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Polyanalytic Toeplitz Operators: Isomorphisms, Symbolic Calculus and Approximation of Weyl Operators

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Abstract
We discuss an extension of Toeplitz quantization based on polyanalytic functions. We derive isomorphism theorem for polyanalytic Toeplitz operators between weighted Sobolev-Fock spaces of polyanalytic functions, which are images of modulation spaces under polyanalytic Bargmann transforms. This generalizes well-known results from the analytic setting. Finally, we derive an asymptotic symbol calculus and present an asymptotic expansion of complex Weyl operators in terms of polyanalytic Toeplitz operators.

Keywords Bargmann transform · Polyanalytic functions · Toeplitz quantization · Sobolev-Fock space · Symbolic calculus · Semiclassical approximation

Mathematics Subject Classification 32A36 · 46E22 · 81Sxx · 81Q20

1 Introduction

Polyanalytic functions and the associated polyanalytic Bargmann transforms have received a lot of attention in Gabor analysis. The main contribution of this paper is the investigation of quantization schemes associated to polyanalytic functions, in particular extensions of Toeplitz quantization using this class of non-analytic functions. We aim to address researchers in microlocal analysis and time-frequency analysis.

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Recall, that Bargmann transforms and Fock spaces provide a widely used language that connects the theory of entire functions with a variety of topics in theoretical and applied mathematics, including signal analysis, quantum mechanics as well as complex geometry and analytic microlocal analysis.

This area of mathematics goes back to the seminal work [7] of Bargmann that has been motivated by applications in quantum mechanics. In microlocal analysis, generalized Bargmann transforms are mostly known as Fourier-Bros-Iagolitzer transforms and were first applied by Bros, Iagolnitzer and Stapp in order to analyze wave front sets, see e.g. [29], or [49] for a more recent and general approach. Janssen established the link between the Bargmann transform and Gabor frames in [30] which allowed him to apply methods from complex analysis to problems in signal analysis. This connection between Gabor frames and complex analysis has turned out to be very fruitful, e.g. for the characterization of the Gabor frame set of a Gaussian in [42,48], or the construction of unconditional bases for Bargmann-Fock spaces in [18].

Toeplitz operators provide a natural framework to describe linear transformations in Fock type spaces that can be interpreted as signal manipulations, quantum observables or pseudodifferential operators, see, e.g., [8,28] for early investigations. In fact, Toeplitz operators are nothing else but the image of anti-Wick or localization operators under the Bargmann transform. Putting it differently, localization operators are in fact Toeplitz operators on the phase space, see [4,9,14]. We would, however, like to mention that in some parts of the literature the terms Toeplitz and anti-Wick quantization are used in interchanged ways. We choose our terminology in alignment with [9,53] and others.

The aim of this paper is to lift the well-established theory of Toeplitz operators to the polyanalytic setting, following initial works of Abreu, Gröchenig and Faustino [1,5,16] as well as [15,23,44]. That is, we introduce multiplication operators on Bargmann-Fock type spaces of polyanalytic functions and, thus, provide a whole new family of quantization schemes. Polyanalytic Toeplitz operators appear as the natural complexification of localization operators with Hermite function windows. Moreover, polyanalytic Bargmann-Fock spaces are precisely the images of the classical modulation spaces under polyanalytic Bargmann transforms.

Polyanalytic functions were first studied by Kolossov more than a century ago. Howoever, it was not until the seminal work of Vasilevski [52] that this generalization of analytic functions has received more attention. The increasing importance in mathematics and signal analysis is due to the link between Gabor superframes generated by Hermite functions which are intrinsically related to polyanalytic spaces [1,22]. Polyanalytic functions appear also in the quantization of a class of magnetic Hamiltonians as its eigenfunctions, known as Landau levels [2], and in the theory of the integer quantum Hall-effect, see also, e.g., [6] and [27] for background, context, and relevance in physics.

In [5] the theory of Bargmann-Fock spaces has been extended to the setting of polyanalytic functions, see also [3] for a survey on these recent developments. One of our main results is a lifting theorem for modulation spaces of Gröchenig-Toft [25,26] to polyanalytic Bargmann-Fock spaces.

Motivated by applications in analytic microlocal analysis and semiclassical quantum theory, in this paper we formulate all results in a semiclassical scaling by assuming that $1 \gg \hbar > 0$ is a small parameter.

This paper is structured as follows: after reviewing some basics about Bargmann transforms and quantization in Sect.2, in Sect. 3 we introduce the idea of true polyanalytic Bargmann transforms as well as polyanalytic Toeplitz quantization $\mathcal{T}_k(m)$ of a symbol $m : \mathbb{C}^d \to \mathbb{C}$, where $k \in \mathbb{N}^d$ indicates the degree of polyanalyticity. Section 4 contains our first main theorem, namely, isomorphism results of the form

$$\mathcal{T}_k(m) : \mathcal{F}_m^{k,p,q}(\mathbb{C}^d) \to \mathcal{F}^{k,p,q}(\mathbb{C}^d)$$

for polyanalytic Toeplitz operators as maps between true polyanalytic Sobolev-Fock spaces $\mathcal{F}_m^{k,p,q}(\mathbb{C}^d)$. These spaces appear as images of the well-known modulation spaces under the true polyanalytic Bargmann transform. In Sect. 5 we present an $\hbar$-dependent asymptotic symbol calculus for localization operators $\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(a)$, where the window φ_k is a Hermite function, as well as for their complex counterparts, namely, polyanalytic Toeplitz operators. For example, we show that the commutator of two Hermite localization operators $\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(a)$ and $\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(b)$ has an asymptotic expansion of the form

$$\tfrac{\mathrm{i}}{\hbar}\left[\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(a), \mathrm{op}_{\mathrm{aw}}^{\varphi_k}(b)\right] = \mathrm{op}_{\mathrm{aw}}^{\varphi_k}\left(\{a, b\}\right) + O(\hbar)$$

with $\{\cdot,\cdot\}$ the usual Poisson bracket on $\mathbb{R}^{2d}$, and thus corresponds to a $O(\hbar)$ deformation of the classical phase space. Finally, in Sect. 6 we apply the new concepts to prove an asymptotic expansion of complex Weyl quantized operators in terms of polyanalytic Toeplitz operators.

In summary, we obtain a whole range of new and related quantization schemes whose combination allows for a refined analysis and more precise approximations. It is the hope of the authors that polyanalytic Toeplitz operators will prove useful in various applications such as manipulation of multiplexed signals, construction and analysis of Gabor superframes and semiclassical quantum dynamics.

2 Background

We start by reviewing some concepts and results that form the basis for the subsequent introduction and investigation of polyanalytic Toeplitz operators. We first recall Bargmann transforms as well as the well-known Toeplitz, Weyl and anti-Wick quantization schemes. Moreover, for the reader's convenience and later reference we recall the spectrogram expansion of Wigner functions from [34].

2.1 Bargmann Transform

The Bargmann transform $\mathcal{B}$—see, e.g., the standard reference [21, §I.6]—maps the usual Hilbert space $L^2(\mathbb{R}^d)$ of quantum mechanics and signal analysis into the Fock space

$$\mathcal{F}(\mathbb{C}^d) := \left\{ F : \mathbb{C}^d \to \mathbb{C} : F \text{ is entire and } \|F\|_{L^2_\Phi} < \infty \right\}$$

which is a closed subspace of the weighted Hilbert space

$$L^2_\Phi(\mathbb{C}^d) := L^2(\mathbb{C}^d, \mathrm{e}^{-|z|^2/2\hbar}\mathrm{d}z).$$

Hence, the Fock space $\mathcal{F}(\mathbb{C}^d)$ consists of entire functions of d variables with controlled growth behaviour at infinity. Analoguously to [1], we define the d-dimensional $\hbar$-rescaled Bargmann transform as

$$\mathcal{B} : L^2(\mathbb{R}^d) \to \mathcal{F}(\mathbb{C}^d), \qquad \mathcal{B}\psi(z) = (2\pi\hbar)^{-d/2} (\pi\hbar)^{-d/4} \int_{\mathbb{R}^d} \psi(x) \mathrm{e}^{(xz - z^2/4 - x^2/2)/\hbar} \mathrm{d}x$$

with $\hbar > 0$ a small parameter. In the language of microlocal analysis, the operator $\mathcal{B}$ corresponds to a particular choice of Fourier-Bros-Iagolnitzer transform, see also Appendix 1. The Bargmann transform $\mathcal{B} : L^2(\mathbb{R}^d) \to \mathcal{F}(\mathbb{C}^d)$ is unitary and the associated orthogonal Bergman projector

$$\mathcal{P} := \mathcal{B}\mathcal{B}^* \tag{1}$$

maps $L^2_\Phi(\mathbb{C}^d)$ into its closed subspace $\mathcal{F}(\mathbb{C}^d)$. One computes its adjoint operator $\mathcal{B}^*$ explicitely as

$$\mathcal{B}^* F(x) = (2\pi\hbar)^{-d/2} (\pi\hbar)^{-d/4} \int_{\mathbb{C}^d} F(w) \mathrm{e}^{-(\overline{w} - x)^2/2\hbar + \overline{w}^2/4\hbar} \mathrm{e}^{-|w|^2/2\hbar} \mathrm{d}w, \quad x \in \mathbb{R}^d,$$

for any function $F \in L^2_\Phi(\mathbb{C}^d)$.

Let us consider the image of an appropriately normalized Hermite function φ_k under the Bargmann transform. Hermite functions appear as the eigenfunctions $\{\varphi_k\}_{k \in \mathbb{N}^d} \subset L^2(\mathbb{R}^d)$ of the harmonic oscillator

$$-\tfrac{\hbar^2}{2}\Delta_q + \tfrac{1}{2}|q|^2, \quad q \in \mathbb{R}^d,$$

and one can show that

$$\mathcal{B}\varphi_k(q + ip) = \frac{1}{(\pi\hbar)^{d/2}\sqrt{2^{|k|+d} k!}} \left(\frac{z}{\sqrt{\hbar}} \right)^k$$

is an analytic monomial, e.g. by invoking the more general formula in [41, Proposition 5]. In particular, $\mathcal{B}\varphi_k$ is normalized and

$$\{\mathcal{B}\varphi_k\}_{k \in \mathbb{N}^d} \subset \mathcal{F}(\mathbb{C}^d) \tag{2}$$

is an orthonormal basis for $\mathcal{F}(\mathbb{C}^d)$ consisting of monomials. This property is characteristic to Hermite functions, see e.g. [32].

The Fock space $\mathcal{F}(\mathbb{C}^d)$ is a reproducing kernel Hilbert space, and the reproducing kernel can be computed explicitly via the Hermite monomial basis (2) as

$$\rho(z,w) = \sum_{k\in\mathbb{N}^d} \overline{\mathcal{B}\varphi_k(z)}\mathcal{B}\varphi_k(w) = (2\pi\hbar)^{-d} \mathrm{e}^{\bar{z}w/2\hbar}. \tag{3}$$

That is, for all $z \in \mathbb{C}^d$ and $F \in \mathcal{F}(\mathbb{C}^d)$ one has the pointwise evaluation property

$$F(z) = \langle F(\circ), \rho(z,\circ)\rangle_{L^2_\Phi(\mathbb{C}^d)} \tag{4}$$

and, as a consequence, one obtains the derivative formula

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k} F(z) = (2\pi\hbar)^{-d}(2\hbar)^{-|k|}\langle F(\circ), \circ^k \rho(z,\circ)\rangle_{L^2_\Phi(\mathbb{C}^d)} \tag{5}$$

for all $k \in \mathbb{N}^d$.

