A kinematic hybrid feedback controller on the unit circle suitable for orientation control of ships

Mathias Marley, Roger Skjetne and Andrew R. Teel

Abstract— **This paper presents a hybrid feedback controller suitable for orientation control of ships. A hybrid kinematic controller on the unit circle is constructed from the gradient of a synergistic potential function, which globally asymptotically stabilizes a desired orientation, with yaw rate viewed as control input. While this idea is not new, the potential function is novel and possesses some desired properties. The kinematic controller generates smooth reference signals for the desired velocity and acceleration, except at instances when the controller switches. Continuity of velocity and acceleration is achieved by controlling the yaw rate through a double integrator. Moreover, the velocity and acceleration converge to their desired values exponentially. The resulting closed-loop system is stable, provided the controller gains satisfy mild constraints. This is shown using a hybrid Lyapunov function.**

I. Introduction

Control of orientations, such as heading of a ship, is analogous to stabilizing a point on the unit circle. Representing the orientation in this manner avoids the need for numerical tricks in implementation, such as mapping a yaw angle to some defined interval (typically $-\pi$ to π radians).

Systems evolving by continuous vector fields on compact manifolds, such as the unit circle, cannot have a globally asymptotically stable equilibrium [1], [2], hence precluding global asymptotic stability by continuous control.. For heading control, this implies the existence of an unstable equilibrium, typically located at the 180 degree error point. The unstable equilibrium may be removed by discontinuous control; however, the resulting controller is not robust towards arbitrarily small disturbances [3], [4]. It may also be trivially removed by not mapping the angle to a defined interval, however this results in the unwinding phenomena [5]. By utilizing a hybrid control structure [6], the unstable equilibrium may be removed using a properly defined switching logic, ensuring both global stability and robustness.

Stability analysis of hybrid systems using multiple Lyapunov functions are proposed in [7]. Various Lyapunov-based design tools for hybrid control systems have since been proposed. Notable examples include patchy control Lyapunov functions [8], synergistic Lyapunov functions [9], [10], hybrid control Lyapunov functions [11], and more recently Lyapunov-based model predictive control [12], [13]. In [14] the authors propose to construct synergistic potential functions on the unit circle, by stretching the unit circle onto itself while keeping the desired equilibrium fixed. This idea is also used in [15] for control of spherical orientations and in [16] for attitude control. Hybrid control systems have been applied for robust global trajectory tracking for underactuated vehicles [17], and underwater vehicles [18].

In this paper we first propose a version of the hybrid kinematic controller given in [14], with yaw rate viewed as control input. The commanded yaw rate, constructed from the gradient of a synergistic potential function, is approximately flat for large angular errors, while it decays smoothly towards zero as the error tends to zero. This property makes it suitable for heading control of ships and similar vehicles, where the convergence rate to a desired orientation is limited by saturation of the angular velocity, rather than acceleration or higher derivatives. The kinematic controller generates smooth references signals for the desired yaw rate, acceleration and jerk (except at instances when the controller switches).

The main contribution of this paper is found in the second part of the control system development; by controlling the yaw rate through a double integrator, continuity of velocity and acceleration is achieved. Furthermore, the control system architecture (where the continuous-time evolution is reminiscent of a cascaded system [19]) simplifies adaptation to more realistic ship models considering propellers and rudders as control input. The error dynamics of the velocity and acceleration admits exponential convergence towards their desired values. Stability is shown using a hybrid Lyapunov function, where the controller gains are chosen such that the sign indefinite terms are dominated (rather than canceling the terms which is often done in hybrid backstepping approaches [9], [10]). While dominating the sign indefinite terms is not new, employing it on a hybrid Lyapunov function places restrictions on the controller gains which are not present for non-hybrid systems. This is because the jumps destroy the cascade structure exhibited by the flows. A simulation study indicates that the constraints are not restrictive for the intended application.

The remainder of this paper is organized as follows: The problem statement is given in Section II, while hybrid control systems are briefly reviewed in Section III. Section IV presents a hybrid controller on the unit circle, with velocity viewed as control input. In Section V the yaw rate and acceleration are rendered continuous by controlling the velocity through a double integrator, while tracking the velocity and acceleration reference signals generated in Section IV. The theoretical results are supported by simulations in Section VI. Finally, concluding remarks are given in Section VII.

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II. Preliminaries and problem formulation

A. Preliminaries

1) Notation: $\mathbb{R}$ is the set of real numbers and $\mathbb{R}^n$ is the n-dimensional Euclidean space. $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ are the set of non-negative and positive numbers, respectively. $\mathbb{N}$ is the set of non-negative integers. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted $|x|$, while (x, y) denotes two column vectors stacked into a column vector, that is, $(x, y) := [x^\top \ y^\top]^\top$. For two vectors $x \in \mathbb{R}^n, y \in \mathbb{R}^n$, we define $\langle x, y\rangle := x^\top y$. For a function $f : \mathbb{R}^n \times Y \to \mathbb{R}^m$, the Jacobian matrix with respect to $x \in \mathbb{R}^n$ is denoted $J_x(f(x, y)) \in \mathbb{R}^{n\times m}$. For a function $f : \mathbb{R}^n \times Y \to R$, $\nabla_x(f(x, y)) := J_x^\top(f(x, y))$ is a column vector. The subscript x is omitted when the argument is clear from context. Finally, $\dot{x}$ is the time derivative of x.

