Hysteretic Control Lyapunov Functions with Application to Global Asymptotic Tracking for Underwater Vehicles

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Abstract— **This paper introduces hysteretic control Lyapunov functions (HCLFs) for hybrid feedback control of a class of continuous-time systems. A family of HCLFs consists of local control Lyapunov functions defined on open domains, and include finite collections of open and closed sets that cover the state-space, implicitly defining a hysteresis-based switching mechanism. Given an HCLF family, we derive sufficient conditions for the existence of globally asymptotically stabilizing hybrid control laws. Moreover, we provide a constructive design procedure for synthesis of optimization-based feedback laws under mild conditions on the objective functions. We construct HCLFs for an underwater vehicle and demonstrate their applicability to hybrid control design for global asymptotic trajectory tracking for underwater vehicles.**

I. Introduction

Topological constraints prevent any continuous state feedback from globally asymptotically stabilizing dynamical systems defined on non-contractible state-spaces, such as mechanical systems with rotational degrees of freedom [1]. As shown in [2], these systems cannot be robustly globally asymptotically stabilized by discontinuous feedback either. However, hybrid feedback with inherent robustness to measurement noise can be employed to achieve global asymptotic stability of compact sets defined on such state-spaces. Examples include feedback derived from patchy control Lyapunov functions [3] or synergistic Lyapunov functions [4], [5]. Synergistic potential functions were introduced in [4] as a tool for achieving global asymptotic stability of rigid-body attitude, and generalized to synergistic Lyapunov functions in [5], [6]. Hybrid feedback based on synergistic potential or Lyapunov functions is related to the seminal work of Branicky [7], and utilizes a family of Lyapunov functions in combination with a hysteresis-based min-switching mechanism selecting the control action corresponding to the Lyapunov function of minimum value. Min-switching hybrid control strategies based on synergistic potential functions have been proposed for systems described on manifolds in [8] and [9].

Control Lyapunov functions (CLFs) are a powerful tool for constructive nonlinear control design, since they can be utilized to determine a stabilizing control law from Lyapunov inequalities [10], [11]. General control laws for stabilization of nonlinear systems using CLFs were first introduced in [12] through Sontag's universal formula, and later in [13]. The control law in [13] is notable in the sense that it pointwise minimizes the norm of the control input with respect to the CLF. More recently, CLFs have been extended to hybrid systems with and without disturbances in [14] and [15], respectively. However, for global asymptotic stabilization of dynamical systems defined on non-contractible state-spaces, there does not exist a continuously differentiable CLF [16].

In this paper, we present a new class of control Lyapunov functions for hybrid feedback control of continuous-time systems, referred to as hysteretic control Lyapunov functions (HCLFs). HCLFs include a hysteresis-based switching mechanism and result in a hybrid control law, transforming the continuous-time system into a hybrid control system. We show that the existence of a family of HCLFs satisfying the small control property implies global stabilizability of a compact set. The hybrid feedback consists of a collection of continuous feedback laws and a hysteresis-based switching mechanism. Moreover, we prove that optimization-based hybrid feedback laws can be constructed under minor assumptions on the objective functions. The collection of optimization-based feedback laws are continuous along flows, implying that the hybrid basic conditions hold such that the stability is robust in the sense of [17].

As a case study, we construct an HCLF family for tracking control of an underwater vehicle through a backstepping approach. The HCLF family is subsequently employed to synthesize a hybrid control law ensuring global asymptotic trajectory tracking. In contrast to traditional backstepping, we find the control input that pointwise minimizes a strictly convex objective function from the set-valued map of stabilizing control inputs defined by the HCLFs. The HCLF construction is reminiscent of the backstepping-based synergistic Lyapunov functions constructed for set-point regulation in [18]. However, we extend the work in [18] to the tracking problem in terms of HCLFs, and exploit inherent stabilizing nonlinear terms through online optimization.

This paper is organized as follows. Section II defines a family of hysteretic CLFs, and proves that a family of continuous feedback laws derived from the feasible set-valued map of control inputs defined by the HCLFs results in global asymptotic stability of any compact set. Then, Section III presents sufficient conditions for the existence of a family of continuous control selections from the feasible set-valued map. Given a collection of radially unbounded and strictly convex objective functions, we present an optimization-based hybrid feedback law that pointwise minimizes the objective functions subject to the stability constraints imposed by the

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HCLFs. Section IV derives the quaternion tracking error dynamics for an underwater vehicle, before a family of HCLFs are constructed in Section V. The HCLF family is subsequently employed for synthesis of a hybrid control law for global asymptotic configuration and velocity tracking. Then, Section VI verifies the theoretical developments through simulations, before Section VII presents our concluding remarks.

Notation

We denote by $\mathbb{R}$ the field of real numbers, by $\mathbb{R}_{\geq 0}$ the non-negative real numbers, and by $\mathbb{Z} \subset \mathbb{R}$ the integers. The Euclidean norm of $x \in \mathbb{R}^n$ is given by $|x| := (x^\mathsf{T} x)^{\frac{1}{2}}$, and $|x|_\mathcal{A} = \inf_{y\in\mathcal{A}} |x - y|$ is the distance of x to a set $\mathcal{A} \subset \mathbb{R}^n$. The unit n-sphere embedded in $\mathbb{R}^{n+1}$ is given by $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, and the closed ball of radius r in $\mathbb{R}^n$ is the set $r\mathbb{B} = \{x \in \mathbb{R}^n : |x| \leq r\}$. The special orthogonal group of order three is defined as $\mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times3} : R^\mathsf{T} R = I, \det R = 1\}$, and the skew-symmetric map that induces the right-handed cross product is written $[\,\cdot\,]_\times : \mathbb{R}^3 \to \mathfrak{so}(3) \subset \mathbb{R}^{3\times3}$. A double arrow denotes set-valued mappings, e.g. $F : \mathcal{X} \rightrightarrows \mathcal{U}$, where $\mathcal{X} \subset \mathbb{R}^n$ is the domain of the mapping (the set where the mapping is not empty-valued), and $\mathcal{U} \subset \mathbb{R}^m$ is its codomain (any set that contains all values F takes in its domain). The graph of F is the set defined as $\mathrm{gph}(F) := \{(x, u) \in \mathcal{X} \times \mathcal{U} : u \in F(x)\}$. A function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{PD}$ if it is zero at zero and positive otherwise. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing, and unbounded.

