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Adaptive set-point regulation of linear $n+1$ hyperbolic systems with uncertain affine boundary condition using collocated sensing and control☆

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ABSTRACT

We solve an adaptive control problem for $n+1$ hyperbolic systems using collocated sensing and control, extending recent results for adaptive control of 2×2 systems and systems with non-collocated sensing and control. The boundary condition has an affine form with both unknown reflective and additive parameters and can be used to model well–reservoir interactions in oil and gas drilling where properties of the reservoir are unknown. Boundedness of the system states in the L_2-norm, and convergence to a steady state profile satisfying a control objective relevant to the drilling application, are proved. The state estimation error is shown to converge to zero in the L_2-norm and one of the boundary parameter estimates (modelling the reservoir pressure in the drilling application) is shown to converge to the true parameter value. The design is illustrated in a simulation example.

1. Introduction

1.1. Motivation

$n+1$ hyperbolic PDE systems, that is, hyperbolic PDEs with n invariants propagating in one direction and one single invariant propagating in the opposite direction, can be used to model multi-phase fluid flow systems or various interconnected single phase flow systems. Although the systems studied can be used to model a wide range of real-world applications [1], our main motivation is a particular application in offshore oil and gas drilling where a drilling fluid is circulated down the drill-string, through the drill-bit and up the annulus to the top of the well where a choke and back-pressure pump are installed to control the top-side pressure. Tight bottom-hole pressure control is important to avoid unwanted incidents such as inflow of oil and gas into the well, called a *kick*, or *loss* of drilling fluid into the reservoir. The problem of controlling the bottom-hole pressure by regulating the top-side pressure and flow is a challenging problem for a number of reasons. (1) The well can be up-to 10 kilometres long and the distributed effects (fluid compressibility) are significant. (2) Flow and pressure measurements are often only available top-side at the rig and the bottom-hole state must be estimated. (3) Properties of the oil and gas reservoir are often unknown. To avoid drilling incidents such as a kick or loss, the control objective is to make the bottom-hole pressure match the reservoir pressure. So, if the reservoir pressure is unknown, the control set-point is itself also unknown.

1.2. Previous work and contributions

In this paper we derive an adaptive estimation and control scheme for $n+1$ hyperbolic systems with unknown boundary parameters in an affine boundary condition on one end and with collocated sensing and control at the opposite end only. The infinite backstepping method has been used extensively to design observers and controllers for hyperbolic systems. The first result for 2×2 systems was presented in [2] and later extended to $n+1$ systems with non-collocated sensing and control in [3], $n+m$ systems in [4], and to $n+1$ systems with collocated sensing and control in [5] (using the backstepping transformation from [4]). For systems with uncertain boundary parameters, adaptive observers for $n+1$ and $n+m$ system with sensing at the same boundary as the uncertain parameters was presented in [6,7] respectively, and for 2×2 systems with sensing only on the opposite boundary as the uncertain parameters in [8]. The extension of the aforementioned papers to output-feedback control to a zero-state was presented in [9–11] respectively. However, all of these extensions used linear boundary conditions, that is only one single unknown reflective boundary parameter. To model well–reservoir interactions with unknown reservoir pressures as

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described in the previous section, the boundary conditions must have an affine form where both the parameters describing the reflective terms and the additive terms are unknown. Extending the result in [11], control of 2×2 hyperbolic systems with unknown affine boundary conditions using collocated sensing and control was presented in [12] and using non-collocated sensing and control in [13].

In this paper, we extend the observer result from [8] to $n+1$ systems, while avoiding time-delayed estimates as in [14]. The method in [8] rely on a set of computationally expensive time-varying backstepping kernels that require continuous, online, re-computation. While a direct extension of the method in [8] to $n+1$ systems using the corresponding time-varying backstepping kernels for $n+1$ systems might be feasible, the well-posedness proof of such time-varying kernels seems to be non-trivial and the computational cost might render the method impractical for use in real-world applications. In this paper, we propose an alternative design where these difficulties (proof of well-posedness and computational cost) are overcome by the use of two cascaded backstepping transformations. The purpose of the first transformation, which is time invariant, is to obtain a static parametric model on which to base the parameter update laws. The second transformation is time variant, and provide output injection kernels that ensure exponential stability at the origin of the state estimation error system. The control part is a generalization of [12] to $n+1$ systems, but using a solution strategy similar to [13]. The formal problem statement is given in Section 1.4. The state observer and parameter adaptive laws are presented in Section 2. The control design with corresponding backstepping target system and reference model is presented in Section 3 with the main result stated in Theorem 1. Stability of the interconnected state estimation error system from Section 2 and tracking error system from Section 3 is given in Section 4 and culminates in the final proof of Theorem 1. An illustrative simulation example is given in Section 5, and some concluding remarks are offered in Fig. 7.

1.3. Notation

For a signal $z : [0,1] \times [0,\infty) \to \mathbb{R}^n$, partial derivatives with respect to i.e. space are denoted z_x or $\partial_x z_i$ for each element $i = 1, \dots, n$. The L_2-norm is denoted

$$\|z\| := \sqrt{\int_0^1 z^T(x,t)z(x,t)dx}. \tag{1}$$

For $f : [0,\infty) \to \mathbb{R}$, we use the vector spaces

$$f \in \mathscr{L}_p \leftrightarrow \left(\int_0^\infty |f(t)|^p dt\right)^{\frac{1}{p}} < \infty \tag{2}$$

for $p \geq 1$ with the particular case

$$f \in \mathscr{L}_\infty \leftrightarrow \sup_{t\geq 0} |f(t)| < \infty. \tag{3}$$

Derivatives with respect to time are denoted $\dot{f}$. If not otherwise stated, a statement for a variable with subscript i refers to all variables with subscript $i = 1, \dots, n$. For two vectors $a, b \in \mathbb{R}^n$, $a \odot b$ denotes the Hadamard product with elements $(a \odot b)_i = a_i b_i$.

1.4. Problem formulation

Consider the system

$$u_t + \Lambda u_x = \Sigma(x)u + \omega(x)v \tag{4a}$$

$$v_t - \mu v_x = \varpi(x)u \tag{4b}$$

where $u : [0,1] \times [0,\infty) \to \mathbb{R}^n$ is the *upward* propagating Riemann invariants, $v : [0,1] \times [0,1) \to \mathbb{R}$ is the single *downward* propagating Riemann invariant, $\Sigma : [0,1] \to \mathbb{R}^{n\times n}$ with diagonal terms being zero, $\omega : [0,1] \to \mathbb{R}^{n\times 1}$, $\varpi : [0,1] \to \mathbb{R}^{1\times n}$, and $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and μ satisfying $-\mu < 0 < \lambda_1 < \cdots < \lambda_n$. The boundary conditions are given by

$$u(0,t) - rv(0,t) = k(\theta - v(0,t)) \tag{5a}$$

$$v(1,t) = U(t) \tag{5b}$$

for some known constant $r = \{r_i\}_{1\leq i\leq n} \in \mathbb{R}^n$, unknown parameters $\theta \in \mathbb{R}$ and $k = \{k_i\}_{1\leq i\leq n} \in \mathbb{R}^n$, but with known lower bounds $\underline{k}_i \leq k_i$, and control signal $U : [0,\infty) \to \mathbb{R}$. The boundary condition (5a) can be written on the affine form $u(0,t) = (r-k)v(0,t) + k\theta$ where $(r-k)v(0,t)$ is the *reflective* term and $k\theta$ the *additive term*. However, since the reflective term contains both the known parameter r and the unknown parameter k, the proceeding stability analysis is simplified by combining the unknown elements in a *bilinear* form as in the right hand side in (5a). For the drilling application this form is particularly useful, since the right hand side of (5a) can be used to model the well–reservoir interaction, which is unknown, and the left hand side of (5a) models the natural boundary reflection in a well isolated from any reservoir.

We assume that system (4) with boundary conditions (5) and some initial conditions

$$u(x,0) =: u_{ic}(x) \tag{6a}$$

$$v(x,0) =: v_{ic}(x) \tag{6b}$$

form a well-posed system. We thus restrict the initial conditions and boundary condition to satisfy certain compatibility conditions. In addition, for a general class of control laws, including the control law U considered in this paper, it was shown in [15, Theorem 1.1] that the growth rate of system (4)–(6) is bounded by an exponential function, meaning that for any finite time, system (4)–(6) has a bounded solution.

The control objective is

$$\lim_{t\to\infty} \int_t^{t+T} |u(0,t) - rv(0,\tau)|\, d\tau = \lim_{t\to\infty} \int_t^{t+T} |\theta - v(0,\tau)|\, d\tau = 0 \tag{7}$$

for some constant $T > 0$. Related to the boundary conditions above, this control objective can be interpreted as *eliminating the effect of the external disturbance*, that is $k(\theta - v(0,t)) \to 0$. For the drilling application this corresponds to stopping any flow of fluids between the well and the reservoir. The integral operator in (7) is included for technical reasons: Since in the following, only solution in the $L_2([0,1])$-norm is studied and not point-wise stability, the control objective is formulated over a time-interval. As a secondary objective we also want to generate reliable estimates of the states with particular interest in $u(0,t)$, $v(0,t)$, and reliable estimates of the unknown parameter θ. That is, for some estimate $\hat{\theta}_i(t)$ we seek

$$\lim_{t\to\infty} \left|\theta - \hat{\theta}_i(t)\right| = 0. \tag{8}$$

For both objectives, we only rely on the boundary measurements

$$y(t) := u(1,t) \tag{9}$$

Remark 1. Identification of the parameter k is intractable since identifiability is lost when the control objective is achieved.

2. State and parameter estimation

To estimate the unknown states we design an observer in terms of the state estimates $(\hat{u}, \hat{v})$. Consider the observer

$$\hat{u}_t + \Lambda\hat{u}_x = \Sigma(x)\hat{u} + \omega(x)\hat{v} + P^+(x,t)(y(t) - \hat{u}(1,t)) \tag{10a}$$

$$\hat{v}_t - \mu\hat{v}_x = \varpi(x)\hat{u} + P^-(x,t)(y(t) - \hat{u}(1,t)) \tag{10b}$$

$$\hat{u}_i(0,t) - r_i\hat{v}(0,t) = \hat{k}_i(t)(\hat{\theta}_i(t) - \hat{v}(0,t)) \tag{10c}$$

$$\hat{v}(1,t) = U(t) \tag{10d}$$

$$\hat{u}(x,0) = \hat{u}_{ic}(x) \tag{10e}$$

$$\hat{v}(x,0) = \hat{v}_{ic}(x) \tag{10f}$$

where P^+ and P^- are spatially and temporary varying injection gains and $\hat{k}(t) = \{\hat{k}_i(t)\}_{1\le i\le n}$ and $\hat{\theta}(t) = \{\hat{\theta}_i(t)\}_{1\le i\le n}$ are estimates of k and θ respectively. The state estimation error $\tilde{u} = u - \hat{u}$, $\tilde{v} = v - \hat{v}$ then satisfies

$$\tilde{u}_t + \Lambda\tilde{u}_x = \Sigma(x)\tilde{u} + \omega(x)\tilde{v} - P^+(x,t)\tilde{u}(1,t) \tag{11a}$$

$$\tilde{v}_t - \mu\tilde{v}_x = \varpi(x)\tilde{u} - P^-(x,t)\tilde{u}(1,t) \tag{11b}$$

$$\tilde{u}(0,t) = (r - \hat{k}(t))\tilde{v}(0,t) + \epsilon(t) \tag{11c}$$

$$\tilde{v}(1,t) = 0. \tag{11d}$$

where we have simplified notation by defining $\epsilon(t) = \{\epsilon_i(t)\}_{1\le i\le n}$,

$$\epsilon_i(t) := k_i(\theta - v(0,t)) - \hat{k}_i(t)(\hat{\theta}_i(t) - v(0,t)). \tag{12}$$

Remark 2. The observer boundary condition (10c) allows the flexibility of using n independent estimates $\hat{\theta}_i$ of the same parameter θ. This flexibility is necessary when proving convergence of the state estimation error system, where we rely on a property of the adaptive law to be designed, namely $\epsilon_i \in \mathscr{L}_2$. This property only follows if a redundant set of parameters $\{\theta_i\}_{1\le i\le n}$ are estimated independently for each boundary condition (10c). Removing this over-parametrization is desirable, but seems to be non-trivial.