2) Unit circle representation of planar rotations: The unit circle and planar rotations are given by [14]

$$\mathcal{S}^1 :=\{z \in \mathbb{R}^2 : z^\top z = 1\} \tag{1}$$

$$SO(2) :=\{R \in \mathbb{R}^{2\times 2} : R^\top R = RR^\top = I, det(R) = 1\}, \tag{2}$$

respectively. We denote the unit vector corresponding to an angle $a \in \mathbb{R}$ as

$$z^a := \begin{bmatrix} z_1^a \\ z_2^a \end{bmatrix} = \begin{bmatrix} \cos(a) \\ \sin(a) \end{bmatrix} \in \mathcal{S}^1, \tag{3}$$

and the corresponding map $R : \mathcal{S}^1 \to SO(2)$ is given by

$$R(z^a) := \begin{bmatrix} z_1^a & -z_2^a \\ z_2^a & z_1^a \end{bmatrix} \in SO(2). \tag{4}$$

Define $\varepsilon_1 = (1, 0)$ and $\varepsilon_2 := (0, 1)$, and let S be the rotation matrix corresponding to the 90 degree counterclockwise rotation

$$S := R(\varepsilon_2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in SO(2). \tag{5}$$

The kinematic equation for motion along the unit circle with angular velocity $\omega_a = \dot{a}$ is given by $\dot{z}^a = \omega_a S z^a$. Note the convenient calculation rules: $z^{a+b} = R(z^b)z^a = R(z^a)z^b$, $z^{a-b} = R(z^b)^\top z^a$, and $R(z^a)R(z^b) = R(z^b)R(z^a)$.

B. Problem formulation

We consider heading control of surface ships and similar vehicles moving in a plane. As such, we adopt marine control terminology [20]. Let z^ψ be the unit orientation vector corresponding to the ship heading, where ψ is referred to as the yaw angle. Furthermore, $\omega_\psi := \dot{\psi}$ is the yaw rate, $\dot{\omega}_\psi$ is the yaw acceleration, and $\ddot{\omega}_\psi$ is the yaw jerk. Let z_d^ψ be the desired orientation vector corresponding to the heading ψ_d, assumed constant, and define $\tilde{z} := z^{\psi-\psi_d} = R(z_d^\psi)^\top z^\psi$. Controlling $\tilde{z} \to \varepsilon_1$ is equivalent to controlling $\psi \to \psi_d$. In the remainder of this paper, z refers to $\tilde{z}$, while ω refers to ω_ψ, that is, omitting the superscript and subscript, respectively.

The main control task is to design a kinematic controller which renders $z = \varepsilon_1$ globally asymptotically stable. Moreover, the kinematic controller shall be suitable for ships. This implies that the kinematic controller admits smooth reference signals for both the yaw rate and yaw acceleration. Furthermore, since convergence to a desired orientation is limited by saturation of ω for large yaw errors, it is desired that the commanded yaw rate is approximately constant and within rate limits for large yaw errors. Define the system

$$\dot{z} = \omega S z, \quad \ddot{\omega} = u, \quad \dot{q} = 0 \tag{6}$$

with states $(z, (\omega, \dot{\omega}), q) \in \mathcal{S}^1 \times \mathbb{R}^2 \times Q$, where q is a logic variable available for feedback control, and $u \in \mathbb{R}$ is the control input. Controlling ω through a double integrator achieves continuity of both the yaw rate and acceleration. We now state the control problem:

Problem statement: Consider the system (6). Design a hybrid feedback controller for u such that $(\omega, \dot{\omega})$ tracks a desired reference $(\omega_d(z, q), \dot{\omega}_d(z, \omega, q))$, where $(\omega_d(z, q), \dot{\omega}_d(z, \omega, q))$ shall render $z = \varepsilon_1$ robustly globally asymptotically stable.

III. Hybrid control systems

A. Modeling framework

In this paper we consider affine control systems of the form

$$\mathcal{H} : \begin{cases} \dot{x} = f(x) + g(x)u & x \in C \\ x^+ = h(x) & x \in D, \end{cases} \tag{7}$$

with states $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$. The continuous-time evolution of x is referred to as flow, while the discrete evolution, denoted x^+, is referred to as jumps [21]. As such, $C \subset \mathbb{R}^n$ is the flow set and $D \subset \mathbb{R}^n$ is the jump map. A solution $x(\cdot, \cdot)$ to $\mathcal{H}$ is parametrized by an ordinary time variable $t \in \mathbb{R}_{\geq 0}$ and a jump variable $j \in \mathbb{N}$, thus existing on a hybrid time domain. A compact set $\mathcal{A} \subset \mathbb{R}^n$ is globally asymptotically stable if all solutions are bounded and $x(t, j) \to \mathcal{A}$ when $(t + j) \to \infty$. See [21] and references therein for a more rigorous treatment of the topic. Note in particular the definition of well-posed hybrid systems [21, Definition 6.29], and the hybrid basic conditions [21, Assumption 6.5].

B. Hybrid feedback and synergistic Lyapunov functions

Consider the system

$$\dot{x} = f(x) + g(x)u, \quad \dot{q} = 0, \tag{8}$$

with states $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, and $q \in Q := \{-1, 1\}$ as a logic variable. The following definition is a slightly modified version of the synergistic Lyapunov functions and feedback pairs presented in [9, Section III]:

Definition 1. *Let $V : \mathbb{R}^n \times Q \to \mathbb{R}$ be a continuously differentiable function. Assume the following conditions hold:*

- *$\forall r \geq 0$, $\{(x, q) \in \mathbb{R}^n \times Q : V(x, q) \leq r\}$ is compact.*
- *V is positive definite with respect to the compact set $\mathcal{A}$.*

If there exists a function $\kappa : \mathbb{R}^n \times Q \to \mathbb{R}^m$ such that

$$\langle \nabla V(x, q), f(x) + g(x)\kappa(x, q)\rangle \leq 0, \tag{9}$$

for all $(x,q) \in \mathbb{R}^n \times Q$*, then* V *is a synergistic Lyapunov function candidate relative to the compact set* $\mathcal{A}$*. Let* $M(x) := \min_{q\in Q} V(x,q)$ *and define*

$$\Psi := \{(x,q) \in \mathbb{R}^n \times Q : \langle \nabla V(x,q), f(x)+g(x)\kappa(x,q)\rangle = 0\} \tag{10}$$

$$\mu := \inf_{(x,q)\in\Psi\setminus\mathcal{A}} \{V(x,q) - M(x)\}. \tag{11}$$

If the synergy gap $\mu > 0$*,* V *is a synergistic Lyapunov function relative to the compact set* $\mathcal{A}$*.*

