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RESEARCH PAPER

A note on the zeros of approximations of the Ramanujan Ξ-function

Andrés Chirre[1] · Oswaldo Velásquez Castañón[2]

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Abstract
In this paper we review the study of the distribution of the zeros of certain approximations for the Ramanujan Ξ-function given by Ki (Ramanujan J 17(1):123–143, 2008), and we provide new proofs of his results. Our approach is motivated by the ideas of Velásquez (J Anal Math 110:67–127, 2010) in the study of the zeros of certain sums of entire functions with some condition of stability related to the Hermite–Biehler theorem.

Keywords Ramanujan zeta function · Riemann hypothesis · Zeros of approximations of the Ramanujan Ξ-function · Distribution of zeros of entire functions

Mathematics Subject Classification 11M26 · 30D10

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✉ Andrés Chirre
carlos.a.c.chavez@ntnu.no

Oswaldo Velásquez Castañón
oswaldo@imca.edu.pe

1 Department of Mathematical Sciences, Norwegian University of Science and Technology, 7491 Trondheim, NO, Norway

2 Instituto de Matemática y Ciencias Afines, Universidad Nacional de Ingeniería, Lima, Peru

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1 Introduction

1.1 Background

Let $\tau(n)$ be the Ramanujan's tau-function, defined by

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n = z \prod_{n=1}^{\infty} (1 - q^n)^{24},$$

where $q = e^{2\pi i z}$, and $\operatorname{Im} z > 0$. It is well known that $\Delta(z)$ spans the space of cusp forms of dimension -12 associated with the unimodular group. The associated Dirichlet series and Euler product for $\Delta(z)$ are given by

$$L(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_{p} \left(1 - \tau(p) p^{-s} + p^{11-2s}\right)^{-1},$$

where the series and the product are absolutely convergent for $\operatorname{Re} s > 13/2$.

Let us define the Ramanujan Ξ-function, denoted by $\Xi_R(s)$, as follows:

$$\Xi_R(s) = (2\pi)^{is-6} L(-is+6)\Gamma(-is+6),$$

where $\Gamma(s)$ is the Gamma function. Another representation for $\Xi_R(s)$ is given by

$$\Xi_R(s) = \int_{-\infty}^{\infty} \phi(t) e^{ist}\, \mathrm{d}t,$$

where

$$\phi(t) = e^{-2\pi \cosh(t)} \prod_{k=1}^{\infty} \left[\left(1 - e^{-2\pi k e^{t}}\right)\left(1 - e^{-2\pi k e^{-t}}\right)\right]^{12}. \tag{1.1}$$

In [1], Hardy highlighted the importance of the location of the zeros of $\Xi_R(s)$ in the strip $|\mathrm{Im}\,(s)| \leq \frac{1}{2}$. The Riemann hypothesis for the Ramanujan zeta function states that all zeros of $\Xi_R(s)$ are real.

1.2 Zeros of the approximations $\Xi_F(s)$

The purpose of this paper is to study the distribution of the zeros of certain approximations for the Ramanujan Ξ-function. Inspired in the representation (1.1), Ki [2] defined these approximations as follows: Let F be a finite sequence of complex numbers $a_0, a_1, \ldots, a_n$ such that at least one of them is different from zero. We define the function

$$\Xi_F(s) = \int_{-\infty}^{\infty} \phi_F(t) e^{ist}\, \mathrm{d}t,$$

where

$$\phi_F(t) = e^{-2\pi \cosh t}\left(\sum_{m=0}^{n} a_m e^{-2\pi m e^t}\right)\left(\sum_{m=0}^{n} \overline{a_m} e^{-2\pi m e^{-t}}\right).$$

We recall that $\overline{\Xi_F(\overline{s})} = \Xi_F(s)$, and one can see that for some sequences F_k, the function $\Xi_{F_k}(s)$ converges uniformly to $\Xi_R(s)$ on all compact subsets of $\mathbb{C}$.