The Bargmann transform $\mathcal{B}$ can be seen as the complex equivalent of a specific short-time Fourier transform, which for a general window function $u \in \mathcal{S}(\mathbb{R}^d)$ is defined as

$$V_u\psi(q,p) = (2\pi\hbar)^{-d/2}\left\langle \psi, M_p T_q u\right\rangle_{L^2(\mathbb{R}^d)} \tag{6}$$

with $(q,p) \in \mathbb{R}^{2d}$ and the standard translation and modulation operators

$$T_q\psi(x) = \psi(x-q), \quad M_p\psi(x) = \mathrm{e}^{\mathrm{i}px/\hbar}\psi(x), \quad \psi \in L^2(\mathbb{R}^d).$$

Namely, for the case of a Gaussian window $g_0 := \varphi_0$ centered in the origin one observes

$$V_{g_0}\psi(q,-p) = \mathrm{e}^{(\mathrm{i}qp-|z|^2/2)/2\hbar}\mathcal{B}\psi(q+\mathrm{i}p). \tag{7}$$

2.2 Toeplitz, Weyl and Anti-Wick Operators

Let us recall the definitions and basic properties of three quantization schemes: Toeplitz, Weyl, and anti-Wick quantization. As briefly discussed in the introduction, the terms Toeplitz, anti-Wick and localization operators are in parts used in interchanged ways within the literature. Classic references include [8,28], while our notation and scaling is, e.g, in accordance with [34,53].

The *Toeplitz operator* $\mathcal{T}(m)$ with symbol $m : \mathbb{C}^d \to \mathbb{C}$ is defined by multiplication with m and subsequent projection down to the Fock space $\mathcal{F}(\mathbb{C}^d)$,

$$\mathcal{T}(m) = \mathcal{P}m\mathcal{P}, \tag{8}$$

or, more explicitly,

$$\mathcal{T}(m)F(w) = (2\pi\hbar)^{-d}\int_{\mathbb{C}^d} m(z)F(z)\mathrm{e}^{\overline{z}w/2\hbar}\mathrm{e}^{-|z|^2/2\hbar}\mathrm{d}z$$

for any $F \in \mathcal{F}(\mathbb{C}^d)$. For $m \in L^\infty(\mathbb{C}^d)$, the quantized operator $\mathcal{T}(m)$ is bounded on the Fock space $\mathcal{F}(\mathbb{C}^d)$. For more general mapping results we refer to Sect. 4.1.

Weyl quantization or canonical quantization appears as the natural quantization scheme connecting classical and quantum mechanics. Here, a function $a : \mathbb{R}^{2d} \to \mathbb{C}$ is associated with the *Weyl quantized operator* $\mathrm{op}(a)$ via

$$(\mathrm{op}(a)\psi)(q) = (2\pi\hbar)^{-d}\int_{\mathbb{R}^{2d}} a(\tfrac{1}{2}(y+q), p)\mathrm{e}^{\mathrm{i}(q-y)p/\hbar}\psi(y)\mathrm{d}y\,\mathrm{d}p \tag{9}$$

where $\mathbb{R}^{2d} \cong T^*\mathbb{R}^d$ is the phase space of classical mechanics. The associated phase space representation of quantum states (or signals) is provided by cross-Wigner functions

$$\mathcal{W}(\psi,\phi)(q,p) = (2\pi\hbar)^{-d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}py/\hbar}\psi(q-\tfrac{y}{2})\overline{\phi}(q+\tfrac{y}{2})\,\mathrm{d}y, \quad (q,p)\in\mathbb{R}^{2d}. \tag{10}$$

That is, for suitable a, ψ and ϕ, one has

$$\langle \mathrm{op}(a)\psi, \phi\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^{2d}} a(z)\mathcal{W}(\psi,\phi)(z)\mathrm{d}z, \tag{11}$$

where we choose the inner product to be left-linear. We note that $\mathcal{W}(\psi,\phi) \in L^2(\mathbb{R}^{2d})$ whenever $\psi, \phi \in L^2(\mathbb{R}^d)$. In the case $\psi = \phi$ we write

$$\mathcal{W}(\psi,\psi) =: \mathcal{W}_\psi$$

for the Wigner function to abbreviate notation.

Despite of their many remarkable properties, Wigner functions $\mathcal{W}_\psi$ exhibit the drawback of attaining negative values whenever ψ is not a Gaussian, see [31,45], and hence typically are not probability densities. However, one can turn $\mathcal{W}_\psi$ into a nonnegative function by convolution with another Wigner function: For all $\psi \in L^2(\mathbb{R}^d)$ and Schwartz class windows $\phi \in \mathcal{S}(\mathbb{R}^d)$ with $\|\psi\|_{L^2(\mathbb{R}^d)} = \|\phi\|_{L^2(\mathbb{R}^d)} = 1$ the convolution

$$S_\psi^\phi := \mathcal{W}_\psi * \mathcal{W}_\phi : \mathbb{R}^{2d} \to \mathbb{R}$$

is a smooth probability density on phase space, as can be deduced from [21, Proposition 1.42]. In time-frequency analysis S_ψ^ϕ is called a *spectrogram* of ψ; see, e.g., the introduction in [20]. Spectrograms constitute a subset of Cohen's class of phase space distributions; see [19, §3.2.1].

A popular window function for spectrograms is provided by the Gaussian wave packet or coherent state

$$g_{(q,p)}(x) = (\pi\varepsilon)^{-d/4} \exp\left(-\tfrac{1}{2\varepsilon}|x-q|^2 + \tfrac{\mathrm{i}}{\varepsilon} p\cdot(x-\tfrac{1}{2}q)\right), \quad (q,p)\in\mathbb{R}^{2d}, \tag{12}$$

centered in (q, p). We denote the Gaussian wave packet centered in the origin $(0, 0)$ by g_0. The corresponding spectrogram

$$S_\psi^{g_0}(z) = \int_{\mathbb{R}^{2d}} \mathcal{W}_\psi(w)(\pi\varepsilon)^{-d}\mathrm{e}^{-|z-w|^2/\varepsilon}\,\mathrm{d}w \tag{13}$$

is known as the *Husimi function* of ψ, first introduced in [28]. Note that

$$\int_{\mathbb{R}^{2d}} a(z)S_\psi^{g_0}(z)\mathrm{d}z = \int_{\mathbb{R}^{2d}} (\mathcal{W}_{g_0} * a)(z)\mathcal{W}_\psi(z)\mathrm{d}z = \left\langle \mathrm{op}_{\mathrm{aw}}(a)\psi, \psi\right\rangle, \tag{14}$$

where $\mathrm{op}_{\mathrm{aw}}(a) = \mathrm{op}(\mathcal{W}_{g_0} * a)$ is the so-called *anti-Wick quantized* operator associated with a; see [21, §2.7]. From [41, Proposition 5] we know that the Husimi functions of L^2-normalized Hermite functions $\{\varphi_k\}_{k\in\mathbb{N}^d}$ are given by the formula

$$S_{\varphi_k}^{g_0}(z) = S_{g_0}^{\varphi_k}(z) = (2\pi\varepsilon)^{-d}\frac{\mathrm{e}^{-|z|^2/2\varepsilon}}{(2\varepsilon)^{|k|}k!}|z|^{2k}.$$

In time-frequency analysis, general anti-Wick type operators $\mathrm{op}_{\mathrm{aw}}^{\varphi}(a)$, (usually) with a Schwartz class window φ, are known as *localization operators*. Here, they are equivalenty defined via multiplication in the image space of the corresponding short-time Fourier transform (6),

$$\mathrm{op}_{\mathrm{aw}}^{\varphi}(a) = V_\varphi^* a V_\varphi, \quad \mathrm{op}_{\mathrm{aw}}^{g_0}(a) = \mathrm{op}_{\mathrm{aw}}(a), \tag{15}$$

where a denotes both the symbol and the multiplication operator. The non-negative phase space density corresponding to this quantization scheme then in turn is given by the spectrogram S_ψ^φ, see [9].

2.3 The Spectrogram Expansion

In the past decades there has been considerable research on the connection between different quantization schemes and their respective calculi, such as the classic comparisons of left, right and Weyl quantization as well as anti-Wick operators, see e.g. [38, §2.3 and §2.4] or [53, §4 and §13] for summaries.

Explicit formulas for the Wigner and Husimi functions of general wave packets have been derived in [41] and subsequently applied in [36] in order to derive second order corrections in the comparison of Wigner and Husimi functions. In [34] these corrections have been generalized to arbitrary order by proving the following spectrogram expansion.

Theorem 1 *(Spectrogram expansion from [34]) Let $\psi \in L^2(\mathbb{R}^d)$, $N \in \mathbb{N}$, and $\hbar > 0$. Then, if one defines the following real-valued phase space function μ_ψ^N in terms of Hermite spectrograms,*

$$\mu_\psi^N(z) = \sum_{j=0}^{N-1}(-1)^j C_{N-1,j} \sum_{\substack{k\in\mathbb{N}^d \\ |k|=j}} S_\psi^{\varphi_k}(z), \quad C_{k,j} = \sum_{m=j}^{k} 2^{-m}\binom{d-1+m}{d-1+j}, \tag{16}$$

for any Schwartz function $a : \mathbb{R}^{2d} \to \mathbb{C}$ there is a constant $C \geq 0$ such that

$$\left| \int_{\mathbb{R}^{2d}} a(z)\mathcal{W}_\psi(z)\mathrm{d}z - \int_{\mathbb{R}^{2d}} a(z)\mu_\psi^N(z)\mathrm{d}z \right| \leq C\hbar^N \|\psi\|_{L^2(\mathbb{R}^d)}^2, \tag{17}$$

where C only depends on bounds on derivatives of a of degree $2N$ and higher. In particular, if a is a polynomial with deg$(a) < 2N$ then (17) vanishes.

Retracing the proof for Theorem 1 in [34] immediately shows that the offdiagonal version of the above approximation holds as well. That is,

$$\begin{aligned} \langle \mathrm{op}(a)\psi, \phi\rangle_{L^2} &= \int a(z)\mathcal{W}(\psi,\phi)(z)\mathrm{d}z \\ &= \int_{\mathbb{R}^{2d}} a(z)\mu^N(\psi,\phi)(z)\mathrm{d}z + O(\hbar^N) \end{aligned} \tag{18}$$

with the off-diagonal phase space representation

$$\mu_{\psi,\phi}^N(z) = \sum_{j=0}^{N-1}(-1)^j C_{N-1,j} \sum_{\substack{k\in\mathbb{N}^d \\ |k|=j}} \mathcal{W}_{\varphi_k} * \mathcal{W}(\psi,\phi)(z) \tag{19}$$

of any two functions $\psi, \phi \in L^2(\mathbb{R}^d)$. We note, however, that $\mu^N(\psi,\phi)$ typically has a non-constant complex phase and, in particular, is not a finite linear combination of probability densities.

In Sect. 6, polyanalytic Toeplitz operators are applied to prove a statement equivalent to Theorem 1 in polyanalytic Bargmann-Fock spaces. This yields a variety of new connections between real and complex Weyl, anti-Wick and Toeplitz type quantization schemes.

3 Polyanalytic Toeplitz Operators

In this section, we first recall the definition of polyanalytic Fock-Bargmann spaces and subsequently introduce and investigate polyanalytic Toeplitz operators which naturally act on these spaces.

3.1 Polyanalytic Bargmann-Fock Spaces

Recall that every polyanalytic function F of order $k \in \mathbb{N}^d$ can be uniquely written as

$$F(z) = \sum_{\ell \leq k} \bar{z}^{\ell} f_{\ell}(z)$$

where f_{ℓ}, $\ell \in \mathbb{N}^d$, are analytic functions and the sum runs over all multiindices with $0 \leq \ell_1 \leq k_1, \ldots, 0 \leq \ell_d \leq k_d$. For all $k \in \mathbb{N}^d$ we denote by

$$\mathfrak{F}^k(\mathbb{C}^d) = \left\{ F : \mathbb{C}^d \to \mathbb{C} : F \text{ polyanalytic of degree } k \in \mathbb{N}^d \text{ and } \|F\|_{L^2_\Phi} < \infty \right\}$$

the polyanalytic Bargmann-Fock space of degree k which, as we will detail later, has an orthogonal decomposition into true polyanalytic Bargmann-Fock spaces as shown by [52]. We consider $\mathfrak{F}^k(\mathbb{C}^d)$ as a subspace of L^2_Φ with the correspondingly inherited norm.