Lemma 1. *The backstepping transformation*

$$\tilde{u}(x,t) = \alpha(x,t) + \int_x^1 P^u(x,\xi)\alpha(\xi,t)d\xi \tag{13a}$$

$$\tilde{v}(x,t) = \beta(x,t) + \int_x^1 P^v(x,\xi)\alpha(\xi,t)d\xi, \tag{13b}$$

with kernels $P^u : \mathscr{T}_1 \to \mathbb{R}^{n\times n}$, $P^v : \mathscr{T}_1 \to \mathbb{R}^{1\times n}$ satisfying

$$P^u_\xi\Lambda + \Lambda P^u_x = \Sigma(x)P^u + \omega(x)P^v \tag{14a}$$

$$P^v_\xi\Lambda - \mu P^v_x = \varpi(x)P^u, \tag{14b}$$

$$\Sigma(x) = P^u(x,x)\Lambda - \Lambda P^u(x,x) \tag{15a}$$

$$\varpi(x) = P^v(x,x)\Lambda + \mu P^v(x,x), \tag{15b}$$

$$P^u_{ij}(0,\xi) = r_iP^v_j(0,\xi),\ 1 \le i \le j \le n \tag{16}$$

and

$$P^u_{ij}(x,1) = \frac{\Sigma_{ij}(x)}{\lambda_j - \lambda_i}, \quad 1 \le j < i \le n \tag{17}$$

defined on the triangular domain $\mathscr{T}_1 = \{(x,\xi)|0 \le x \le \xi \le 1\}$, is invertible and maps target system

$$\alpha_t(x,t) + \Lambda\alpha_x(x,t) = \omega(x)\beta(x,t) - M^+(x,t)\alpha(1,t) - \int_x^1 D^+(x,\xi)\beta(\xi,t)d\xi \tag{18a}$$

$$\beta_t(x,t) - \mu\beta_x(x,t) = -\int_x^1 d^-(x,\xi)\beta(\xi,t)d\xi \tag{18b}$$

$$\alpha(0,t) = \int_0^1 H(\xi,t)\alpha(\xi,t)d\xi + (r - \hat{k}(t))\beta(0,t) + \epsilon(t) \tag{18c}$$

$$\beta(1,t) = 0 \tag{18d}$$

where $M^+ = \{m^+_{ij}(x,t)\}_{1\le i\le j\le n}$ is an upper triangular matrix to be decided, D^+ and d^- satisfy

$$D^+(x,\xi) = P^u(x,\xi)\omega(\xi) - \int_x^\xi P^u(x,s)D^+(s,\xi)ds \tag{19a}$$

$$d^-(x,\xi) = P^v(x,\xi)\omega(\xi) - \int_x^\xi P^v(x,s)D^+(s,\xi)ds, \tag{19b}$$

and $H(\xi,t) = \{h_{ij}(\xi,t)\}_{1\le i,j\le n}$ is defined by

$$H(\xi,t) := -P^u(0,\xi) + (r - \hat{k}(t))P^v(0,\xi), \tag{20}$$

into the error system (11) with

$$P^+(x,t) = P^u(x,1)\Lambda + M^+(x,t) + \int_x^1 P^u(x,\xi)M^+(\xi,t)d\xi \tag{21a}$$

$$P^-(x,t) = P^v(x,1)\Lambda + \int_x^1 P^v(x,\xi)M^+(\xi,t)d\xi. \tag{21b}$$

Moreover, the kernel equation (14)–(17) has a unique solution.

The target system (18), but without the $M^+(x,t)\alpha(1,t)$ term, and injection gains $P^+(x) = P^u(x,1)\Lambda$ and $P^-(x) = P^v(x,1)\Lambda$, was first used in [5] for collocated observer design for $n+1$ systems, which itself was a straightforward application of the kernel equations derived in [4]. The effect of including the $M^+(x,t)\alpha(1,t)$ term in the target system can be seen by substituting $M^+(x,t)\alpha(1,t)$ for $\alpha(x,t)$ in (13) showing the origin of the injection gains (21). The proof of Lemma 1 is therefore similar to [4] and omitted.

The advantage of transforming the error system to the form (18) is that, for $t \ge \mu^{-1}$, $\beta \equiv 0$ and the α-dynamics can be solved for each element α_i. This solution is exploited in the next lemma to obtain a bilinear parametric model relating the unknown parameters to known signals.

Lemma 2. *Let $\lambda := \min_i \lambda_i = \lambda_1$. For $t \ge t_F := \mu^{-1} + 2\lambda^{-1}$,*

$$\vartheta_i(t) = k_i(\theta + \psi(t)), \quad 1 \le i \le n \tag{22}$$

where ϑ and ψ are known signals defined by

$$\begin{aligned}\vartheta_i(t) = {} & \tilde{y}_i(t + \lambda_i^{-1} - \lambda^{-1}) \\ & + \hat{k}_i(t - \lambda^{-1})\left(\hat{\theta}_i(t - \lambda^{-1}) + \psi(t)\right) \\ & - \sum_{j=i}^n \int_{t-\lambda^{-1}}^{t+\lambda_i^{-1}-\lambda^{-1}} m^+_{ij}((\tau - t + \lambda^{-1})\lambda_i, \tau)\tilde{y}_j(\tau)d\tau \\ & - \sum_{j=1}^n \int_0^1 h_{ij}(\xi, t - \lambda^{-1})\Bigg(\tilde{y}_j(t + \lambda_j^{-1}(1-\xi) - \lambda^{-1}) \\ & - \sum_{l=j}^n \int_{t-\lambda^{-1}}^{t+\lambda_j^{-1}(1-\xi)-\lambda^{-1}} m^+_{jl}(\xi + \lambda_j(\tau + \lambda^{-1} - t), \tau) \\ & \times \tilde{y}_l(\tau)d\tau\Bigg)d\xi,\end{aligned} \tag{23}$$

$$\psi(t) \equiv -\hat{v}(0, t-\lambda^{-1}) + \sum_{i=1}^{n} \int_0^1 P_i^v(0,\xi) \times \left(\sum_{j=1}^{n} \int_{t-\lambda^{-1}}^{t+\lambda_i^{-1}(1-\xi)-\lambda^{-1}} m_{ij}^+(\xi + \lambda_i(\tau + \lambda^{-1} - t), \tau) \times \tilde{y}_j(\tau)d\tau - \tilde{y}_i(t + \lambda_i^{-1}(1-\xi) - \lambda^{-1}) \right) d\xi, \tag{24}$$

and

$$\tilde{y}_i(t) := \alpha_i(1,t) = y_i(t) - \hat{u}_i(1,t). \tag{25}$$

Proof. Since $\beta \equiv 0$ for $t \geq \mu^{-1}$, we have on component form

$$\partial_t \alpha_i + \lambda_i \partial_x \alpha_i = \sum_{j=i}^{n} m_{ij}^+(x,t)\tilde{y}_j(t) \tag{26a}$$

$$\alpha_i(0,t) = \epsilon_i(t) + \sum_{j=1}^{n} \int_0^1 h_{ij}(\xi,t)\alpha_j(\xi,t)d\xi \tag{26b}$$

which, by integrating along the characteristic lines, can be shown to have the *implicit* solution

$$\alpha_i(x,t) = \sum_{j=i}^{n} \int_{t+\lambda_i^{-1}(x_0-x)}^{t} m_{ij}^+(x + \lambda_i(\tau - t), \tau)\tilde{y}_j(\tau)d\tau + \alpha_i(x_0, t + \lambda_i^{-1}(x_0 - x)) \tag{27}$$

valid for all $t \geq \mu^{-1} + \lambda_i^{-1}$ and some $x_0 \in [0,1]$. Selecting $x_0 = 0$ and inserting (18c) yield

$$\alpha_i(x,t) = \sum_{j=i}^{n} \int_{t-\lambda_i^{-1}x}^{t} m_{ij}^+(x + \lambda_i(\tau - t), \tau)\tilde{y}_j(\tau)d\tau + \sum_{j=1}^{n} \int_0^1 h_{ij}(\xi, t - \lambda_i^{-1}x)\alpha_j(\xi, t - \lambda_i^{-1}x)d\xi + \epsilon_i(t - \lambda_i^{-1}x). \tag{28}$$

Selecting $x_0 = 1$ and inserting (25) yield

$$\alpha_i(x,t) = -\sum_{j=i}^{n} \int_{t}^{t+\lambda_i^{-1}(1-x)} m_{ij}^+(x + \lambda_i(\tau - t), \tau)\tilde{y}_j(\tau)d\tau + \tilde{y}_i(t + \lambda_i^{-1}(1-x)). \tag{29}$$

We have from (13b) for $t \geq \mu^{-1}$, and (29) that

$$\begin{aligned} v(0,t) &= \hat{v}(0,t) + \int_0^1 P^v(0,\xi)\alpha(\xi,t)d\xi \\ &= \hat{v}(0,t) + \sum_{i=1}^{n} \int_0^1 P_i^v(0,\xi) \Bigg(\tilde{y}_i(t + \lambda_i^{-1}(1-\xi)) \\ &\quad - \sum_{j=i}^{n} \int_{t}^{t+\lambda_i^{-1}(1-\xi)} m_{ij}^+(\xi + \lambda_i(\tau - t), \tau)\tilde{y}_j(\tau)d\tau \Bigg) d\xi. \end{aligned} \tag{30}$$

Thus,

$$v(0, t - \lambda^{-1}) = -\psi(t). \tag{31}$$

Next, inserting the right hand side of (29) into the left hand side of (28) evaluated at $x = 1$ and $t = t + \lambda_i^{-1} - \lambda^{-1} \leq t$ yields

$$\begin{aligned} \tilde{y}_i(t + \lambda_i^{-1} - \lambda^{-1}) &= \epsilon_i(t - \lambda^{-1}) \\ &+ \sum_{j=i}^{n} \int_{t-\lambda^{-1}}^{t+\lambda_i^{-1}-\lambda^{-1}} m_{ij}^+((\tau - t + \lambda^{-1})\lambda_i, \tau)\tilde{y}_j(\tau)d\tau \\ &+ \sum_{j=1}^{n} \int_0^1 h_{ij}(\xi, t - \lambda^{-1})\alpha_j(\xi, t - \lambda^{-1})d\xi \\ &= \sum_{j=i}^{n} \int_{t-\lambda^{-1}}^{t+\lambda_i^{-1}-\lambda^{-1}} m_{ij}^+((\tau - t + \lambda^{-1})\lambda_i, \tau)\tilde{y}_j(\tau)d\tau \\ &+ k_i(\theta - v(0, t - \lambda^{-1})) \\ &- \hat{k}_i(t - \lambda^{-1})(\hat{\theta}_i(t - \lambda^{-1}) - v(0, t - \lambda^{-1})) \\ &+ \sum_{j=1}^{n} \int_0^1 h_{ij}(\xi, t - \lambda^{-1}) \Bigg(\tilde{y}_j(t + \lambda_j^{-1}(1-\xi) - \lambda^{-1}) \\ &- \sum_{l=j}^{n} \int_{t-\lambda^{-1}}^{t+\lambda_j^{-1}(1-\xi)-\lambda^{-1}} m_{jl}^+(\xi + \lambda_j(\tau + \lambda^{-1} - t), \tau) \\ &\times \tilde{y}_l(\tau)d\tau \Bigg) d\xi \end{aligned} \tag{32}$$

which is equivalent to (22) in view of (23)–(25) and (31).