Throughout this paper, we will study the distribution of the zeros of the function $C_F(s) := \Xi_F(-is)$. Note that the zeros of $C_F(s)$ are symmetric respect to the line $\operatorname{Re} s = 0$. Using the argument principle, Ki [2, Theorem 1] established for $T \geq 2$ that[1]

$$N(T, C_F) = \frac{T}{\pi} \log \frac{T}{e\pi} + O(\log T),$$

where $N(T, C_F)$ stands for the number of zeros of $C_F(s)$ such that $1 \leq \operatorname{Im} s < T$, counting multiplicity. In the lower half-plane a similar result holds. Moreover, using the method developed by Levinson [3], he stated that

$$\overline{N}(T, C_F) - \overline{N_1}(T, C_F) = O(T), \tag{1.2}$$

where $\overline{N}(T, C_F)$ stands for the number of zeros of $C_F(s)$ such that $|\operatorname{Im} s| < T$, counting multiplicity and $\overline{N_1}(T, C_F)$ denotes the number of simple zeros such that $|\operatorname{Im} s| < T$ and $\operatorname{Re} s = 0$. In a sense, it means that almost all zeros of $C_F(s)$ lie on the line $\operatorname{Re} s = 0$ and are simple. Our first goal is to establish a refinement of (1.2).

Theorem 1 *For $T \geq 2$ we have*

$$0 \leq \overline{N}(T, C_F) - \overline{N_1}(T, C_F) \leq \left(32n + \frac{32\ln(2n+1)}{\pi}\right)T + O(1).$$

On the other hand, Ki [2, Theorem 2] found a result about the vertical distribution of the zeros of $C_F(s)$, based on the zeros of the function $\psi_F(s)$, defined by

$$\psi_F(s) = \pi^{-s}\sum_{m=0}^{n} a_m (2m+1)^{-s}. \tag{1.3}$$

Let $k \geq 0$ be an integer such that $P(1) = P'(1) = \cdots = P^{(k-1)}(1) = 0$ and $P^{(k)}(1) \neq 0$, where $P(y) = \sum_{m=0}^{n} a_m y^m$.

Theorem 2 *Let $\Delta_* < \Delta_{**}$ be positive real numbers. Suppose that $\psi_F(s-k)$ has finitely many zeros in $-\Delta_{**} < \operatorname{Re} s < \Delta_*$. Let δ be such that $0 < \delta < \Delta_*$. Then all but finitely many zeros of $C_F(s)$ which lie in $|\operatorname{Re} s| \leq \delta$ are on the line $\operatorname{Re} s = 0$. In particular, all but finitely many zeros of $C_F(s)$ are on the line $\operatorname{Re} s = 0$, if $\psi_F(s-k)$ has finitely many zeros in $\operatorname{Re} s > -\Delta_{**}$.*

1 Throughout the paper we use the Vinogradov's notation $f = O(g)$ (or $f \ll g$) to mean that $|f(t)| \leq C|g(t)|$ for a certain constant $C > 0$ and t sufficiently large.

Ki included a second proof for the second part of Theorem 2. In particular, this second proof gave information about the simplicity of the zeros of $C_F(s)$. Anyway, Ki conjectured that second case for $\psi_F(s-k)$ is not possible. On the other hand, using (2.7) it is clear that $\psi_F(s-k)$ has the same set of zeros of a Dirichlet polynomial in the framework of [4, Sect. 12.5]. The set of zeros of a Dirichlet polynomial is quasi-periodic (see [5, Appendix 6, p. 449]). Then, if $s_0 = \sigma_0 + i\tau_0$ is a zero of the Dirichlet polynomial, for any $\varepsilon > 0$ we can construct a sequence $\{s_n = \sigma_n + i\tau_n\}_{n\in\mathbb{N}}$ of zeros, such that $\sigma_n \in]\sigma_0-\varepsilon, \sigma_0+\varepsilon[$ for all $n \in \mathbb{N}$ and $\tau_n \to \pm\infty$. This implies that each open vertical strip has no zeros or has infinite zeros. Therefore, the hypothesis in Theorem 2 is reduced to $\psi_F(s-k)$ having no zeros in $-\Delta_{**} < \operatorname{Re} s < \Delta_*$. Our second goal in this paper is to give a new proof of this result.

