# Ola Bratteli and His Diagrams

*Tone Bratteli, Trond Digernes, George A. Elliott, David E. Evans, Palle E. T. Jorgensen, Aki Kishimoto, Magnus B. Landstad, Derek W. Robinson, and Erling Størmer*

## Introduction

### Magnus B. Landstad and George A. Elliott

Ola Bratteli is known first and foremost for what are now called Bratteli diagrams, a kind of infinite, bifurcating, graded graph. He showed how these diagrams (cousins of Coxeter–Dynkin diagrams) can be used to study algebras that are infinite increasing unions of direct sums of matrix algebras. They turned out to be very useful tools, giving a large class of examples, and led later to a *K*-theoretical classification both of the algebras just mentioned and, more recently, of an enormously larger class (all "well-behaved" simple amenable C*-algebras). According to MathSciNet, Ola Bratteli has 113 publications with 21 coauthors and he received various awards. He was a member of the Norwegian Academy of Science and Letters and a member of the AMS for forty-three years.

As to his biography, Ola Bratteli graduated with distinction from the University of Oslo in 1971 and took his doctorate there in May 1974. He was a research fellow at New York University 1971–73, had various postdoc positions 1973–77, was an associate professor at the University of Oslo 1978–79, a full professor at the University of Trondheim (now NTNU) 1980–91, and since 1991 at the University of Oslo.

**Figure 1.** Ola Bratteli (1946–2015).

Ola's father, Trygve Bratteli, was a Norwegian politician from the Labour Party and prime minister of Norway in 1971–72 and 1973–76. During the Nazi invasion of Norway, he was arrested in 1942, and was a *Nacht und Nebel* prisoner in various German concentration camps from 1943 to 1945 but miraculously survived. Ola's mother, Randi Bratteli, was a respected journalist and author of several books. Ola was born October 24, 1946, and died February 8, 2015. He is survived by his wife, Rungnapa (Wasana), and their son, Kitidet.

For a detailed biography, see the MacTutor History of Mathematics Archive: `www-history.mcs.st-and.ac.uk/Biographies/Bratteli.html`.

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## Bratteli Diagrams

George A. Elliott

One of Bratteli's most important discoveries, I think, was what is now called a Bratteli diagram, which he found as a way of codifying the data of an approximately finite-dimensional (AF) C*-algebra (the completion of an increasing sequence of finite-dimensional C*-algebras, i.e., finite direct sums of full matrix algebras), in a far-reaching extension of the thesis of Glimm. (Glimm considered the case of simple finite-dimensional C*-algebras, with unital embeddings. The nonunital case was later studied by Dixmier.)

In one sense, Bratteli diagrams had perhaps already been invented, as, for one thing, the idea is so simple—a vertically arranged sequence of horizontal rows of points, with numbered lines connecting the points of each row to the points of the row below, these numbers recording the multiplicities of the partial embeddings of the simple direct summands at one stage of the sequence of algebras into those at the next stage.

In the unital case, with unital embeddings, the orders of the simple direct summands at each stage, which can be written as numbers accompanying the corresponding points in the diagram, are determined in a simple way by the multiplicities, assuming that the first stage is just the complex numbers (which clearly is no loss of generality).

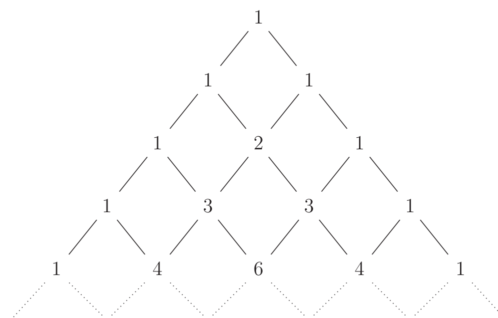
Pascal's triangle is a Bratteli diagram—with the multiplicities equal to one, and the numbers appearing in the rows being of course the successive degrees of binomial coefficients. The unital AF C*-algebra this diagram encodes arises in physics as the gauge-invariant subalgebra of the so-called CAR algebra, the C*-algebra of the canonical anticommutation relations (also AF). (This well-known subalgebra is referred to as the GICAR algebra.)