Note, that polyanalytic functions satisfy a generalized Cauchy-Riemann equation of the form

$$\partial_{\bar{z}_1}^{k_1+1} \cdots \partial_{\bar{z}_d}^{k_d+1} F(z) = 0 \Longleftrightarrow F : \mathbb{C}^d \to \mathbb{C} \text{ is polyanalytic of degree } k.$$

For later reference let us define "translations" in Bargmann-Fock spaces by

$$\Theta_z \mathcal{B} f(w) = \mathcal{B} M_p T_q f(w), \quad z = q + ip, \tag{20}$$

such that

$$\Theta_z F(w) = (2\pi\hbar)^{-d/2} \mathrm{e}^{\mathrm{i}pq/2\hbar - |z|^2/4\hbar + zw/2\hbar} F(w - \bar{z}), \quad w \in \mathbb{C}^d.$$

By once again closely following [1], we then define true polyanalytic Bargmann transforms as follows.

Definition 1 (True polyanalytic Bargmann transform) For $k \in \mathbb{N}^d$, the *true polyanalytic Bargmann transform* $\mathcal{B}_k : L^2(\mathbb{R}^d) \to \mathfrak{F}^k(\mathbb{C}^d)$ of degree k is defined as

$$\mathcal{B}_k f(z) := \frac{1}{\sqrt{k!}(2\hbar)^{|k|/2}} \mathrm{e}^{|z|^2/2\hbar} \frac{\mathrm{d}^k}{\mathrm{d}z^k} \left(\mathrm{e}^{-|z|^2/2\hbar} \mathcal{B} f(z) \right)$$

in analogy to the definition of Hermite polynomials via their generating function.

As a next step, let us compare $\mathcal{B}^k$ with the short-time Fourier transform associated with the k-th Hermite function as window just as the zero'th order comparison (7). We include a proof for the convenience of the reader.

Lemma 1 *(see e.g. [1]) For all $k \in \mathbb{N}^d$ it holds*

$$V_{\varphi_k} f(q, -p) = \mathrm{e}^{\mathrm{i}pq/2\hbar - |z|^2/4\hbar} \mathcal{B}_k f(z)$$

with $z = q + \mathrm{i}p$. In particular, $\mathcal{B}_k : L^2(\mathbb{R}^d) \to \mathfrak{F}^k(\mathbb{C}^d)$ is a partial isometry.

Proof By utilizing the partial isometry property of the Bargmann transform and recalling the translation formula (20), for $z = q + \mathrm{i}p$ we compute

$$\begin{aligned}
V_{\varphi_k} f(q, -p) &= \left\langle f, M_{-p} T_q \varphi_k \right\rangle_{L^2(\mathbb{R}^d)} \\
&= \langle \mathcal{B} f, \Theta_{\bar{z}} \mathcal{B} \varphi_k \rangle_{L^2_\Phi(\mathbb{C}^d)} \\
&= \frac{(2\pi\hbar)^{-d}}{(2\hbar)^{|k|/2}\sqrt{k!}} \mathrm{e}^{\mathrm{i}pq/2\hbar - |z|^2/4\hbar} \left\langle \mathcal{B} f(w), \mathrm{e}^{\bar{z}w} \, (w - z)^k \right\rangle_{L^2_\Phi} \\
&= \frac{\mathrm{e}^{\mathrm{i}pq/2\hbar - |z|^2/4\hbar}}{(2\pi\hbar)^d (2\hbar)^{|k|/2}\sqrt{k!}} \sum_{0 \le \ell \le k} \binom{k}{\ell} (-\bar{z})^{k-\ell} \left\langle \mathcal{B} f(w), \mathrm{e}^{\bar{z}w/2\hbar} w^\ell \right\rangle_{L^2_\Phi} = (\star)
\end{aligned}$$

which by means of the differentiation formula (5) leads to the desired result

$$\begin{aligned}
(\star) &= \frac{(2\hbar)^{|k|/2}}{\sqrt{k!}} \mathrm{e}^{\mathrm{i}pq/2\hbar - |z|^2/4\hbar} \sum_{0 \le \ell \le k} \binom{k}{\ell} (-\bar{z})^{k-\ell} \mathcal{B} f^{(\ell)}(z) \\
&= \frac{1}{\sqrt{k!}(2\hbar)^{|k|/2}} \mathrm{e}^{\mathrm{i}pq/2\hbar - |z|^2/4\hbar} \mathrm{e}^{|z|^2/2\hbar} \frac{d^k}{dz^k} \left(\mathrm{e}^{-|z|^2/2\hbar} \mathcal{B} f(z) \right) \\
&= \mathrm{e}^{\mathrm{i}pq/2\hbar - |z|^2/4\hbar} \mathcal{B}_k f(z)
\end{aligned}$$

with standard multiindex notation. Since $\mathcal{B} f^{(\ell)}$ is analytic for all $\ell \in \mathbb{N}^d$, $\mathcal{B}_k f$ is polyanalytic of degree k and the partial isometry property of the polyanalytic Bargmann transforms $\mathcal{B}_k$ follows directly from the corresponding property of the STFT. □

Note that Hermite functions can be used to construct orthonormal bases for polyanalytic function spaces. Namely, the set of transformed Hermite functions

$$\{\mathcal{B}_\ell \varphi_m\}_{\ell \le k, m \in \mathbb{N}^d}, \quad \mathcal{B}_\ell \varphi_m(z) \propto z^{m-\ell} \prod_{j=1}^d L_{\ell_j}^{(m_j - \ell_j)}\left(\tfrac{1}{2\hbar}|z_j|^2\right) \quad \text{for } m \ge \ell, \tag{21}$$

and analogously for $m \le \ell$ is an orthonormal basis of $\mathfrak{F}^k(\mathbb{C}^d)$ for all $k \in \mathbb{N}^d$, where $L_n^{(m)}$ denote the Laguerre polynomials, see e.g. [1]. Formula (21) can be proven by using the Laguerre connection for overlap integrals of two shifted Hermite functions similar as for the computation of Wigner transforms of Hermite functions, see e.g. [41]. The polynomials in (21) are particular examples of so-called *special Hermite functions*, see also [43].

The polyanalytic Bargmann-Fock spaces admit a decomposition in terms of *true polyanalytic Bargmann-Fock spaces*

$$\mathcal{F}^k(\mathbb{C}^d) := \mathrm{Span}\{\mathcal{B}_k \varphi_m\}_{m \in \mathbb{N}^d}, \quad \mathcal{F}^0(\mathbb{C}^d) = \mathcal{F}(\mathbb{C}^d),$$

namely as the orthogonal sum

$$\mathfrak{F}^k(\mathbb{C}^d) = \bigoplus_{\ell \in \mathbb{N}^d, \ell \leq k} \mathcal{F}^\ell(\mathbb{C}^d).$$

In particular, recalling (21) we know that for all $m \in \mathbb{N}^d$ the basis function $\mathcal{B}_\ell \varphi_m$ is a polynomial of degree ℓ in $\overline{z}$ which implies that all nonzero elements of $\mathcal{F}^k(\mathbb{C}^d)$ share this property as well.

The true polyanalytic Bargmann transform $\mathcal{B}^k$ acts as an isometric isomorphism

$$\mathcal{B}_k : L^2(\mathbb{R}^d) \to \mathcal{F}^k(\mathbb{C}^d)$$

and in analogy to (1), the map

$$\mathcal{P}_k := \mathcal{B}_k \mathcal{B}_k^* : L_\Phi(\mathbb{C}^d) \to \mathcal{F}^k(\mathbb{C}^d), \quad \mathcal{P} = \mathcal{P}_0$$

is the *polyanalytic Bergman projector* and its kernel the *polyanalytic Bergman kernel*. The reproducing kernel of $\mathcal{F}^k(\mathbb{C}^d)$ is given by

$$\rho^k(z, w) = (2\pi\hbar)^{-d} \prod_{j=1}^{d} L_{k_j}\left(\tfrac{1}{2\hbar}|z_j - w_j|^2\right) \mathrm{e}^{\overline{z}w/2\hbar}$$

where L_k denotes the kth Laguerre polynomial.

3.2 Polyanalytic Toeplitz Quantization

Recall from (15) that general anti-Wick or localization operators are given by

$$\begin{aligned} \mathrm{op}_{\mathrm{aw}}^{\varphi}(a)\psi &= \mathrm{op}(\mathcal{W}_\varphi * a)\psi \\ &= V_\varphi^* a V_\varphi \psi \end{aligned}$$

where a here denotes both the phase space function a and the operator of multiplication with a. Expectation values of anti-Wick operators are computed on the phase space via the corresponding spectrogram:

$$\left\langle \mathrm{op}_{\mathrm{aw}}^{\varphi}(a)\psi, \psi \right\rangle = \int_{\mathbb{R}^{2d}} a(z) S_\psi^\varphi(z) \mathrm{d}z.$$

In the following, we extend the concept of Toeplitz operators as, e.g., defined in [53, §13] from (8) to the d-dimensional polyanalytic setting, see also [16] for discussions in the one-dimensional case. For defining the quantization, we utilize the polyanalytic Bergman projectors defined in Sect. 3.1.

Definition 2 (True polyanalytic Toeplitz quantization) Let $k \in \mathbb{N}^d$ and $f \in L^\infty(\mathbb{C}^d)$. Then, the bounded operator

$$\mathcal{T}_k(f) := \mathcal{P}_k f \mathcal{P}_k, \quad \mathcal{T}_k(f) : \mathcal{F}^k(\mathbb{C}^d) \to \mathcal{F}^k(\mathbb{C}^d)$$

is called the *true polyanalytic Toeplitz quantization of degree* k.

For the quantization of more general symbols f one needs to introduce corresponding Sobolev type subspaces of $\mathcal{F}^k(\mathbb{C}^d)$ with stronger decay conditions, as we discuss in Sect. 4.1.

Note that the Bergman projector on the right-hand side of the multiplication operator in Definition 2 can be safely ommited when acting on polyanalytic Bargmann-Fock spaces. It is included in order to support the intuition that real-valued symbols $f \in L^\infty(\mathbb{C}^d, \mathbb{R})$ give rise to self-adjoint operators.

For later reference, we also define an off-diagonal type polyanalytic Toeplitz quantization by multiplication in the polyanalytic space $\mathcal{F}^k(\mathbb{C}^d)$ and projection back to the usual Fock space $\mathcal{F}(\mathbb{C}^d)$.

Definition 3 (Projected polyanalytic Toeplitz quantization) Let $k \in \mathbb{N}^d$ and $f \in L^\infty(\mathbb{C}^d)$. Then, the bounded operator

$$\mathcal{T}_{k,0}(f) := \mathcal{B}\mathcal{B}_k^* f \mathcal{B}_k \mathcal{B}^*, \quad \mathcal{T}_{k,0}(f) : \mathcal{F}(\mathbb{C}^d) \to \mathcal{F}(\mathbb{C}^d)$$

is called the *k-projected polyanalytic Toeplitz quantization* of f.

Polyanalytic Toeplitz operators and anti-Wick quantization are closely related in the following way: Let $f \in L^\infty(\mathbb{C}^d)$ and $u, v \in \mathcal{F}^k(\mathbb{C}^d)$, $\mathcal{B}_k^* u =: \phi$ and $\mathcal{B}_k^* v =: \psi$ where $\phi, \psi \in L^2(\mathbb{R}^d)$. Then, one computes

$$\begin{aligned}\langle u, \mathcal{P}_k f \mathcal{P}_k v\rangle_{L^2_\Phi(\mathbb{C}^d)} &= \left\langle \phi, \mathcal{B}_k^* f \mathcal{B}_k \psi\right\rangle_{L^2(\mathbb{R}^d)} \\ &= (-1)^d \left\langle \phi, V_{\varphi_k}^* \check{f} V_{\varphi_k} \psi\right\rangle_{L^2(\mathbb{R}^d)} \end{aligned} \tag{22}$$

where we define

$$\check{f}(q, p) := f(q - ip). \tag{23}$$

For later purposes, let us also define the "inverse action" of this map as

$$\widehat{u}(z) := u(q, -p), \quad z = q + ip \in \mathbb{C}^d, \quad u : \mathbb{R}^{2d} \to \mathbb{C}. \tag{24}$$

Relation (22) supports the intuition that the localization quantization rule (15) with Hermite functions as windows can be seen as the real-valued equivalent of polyanalytic Toeplitz quantization, see also [16].