II. Hysteretic Control Lyapunov Functions

In this section, we define hysteretic control Lyapunov functions for the following class of continuous-time systems

$$\mathcal{N} : \begin{cases} \dot{x} = f(x, e, u) \\ \dot{e} \in c\mathbb{B} \end{cases} \quad (x, e, u) \in \mathcal{X} \times \mathcal{E} \times \mathcal{U}, \tag{1}$$

where $c > 0$, $u \in \mathcal{U}$ describes the input, and $e \in \mathcal{E}$ describes a known exogenous reference signal. It is assumed that $\mathcal{N}$ satisfies

(N1) the state space $\mathcal{X} \subset \mathbb{R}^n$ is closed;
(N2) the input space $\mathcal{U} \subset \mathbb{R}^m$ is closed and convex;
(N3) the exogenous reference space $\mathcal{E} \subset \mathbb{R}^k$ is compact;
(N4) the mapping $f : \mathcal{X} \times \mathcal{E} \times \mathcal{U} \to \mathbb{R}^n$ is continuous.

Systems of this form adequately describe a wide range of tracking problems for mechanical systems.

Definition 1 (HCLF Family)**.** *Let $\mathcal{A} \subset \mathcal{X}$ be compact and $\mathcal{R} \subset \mathbb{Z}$ be finite. A collection of functions $\{V_r\}_{r\in\mathcal{R}}$ is a family of hysteretic control Lyapunov functions for $(\mathcal{N}, \mathcal{A})$ with negativity margins $\{\sigma_r\}_{r\in\mathcal{R}}$, if there exists collections of sets $\{\mathcal{I}_r\}_{r\in\mathcal{R}}$, $\{\mathcal{O}_r\}_{r\in\mathcal{R}}$, and $\{\mathcal{M}_r\}_{r\in\mathcal{R}}$, class-$\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$, and a class-$\mathcal{PD}$ function ρ, such that*

(H1) *$\{\mathcal{I}_r\}_{r\in\mathcal{R}}$ covers $\mathcal{X}$, and for each $r \in \mathcal{R}$, $\mathcal{I}_r$ is closed in $\mathcal{X}$, $\mathcal{O}_r$ is open in $\mathcal{X}$, $\mathcal{M}_r$ is closed in $\mathcal{X}$, and $\mathcal{I}_r \subset \mathcal{O}_r \subset \mathcal{M}_r$;*
(H2) *for each $r \in \mathcal{R}$, V_r is continuously differentiable on an open set containing $\mathcal{M}_r$, and for all $x \in \mathcal{M}_r$,*

$$\underline{\alpha}(|x|_\mathcal{A}) \leq V_r(x) \leq \overline{\alpha}(|x|_\mathcal{A}); \tag{2}$$

(H3) *for all $(r, s) \in \mathcal{R} \times \mathcal{R}$ and all $x \in \mathcal{M}_r \setminus \mathcal{O}_r \cap \mathcal{I}_s$,*

$$V_s(x) < V_r(x); \tag{3}$$

(H4) *for all $(e, r) \in \mathcal{E} \times \mathcal{R}$ and all $x \in \mathcal{M}_r$, $\sigma_r : \mathcal{M}_r \times \mathcal{E} \to \mathbb{R}_{\geq 0}$ is continuous, $\sigma_r(x, e) \leq \rho(|x|_\mathcal{A})$, and*

$$\inf_{u\in\mathcal{U}} \nabla V_r(x)^\mathsf{T} f(x, e, u) + \sigma_r(x, e) \leq -\rho(|x|_\mathcal{A}). \tag{4}$$

A family of HCLFs for $(\mathcal{N}, \mathcal{A})$ is a tool for the design of a hybrid controller of the form

$$\mathcal{C} : \begin{cases} u = \kappa_r(x, e) & (x, e) \in \mathcal{C}_r \\ r^+ \in \mathcal{G}_r(x) & (x, e) \in \mathcal{D}_r, \end{cases} \tag{5}$$

where $\{\kappa_r\}_{r\in\mathcal{R}}$ is a collection of feedback control laws, and the flow set, jump set, and jump map are defined as

$$\mathcal{C}_r := \mathcal{M}_r \times \mathcal{E}, \tag{6}$$
$$\mathcal{D}_r := \mathcal{X} \setminus \mathcal{O}_r \times \mathcal{E}, \tag{7}$$
$$\mathcal{G}_r(x) := \{s \in \mathcal{R} : x \in \mathcal{I}_s \setminus \mathcal{O}_r\}, \tag{8}$$

respectively. Additionally, for a given $r \in \mathcal{R}$, we also write

$$\mathcal{B}_r := (\mathcal{A} \times \mathcal{E}) \cap \mathcal{C}_r. \tag{9}$$

Applying the hybrid controller $\mathcal{C}$ to the system $\mathcal{N}$, results in the hybrid closed-loop system of the form

$$\mathcal{H} : \begin{cases} \left.\begin{aligned} \dot{x} &= f(x, \kappa_r(x, e), e) \\ \dot{e} &\in c\mathbb{B} \end{aligned}\right\} & (x, e) \in \mathcal{C}_r \\ r^+ \in \mathcal{G}_r(x) & (x, e) \in \mathcal{D}_r. \end{cases} \tag{10}$$

For convenience of notation, the flow map (jump map) of a state in a hybrid system is omitted if it remains unchanged along flows (across jumps). When the the compact set $\mathcal{A} \times \mathcal{E} \times \mathcal{R}$ is globally asymptotically stable for the system $\mathcal{H}$, we shall say that $\mathcal{C}$ globally asymptotically stabilizes $\mathcal{A}$ for $\mathcal{N}$.

The definition of an HCLF family naturally leads to a collection of feasible set-valued mappings for the input. These mappings can, for each $r \in \mathcal{R}$, be defined as $F_r : \mathcal{C}_r \rightrightarrows \mathcal{U}$,

$$F_r(x, e) := \{u \in \mathcal{U} : \nabla V_r(x)^\mathsf{T} f(x, e, u) + \sigma_r(x, e) \leq 0\}. \tag{11}$$

The fact that the domain of F_r is $\mathcal{C}_r$ follows readily from (H4). Moreover, for each $r \in \mathcal{R}$ and all $(x, e) \in \mathcal{C}_r$, any input $u \in F_r(x, e)$ results in a rate of change of V_r at (x, e) less than or equal to $-\sigma_r(x, e)$ while flowing. The negativity margins should therefore be viewed as design parameters.