Theorem 3 *Let $\Delta_* < \Delta_{**}$ be positive real numbers. Suppose that $\psi_F(s-k)$ has no zeros in $-\Delta_{**} < \operatorname{Re} s < \Delta_*$. Let δ be such that $0 < \delta < \Delta_*$. Then all but finitely many zeros of $C_F(s)$ which lie in $|\operatorname{Re} s| \le \delta$ are on the line $\operatorname{Re} s = 0$ and are simple.*

We highlight that our proof includes information about the simplicity of the zeros for the first case. The key relation between the functions $C_F(s)$ and $\psi_F(s-k)$ is due to de Bruijn [6, p. 225], who showed that

$$C_F(s) = \sum_{m=k}^{\infty} b_m \psi_F(s-m)\Gamma(s-m) + \sum_{m=k}^{\infty} \overline{b_m\, \psi_F(-s-m)}\Gamma(-s-m),$$

where b_m are complex numbers and $b_k \neq 0$.

1.3 Strategy outline

Our approach is motivated by a result of Velásquez [7, Theorem 36], about the distribution of the zeros of a function of the form $f(s) = h(s) + h^*(2a-s)$, where $h(s)$ is a meromorphic function[2], and $a \in \mathbb{R}$. This result can be regarded as a generalization of the necessary condition of stability for the function $h(s)$, in the Hermite–Biehler theorem [5, 21, Part III, Lecture 27]. In our case, using an auxiliary function $W_F(s)$, we have the representation $C_F(s) = h(s) + h^*(-s)$, where $h(s) = W_F(-is - i/2)$. Some estimates of $h(s)$ due to Ki [2, Theorem 2.1] play an important role to establish the necessary growth conditions in [7, Theorem 36]. On the other hand, the strong relation between the zeros of $h(s)$ and ψ_F(s) (see (2.5)) implies that one study the distribution of zeros of ψ_F(s), as a set of zeros of a Dirichlet polynomial.

Throughout the paper, we fix a sequence F. For a function $f(s)$ and the parameters $\sigma_1 < \sigma_2$, and $T_1 < T_2$, we denote the counting function

$$N(\sigma_0, \sigma_1, T_1, T_2, f) = \#\{s \in \mathbb{C} : f(s) = 0,\ \sigma_0 < \sigma < \sigma_1,\ T_1 < \tau < T_2\},$$
$$\widehat{N}(\sigma_0, \sigma_1, T_1, T_2, f) = \#\{s \in \mathbb{C} : f(s) = 0,\ \sigma_0 \le \sigma \le \sigma_1,\ T_1 < \tau < T_2\},$$

[2] For a meromorphic function $h(s)$, we define the function $h^*(s) = \overline{h(\overline{s})}$.

where, in both cases, the counts are with multiplicity, and

$$N_0^{'}(T, g) = \#\{s \in \mathbb{C} : g(s) = 0,\ \operatorname{Re} s = 0,\ |\operatorname{Im} s| < T\},$$

where the count is without multiplicity.

