$, where $\mathbb{A}_3^{\mathrm{bip}}$ is the Dynkin quiver of type A with three vertices and bipartite orientation.

We provide more information on the preprojective algebra $\Pi(\Lambda)$ of a 2-hereditary algebra. It is a Jacobian algebra which is selfinjective in the case when Λ is 2-representation-finite, and 3-Calabi–Yau in the case when Λ is 2-representation-infinite. We give a brief overview of these useful facts. They are key in our classification result.

Definition 4.2. Let Q be a quiver and $\mathcal{J}$ be the ideal generated by arrows. A *potential* W is an element in $\widehat{kQ}/[\widehat{kQ}, \widehat{kQ}]$, where $\widehat{kQ}$ is the completion of the path algebra with respect to the $\mathcal{J}$-adic topology.

Definition 4.3. Let (Q, W) be a quiver with potential. The *Jacobian algebra* of (Q, W) is defined as
$$\mathcal{P}(Q, W) := \widehat{kQ}/\langle \delta_a W \mid a \in Q_1 \rangle.$$

Every 3-preprojective algebra is a Jacobian algebra.

Theorem 4.4. [Kel11, Theorem 6.10] *Let Λ be a finite-dimensional algebra of global dimension 2. Then there exists a quiver Q_Λ and a potential W_Λ such that $\Pi(\Lambda) \cong \mathcal{P}(Q_\Lambda, W_\Lambda)$.*

Let M be a minimal set of relations in Λ. The quiver of Q_Λ is given by adding new arrows $c_\rho : i \to j$ for every relation $\rho : j \to i$ in M. The potential W_Λ is given by
$$W_\Lambda = \sum_{\rho \in M} \rho c_\rho.$$
In particular, if Λ is quadratic, then $\Pi(\Lambda)$ is quadratic as well.

One important assumption for the main result of this section is that $\Pi(\Lambda)$ is a *planar* quiver algebra with potential. In fact, we give at the end of this subsection an example of a 2-hereditary quadratic monomial algebra whose preprojective algebra does not satisfy this property. We provide the definition here.

Definition 4.5. Let Q be a quiver without loops or 2-cycles. An *embedding* $\epsilon : Q \to \mathbb{R}^2$ is a map which is injective on the vertices, sends arrows $a : i \to j$ to the open line segment l_a from $\epsilon(i)$ to $\epsilon(j)$, and satisfies

- $\epsilon(i) \notin l_a$ for every $i \in Q_0$ and $a \in Q_1$ and
- $l_a \cap l_b = \varnothing$ for all $a \neq b \in Q_1$.

The pair (Q, ϵ) is called a *plane quiver*. A *face* of (Q, ϵ) is a bounded component of $\mathbb{R}^2 \setminus \epsilon(Q)$ which is an open polygon.

Definition 4.6. Let (Q, ϵ) be a plane quiver such that every bounded connected component of $\mathbb{R}^2 \setminus \epsilon(Q)$ is a face and the arrows bounding every face are cyclically oriented. The *potential induced* from (Q, ϵ) is the linear combination W of the bounding cycles of all faces. The quiver with potential (Q, W) is called the *planar QP induced from* (Q, ϵ) and any quiver with potential obtained in this way is called a *planar QP*.

Remark 4.7. A quiver with potential whose underlying quiver is planar is not necessarily a planar QP. In fact, the planarity has to be compatible with the potential, that is, each face is bounded by an oriented cycle.

We can obtain algebras of global dimension at most 2 from Π by using cuts. In fact, let (Q, W) be a quiver with potential and $\mathcal{C} \subset Q_1$ be a subset. We define a grading $g_{\mathcal{C}}$ on Q by setting

$$g_{\mathcal{C}}(a) := \begin{cases} 1 & a \in \mathcal{C} \\ 0 & a \notin \mathcal{C} \end{cases}$$

for each $a \in Q_1$.

Definition 4.8. A subset $\mathcal{C} \subset Q_1$ is called a *cut* if W is homogeneous of degree 1 with respect to $g_{\mathcal{C}}$.

When $\mathcal{C}$ is a cut, there is an induced grading on $\mathcal{P}(Q, W)$. We denote by $\mathcal{P}(Q, W)_{\mathcal{C}}$ the degree 0 part with respect to this grading.

Definition 4.9. A cut $\mathcal{C}$ is called *algebraic* if it satisfies the following properties:

(1) $\mathcal{P}(Q, W)_{\mathcal{C}}$ is a finite-dimensional k-algebra with global dimension at most two;
(2) $\{\delta_c W\}_{c \in \mathcal{C}}$ is a minimal set of generators in the ideal $\langle \delta_c W \mid c \in \mathcal{C} \rangle$.