Using the bilinear parametric model (22) we can now design adaptive update laws for the parameter estimates based on the gradient method [16, Theorem 4.5.2]

Lemma 3. *Let $\tilde{\theta}_i(t) = \theta - \hat{\theta}_i(t)$, $\tilde{k}_i(t) = k - \hat{k}_i(t)$ and $\tilde{\vartheta}_i(t) := \vartheta_i(t) - \hat{k}_i(t)(\hat{\theta}_i(t) + \psi(t))$. The adaptive laws*

$$\dot{\hat{\theta}}_i = \gamma_{\theta_i} \frac{\tilde{\vartheta}_i(t)}{1 + \psi^2(t)} \tag{33a}$$

$$\dot{\hat{k}}_i = \gamma_{k_i} \left[\hat{\theta}_i(t) + \psi(t) \right] \frac{\tilde{\vartheta}_i(t)}{1 + \psi^2(t)} \tag{33b}$$

for $t \geq t_F$ and $\dot{\hat{\theta}}_i = \dot{\hat{k}}_i = 0$ otherwise, where $\gamma_{\theta_i}, \gamma_{k_i} > 0$ are the adaptation gains, have the following properties:

(1) $\hat{\theta}_i, \hat{k}_i \in \mathscr{L}_\infty$.
(2) $\dot{\hat{\theta}}_i, \dot{\hat{k}}_i \in \mathscr{L}_2 \cap \mathscr{L}_\infty$.
(3) $\pi_i := \frac{\epsilon_i}{\sqrt{1+v^2(0,\cdot)}} \in \mathscr{L}_\infty \cap \mathscr{L}_2$.
(4) If ψ is bounded for almost all $t \geq 0$ and $(\hat{\theta}_i + \psi) \in \mathscr{L}_2$, then $\tilde{\theta} \in \mathscr{L}_2$, $\hat{\theta}_i$ converges to θ and $\hat{k}_i$ converges to some constant.

Proof. Differentiating

$$V_{0_i}(t) = \frac{k_i}{2\gamma_{\theta_i}} \tilde{\theta}_i^2(t) + \frac{1}{2\gamma_{k_i}} \tilde{k}_i^2(t) \tag{34}$$

with respect to time for $t \geq t_F$ and inserting (33) yield

$$\begin{aligned} \dot{V}_{0_i}(t) &= -k_i \frac{1}{\gamma_{\theta_i}} \tilde{\theta}_i(t)\dot{\hat{\theta}}_i(t) - \frac{1}{\gamma_{k_i}} \tilde{k}_i(t)\dot{\hat{k}}_i(t) \\ &= -k_i \tilde{\theta}_i(t) \frac{\tilde{\vartheta}_i(t)}{1 + \psi^2(t)} - \tilde{k}_i(t) \left[\hat{\theta}_i(t) + \psi(t) \right] \frac{\tilde{\vartheta}_i(t)}{1 + \psi^2(t)} \\ &= -\frac{\tilde{\vartheta}_i(t)}{1 + \psi^2(t)} \left(k_i \tilde{\theta}_i(t) + \tilde{k}_i(t) \left[\hat{\theta}_i(t) + \psi(t) \right] \right) \\ &= -\frac{\tilde{\vartheta}_i^2(t)}{1 + \psi^2(t)} \leq 0 \end{aligned} \tag{35}$$

implying $V_{0_i} \in \mathscr{L}_\infty$ and Property 1. Integrating $\dot{V}_{0_i}$ from $t = 0$ to $t = \infty$ yields

$$\int_0^\infty \frac{\tilde{\vartheta}_i^2(\tau)}{1 + \psi^2(\tau)} d\tau = V_{0_i}(0) - V_{0_i}(\infty). \tag{36}$$

Since V_{0_i} is a non-increasing function of time and bounded below, $V_{0_i}(\infty)$ is finite. From the adaptive law (33a) we immediately see

that $\dot{\hat{\theta}}_i \in \mathscr{L}_\infty \cap \mathscr{L}_2$. For the $\hat{k}_i$ update law for $t \geq t_F$, we have

$$\dot{\hat{k}}_i(t) \leq \gamma_{k_i} \left| \frac{\hat{\theta}(t) + \psi(t)}{\sqrt{1+\psi^2(t)}} \right| \left| \frac{\tilde{\vartheta}_i(t)}{\sqrt{1+\psi^2(t)}} \right| \tag{37}$$

which shows that also $\dot{\hat{k}}_i \in \mathscr{L}_\infty \cap \mathscr{L}_2$. That $\pi_i \in \mathscr{L}_\infty$ follows from Property 1. To show that $\pi_i \in \mathscr{L}_2$, let

$$\tilde{\Theta}_i(t) =: \left[k_i - \hat{k}_i(t), k_i\theta - \hat{k}_i(t)\hat{\theta}_i(t)\right]^T \tag{38a}$$

$$\Phi_i(t) =: \frac{1}{\sqrt{1+\psi^2(t)}} [\psi(t), 1]^T \tag{38b}$$

so that $\tilde{\vartheta}_i(t)(1+\psi^2(t))^{-\frac{1}{2}} = \Phi_i^T(t)\tilde{\Theta}_i(t)$. We have

$$\Phi_i^T(t)\tilde{\Theta}_i(t) = \Phi_i^T(t)\left(\int_{t-\lambda^{-1}}^{t} \dot{\tilde{\Theta}}_i(\tau)d\tau + \tilde{\Theta}_i(t-\lambda^{-1})\right) \tag{39}$$

which after rearranging and squaring both sides give the inequality

$$\begin{aligned}
&(\Phi_i^T(t)\tilde{\Theta}_i(t-\lambda^{-1}))^2 \\
&\leq 2(\Phi_i^T(t)\tilde{\Theta}_i(t))^2 + 2\left(\Phi_i^T(t)\int_{t-\lambda^{-1}}^{t} \dot{\tilde{\Theta}}_i(\tau)d\tau\right)^2 \\
&\leq 2(\Phi_i^T(t)\tilde{\Theta}_i(t))^2 + c\int_{t-\lambda^{-1}}^{t} \dot{\tilde{\theta}}_i^2(\tau) + \dot{\tilde{k}}_i^2(\tau)d\tau
\end{aligned} \tag{40}$$

for some constant $c > 0$. From (36), the first term is clearly integrable. For the second term, we have by changing the order of integration

$$\begin{aligned}
&\lim_{T\to\infty} \int_{\lambda^{-1}}^{T}\int_{t-\lambda^{-1}}^{t} \dot{\tilde{\theta}}_i^2(\tau)d\tau dt \\
&= \lim_{T\to\infty}\left[\int_0^{\lambda^{-1}}\int_{\lambda^{-1}}^{\tau+\lambda^{-1}} + \int_{\lambda^{-1}}^{T-\lambda^{-1}}\int_{\tau}^{\tau+\lambda^{-1}} + \int_{T-\lambda^{-1}}^{T}\int_{\tau}^{T}\right] \\
&\quad \times dt\dot{\tilde{\theta}}_i^2(\tau)d\tau.
\end{aligned} \tag{41}$$

The first inner integral evaluates to τ with $\tau \in [0, \lambda^{-1}]$, the second inner integral evaluates to λ^{-1}, and the third inner integral evaluates to $(T-\tau)$ with $\tau \in [T-\lambda^{-1}, T]$. Since all the inner integrals evaluate to λ^{-1} or less,

$$\lim_{T\to\infty}\int_{\lambda^{-1}}^{T}\int_{t-\lambda^{-1}}^{t} \dot{\tilde{\theta}}_i^2(\tau)dtd\tau \leq \lim_{T\to\infty} \lambda^{-1}\int_{\lambda^{-1}}^{T} \dot{\tilde{\theta}}_i^2(\tau)d\tau \tag{42}$$

which by Property 2 is bounded. The term involving $\dot{\tilde{k}}_i$ can similarly be shown to be integrable, showing that the left hand side of (40) is integrable, which after substituting (38a) back into the left hand side of (40) and using the definition (12) and (31) completes the proof of Property 3. The adaptive law (33a) is rewritten as

$$\dot{\tilde{\theta}}_i(t) = -f_i(t)\left(k_i\tilde{\theta}_i(t) + \tilde{k}_i(t)\left(\hat{\theta}_i(t) + \psi(t)\right)\right) \tag{43}$$

where $f_i(t) = \gamma_{\theta_i}/(1+\psi^2(t)) > 0$ for all $t > t_F$. Forming $V_{\theta_i}(t) = \frac{1}{2}\tilde{\theta}_i^2(t)$, time differentiating and applying Young's inequality to the cross term, we get

$$\begin{aligned}
\dot{V}_{\theta_i}(t) &= -f_i(t)k_i\tilde{\theta}_i^2(t) - \tilde{\theta}_i(t)f_i(t)\tilde{k}_i(t)\left(\hat{\theta}_i(t) + \psi_i(t)\right) \\
&\leq -\frac{k_i}{2}f_i(t)\tilde{\theta}_i^2(t) + \frac{1}{2k_i}f_i(t)\tilde{k}_i^2(t)\left(\hat{\theta}_i(t) + \psi_i(t)\right)^2.
\end{aligned} \tag{44}$$

Since by assumption for Property 4, ψ_i is bounded for almost all $t \geq 0$, it follows that $\operatorname{ess\,inf}_{t\geq 0} f_i(t) > 0$, which along with Property 1 and boundedness of $f_i(t)$, provide the existence of constants b_i and $c_i > 0$ such that

$$\dot{V}_{\theta_i}(t) \leq -c_i\tilde{\theta}_i^2(t) + g_i(t)\tilde{\theta}_i^2(t) + b_i\left(\hat{\theta}_i - \psi_i(t)\right)^2, \tag{45}$$

where $g_i(t) = 0$ almost everywhere and therefore $g_i(t) \in \mathscr{L}_1$. Since $(\hat{\theta}_i - \psi_i)^2 \in \mathscr{L}_1$, it follows from [17, Lemma D.6] (Lemma 10 in Appendix C) that $V_{\theta_i} \in \mathscr{L}_1 \cap \mathscr{L}_\infty$ which together with [18, Lemma 2.7] (Lemma 11 in Appendix C) imply $V_{\theta_i}, \tilde{\theta}_i \to 0$. Convergence in $\hat{k}_i$ to some constant can be shown by integrating (33b) from $t = 0$ to $T = \infty$ and applying Cauchy–Schwarz' inequality

$$\begin{aligned}
\int_0^T \left|\dot{\hat{k}}_i(\tau)\right| d\tau \leq &\gamma_{k_i}\sqrt{\int_0^T \left|\hat{\theta}_i(\tau) + \psi_i(\tau)\right|^2 d\tau} \\
&\times \sqrt{\int_0^T \left|\frac{\tilde{\vartheta}_i(\tau)}{1+\psi_i^2(\tau)}\right|^2 d\tau} < \infty
\end{aligned} \tag{46}$$

which, by Property 2 and $(\hat{\theta}_i + \psi_i)^2 \in \mathscr{L}_1$, shows that $\dot{\hat{k}}_i \in \mathscr{L}_1$. Then for any $\delta_i > 0$ there exists a T_i such that

$$\int_{T_i}^{\infty} \left|\dot{\hat{k}}_i(\tau)\right| d\tau < \delta_i. \tag{47}$$

Therefore,

$$\left|\hat{k}_i(t) - \hat{k}_i(T)\right| \leq \left|\int_{T_i}^{t} \dot{\hat{k}}_i(\tau)d\tau\right| \leq \int_{T_i}^{\infty} \left|\dot{\hat{k}}_i(\tau)\right| d\tau < \delta_i \tag{48}$$

which shows that $\hat{k}_i(t)$ has a limit as $t \to \infty$ and the second part of the proof of Property 4 is complete.