Select $\delta \in \mathbb{R}_{>0}$ and define the closed-loop system

$$\mathcal{H} : \begin{cases} \dot{x} = f(x) + g(x)\kappa(x,q),\ \dot{q} = 0 & (x,q) \in C \\ q^+ = -q,\ x^+ = x & (x,q) \in D, \end{cases} \tag{12}$$

$$C := \{(x,q) \in \mathbb{R}^n \times Q : M(x) - V(x,q) \geq -\delta\} \tag{13}$$

$$D := \{(x,q) \in \mathbb{R}^n \times Q : M(x) - V(x,q) \leq -\delta\}. \tag{14}$$

The following theorem states global asymptotic stability of $\mathcal{H}$, subject to the constraint $\delta < \mu$ [9, Theorem 7]:

Theorem 1. *For the system (8), assume* V *is a synergistic Lyapunov function relative to the set* $\mathcal{A}$ *with synergy gap* μ*. If* $\delta < \mu$*,* $\mathcal{A}$ *is globally asymptotically stable for the closed-loop system (12).*

Proof. Since C and D are constructed such that $(\Psi\setminus\mathcal{A})\cap C = \emptyset$ when $\delta < \mu$, the inequality (9) is strict $\forall (x,q) \in C\setminus\mathcal{A}$, that is, V is strictly decreasing during flow. Moreover, V is strictly decreasing during jumps since $V(x,-q) - V(x,q) \leq -\delta < 0, \forall (x,q) \in D$. Then, the conditions of [21, Theorem 3.18] are satisfied, which states global asymptotic stability. ☐

IV. Control System Design: Part I

In this section we develop a controller for the reduced system

$$\dot{z} = \omega_d S z, \quad \dot{q} = 0, \tag{15}$$

with states $(z,q) \in \mathcal{S}^1 \times Q$, and ω_d viewed as input. We start with a non-hybrid control system, which later serves as basis for a hybrid control system.

A. Almost globally stabilizing non-hybrid controller

We construct a non-hybrid control law for ω_d from the gradient of a potential function which is positive definite with respect to $z = \varepsilon_1$. In the following, recall that $z_1^2 + z_2^2 = 1$. Define $P : \mathcal{S}^1 \to [0\ 1]$ as

$$P(z) := L(\arccos(\lambda z_1) - \arccos(\lambda)) \tag{16}$$

$$L := \frac{1}{\arccos(-\lambda) - \arccos(\lambda)}, \tag{17}$$

with $0 < \lambda < 1$. Here, λ is a regularization parameter such that the gradient of $P(z)$ on $\mathcal{S}^1$, given by

$$\langle \nabla P(z), Sz\rangle = \begin{bmatrix} \frac{-L\lambda}{\sqrt{1-\lambda^2 z_1^2}} & 0 \end{bmatrix} Sz = \frac{L\lambda z_2}{\sqrt{1-\lambda^2 z_1^2}}, \tag{18}$$

is continuous and without singularities (note that for $\lambda = 1$, (18) has singularities for $z_1 = \pm 1$; any $0 < \lambda < 1$ avoids this). Define $\sigma_0 : \mathcal{S}^1 \to [-1, 1]$ as

$$\sigma_0(z) := \frac{z_2}{\sqrt{1-\lambda^2 z_1^2}}, \tag{19}$$

which is a smooth approximation of $\operatorname{sign}(z_2)$. With sufficiently high $\lambda < 1$, $|\sigma_0(z)|$ is approximately constant and close to unity for most $z \in \mathcal{S}^1$, while converging smoothly towards zero at the critical points $z = \pm\varepsilon_1$. Hence, choosing $\omega_d := -K_0\sigma_0(z)$, with $K_0 \in \mathbb{R}_{>0}$ (slightly) less than the maximum yaw rate of the vehicle, yields a velocity profile suitable for vehicles where the convergence rate to the desired orientation is governed by saturation of ω. Noting that

$$\langle \nabla P(z), Sz\rangle(-K_0\sigma_0(z)) = -K_0 L\lambda\sigma_0(z)^2 \leq 0, \tag{20}$$

and $\sigma_0(z) = 0$ if and only if $z = \pm\varepsilon_1$, all solutions starting in $z \in \mathcal{S}^1 \setminus \{z = -\varepsilon_1\}$ converge to $z = \varepsilon_1$.

The first and second directional derivatives of $\sigma_0(z)$ on $\mathcal{S}^1$ are given by

$$\sigma_1(z) := \langle \nabla\sigma_0(z), Sz\rangle = \frac{z_1(1-\lambda^2)}{(1-\lambda^2 z_1^2)^{\frac{3}{2}}} \tag{21}$$

$$\sigma_2(z) := \langle \nabla\sigma_1(z), Sz\rangle = (\lambda^2 - 1)z_2\left(\frac{3\lambda^2 z_1^2}{(1-\lambda^2 z_1^2)^{\frac{5}{2}}} + \frac{1}{(1-\lambda^2 z_1^2)^{\frac{3}{2}}}\right), \tag{22}$$

from which we obtain $\dot{\omega}_d = -K_0\sigma_1(z)\omega_d$ and $\ddot{\omega}_d = -K_0\sigma_2(z)\omega_d^2 - K_0\sigma_1(z)\dot{\omega}_d$, where $\ddot{\omega}_d$ is required to track $\dot{\omega}_d$.