All truncated Jacobian algebras $\mathcal{P}(Q, W)_{\mathcal{C}}$ given by algebraic cuts $\mathcal{C}$ are cluster equivalent [HI11b, Proposition 7.6]. These are related to 2-APR tilts [IO11].

When Λ is 2-hereditary, Π enjoys some additional characteristics.

Proposition 4.10. *Let Λ be a k-algebra such that* $\operatorname{gl.dim} \Lambda \leq 2$.

- [HIO14, Theorem 5.6] *The following are equivalent.*
  (1) $\Pi(\Lambda) = \mathcal{P}(Q, W)$ is a bimodule 3-Calabi–Yau Jacobian algebra of Gorenstein parameter 1;
  (2) $\Pi(\Lambda)_{\mathcal{C}}$ is a 2-representation-infinite algebra for every cut $\mathcal{C} \subset Q_1$.
- [HI11b, Proposition 3.9] *The following are equivalent.*
  (1) $\Pi(\Lambda) = \mathcal{P}(Q, W)$ is a finite-dimensional selfinjective Jacobian algebra;
  (2) $\Pi(\Lambda)_{\mathcal{C}}$ is a 2-representation-finite algebra for every cut $\mathcal{C} \subset Q_1$.

This characterisation allows us to work with the following exact sequences.

Theorem 4.11. *Let $\Pi = \mathcal{P}(Q, W)$ be a Jacobian algebra.*

- [Boc08, Proof of Theorem 3.1] Π *is 3-Calabi–Yau if and only if the following complex of left Π-modules is exact for every simple module S_i:*

$$0 \to P_i \xrightarrow{[a]} \bigoplus_{\substack{a \in Q_1 \\ h(a)=i}} P_{t(a)} \xrightarrow{[\delta_{(a,b)} W]} \bigoplus_{\substack{b \in Q_1 \\ t(b)=i}} P_{h(b)} \xrightarrow{[b]} P_i \to S_i \to 0, \tag{4.1}$$

where $P_j := \Pi e_j$ *and* $\delta_{(a,b)}W := \delta_a^{\mathcal{L}} \circ \delta_b^{\mathcal{R}} W$.

- [HI11b, Theorem 3.7] Π *is selfinjective if and only if it is finite-dimensional and the following complex of left* Π*-modules is exact for every simple module* S_i*:*

$$P_i \xrightarrow{[a]} \bigoplus_{\substack{a\in Q_1\\ h(a)=i}} P_{t(a)} \xrightarrow{[\delta_{(a,b)}W]} \bigoplus_{\substack{b\in Q_1\\ t(b)=i}} P_{h(b)} \xrightarrow{[b]} P_i \to S_i \to 0. \tag{4.2}$$

We also need a couple of additional definitions to treat the case when Λ is in addition a quadratic monomial algebra. Let Π be a Jacobian algebra with potential W. We say that Π admits a *monomial cut* $\mathcal{C}$ if $\Pi_{\mathcal{C}}$ is a monomial algebra. Also, an arrow a in the quiver of Π is called a *border* if a is part of exactly one summand of W. It is clear that if Π is quadratic, then W is a sum of cyclic paths of length 3, since it is homogeneous of degree 1. We call those summands *triangles*. By ideas similar to Lemma 2.13, every arrow is part of at least one triangle.

The following lemma is clear.

Lemma 4.12. *A cut* $\mathcal{C}$ *is monomial if and only if the arrows in degree* 1 *are borders.*

In particular, the existence of a monomial cut in Π implies that there is at least one border in each summand. An important step in our classification proof is to show that there is exactly one border, unless there is only one summand.

The following lemma is elementary, but we include a proof for the convenience of the reader.

Lemma 4.13. *Let* $\Pi = \mathcal{P}(Q, W)$ *be a Jacobian algebra which is either selfinjective or Calabi–Yau. The matrix* $[\delta_{(a,b)}W]$ *in the complexes (4.1) and (4.2) is indecomposable, that is, it is not similar to a block matrix.*