4 Polyanalytic Sobolev-Fock Spaces and Isomorphism Theorems

In this section, we first provide a short overview on Sobolev-Fock and modulation spaces that serve as a general class of spaces with natural mapping properties for Toeplitz and localization operators, respectively. Afterwards, we present the polyanalytic generalizations of those spaces and, as a main result, prove an isomorphism theorem for polyanalytic Toeplitz operators.

4.1 Modulation Spaces and Sobolev-Fock Spaces

Let us briefly review *modulation spaces* and and their images under the Bargmann transform, the so-called *Sobolev-Fock spaces*. Modulation spaces form a natural framework for the calculus of localization operators in the same way as Sobolev-Fock spaces do for Toeplitz operators.

Following usual conventions, see e.g. [17,26], we call a locally bounded weight function $m : \mathbb{R}^{2d} \to (0, \infty)$ *moderate* if

$$\sup_{z \in \mathbb{R}^{2d}} \left(\frac{m(z+y)}{m(z)}, \frac{m(z-y)}{m(z)} \right) =: v(y) < \infty \quad \text{for all } y \in \mathbb{R}^{2d}.$$

As a result, v is a submultiplicative function and m satisfies

$$m(z+y) \leq m(z)v(y) \quad \text{for all } z, y \in \mathbb{R}^{2d}.$$

We restrict ourselves to weights of polynomial growth and call a weight function *admissable* if it is moderate, continuous and at most of polynomial growth. For any fixed submultiplicative weight function $v : \mathbb{R}^{2d} \to [1, \infty)$ we define the set of v-admissable weights as

$$\mathcal{M}_v = \{m \in L^\infty_{loc}(\mathbb{R}^{2d}) \text{ admissable and } 0 < m(z+y) \leq m(z)v(y) \ \forall\, y, z \in \mathbb{R}^{2d}\}.$$

Then, the modulation spaces with admissible weight m are defined as

$$M^{p,q}_m(\mathbb{R}^d) = \left\{ f \in \mathcal{S}(\mathbb{R}^d)' : \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_{g_0} f(x,\xi)|^p m(x,\xi)^p \mathrm{d}x \right)^{q/p} \mathrm{d}\xi \right)^{1/q} < \infty \right\},$$

$1 \leq p, q \leq \infty$, and contain functions (or distributions) that show controlled growth properties together with their Fourier transforms. We note that modulation spaces do not change if we replace the Gaussian window g_0 by a different Schwartz function, see e.g. [24, §11].

Similarly as the classical Fock space $\mathcal{F}(\mathbb{C}^d)$ is the image of $L^2(\mathbb{R}^d)$ under the Bargmann transform, one can look at Fock-type spaces that are the equivalents of modulation spaces in the complex setting. We use the notation from [26] and write $\mathcal{M}^{\mathbb{C}}_v$ for complex v-admissable weights with $v : \mathbb{C}^d \to [1, \infty)$ moderate. We introduce for any complex moderate weight m the Sobolev-Fock spaces

$$\mathcal{F}_m^{p,q}(\mathbb{C}^d) = \left\{ F : \mathbb{C}^d \to \mathbb{C} \text{ entire and } \|F\|_{L^{p,q}_{\Phi,m}} < \infty \right\}$$

that are complete subspaces of the Banach spaces $L^{p,q}_{\Phi,m}$ with the weighted mixed p, q-norm

$$\|F\|_{L^{p,q}_{\Phi,m}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(z)|^p m(z)^p \mathrm{e}^{-p|z|^2/4\hbar} \mathrm{dRe}(z) \right)^{q/p} \mathrm{dIm}(z) \right)^{1/q}$$

consisting of entire functions. In particular, $\mathcal{F}_1^{2,2}(\mathbb{C}^d) = \mathcal{F}(\mathbb{C}^d)$ gives the usual Fock space. It is well-known, see e.g. [25,26], that the Bargmann transform $\mathcal{B}$ maps the modulation space $M_m^{p,q}(\mathbb{R}^d)$ isometrically to the Sobolev-Fock space $\mathcal{F}_{\breve{m}}^{p,q}(\mathbb{C}^d)$, where we employ the notation from (23). In particular, from [18] we are able to rephrase the following result, see also [25, Theorem 5.4] and [47,51].

Lemma 2 *Let $\mu \in \mathcal{M}_w^{\mathbb{C}}$ and $m \in \mathcal{M}_v^{\mathbb{C}}$. Then, for all $1 \le q, p \le \infty$, the Toeplitz operator $\mathcal{T}(m)$ is a bounded, invertible map from $\mathcal{F}_\mu^{p,q}(\mathbb{C}^d)$ to $\mathcal{F}_{\mu/m}^{p,q}(\mathbb{C}^d)$.*

4.2 Polyanalytic Sobolev-Fock Spaces

Based on the analytic Sobolev-Fock space theory suitable for Toeplitz operators from Sect. 4.1 one can define similar function spaces in the polyanalytic setting. For any $k \in \mathbb{N}^d$ we closely follow the definitions in [5] and define *true polyanalytic Sobolev-Fock spaces* with mixed p, q-norms as

$$\mathcal{F}_m^{k,p,q}(\mathbb{C}^d) = \left\{ F : \mathbb{C}^d \to \mathbb{C} \text{ true polyanalytic of degree } k \text{ and } \|F\|_{L^{p,q}_{\Phi,m}} < \infty \right\}$$

where $\mathcal{F}_m^{0,p,q}(\mathbb{C}^d) = \mathcal{F}_m^{p,q}(\mathbb{C}^d)$. As we summarize in the following Lemma 3, true polyanalytic Fock-Sobolev spaces are precisely the image of the usual modulation spaces under the true polyanalytic Bargmann transform.

Lemma 3 *For all $1 \le p, q \le \infty$, $k \in \mathbb{N}^d$ and $m \in \mathcal{M}_v$, the true polyanalytic Bargmann transform $\mathcal{B}_k$ is an isomorphism*

$$\mathcal{B}_k : M_m^{p,q}(\mathbb{R}^d) \to \mathcal{F}_{\breve{m}}^{k,p,q}(\mathbb{C}^d).$$

Proof For $k = 0$ this result is well-known, see e.g. [24–26]. For $k \neq 0$ the results follow from Lemma 1 by observing that the modulation space $M_m^{p,q}(\mathbb{R}^d)$ can be defined without harm with the Hermite window φ_k instead of g. □

Remark 1 We note that—as we stick to weight functions of polynomial growth—the Schwartz space is contained in all considered modulation spaces. This in particular implies that the span of special Hermite functions

$$\mathrm{Span}\{\mathcal{B}_\ell \varphi_m\}_{|\ell|=n, m\in\mathbb{N}^d} = \mathrm{Span}\Big\{ z^{m-\ell} \prod_{j=1}^d L_{\ell_j}^{(m_j-\ell_j)}(\tfrac{1}{2\hbar}|z_j|^2) \Big\}$$

is a dense subset of the direct sum $\bigoplus_{|k|=n} \mathcal{F}_m^{k,p,q}(\mathbb{C}^d)$ of all true polyanalytic Fock spaces of total degree n, see also [43]. Moreover, the basis functions $\mathcal{B}_\ell\varphi_m$ are orthogonal if m is radial in each component, that is, $m(z_1, \dots, z_d) = \tilde{m}(|z_1|, \dots, |z_d|)$ for some $\tilde{m}$, see also [26].

4.3 Isomorphism Results

In the following, we generalize the isomorphism result from Lemma 2 to the polyanalytic context. For this purpose, we investigate the mapping properties of polyanalytic Toeplitz operators on their respective Sobolev-Fock spaces. This constitutes a main result of this paper.

Theorem 2 *Let $1 \le p, q \le \infty$, $k \in \mathbb{N}^d$, $\mu \in \mathcal{M}_w^{\mathbb{C}}$ and $m \in \mathcal{M}_v^{\mathbb{C}}$ be continuous. Then the true polyanalytic Toeplitz operator $\mathcal{T}_k(m)$ constitutes an isomorphism as a map*

$$\mathcal{T}_k(m) : \mathcal{F}_\mu^{k,p,q}(\mathbb{C}^d) \to \mathcal{F}_{\mu/m}^{k,p,q}(\mathbb{C}^d)$$

and the k-projected polyanalytic Toeplitz operator $\mathcal{T}_{k,0}(m)$ is an isomorphism

$$\mathcal{T}_{k,0}(m) : \mathcal{F}_\mu^{p,q}(\mathbb{C}^d) \to \mathcal{F}_{\mu/m}^{p,q}(\mathbb{C}^d).$$

Proof By Lemma 3 the polyanalytic Bargmann transform $\mathcal{B}_k$ is an isomorphism as a map

$$\mathcal{B}_k : M_{\breve{m}}^{p,q}(\mathbb{R}^d) \to \mathcal{F}_m^{k,q,p}(\mathbb{C}^d).$$

and the isomorphism property for the localization operator $\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(\breve{f})$ with Hermite function window as a map

$$\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(\breve{m}) : M_{\breve{\mu}}^{p,q}(\mathbb{R}^d) \to M_{\breve{\mu}/\breve{m}}^{p,q}(\mathbb{R}^d)$$

is well-known, see e.g. [26, Theorem 4.3]. Moreover, from (22) we infer that

$$\mathcal{B}_k^* \mathcal{T}_k(m) \mathcal{B}_k = (-1)^d \mathrm{op}_{\mathrm{aw}}^{\varphi_k}(\breve{m}).$$

Hence, $\mathcal{T}_k(m)$ can be written as composition of three isomorphisms

$$\begin{array}{ccc}
\mathcal{F}_\mu^{k,p,q} & \xrightarrow{\mathcal{T}_k(m)} & \mathcal{F}_{\mu/m}^{k,p,q} \\
\uparrow{\scriptstyle \mathcal{B}_k} & & {\scriptstyle \mathcal{B}_k}\uparrow \\
M_{\breve{\mu}}^{p,q} & \xrightarrow{(-1)^d \mathrm{op}_{\mathrm{aw}}^{\varphi_k}(\breve{m})} & M_{\breve{\mu}/\breve{m}}^{p,q}
\end{array}$$

which completes the proof for the first part of the assertion. For the second part one similarly obtains the diagram

$$\begin{array}{ccc} \mathcal{F}^{p,q}_{\mu} & \xrightarrow{\mathcal{T}_{k,0}(m)} & \mathcal{F}^{p,q}_{\mu/m} \\ \uparrow \mathcal{B} & & \mathcal{B} \uparrow \\ M^{p,q}_{\breve{\mu}} & \xrightarrow{(-1)^d \operatorname{op}^{\varphi_k}_{\mathrm{aw}}(\breve{m})} & M^{p,q}_{\breve{\mu}/\breve{m}} \end{array}$$

for showing the isomorphism property. □

5 Symbol Calculus

After we presented the basic concept of polyanalytic Toeplitz operators and their natural action on polyanalytic Sobolev-Fock spaces in the previous sections, we now turn towards a basic symbolic operator calculus for Hermite localization operators as well as polyanalytic Toeplitz operators by providing expansions for compositions and commutators.

For localization operators with symbols in modulation spaces, composition formulas and Fredholm properties have been derived in great generality in [12]. Our aim is to obtain more explicit expressions and expansions for small $\hbar$. We start by presenting asymptotic expansions of localization operators with Hermite windows and their compositions as $\hbar \to 0$, before moving on to polyanalytic spaces and operators.

5.1 Weyl Expansion of Hermite Localization Operators

By observing that localization operators are in fact smoothed Weyl operators,

$$\operatorname{op}^{\varphi}_{\mathrm{aw}}(a) = \operatorname{op}(\mathcal{W}_{\varphi} * a)$$

one can Taylor expand the convolution and use the Moyal product expansion in order to derive asymptotic expansions of compositions of localization operators.

For the standard case of a Gaussian window we recall the following formula from [35, Lemma 1] that originated from [21, Proposition 2.96].