B. Globally stabilizing hybrid controller

To remove the unstable equilibrium located at $z = -\varepsilon_1$ we adopt the procedure proposed in [14], where synergistic Lyapunov functions are constructed from a non-hybrid potential function by angular stretching of the unit circle. For the system (15), define $Q := \{-1, 1\}$. The control task is to render the set $\mathcal{A}_0 := \{(z,q) \in \mathcal{S}^1 \times Q : z = \varepsilon_1\}$ globally asymptotically stable.

1) Synergistic Lyapunov function: Define the angle $\phi := qkP(z)$ and $\bar{z} : \mathcal{S}^1 \times Q \to \mathcal{S}^1$ as

$$\bar{z}(z,q) := R(z^\phi)z, \tag{23}$$

where $k \in \mathbb{R}_{>0}$ is a constant gain. If k satisfies

$$k < \frac{1}{\max(|\nabla P(z)|)} = \frac{\sqrt{1-\lambda^2}}{L\lambda} < \pi, \tag{24}$$

$\bar{z}$ is a diffeomorphism which stretches the unit circle onto itself while keeping ε_1 fixed [14, Theorem 4.3]. In particular, each point is given the $\phi = qkP(z)$ counterclockwise rotation. Define the synergistic Lyapunov function candidate $V_0 : \mathcal{S}^1 \times Q \to [0\ 1]$ as

$$V_0(z,q) := P(\bar{z}(z,q)). \tag{25}$$

Lemma 4.1 in [14] states that

$$J_{\bar{z}(z,q)}(\bar{z}(z,q))Sz = (1 + qk\langle \nabla P(z), Sz\rangle)S\bar{z}(z,q) \tag{26}$$

Using (26) the gradient of V_0 on $\mathcal{S}^1$ is obtained as

$$\langle \nabla V_0(z,q), Sz\rangle = (1+qkL\lambda\sigma_0(z))L\lambda\bar{\sigma}_0(z,q), \quad (27)$$

where $\bar{\sigma}_0(z,q) := \sigma_0(\bar{z}(z,q))$. Define the set of critical points where the gradient of V_0 on $\mathcal{S}^1$ vanishes, by

$$\Psi_0 := \{(z,q) \in (\mathcal{S}^1 \times Q) : \langle \nabla V_0(z,q), Sz\rangle = 0\}. \quad (28)$$

In addition to $z = \varepsilon_1$, the point $z = -\varepsilon_1$ is now split and shifted into two new critical points – one for each value of q. Noting the bound (24), and $|q\sigma_0(z)| \leq 1$, we get that

$$(1+qkL\lambda\sigma_0(z)) \geq 1 - kL\lambda > 1 - \frac{\sqrt{1-\lambda^2}}{L\lambda}L\lambda > 0 \quad (29)$$

holds for all $(z,q) \in \mathcal{S}^1 \times Q$. It follows that

$$-\langle \nabla V_0(z,q), Sz\rangle\bar{\sigma}_0(z,q) = -(1+qkL\lambda\sigma_0(z))L\lambda\bar{\sigma}_0(z,q)^2 \quad (30)$$

is non-positive for all $(z,q) \in \mathcal{S}^1 \times Q$, and strictly negative for all $(z,q) \in (\mathcal{S}^1 \times Q) \setminus \Psi_0$. Since V_0 has a non-zero synergy gap [14], it is a valid synergistic Lyapunov function.

2) Virtual control law: Define $\kappa_0 : \mathcal{S}^1 \times Q \to \mathbb{R}$ as

$$\kappa_0(z,q) := -K_0\bar{\sigma}_0(z,q). \quad (31)$$

Then the compact set $\mathcal{A}_0$ is asymptotically stable for

$$\dot{z} = \kappa_0(z,q)Sz, \quad \dot{q} = 0, \quad (32)$$

with region of attraction $(\mathcal{S}^1 \times Q) \setminus (\Psi_0 \setminus \mathcal{A}_0)$.

For future use, we calculate the first and second directional derivatives of $\bar{\sigma}_0(z)$ on $\mathcal{S}^1$. Repeated application of (26) yields

$$\begin{aligned}\bar{\sigma}_1(z,q) :=& \langle \nabla\bar{\sigma}_0(z,q), Sz\rangle \\ =& (1+qkL\lambda\sigma_0(z))\sigma_1(\bar{z}(z,q))\end{aligned} \quad (33)$$

$$\begin{aligned}\bar{\sigma}_2(z,q) :=& \langle \nabla\bar{\sigma}_1(z,q), Sz\rangle \\ =& (1+qkL\lambda\sigma_0(z))^2\sigma_2(\bar{z}(z,q)) \\ &+ qkL\lambda\sigma_1(z)\bar{\sigma}_1(z,q).\end{aligned} \quad (34)$$

The desired velocity and resulting acceleration and jerk profiles are obtained as before, simply replacing $\sigma_i(z)$ with $\bar{\sigma}_i(z,q)$ for $i = 0,1,2$:

$$\omega_d := -K_0\bar{\sigma}_0(z,q) \quad (35)$$

$$\dot{\omega}_d = -K_0\bar{\sigma}_1(z,q)\omega_d \quad (36)$$

$$\ddot{\omega}_d = -K_0\bar{\sigma}_2(z,q)\omega_d^2 - K_0\bar{\sigma}_1(z,q)\dot{\omega}_d. \quad (37)$$

See Figure 1 for illustration (the parameters θ and Δ are introduced shortly). Fixing q, it is trivial to numerically verify that $\dot{\omega}_d$ and $\ddot{\omega}_d$ does not exceed the dynamic constraints of the vehicle, for a given set of parameters λ, k, and K_0.