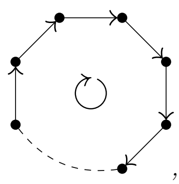
Proof. Suppose by contradiction that the complexes can be written as

$$P_i \xrightarrow{\begin{bmatrix}[a']\\ [a'']\end{bmatrix}} \bigoplus_{\substack{a\in Q_1\\ h(a)=i}} P_{t(a)} \xrightarrow{\begin{bmatrix}[\delta_{(a',b')}W] & [0]\\ [0] & [\delta_{(a'',b'')}W]\end{bmatrix}} \bigoplus_{\substack{b\in Q_1\\ t(b)=i}} P_{h(b)} \xrightarrow{\begin{bmatrix}[b'] & [b'']\end{bmatrix}} P_i \to S_i \to 0,$$

for some vectors of arrows $[a'], [a''], [b'], [b'']$. Then the element $([0], [a'']) \in \bigoplus_{\substack{a\in Q_1\\ h(a)=i}} P_{t(a)}$ is a cycle which is not a boundary, contradicting the exactness of the complex. $\square$

Using this, we now show that we can exclude an important class of examples, namely those coming from truncated Nakayama algebras.

Lemma 4.14. *Let* $\Pi := \mathcal{P}(Q, W)$ *be a Jacobian algebra which is either selfinjective or Calabi–Yau. Suppose that there exists a summand* W' *of* W *in which every arrow is a border. Then* Π *is given by the quiver*



,

with potential the one obtained by summing over every cyclic rotation of the complete cycle.

Proof. Let $W' = x_n \cdots x_1$ be the summand in which every arrow is a border. By Lemma 4.13, the matrix

$$[\delta_{(a,b)}W] : \bigoplus_{\substack{a\in Q_1\\ h(a)=h(x_n)}} P_{t(a)} \to \bigoplus_{\substack{b\in Q_1\\ t(b)=h(x_n)}} P_{t(b)},$$

is indecomposable. However, the column

$$[\delta_{(a,x_1)}W]_{h(a)=h(x_n)}$$

and the row

$$[\delta_{(x_n,b)}W]_{t(b)=h(x_n)}$$

each only have one non-zero element, since x_1 and x_n are borders. Thus, $[\delta_{(a,b)}W]$ can only be indecomposable if its dimension is 1×1. This means that x_n is the only arrow ending at $h(x_n)$ and x_1 is the only arrow starting at $h(x_n)$. Repeating the argument with x_1, x_2, …, x_{n-1}, we deduce that there is also only one arrow ending and one starting at $h(x_i)$, for $i = 1, \dots, n-1$, as well. Since the quiver Q is connected, Π must be given by the QP described in the statement. $\square$

Now assume that Π is the preprojective algebra of a 2-hereditary algebra. If Π admits a monomial cut, then we show that summands of the potential cannot have two borders either. For this, we need the following proposition. It is shown in [HI11b] in the case where $\mathcal{P}(Q,W)$ is a selfinjective algebra, but the same proof also works in the case when $\mathcal{P}(Q,W)$ is a 3-Calabi–Yau algebra.

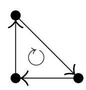
Proposition 4.15 ([HI11b, Proposition 3.10])**.** *Let $\mathcal{P}(Q,W)$ be a preprojective algebra over a 2-hereditary algebra. Then every cut $\mathcal{C} \subset Q_1$ is algebraic.*

Lemma 4.16. *Let $\Pi = \mathcal{P}(Q,W)$ be a quadratic Jacobian algebra which is either selfinjective or 3-Calabi–Yau. Suppose that Π admits a monomial cut. Then there does not exist a triangle W' of W which has exactly two borders.*

Proof. Suppose by contradiction that $W' = xyz$ is a triangle of W such that x and y are both borders, and z is not. Then there is another summand $W'' = uvz$ containing z. Let $\mathcal{C}$ be the monomial cut on Π, in which we may assume without loss of generality that x is in degree 1. Then, the grading $\mathcal{C}'$ obtained from $\mathcal{C}$ by putting x in degree 0 and y in degree 1 is also a cut, since x and y are borders which do not appear in other triangles. Now, suppose that v is in degree 1. Then, in $\Pi_{\mathcal{C}}$, we have that $zu = 0$ and $yz = 0$. Thus gl.dim $\Pi_{\mathcal{C}} \geq 3$, and $\mathcal{C}$ is not algebraic, contradicting Proposition 4.15. Similarly, if u is in degree 1, then $\mathcal{C}'$ is a non algebraic cut. As z cannot be in degree 1 in a monomial cut, W'' cannot be put in degree 1 in $\mathcal{C}$. $\square$

Combining the previous two lemmas, we obtain the following corollary.

Corollary 4.17. *Let $\Pi = \mathcal{P}(Q,W)$ be a quadratic Jacobian algebra which is selfinjective or 3-Calabi–Yau and admits a monomial cut. Then either every summand of W has exactly one border, or Π is the quiver algebra with potential with a unique triangle:*



Note that the latter case is the preprojective algebra of $k\mathbb{A}_3/\mathcal{J}^2$, the first example in our main theorem.