Lemma 4 *Let* $a : \mathbb{R}^{2d} \to \mathbb{C}$ *be a Schwartz function,* $\hbar > 0$, $N \in \mathbb{N}$. *Then,*

$$\operatorname{op}_{\mathrm{aw}}(a) = \operatorname{op}\left(a + \sum_{k=1}^{N-1} \frac{(\hbar\Delta)^k}{4^k k!} a\right) + \hbar^N \operatorname{op}(r_\hbar)$$

with a family $r_\hbar$ *of Schwartz functions satisfying* $\sup_{\hbar>0} \|\operatorname{op}(r_\hbar)\|_{L^2\to L^2} < \infty$.

Let us generalize this formula for higher order Hermite functions. We do this by similar means as applied in [34] for deriving the expansion with Hermite spectrograms.

For this purpose, let us recall the formula for Wigner transforms of Hermite functions,

$$\mathcal{W}_{\varphi_k}(z) = (\pi\hbar)^{-d} \mathrm{e}^{-|z|^2/\hbar} (-1)^{|k|} \prod_{j=1}^{d} L_{k_j}(\tfrac{2}{\hbar}|z_j|^2), \tag{25}$$

where $z = (q, p) \in \mathbb{R}^{2d}$, $z_j = (q_j, p_j) \in \mathbb{R}^2$ and

$$L_k(x) = \sum_{j=0}^{k} \binom{k}{k-j} \frac{(-x)^j}{j!}, \quad x \in \mathbb{R},\ k \in \mathbb{N},$$

is the kth Laguerre polynomial, see, e.g., [21, §1.9]. In order to generalize Lemma 4 to arbitrary Hermite function windows we have to first get a better understanding of higher order moments of the Wigner transforms of Hermite functions. Note, that due to the symmetry of $\mathcal{W}_{\varphi_k}$ only even moments are different from zero.

Proposition 1 *Let $\alpha, \beta, k \in \mathbb{N}$ be arbitrary.Then,*

$$\int_{\mathbb{R}^2} x^{2\alpha} \xi^{2\beta} \mathcal{W}_{\varphi_k}(x, \xi) \mathrm{d}x \mathrm{d}\xi = {}_2F_1(\alpha + \beta + 1, -k; 1; 2)(-1)^k \frac{\hbar^{\alpha+\beta} (2\alpha)!(2\beta)!}{4^{\alpha+\beta} \alpha! \beta!}$$

where ${}_2F_1$ is the hypergeometric function.

For the proof of Proposition 1, which is mainly built on relations of binomial sums and Gamma functions, we refer to Appendix 1. Now, we are ready to generalize Lemma 4 as follows.

Lemma 5 *Let $k, N \in \mathbb{N}^d$, $\hbar > 0$ and a being a Schwartz function. Then,*

$$\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(a) = \mathrm{op}\left(\sum_{m=0}^{N-1} \frac{\hbar^m D_m}{4^m m!} a \right) + \hbar^N \mathrm{op}(r_\hbar^k)$$

with a family $r_\hbar^k$ of functions satisfying $\sup_{\hbar > 0} \|\mathrm{op}(r_\hbar^k)\|_{L^2 \to L^2} < \infty$ and the order $2m$ phase space differential operator D_m given by

$$D_m a(z) = (-1)^{|k|} m! \sum_{|\alpha|=m} c_\alpha^{(k)} \partial^{2\alpha} a(z)$$

that is a sum of total order $2m$ differential operators with constant coefficients

$$c_\alpha^{(k)} = \frac{1}{\alpha!} \prod_{j=1}^{d} {}_2F_1(\alpha_j + \alpha_{j+d} + 1, -k_j; 1; 2).$$

Proof We can basically retrace the proof idea of [35, Lemma 1] by writing

$$a * \mathcal{W}_{\varphi_k}(z) = \int_{\mathbb{R}^{2d}} a(\zeta)\mathcal{W}_{\varphi_k}(z-\zeta)\mathrm{d}\zeta \tag{26}$$

and using a Taylor expansion of a around z,

$$\begin{aligned} a(\zeta) = &\sum_{|\alpha|=0}^{2N-1} \frac{(\zeta - z)^\alpha}{\alpha!}(\partial^\alpha a)(z) \\ &+2N \sum_{|\alpha|=2N} \frac{(\zeta - z)^\alpha}{\alpha!} \int_0^1 (1-\theta)^{2N-1}(\partial^\alpha a)(z+\theta(\zeta - z))\mathrm{d}\theta. \end{aligned}$$

Since the symmetry of (25) implies that

$$\int_{\mathbb{R}^{2d}} f(z)\mathcal{W}_{\varphi_k}(z)\mathrm{d}z = 0$$

whenever f is an odd function, the derivatives of odd degree in the Taylor expansion of a do not contribute to the integral (26). For the even degree polynomials we apply Proposition 1 and compute

$$\begin{aligned} &\sum_{|\alpha|=m} \int_{\mathbb{R}^{2d}} \frac{(\zeta - z)^{2\alpha}}{2\alpha!}(\partial^{2\alpha} a)(z)\mathcal{W}_{\varphi_k}(z-\zeta)\mathrm{d}\zeta \\ &= \sum_{|\alpha|=m} \frac{(\partial^{2\alpha} a)(z)}{2\alpha!} \int_{\mathbb{R}^{2d}} \zeta^{2\alpha}\mathcal{W}_{\varphi_k}(\zeta)\mathrm{d}\zeta \\ &= (-1)^k \frac{\hbar^m}{4^m} \sum_{|\alpha|=m} \frac{(\partial^{2\alpha} a)(z)}{\alpha!} \prod_{j=1}^{d} {}_2\mathrm{F}_1(\alpha_j + \alpha_{j+d} + 1, -k_j; 1; 2) \end{aligned}$$

by utilizing the fact that the Wigner function factorizes in the form (25). Hence,

$$a * \mathcal{W}_{\varphi_k}(z) = \sum_{m=0}^{N-1} \frac{\hbar^m D_m}{4^m m!} a + \hbar^N r_\hbar^k \tag{27}$$

which completes the proof as the Calderon-Vaillancourt theorem implies the uniform boundedness of $r_\hbar^k$. □

We would like to stress that due to the fact that the coefficients c_α are varying in α it is not straight-forward to write down an inverse expansion as in general $D_m D_n \not\propto D_{m+n}$ unless for the Husimi case $k = 0$.

5.2 Compositions and Commutators of Hermite Localization Operators

Recall that the composition of two Weyl quantized operators is a Weyl quantized operator again, with the symbol given by the famous *Moyal product* $\sharp$ of the two symbols,

$$\mathrm{op}(a)\mathrm{op}(b) = \mathrm{op}(a\sharp b), \tag{28}$$

see, e.g., [53, §4.3]. In contrast, the product of two localization operators typically is not a localization operator again. However, the product can be expanded as a sum of localization operators with a regularizing operator as error term that becomes arbitrary small as $\hbar \to 0$, see [12].

Based on the expansion from Lemma 5, we obtain the following Weyl composition formula for two localization operators that employs the operator $A(\nabla)$,

$$A(\nabla)f(z,w) = \tfrac{1}{2}\sigma(\nabla_w, \nabla_z)f(z,w), \quad z, w \in \mathbb{R}^{2d},$$

acting on functions on the doubled phase space $\mathbb{R}^{4d}$, where σ denotes the standard symplectic form and the subscript gradients ∇_z, ∇_w denote $2d$-dimensional gradients in the indicated coordinates. The notation $A(\nabla)$ in accordance with [53] aims to support the reader's intution as $A(\nabla)$ is the generator of bidifferential operators

$$\alpha_n(a,b) = \Big[A(\nabla)^n(a\otimes b)\Big]_{\mathrm{diag}}, \quad n \in \mathbb{N}, \tag{29}$$

that define the Moyal product expansion.

Proposition 2 *(Composition of localization operators) Let $k, N \in \mathbb{N}^d$, $\hbar > 0$ and a and b be Schwartz functions. Then,*

$$\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(a)\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(b) = \mathrm{op}\left(\sum_{j=0}^{N-1}\frac{\hbar^j}{4^j}\left(\sum_{n+m+\ell=j}\frac{C_{n,m,\ell}(a,b)}{m!n!\ell!}\right)\right) + \hbar^N \mathrm{op}(\rho_\hbar^k)$$

with a family $\rho_\hbar^k$ of Schwartz functions satisfying $\sup_{\hbar>0}\|\mathrm{op}(\rho_\hbar^k)\|_{L^2\to L^2} < \infty$ and the total order $2(n+m+\ell)$ bidifferential operators

$$C_{n,m,\ell}(a,b) = [(-4\mathrm{i})^\ell \alpha_\ell(D_m a, D_n b)]_{diag}$$

where D_n has been defined in Lemma 5 and α_ℓ in (29).

Proof We apply Lemma 5 and the expansion of the Moyal product $\sharp$ to compute

$$\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(a)\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(b) = \mathrm{op}\Big(\sum_{m+n=0}^{N-1}\frac{\hbar^{n+m}}{4^{m+n}m!n!}D_m a\sharp D_n b\Big) + \hbar^N \mathrm{op}(\varrho_\hbar^k)$$

$$= \sum_{j=0}^{N-1} \frac{\hbar^j}{4^j} \mathrm{op}\left(\sum_{n+m+\ell=j} \frac{(-4\mathrm{i})^\ell}{m!n!\ell!} [A(\nabla)^\ell (D_m a \otimes D_n b)]_{\mathrm{diag}} \right) + \hbar^N \mathrm{op}(\rho_\hbar^k)$$

where $\varrho_\hbar^k$ and $\rho_\hbar^k$ are families of Schwartz functions giving rise to uniformly bounded operator families. □

For illustration purposes, let us look at the general expansion from Proposition 2 in the case of second order errors. We compute

$$D_1 a(z) = \sum_{j=1}^{d} (2k_j + 1)(\partial_j^2 a(z) + \partial_{j+d}^2 a(z))$$

and observe that D_1 is a diagonally weighted Laplace operator on $\mathbb{R}^{2d}$,

$$D_1 a(z) = \langle \nabla_z, \mathrm{diag}(2k+1, 2k+1)\nabla_z \rangle\, a(z) =: \Delta_{(k)} a(z),$$

where $(2k+1, 2k+1) := (2k_1+1, 2k_1+1, \ldots, 2k_d+1) \in \mathbb{R}^{2d}$. Hence, we obtain the following second order composition formula for two localization operators in terms of a Weyl operator:

Lemma 6 *Let $k \in \mathbb{N}^d$, $\hbar > 0$ and a and b be Schwartz functions. Then,*

$$\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(a)\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(b) = \mathrm{op}\left(ab + \tfrac{\hbar}{2}\left(-\mathrm{i}\sigma(\nabla a, \nabla b) + \tfrac{1}{2} b \Delta_{(k)} a + \tfrac{1}{2} a \Delta_{(k)} b \right) \right) + O(\hbar^2).$$

In fact, if we allow for second order error terms, the expansion from Lemma 5 can be approximately inverted via

$$\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(a - \tfrac{\hbar}{4}\Delta_{(k)} a) = \mathrm{op}(a) + O(\hbar^2). \tag{30}$$

This observation in turn implies the following composition formula for Hermite window localization operators.

Theorem 3 *Let $k \in \mathbb{N}^d$, $\hbar > 0$ and a and b be Schwartz functions. Then, it holds*

$$\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(a)\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(b) = \mathrm{op}_{\mathrm{aw}}^{\varphi_k}\left(ab - \tfrac{\hbar}{2}\left(\mathrm{i}\sigma(\nabla a, \nabla b) + \left\langle \nabla_{(k)} a, \nabla_{(k)} b \right\rangle \right) \right) + \hbar^2 \mathrm{op}(\theta_\hbar^k).$$

with a family $\theta_\hbar^k$ of Schwartz functions satisfying $\sup_{\hbar>0} \|\mathrm{op}(\theta_\hbar^k)\|_{L^2 \to L^2} < \infty$ and the weighted gradient

$$\nabla_{(k)} = (\sqrt{2k_1+1}\,\partial_1, \sqrt{2k_1+1}\,\partial_2, \ldots, \sqrt{2k_d+1}\,\partial_{2d}).$$

Proof By combining Proposition 6 and (30) we obtain

$$\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(a)\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(b) = \mathrm{op}_{\mathrm{aw}}^{\varphi_k}\left(ab + \tfrac{\hbar}{2}(-\mathrm{i}\sigma(\nabla a, \nabla b) + \tfrac{1}{2} b \Delta_{(k)} a + \tfrac{1}{2} a \Delta_{(k)} b - \tfrac{1}{2}\Delta_{(k)}(ab)) \right)$$

$$+\hbar^2 \mathrm{op}(\theta^k_\hbar)$$

where, by the Calderon-Vailloncourt theorem, the the second order terms in $\hbar$ have a Schwartz class symbol with the desired boundedness properties. Then, calculating

$$\Delta_{(k)}(ab) = \Delta_{(k)}ab + a\Delta_{(k)}b + 2\left\langle \nabla_{(k)}a, \nabla_{(k)}b \right\rangle$$

implies the result. □

From Theorem 3 we directly infer that the commutator of two localization operators exhibits the same Poisson bracket property as the Moyal bracket for Weyl operators with the difference that the error is of second instead of third order in $\hbar$.