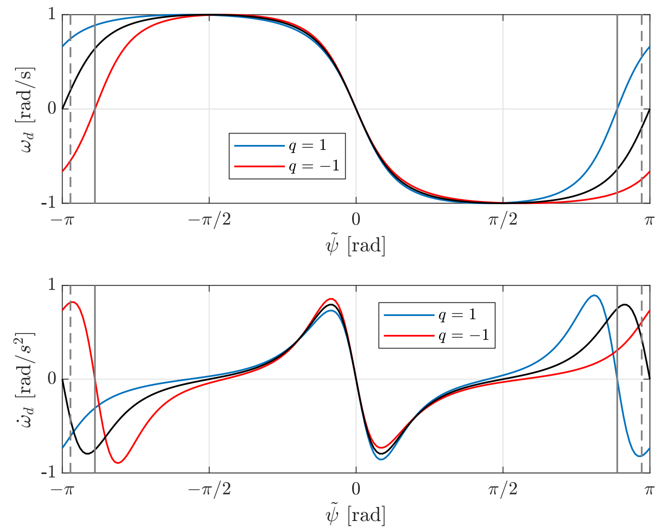

Fig. 1. Velocity (top) and acceleration (bottom) profile versus heading error for $\lambda = 0.9$, $K_0 = 1$, $\theta = \frac{\pi}{9}$, and $\Delta = \frac{\pi}{36}$ (which is not a parameter in $\bar{\sigma}_0$). Non-hybrid counterpart shown in black. Solid grey lines at $\pi \pm \theta$ and dashed grey lines at $\pi \pm \Delta$.

3) Switching logic: The set $\Psi_0 \setminus \mathcal{A}_0$ corresponds to the points where $V_0(z,q) = 1$. Let $\theta \in \mathbb{R}_{>0}$ set an angular shift of these critical points away from $z = -\varepsilon_1$, obtained by choosing

$$k = \frac{\theta}{P(-R(z^\theta)\varepsilon_1)}. \quad (38)$$

Due to the bound in (24), θ must satisfy $\theta \leq \theta_{max} < \frac{\pi}{2}$. Define $M_0(z) := \min_{q\in Q} V_0(z,q)$ and

$$\mu := \inf_{(z,q)\in\Psi_0\setminus\mathcal{A}_0} \{V_0(z,q) - M_0(z)\}, \quad (39)$$

which gives $\mu = 1 - V_0(-R(z^\theta)\varepsilon_1, 1)$. Next, select a desired hysteresis half-width $\Delta \in \mathbb{R}_{>0}$ at the 180-degree error point, satisfying $\Delta < \theta$, and define

$$\delta := |V_0(-R(z^\Delta)\varepsilon_1, 1) - V_0(-R(z^\Delta)\varepsilon_1, -1)| < \mu. \quad (40)$$

From this we define the flow set and jump set as

$$C_0 := \{(z,q) \in \mathcal{S}^1 \times Q : M_0(z) - V_0(z,q) \geq -\delta\} \quad (41)$$

$$D_0 := \{(z,q) \in \mathcal{S}^1 \times Q : M_0(z) - V_0(z,q) \leq -\delta\}, \quad (42)$$

which achieves global asymptotic stability of $\mathcal{A}_0$ for the closed-loop system

$$\mathcal{H}_0 : \begin{cases} \dot{z} = \kappa_0(z,q)Sz, \ \dot{q} = 0 & (z,q) \in C_0 \\ q^+ = -q, \ z^+ = z & (z,q) \in D_0. \end{cases} \quad (43)$$

V. Control System Design: Part II

The reference signals $(\omega_d, \dot{\omega}_d)$ in (35)-(36) have discontinuities when q is switched. Continuity of $(\omega, \dot{\omega})$ is achieved by controlling ω through a double integrator. To this end, define the error variables $e := (e_1, e_2) \in \mathbb{R}^2$ as

$$e_1 := \omega - \kappa_0(z,q), \quad e_2 := \dot{\omega} - \kappa_1(z, e_1, q), \quad (44)$$

where $\kappa_1 : \mathcal{S}^1 \times \mathbb{R} \times Q \to \mathbb{R}$ is a virtual control law for $\dot{\omega}$ to be selected.

Remark 1. *At this point, we could follow one of the backstepping procedures proposed in [10]. However, this precludes convergence of ω to ω_d except at $z = \varepsilon_1$. To illustrate this, consider the system*

$$\begin{bmatrix} \dot{z} \\ \dot{e}_1 \end{bmatrix} := \begin{bmatrix} (e_1 + \omega_d)Sz \\ -\dot{\omega}_d \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u =: a(z, e_1, q) + bu, \quad (45)$$

with ω_d given in (35), while the time derivative becomes $\dot{\omega}_d = -K_0\bar{\sigma}_1(z,q)(e_1+\omega_d)$, that is replacing ω_d with $e_1+\omega_d$ in (36) (this also applies to $\ddot{\omega}_d$ in (37)). Define the control Lyapunov function $V : \mathcal{S}^1 \times \mathbb{R} \times Q \to \mathbb{R}$ as $V(z, e_1, q) := V_0(z, q) + \frac{\gamma}{2}e_1^2$ with $\gamma \in \mathbb{R}_{>0}$. This gives

$$\begin{aligned} &\langle \nabla V(z, e_1, q), a(z, e_1, q) + bu \rangle \\ &\quad = \langle \nabla V_0(z, q), Sz \rangle (e_1 + \omega_d) + \gamma e_1(-\dot{\omega}_d + u). \quad (46) \end{aligned}$$

Choosing $u = -K_1 e_1 + \dot{\omega}_d - \frac{1}{\gamma}\langle \nabla V_0(z, q), Sz \rangle$ ensures that (46) is non-positive for all $K_1 \in \mathbb{R}_{>0}$, but results in $\dot{e}_1 = -K_1 e_1 - \frac{1}{\gamma}\langle \nabla V_0(z, q), Sz \rangle$, where the last term is non-zero for all $(z, q) \in \mathcal{S}^1 \times Q \setminus \Psi_0$. Also note that it is not possible to dominate the last term by choosing a very large γ, since γ must satisfy a rather strict upper bound to ensure that the critical points where $V_0(z, q) = 1$ are contained in D.