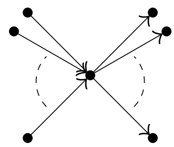
The vanishing-of-Ext condition also gives information about the quiver of 2-hereditary quadratic monomial algebras. We have the following corollary to Proposition 3.1.

Corollary 4.18. *Let Λ be a quadratic monomial algebra with $\operatorname{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) = 0$. Then for every relation $\rho = ba$, the vertex $h(b)$ is a sink and the vertex $t(a)$ is a source.*

Proof. Since gl.dim $\Lambda = 2$, every arrow is either the start or the end of every relation they divide. The result thus follows directly from Proposition 3.1. □

This leads to the following definition.

Definition 4.19. Let $r, s \in \mathbb{Z}_{\geq 1}$. The (r, s)-*star quiver*, denoted by $S_{(r,s)}$, is the quiver



with $r + s + 1$ vertices and a central vertex z which is the head of r arrows and the tail of s arrows. We always denote the arrows $i \to z$ by a_i and the arrows $z \to j$ by b_j.

We conclude that every quadratic monomial 2-hereditary algebra is a bound quiver algebra over a star quiver.

Corollary 4.20. *Let Λ be a quadratic monomial algebra with $\operatorname{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) = 0$. Then the quiver of Λ is an (r, s)-star quiver. In particular, 2-hereditary quadratic monomial algebras are given by quotients of (r, s)-star quiver algebras.*

Proof. By Corollary 4.18, every relation is a path of length 2 which starts at a source and ends at a sink. Furthermore, Proposition 3.3 implies that these vertices can only be source and sink to one arrow. This means that the quiver of Λ is made of paths of length 2 which all intersect at a common middle vertex. □

Before completing the proof of the main theorem of this section, we explore further some quick restrictions on the relations which are imposed by the vanishing-of-Ext condition. From now on in this section, we let Λ be a bound (r, s)-star quiver algebra. For each arrow a such that $h(a) = z$, we define

$$\mathcal{Z}_a := \{b : z \to j \mid ba = 0\}$$

and we define a set $\mathcal{Z}_b$ similarly for arrows b such that $t(b) = z$. By Lemma 4.16, we have that $|\mathcal{Z}_a|$ and $|\mathcal{Z}_b|$ are greater than or equal to 2, unless $(r, s) = (1, 1)$.

Lemma 4.21. *Let Λ be as above. If there are two distinct arrows a and a' such that $\mathcal{Z}_a \subset \mathcal{Z}_{a'}$, then $\operatorname{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) \neq 0$.*

Proof. Suppose that the two arrows in the statement are such that $h(a) = h(a') = z$, the case where $t(a) = t(a') = z$ being similar. Consider then the element

$$(e_{t(a')} \otimes_k e_z a)_{a'} \in \Lambda e_{t(a')} \otimes_k e_z \Lambda.$$

This is a cocycle since $\mathcal{Z}_a \subset \mathcal{Z}_{a'}$. It is however not a coboundary, by the same principles as in section 3. □

We obtain the following corollary as a particular case.

Corollary 4.22. *Let Λ be as above and assume that $\operatorname{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) = 0$. Suppose that $s \geq 2$ and let a be an arrow such that $h(a) = z$. Then $|\mathcal{Z}_a| \leq s - 1$. Similarly, if $r \geq 2$ and b is an arrow such that $t(b) = z$, then $|\mathcal{Z}_b| \leq r - 1$.*

We now show that the upper bound on $|\mathcal{Z}_a|$ is even smaller if Λ is 2-RF.

Lemma 4.23. *Let Λ be as above and suppose that Λ is 2-RF. Suppose that $s \geq 2$ and let a be an arrow such that $h(a) = z$. Then $|\mathcal{Z}_a| \leq s-2$. Similarly, if $r \geq 2$ and b is an arrow such that $t(b) = z$, then $|\mathcal{Z}_b| \leq r-2$.*

Proof. We can use a Loewy length ($\ell\ell$) argument as follows. Suppose that there is an arrow $a_i : i \to z$ such that $|\mathcal{Z}_{a_i}| = s-1$. Consider the preprojective algebra Π over Λ. We refer to its description below Theorem 4.4. Let $P_i := \Pi e_i$. We show that $\ell\ell(P_i) = 3$, whereas $\ell\ell(P_z) \geq 4$. Since Π is selfinjective, this contradicts [MV99, Theorem 3.3].