The following theorem proves that a selection of continuous feedback laws from the feasible set-valued mapping F_r renders $\mathcal{A}$ globally asymptotically stable for the system $\mathcal{N}$. This stability is robust to perturbations in the sense of [17, Definition 7.15], as seen from [17, Proposition 7.21].

Theorem 1. *Let $\{V_r\}_{r\in\mathcal{R}}$ be an HCLF family for $(\mathcal{N}, \mathcal{A})$ with negativity margins $\{\sigma_r\}_{r\in\mathcal{R}}$. If there exists a collection of feedback control laws $\{\kappa_r\}_{r\in\mathcal{R}}$ such that, for each $r \in \mathcal{R}$, $\kappa_r : \mathcal{C}_r \to \mathcal{U}$ is continuous, and for all $(x, e) \in \mathcal{C}_r$, $\kappa_r(x, e) \in F_r(x, e)$, then the controller $\mathcal{C}$ renders $\mathcal{A}$ globally asymptotically stable for $\mathcal{N}$.*

Proof. Let $\mathcal{H}^\dagger$ denote the hybrid system $\mathcal{H}$ with each jump set D_r replaced by $\hat{D}_r = C_r \cap D_r$. Since $\mathcal{R}$ is finite and each κ_r is continuous, it is straightforward to verify that $\mathcal{H}$ and $\mathcal{H}^\dagger$ satisfy the hybrid basic conditions [17, Assumption 6.5]. For each $r \in \mathcal{R}$ and all $(x,e) \in \hat{D}_r$, we find that $s \in G_r(x)$ implies $(x,e) \in C_s \setminus \hat{D}_s$. It now follows from [19, Lemma 2.7] that, for each bounded solution to $\mathcal{H}^\dagger$, there exists a positive scalar that bounds the time of flow after each jump from below. We now define a Lyapunov function candidate $V : (x,r) \mapsto V_r(x)$. From (H2), (H3), and the non-increase along flows by definition of the feasible set-valued mapping (11), uniform global stability of $\mathcal{A} \times \mathcal{E} \times \mathcal{R}$ for $\mathcal{H}^\dagger$, and hence boundedness of solutions, can be concluded. One may now apply [17, Proposition 3.27] to conclude global asymptotic stability of $\mathcal{A} \times \mathcal{E} \times \mathcal{R}$ for $\mathcal{H}^\dagger$. Solutions to $\mathcal{H}$ that are not solutions to $\mathcal{H}^\dagger$ are those with initial values (x^*, e^*, r^*) such that $r^* \in \mathcal{R}$ and $(x^*, e^*) \in D_{r^*} \setminus C_{r^*}$. Such solutions immediately jump from r^* to some $s \in G_{r^*}(x^*)$, after which they coincide with a solution to $\mathcal{H}^\dagger$ initiated in (x^*, e^*, s). It is therefore clear that $\mathcal{A} \times \mathcal{E} \times \mathcal{R}$ is globally asymptotically stable for $\mathcal{H}$. □

III. Hysteretic Feedback Control Design

Let $\{V_r\}_{r\in\mathcal{R}}$ be an HCLF family for $(\mathcal{N}, \mathcal{A})$ with negativity margins $\{\rho_r\}_{r\in\mathcal{R}}$. The following theorem provides sufficient conditions for the existence of a hybrid control law for $\mathcal{N}$ with inherent robustness properties.

Theorem 2 (Continuous Selection)**.** *Let $\{V_r\}_{r\in\mathcal{R}}$ be an HCLF family for $(\mathcal{N}, \mathcal{A})$ with negativity margins $\{\rho_r\}_{r\in\mathcal{R}}$. If it holds that,*

(C1) *for each $r \in \mathcal{R}$ and all $(x,e) \in C_r$, the mapping*

$$u \mapsto \nabla V_r(x)^\top f(x,e,u) + \rho_r(x,e), \qquad (12)$$

is convex on $\mathcal{U}$;

(C2) *there exists a collection of control laws $\{\kappa_r\}_{r\in\mathcal{R}}$, where for each $r \in \mathcal{R}$, $\kappa_r : C_r \to \mathcal{U}$ is continuous and the set-valued mapping $\hat{F}_r : C_r \rightrightarrows \mathcal{U}$ defined by*

$$\hat{F}_r(x,e) := \begin{cases} \kappa_r(x,e) , & \text{if } (x,e) \in B_r \\ F_r(x,e), & \text{if } (x,e) \in C_r \setminus B_r, \end{cases} \qquad (13)$$

is lower semicontinuous for all $(x,e) \in B_r$,

then there exists a collection of feedback control laws $\{\kappa_r\}_{r\in\mathcal{R}}$ such that, for each $r \in \mathcal{R}$, $\kappa_r : C_r \to \mathcal{U}$ is continuous, and the hybrid controller $\mathcal{C}$ renders the set $\mathcal{A}$ globally asymptotically stable for the system $\mathcal{N}$.

Proof. Since f is continuous, each ∇V_r and ρ_r are continuous, and each C_r is closed, it follows from [13, Corollary 2.13] that each $F_r^\circ : C_r \setminus B_r \rightrightarrows \mathcal{U}$ defined as

$$F_r^\circ(x,e) = \left\{ u \in \mathcal{U} : \nabla V_r(x)^\top f(x,e,u) + \rho_r(x,e) < 0 \right\},$$

is lower semicontinuous. From (C1), [20, Theorem 7.6], and the fact that taking closures preserves lower semicontinuity, it follows that for each $r \in \mathcal{R}$ and all $(x,e) \in C_r \setminus B_r$

$$\begin{aligned} \overline{F_r^\circ(x,e)} &= \left\{ u \in \mathcal{U} : \nabla V_r(x)^\top f(x,e,u) + \rho_r(x,e) \le 0 \right\} \\ &= F_r(x,e), \end{aligned}$$

is closed-convex-valued and lower semicontinuous. Now, it follows from (C2) that each $\hat{F}_r$ is lower semicontinuous. Then, the Michael selection theorem [13, Theorem 2.18] implies the existence of a collection of functions $\{\hat{\kappa}_r\}_{r\in\mathcal{R}}$ such that $\hat{\kappa}_r : C_r \to \mathcal{U}$ is continuous and $\hat{\kappa}_r(x,e) \in \hat{F}_r(x,e)$ for each $r \in \mathcal{R}$ and all $(x,e) \in C_r$. The rest of the proof follows from Theorem 1 because $\hat{F}_r(x,e) \subset F_r(x,e)$ for each $r \in \mathcal{R}$ and all $(x,e) \in C_r$. □

Condition (C1) always holds when the mapping $u \mapsto f(x,e,u)$ is affine for all $(x,e) \in \mathcal{X} \times \mathcal{E}$. Additionally, (C2) is recognized as the the small control property [13].