A. Exponentially stabilizing the error dynamics

The time derivative of κ_0, expressed in terms of the error variables, is given by

$$\alpha_1(z, e_1, q) := \dot{\kappa}_0 = -K_0\bar{\sigma}_1(z, q)(e_1 - K_0\bar{\sigma}_0(z, q)) \quad (47)$$

Selecting

$$\kappa_1(z, e_1, q) := -K_1 e_1 + \alpha_1(z, e_1, q) \quad (48)$$

yields $\dot{e}_1 = -K_1 e_1 + e_2$, which is exponentially stable during flow when $e_2 = 0$ (note that e_1 may still increase during jumps). Differentiating κ_1 with respect to time yields

$$\begin{aligned} \alpha_2(z, e, q) := \dot{\kappa}_1 = &- K_1(-K_1 e_1 + e_2) \\ &- K_0\bar{\sigma}_1(z, q)(e_2 + \kappa_1(z, e_1, q)) \quad (49) \\ &- K_0\bar{\sigma}_2(z, q)(e_1 - K_0\bar{\sigma}_0(z, q))^2. \end{aligned}$$

The control law $\kappa_2 : \mathcal{S}^1 \times \mathbb{R}^2 \times Q \to \mathbb{R}$ for u is chosen as

$$\kappa_2(z, e, q) := -K_2 e_2 + \alpha_2(z, e, q), \quad (50)$$

resulting in $\dot{e}_2 = -K_2 e_2$, achieving exponential stability during flow (as before e_2 may still increase during jumps).

B. Closed-loop system and dynamic switching logic

The closed-loop system, expressed in terms of error variables, with κ_1 defined in (48) and $u = \kappa_2$ defined in (50), is given by

$$\mathcal{H} : \begin{cases} \begin{bmatrix} \dot{z} \\ \dot{e} \end{bmatrix} = f(z, e, q), \ \dot{q} = 0 & (z, e, q) \in C \\ \begin{bmatrix} e^+ \\ q^+ \end{bmatrix} = \begin{bmatrix} h(z, e, q) \\ -q \end{bmatrix}, \ z^+ = z & (z, e, q) \in D, \end{cases} \quad (51)$$

where

$$f(z, e, q) := \begin{bmatrix} (e_1 + \kappa_0(z, q))Sz \\ -K_1 e_1 + e_2 \\ -K_2 e_2 \end{bmatrix}, \quad (52)$$

$$h(z, e, q) := \begin{bmatrix} e_1 + \kappa_0(z, q) - \kappa_0(z, -q) \\ e_2 + \kappa_1(z, q) - \kappa_1(z, -q) \end{bmatrix}. \quad (53)$$

The updated flow and jump sets will be defined shortly. Note that the evolution during flow exhibits a cascaded structure, while the evolution during jumps is not a cascade (since e^+ depends on z).

For the system $\mathcal{H}$, define the sets $\mathcal{A} = \mathcal{A}_0 \times \{e \in R^2 : e = 0\}$ and $\Psi = \Psi_0 \times \{e \in R^2 : e = 0\}$. Let $X := \mathcal{S}^1 \times \mathbb{R}^2 \times Q$, and define the Lyapunov function candidate $V : X \to \mathbb{R}$ as

$$V(z, e, q) := V_0(z, q) + \frac{1}{2}\gamma_1 e_1^2 + \frac{1}{2}\gamma_2 e_2^2, \quad (54)$$

which is positive definite with respect to $\mathcal{A}$ for $(\gamma_1, \gamma_2) \in \mathbb{R}^2_{>0}$. Further define $M(z, e) := \min_{q \in Q} V(z, e, q)$, and

$$C := \{(z, e, q) \in X : M(z, e) - V(z, e, q) \geq -\delta\} \quad (55)$$

$$D := \{(z, e, q) \in X : M(z, e) - V(z, e, q) \leq -\delta\}. \quad (56)$$

Since M depends not only on z, but also on e, this results in a dynamic switching condition which depends both on the yaw rate and yaw acceleration. The switching is influenced by the parameters (γ_1, γ_2), which we refer to as dynamic jump parameters.

C. Stability of the closed-loop system

Redefining the system $\mathcal{H}$ in (53) in terms of the original states $(z, \omega, \dot{\omega})$ yields a system of the form given in (12), without changing the nature of the system. Noting that $V, M, \Psi, \mathcal{A}$ may also be expressed in terms of the original states, $\mathcal{A}$ is globally asymptotically stable if $(\Psi \setminus \mathcal{A}) \cap C = \emptyset$, and

$$\langle \nabla V(z, e, q), f(z, e, q) \rangle < 0, \ \forall (z, e, q) \in X \setminus \Psi. \quad (57)$$

1) Ensuring non-increase during flow: The evolution of V during flow may be expressed as

$$\langle \nabla V(z, e, q), f(z, x, q) \rangle = -\phi_1^\top \Phi_1 \phi_1 - e^\top \Phi_2 e \quad (58)$$

where $\phi_1 := [\bar{\sigma}_0(z, q) \ e_1]^\top$ and

$$\Phi_1 := \begin{bmatrix} (1 + qkL\lambda\sigma_0(z))L\lambda K_0 & -\frac{(1+qkL\lambda\sigma_0(z))L\lambda}{2} \\ -\frac{(1+qkL\lambda\sigma_0(z))L\lambda}{2} & c_0\gamma_1 K_1 \end{bmatrix} \quad (59)$$

$$\Phi_2 := \begin{bmatrix} (1 - c_0)\gamma_1 K_1 & -\frac{1}{2}\gamma_1 \\ -\frac{1}{2}\gamma_1 & \gamma_2 K_2 \end{bmatrix}. \quad (60)$$