The last lemma in this section is related to stability of the estimation error system (11) which is achieved by specifying the matrix M^+, which appears in the output injection gains P^+ and P^-.

Lemma 4. *For $t \geq \mu^{-1}$, the backstepping transformation*

$$\alpha(x,t) = \bar{\alpha}(x,t) + \int_x^1 F(x,\xi,t)\bar{\alpha}(\xi,t)d\xi \tag{49}$$

where $F = \{f_{ij}\}_{1\leq i,j\leq n}$ is given by

$$f_{ij}(x,\xi,t) := \begin{cases} \bar{f}_{ij}(\xi - \lambda_j\lambda_i^{-1}x, t), & (j \geq i) \cap (\lambda_i\xi - \lambda_j x \geq 0) \\ 0, & \text{otherwise} \end{cases} \tag{50}$$

and $\bar{f}_{ij}$, defined for $j \geq i$, is the on-line solution to the Volterra integral equation

$$\bar{f}_{ij}(\xi,t) = h_{ij}(\xi,t) + \sum_{l=1}^{j}\int_0^{\xi} \frac{h_{il}(\lambda_l\lambda_j^{-1}s, t)}{\lambda_j\lambda_l^{-1}}\bar{f}_{lj}(\xi - s, t)ds, \tag{51}$$

which for every bounded $\hat{k}$ (recall the definition of h_{ij} in (20)) has a unique, bounded solution, is invertible and maps the sub-system (18a) and (18c) (recall that $\beta \equiv 0$ for $t \geq \mu^{-1}$) with

$$M^+(x,t) = F(x,1,t)\Lambda \tag{52}$$

into the target system

$$\bar{\alpha}_t(x,t) + \Lambda\bar{\alpha}_x(x,t) = \int_x^1 A(x,\xi,t)\bar{\alpha}(\xi,t)d\xi \tag{53a}$$

$$\bar{\alpha}(0,t) = \epsilon(t) + \int_0^1 \bar{H}(\xi,t)\bar{\alpha}(\xi,t)d\xi \tag{53b}$$

where $\bar{H}$ is the strictly lower triangular matrix

$$\bar{H}(\xi,t) := H(\xi,t) - F(0,\xi,t) + \int_0^{\xi} H(s,t)F(s,\xi,t)ds \tag{54}$$

and A is the solution to the Volterra integral equation

$$A(x,\xi,t) = -F_t(x,\xi,t) - \int_x^{\xi} F(x,s)A(s,\xi,t)ds \tag{55}$$

which has the property $\|A\| \in \mathscr{L}_2 \cap \mathscr{L}_\infty$.

Proof. Differentiating (49) with respect to time and space, inserting the dynamics (53), and integrating by parts yield

$$\begin{aligned}
&\alpha_t(x,t)+\Lambda\alpha_x(x,t)+M^+(x,t)\alpha(1,t)\\
&=\bar\alpha_t(x,t)+\Lambda\bar\alpha_x(x,t)+M^+(x,t)\alpha(1,t)\\
&\quad+\int_x^1 F_t(x,\xi,t)\bar\alpha(\xi,t)d\xi\\
&\quad+\int_x^1 F(x,\xi,t)\int_\xi^1 A(\xi,s,t)\bar\alpha(s,t)dsd\xi\\
&\quad+F(x,x,t)\Lambda\bar\alpha(x,t)-F(x,1,t)\Lambda\bar\alpha(1,t)\\
&\quad+\int_x^1 F_\xi(x,\xi,t)\Lambda\bar\alpha(\xi,t)d\xi\\
&\quad-\Lambda F(x,x,t)\bar\alpha(x,t)+\int_x^1\Lambda F_x(x,\xi,t)\bar\alpha(\xi,t)d\xi. \qquad (56)
\end{aligned}$$

From the definition (50), we have that f_{ij} satisfies

$$\lambda_j\partial_\xi f_{ij}+\lambda_i\partial_x f_{ij}=0 \qquad (57)$$

for all $(x,\xi)\in\mathscr{T}_1$. This fact together with (52) and (55) and $\alpha(1,t)=\bar\alpha(1,t)$ shows that the right hand side of (56) is zero, which verifies (18) (with $\beta\equiv 0$). For the boundary condition, we have

$$\begin{aligned}
&\bar\alpha(0,t)=\epsilon(t)-\int_0^1 F(0,\xi,t)\bar\alpha(\xi,t)d\xi\\
&+\int_0^1 H(\xi,t)\left(\bar\alpha(\xi,t)+\int_\xi^1 F(\xi,s,t)\bar\alpha(s,t)ds\right)d\xi\\
&=\int_0^1\left(H(\xi,t)-F(0,\xi,t)+\int_0^\xi H(s,t)F(s,\xi,t)ds\right)\\
&\quad\times\bar\alpha(\xi,t)d\xi+\epsilon(t), \qquad (58)
\end{aligned}$$

which by $\bar H$ as in (54) equals (53b). Substituting the solution (50) into the integral on the right hand side of (51) and $\bar f_{ij}(\xi,t)=f_{ij}(0,\xi,t)$ for $j\ge i$ on the left hand side yield

$$f_{ij}(0,\xi,t)=h_{ij}(\xi,t)+\sum_{l=1}^n\int_0^\xi h_{il}(s,t)f_{lj}(s,\xi,t)ds \qquad (59)$$

for $j\ge i$, which from the definition (54) shows that $\bar H$ is strictly lower triangular. Existence of a unique, bounded solution to (51), as well as the existence of an inverse transformation $\bar\alpha\to\alpha$ are shown in Lemma 9 (Appendix A). Lemma 8 (Appendix A) gives an upper bound for A in (55) in terms of $\|F(\cdot,t)\|$ and $\|F_t(\cdot,t)\|$ which again by Lemma 9 is bounded by $\hat k(t)$ and $\dot{\hat k}(t)$. By Lemma 3, $\|A\|\in\mathscr{L}_2\cap\mathscr{L}_\infty$ follows.

If $v(0,\cdot)$ is bounded, it follows from Lemma 3, Property 3 that $\epsilon_i\in\mathscr{L}_2\cap\mathscr{L}_\infty$. Using the fact that $\|A\|\in\mathscr{L}_2\cap\mathscr{L}_\infty$ and $\bar H$ is lower triangular, it is then possible to show that the target system (53) is exponentially stable and in turn that the state estimation error system (11) is exponentially stable. This stability result is proved in Section 4 in conjunction with a proof of stability for the closed loop tracking error discussed in the next section.

3. Stabilization and reference tracking

The stabilizing part of the control law and the reference tracking part is designed independently in the next to lemmas. Let

$$U(t)=U_{stab}(t)+U_{track}(t). \qquad (60)$$

Lemma 5. *The backstepping transformation*

$$w(x,t)=\hat u(x,t) \qquad (61a)$$

$$z(x,t)=\hat v(x,t)-\int_0^x K^u(x,\xi)\hat u(\xi,t)d\xi-\int_0^x K^v(x,\xi)\hat v(\xi,t)d\xi, \qquad (61b)$$

$$\zeta(x,t)=z(x,t)-\mu^{-1}\int_0^x G(x-\xi,t)z(\xi,t)d\xi \qquad (61c)$$

with kernels $K^u:\mathscr{T}_2\to\mathbb{R}^{1\times n}$, $K^v:\mathscr{T}_2\to\mathbb{R}$ and $G:\mathscr{T}_2\times[0,\infty)\to\mathbb{R}$ satisfying

$$\mu K^u_x-K^u_\xi\Lambda=K^u\Sigma+K^v\varpi \qquad (62a)$$

$$\mu K^v_x+K^v_\xi\mu=K^u\omega \qquad (62b)$$

$$\varpi=-K^u(x,x)\Lambda-\mu K^u(x,x) \qquad (62c)$$

$$K^v(x,0)\mu=K^u(x,0)\Lambda r, \qquad (62d)$$

$$G(x,t)=K^u(x,0)\Lambda\hat k(t)-\mu^{-1}\int_0^x G(x-\xi,t)K^u(\xi,0)\Lambda\hat k(t)d\xi \qquad (62e)$$

over the triangular domain $\mathscr{T}_2=\{(x,\xi)|0\le\xi\le x\le 1\}$, which has a unique, bounded solution for every bounded $\hat k$, and the control law (60) with

$$\begin{aligned}
U_{stab}(t)=&\int_0^1 K^u(1,\xi)\hat u(\xi,t)d\xi\\
&+\int_0^1 K^v(1,\xi)\hat v(\xi,t)d\xi\\
&+\mu^{-1}\int_0^1 G(1-\xi,t)z(\xi,t)d\xi, \qquad (63)
\end{aligned}$$

map the observer system (10) into the target system

$$\begin{aligned}
w_t(x,t)+\Lambda w_x(x,t)=&\Sigma(x)w(x,t)+\omega(x)z(x,t)\\
&+P^+(x,t)\bar\alpha(1,t)\\
&+\int_0^x C^+(x,\xi)w(\xi,t)d\xi\\
&+\int_0^x C^-(x,\xi)z(t,\xi)d\xi \qquad (64a)\\
\zeta_t(x,t)-\mu\zeta_x(x,t)=&\Omega^T(x,t)\alpha(1,t)\\
&+\Psi^T(x,t)(\hat k(t)\odot\hat\theta(t))\\
&+\int_0^x B(x,\xi,t)\zeta(\xi,t)d\xi \qquad (64b)\\
w_i(0,t)-r_i\zeta(0,t)=&\hat k_i(t)(\hat\theta_i(t)-\zeta(0,t)) \qquad (64c)\\
\zeta(1,t)=&U_{track}(t) \qquad (64d)
\end{aligned}$$

where $C^+:\mathscr{T}_2\to\mathbb{R}^{n\times n}$, $C^-:\mathscr{T}_2\to\mathbb{R}^n$, $\Omega:[0,1]\to\mathbb{R}^n$, $B:\mathscr{T}_2\times[0,\infty)\to\mathbb{R}$ and $\Psi:[0,1]\to\mathbb{R}^n$ are characterized in the proof, and B has the property $B(x,\xi,\cdot)\in\mathscr{L}_2\cap\mathscr{L}_\infty$.