Let $b_j : z \to j$ be the only arrow such that $b_j a_i \neq 0$. Let $\rho = b_j a_x$ be a relation in Λ and c_ρ be the corresponding arrow in Π. Then $c_\rho b_j a_i = -\sum c_{\rho'} b_{j'} a_i = 0$, where the sum is taken over relations of the form $\rho' := b_{j'} a_x$ in Λ which are not equal to ρ. The sum is not empty since $s \geq 2$. This shows that $\ell\ell(P_i) = 3$. Now, since a path of the form $a_\nu c_\rho b_\mu$ is never 0 in Π for any vertices ν, μ and relations ρ, we have $\ell\ell(P_z) \geq 4$. Here, we have used the fact that $|\mathcal{Z}_{b_\mu}|, |\mathcal{Z}_{a_\nu}| \geq 2$. The argument is dual for an arrow $b : z \to i$ such that $|\mathcal{Z}_b| = r-1$. ☐

In particular, this implies that, if Λ is 2-RF, then either $(r,s) = (1,1)$, or $r, s \geq 4$.

We now have plenty of tools to give a full classification of the monomial 2-hereditary algebras whose preprojective algebras is a planar quiver with potential. We prove the main theorem of this section. Note that, for the previous results of this section, we have not assumed that the preprojective algebra is a planar QP. We need the hypothesis now.

Proof of Theorem 4.1. By reasons given above, one can easily check that the two bound quiver algebras described in (1.1) are 2-RF. Assume that Λ is a 2-hereditary quadratic monomial algebra whose preprojective algebra is a planar quiver with potential. We prove that they are the only ones coming from a planar quiver with potential.

By Corollary 4.20, Λ is an (r,s)-star quiver. By [Pet19, Proposition 3.15], the planarity assumption allows us to conclude that every arrow in $\Pi(\Lambda)$ is contained in at most two summands of the potential W. Combining this with Lemma 4.16, we see that every arrow in Λ is part of exactly 2 relations. Therefore, the quiver of $\Pi(\Lambda)$ is given by the intersection of oriented triangles which all share a common vertex z, thus forming a regular polygon shape. In particular, $r = s$. In addition, if Λ is 2-RF, then we have that $r, s \geq 4$ by Lemma 4.23, unless $(r,s) = (1,1)$. If Λ is 2-RI, then we also obtain the same conclusion, since in the case $r = 2$ or $r = 3$, the preprojective algebra is clearly finite-dimensional. If $r = 1$ or $r = 4$, then we recover the bound quiver algebras described in (1.1).

Assume that $r \geq 5$. We show that $\mathrm{Ext}^1_{\Lambda^e}(\Lambda, \Lambda^e) \cong \mathrm{Ext}^1_\Lambda(D\Lambda, \Lambda) \neq 0$. Let I_m be the injective module associated to a sink m and $b : z \to m$. Also recall that $\mathcal{Z}_b^{\complement} := Q_1 \setminus \mathcal{Z}_b$. Then $|\mathcal{Z}_b^{\complement}| = r-2$. Let $a_1, \dots, a_{r-2}$ be the arrows in $\mathcal{Z}_b^{\complement}$ and define $t_i := t(a_i)$ for $i = 1, \dots, r-2$. Without loss of generality, we can assume that we ordered the arrows so that $|\mathcal{Z}_{a_i} \cap \mathcal{Z}_{a_{i+1}}| = 1$ for $i = 1, \dots, r-3$. This is due to the planarity assumption on $\Pi(\Lambda)$. We call b_i the arrow in this intersection for $i = 1, \dots, r-3$ and define $h_i := h(b_i)$. Then the projective resolution of I_m is given by

$$0 \to \bigoplus_{i=1,\dots,r-3} P_{h_i} \to P_z^{r-3} \to \bigoplus_{i=1,\dots,r-2} P_{t_i} \to 0.$$

Applying $\mathrm{Hom}_\Lambda(-, \Lambda e_{t_2})$, we obtain a complex

$$0 \to \bigoplus_{i=1,\dots,r-2} e_{t_i} \Lambda e_{t_2} \to (e_z \Lambda e_{t_2})^{r-3} \to \bigoplus_{i=1,\dots,r-3} e_{h_i} \Lambda e_{t_2} \to \mathrm{Ext}^2_\Lambda(I_m, \Lambda e_{t_2}) \to 0.$$

This complex is not exact at $(e_1 \Lambda e_{t_2})^{r-3}$ since $\dim_k(\bigoplus_{i=1,\dots,r-2} e_{t_i} \Lambda e_{t_2}) = 1$, $\dim_k((e_z \Lambda e_{t_2})^{r-3}) = r-3$ and $\dim_k(\bigoplus_{i=1,\dots,r-3} e_{h_i} \Lambda e_{t_2}) = r-5$. The last equality can be explained by the fact that $ba_2 = 0$ for $b \in Q_1$ if and only if $b = b_1$ or b_2.