Bratteli was not content just to look at the diagrams—he isolated the equivalence relation between them that is determined by isomorphism of the associated C*-algebras. This was prophetic, as he in fact was noticing that the diagrams formed a category, in which his equivalence is just isomorphism. It was later noticed that this category is equivalent to the category of ordered groups arising from the algebras in question via the *K*-functor. This led eventually to a *K*-theoretical classification of an enormous—still evolving!—class of amenable C*-algebras, analogous to the classification by Connes and Haagerup and others of amenable von Neumann algebras.

Bratteli diagrams arose in a fundamental way in Jones's theory of subfactors. Given a subfactor of Jones index less than four, the increasing sequence of relative commutants in the Jones tower are finite-dimensional and so give rise to an AF algebra. Its Bratteli diagram is periodic with period two, with the step from the second row to the third being just the reflection of the step from the first row to the second. Both these one-step Bratteli diagrams are obtained from a single Coxeter–Dynkin diagram by pleating it in the two different possible ways.

Bratteli diagrams with an order structure were introduced by Vershik to describe a measurable transformation, and were adapted by Herman, Putnam, and Skau to describe a minimal transformation of the Cantor set. Using this description, Giordano, Putnam, and Skau classified the orbit structures of such transformations, the invariant being (in the generic case) the ordered *K*-group of the associated C*-algebra, also classified by this ordered group.

Below is a Bratteli diagram representing the GICAR algebra (see above). Note the resemblance to Pascal's triangle.




The corresponding inductive chain system (depicted vertically) is

$$\begin{array}{c} \mathbb{C} \\ \downarrow \phi_0 \\ \mathbb{C} \oplus \mathbb{C} \\ \downarrow \phi_1 \\ \mathbb{C} \oplus M_2(\mathbb{C}) \oplus \mathbb{C} \\ \downarrow \phi_2 \\ \mathbb{C} \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \\ \downarrow \phi_3 \\ \mathbb{C} \oplus M_4(\mathbb{C}) \oplus M_6(\mathbb{C}) \oplus M_4(\mathbb{C}) \oplus \mathbb{C} \\ \downarrow \phi_4 \\ \vdots \end{array}$$

where the (injective) connecting homomorphisms are given by

$$\begin{aligned} \phi_0(a) &= a \oplus a, \\ \phi_1(a,b) &= a \oplus \begin{pmatrix} a & \\ & b \end{pmatrix} \oplus b, \\ \phi_2(a,B,c) &= a \oplus \begin{pmatrix} a & \\ & B \end{pmatrix} \oplus \begin{pmatrix} B & \\ & c \end{pmatrix} \oplus c, \\ \phi_3(a,B,C,d) &= a \oplus \begin{pmatrix} a & \\ & B \end{pmatrix} \oplus \begin{pmatrix} B & \\ & C \end{pmatrix} \oplus \begin{pmatrix} C & \\ & d \end{pmatrix} \oplus d, \\ &\vdots \end{aligned}$$

where blank means one or more zeros; a, b, c, d are complex numbers; and B, C are matrices.

Ola was always a fountain, or mountain, of good sense. Once, when he was visiting Toronto, we went to the gym to go swimming. I was a member, but it was a little murky what guest privileges I had. I thought I had them as a faculty member and proceeded to explain that. This met with resistance, which gradually became more protracted. In the meantime, Ola slipped past the desk, picked up a towel, and half an hour later had finished his swim—perhaps even had a sauna too. At that point I gave up and we left.

## Ola—A Child of Peace, a Man of the Outdoors, and a Family Man

### Tone Bratteli

On United Nations Day, October 24, 1946, a child of peace was born in a crowded maternity department in Oslo. He was one of many in the baby boom that followed World War II.

This event was no matter of course. My father came home in 1945 after years in extermination camps in Germany. He survived by a hair's breadth. Soon after his return he met my mother. Her father had also come home from concentration camps. The two found each other quickly, despite my father's shyness. My mother's sociable nature made up for that. And in October 1946 he could lift his son up in the air in joy. He had not been sure whether he would be able to have children after the appalling treatment in the camps.

A year and a half later, May 8, 1948, I was born. And after another three years, May 20, 1951, our little sister, Marianne, arrived. Ola's sometimes insistent little sister.