2 Preliminary results

In this section we collect several results for our proof. We highlight that in [2, Proposition 2.3], Ki showed that there is a constant $\beta_0 > 0$ such that $C_F(s) \neq 0$, for $|\operatorname{Re} s| \geq \beta_0$. This implies that for $\beta \geq \beta_0$,

$$\overline{N}(T, C_F) = N(-\beta, \beta, -T, T, C_F). \tag{2.1}$$

Therefore, we can restrict our analysis of the zeros in vertical strips. Now, let us start to find a new representation for $C_F(s)$. We define the entire function

$$W_F(s) = \int_{-\infty}^{\infty} \tilde{\phi}_F(t) e^{ist}\, \mathrm{d}t,$$

where

$$\tilde{\phi}_F(t) = \frac{e^{-2\pi \cosh t}}{e^{t/2} + e^{-t/2}} \left(\sum_{m=0}^{n} a_m e^{-2\pi m e^t} \right) \left(\sum_{m=0}^{n} \overline{a_m} e^{-2\pi m e^{-t}} \right).$$

Then, we obtain the following relation:

$$C_F(s) = W_F\left(-is - \frac{i}{2} \right) + W_F\left(-is + \frac{i}{2} \right). \tag{2.2}$$

If we denote by

$$h(s) = W_F\left(-is - \frac{i}{2} \right), \tag{2.3}$$

we rewrite (2.2) as

$$C_F(s) = h(s) + h^*(-s).$$

This representation allows us to use the following result (see [7, Theorem 36]).

Theorem 4 *Let $\sigma_0 > 0$ be a parameter and $h(s)$ be an entire function such that $h(s) \neq 0$ for* $\operatorname{Re} s = \sigma_0$. *We define the entire function*

$$f(s) = h(s) + h^*(-s).$$

Suppose that the function

$$F(s)=\frac{h^*(-s)}{h(s)}$$

satisfies the following conditions.

(i) $F(s)\neq\pm1$ *on the line* $\operatorname{Re}s=\sigma_0$, *and for some* $\tau_0>0$ *we have* $|F(s)|<1$ *for* $s=\sigma_0+i\tau$ *with* $|\tau|\geq\tau_0$.

(ii) *There exist an increasing function* $\varphi:\mathbb{R}\to\mathbb{R}$, *a constant* $K>0$ *and sequences* $\{T_m\}_{m\in\mathbb{N}}$, $\{T_m^*\}_{m\in\mathbb{N}}$ *such that* $\lim\limits_{m\to\infty}T_m=\lim\limits_{m\to\infty}T_m^*=\infty$,

$$T_m\leq T_{m+1}\leq\varphi(T_m),\quad T_m^*\leq T_{m+1}^*\leq\varphi(T_m^*)\quad \textit{for } m\in\mathbb{N},$$

and $|F(s)|<e^{K|s|}$, *for* $s=\sigma+i\tau$ *with* $0\leq\sigma\leq\sigma_0$ *and* $\tau=T_m$, $\tau=-T_m^*$, *for* $m\in\mathbb{N}$.

Then, for $T\geq2$, *we have that*

$$N(-\sigma_0,\sigma_0,-T,T,f)-N_0'(T,f)\leq4\widehat{N}(0,\sigma_0,-\varphi(2T),\varphi(2T),h)+O(1),\quad (2.4)$$

To prove that the function $h(s)$ defined in (2.3) satisfies the conditions of the previous theorem, we will use the estimates used by Ki. By [2, Eq. (2.1)], using the change of variable $s\mapsto -is-i/2$, we have that

$$h(s)=\Gamma(s-k)\Big(b_k\psi_{F,k}(s)+O\big(|s|^{-1/2}\big)\Big)\tag{2.5}$$

holds uniformly on the half-plane $\operatorname{Re}s\geq-1/4$ and $|s|$ sufficiently large. On the other hand, by [2, Theorem 2.1] it follows using the change of variable $s\mapsto -is+i/2$: for $\Delta>0$ sufficiently large,

$$\frac{h^*(-s)}{\Gamma(s-k-1)|\tau|^{\mu(\sigma)}}=O(1),\tag{2.6}$$

for $s=\sigma+i\tau$ with $0\leq\sigma\leq\Delta$ and $|\tau|\geq1$, and the function $\mu(\sigma)$ is given by

$$\mu(\sigma)=\begin{cases}1-\sigma, & \text{si}\quad 0\leq\sigma\leq1,\\ 0, & \text{si}\quad \sigma>1.\end{cases}$$