Thus $\mathrm{Ext}^1_\Lambda(D\Lambda, \Lambda) \neq 0$. Note that we could have chosen to take $\mathrm{Hom}_\Lambda(-, \Lambda e_{t_\mu})$ for any $\mu = 2, \dots, r-3$ and still obtain the same conclusion. □

Example 4.24. We give an example of a quadratic monomial 2-RF algebra whose 3-preprojective algebra is a non-planar selfinjective quiver with potential. If $(r, s) = (9, 6)$ with arrows $a_i \colon i \to z$ for $i = 1, \dots, 9$ and $b_j \colon z \to j$, for $j = 1, \dots, 6$ one gets an example with relations

$$b_1a_1, b_4a_1, b_1a_2, b_5a_2, b_1a_3, b_6a_3, b_2a_4, b_4a_4, b_2a_5, b_5a_5, b_2a_6, b_6a_6, b_3a_7, b_4a_7, b_3a_8, b_5a_8, b_3a_9, b_6a_9.$$

This can be seen to be a cut of $\Pi(\mathbb{D}_4 \otimes \mathbb{D}_4)$, where both copies of $\mathbb{D}_4$ are oriented with arrows going out of the central vertex. Since $\mathbb{D}_4$ with this orientation is ℓ-homogeneous, [HI11a, Proposition 1.4] implies that the tensor product is 2-RF, and hence, this example is also 2-RF. As the quiver of $\Pi(\mathbb{D}_4 \otimes \mathbb{D}_4)$ is a non-planar graph, the example is non-planar, too.

We note that by observing that to get a quadratic monomial cut the QP must have "enough" borders, it is not too hard to see that the only possible tensor products of Dynkin diagrams that have 3-preprojective algebras with such cuts involve $\mathbb{A}_3$ and $\mathbb{D}_4$ with bipartite orientation. Moreover, $\mathbb{A}_3 \otimes \mathbb{D}_4$ can be checked to not be 2-RF, and $\mathbb{A}_3 \otimes \mathbb{A}_3$ yields the planar example.

Remark 4.25. One should note that many natural constructions on algebras that preserve the property of being n-hereditary do not necessarily preserve being monomial. For instance, this includes tensor products and certain skew-group ring constructions.

Moreover, while there exist other non-planar examples, the ones we know of are all fairly large and are somewhat more complicated than the one mentioned above.

4.2. **The case $n \geq 3$.** We classify all n-representation-finite quadratic monomial algebras of global dimension higher than 2. Note that we do not assume that the preprojective algebra is a planar QP.

Theorem 4.26. *With the exception of $k\mathbb{A}_{n+1}/\mathcal{J}^2$, there are no quadratic monomial n-RF algebras for $n \geq 3$.*

Proof. To begin with, we observe that, by Proposition 3.1, every arrow in Λ lies on some maximal path

$$0 \to 1 \to 2 \to \cdots \to i \to i+1 \to \cdots \to n-2 \to n-1 \to n$$

in which every two consecutive arrows are a relation. Also note that 0 must be a source and n a sink.

We begin by showing that there cannot exist an arrow in Λ different from the one in the diagram above leaving a vertex i with $i < n-1$. Indeed, let $a \colon i \to i+1$ be the arrow in the diagram and assume there was some other arrow $a' \colon i \to j$. Since $\Pi(\Lambda)$ is selfinjective, the projective at i over $\Pi(\Lambda)$ cannot have a non-simple socle. Hence, there must be a commutation relation in $\Pi(\Lambda)$ starting at i of the form $ra + \Sigma_k \alpha_k r_k b_k$ with arrows r and r_k in $\Pi(\Lambda)$ (but not in Λ), arrows b_k, with α_k scalars, and $\alpha_k \neq 0$ for some arrow $b_k \neq a$.