My father was on his way into politics, with the result that when Ola was five years old we moved to a so-called official residence in which our parents could also carry out social duties. On the day we moved, Ola and I scrambled around searching for our old home. The little we had with us was swallowed up by the huge rooms. Many leading politicians from other countries came to this apartment. We children hid away in our rooms and were not especially eager to introduce ourselves. Ola was a quiet boy, but one evening he came in to join the houseguests and started tugging on the men's ties. This was a big surprise. What had happened to Ola? He had been in the kitchen, where some half-empty wine glasses had caught his eye. After tasting that juice, the shy, self-conscious boy was briefly transformed into a party animal.

But Ola was soon ready for Bolteløkka School and a meeting with the subject that became his passion—mathematics. My mother was rather taken aback the day she discovered that Ola had wallpapered his room with equations. He did not play football or hang out with friends. Instead he solved equations and went for very long walks in the forests and the mountains. He also did my maths homework for me, because to me that subject was a struggle.

**Figure 2.** The Bratteli family, 1952.

Ola achieved the best grade in mathematics; mine was the worst.

As a young man, Ola spent many hours skiing, and the trips could easily reach 50 to 60 kilometres. He also tried to find detours to make them even longer. When he came home, a slice of bread or two was not enough. He ate the whole loaf.

Early on, Ola showed an interest in music and visual arts as well. He made an attempt to teach himself to play the cello. It was not a success. But he spent a lot of time listening to music and he went to exhibitions. Our sister, Marianne, is an artist, so my brother and sister had something in common there. Later in life, when I travelled to several continents in my job as a journalist and in other connections, Ola always knew which gallery I should head for if there was only time for one. The composition of a picture and the solution to a mathematical puzzle must have something in common.

Ola graduated from the University of Oslo in 1971. He was awarded the best possible grade and was what we call "reported to the King." That meant that the entire government was informed about his academic accomplishments. It was a big day for everyone in the family, but perhaps most of all for my father. He was the prime minister who came from such a poor background that he never completed his education himself, but was self-taught. Now he was to inform the king and government about his son's academic triumph. As usual, Ola was unassuming and self-conscious, but he was no doubt satisfied.

It was difficult to be completely anonymous and work undisturbed with his own research in Oslo. Being the child of a prime minister has many sides. You have to take a variety of comments and media coverage in your stride. When Ola went to New York in 1971, it was a kind of escape. From his apartment in Greenwich Village, it was easy to get to theatres and exhibitions, and Ola soaked up everything he could get to. One day when it snowed in New York—that does happen, after all—he skied up and down Fifth Avenue. Finally, skiing weather in the Big Apple....

Ola returned home in 1973 and took his doctorate in 1974. This was during Dad's second term as prime minister and also led to coverage in the media.

In his personal life, Ola was simply a very kind, generous, caring and family-loving man. With a twinkle in his eye, it was easy for him to establish rapport with children.

During the years he lived abroad, he kept in touch with our mother by phone and wanted to know how we were doing.

Before Ola moved back to Norway, he was fortunate enough to meet Rungnapa (Wasana). He was to share almost half his life with her. Kitidet—his son—was his pride and joy.

Rungnapa and Ola travelled widely all over the world. There were holidays, but she also accompanied him to conferences and on visits to universities. After a while, they settled down in Norway. They enjoyed good years together. In the last years of his life, Ola's health deteriorated. Rungnapa was enormously supportive and did all she could to make it possible for him to live at home as long as possible. He was a lucky man.

Now Rungnapa and Kitidet have moved back to Phitsanulok in Thailand. Ola and Rungnapa built a house in her home city several years ago and had no doubt planned to spend the winters there eventually. It did not turn out that way; Ola died so early. But outside the house there is a small temple for Ola. So in a way he is there too.

**Figure 3.** Wedding in Trondheim, 1986.

## Reminiscences of Ola

### Trond Digernes

I first met Ola around 1970 when we were both Master's students at the University of Oslo. At that time Ola was a slender young man with a passion for the outdoors, especially mountain hiking.