Finally, we will need to establish bounds for the right-hand side of (2.4) to estimate the number of zeros of $h(s)$. The relation (2.5) tells us that we must study the behavior of the zeros of $\psi_F(s)$. We define $\psi_{F,k}(s):=\psi_F(s-k)$. Thus, using (1.3) this function can be written as

$$\psi_{F,k}(s)=\sum_{m=0}^{n}a_me^{-\ln((2m+1)\pi)(s-k)}=e^{-\ln((2n+1)\pi)(s-k)}\left[\sum_{m=0}^{n}p_me^{\beta_ms}\right],\tag{2.7}$$

where $p_m = (a_{n-m})e^{-\beta_m k}$ and $\beta_m = \ln((2n+1)/(2(n-m)+1))$, for $0 \le m \le n$. The sum on the right-hand side of (2.7) is a Dirichlet polynomial in the framework [4, Sect. 12.5].

Proposition 5 *Let $Z(\psi_{F,k})$ denote the set of zeros of $\psi_{F,k}(s)$.*

(1) *There is a positive real number c_0 such that $Z(\psi_{F,k}) \subset \{s \in \mathbb{C} : |\mathrm{Re}\, s| < c_0\}$.*
(2) *For $T_1 < T_2$ and $c \ge c_0$, we have that*

$$N(-c, c, T_1, T_2, \psi_{F,k}) \le n + \frac{\ln(2n+1)}{2\pi}(T_2 - T_1).$$

(3) *Let $K \subset \mathbb{C}$ such that $|\mathrm{Re}\, s| \le M$ for $s \in K$, and some $M > 0$. Suppose that K is uniformly bounded from the zeros of $\psi_{F,k}(s)$, i.e.,*

$$\inf\{|s - z| : s \in K, z \in Z(\psi_{F,k})\} > 0.$$

Then, $\inf\{|\psi_{F,k}(s)| : s \in K\} > 0$.

Proof See [4, Theorems 12.4, 12.5 and 12.6]. □

3 Proofs of Theorems 1 and 3

3.1 Proof of Theorem 1

Let us define the function

$$F(s) = \frac{h^*(-s)}{h(s)}. \tag{3.1}$$

Since that $h(s)$ and $h^*(-s)$ are entire functions, we can choose $\sigma_0 > 0$ sufficiently large such that $F(s) \ne \pm 1$ and $h(s) \ne 0$ on the line $\mathrm{Re}\, s = \sigma_0$. Using (2.5) and (2.6) we get for $s = \sigma + i\tau$ with $0 \le \sigma \le \sigma_0$ and $|\tau|$ sufficiently large,

$$\begin{aligned} &F(s)\\ &\quad = \frac{O(1)\Gamma(s-k-1)|\tau|^{\mu(\sigma)}}{\Gamma(s-k)\big(b_k\psi_{F,k}(s) + O\big(|s|^{-1/2}\big)\big)}\\ &\quad = \frac{O(1)|\tau|^{\mu(\sigma)}}{(s-k-1)\big(b_k\psi_{F,k}(s) + O\big(|s|^{-1/2}\big)\big)}. \end{aligned} \tag{3.2}$$

Now, we analyze the behavior of $F(s)$ on the line $\mathrm{Re}\, s = \sigma_0$. Note that $\mu(\sigma_0) = 0$. On another hand, the line $\mathrm{Re}\, s = \sigma_0$ is uniformly bounded from the zeros of $\psi_{F,k}(s)$. Then, recalling that $b_k \ne 0$, by Proposition 5 and the triangle inequality we get

$$\big|b_k\psi_{F,k}(s) + O\big(|s|^{-1/2}\big)\big| \gg 1, \tag{3.3}$$

for $s = \sigma_0 + i\tau$, with $|\tau|$ sufficiently large. Inserting this in (3.2), it follows

$$|F(s)| \ll \frac{1}{|s-k-1|}.$$

Therefore, for $s = \sigma_0 + i\tau$ with $|\tau|$ sufficiently large we conclude that $|F(s)| < 1$. This implies (i) of Theorem 4. Let us prove (ii) of Theorem 4. For each $m \in \mathbb{Z}$ we consider the rectangle

$$R_m = \{s \in \mathbb{C} : -\sigma_0 < \operatorname{Re} s < \sigma_0, \;\; m < \operatorname{Im} s < m+1\}.$$