Indeed, to see that the latter claim must hold, let $\Pi(\Lambda) = kQ/I$ with $I = \langle \rho_l \rangle$ where $\{\rho_l\}$ is a set of relations which we can assume to be the one obtained via Proposition 2.7. Then, if we have paths $pa, qa' \in \operatorname{soc} \Pi(\Lambda)e_i$ non-zero in $\Pi(\Lambda)$, we have

$$pa - qa' \in \langle \rho_l \rangle,$$

at least provided we adjust, say, q by a scalar. In other words, we get

$$pa - qa' = \Sigma_l u_l \rho_l v_l$$

where u_l and v_l can be assumed to be paths up to scalars. Using that each v_l has either first arrow equal to a, a' or neither, we can rewrite this as

$$\begin{aligned} pa - qa' &= \Sigma_l u_l \rho_l v_l \\ &= \Sigma_m u_m \rho_m v_m + \Sigma_m u'_m \rho'_m v'_m a + \Sigma_m u''_m \rho''_m v''_m a' \end{aligned}$$

with v'_m, v''_m paths and v_m some path beginning with neither a nor a'. We see that if the paths pa, qa' are non-zero in $\Pi(\Lambda)$ and $a \neq a'$, then

$$\Sigma_m u_m \rho_m v_m = pa - qa' - \Sigma_m u'_m \rho'_m v'_m a - \Sigma_m u''_m \rho''_m v''_m a' \neq 0$$

as otherwise $pa = \Sigma_m u'_l \rho'_m v'_m a \in \langle \rho_j \rangle$ and pa would be zero in $\Pi(\Lambda)$. Moreover, we observe that some v_l equals e_i up to scalars and some ρ_l occurring in a term of $\Sigma_m u_m \rho_m v_m$ must be of the form $\alpha ra + \Sigma_k \alpha_k r_k b_k$ as stated above. In particular, if there is no term $ra + \Sigma_k \alpha_k r_k b_k$ with $\alpha_k \neq 0$ for some arrow $b_k \neq a$ we would again have pa zero in $\Pi(\Lambda)$. This establishes the claim.

Note that Λ is Koszul, so by Proposition 2.7, we know that such a commutation relation and new arrows beginning in vertices $i+1$ and j_k in the preprojective correspond to elements in K^n ending with arrows $a\colon i \to i+1$ and $b_k\colon i \to j_k$, and differing only in the final arrow. However, there can be no such element ending in $i+1$ as $i+1 < n$ is not a sink.

Since Λ is n-RF if and only if Λ^{op} is n-RF, we have also shown that there are no arrows ending in i with $1 < i$. Hence, without loss of generality, we can assume that if Λ has quiver different from linearly oriented $\mathbb{A}_n$, then there must be at least two distinct arrows starting in $n-1$.

Yet, if this was the case, the $\Pi(\Lambda)$-projective at $n-1$ would be of Loewy length ≥ 3 whereas a $\Pi(\Lambda)$-projective at $i < n-2$ would be of Loewy length ≤ 2, as there cannot be any new arrows in $\Pi(\Lambda)$ not in Λ going out of i or $i+1$ as they are not sinks in Λ. Of course, by what we have shown above, there are also no arrows in Λ going out of i or $i+1$ other than those in the diagram. This yields a contradiction by the fact that $\Pi(\Lambda)$ has homogeneous relations and [MV99, Theorem 3.3]. By using Vaso's classification ([Vas19]) of n-RF algebras that are quotients of Nakayama algebras, we are done. □

References

[AIR15] Claire Amiot, Osamu Iyama, and Idun Reiten. Stable categories of Cohen-Macaulay modules and cluster categories. *Amer. J. Math.*, 137(3):813–857, 2015.

[ARS97] Maurice Auslander, Idun Reiten, and Sverre O. Smalø. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. Corrected reprint of the 1995 original.

[Aus71] Maurice Auslander. Representation dimension of Artin algebras. Lecture notes, Queen Mary College, London, 1971.

[Bar97] Michael J. Bardzell. The alternating syzygy behavior of monomial algebras. *J. Algebra*, 188(1):69–89, 1997.

[BGL87] Dagmar Baer, Werner Geigle, and Helmut Lenzing. The preprojective algebra of a tame hereditary Artin algebra. *Comm. Algebra*, 15(1-2):425–457, 1987.

[BGS96] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel. Koszul duality patterns in representation theory. *J. Amer. Math. Soc.*, 9(2):473–527, 1996.

[BHon] Ragnar-Olaf Buchweitz and Lutz Hille, In preparation.

[Boc08] Raf Bocklandt. Graded Calabi Yau algebras of dimension 3. *J. Pure Appl. Algebra*, 212(1):14–32, 2008.

[DHZL08] A. Dugas, B. Huisgen-Zimmermann, and J. Learned. Truncated path algebras are homologically transparent. In *Models, modules and abelian groups*, pages 445–461. Walter de Gruyter, Berlin, 2008.