Proof. The w-dynamics (64a) with

$$C^+(x,\xi)=\omega(x)K^u(x,\xi)+\int_\xi^x C^-(x,s)K^u(s,\xi)d\xi \qquad (65a)$$

$$C^-(x,\xi)=\omega(x)K^v(x,\xi)+\int_\xi^x C^-(x,s)K^v(s,\xi)d\xi \qquad (65b)$$

is identical to the target system in [4] and is verified by inserting (61) into (64a), using the dynamics (10a) and changing the order of integration. For (64b), differentiating (61b) with respect to time and space, inserting (10) and integrating by parts and collecting

similar terms yield

$$\begin{aligned}
&z_t(x,t)-\mu z_x(x,t)=\hat{v}_t(x,t)-\mu\hat{v}_x(x,t)\\
&-\int_0^x K^u(x,\xi)\big(-\Lambda\hat{u}_\xi(\xi,t)+\Sigma\hat{u}(\xi,t)+\omega\hat{v}(x,t)\\
&\qquad\qquad+P^+(\xi,t)\bar{\alpha}(1,t)\big)\,d\xi\\
&-\int_0^x K^v(x,\xi)\big(\mu\hat{v}_\xi(\xi,t)+\varpi\hat{u}(\xi,t)\\
&\qquad\qquad+P^-(\xi,t)\bar{\alpha}(1,t)\big)\,d\xi\\
&+\mu K^u(x,x)\hat{u}(x,t)+\mu\int_0^x K^u_x(x,\xi)\hat{u}(\xi,t)d\xi\\
&+\mu K^v(x,x)\hat{v}(x,t)+\mu\int_0^x K^v_x(x,\xi)\hat{v}(\xi,t)d\xi\\
&=\Omega_0^T(x,t)\bar{\alpha}(1,t)+\Psi_0^T(x)(\hat{k}(t)\odot\hat{\theta}(t)-\hat{k}(t)z(0,t))
\end{aligned}\tag{66}$$

provided K^u and K^v satisfy (62a)–(62d),

$$\begin{aligned}
\Omega_0^T(x,t):=&P^-(x)-\int_0^x K^u(x,\xi)P^+(\xi,t)d\xi\\
&-\int_0^x K^v(x,\xi)P^-(\xi,t)d\xi
\end{aligned}\tag{67}$$

and $\Psi_0^T(x)=-K^u(x,0)\Lambda$ gives (64b). Differentiating (61c) with respect to time and space, inserting (66), integrating by parts, using (62e), defining

$$\Omega^T(x,t):=\Omega_0^T(x)-\mu^{-1}\int_0^x G(x-\xi,t)\Omega_0^T(\xi)d\xi \tag{68a}$$

$$\Psi^T(x,t):=\Psi_0^T(x)-\mu^{-1}\int_0^x G(x-\xi,t)\Psi_0^T(\xi)d\xi \tag{68b}$$

and

$$\begin{aligned}
B(x,\xi,t):=&-\mu^{-1}G_t(x-\xi)\\
&-\mu^{-1}\int_\xi^x G_t(x-s,t)G_0(s,\xi)d\xi
\end{aligned}\tag{69}$$

where G_0 is the kernel of the inverse transformation (A.6) in Lemma 9, finally yields (64b). The details can be found in [11, Appendix D]. The boundary conditions (64c) and (64d) follow trivially by evaluating (61) at $x=0$ and $x=1$ and applying (63) and (62d). The existence of a unique solution (K^u,K^v) to (62a)–(62d) is proved in [4]. The existence of a bounded solution to (62e) follows from Lemma 9. Furthermore, Lemma 9 gives an upper bound for G_t in terms of $\dot{\hat{k}}$ which together with boundedness of the inverse transformation kernel G_0 and boundedness and integrability of $\dot{\hat{k}}$ from Lemma 3 yield $B(x,\xi,\cdot)\in\mathscr{L}_2\cap\mathscr{L}_\infty$.

We claim that solutions of system (64) are bounded in the L_2-sense. Our strategy is now to design a simple stable reference model ϕ that can be solved explicitly and shown to satisfy a tracking objecting relevant to (7). The tracking performance of (64) can then in turn be inferred from the stability of the tracking error $\eta:=\zeta-\phi$.

Lemma 6. *Consider the reference model*

$$\phi_t-\mu\phi_x=\Psi^T(x,t)(\hat{k}(t)\odot\hat{\theta}(t)) \tag{70a}$$

$$\phi(1,t)=U_{track}(t) \tag{70b}$$

where $\phi:[0,1]\times[0,\infty]\to\mathbb{R}$, $\hat{k}$ and $\hat{\theta}$ are generated from (33) and $U_{track}\in\mathscr{L}_\infty$. Solutions of (70) are bounded for all $x\in[0,1]$. If in addition

$$U_{track}(t)=\hat{\theta}_i(t)-\mu^{-1}\int_0^1\Psi^T(\xi,t)(\hat{k}(t)\odot\hat{\theta}(t))d\xi \tag{71}$$

for some $i=1,\dots,n$, then

$$(\phi(0,\cdot)-\hat{\theta}_i)\in\mathscr{L}_2. \tag{72}$$

Proof. Solving (70) along its characteristics for $t\geq\mu^{-1}$ yields

$$\begin{aligned}
\phi(x,t)=&U_{track}(t-\mu^{-1}(1-x))+\\
&\mu^{-1}\int_x^1\Psi^T(\xi,t-\mu^{-1}(\xi-x))\\
&\times(\hat{k}\odot\hat{\theta})(t-\mu^{-1}(\xi-x))d\xi.
\end{aligned}\tag{73}$$

For point-wise boundedness, by Property 1 in Lemma 3, $\hat{k},\hat{\theta}\in\mathscr{L}_\infty$ implying that the kernel equation (62e) has a unique solution $G(x,\cdot)\in\mathscr{L}_\infty$. So if $U_{track}\in\mathscr{L}_\infty$, then $\phi(x,\cdot)\in\mathscr{L}_\infty$ for all $x[0,1]$. Towards establishing (72), evaluating (73) at $x=0$, inserting (71) and rearranging yield

$$\begin{aligned}
&\phi(0,t)-\hat{\theta}_i(t)\\
=&\mu^{-1}\int_0^1\Psi^T(\xi,t-\mu^{-1}\xi)(\hat{k}\odot\hat{\theta})(t-\mu^{-1}\xi)d\xi\\
&-\mu^{-1}\int_0^1\Psi^T(\xi,t-\mu^{-1})(\hat{k}\odot\hat{\theta})(t-\mu^{-1})d\xi\\
&+\hat{\theta}_i(t-\mu^{-1})-\hat{\theta}_i(t).\\
\leq&\mu^{-1}\int_0^1\int_{t-\mu^{-1}}^{t-\mu^{-1}\xi}\frac{d}{d\tau}\Big(\Psi^T(\xi,\tau)(\hat{k}\odot\hat{\theta})(\tau)\Big)d\tau d\xi\\
&-\int_{t-\mu^{-1}}^t\dot{\hat{\theta}}_i(\tau)d\tau
\end{aligned}\tag{74}$$

From Lemma 9, G and G_t are bounded by $\hat{k}$ and $\dot{\hat{k}}$, respectively. It then follows from the definitions (68b) that

$$\sup_{x\in[0,1]}|\Psi(x,t)|\leq c_3\mu+c_4\mu|\hat{k}(t)| \tag{75a}$$

$$\sup_{x\in[0,1]}|\Psi_t(x,t)|\leq c_5\mu|\dot{\hat{k}}(t)| \tag{75b}$$

for some $c_3,c_4,c_5>0$. We then obtain the upper bound

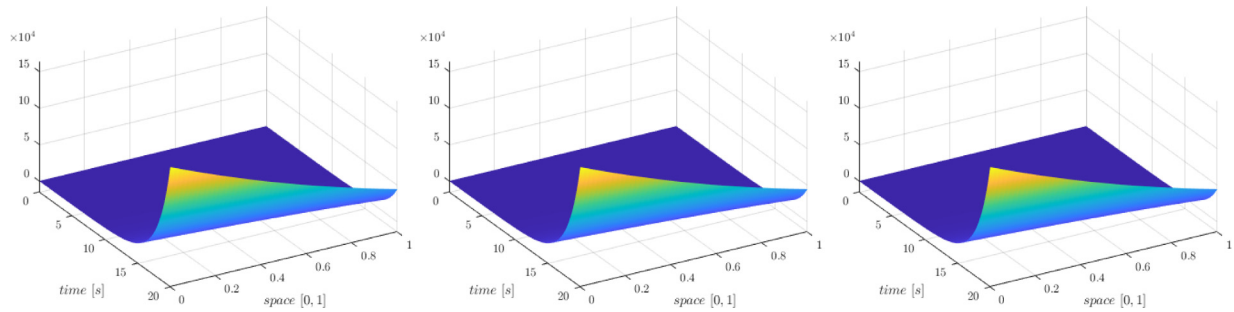
$$\begin{aligned}
&|\phi(0,t)-\hat{\theta}_i(t)|\\
\leq&(c_3+\sup_{t\geq0}|\hat{k}(t)|c_4)\sup_{t\geq0}|\hat{k}(t)|\odot\int_0^1\int_{t-\mu^{-1}}^{t-\mu^{-1}\xi}|\dot{\hat{\theta}}(\tau)|d\tau d\xi\\
&+(c_3+\sup_{t\geq0}|\hat{k}(t)|c_4)\sup_{t\geq0}|\hat{\theta}(t)|\odot\int_0^1\int_{t-\mu^{-1}}^{t-\mu^{-1}\xi}|\dot{\hat{k}}(\tau)|d\tau d\xi\\
&+\sup_{t\geq0}|(\hat{k}\odot\hat{\theta})(t)|c_5\int_0^1\int_{t-\mu^{-1}}^{t-\mu^{-1}\xi}|\dot{\hat{k}}(\tau)|d\tau d\xi\\
&+\int_{t-\mu^{-1}}^t|\dot{\hat{\theta}}_i(\tau)|d\tau.
\end{aligned}\tag{76}$$

Following the same steps as in the proof of Property 3 in Lemma 3, it is possible to show that all integrals on the right hand side of (74) are square integrable and (72) follows.

4. Stability proof

We first study stability in terms of the *target tracking error* $\eta:=\zeta-\phi$ and target state estimation error $\bar{\alpha}$. Differentiating the *tracking error* η with respect to time and space and inserting the dynamics (66) and (70), and combining the result with the state estimation error $\bar{\alpha}$ with dynamics (53) for $t\geq\mu^{-1}$, gives the combined dynamics

$$\bar{\alpha}_t=-\Lambda\bar{\alpha}_x+\int_x^1 A(x,\xi,t)\bar{\alpha}(\xi,t)d\xi \tag{77a}$$



(a) $u_1(x,t)$ (b) $u_2(x,t)$ (c) $v(x,t)$

Fig. 1. System states open loop.

$$\eta_t = \mu\eta_x + \Omega^T(x)\bar{\alpha}(1,t) + \int_0^x B(x,\xi,t)(\eta(\xi,t)+\phi(\xi,t))d\xi \tag{77b}$$

$$\bar{\alpha}(0,t) = \epsilon(t) + \int_0^1 \bar{H}(\xi,t)\bar{\alpha}(\xi,t)d\xi \tag{77c}$$

$$\eta(1,t) = 0 \tag{77d}$$

Our strategy is to show boundedness and convergence of the combined system (77) and relate those results back to the original closed loop system (4)–(5) (with the appropriate control law $U(t) = U_{stab}(t) + U_{track}(t)$) and the ultimate control objective (7).