During the year 1970–71 there were three of us who spent much time together: Ola, John Erik Fornæss, and myself. We played bridge, took a skiing vacation in the Norwegian mountains, and went mountain hiking in the summer. At the end of summer 1971 our roads parted. We all went to the US for PhD studies, but to different institutions: Ola to the Courant Institute, John Erik to the University of Washington, and I to UCLA. Other adventures with Ola in the early seventies included a multiday hike in the Sierra Nevadas in the summer of 1972, and a trip by Jeep through the roadless interior of Iceland in the summer of 1973. The latter involved getting stuck in rivers and sleeping out in the open. After Iceland, Ola returned to New York, whereas I was headed for a year's stay at CPT/CNRS, Marseille.

At CPT/CNRS, Marseille, the year 1973–74 was organized as a special year dedicated to operator algebras and mathematical physics. It attracted several high-powered researchers, among them Alain Connes and Masamichi Takesaki, and Derek Robinson and Daniel Kastler were already there. Ola joined the Marseille group in January 1974, and this was also when he started his long-time collaboration with Derek Robinson. During the decade 1980–90 I joined Ola and Derek on several occasions for discussions

**Figure 4.** Ola and Trond in Trondheim, 1983.

at ANU, Canberra, and Derek also visited Trondheim. This resulted in a few joint publications, sometimes also with other coauthors.

I was involved in only a fraction of Ola's mathematical work, but since we spent much time together, we had many interesting discussions. Given Ola's incisive mind and deep understanding of everything he was involved in, it was always a rewarding experience to exchange ideas with him. He is dearly missed, both as friend and colleague.

## Ola Bratteli, Friend and Mathematician

### Erling Størmer

I met Ola for the first time when he was about to start on his Master's thesis. Then he was a dark-haired lad with a beard, radiating health and fitness, who often went for very long skiing trips. It soon struck me that he was a highly effective person who could pick up new theory extremely quickly. By then he had taught himself a great deal about the field of operator algebras, which he wanted to work on. In the late 1960s the physicists became interested, and operator algebras became a popular field. So Ola's timing was excellent when he passed the Master's examination in 1971 with top grades. When it became clear to me how good the thesis was, I said to Ola that we had made a big mistake; this thesis should have been used for a doctoral degree. And it was precisely the results here ([1]) that made Ola well known as a mathematician from an early stage.

In 1959, James Glimm studied operator algebras that were achieved by taking an infinite union of an increasing family of $n\times n$ matrices. This became a famous piece of work, and Ola quickly discovered that he could generalize Glimm's work by studying infinitely increasing unions of matrix algebras. He then ended up with an infinitely large diagram that described all the inclusions, which thus also described how the operator algebra was constructed. His main finding was that this diagram fully described the operator algebra, enabling a classification of all such operator algebras, now known as AF algebras. This result proved far more important than Ola and I had anticipated, and the figures are now called Bratteli diagrams.

After graduating, Ola studied for two years at New York University, where Glimm was. There was here an excellent environment in mathematical physics but also for cultural life and food. His appearance changed in some ways; much of his hair disappeared, his beard was gone, and he put on so much weight that when I met him later, I did not recognize him, so I introduced myself to him.

Ola returned home and took his doctorate in Oslo in 1974. After that, he went to Marseilles, where there was a very active and high-quality environment led by physicists who developed the theory of quantum physics in operator algebras. There he met Derek Robinson, and they started a cooperation that lasted for the rest of Ola's career. Robinson moved to Australia, and on one of Ola's journeys to Australia he stopped in Thailand. Here he was fortunate enough to meet Wasana, whom he married. Together, they had a good life.

Ola had many more coauthors and was altogether a very popular person to work with. Everyone liked Ola. When there were several of us together, he was not a man of many words, but in private he would talk. Ola radiated a good spirit, and it was easy to become fond of him. He was a person with a warm heart and a subtle humor that will stay with us for the rest of our lives.

Ola Bratteli passed away at the age of sixty-eight after several years of steadily declining health. It was an extremely sad experience to see how he had become weaker each time I saw him in recent years. His strength began to fail at a fairly early stage. His last research articles were published in 2008, and after that he had little energy to do more. So his brilliant career as a mathematician came to an end far too early. Ola will be remembered and missed for a long time, both as a mathematician and as the wonderful person he was.