[DJW19] Tobias Dyckerhoff, Gustavo Jasso, and Tashi Walde. Simplicial structures in higher Auslander-Reiten theory. *Adv. Math.*, 355:106762, 73, 2019.

[GI19] Joseph Grant and Osamu Iyama. Higher preprojective algebras, Koszul algebras, and superpotentials, 2019. arXiv:1902.07878.

[Hap89] Dieter Happel. Hochschild cohomology of finite-dimensional algebras. In *Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année (Paris, 1987/1988)*, volume 1404 of *Lecture Notes in Math.*, pages 108–126. Springer, Berlin, 1989.

[HI11a] Martin Herschend and Osamu Iyama. n-representation-finite algebras and twisted fractionally Calabi-Yau algebras. *Bull. Lond. Math. Soc.*, 43(3):449–466, 2011.

[HI11b] Martin Herschend and Osamu Iyama. Selfinjective quivers with potential and 2-representation-finite algebras. *Compos. Math.*, 147(6):1885–1920, 2011.

[HIMO14] Martin Herschend, Osamu Iyama, Hiroyuki Minamoto, and Steffen Oppermann. Representation theory of Geigle-Lenzing complete intersections, 2014. arXiv:1409.0668.

[HIO14] Martin Herschend, Osamu Iyama, and Steffen Oppermann. n-representation infinite algebras. *Adv. Math.*, 252:292–342, 2014.

[IJ17] Osamu Iyama and Gustavo Jasso. Higher Auslander correspondence for dualizing R-varieties. *Algebr. Represent. Theory*, 20(2):335–354, 2017.

[IO11] Osamu Iyama and Steffen Oppermann. n-representation-finite algebras and n-APR tilting. *Trans. Amer. Math. Soc.*, 363(12):6575–6614, 2011.

[IO13] Osamu Iyama and Steffen Oppermann. Stable categories of higher preprojective algebras. *Adv. Math.*, 244:23–68, 2013.

[IW14] Osamu Iyama and Michael Wemyss. Maximal modifications and Auslander-Reiten duality for non-isolated singularities. *Invent. Math.*, 197(3):521–586, 2014.

[Iya07a] Osamu Iyama. Auslander correspondence. *Adv. Math.*, 210(1):51–82, 2007.

[Iya07b] Osamu Iyama. Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories. *Adv. Math.*, 210(1):22–50, 2007.

[Iya11] Osamu Iyama. Cluster tilting for higher Auslander algebras. *Adv. Math.*, 226(1):1–61, 2011.

[JK19] Gustavo Jasso and Julian Külshammer. Higher Nakayama algebras I: Construction. *Adv. Math.*, 351:1139–1200, 2019. With an appendix by Külshammer and Chrysostomos Psaroudakis and an appendix by Sondre Kvamme.

[Kel11] Bernhard Keller. Deformed Calabi-Yau completions. *J. Reine Angew. Math.*, 654:125–180, 2011. With an appendix by Michel Van den Bergh.

[MM11] Hiroyuki Minamoto and Izuru Mori. The structure of AS-Gorenstein algebras. *Adv. Math.*, 226(5):4061–4095, 2011.

[MV99] Roberto Martínez-Villa. Graded, selfinjective, and Koszul algebras. *J. Algebra*, 215(1):34–72, 1999.

[Pas20] Andrea Pasquali. Self-injective Jacobian algebras from Postnikov diagrams. *Algebr. Represent. Theory*, 23(3):1197–1235, 2020.

[Pet19] Samuel Pettersson. Searching for self-injective planar quivers with potential. Master's thesis, Department of Mathematics, Uppsala University, 2019.

[Thi20] Louis-Philippe Thibault. Preprojective algebra structure on skew-group algebras. *Adv. Math.*, 365:107033, 2020.

[Vas19] Laertis Vaso. n-cluster tilting subcategories of representation-directed algebras. *J. Pure Appl. Algebra*, 223(5):2101–2122, 2019.

Institutt for matematiske fag, NTNU, 7491 Trondheim, Norway
Email address: `mads.sandoy@ntnu.no`

Institutt for matematiske fag, NTNU, 7491 Trondheim, Norway
Email address: `lp.thibault@mail.utoronto.ca`

Paper 4

SKEW GROUP ALGEBRAS, THE (FG) PROPERTY AND SELF-INJECTIVE RADICAL CUBE ZERO ALGEBRAS

Mads Hustad Sandøy

ISBN 978-82-326-6472-6 (printed ver.)
ISBN 978-82-326-6314-9 (electronic ver.)
ISSN 1503-8181 (printed ver.)
ISSN 2703-8084 (online ver.)

NTNU

Norwegian University of
Science and Technology