Corollary 1 *Let $k \in \mathbb{N}^d$, $\hbar > 0$ and a and b be Schwartz functions. Then, for the commutator of Hermite window localization operators it holds*

$$\left[\mathrm{op}^{\varphi_k}_{\mathrm{aw}}(a), \mathrm{op}^{\varphi_k}_{\mathrm{aw}}(b)\right] = \tfrac{\hbar}{\mathrm{i}}\mathrm{op}^{\varphi_k}_{\mathrm{aw}}\left(\{a, b\}\right) + O(\hbar^2)$$

where $\{a, b\}$ denotes the Poisson bracket.

Remark 2 (Hermite star products) The Hermite star products $\star_k$ can be formally defined as

$$\mathrm{op}^{\varphi_k}_{\mathrm{aw}}(a)\mathrm{op}^{\varphi_k}_{\mathrm{aw}}(b) =: \mathrm{op}^{\varphi_k}_{\mathrm{aw}}(a\star_k b)$$

on the algebra $\mathcal{C}^\infty(\mathbb{R}^{2d})[[\hbar]]$ of formal power series in $\hbar$ with smooth coefficients. Corollary 1 illustrates that—just as the Moyal product $\sharp$— all Hermite star products $\star_k$ are compatible with the canonical Poisson structure on phase space. Moreover, the expansion from Lemma 5 implies that the differential star products $\star_k$ and $\sharp$ are equivalent for all k in the sense of deformation quantization, see, e.g., [11,37,46]. In particular, we note that the bidifferential operator $\mathrm{i}\alpha_1(a, b) + \frac{1}{2}\left\langle \nabla_{(k)}a, \nabla_{(k)}b \right\rangle$ from Theorem 3 defines the same 2-cocycle as the Moyal bidifferential operator $\mathrm{i}\alpha_1(a, b)$ in the Hochschild cochain complex over $\mathcal{C}^\infty(\mathbb{R}^{2d})[[\hbar]]$ and only differs by the symmetrical coboundary term.

Remark 3 (Anti-Wick star product) In the Husimi case $k = 0$ one has $D_n = \Delta^n$ and can explicitly derive higher order versions of Theorem 3. In analogy to the Moyal expansion a simple but tedious calculation yields

$$\mathrm{op}^{\varphi_0}_{\mathrm{aw}}(a)\mathrm{op}^{\varphi_0}_{\mathrm{aw}}(b) = \mathrm{op}^{\varphi_0}_{\mathrm{aw}}(a\star_0 b) = \sum_{j=0}^{\infty} \mathrm{op}^{\varphi_0}_{\mathrm{aw}}\left(\hbar^n \beta_n(a, b)\right)$$

with the bidifferential operators

$$\beta_n(a, b) = \frac{(-1)^n}{2^n n!}\Big[\left(\mathrm{i}\sigma(\nabla_z, \nabla_w) + \langle \nabla_z, \nabla_w\rangle\right)^n a(z)b(w)\Big]_{\mathrm{diag}},$$

see [33]. In other words, the operator $B(\nabla) = A(\nabla) + \frac{1}{2}\langle \nabla_z, \nabla_w \rangle$ generates the bidifferential operators β_n that charaterize the anti-Wick star product $\star_0$.

The symmetric term $\left\langle \nabla_{(k)} a, \nabla_{(k)} b \right\rangle$ creates coboundary terms in the bidifferential operators defining Hermite star products and implies that the $O(\hbar^2)$ error for the commutator expansion in Corollary 1 in general is sharp. In contrast, for the Moyal case the antisymmetry of $A(\nabla)$ causes $O(\hbar^3)$ errors for the commutator which is the main ingredient for the the well-known Egorov theorem that allows to link quantum and quasi-classical dynamics with $O(\hbar^2)$ errors, see [10,39].

We conclude this section by stressing again that the Weyl operator error term $\mathrm{op}(\theta_\hbar^k)$ in Theorem 3 is in general not a localization operator itself. This makes the composition formula purely asymptotic in nature.

5.3 Calculus of Polyanalytic Toeplitz Operators

The formulas from Sect. 5.2 also imply composition rules for polyanalytic Toeplitz operators as they appear as the complex equivalents of corresponding localization operators with Hermite function windows.

From (22) we first recall the translation formulas

$$\mathcal{B}_k^* \mathcal{T}_k(f) \mathcal{B}_k = (-1)^d \mathrm{op}_{\mathrm{aw}}^{\varphi_k}(\check{f}), \quad \mathcal{B}^* \mathcal{T}_{k,0}(f) \mathcal{B} = (-1)^d \mathrm{op}_{\mathrm{aw}}^{\varphi_k}(\check{f}) \tag{31}$$

between localization operators acting on real-valued signals and polyanalytic Toeplitz operators acting in the complex domain. Moreover, for convenience of notation we introduce the operators

$$\Xi_{(k)}(\nabla) f(z, w) = \mathrm{i}\sigma(\nabla_w, \nabla_z) f(z, w) + \left\langle \nabla_{(k),z} \nabla_{(k),w} \right\rangle f(z, w), \quad z, w \in \mathbb{R}^{2d},$$

and their complex counterpart

$$\widehat{\Xi}_{(k)}(\partial, \overline{\partial}) F(z, w) = \left(4\mathrm{i}\partial_z \overline{\partial}_w + \left\langle \nabla_{\mathrm{Re}z,\mathrm{Im}z}, d_k \nabla_{\mathrm{Re}w,\mathrm{Im}w} \right\rangle_{\mathbb{R}^{2d}}\right) F(z, w), \quad z, w \in \mathbb{C}^d,$$

where $\partial_z, \overline{\partial}_z$ as usual denote complex Wirtinger differentials and the diagonal matrix $d_k = \mathrm{diag}(2k, 2k) \in \mathbb{R}^{2d \times 2d}$. Note that for $k = 0$ the second term vanishes and we obtain the simple expression

$$\widehat{\Xi}_{(0)}(\partial, \overline{\partial}) F(z, w) = 4\mathrm{i}\partial_z \overline{\partial}_w F(z, w).$$

The operator $\widehat{\Xi}_{(k)}(\partial, \overline{\partial})$ represents $\Xi_{(k)}(\nabla)$ in the complex domain in the following way.

Lemma 7 *Let $m, \mu : \mathbb{C}^d \to \mathbb{C}$ be smooth and $k \in \mathbb{N}^d$. Then, it holds*

$$\widehat{\Xi}_{(k)}(\partial, \overline{\partial})(m \otimes \mu)(\bar{z}, \bar{w}) = \Xi_{(k)}(\nabla)(\check{m} \otimes \check{\mu})(z_{\mathbb{R}}, w_{\mathbb{R}}),$$

where $z, w \in \mathbb{C}^d$ and $z_{\mathbb{R}} = (\mathrm{Re}z, \mathrm{Im}z)$, $w_{\mathbb{R}} = (\mathrm{Re}w, \mathrm{Im}w) \in \mathbb{R}^{2d}$.

Proof The proof is a simple calculation that only uses the definition of the complex Wirtinger differentials $\partial_z f(z) = \frac{1}{2}(\partial_{\mathrm{Re}z} f(z) - \mathrm{i}\partial_{\mathrm{Im}z}) f(z)$ and $\overline{\partial} f(z) = \frac{1}{2}(\partial_{\mathrm{Re}z} f(z) + \mathrm{i}\partial_{\mathrm{Im}z})$. □

With this notation in place, we arrive at the following composition and commutator formulas for polyanalytic Toeplitz operators.

Theorem 4 *Let $k \in \mathbb{N}^d$, $\hbar > 0$ and $m, \mu : \mathbb{C}^d \to \mathbb{C}$ be Schwartz class functions. Then, it holds*

$$\mathcal{T}_k(m)\mathcal{T}_k(\mu) = \mathcal{T}_k(m\mu - \tfrac{\hbar}{2}[\widehat{\Xi}_{(k)}(\nabla)(m \otimes \mu)]_{diag}) + \hbar^2 \mathcal{B}_k^* \mathrm{op}(\theta_\hbar^k)\mathcal{B}_k.$$

with a family $\theta_\hbar^k$ of Schwartz functions satisfying $\sup_{\hbar>0} \|\mathrm{op}(\theta_\hbar^k)\|_{L^2 \to L^2} < \infty$.

Proof We calculate

$$\begin{aligned} \mathcal{T}_k(m)\mathcal{T}_k(\mu) &= \mathcal{B}_k^* \mathrm{op}_{\mathrm{aw}}^{\varphi_k}(\breve{m})\mathrm{op}_{\mathrm{aw}}^{\varphi_k}(\breve{\mu})\mathcal{B}_k \\ &= \mathcal{B}_k^* \mathrm{op}_{\mathrm{aw}}^{\varphi_k}(\breve{m}\breve{\mu} - \tfrac{\hbar}{2}[\Xi_{(k)}(\nabla)(\breve{m} \otimes \breve{\mu})]_{\mathrm{diag}})\mathcal{B}_k + O(\hbar^2) \end{aligned}$$

by using (31) and applying Theorem 3. From Lemma 7 we then obtain

$$[\Xi_{(k)}(\nabla)(\breve{m} \otimes \breve{\mu})]_{\mathrm{diag}}(q, p) = [\widehat{\Xi}_{(k)}(\nabla)(m \otimes \mu)]_{\mathrm{diag}}(q - ip)$$

which completes the proof. □

Let us remark here, that in the usual Toeplitz quantization case $k = 0$ this composition formula beautifully reduces to

$$\mathcal{T}(m)\mathcal{T}(\mu) = \mathcal{T}(m\mu - 2\mathrm{i}\hbar\, \partial m\, \overline{\partial}\mu) + O(\hbar^2).$$

Ignoring all growth restrictions, this shows that whenever μ is analytic (or m antianalytic) the product $m\mu$ is the appropriate Toeplitz symbol of the composition up to second order errors.

6 Weyl Quantization and Polyanalytic Toeplitz Operators

Let us revisit the spectrogram expansion from Theorem 1 with the objects we have defined and investigated so far. By recalling the phase space integral formulas (11) and (14) we first observe that Theorem 1 in fact can be read as a weak approximation of Weyl operators in terms of localization operators with Hermite function windows. In this section, we derive expansions of complex Weyl operators in terms of Bargmann quantized operators and, thus, prove a complex version of Theorem 1.