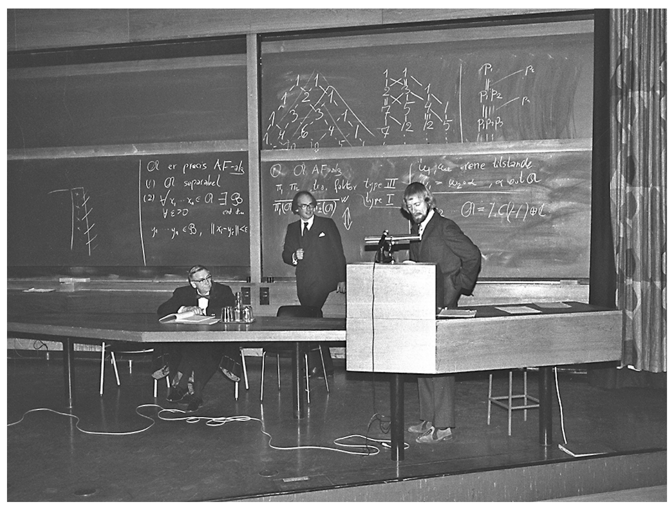


**Figure 5.** Ola's PhD defense, May 1974. From left: the dean, Ola, Gert K. Pedersen.

## Life with Ola

### Derek W. Robinson

I met Ola at the beginning of 1974 in Marseille. He was introduced to me by Trond Digernes. It was to be a pivotal moment in each of our lives, although we had no premonition of this at the time. There were many unpredictable consequences: a few years later Trond was to meet his wife, Hallie, at an open air opera in Sydney; Ola became the owner of a mushroom farm in northern Thailand; and Ola and I were to write a book that is still bought, read, and regularly cited thirty-five years later. At that time I was professor of physics at the University of Marseille, where, under the influence of the late Daniel Kastler, there was a strong visitors program in mathematical physics. This explained Ola's and Trond's presence.

My collaboration with Ola began the day we met. I explained to him some of the ideas I had about quantum dynamics and derivations on C*-algebras, and shortly after we wrote our first paper on these topics. By 1976 we had coauthored three other papers. When we met we had different interests, different backgrounds, and quite different personalities. Ola was quiet, well organized, and introspective, characteristics I did not share. Ola rapidly assimilated the Mediterranean lifestyle—the sun, the sand, and the seafood, especially the seafood. It was a time of calm cooperation amid the chaos of French academic life. In 1975 I began to think about writing a book on operator algebras and their applications in physics. I discussed the idea in spring 1976 with Ola, thinking that with our disparate backgrounds and successful working relationship we would be able to do justice to the subject and its recent developments. I was pleased he did not dismiss the idea immediately.

Initially the book was intended as a relatively short-term project: a monograph of 300–400 pages with the early chapters on mathematical background and the later chapters on applications to quantum statistical mechanics. We quickly realized that we would exceed the estimated length, so the short-term project turned out to be a long-term project, and the book changed from one volume to two.

We began each chapter with a tentative sketch of the intended sections. After discussing the general presentation of the material in each section, we began to draft alternate sections. We then exchanged drafts and edited each other's work. This process would be repeated until we were each satisfied with the outcome. We often had different notions of the relative significance of the material and the emphasis to be given to various statements and results. At times my first draft would be completely changed by Ola and vice versa. Somehow the process always reached equilibrium after a reasonably short time, with one exception, the section on modular theory. This took seven exchanges before we were both satisfied. This procedure had various advantages. It naturally introduced a uniformity of style. It also gave a fairly foolproof method of avoiding error, although we were not completely successful in that respect.

**Figure 6.** Derek and Ola, 1988.

The first volume of the book was completed by September 1977, which left three months to complete the second volume of the book before I left France to take up a position in Australia. In that time we managed to write about 40 percent of Volume 2, aided by discussions with Akitaka Kishimoto, who had just arrived in Marseille as a postdoc. I returned to Marseille in June 1978, and work began again. I then returned to Sydney and Ola moved to Trondheim, but we returned to Marseille in June 1979 to tackle the final work. We completed the book by working nonstop for three and a half weeks. The second volume finally appeared in 1981, so the total operation of writing and publishing the 1000-page, two-volume book took about three years. That was not the end, however; we returned to it again twice, preparing the second edition, but that is a different story.

After the book was finished, our collaboration continued, with Ola visiting Australia almost every year. He was lucky to survive his first visit. Whilst touring the almost deserted roads of a national park, he reverted, European style, to driving on the right side, the wrong side for Australia. This led to a head-on collision that wrecked both cars, fortunately without any personal injuries.