Lemma 7. *The target tracking error η and target state estimation error $\bar{\alpha}$ specified by (77) have the properties*

$$\|\bar{\alpha}\|, \|\eta\| \in \mathscr{L}_\infty \cap \mathscr{L}_2, \tag{78}$$

$$\bar{\alpha}(0,\cdot), \eta(0,\cdot) \in \mathscr{L}_2 \tag{79}$$

and

$$\|\bar{\alpha}\|, \|\eta\| \to 0. \tag{80}$$

The proof of Lemma 7 is given in Appendix B.

Theorem 1. *Consider the closed loop system (4)–(5b) with control law*

$$U(t) = \int_0^1 K^{vu}(1,\xi)\hat{u}(\xi,t)d\xi + \int_0^1 K^{vv}(1,\xi)\hat{v}(\xi,t)d\xi + \mu^{-1}\int_0^1 G(1-\xi,t)\left(z(\xi,t)-\hat{\theta}(t)\right)d\xi + \hat{\theta}_1(t). \tag{81}$$

and state and parameter estimates generated by the observer (10) with injection gains (21) and (52), transformation (61b) and adaptive laws (33). All signals in the closed loop are bounded and the objective (7) is achieved. Moreover, $\hat{\theta}_1 \to \theta$ and $\hat{k}_1$ converges to some constant.

Proof. Invertibility of the transformations (13), (49) and (61) together with boundedness of the adaptive laws from Lemma 3 give $\|u\|, \|v\|, \|\hat{u}\|, \|\hat{v}\| \in \mathscr{L}_\infty$ and $\|\tilde{u}\|, \|\tilde{v}\| \in \mathscr{L}_2$. For the tracking objective, we have

$$\begin{aligned} v(0,t) - \hat{\theta}_1(t) &= \hat{v}(0,t) - \hat{\theta}_1(t) + \tilde{v}(0,t) \\ &= \zeta(0,t) - \hat{\theta}_1(t) + c_7\|\bar{\alpha}(\cdot,t)\|^2 \\ &= \eta(0,t) + \phi(0,t) - \hat{\theta}_1(t) + c_7\|\bar{\alpha}(\cdot,t)\|^2 \\ &\le |\eta(0,t)| + |\phi(0,t) - \hat{\theta}_1(t)| + c_7\|\bar{\alpha}(\cdot,t)\|^2 \end{aligned} \tag{82}$$

for some $c_7 > 0$. By Lemmas 6 and 7 the right hand side is bounded, implying $(v(0,\cdot) - \hat{\theta}_1) \in \mathscr{L}_2$. Now since $|v(0,t) - \hat{\theta}_1(t)| \le |v(0,t) - \theta| + |\tilde{\theta}_1(t)|$, the objective is satisfied if $\tilde{\theta}_1 \in \mathscr{L}_2$ which by Property 4 in Lemma 3 is the case if ψ is bounded and $(\hat{\theta}_1 + \psi) \in \mathscr{L}_2$. Boundedness of ψ follows from boundedness of η, α, P_i^v and m_{ij}^+. For the second condition, we have

$$\begin{aligned} \hat{\theta}_1(t+\lambda^{-1}) + \psi(t+\lambda^{-1}) &= \hat{\theta}_1(t+\lambda^{-1}) - v(0,t) \\ &= (\hat{\theta}_1(t) - v(0,t)) + \int_t^{t+\lambda^{-1}} \dot{\hat{\theta}}_1(\tau)d\tau \end{aligned} \tag{83}$$

where the last term, similarly to the proof of Property 3 in Lemma 3, can be shown to be integrable. By Property 4 in Lemma 3, we then also have that $\hat{\theta}_1 \to \theta$ and $\hat{k}_1$ converges to some constant, and the proof is complete.

5. Simulations

The system (4) with control law (81), observer (10) with injection gains (21), was simulated in MATLAB using the *method of lines* with the `ode45` solver and $N = 100$ spatial discretization points. The kernels equations (14)–(17) and (62) were solved by *successive approximation* of the corresponding integral equations. The following set of parameters were used.

$$\begin{bmatrix} \Lambda & 0_{2\times1} \\ 0_{1\times2} & \mu \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{84a}$$

$$\begin{bmatrix} \Sigma & \omega \\ \varpi & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.4 & 0 \\ -0.7 & 0 & 0.1 \\ 0.5 & -0.1 & 0 \end{bmatrix} \tag{84b}$$

$$r = \begin{bmatrix} 4 & -4 \end{bmatrix}^T \tag{84c}$$

$$k = \begin{bmatrix} 1 & -1 \end{bmatrix}^T \tag{84d}$$

$$\theta = 2. \tag{84e}$$

The initial condition were selected as

$$u_{ic}(x) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \quad \forall x \in [0,1] \tag{85a}$$

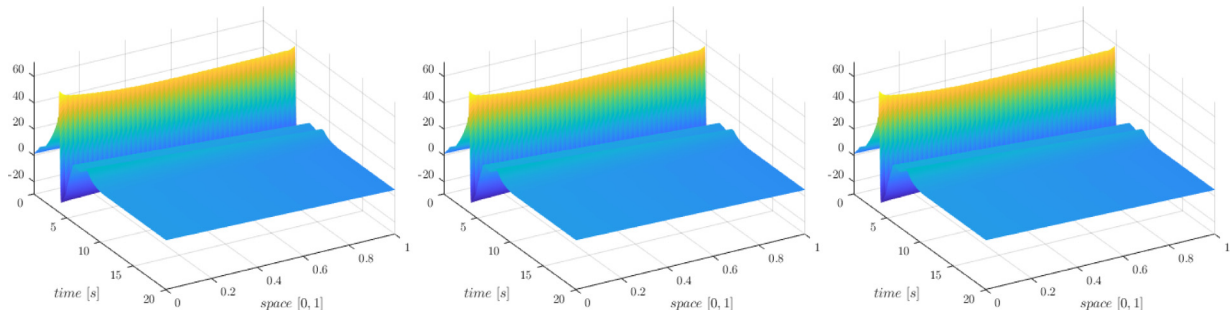
$$v_{ic}(x) = \sin(2\pi x) \tag{85b}$$

$$\hat{u}_{ic}(x) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T, \quad \forall x \in [0,1] \tag{85c}$$

$$\hat{v}_{ic}(x) = 0, \quad \forall x \in [0,1] \tag{85d}$$

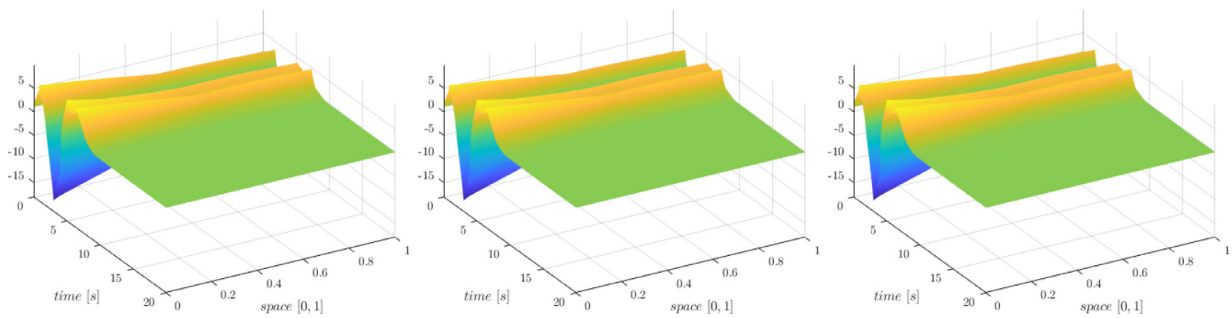
$$\hat{k}(0) = \hat{\theta}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T. \tag{85e}$$

This system is open loop unstable, as can be seen in Fig. 1. In closed loop with the observer (10), control law (60) and adaptive law (33), Figs. 2 and 7(a) show that the system states (u, v) and the reacting control signal U are bounded and converge to a non-zero steady state profile. The observer is able to correctly estimate the system states and the state estimation error converge to zero



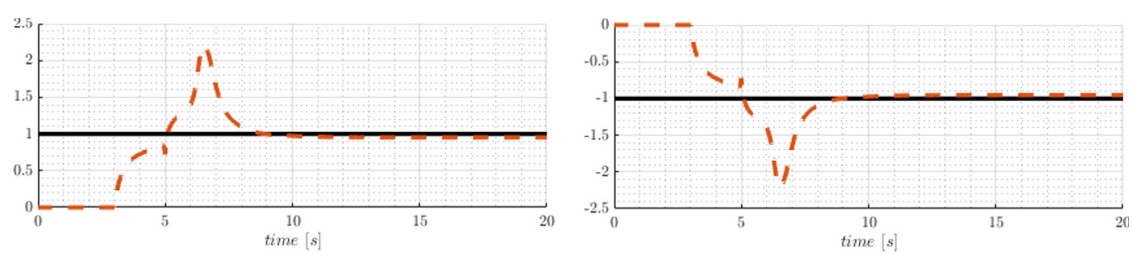
(a) $u_1(x,t)$ (b) $u_2(x,t)$ (c) $v(x,t)$

Fig. 2. System states controlled case.



(a) $\tilde{u}_1(x,t)$ (b) $\tilde{u}_2(x,t)$ (c) $\tilde{v}(x,t)$

Fig. 3. State estimation error.



(a) $\hat{k}_1(t)$ (b) $\hat{k}_2(t)$

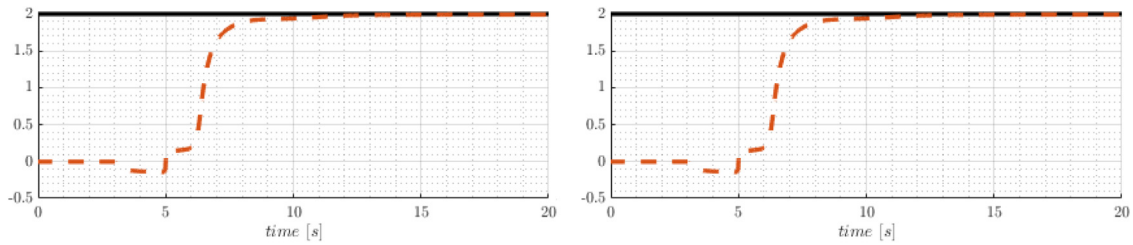
Fig. 4. Parameter estimates $\hat{k}(t)$.

as can be seen in Fig. 3. Fig. 7(b) shows that the objective (7) is achieved and, in-line with Theorem 1, Figs. 4 and 5 show that the parameter estimates $\hat{\theta}$ converge to the true parameter θ and $\hat{k}$ converge to some constants. The dynamic injection gains (21) are shown in Fig. 6. For benchmarking, the corresponding static injection gains formed by replacing r with $(r-k)$ in (16) which would be off-line computable if k is a-priori known, is also included in (21). The figure shows that the upper triangular elements of the dynamic injection gains P^+ and P^- approximates the static injection gains as better estimates $\hat{k}$ close to k are used to compute (20).