Theorem 2 implies the existence of a collection of continuous control laws rendering the compact set $\mathcal{A}$ globally asymptotically stable for the system $\mathcal{N}$. However, it is neither constructive nor optimal. The following theorem enables us to take continuous selections from $\hat{F}_r(x,e)$ minimizing a specified objective function.

Theorem 3 (Optimal Selection)**.** *Let $\{V_r\}_{r\in\mathcal{R}}$ be an HCLF family for $(\mathcal{N}, \mathcal{A})$ with negativity margins $\{\rho_r\}_{r\in\mathcal{R}}$ satisfying the assumptions of Theorem 2. If $\{h_r\}_{r\in\mathcal{R}}$ is a collection of functions satisfying,*

(O1) *for each $r \in \mathcal{R}$ and for all $(x,e) \in C_r$, $h_r : \mathcal{X} \times \mathcal{E} \times \mathcal{U} \to \mathbb{R}_{\geq 0}$ is continuous, and also strictly convex in its third argument;*

(O2) *there exist class-$\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$ such that, for each $r \in \mathcal{R}$ and for all $(x,e) \in C_r$,*

$$\underline{\alpha}(|u - \kappa_r(x,e)|) \le h_r(x,e,u) \le \overline{\alpha}(|u - \kappa_r(x,e)|), \quad (14)$$

then there exists a family of feedback control laws $\{\kappa_r\}_{r\in\mathcal{R}}$, such that for each $r \in \mathcal{R}$, $\kappa_r : C_r \to \mathcal{U}$ is continuous and defined by

$$\kappa_r(x,e) = \operatorname*{arg\,min}_{u\in\hat{\mathcal{F}}_r(x,e)} h_r(x,e,u), \qquad (x,e) \in C_r, \qquad (15)$$

such that the hybrid control law $\mathcal{C}$, renders the set $\mathcal{A}$ globally asymptotically stable for the system $\mathcal{N}$.

Proof. Theorem 2 establishes that each F_r is nonempty, closed-convex-valued and lower semicontinuous for all $(x,e) \in C_r \setminus B_r$. Additionally, F_r is upper semicontinuous as it is closed-valued and $\mathcal{U}$ is closed [21, Example 5.8]. Hence, by [22, Proposition 5.2.18], $\operatorname{gph} F_r$ is closed for each $r \in \mathcal{R}$. Then, (14) and continuity of each h_r on $\operatorname{gph} F_r$, implies that for every compact set $K \subset C_r \setminus B_r$ and all $\gamma \in \mathbb{R}$, the sets

$$\left\{ f(x,e,u) : (x,e) \in K, u \in F_r(x,e), h_r(x,e,u) \le \gamma \right\},$$

are compact. By [23, Theorem 1.4], each function

$$c_r(x,e) = \min_{u\in\mathcal{F}_r(x,e)} h_r(x,e,u), \quad (x,e) \in C_r \setminus B_r,$$

is continuous, and each set-valued mapping $P_r : C_r \setminus B_r \rightrightarrows \mathcal{U}$ of minimal solutions, defined as

$$P_r(x,e) := \operatorname*{arg\,min}_{u\in\mathcal{F}_r(x,e)} h_r(x,e,u),$$

is upper semicontinuous and compact-valued for each $r \in \mathcal{R}$. Now, F_r is nonempty and closed-convex-valued for each

$r \in \mathcal{R}$, for every $(x,e) \in C_r$, and each function h_r is strictly convex in u for all $u \in F_r(x,e)$. It follows from [21, Theorem 2.6] that P_r is single-valued, such that it is possible to set $\kappa_r(x,e) := P_r(x,e)$ for each $r \in \mathcal{R}$ and for all $(x,e) \in C_r \setminus B_r$. Consequently, by [21, Corollary 5.20], κ_r is continuous for all $(x,e) \in C_r \setminus B_r$ as it is upper semicontinuous in the sense of a set-valued mapping.

To show continuity of each κ_r in B_r, rewrite (15) as

$$\kappa_r(x,e) := \begin{cases} \gamma_r(x,e) & \text{if } (x,e) \in B_r \\ \displaystyle\arg\min_{u \in \mathcal{F}_r(x,e)} h_r(x,e,u) & \text{if } (x,e) \in C_r \setminus B_r, \end{cases}$$

which follows from (14) and the fact that $F_r(x,e) = U$ when $(x,e) \in B_r$. Now, $\hat{F}_r(x,e)$ is lower semicontinuous for all $(x,e) \in C_r$ by (C2). Therefore, there exists a family of continuous selections $\phi_r(x,e) \in \hat{F}_r(x,e)$ with $\phi_r(x,e) = \gamma_r(x,e)$, for all $(x,e) \in B_r$. It follows from (14) that for each $r \in \mathcal{R}$, and for all $(x,e) \in C_r$,

$$0 \le |\kappa_r(x,e) - \gamma_r(x,e)| \le |\phi_r(x,e) - \gamma_r(x,e)|. \tag{16}$$

From continuity of each ϕ_r and γ_r it follows that each κ_r is continuous. The rest of the proof follows from Theorem 1. □

IV. Trajectory Tracking for Underwater Vehicles

In the remaining part of the paper we will illustrate how the results of the previous sections can be applied. Specifically, we will construct a family of HCLFs and synthesize a hybrid control law ensuring global asymptotic tracking for an underwater vehicle. This section provides kinematic and dynamic models of an underwater vehicle, before the tracking error dynamics are derived.