6.1 An Anti-Wick Expansion of Weyl Operators

By employing the off-diagonal convolution formula

$$\mathcal{W}_u * \mathcal{W}(\psi, \phi) = V_u \psi \, \overline{V_u \phi}. \tag{32}$$

and the definition (15) of localization operators we can rewrite (18) as

$$\begin{aligned} \int_{\mathbb{R}^{2d}} a(z) \mu^N(\psi, \phi)(z) \mathrm{d}z &= \int_{\mathbb{R}^{2d}} a(z) \sum_{j=0}^{N-1} (-1)^j C_{N-1,j} \sum_{\substack{k \in \mathbb{N}^d \\ |k|=j}} \mathcal{W}_{\varphi_k} * \mathcal{W}(\psi, \phi)(z) \, \mathrm{d}z \\ &= \int_{\mathbb{R}^{2d}} a(z) \sum_{j=0}^{N-1} (-1)^j C_{N-1,j} \sum_{\substack{k \in \mathbb{N}^d \\ |k|=j}} V_{\varphi_k} \psi(z) \, \overline{V_{\varphi_k} \phi(z)} \, \mathrm{d}z \\ &= \sum_{j=0}^{N-1} (-1)^j C_{N-1,j} \sum_{\substack{k \in \mathbb{N}^d \\ |k|=j}} \left\langle V_{\varphi_k}^* a V_{\varphi_k} \psi, \phi \right\rangle_{L^2(\mathbb{R}^d)} \\ &= \sum_{j=0}^{N-1} (-1)^j C_{N-1,j} \sum_{\substack{k \in \mathbb{N}^d \\ |k|=j}} \left\langle \mathrm{op}_{\mathrm{aw}}^{\varphi_k}(a) \psi, \phi \right\rangle_{L^2(\mathbb{R}^d)} . \end{aligned}$$

Hence, the spectrogram approximation from Theorem 1 can be rewritten in the following operator form:

Proposition 3 *Let $N \in \mathbb{N}$ an $\hbar > 0$. Then, for all $\hbar$-independent Schwartz class functions $a : \mathbb{R}^{2d} \to \mathbb{C}$ it holds*

$$\mathrm{op}(a) = \sum_{j=0}^{N-1} (-1)^j C_{N-1,j} \sum_{\substack{k \in \mathbb{N}^d \\ |k|=j}} \mathrm{op}_{\mathrm{aw}}^{\varphi_k}(a) + O(\hbar^N)$$

in the operator norm topology on $L^2(\mathbb{R}^d)$.

One can generalize Proposition 3 in the usual sense by allowing for more general symbol classes. In particular, Proposition 3 remains true as long as a belongs to a suitable Shubin class $\Gamma_\rho^{2N(1-\rho)}(\mathbb{R}^{2d})$ of symbols, where

$$\Gamma_\rho^m(\mathbb{R}^{2d}) = \left\{ a \in \mathbb{C}^\infty(\mathbb{R}^{2d}, \mathbb{C}) : |\partial_z^\alpha a(z)| \leq C_\alpha \langle z \rangle^{m-\rho|\alpha|} \quad \forall z \in \mathbb{R}^{2d}, \quad \alpha \in \mathbb{N}^{2d} \right\} \tag{33}$$

with $\langle z\rangle = (1+|z|^2)^{1/2}$. Note that the Weyl quantization of a symbol $a \in \Gamma_\rho^m(\mathbb{R}^{2d})$ creates a bounded operator from the Shubin-Sobolev space

$$Q^m(\mathbb{R}^d) = \left\{\psi \in \mathcal{S}'(\mathbb{R}^d) : (1+|x|^2-\Delta)^{-m/2}\psi \in L^2(\mathbb{R}^d)\right\}$$

into $L^2(\mathbb{R}^d)$, and it is known that $Q^m(\mathbb{R}^d)$ actually coincides (with equivalent norm) with the modulation space $M^2_{\langle z\rangle^m}(\mathbb{R}^d)$, see [9,40]. For example, a more general version of Proposition 3 can be formulated as follows.

Corollary 2 *Let $N \in \mathbb{N}$, $\hbar > 0$ and assume $a \in \Gamma_\rho^m(\mathbb{R}^{2d})$ for $m \in \mathbb{R}$, $\rho \geq 0$. Then,*

$$\mathrm{op}(a) - \sum_{j=0}^{N-1}(-1)^j C_{N-1,j} \sum_{\substack{k\in\mathbb{N}^d\\ |k|=j}} \mathrm{op}_{\mathrm{aw}}^{\varphi_k}(a) = O(\hbar^N)$$

as a bounded operator from $M^2_{m-2N\rho}(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d)$.

We note that Corollary 2 directly implies the famous *Sharp Gårding* inequality, see, e.g., [53, §4.7], in the sense that for any suitable symbol $a \geq 0$ one has

$$\mathrm{op}(a) + O(\hbar) \geq 0$$

since any anti-Wick quantization of a non-negative symbol yields a non-negative operator. In fact, often the introduction of Husimi functions and anti-Wick operators is mainly motivated by this property. Unfortunately, one does not as easily obtain a Fefferman-Phong inequality with second order errors in $\hbar$ since the phase space density that is underlying the second order approximation

$$(1+\tfrac{d}{2})\mathrm{op}_{\mathrm{aw}}^{\varphi_0} - \tfrac{1}{2}\sum_{j=1}^{d} \mathrm{op}_{\mathrm{aw}}^{\varphi_{e_j}}(a) = \mathrm{op}(a) + O(\hbar^2)$$

which has been investigated in [36] in general takes negative values—as does the Wigner function.

6.2 Polyanalytic Bargmann Representation of Anti-holomorphic Weyl Quantized Polynomials

The close connection between polyanalytic Bargmann transforms and short-time Fourier transforms with Hermite windows allows to rephrase Proposition 3 in a Fock space setting.

For the convenience or the reader, we use the notion of *polyanalytic Bargmann-Fock spaces of total degree $n \in \mathbb{N}$* that we define as the direct sum

$$\mathbf{F}^n(\mathbb{C}^d) := \bigoplus_{|k|=n} \mathcal{F}^k(\mathbb{C}^d),$$

and the corresponding Bargmann transforms

$$\mathbf{B}_n : L^2(\mathbb{R}^d) \to \mathbf{F}^n(\mathbb{C}^d), \quad \mathbf{B}_n := \sum_{|k|=n} \mathcal{B}_k$$

on the span of the true polyanalytic functions of total degree $n \in \mathbb{N}$. We want to highlight that $\mathbf{B}_n$ is different from the polyanalytic Bargmann transform for vector-valued signals defined and analyzed in [1] with applications to multiplexing. We denote the corresponding *polyanalytic Bergman projector of total degree* $n \in \mathbb{N}$ by

$$\mathbf{P}_n := \mathbf{B}_n \mathbf{B}_n^*, \quad n \in \mathbb{N}.$$

Note that one has the following property:

Lemma 8 *The polyanalytic Bergman projector of total degree $n \in \mathbb{N}$ satisfies*

$$\boldsymbol{P}_n \neq \sum_{|k|=n} \mathcal{P}_k$$

This follows since in general for the mixed terms it holds $\mathcal{B}_k \mathcal{B}_\ell^* \neq 0$ though for $k \neq \ell$ one still has the orthogonality property $\mathcal{B}_k^* \mathcal{B}_\ell = 0$.

The important property of almost-invariance of polyanalytic Fock spaces under multiplication with holomorphic polynomials allows to prove the following result that might allow new insights about the manipulation of signals in a multiplexing setup, see [1,5].

Proposition 4 *(Polyanalytic Bargmann representation of antiholomorphic Weyl operators) Let $N \in \mathbb{N}$, $\hbar > 0$ and $p : \mathbb{C}^d \to \mathbb{C}$ be a (holomorphic) polynomial of degree $N-1$. Then, one has*

$$\mathrm{op}(\breve{p}) = \sum_{j=0}^{N-1} (-1)^j C_{N-1,j} \mathbf{B}_j^* p \mathbf{B}_j$$

where $\breve{p}(q,p) := p(q - ip)$ is the transformation from (23) and $\mathbf{B}_j = \sum_{|k|=j} \mathcal{B}_k$.

Proof We start by rewriting the generalized anti-Wick operators in the anti-Wick expansion from Proposition 3 in terms of polyanalytic Bargmann transforms,

$$\mathrm{op}(\breve{p}) = \sum_{j=0}^{N-1} (-1)^j C_{N-1,j} \sum_{\substack{k \in \mathbb{N}^d \\ |k|=j}} \mathrm{op}_{\mathrm{aw}}^{\varphi_k}(\breve{p}) = \sum_{j=0}^{N-1} (-1)^j C_{N-1,j} \sum_{\substack{k \in \mathbb{N}^d \\ |k|=j}} \mathcal{B}_k^* p \mathcal{B}_k,$$

where the error vanishes because p is of sufficiently low degree, see Theorem 1. Since p is a holomorphic polynomial, for each true polyanalytic Fock space $\mathcal{F}^k(\mathbb{C}^d)$ multiplication by p leaves a dense subset of $\mathcal{F}^k(\mathbb{C}^d)$ consisting of true polyanalytic polynomials invariant. Moreover, true polyanalytic Fock spaces are othogonal: for any $u \in \mathcal{F}^k(\mathbb{C}^d)$ and $v \in \mathcal{F}^\ell(\mathbb{C}^d)$ with $k \neq \ell$ it holds

$$\langle u, v \rangle_{L^2_\Phi} = 0.$$

Thus, the result follows from observing

$$\sum_{j=0}^{N-1} (-1)^j C_{N-1,j} \sum_{\substack{k \in \mathbb{N}^d \\ |k|=j}} \mathcal{B}_k^* p \mathcal{B}_k = \sum_{j=0}^{N-1} (-1)^j C_{N-1,j} \mathbf{B}_j^* p \mathbf{B}_j.$$

$\square$

We can revisit this result in the context of multiplexing as e.g. considered in [5]. Namely, polyanalytic Bargmann transforms allow to transform n signals $(\psi_0, \ldots, \psi_{n-1})$ into the single signal

$$\mathcal{B}\psi_0 + \mathbf{B}_1\psi_1 + \ldots + \mathbf{B}_{n-1}\psi_{n-1} : \mathbb{C}^d \to \mathbb{C}$$

that now can jointly be transmitted or manipulated. Afterwards, the n original signals can be recovered via orthogonal projection by using the suitable (polyanalytic) Bergman projectors. This is an implication of the orthogonality of polyanalytic Fock spaces of different degree.

In other words, Proposition 4 can be understood in the sense that the polynomial manipulation of a single multiplexed signal with arbitrary number of "multiplexing copies" can be expressed in terms of the action of usual Weyl operators or, conversely, that the action of a Weyl quantized polynomial on any signal can be expressed via a linear combination of multiplications in polyanalytic Bargmann-Fock spaces of suitable degree. For more general manipulations the error terms from the spectrogram expansion can be used when approximating the multi-level Bargmann multiplier by a Weyl operator.

6.3 A Polyanalytic Toeplitz Expansion of Complex Weyl Operators

The aim of this section is to provide a version of the anti-Wick expansion from Proposition 3 in the complex setting. That is, instead of anti-Wick operators we employ the earlier defined polyanalytic Toeplitz operators and relate them to complex Weyl operators as considered in [53, §13].