We divide this rectangle into $2n+1$ subrectangles $R_{m,j}$ defined by

$$R_{m,j} = \left\{s \in \mathbb{C} : -\sigma_0 < \operatorname{Re} s < \sigma_0, \;\; m + \frac{j-1}{2n+1} < \operatorname{Im} s < m + \frac{j}{2n+1}\right\},$$

for $j \in \{1, 2, \ldots, 2n+1\}$. By Proposition 5 we have that $N(-\sigma_0, \sigma_0, m, m+1, \psi_{F,k}) \leq 2n$. So, there exists j_0 such that $\psi_{F,k}(s)$ does not vanish in R_{m,j_0}. Let us write

$$T_m = m + \frac{j_0 - \frac{1}{2}}{2n+1}.$$

Note that $m < T_m < m+1$. Then, if we define $\varphi(x) = x + 2$, we have that

$$m < T_m < m+1 < T_{m+1} < m+2 < T_m + 2 = \varphi(T_m).$$

Let $K = \{s \in \mathbb{C} : -\sigma_0 < \operatorname{Re} s < \sigma_0, \;\; \operatorname{Im} s = T_m, m \in \mathbb{Z}\}$. For any $s \in K$, we have that $|s - z| \geq 1/2(2n+1)$, for all $z \in Z(\psi_{F,k})$. Then K is uniformly bounded from the zeros of $\psi_{F,k}(s)$. Using Proposition 5 we see that (3.3) holds for $s \in K$ with $|m|$ sufficiently large. Therefore, in (3.2) we obtain that for $s = \sigma + i\tau$ with $0 \leq \sigma \leq \sigma_0$ and $\tau = T_m$ ($|m|$ sufficiently large) it follows

$$F(s) \ll \frac{|\tau|^{\mu(\sigma)}}{|s-k-1|}.$$

Using the fact that $\mu(\sigma) \leq 1$, we conclude that

$$|F(s)| \ll 1 < e^{|s|}.$$

Now, we choose $T_m^* = -T_{-m}$, for all $m \in \mathbb{N}$. Thus, we obtain (ii) of Theorem 4. Therefore

$$N(-\sigma_0, \sigma_0, -T, T, C_F) - N_0^{'}(T, C_F) \leq 4\widehat{N}(0, \sigma_0, -\varphi(2T), \varphi(2T), h) + O(1). \tag{3.4}$$

To conclude we need to bound $\widehat{N}(0, \sigma_0, -\varphi(2T), \varphi(2T), h)$. Firstly, we choose $0 < \varepsilon < 1/4$ such that $h(s)$ and $\psi_{F,k}(s)$ do not vanish on $\operatorname{Re} s = -\varepsilon_0$. The definition of T_m implies that