A. Kinematics

The position and attitude of a rigid underwater vehicle are uniquely described by a vector $p \in \mathbb{R}^3$ specifying the position of the body frame origin with respect to the inertial frame origin, and a rotation matrix $R \in \mathrm{SO}(3)$ specifying the body frame axes projected onto the inertial frame axes. The rate of change of these quantities is related to the linear and angular body velocities, $v \in \mathbb{R}^3$ and $\omega \in \mathbb{R}^3$, respectively, by

$$\begin{aligned} \dot{p} &= Rv \\ \dot{R} &= R[\omega]_\times. \end{aligned} \tag{17}$$

It is well-known that no three-parameter parametrization of $\mathrm{SO}(3)$ is globally non-singular [24], which is why a four-parameter unit quaternion representation is often preferred for control design. A unit quaternion is written as a vector $q = (\eta, \epsilon) \in \mathbb{S}^3$, where $\eta \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^3$, respectively describe the real and imaginary component of the quaternion. Any unit quaternion maps to a rotation matrix through the surjective map $R\colon \mathbb{S}^3 \to \mathrm{SO}(3)$ defined by

$$R(q) := I_3 + 2\eta[\epsilon]_\times + 2[\epsilon]_\times^2. \tag{18}$$

The quaternion kinematic equation is given by

$$\dot{q} = T(q)\omega, \tag{19}$$

where $T\colon \mathbb{S}^3 \to \mathbb{R}^{4\times 3}$ is defined by

$$T(q) := \frac{1}{2}\begin{bmatrix} -\epsilon^\top \\ \eta I_3 + [\epsilon]_\times \end{bmatrix}. \tag{20}$$

Let $q_d = (\eta_d, \epsilon_d) \in \mathbb{S}^3$ represent the desired quaternion. The error quaternion corresponding to $R(\tilde{q}) = \tilde{R} = R_d^\top R$ is

$$\tilde{q} = q_d^{-1} \otimes q = (\tilde{\eta}, \tilde{\epsilon}), \tag{21}$$

where $\otimes$ denotes the quaternion product. Note that the map defined in (18) is not injective, since it maps unit quaternions representing antipodal points in $\mathbb{S}^3$ to the same element in $\mathrm{SO}(3)$. Hence, the set of unit quaternions corresponding to $R(\tilde{q}) = I_3$ is $\tilde{q} = \pm e_1 = \pm(1,0,0,0)$.

Defining $\eta := (p,q) \in \mathbb{R}^3 \times \mathbb{S}^3$, and collecting the linear and angular velocities in the vector $\nu = (v, \omega) \in \mathbb{R}^6$ results in the kinematic equation

$$\dot{\eta} = \begin{bmatrix} R(q) & 0_{3\times 3} \\ 0_{4\times 3} & T(q) \end{bmatrix}\nu := \mathcal{T}(q)\nu. \tag{22}$$

B. Dynamics

The dynamics of an underwater vehicle is modeled as [25]

$$M\dot{\nu} + F(\nu)\nu + g(R) = Bu, \tag{23}$$

where $M \in \mathbb{R}^{6\times 6}$ is the inertia matrix, including hydrodynamic mass, $F\colon \mathbb{R}^6 \to \mathbb{R}^{6\times 6}$ describes velocity dependent inertia and damping terms, $g\colon \mathrm{SO}(3) \to \mathbb{R}^6$ comprises the acting weight and buoyancy forces, $B \in \mathbb{R}^{6\times m}$ is the actuator configuration matrix and $u \in U = \mathbb{R}^m$ is the vector of actuator control inputs. We make the following assumptions on these quantities

(A1) $M = M^\top = \begin{bmatrix} M_1 & M_2 \\ M_2^\top & M_3 \end{bmatrix} > 0$;

(A2) *F and g are continuous;*

(A3) *the actuator configuration matrix B has full rank;*

C. Tracking Error Dynamics using Quaternions

A bounded reference trajectory for the vehicle configuration, velocity and acceleration is generated from the exogenous system

$$\left.\begin{aligned} \dot{p}_d &= R_d v_d \\ \dot{R}_d &= R_d[\omega_d]_\times \\ \dot{\nu}_d &= e \\ \dot{e} &\in c\mathbb{B} \end{aligned}\right\} \; (p_d, R_d, \nu_d, e) \in \Omega_1 \times \mathrm{SO}(3) \times \Omega_2 \times E, \tag{24}$$

where $c > 0$, and $\Omega_1 \subset \mathbb{R}^3$, $\Omega_2 \subset \mathbb{R}^6$ and $E \subset \mathbb{R}^6$ are compact. Let $\eta_d = (p_d, q_d) \in \Omega_1 \times \mathbb{S}^3 := \Omega$ represent the desired position vector and unit quaternion, and define the configuration error by $\tilde{\eta} := (\tilde{p}, \tilde{q}) \in \mathbb{R}^3 \times \mathbb{S}^3 := \mathcal{E}$, where $\tilde{p} = R(q_d)^\top(p - p_d)$ is the natural position error. The error kinematics are given by

$$\dot{\tilde{\eta}} = \begin{bmatrix} R(\tilde{q}) & 0_{3\times 3} \\ 0_{4\times 3} & T(\tilde{q}) \end{bmatrix}\tilde{\nu} := \mathcal{T}(\tilde{q})\tilde{\nu}, \tag{25}$$

where $\tilde\nu = \nu - H(\tilde\varphi)\nu_d$ is the body velocity error, and $H : \mathcal{E} \to \mathbb{R}^{6\times 6}$ is defined by

$$H(\tilde\varphi) := \begin{bmatrix} R(\tilde q)^T & R(\tilde q)^T[\tilde p]_\times \\ 0_{3\times 3} & R(\tilde q)^T \end{bmatrix}. \tag{26}$$

The error dynamics are then given by

$$\begin{aligned} \dot{\tilde\nu} &= M^{-1}(Bu - F(\nu) - g(q)) - H(\tilde\varphi)\dot\nu_d - \dot H(\tilde\varphi,\tilde\nu)\nu_d \\ &= M^{-1}Bu + f(\tilde\varphi,\tilde\nu,e). \end{aligned} \tag{27}$$

Defining the extended state-space

$$\mathcal{X} := \mathcal{E} \times \mathbb{R}^6 \times \mathcal{R}_2, \tag{28}$$

with state vector $x = (\tilde\xi, \tilde\nu, \varphi, r) \in \mathcal{X}$, results in the following quaternion representation of the kinematic and dynamic tracking error equations

$$\mathcal{N}: \begin{cases} \dot x = f(x,e,u) \\ e \in c\mathcal{B} \end{cases} \quad (x,e,u) \in \mathcal{X}\times\mathcal{E}\times\mathcal{U}, \tag{29}$$

where the continuous map $f : \mathcal{X}\times\mathcal{E}\times\mathcal{U} \to \mathbb{R}^{26}$ is given by

$$f(x,e,u) = \begin{pmatrix} T(\tilde q)\tilde\nu \\ f(\tilde\varphi,\tilde\nu,e) + M^{-1}Bu \\ T(q)\nu \\ e \end{pmatrix}. \tag{30}$$

The tracking control objective is global asymptotic stabilization of the compact set

$$\begin{aligned} \mathcal{A} &= \{x \in \mathcal{X} : \tilde p = 0, R(\tilde q) = I_3, \tilde\nu = 0\} &(31)\\ &= \{x \in \mathcal{X} : \tilde p = 0, \tilde q = \pm e_1, \tilde\nu = 0\}. &(32) \end{aligned}$$

V. HCLF-BASED HYBRID CONTROL DESIGN

This section constructs HCLFs for trajectory tracking of an underwater vehicle. The HCLFs are subsequently employed to synthesize an optimization-based hybrid feedback control law.