6. Concluding remarks

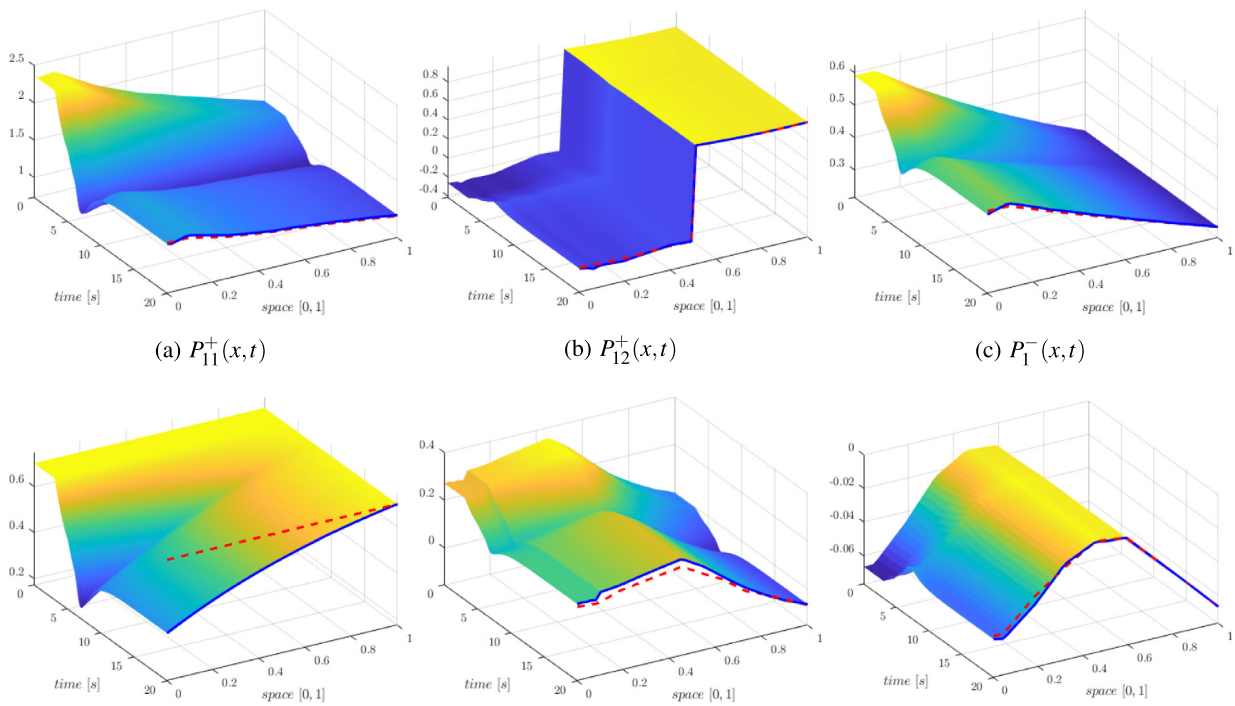
We have designed an estimation and control scheme for $n+1$ hyperbolic systems utilizing only collocated sensing and control at one boundary and with unknown parameters appearing in an affine form at the opposite boundary. The observer injection gains are computationally expensive. However, by separating the known part of the boundary condition from the unknown, the backstepping kernel can be computed off-line, leaving only a simple Volterra integral equation to be solved on-line each time a new parameter estimate is generated. Similarly for the controller, the backstepping integrals can be solved off-line leaving only the part of the controller dependent on parameter estimates to be solved on-line. A reference model and reference signal based on parameter estimates was designed. Reference tracking together with parameter convergence was shown to guarantee the overall control objective. The resulting parameter convergence properties are strong. Parameter convergence for one boundary parameter (the parameter modelling reservoir pressure in the drilling application) is guaranteed if the control objective is satisfied. This property is very useful in offshore oil and gas drilling, where good estimates of the reservoir pressure are important in itself.

For many application, the observer and/or controller is stable even if the on-line computed parts are ignored. For instance, this seems to be the case with most realistic drilling parameters. Further work includes studying conditions for when the off-line computable elements are sufficient to guarantee stability.



(a) $\hat{\theta}_1(t)$ (b) $\hat{\theta}_2(t)$

Fig. 5. Parameter estimates $\hat{\theta}(t)$.



(a) $P^+_{11}(x,t)$ (b) $P^+_{12}(x,t)$ (c) $P^-_1(x,t)$

(d) $P^+_{21}(x,t)$ (e) $P^+_{22}(x,t)$ (f) $P^-_2(x,t)$

Fig. 6. Injection gains $P^+(x,t)$ and $P^-(x,t)$ together with static injection gains (red, dashed line).

CRediT authorship contribution statement

Haavard Holta: Conceptualization, Methodology, Writing - review & editing. **Ole Morten Aamo:** Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Existence, boundedness and invertibility of transformations

Lemma 8 (*Lemma 9 from [11]*). *Consider a function $F(x,\xi,t)$. Assume F satisfies the following Volterra equation in x and ξ*

$$F(x,\xi,t) = f(x,\xi,t) + \int_x^{\xi} G(x,s,t)F(s,\xi,t)ds \tag{A.1}$$

for some bounded, known functions $f(x,\xi,t)$ and $G(x,\xi,t)$. For any given time t, Eq. (A.1) has a unique, bounded solution $F(x,\xi,t)$, with an upper bound given by

$$|F(x,\xi,t)| \le \bar{f}e^{\bar{G}x} \tag{A.2}$$

where $\bar{f}$ and $\bar{G}$ are upper bounds of f and G, respectively.

Although Lemma 8 are given for scalar F, f and G in [8], the result is trivially extended to functions mapping to $\mathbb{R}^n$.

Lemma 9. *For every bounded $\hat{k}$ there exists a unique bounded solution F to (50)–(51) and G to (62e), with bounds on the form*

$$\sup_{x\in[0,1]} |F(x,t)| \le c_1|\hat{k}(t)|, \quad \sup_{x\in[0,1]} |F_t(x,t)| \le c_2|\dot{\hat{k}}(t)| \tag{A.3}$$

and

$$\sup_{x\in[0,1]} |G(x,t)| \le c_3|\hat{k}(t)|, \quad \sup_{x\in[0,1]} |G_t(x,t)| \le c_4|\dot{\hat{k}}(t)| \tag{A.4}$$

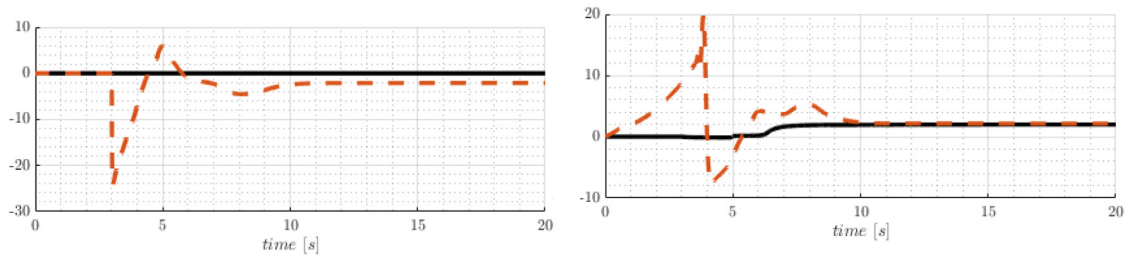
for some $c_1, c_2, c_3, c_4 > 0$. Moreover, the transformations $\alpha \to \bar{\alpha}$ in (49) and $z \to \zeta$ in (61c) are invertible with inverses

$$\bar{\alpha}(x,t) = \alpha(x,t) + \int_x^1 F_0(\xi,t)\alpha(\xi,t)d\xi \tag{A.5}$$

and

$$z(x,t) = \zeta(x,t) + \int_0^x G_0(x,\xi,t)\zeta(\xi,t)d\xi \tag{A.6}$$

where F_0 and G_0 have a bounded unique solution for every bounded $\hat{k}$.



(a) Control signal $U(t)$.

(b) Control objective: $v(0,t)$ (red dashed) vs. $\hat{\theta}_1(t)$ (solid black).

Fig. 7. Control signal and control objective.

Proof. Let F_j and H_j denote the jth column with row elements $i \leq j$ of F and H, respectively, and let $\breve{H}_j$ denote the matrix with columns $1, \ldots, j$ of H. That is, $F_j(x,\xi,t) = [f_{1j}(x,\xi,t), \ldots, f_{jj}(x,\xi,t)]^T$, $H_j(x,\xi,t) = [h_{1j}(x,\xi,t), \ldots, h_{jj}(x,\xi,t)]^T$ and $\breve{H}_j(s,t) = [H_1, \ldots, H_j]$. For each $j \in [1,n]$, the Volterra integral equation (51) can be written as (using the intermediate step (59))

$$F_j(0,\xi,t) = H_j(\xi,t) + \int_0^{\xi} \breve{H}_j(s,t)F_j(s,\xi,t)ds, \tag{A.7}$$

which is on the form (A.1), and we have that F is bounded by H. Differentiating (A.7) with respect to time give similarly an upper bound in terms of H_t. From the definition of H in (20) we obtain the upper bounds (A.3) in terms of $\hat{k}$ and $\dot{\hat{k}}$, showing that $F(x,\xi,t)$ has a unique bounded solution for every bounded $\hat{k}$. For invertibility of the transformation, substituting the right hand side of (A.5) into (49), rearranging and changing the order of integration yields a Volterra integral equation for G_0 for which boundedness follows from Lemma 8. Similar arguments can be made for (A.4) and the inverse transformation (A.6).

Appendix B. Proof of Lemma 7

Proof. Define the Lyapunov function candidates

$$V_1(t) = \int_0^1 e^{-x}\bar{\alpha}^T(x,t)\Pi\bar{\alpha}(x,t)dx \tag{B.1a}$$

$$V_2(t) = \mu^{-1}\int_0^1 e^{\sigma x}\eta^2(x,t)dx \tag{B.1b}$$

where Π is a positive definite diagonal matrix and $\sigma > 0$. Differentiating (B.1a) with respect to time, inserting the system dynamics (77a) and integrating by parts give the upper bound

$$\begin{aligned}\dot{V}_1(t) \leq &-\int_0^1 \bar{\alpha}^T(x,t)\left[\lambda_1 e^{-1}\Pi - \lambda_n \bar{H}^T(x,t)\Pi\bar{H}(x,t) - 1\right] \\ &\times \bar{\alpha}(x,t)dx \\ &- c_1\bar{\alpha}^T(1,t)\bar{\alpha}(1,t) + c_2\epsilon^T(t)\epsilon(t) \\ &+ c_3\|A(\cdot,\cdot,t)\|^2 \parallel V_1(t)\end{aligned} \tag{B.2}$$

for some positive constants c_1, c_2, c_3. Since $\bar{H}(x,t)$, as defined in Lemma 4, is strictly lower triangular, and by Property 1 in Lemma 3 bounded for all $t \geq 0$, it is possible to (recursively) select Π such that $\lambda_1 e^{-1}\Pi - \lambda_n\bar{H}^T(x,t)\Pi\bar{H}(x,t) - 1 \succ 0$ yielding

$$\begin{aligned}\dot{V}_1(t) \leq &- c_4V_1(t) + c_3\|A(\cdot,\cdot,t)\|^2V_1(t) \\ &- c_1\bar{\alpha}^T(1,t)\bar{\alpha}(1,t) + c_2\epsilon^T(t)\epsilon(t)\end{aligned} \tag{B.3}$$

for some positive constant c_4. From (13b), (61b), (49) and $z(0,t) = \eta(0,t) + \phi(0,t)$ we obtain the upper bound

$$v^2(0,t) \leq 4\eta^2(0) + 4\phi(0,t) + c_5V_1(t), \tag{B.4}$$

for some $c_5 > 0$, so that for the last term in (B.3), we have

$$\begin{aligned}c_2\epsilon^T(t)\epsilon(t) &\leq c_2\pi^T(t)\pi(t)(1 + v^2(0,t)) \\ &\leq c_2\pi^T(t)\pi(t)(1 + 4\eta^2(0) + 4\phi(0,t) + c_5V_1(t))\end{aligned} \tag{B.5}$$

where $\pi := \{\pi_i\}_{1\leq i\leq n}$ is defined in Property 3 in Lemma 3, and therefore

$$\begin{aligned}\dot{V}_1(t) \leq &- c_4V_1(t) + c_3\|A(\cdot,\cdot,t)\|^2V_1(t) \\ &- c_1\bar{\alpha}^T(1,t)\bar{\alpha}(1,t) \\ &+ c_2\pi^T(t)\pi(t)(1 + 4\eta^2(0) + 4\phi(0,t) + c_5V_1(t)).\end{aligned} \tag{B.6}$$