Ola realized that by timing his Australian visits correctly he could ensure that it was almost always summer. It also had the advantage that we could spend a maximum of time working at the family beach house. Ola would then take his daily swim before heading to the local oyster shop. He bought freshly harvested oysters by the bag to be opened, seasoned with a lemon from our garden, and savoured as the sun went down.

"Those were the days my friend, I thought they'd never end."

## A Tribute to Ola Bratteli

### Aki Kishimoto

Before I first met him in September of 1977, I must have read his early paper ([1]) of 1972 and his more recent series of papers on unbounded derivations, mostly written with D. W. Robinson, because his image had been firmly established in my mind as a formidable mathematician with whom I could hardly be compared. And he was, and I think I was quite lucky to come to know him in the earliest possible days, which brought me chances to collaborate with him for three decades.

Ola and Derek were writing the book in Marseille, a kind of book I love and could get familiar with, when I visited in 1977. Derek soon left for Sydney, but Ola and I spent almost a year together there. Though he was just one year my senior, he was a kind of mentor in mathematics and everything else during the stay. He also arranged for me to attend two conferences, sort of encouraging me against my timidity. In all these activities when transportation was required, Ola was in charge, which eliminated a practical worry for me.

In one outing I remember we hiked on rocky coastal paths, where I learned to mumble "Bonjour" to strangers we encountered. (Later I found this practice ubiquitous, and Ola would say "Konnichiwa" awkwardly when we hiked in Japan.) We were not sports types, but he was better than me in everything. So when we had to descend a steep slope made of gravel at one point, I was surprised to find myself rather enjoying skidding down while Ola tried to walk down steadily, as if reflecting his meticulous style of doing mathematics.

Ola's paper ([1]) on AF algebras became an inspirational source for me. This class might look rather special, but now we might say *if a* C**-algebra is not obviously not AF, then it would be AF*. As a touchstone of this credo Ola and I examined the fixed point algebra $F_\theta$ of a C*-algebra $C_\theta(u,v)$ generated by two unitaries $u,v$ with $uv=e^{2\pi i\theta}vu$ with $\theta$ irrational, under the period two automorphism σ: $\sigma(u)=u^{-1}$, $\sigma(v)=v^{-1}$. ($C_0(u,v)$ with $\theta=0$ still can be defined and comprises the continuous functions on the torus, while $F_0$ is on the sphere or rather a pillow with four corner points.) We managed to show $F_\theta$, a noncommutative pillow, is AF (when $\theta$ is irrational). Another notable result in [1] was that any two irreducible representations of a simple AF algebra are bridged by an automorphism, which we established in a different setting, published as "Homogeneity of the pure state space of the Cuntz algebra" in *J. Funct. Anal.* **171** (2000). This gave me hope of successfully attacking a more general case, and I did it with some help from others.

Ola had many works; I cannot touch on all of them. Among them the books [3,4] are a good reference for those in the field of mathematical physics, me included. Our last work "Approximately inner derivations," *Math. Scand.* **103** (2008) with Derek is an attempt to shed light on a topic dealt with there, which yet haunts me to this day.

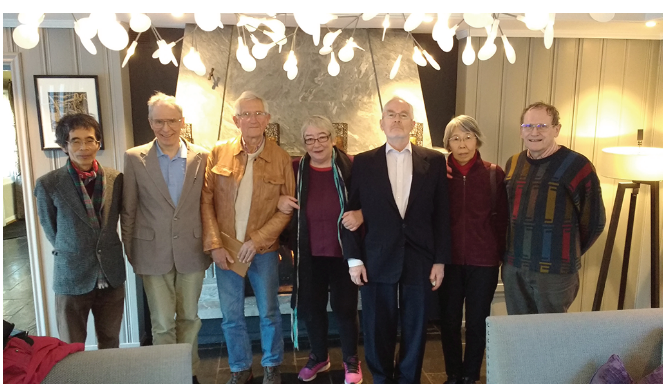


**Figure 7.** Lysebu, Oslo, 2017. From left: Aki Kishimoto, Palle Jorgensen, Derek Robinson, Tone Bratteli, George Elliott, Reiko Kishimoto, and David Evans.