From microlocal analysis it is known that the Bargmann transform can be recovered as a Fourier-Bros-Iagolitzer transform characterized by the holomorphic quadratic phase function

$$\theta(z, w) = \tfrac{\mathrm{i}}{2}((z - w)^2 - z^2/2), \tag{34}$$

see also Appendix 1 for more context. In fact, θ gives rise to the complex symplectic map

$$\kappa : \mathbb{C}^{2d} \to \mathbb{C}^{2d}, \quad \kappa(z, w) \mapsto (\mathrm{i}w - z, \tfrac{1}{2}(\mathrm{i}z - w)) \tag{35}$$

by means of the implicit generating function type definition

$$\kappa(w, -\partial_w \theta(z, w)) = (z, \partial_z \theta(z, w)), \quad z, w \in \mathbb{C}^d.$$

One can show that κ is a bijection as a map from $\mathbb{R}^{2d}$ on the Lagrangian subspace

$$\Lambda = \{(z, -\tfrac{i}{2}\overline{z}) : z \in \mathbb{C}^d\} \subset \mathbb{C}^{2d} \tag{36}$$

of real dimension $2d$. The subspace Λ is Im-Lagrangian and Re-symplectic with respect to the complex symplectic form $\sigma_{\mathbb{C}} = \sum_{j=1}^{d} \mathrm{d}w \wedge \mathrm{d}z$ on $\mathbb{C}^n \times \mathbb{C}^n$, that is,

$$\mathrm{Im}\sigma_{\mathbb{C}} \restriction_{\Lambda} = 0 \quad \text{and} \quad \mathrm{Re}\sigma_{\mathbb{C}} \restriction_{\Lambda} \ \text{is nondegenerate.}$$

In particular Λ is only $\mathbb{R}$-linear but not $\mathbb{C}$-linear and, hence, is not of the type of complex Lagrangian subspaces usually considered in the parametrization of generalized coherent states, see, e.g., [13]. In other words, Λ is an isotropic subspace of maximal dimension, but the Hermitian form

$$\mathbb{C}^{2d} \ni z \mapsto \tfrac{\mathrm{i}}{2} \langle z, \Omega z\rangle_{\mathbb{C}^{2d}} = \tfrac{\mathrm{i}}{2}\overline{z} \cdot \Omega z$$

with the standard symplectic matrix

$$\Omega = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$$

is neither positive nor negative definite on Λ, since one computes

$$\langle z, \Omega z\rangle_{\mathbb{C}^{2d}} = \tfrac{1}{2}(\mathrm{Im}(\zeta)^2 - \mathrm{Re}(\zeta)^2) \quad \forall z = (\zeta, -\tfrac{\mathrm{i}}{2}\overline{\zeta}) \in \Lambda.$$

In [53, §13] the symplectic mapping κ from (35) is used to introduce a complex Weyl quantization on the Bargmann transform side. Namely, the bijection κ can be used to identify $\mathbb{C}^d$ with the Lagrangian subspace $\Lambda \subset \mathbb{C}^{2d}$ and for a Schwartz function $a : \Lambda \to \mathbb{C}$ we define its Weyl quantization

$$\mathrm{op}_{\Phi}(a) : L^2_{\Phi}(\mathbb{C}^d) \to L^2_{\Phi}(\mathbb{C}^d) \tag{37}$$

via the usual Fourier integral formalism

$$\mathrm{op}_{\Phi}(a) f(z) = (2\pi\hbar)^{-d} \int_{\Gamma_{\Phi}(z)} a(\tfrac{z+w}{2}) \mathrm{e}^{\mathrm{i}(z-w)\zeta/\hbar} f(w) \mathrm{d}\zeta \wedge \mathrm{d}w \tag{38}$$

along the z-dependent contour

$$\Gamma_\Phi(z) : w \mapsto \tfrac{2}{\mathrm{i}}\partial_z \Phi(\tfrac{z+w}{2}) = -\tfrac{i}{2}\overline{\tfrac{w+z}{2}}.$$

One can check that $\mathrm{op}_\Phi(a)$ defines a bounded operator both on $L^2_\Phi(\mathbb{C}^d)$ and the Fock space $\mathcal{F}(\mathbb{C}^d)$. Now, the Bargmann transform appears as the appropriate translation between real and complex Weyl quantization.

Lemma 9 *(see Theorem 13.9 from [53]) For any Schwartz function $a : \Lambda \to \mathbb{C}$ one has*

$$\mathcal{B}^* \mathrm{op}_\Phi(a) \mathcal{B} = \mathrm{op}(\kappa^\star a)$$

where $\kappa^\star$ denotes the pull-back by κ.

Note that Lemma 9 naturally extends to larger symbol classes, in particular to Shubin classes $\Gamma^m_\rho(\Lambda)$ that consist of functions a for which $\kappa^\star a \in \Gamma^m_\rho(\mathbb{R}^{2d})$, see also (33). We apply Lemma 9 to obtain an expansion of complex Weyl quantized operators in terms of $k, 0$-polyanalytic Toeplitz operators and, thus, provide a complex version of Proposition 3.

Theorem 5 *Let $N \in \mathbb{N}$, $\hbar > 0$ and assume $a \in \Gamma^m_\rho(\Lambda)$ for $m \in \mathbb{R}$, $\rho \geq 0$. Then, one has the approximation*

$$\Big\| \mathrm{op}_\Phi(a) - \sum_{j=0}^{N-1} (-1)^j C_{N-1,j} \sum_{\substack{k\in\mathbb{N}^d\\ |k|=j}} \mathcal{T}_{k,0}\big(\widehat{\kappa^\star a}\big) \Big\|_{\mathcal{F}^{m-2N\rho}(\mathbb{C}^d)\to\mathcal{F}(\mathbb{C}^d)} = O(\hbar^N).$$

where $\widehat{\kappa^\star a}(z) = \kappa^\star a(q, -p)$ with $z = q + ip \in \mathbb{C}^d$.

Proof We have to show that $\mathrm{op}_\Phi(a) : \mathcal{F}^{m-2N\rho}(\mathbb{C}^d) \to \mathcal{F}(\mathbb{C}^d)$ which then implies $\mathcal{P}\mathrm{op}_\Phi(a)\mathcal{P} = \mathrm{op}_\Phi(a)$ as an operator on $\mathcal{F}^{m-2N\rho}(\mathbb{C}^d)$. For any $u \in \mathcal{F}_{m-2N\rho}(\mathbb{C}^d)$, we can rewrite

$$\mathrm{op}_\Phi(a)u(z) = \int_{\mathbb{C}^d} K_a(w, z)u(w)\mathrm{d}w$$

with the Schwartz kernel

$$K_a(w, z) = (2\pi\hbar)^{-d} a(\tfrac{z+w}{2})\mathrm{e}^{(z-w)\overline{(z+w)}/4\hbar} f(w)$$

where

$$\frac{\mathrm{d}}{\mathrm{d}\overline{z}} K_a(w, z) = \frac{\mathrm{d}}{\mathrm{d}\overline{w}} K_a(w, z).$$

Then, the holomorphy of $\mathrm{op}_\Phi(a)u$ follows directly by the holomorphy of u since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\overline{z}}\mathrm{op}_\Phi(a)u(z) &= \frac{\mathrm{d}}{\mathrm{d}\overline{z}}\int_{\mathbb{C}^d} K_a(w,z)u(w)\mathrm{d}w \\ &= \int_{\mathbb{C}^d} \frac{\mathrm{d}}{\mathrm{d}\overline{w}} K_a(w,z)u(w)\mathrm{d}w \\ &= \int_{\mathbb{C}^d} K_a(w,z)(-\tfrac{\mathrm{d}}{\mathrm{d}\overline{w}}u)(w)\mathrm{d}w = 0.\end{aligned}$$

The appropriate decay of $\mathrm{op}_\Phi(a)u$ can be inferred from the intertwining property in Lemma 9 and the maping properties of usual Weyl quantized operators on modulation spaces. Finally, the approximation order follows from Corollary 2. □

7 Outlook

The concept of polyanalytic Toeplitz operators we propose in this paper appears quite straight-forward once written down and naturally exhibits all the favorable mapping qualities that are known from the analytic Bargmann setting. However, by the connection to short-time Fourier transforms and, via the spectrogram expansion, to Weyl operators this new concept allows to formulate profound transition and approximation formulas for the whole range of real and complex Weyl, Toeplitz as well as localization operators.

We believe that polyanalytic Toeplitz quantization might prove a useful concept in a variety of areas, including the deeper investigation and approximation of multiplexed signals, the analysis and generalization of complex quantization theories and a geometrically satisfying complex generalization of coherent state approximations and dynamics.

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Appendix A. Elements from Microlocal Analysis

The Bargmann transform considered in this paper is a special form of a so-called *Fourier-Bros-Iagolitzer transform* that are considered in microlocal analysis. In the analytic microlocal setup from [53, §13], the operator $\mathcal{B}$ corresponds to the Fourier-Bros-Iagolnitzer transform associated with the holomorphic quadratic phase θ defined in 34 and the corresponding strictly plurisubharmonic exponential weight function

$$\Phi(z) = -\max_{x\in\mathbb{R}^d} \operatorname{Im}\theta(z,x)$$

that gives rise to a weighted Hilbert space $L^2_\Phi(\mathbb{C}^d)$. For the Bargmann transform $\mathcal{B}$ considered in this paper the corresponding choice for Φ is

$$\Phi(z) = \tfrac{1}{4}|z|^2$$

and the corresponding induced norm on the weighted space L^2_Φ.

Appendix B. Moments of Special Hermite Functions

The Wigner transforms $\mathcal{W}_{\varphi_k}$ of Hermite functions are also known as special Hermite functions, see, e.g., [50]. Moments of these functions are of special interest as they resemble the quantum expectation values of quantized monomials in the kth harmonic oscillator eigenstate. That is,

$$\int_{\mathbb{R}^{2d}} z^\alpha \mathcal{W}_{\varphi_k}(z)\mathrm{d}z = \langle \varphi_k, \operatorname{op}(z^\alpha)\varphi_k\rangle$$

with standard multiindex notation, where $\alpha \in \mathbb{N}^{2d}$. As the Wignerfunctions of mulitdimensional Hermite functions factorize into 2-dimensional Wigner functions, in the following we only compute formulas for this case by proving Proposition 1.

Proof of Proposition 1 We start by computing

$$\begin{aligned}
\int_{\mathbb{R}^2} x^{2\alpha}\xi^{2\beta}\mathcal{W}_{\varphi_k}(x,\xi)\mathrm{d}x\mathrm{d}\xi &= (\pi\hbar)^{-1}\int_{\mathbb{R}^2} x^{2\alpha}\xi^{2\beta}\mathrm{e}^{-(x^2+\xi^2)/\hbar}(-1)^k L_k(\tfrac{2}{\hbar}(x^2+\xi^2))\mathrm{d}x\mathrm{d}\xi \\
&= \pi^{-1}\hbar^{\alpha+\beta}\int_{\mathbb{R}^2} x^{2\alpha}\xi^{2\beta}\mathrm{e}^{-(x^2+\xi^2)^2}(-1)^k L_k(2(x^2+\xi^2))\mathrm{d}x\mathrm{d}\xi \\
&= \pi^{-1}\hbar^{\alpha+\beta}\int_{\mathbb{R}^2} x^{2\alpha}\xi^{2\beta}\mathrm{e}^{-(x^2+\xi^2)^2}(-1)^k \sum_{j=0}^{k}\binom{k}{k-j}\frac{(-2(x^2+\xi^2))^j}{j!}\mathrm{d}x\mathrm{d}\xi = (*)
\end{aligned}$$

by using the Laguerre formula (25) for the Wigner function $\mathcal{W}_{\varphi_k}$. Expanding the polynomial in x and ξ by the binomial theorem and using the definition of the Gamma function we get

$$\begin{aligned}
(*) &= \pi^{-1}\hbar^{\alpha+\beta}(-1)^k\sum_{j=0}^{k}\frac{(-2)^j}{j!}\binom{k}{k-j}\sum_{n=0}^{j}\binom{j}{n}\int_{\mathbb{R}^2}\mathrm{e}^{-x^2-\xi^2}x^{2(n+\alpha)}\xi^{2(j-n+\beta)}\mathrm{d}x\mathrm{d}\xi \\
&= \pi^{-1}\hbar^{\alpha+\beta}(-1)^k\sum_{j=0}^{k}\frac{(-2)^j}{j!}\binom{k}{k-j}\sum_{n=0}^{j}\binom{j}{n}\Gamma(\tfrac{1}{2}+n+\alpha)\Gamma(\tfrac{1}{2}+j-n+\beta).
\end{aligned}$$

Finally, using binomial sum theorems for Gamma functions and the hypergeometric function ${}_2\mathrm{F}_1$ we compute

$$
\begin{aligned}
(*) &= \pi^{-1}\hbar^{\alpha+\beta}(-1)^k \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})}{\Gamma(\alpha+\beta+1)} \sum_{j=0}^{k} \frac{(-2)^j}{j!}\binom{k}{k-j}\Gamma(\alpha+\beta+1+j) \\
&= \pi^{-1}\hbar^{\alpha+\beta}(-1)^k\Gamma(\alpha+\tfrac{1}{2})\Gamma(\beta+\tfrac{1}{2}){}_2\mathrm{F}_1(\alpha+\beta+1,-k;1;2) \\
&= \hbar^{\alpha+\beta}(-1)^k\Gamma(\alpha+\tfrac{1}{2})\Gamma(\beta+\tfrac{1}{2}){}_2\mathrm{F}_1(\alpha+\beta+1,-k;1;2) \\
&= {}_2\mathrm{F}_1(\alpha+\beta+1,-k;1;2)(-1)^k\frac{\hbar^{|\alpha|}(2\alpha)!}{4^{|\alpha|}\alpha!}
\end{aligned}
$$

which completes the proof. □

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