Differentiating (B.1b) with respect to time, inserting the system dynamics (77b) and integrating by parts give the upper bound

$$\begin{aligned}\dot{V}_2(t) \leq &-\eta^2(0,t) - \mu\sigma V_2 \\ &+ 2\mu^{-1}\int_0^1 e^{\sigma x}\eta(x,t)\int_0^x B(x,\xi,t)(\eta(\xi,t) + \phi(\xi,t))d\xi dx \\ &+ 2\mu^{-1}\int_0^1 e^{\sigma x}\eta(x,t)\Omega^T(x,t)dx\bar{\alpha}(1,t).\end{aligned} \tag{B.7}$$

The third and fourth terms in (B.7) are bounded by sequentially applying Young's inequality and Cauchy–Schwarz' inequality, yielding

$$\begin{aligned}\dot{V}_2(t) \leq &-\eta^2(0,t) + (\mu\sigma)^{-1}\bar{\Omega}(e^{\sigma} - 1)\bar{\alpha}^T(1,t)\bar{\alpha}(1,t) \\ &- (\mu\sigma - 2 - \bar{\Omega})V_2(t) \\ &+ \|B(\cdot,\cdot,t)\|^2V_2 + \|B(\cdot,\cdot,t)\|^2e^{\sigma}\mu^{-1}\|\phi(\cdot,t)\|^2\end{aligned} \tag{B.8}$$

where $\bar{\Omega}$ upper bounds all elements in $\Omega(x,t)$ for all $(x,t) \in [0,1] \times [0,1)$. Selecting $\sigma > \mu^{-1}(2 + \bar{\Omega})$ ensures that the third term in (B.8) is negative semidefinite. Now forming $V_3(t) = a_1V_1(t) + V_2(t)$ with $a_1 = (\mu\sigma)^{-1}\bar{\Omega}(e^{\sigma} - 1)c_1^{-1}$gives

$$\begin{aligned}\dot{V}_3(t) \leq &- c_6V_3 + l_1(t)V_3 + l_2(t) \\ &- \left(1 - 4c_2a_1\pi^T(t)\pi(t)\right)\eta^2(0,t)\end{aligned} \tag{B.9}$$

with

$$l_1(t) = \|B(\cdot,\cdot,t)\|^2 + c_2a_1\pi^T(t)\pi(t) \tag{B.10a}$$

$$\begin{aligned}l_2(t) = &c_2a_1\pi^T(t)\pi(t)(1 + 4\phi(0,t)) + c_3\|A(\cdot,\cdot,t)\|^2 \\ &+ \|B(\cdot,\cdot,t)\|^2e^{\sigma}\mu^{-1}\|\phi(\cdot,t)\|^2\end{aligned} \tag{B.10b}$$

for some $c_6 > 0$. By Lemmas 3–6 we have that respectively $\pi^2 \in \mathscr{L}_1$, $\|A(\cdot,\cdot,t)\|^2 \in \mathscr{L}_1 \cap \mathscr{L}_\infty$, $\|B(\cdot,\cdot,t)\|^2 \in \mathscr{L}_1 \cap \mathscr{L}_\infty$ and that $\sup_{t\geq 0}\|\phi(\cdot,t)\|^2$ exists. In addition, it can be shown that the growth rate of (4)–(6) and (10) are exponentially bounded (see e.g. [15, Theorem 1.1]) and in turn, since all transformations are bounded, that the growth rate of (77) is exponentially bounded. We then have that V_3 with the upper bound (B.9) together with V_0 with the upper bound (35) in the proof of Lemma 3 satisfy the conditions in Lemma 10 in Appendix C and V_3, $\|\eta\|^2$, $\|\bar{\alpha}\|^2$

$\in \mathscr{L}_1 \cap \mathscr{L}_\infty$ follows. Rearranging (B.9) and integrating from $t = 0$ to $t = T$ yield

$$\left(1 - c_2 4 \sup_{t\geq 0} \pi^T(t)\pi(t)\right) \int_0^T \eta^2(0,t)dt$$
$$\leq V(0) - V(T) + \int_0^T l_1(t)V_3(t)dt + \int_0^T l_2(t)dt \tag{B.11}$$

which when taking the limit as $T \to \infty$ shows that $\eta^2(0,\cdot) \in \mathscr{L}_1$. Similarly, integrating (B.3) with the bound (B.5) from $t = 0$ to $t = T$ where $T \to \infty$, shows that $\bar{\alpha}^2(0,\cdot) \in \mathscr{L}_1$. Now, (B.9) has the form considered in Lemma 11 in Appendix C and $V_3, \|\bar{\alpha}\|, |\eta\ \| \to 0$ follows.

Appendix C. Additional stability and convergence lemmas

Lemma 10 (*Lemma 8 from [9]*). *Let $V_3(t)$, $V_0(t)$, $l_1(t)$, $l_2(t)$ and $f(t)$ be real-valued functions and $G(t)$ a real-valued matrix of dimension $n \times n$ defined for $t \geq 0$, with*

$$V_0(t) = \frac{1}{2}\nu^T(t)\nu(t) \tag{C.1}$$

for a signal vector ν of length n. Suppose

$$0 \leq V_0(t), V_3(t), l_1(t), l_2(t), f(t) \ \ \forall t \geq 0 \tag{C.2a}$$
$$l_1, l_2 \in \mathscr{L}_1 \tag{C.2b}$$
$$|\nu| \in \mathscr{L}_\infty \tag{C.2c}$$
$$0 \leq G(t) = G^T(t) \leq I_{n\times n} \tag{C.2d}$$
$$\int_0^t f(s)ds \leq Ae^{Bt} \tag{C.2e}$$
$$\dot{V}_0 \leq -\nu^T(t)G(t)\nu(t) \tag{C.2f}$$
$$\dot{V}_3 \leq -cV_3(t) + l_1(t)V_3(t) + l_2(t) - a\left(1 - b\nu^T(t)G(t)\nu(t)\right)f(t) \tag{C.2g}$$

for some positive constants A, B, a, b and c. Then $V_3 \in \mathscr{L}_1 \cap \mathscr{L}_\infty$.

Lemma 11 (*Lemma 2.17 from [18]*). *Consider a signal g satisfying*

$$\dot{g}(t) = -ag(t) + bh(t) \tag{C.3}$$

for a signal $h \in \mathscr{L}_1$ and some constants $a, b > 0$. Then

$$g \in \mathscr{L}_\infty \tag{C.4}$$

and

$$\lim_{t\to\infty} g(t) = 0. \tag{C.5}$$

References

[1] Georges Bastin, Jean-Michel Coron, Stability and Boundary Stabilization of 1-D Hyperbolic Systems, volume 88, Springer, 2016.

[2] Rafael Vazquez, Miroslav Krstic, Jean-Michel Coron, Backstepping boundary stabilization and state estimation of a 2×2 linear hyperbolic system, in: Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference, 2011, pp. 4937–4942.

[3] Florent Di Meglio, R. Vazquez, M. Krstic, Stabilization of a system of $n + 1$ coupled first-order hyperbolic linear PDEs with a single boundary input, IEEE Trans. Automat. Control 58 (12) (2013) 3097–3111.

[4] Long Hu, Florent Di Meglio, Rafael Vazquez, Miroslav Krstic, Control of homodirectional and general heterodirectional linear coupled hyperbolic PDEs, IEEE Trans. Automat. Control 61 (11) (2016) 3301–3314.

[5] Michelangelo Bin, Florent Di Meglio, Boundary estimation of parameters for linear hyperbolic PDEs, IEEE Trans. Automat. Control 62 (8) (2017) 3890–3904.

[6] Henrik Anfinsen, Mamadou Diagne, Ole Moren Aamo, Miroslav Krstic, An adaptive observer design for $n + 1$ coupled linear hyperbolic PDEs based on swapping, IEEE Trans. Automat. Control 61 (12) (2016) 3979–3990.

[7] Henrik Anfinsen, Mamadou Diagne, Ole Morten Aamo, Miroslav Krstic, Estimation of boundary parameters in general heterodirectional linear hyperbolic systems, Automatica 79 (2017) 185–197.

[8] Henrik Anfinsen, Ole Morten Aamo, Boundary parameter and state estimation in 2×2 linear hyperbolic PDEs using adaptive backstepping, in: Proceedings of the 55th Conference on Decision and Control, 2016, pp. 2054–2060.

[9] Henrik Anfinsen, Ole Morten Aamo, Adaptive stabilization of n+1 coupled linear hyperbolic systems with uncertain boundary parameters using boundary sensing, Systems Control Lett. 99 (2017) 72–84.

[10] Henrik Anfinsen, Ole Morten Aamo, Adaptive output feedback stabilization of $n + m$ coupled linear hyperbolic PDEs with uncertain boundary conditions, SIAM J. Control Optim. 55 (6) (2017) 3928–3946.

[11] Henrik Anfinsen, Ole Morten Aamo, Adaptive stabilization of 2×2 linear hyperbolic systems with an unknown boundary parameter from collocated sensing and control, IEEE Trans. Automat. Control 62 (12) (2017) 6237–6249.

[12] Haavard Holta, Henrik Anfinsen, Ole Morten Aamo, Adaptive set-point regulation of linear 2×2 hyperbolic systems with uncertain affine boundary condition using collocated sensing and control, in: Proceedings of the 2017 Asian Control Conference, ASCC, 2017, pp. 2766–2771.

[13] Haavard Holta, Henrik Anfinsen, Ole Morten Aamo, Adaptive set-point regulation of linear 2x2 hyperbolic systems with application to the kick and loss problem in drilling, Automatica 119 (2020) 109078.

[14] Henrik Anfinsen, F. Di Meglio, O.M. Aamo, Estimating the left boundary condition of coupled 1-D linear hyperbolic PDEs from right boundary sensing, in: Proceedings of the European Control Conference, 2016, pp. 2179–2184.

[15] Henrik Anfinsen, Ole Morten Aamo, Adaptive Control of Hyperbolic PDES (Communications and Control Engineering), Springer, 2019.

[16] Petros A. Ioannou, Jing Sun, Robust Adaptive Control, Courier Corporation, 2012.

[17] Miroslav Krstic, Ioannis Kanellakopoulos, Peter V. Kokotovic, Nonlinear and Adaptive Control Design, Wiley, 1995.

[18] Gang Tao, Adaptive Control Design and Analysis, John Wiley & Sons, Inc., New York, NY, USA, 2003.