Consider the candidate family of kinematic HCLFs

$$V_{r,1}(\tilde\xi) = \underbrace{2k(1 - r\tilde\eta)}_{V_r(\tilde q)} + \frac{1}{2}\tilde p^T K_p \tilde p, \tag{33}$$

Differentiating (33) along the error kinematics yields

$$\begin{aligned} \langle \nabla V_{r,1}(\tilde\xi), T(\tilde q)\tilde\nu\rangle &= kr\tilde\epsilon^T\tilde\omega + \tilde p^T K_p\left(R(\tilde q)v - v_d - [\omega_d]_\times \tilde p\right) \\ &= \tilde\psi^T K_\psi G_r(\tilde q)^T \tilde\nu, \end{aligned} \tag{34}$$

where $\tilde\psi = (\tilde p, \tilde\epsilon)$ and

$$K_\psi = \begin{bmatrix} K_p & 0_{3\times3} \\ 0_{3\times3} & kI_3 \end{bmatrix}, \quad G_r(\tilde q)^T = \begin{bmatrix} R(\tilde q) & 0_{3\times3} \\ 0_{3\times3} & rI_3 \end{bmatrix}. \tag{35}$$

Define the backstepping variable

$$z := \tilde\nu - \alpha_r(\tilde\psi), \tag{36}$$

and rewrite (34) as

$$\langle \nabla V_{r,1}(\tilde\xi), T(\tilde q)\tilde\nu\rangle = \tilde\psi^T K_\psi G_r(\tilde q)^T \alpha_r(\tilde\psi) + \tilde\psi^T K_\psi G_r(\tilde q)^T z.$$

The stabilizing function $\alpha_r(\tilde\psi)$ for $\tilde\nu$ is chosen as

$$\alpha_r(\tilde\psi) = -G_r(\tilde q)\tilde\psi. \tag{37}$$

Since $r^2 = 1$ for all $r \in \mathcal{R}$, it holds that $G_r(\tilde q)^T G_r(\tilde q) = I$, which results in

$$\langle \nabla V_{r,1}(\tilde\xi), T(\tilde q)\tilde\nu\rangle = -\tilde\psi^T K_\psi \tilde\psi + \tilde\psi^T K_\psi G_r(\tilde q)^T z. \tag{38}$$

Augmenting $V_{r,1}$ with a positive definite term in z yields

$$V_r(x) = V_{r,1}(\tilde\xi) + \frac{1}{2}z^T M z, \tag{39}$$

which has compact sublevel sets and is positive definite with respect to the compact set

$$\mathcal{A} = \{(x,r) \in \mathcal{X}\times\mathcal{R} : \tilde p = 0, \tilde q = re_1, z = 0\}. \tag{40}$$

Differentiating V_r along flows yields

$$\begin{aligned} \langle \nabla V_r(x), f(x,e,u)\rangle &= -\tilde\psi^T K_\psi \tilde\psi \\ &+ z^T\Big[G_r(\tilde q)K_\psi\tilde\psi + M\dot G_r(\tilde q,\tilde\omega)\tilde\psi + MG_r(\tilde q)ST(\tilde q)\tilde\nu \\ &+ Bu - F(\nu) - g(q) - M\left[H(\tilde\varphi)e + \dot H(\tilde\varphi,\tilde\nu)\nu_d\right]\Big], \end{aligned} \tag{41}$$

where

$$\dot G_r(\tilde q,\tilde\omega) = \begin{bmatrix} -[\tilde\omega]_\times R(\tilde q)^T & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}, \tag{42}$$

$$S = \begin{bmatrix} I_{3\times3} & 0_{3\times4} \\ 0_{3\times4} & I_{3\times3} \end{bmatrix}, \quad \dot{\tilde\psi} = ST(\tilde q)\tilde\nu. \tag{43}$$

Note that the set $\mathcal{R}$ only consists of two elements. Thus, the only possible switching strategy for the logic variable is $r^+ = -r$. In order to derive the sets $\{\mathcal{I}_r\}_{r\in\mathcal{R}}$, $\{\mathcal{O}_r\}_{r\in\mathcal{R}}$ and $\{\mathcal{M}_r\}_{r\in\mathcal{R}}$, defining the flow and jump sets, we calculate the change in V_r along jumps as

$$\begin{aligned} (V_{r^+} - V_r)(x) &= 4kr\tilde\eta + \tilde\nu^T M\left(G_{r^+}(\tilde q) - G_r(\tilde q)\right)K_\psi\tilde\psi \\ &+ \frac{1}{2}\tilde\psi^T K_\psi\left(G_{r^+}(\tilde q)^T M G_{r^+}(\tilde q) - G_r^T(\tilde q)MG_r(\tilde q)\right)K_\psi\tilde\psi, \end{aligned}$$

where

$$G_{r^+}(\tilde q) - G_r(\tilde q) = -2rI \tag{44}$$

$$G_{r^+}(\tilde q)^T M G_{r^+}(\tilde q) - G_r(\tilde q)^T M G_r(\tilde q) = -2r\Gamma(\tilde q), \tag{45}$$

and

$$I = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & I_3 \end{bmatrix}, \quad \Gamma(\tilde q) = \begin{bmatrix} 0_{3\times3} & R(\tilde q)M_2 \\ M_2^T R(\tilde q)^T & 0_{3\times3} \end{bmatrix}. \tag{46}$$