**Figure 8.** Sjusjøen, Lillehammer, Easter 2004.

## Mathematics Collaborations on Three Continents

### Palle E. T. Jorgensen

Ola Bratteli had a profound influence on modern analysis, especially themes connected with operator algebras, classification, noncommutative harmonic analysis, and representation theory. While Ola's first paper was solo, almost all that followed were joint.

In January of 2017, a weeklong conference was organized in Oslo, with the aim of presenting some of the many collaborative advances in mathematics and its applications involving Ola.

My own collaboration with Ola started by chance, dates back to the mid-1970s, and lasted for four decades. Our early work was in noncommutative geometry, and our later research moved in a diverse number of directions.

My own collaborations involved my visiting Ola in Oslo. In addition, we both made research visits to Derek Robinson, Dai Evans, and George Elliott. In all, I have thirty joint research publications (including two AMS Memoirs) with Ola, and a book.

Early joint research includes the themes Lie algebras of operators, smooth Lie group actions on noncommutative tori, and a study of decomposition of unbounded derivations into invariant and approximately inner parts. These topics are part of a systematic analysis of unbounded $*$-derivations as infinitesimal generators in operator algebras, with direct connections to quantum statistical mechanics. Other applications include noncommutative geometry, as envisioned by Alain Connes. Our later joint research

**Figure 9.** Ola preparing bouillabaisse in Trondheim.

moved more in the direction of representations and certain applications.

My most recent, and substantial, joint work with Ola was wavelets. That part includes a book *Wavelets through a Looking Glass: The World of the Spectrum*, which presents the subject from a representation theoretic viewpoint.

A common theme in my joint research with Ola is my insistence on the central role to be played by representations and decomposition theory. Some of our early work dealt with representations of Lie groups, of C*-algebras, and of multiscale systems.

A quite different representation theoretic theme was the theory of numerical AF invariants, representations and centralizers of certain states on the Cuntz algebras, and a related but different study of combinatorial notions we called iterated function systems and permutation representations of Cuntz algebras. They play a key role in our understanding of such multilevel systems as wavelet multiresolution scales, in addition to multiband filters in signal processing.

Later joint work between Ola, me, K.-H. Kim, and F. Roush was inspired by Bratteli diagrams. It is known that the nonstationary case defies classification (order-isomorphism is undecidable), but we discovered that the stationary case could be decided by explicit classification numbers and associated finite algorithms. In this work, for the stationary dimension groups, we obtained explicit computation of numerical isomorphism invariants. We proved decidability of the isomorphism problem for stationary AF algebras and the associated ordered simple dimension groups.

Sadly, Ola's health declined towards the end, but I am happy to have had the benefit of many intense research experiences from the early part of our careers.

## Ola and Orbifolds

### David E. Evans

I first briefly met Ola in the new year of 1977 when I was a postdoc in Oslo. Later I got to know him very well, not only through our work and joint papers (fifteen altogether) but also through holidays taken together, particularly skiing ones in Rondane and Sjusjøen. He spent six months with me in Warwick in 1982, which was the start of our collaboration. Our first conference together was at Arco Felice, Naples, in March 1978. Ola and Aki drove there from Marseille in his Citroën deux chevaux. This was a meeting organized by Vittorio Gorini on Mathematical Problems in Quantum Theory of Irreversible Processes that brought together our mutual interests in derivations and generators of one parameter semigroups of positive maps, on which we would later collaborate. At a meeting in Chennai organized by Sunder, Ola took me on an expedition through the local markets to find saffron, which was later put to good use in his bouillabaisse, the finest I have ever had, with fresh fish from the harbor back in Norway.

In his PhD thesis ([1]), Ola classified AF C*-algebras in terms of what are now known as Bratteli diagrams. This work was not only pivotal in Elliott's classification of AF algebras through *K*-theory; it is ubiquitous in operator algebras, dynamical systems, and in Jones's theory of subfactors.