$$\frac{1}{2n+1} \leq T_{m+1} - T_m \leq 2, \tag{3.5}$$

and using Proposition 5 we obtain $N(-\varepsilon, \sigma_0, T_m, T_{m+1}, \psi_{F,k}) \leq 2n$. Let us divide the rectangle $\{s \in \mathbb{C} : -\varepsilon < \operatorname{Re} s < 0 \text{ and } T_m < \operatorname{Im} s < T_{m+1}\}$ into $2n+1$ vertical subrectangles with horizontal length $\varepsilon/(2n+1)$. So, one of these rectangles, denoted by I_m, has no zeros of $\psi_{F,k}(s)$ and $h(s)$. Suppose that the right vertical side of I_m is contained on the line $\operatorname{Re} s = -\varepsilon_m$ that we can suppose without loss of generality that does not contain a zero of $\psi_{F,k}(s)$. Now, if we place a circle of radius $\delta > 0$ sufficiently small (for instance $\delta < 1/(2n+1)(16n)$) we can enclosed the zeros of the rectangle $J_m = \{s \in \mathbb{C} : -\varepsilon_m < \operatorname{Re} s < \sigma_0 \text{ and } T_m < \operatorname{Im} s < T_{m+1}\}$ in a contour C_m such that the distance between C_m and J_m is at least $1/(2n+1)(16n)$ and C_m is distanced at least $1/(2n+1)(32n)$ from the zeros of $\psi_{F,k}(s)$. We set $K = \bigcup_m \partial J_m \quad (\partial J_m = \text{the boundaries of } J_m)$. Then, applying (2.5) and Proposition 5, we can find $M > 0$ such that

$$\left| b_k \psi_{F,k}(s) - \frac{h(s)}{\Gamma(s-k)} \right| < M < |b_k \psi_{F,k}(s)|$$

for $s \in K$, with $|m|$ sufficiently large. If we denote $w(s) = h(s)/\Gamma(s-k)$, applying Rouché's theorem we obtain that there is $m_0 \in \mathbb{N}$ sufficiently large such that

$$N(-\varepsilon_m, \sigma_0, T_m, T_{m+1}, w) = N(-\varepsilon_m, \sigma_0, T_m, T_{m+1}, \psi_{F,k}), \tag{3.6}$$

and

$$N(-\varepsilon_{-m-1}, \sigma_0, T_{-m-1}, T_{-m}, w) = N(-\varepsilon_{-m-1}, \sigma_0, T_{-m-1}, T_{-m}, \psi_{F,k}) \tag{3.7}$$

for $m \geq m_0$.

Finally, let T be a positive real parameter. If $T < T_{m_0}$ we obtain $N(0, \sigma_0, 0, T, h) = O(1)$. If $T \geq T_{m_0}$, we choose $m_1 \geq m_0 \geq 1$ such that $m_1 < T_{m_1} \leq T < T_{m_1+1} < m_1 + 2$. Since that the zeros of $1/\Gamma(s)$ are the non-positive integers, by (3.6) , Proposition 5 and (3.5), we get

$$\begin{aligned} \widehat{N}(0, \sigma_0, 0, T, h) &\leq \sum_{j=m_0}^{m_1} N(-\varepsilon_j, \sigma_0, T_j, T_{j+1}, h) + \widehat{N}(0, \sigma_0, 0, T_{m_0} + 1, h) \\ &= \sum_{j=m_0}^{m_1} N(-\varepsilon_j, \sigma_0, T_j, T_{j+1}, w) + O(1) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=m_0}^{m_1} N(-\varepsilon_0, \sigma_0, T_j, T_{j+1}, \psi_{F,k}) + O(1) \\
&\leq \sum_{j=m_0}^{m_1} \left(n + \frac{\ln(2n+1)}{2\pi} (T_{j+1} - T_j) \right) + O(1) \\
&\leq \left(n + \frac{\ln(2n+1)}{\pi} \right) T + O(1).
\end{aligned}
$$

Similarly, for $T < 0$ we use (3.7) to obtain a similar bound. Thus, we obtain for $T > 0$ that

$$
\widehat{N}(0, \sigma_0, -T, T, h) \leq \left(2n + \frac{2\ln(2n+1)}{\pi} \right) T + O(1).
$$

We replace T by $\varphi(2T)$ in the above expression, and inserting in (3.4), and one can see that

$$
N(-\sigma_0, \sigma_0, -T, T, C_F) - N_0^{'}(T, C_F) \leq \left(16n + \frac{16\ln(2n+1)}{\pi} \right) T + O(1). \tag{3.8}
$$