Let

$$\rho(x) := \tilde\eta - \frac{1}{2k}\tilde\nu^T M I K_\psi\tilde\psi - \frac{1}{4k}\tilde\psi^T K_\psi\Gamma(\tilde q)K_\psi\tilde\psi, \tag{47}$$

such that $V_{r^+}(x) - V_r(x) = 4kr\rho(x)$. Define the sets

$$\begin{aligned} \mathcal{I}_r &:= \{x \in \mathcal{X} : r\rho(x) \ge 0\}, \\ \mathcal{O}_r &:= \{x \in \mathcal{X} : r\rho(x) < \delta\}, \\ \mathcal{M}_r &:= \{x \in \mathcal{X} : r\rho(x) \ge \delta\}, \end{aligned} \tag{48}$$

where $\delta \in (0,1)$ is the hysteresis half-width. The flow and jump sets can now be constructed according to (6) and (7), respectively. Then, (H3) holds (strictly) by construction of $\mathcal{O}_r$ and $\mathcal{I}_r$ since $V_{r^+}(x) - V_r(x) = 4kr\rho(x) < 0$ for all $r \in \mathcal{R}$ and all $x \in (\mathcal{M}_r \setminus \mathcal{O}_r) \cap \mathcal{I}_{r^+}$. From (41), it is straightforward to verify that each V_r satisfies (H4) with $\sigma_r(x) = \tilde\psi^T K_\psi\tilde\psi +$

$z^T K_z z$ for all $x \in M_r$ and some $K_z = K_z^T > 0$. Hence, by De nition 1, $\{V_r\}_{r \in R}$ is a family of HCLFs for $(N; A)$.

In order to use Theorem 3 to synthesize an optimization-based hybrid control law, consider the set-valued map F_r de ned in (13). We choose $_r$ such that it renders A forward invariant, i.e. that $\underline{\ } = 0$ when $(x; r) \in A$. Inspection of (27) yields

$$_r(x; e) = B^y (F( ) + g(q) + Me); \quad (49)$$

where $B^y \in R^{m \; 6}$ is the Moore-Penrose inverse of B. In order to show lower semicontinuity of F_r, consider the continuous feedback control law

$$_r(x; e) = B^y \; F( ) + g(q) + M \; H('~)e + H(';~ ~)$$

$$MG_r(e)ST(e)~ \; MG(e;!~)\# \; G_r(e)K_{\#}\# \; K_z z \; ; \quad (50)$$

which results in

$$\nabla V_r(x)^T f(x; e; \; _r(x; e)) + \; _r(x) = 0; \quad (51)$$

for all $(x; e) \in C_r$. Hence, $_r(x; e) \in F_r(x; e)$ for all $(x; e) \in C_r \setminus B_r$. Moreover, for $(x; e) \in B_r$ it holds that

$$_r(x; e) = B^y (F( ) + g(q) + Me) = \; _r(x; e): \quad (52)$$

Consequently, $_r(x; e)$ is a continuous single-valued selection of F_r since $_r(x; e) \in F_r(x; e)$ for all $(x; e) \in C_r$. By [26, Proposition 2.2], F_r is lower semicontinuous for all $(x; e) \in C_r$. Hence, by de ning the objective function

$$h_r(x; e; u) := |u \; _r(x; e)|^2; \quad (53)$$

all of the conditions in Theorem 3 are satis ed. Consequently, the set A can be rendered globally pre-asymptotically stable for the system N by the hybrid control law

$$C: \begin{cases} u = \; _r(x; e) & (x; e) \in C_r \\ r^+ = \; r & (x; e) \in D_r; \end{cases} \quad (54)$$

where $_r : C_r \to U$ is obtained from the quadratic program

$$_r(x; e) = \arg\min_{u \in R^m} \; u^T u \; \; 2u^T \; _r(x; e)$$

subject to (55)

$$z^T \; Bu + M\hat{f}_r(x) + G_r(e)K_{\#}\# + K_z z \; \; 0;$$

and where

$$\hat{f}_r(x) = f(';~ ~; e) + G(e;!~)\# + G_r(e)ST(e)~: \quad (56)$$

Proposition 1. The hybrid control law (54) renders the compact set A de ned in (32) globally asymptotically stable for the system (29).

Proof. We have shown that the HCLF family $\{V_r\}_{r \in R}$ de ned in (39) together with the collection of objective functions $\{h_r\}_{r \in R}$ satisfy the conditions of Theorem 3. It follows that the hybrid control law (54)-(55) renders the compact set A de ned in (40) globally asymptotically stable. Observe from (36) and (37) that A is equivalent to A, de ned in (32), which implies that the control law (54) results in global asymptotic stability of A. □

Fig. 1. The position p, the desired position p and the unit quaternion orientation error $q = (~; ~)$.

VI. NUMERICAL SIMULATION

In this section, we verify the theoretical results in simulation for the 6-DOF underwater vehicle ODIN, we refer to [27] for the model parameters. The system is initialized at the con guration $'_0 = (0_{3\;1}; \; _0)$, $_0 = \frac{1}{\sqrt{50}}(3; 4; 5)$, with the initial velocity $_0 = (0_{3\;1}; 1{:}2 \; _0)$. The desired position and orientation is obtained from the exogenous system in (24), initialized at $p = 0$, $R = I$. The desired acceleration e is generated from the low-pass lter

$$T\underline{e} + e = e_r; \quad (57)$$

with time constant $T = 15$ s and the reference acceleration

$$e_r = \begin{cases} (0{:}1; 0{:}06; \; 0{:}07; 0; 0; 0); & 0 \; t \; 5 \\ 0_{6\;1}; & 5 < t \; 10 \\ (0_{3\;1}; 0{:}05; \; 0{:}1; 0{:}02) & 10 < t \; 15 \\ (0_{4\;1}; \; 0{:}1; 0{:}02) & t > 15: \end{cases} \quad (58)$$

The control gains are chosen as $k = 1$; $K_p = I_3$ and $K_z = \frac{1}{2} I_6$. The system is simulated with Simulink, using the ode15 solver with a maximum step-size of 0.01. Simulation results are presented in Figs. 1 to 3. Observe that the only jump occurs at $t = 0$, which is due to the initial angular velocity. Moreover, note that the control input is continuous for all $t > 0$. To emphasize the necessity of (14) for continuity of the control law along ows, Fig. 4 depicts the control inputs for the same control scenario with $h = |u|^2$, which clearly does not satisfy (14). From Fig. 4, it is apparent that the control input exhibits signi cant discontinuities for $t > 20$ s, despite the fact that no jumps occur as observed from the logic variable r.

In order to highlight the bene ts of the optimization-based control law obtained from (55), Figs. 5 to 7 depict simulation results for the same control scenario using $u = \; _r(x; e)$ given by (50). From Figs. 1, 2, 5 and 6, it is clear that the optimization-based control law achieves faster convergence to the desired orientation with less control effort.