Here, $c_0 \in \mathbb{R}_{>0}$ may be selected in the interval $0 < c_0 < 1$. Note the off-diagonal terms which are intentionally not canceled since this would preclude exponential convergence of e (during flow). Instead we achieve stability by selecting (K_0, K_1, K_2) and (γ_1, γ_2) such that Φ_1 and Φ_2 are positive definite. Since $0 < (1 + qkL\lambda\sigma_0(z))L\lambda \leq (1 + kL\lambda)L\lambda$, it follows that Φ_1 is positive definite if the following inequality is satisfied:

$$c_0\gamma_1 K_0 K_1 > \frac{(1+kL\lambda)L\lambda}{4}. \tag{61}$$

Similarly, Φ_2 is positive definite if

$$(1-c_o)\gamma_2 K_1 K_2 > \frac{\gamma_1}{4}. \tag{62}$$

2) Bound on dynamic jump parameters: It is possible to satisfy (61) and (62) for any value of (K_0, K_1, K_2) by selecting (γ_1, γ_2) sufficiently large. Simultaneously satisfying $(\Psi \setminus \mathcal{A}) \cap C = \emptyset$ is more challenging. It is trivial to verify that $e_1 = \omega = \kappa_0(z,q) = 0,\ \ \forall (z,e,q) \in \Psi$. Since $e_2 = 0 \implies \dot{\omega} = \kappa_1(z,q) = 0$, and $e_1 = \omega = \kappa_0(z,q) \implies \kappa_1 = 0$, we conclude that $(\omega, \dot{\omega}) = 0,\ \forall (z,e,q) \in \Psi$ (this result is expected since the control law vanishes in Ψ.) For jumps leaving Ψ, the following holds $\forall (z,e,q) \in \Psi$:

$$|e_1^+|^2 - |e_1|^2 = |\kappa_0(z,-q)|^2 \le K_0^2. \tag{63}$$

$$|e_2^+|^2 - |e_2|^2 = |\kappa_1(z,e_1,-q)|^2 \le K_1^2 K_0^2. \tag{64}$$

Using (63,64) and (39) gives, $\forall (z,e,q) \in \Psi \setminus \mathcal{A}$:

$$V(z,e,-q) - V(z,e,q) \le -\mu + \frac{1}{2}K_0^2\left(\gamma_1 + \gamma_2 K_1^2\right). \tag{65}$$

Hence, $(\Psi \setminus \mathcal{A}) \cap C = \emptyset$ if (K_0, K_1) and (γ_1, γ_2) satisfy

$$\frac{1}{2}\gamma_1 K_0^2 + \frac{1}{2}\gamma_2 K_1^2 K_0^2 < \mu - \delta. \tag{66}$$

The inequality (66) makes out sufficient conditions for allowable values of λ and θ. The condition may be relaxed by calculating a tighter bound in (63).

3) Selecting the parameters: To show that it is always possible to simultaneously satisfy (61,62) and (66), we propose the following procedure for systematically selecting the design parameters: First, select $0 < c_1 < 1$ as the fraction of the "remaining" synergy gap $(\mu - \delta)$ to be used on e_1, and let

$$\gamma_1 = 2c_1 \frac{\mu - \delta}{K_0^2}. \tag{67}$$

Inserting (67) into (61), followed by some algebraic manipulations, gives the following bound on K_1:

$$K_1 > K_0 \frac{(1+kL\lambda)L\lambda}{8c_o c_1(\mu - \delta)}. \tag{68}$$

Next, select $0 < c_2 < 1$ as the fraction of the "remaining" synergy gap $(\mu - \delta - \frac{1}{2}\gamma_1 K_0^2)$ to be used on e_2, and let

$$\gamma_2 = 2c_2 \frac{(1-c_1)(\mu-\delta)}{K_1^2 K_0^2}. \tag{69}$$

Finally, inserting (69) into (62), followed by some algebraic manipulations, we arrive at

$$K_2 > \frac{c_1 K_1}{4(1-c_o)c_2(1-c_1)}. \tag{70}$$

This reduces the design task to selecting three parameters (c_0, c_1, c_2) in the interval 0 to 1. Furthermore, (68) and (70) show that the controller gains (K_0, K_1, K_2) are linearly dependent, as one would expect.

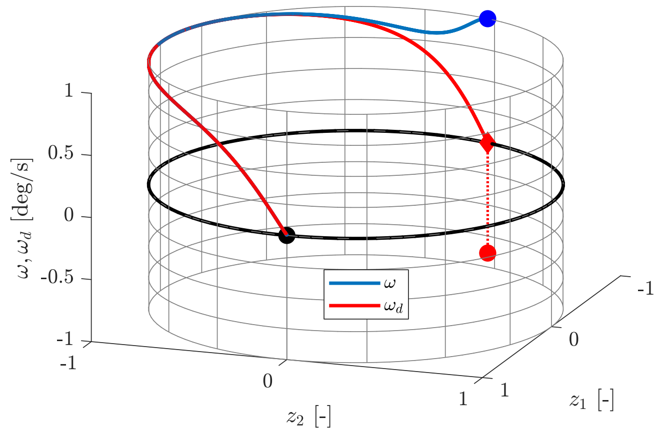

Fig. 2. Trajectory of ω and ω_d projected onto $\mathcal{S}^1 \times \mathbb{R}$. Initial values $z = z^{\pi-\theta}$, $\omega = 1$ deg/s, $\dot{\omega} = 0$ and $q = -1$. This results in $\omega_d(0,0) = -0.8874$ deg/s (red dot), followed by a jump to $\omega_d(0,1) = 0$ (red diamond). The yaw rate ω starts at the blue dot, converges to the desired value ω_d, and then follows the trajectory generated by ω_d to $z = \varepsilon_1$, $\omega = 0$ (black dot).