To obtain our desired result we will use an argument of Ki in [2, p. 131]. Following his idea, for $T > 0$ we get that

$$
\begin{aligned}
&N(-\sigma_0, \sigma_0, -T, T, C_F) - \overline{N_1}(T, C_F) \\
&\quad \leq 2\left(N(-\sigma_0, \sigma_0, 0, T, C_F) - \sum_{k=1}^{\infty} \overline{N_k}(T, C_F) \right),
\end{aligned} \tag{3.9}
$$

where $\overline{N_k}(T, C_F)$ denotes the number of zeros of C_F with multiplicity k with $|\mathrm{Im}\, s| < T$ and $\mathrm{Re}\, s = 0$, counting with multiplicity. Note that

$$
N_0^{'}(T, C_F) \leq \sum_{k=1}^{\infty} \overline{N_k}(T, C_F). \tag{3.10}
$$

We conclude combining (3.8), (3.9), (3.10), and recalling by (2.1) that $\overline{N}(T, C_F) = N(-\sigma_0, \sigma_0, -T, T, C_F)$.

3.2 Proof of Theorem 3

The proof is similar to the previous case. Using the function defined in (3.1), without loss of generality we can choose $\delta > 0$ in such a way that $F(s) \neq \pm 1$, $h(s) \neq 0$ and $C_F(s) \neq 0$ when $\sigma = \delta$. By (2.5) and (2.6) it follows for $s = \sigma + i\tau$ with $0 \leq \sigma \leq \delta$ and $|\tau|$ sufficiently large

$$F(s)=\frac{O(1)|\tau|^{\mu(\sigma)}}{(s-k-1)\big(b_k\psi_{F,k}(s)+O\big(|s|^{-1/2}\big)\big)}.$$

Using the fact that the $\psi_{F,k}(s)$ has no zeros in the strip $-\Delta_{**}<\operatorname{Re}s<\Delta_*$, by Proposition 5 we get

$$\left|b_k\psi(s-k)+O\big(|s|^{-1/2}\big)\right|\gg 1, \tag{3.11}$$

for $s=\sigma+i\tau$, with $0\le\sigma\le\delta$ and $|\tau|$ sufficiently large. Therefore

$$|F(s)|\ll\frac{|\tau|^{\mu(\sigma)}}{|s-k-1|}. \tag{3.12}$$

Using the fact that $\mu(\delta)<1$, then

$$|F(s)|\ll\frac{|\tau|^{\mu(\delta)}}{|s-k-1|}\ll\frac{1}{|\tau|^{1-\mu(\delta)}}<1,$$

for $s=\delta+i\tau$, with $|\tau|$ sufficiently large. Further, we have that $\mu(\sigma)\le 1$, which implies in (3.12) that

$$|F(s)|\ll\frac{|\tau|^{\mu(\sigma)}}{|s-k-1|}\ll 1<e^{|s|},$$

for $s=\sigma+i\tau$ with $0\le\sigma\le\delta$ and $|\tau|$ sufficiently large. Choosing $\varphi(x)=x+2$ and $T_m=T_m^*=m$, for m sufficiently large, we get that the hypotheses in Theorem 4 are satisfied. Then

$$N(-\delta,\delta,-T,T,C_F)-N_0^{'}(T,C_F)\le 4\widehat{N}(0,\delta,-\varphi(2T),\varphi(2T),h)+O(1). \tag{3.13}$$

Combining (2.5) and (3.11), we get a constant $L>0$ such that $|h(s)|\ge L|\Gamma(s-k)|$ for $s=\sigma+i\tau$ with $0\le\sigma\le\delta$ and $|\tau|$ sufficiently large. Then, $h(s)$ only has finitely many zeros on the strip $0\le\sigma\le\delta$, because all possible zeros are contained in a compact set. Therefore, the right-hand side in (3.13) is bounded and this implies our desired result.

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