My own work with Ola started with dynamical systems of one-parameter semigroups of positive maps in 1982. Later visits to Swansea led to the collaboration with George Elliott and Akitaka Kishimoto on noncommutative spheres, the irrational rotation algebra, and the *K*-theoretic obstructions to classifying such amenable C*-algebras and their dynamical systems. This was particularly motivated

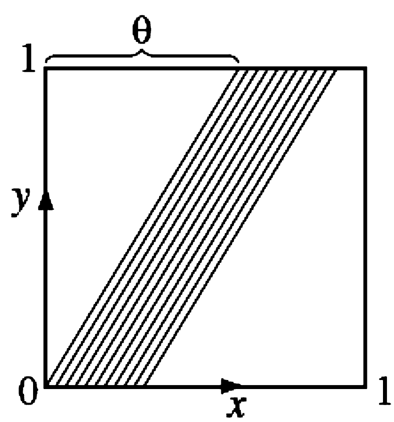



**Figure 10.** Kronecker flow on a torus. The flip on the irrational or Kronecker flow on the torus yields a nonsingular flow on the sphere which is zero-dimensional in a strong sense. The corresponding C*-algebra is approximately finite-dimensional.

by the remarkable construction by Blackadar (*Annals of Math*, 1990) of a $\mathbb{Z}_2$-symmetry on the Fermion algebra with non-AF fixed point algebra, and the subsequent work of Kumjian in showing that a $\mathbb{Z}_2$-symmetry of a Bunce–Deddens algebra and the corresponding crossed product $C(\mathbb{T}) \rtimes$ dyadic rotations $\rtimes \mathbb{Z}_2$ yielded an AF algebra. The issue of existence of such symmetries had been a well-known open problem.

This led us to consider the $\mathbb{Z}_2$-symmetry on the rotation algebra with the *flip* $\sigma$: $\sigma(u)=u^{-1}$, $\sigma(v)=v^{-1}$ on the generators. On the classical two torus, $\mathbb{T}^2$, this yields a singular orbifold $\mathbb{T}^2/\mathbb{Z}_2$ which can be thought of as a tetrahedron, topologically a sphere, but with four singular vertices. This led us to refer to the fixed point algebras and crossed products of rotation algebras as noncommutative spheres, but as Alain Connes pointed out, they are better described as noncommutative toroidal orbifolds, as they do not have the $K$-theory of a sphere.

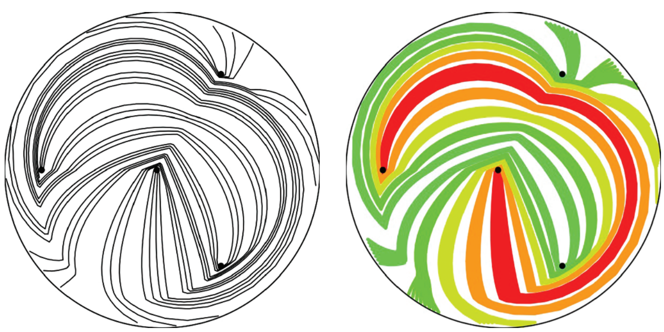


**Figure 11.** Non-commutative orbifold from folding the Kronecker flow on a torus.

Taking matrix-valued functions constrained at the singular points to have dimension drops and their inductive limits led us on a path towards studying group actions on approximately finite-dimensional AF algebras with non-AF fixed point algebras and crossed product algebras.

This way we also showed the existence of non-AF C*-algebras that when tensored with a certain UHF algebra (i.e., an infinite tensor product of matrix algebras) become UHF as well, that the fixed point algebra of an irrational rotation algebra by the flip is AF, and that the irrational rotation algebras are inductive limits of sums of matrix algebras over the continuous functions on a circle.

The work described above laid the foundations for subsequent work over the last twenty-five years on the classification of amenable C*-algebras by $K$-theoretic data, before which the classification was out of sight and did not appear feasible. Our work on orbifolds also directly led to the study of orbifold subfactors (Evans and Kawahigashi, *Comm. Math. Phys.*, 1994), as reported in the first Danish-Norwegian Workshop on Operator Algebras at Røros in 1991.

Ola had a long connection and affection for Thailand, making regular visits; his wife for more than thirty years, Wasana, was from Phitsanulok. In 2016, Paulo Bertozzini initiated at Thammasat University in Bangkok the Ola Bratteli Mathematical Physics and Mathematics in Thailand Colloquium, where I was honored to give the first talk.

Ola had a generous spirit and integrity. We will miss his presence and friendship.

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