In Fig. 8, we compare our HCLF approach with the local CLF $V = 2(1 \; ~)$, corresponding to $r \; 1$. The system con guration is initialized at $'_0 = (0_{3\;1}; q_0)$, $q_0 =$

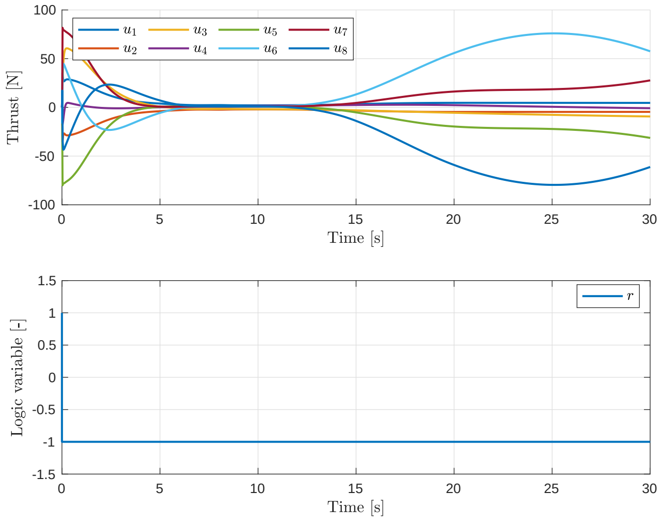

Fig. 2. The thruster control inputs u, and the logic variable r.

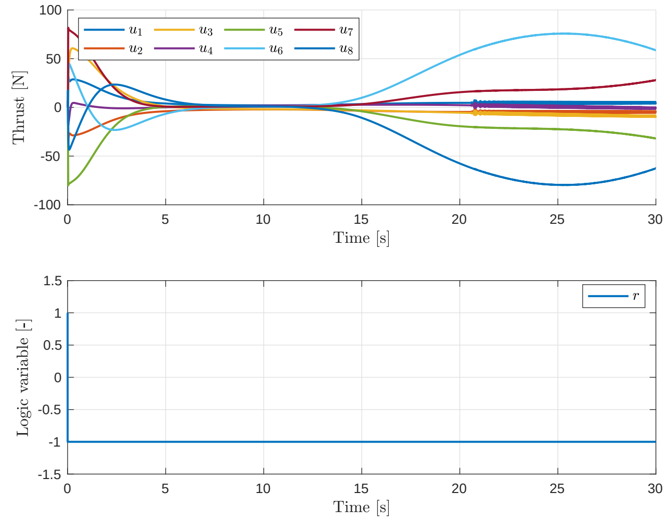

Fig. 4. The thruster control inputs u with $h_r = |u|^2$.

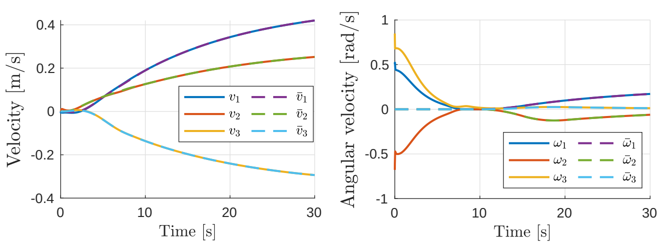

Fig. 3. The linear and angular velocities v and ω, and their desired values $\bar{v}$ and $\bar{\omega}$, respectively.

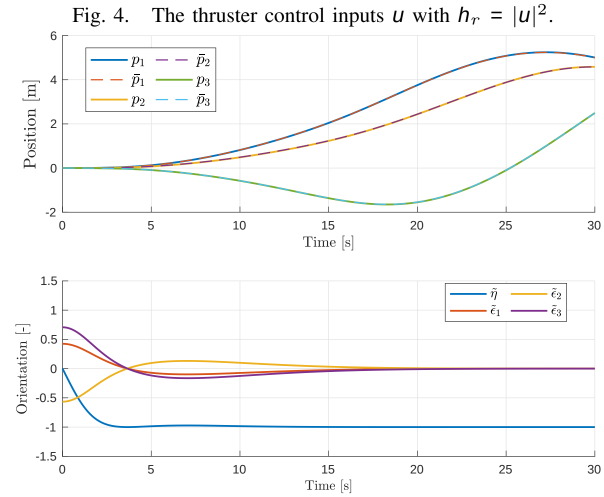

Fig. 5. The position p, the desired position $\bar{p}$ and the unit quaternion orientation error $\tilde{q} = (\tilde{\eta}, \tilde{\epsilon})$ using (50).

$(-0.95, 0, \sqrt{1-0.95^2}, 0)$ and the desired configuration is initialized at $\bar{p} = 0, \bar{R} = I$. Since $V = 0$ if and only if $\tilde{\eta} = 1$, the control law synthesized from V stabilizes $q = +e_1$ and leaves $q = -e_1$ unstable, despite the fact that both points correspond to the same physical rotation [28]. This can be observed in Fig. 8, where the control law (unnecessarily) performs a full rotation of the rigid body. A naïve solution to this problem is to employ the CLF $V = 2(1 - |\tilde{\eta}|)$ with the goal of rendering $q = \pm e_1$ asymptotically stable. However, this leads to a discontinuous control law with no robustness to measurement noise. In fact, it can be shown that arbitrarily small measurement noise can destroy any global attractivity property [29].

Another well-known CLF, albeit local, is $V = 2(1 - \tilde{\eta}^2)$, which achieves almost global asymptotic stability of the set $\{q : q = \pm e_1\}$. However, since the gradient of V vanishes at $\tilde{\eta} = 0$, control laws synthesized from this CLF exhibit poor convergence properties around $\tilde{\eta} = 0$. This is demonstrated through simulation in Fig. 9.

VII. Conclusions

This paper has presented a new class of control Lyapunov functions, referred to as hysteretic control Lyapunov functions (HCLFs). We have stated sufficient conditions for the existence of a collection of continuous feedback laws, which together with the hysteresis-based switching mechanism defined by the HCLFs lead to a hybrid feedback law. This hybrid feedback law globally asymptotically stabilizes compact sets for a class of continuous-time systems defined on state-spaces that are not necessarily contractible. Moreover, we have shown how a collection of optimization-based feedback laws can be derived from a family of HCLFs under mild assumptions on the objective function. As a result, HCLFs can serve as a tool for synthesis of optimal feedback laws ensuring global asymptotic tracking of spatial rigid-bodies such as underwater vehicles and satellites. Finally, we have derived a family of HCLFs for configuration and velocity control of an underwater vehicle through backstepping, and synthesized a globally asymptotically stabilizing optimization-based control law from the derived HCLFs.

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