VI. Simulations

In this section a small case study is presented. We select $K_0 = \frac{\pi}{180}$ (corresponding to 1 deg/s), which is a reasonable heading velocity for ships. Further discussion of the design parameters with respect to ship dynamics is outside the scope of this paper. A brief discussion is provided in the following:

Parameters for the controller generating ω_d: Selecting large regularization parameter λ yields faster convergence, at the cost of large commanded accelerations in the vicinity of $z = \varepsilon_1$. Increasing the angular shift of the critical points θ yields a larger synergy gap μ. While the peak value of ω_d is not influenced by θ, the peak values of $(\dot{\omega}_d, \ddot{\omega}_d)$ increase with larger values of θ. This effect diminishes close to $z = \varepsilon_1$, but is still undesired. Large hysteresis half-width δ increases the robustness margin towards disturbances of z, at the cost of stricter bounds on (K_1, K_2) (since $\mu - \delta$ is reduced). We select $\lambda = 0.9$, $\theta = \frac{\pi}{9}$ (20 degrees), and $\Delta = \frac{\pi}{36}$ (5 degrees).

Parameters for the smoothing controller: Increasing (c_1, c_2) relaxes the constraints on the controller gains (K_1, K_2), at the cost of reduced robustness margin, while c_0 is a trade-off between the bounds on K_1 and K_2. We select $c_0 = c_1 = c_2 = 0.5$, which gives $\gamma_1 = 375.2$ and $K_1 > 0.0352$. We select $K_1 = 0.1\ s^{-1}$, which gives $\gamma_2 = 18760$ and $K_2 > 0.1$. Finally, we select $K_2 = 0.2\ s^{-1}$.

A simulation is initialized with $z(0,0) = z^{\pi-\theta}$, where the arguments are the time variable t and jump variable j. Furthermore, we select $\omega(0,0) = 1$ deg/s and $\dot{\omega}(0,0) = 0$. Note that $z(0,0) = z^{\pi-\theta}$ is close to the critical point for $q = 1$, $e = 0$. We initialize with $q(0,0) = -1$, resulting in $\omega_d(0,0) = -0.8874$ deg/s, corresponding to traversing the circle in the clockwise direction. However, since the velocity already has a large value in the counterclockwise direction, q is automatically toggled; V_0 increases but V decreases overall due to the decrease in e_1. This results in a jump to $\omega_d(0,1) = 0$. Following this, z flows to ε_1 in the

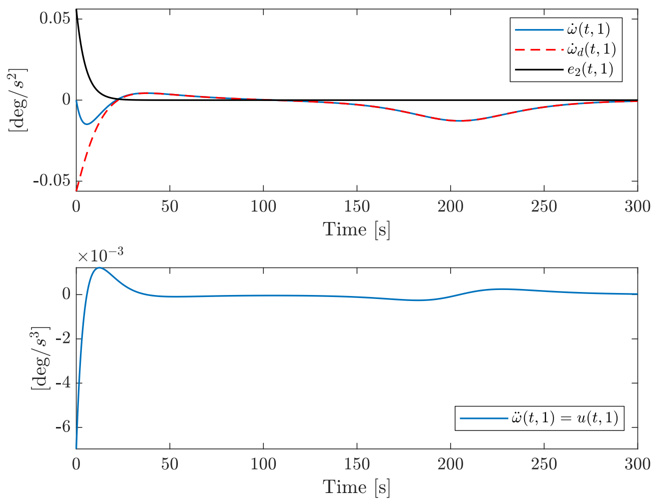

Fig. 3. Acceleration signals (top) and control input (bottom) corresponding to the trajectory in Figure 2. Only the values after the initial jump is shown.

counterclockwise direction, with $\omega(t, j)$ converging towards $\omega_d(t, j)$. The resulting trajectory in the space of $\mathcal{S}^1 \times \mathbb{R}$, with yaw rate as the vertical axis, is shown in Figure 2, while the corresponding yaw acceleration and control input is shown in Figure 3. Note the moderate convergence rate of the acceleration error e_2 to zero, despite selecting K_1 and K_2 significantly larger than their lower bounds. This indicates that the bounds on the controller gains are not restrictive for the intended application.

Convergence of $\tilde{\psi}$ within 1 degree is achieved in 300 seconds. For this example, disabling the switching and thus changing the direction of motion yields convergence within 1 degree in 265 seconds. However, changing the direction of motion in such a manner is unpredictable for neighboring ships. For autonomous ships, decisiveness and clearly showing the intentions is a desired property [22]. Decisiveness is an inherent property of the hybrid control system presented herein.

VII. Conclusions and Future Work

This paper presented a hybrid feedback controller which globally and robustly stabilizes a point on the unit circle, where continuity of velocity and acceleration is achieved by controlling the velocity through a double integrator. The velocity and acceleration converges exponentially to desired reference values. Stability is shown using a hybrid Lyapunov function. The parameters of the Lyapunov function must satisfy certain bounds to ensure that the critical points where the control law vanishes are contained in the jump set. This results in mild constraints on the controller gains. A constructive way of choosing the controller gains is proposed, which also shows that the controller gains are linearly dependent.

The control system architecture, reminiscent of a cascaded system, has some desired properties which makes it suitable for ships and vehicles with similar dynamics. In future work, the controller will be adopted for heading control of ships, using the thrusters or rudders as control inputs, and considering uncertainty in the ship dynamics. Applications may for instance be a rudder-actuated ship, where continuity of the commanded rudder angle is desired.

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