The Translation between the Required Return on Unlevered and Levered Equity for Explicit Cash Flows and Fixed Debt Financing

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Abstract

Purpose: The primary purpose is to develop the translation formula between the required return on unlevered and levered equity for the specific case where cash flows have a finite lifetime and the flow to debt is prespecified.

The secondary purpose is to underpin the importance of the type of stochasticity of cash flows for translation formulas. A general derivation of such formulas and the discount rate in the free cash flow approach is shown.

Design/methodology/approach: The paper starts with the same assumptions that have been applied by Modigliani and Miller (1963), Miles and Ezzell (1980), and other researchers. Then the paper develops the mathematical foundations to apply a deterministic backward-iterative scheme for valuing cash flows. After stating the valuation formulas for levered and unlevered equity, debt and tax shields, we mathematically derive the relationship between the unlevered return and levered return on equity.

Findings: Conventional translation formulas apply to very special cases. They can generally not be used for projects with non-constant leverage and a finite lifetime. In general, translation formulas depend on continuing values, cash flows, leverage, taxation, risk-free rate, etc. In our special case, the translation depends on the structure of the debt in addition to the well-known parameters in conventional formulas. Our formula contains the Modigliani-Miller translation formula as a special case.

Originality/value: We develop a novel formula for the translation of the required return on unlevered to levered equity. With this formula we offer a solution for the consistent valuation of cash flows with a limited lifetime and given debt financing.

1 Introduction

Discounted-cash-flow-based valuation is one of the most widely used approaches for firms and investment projects in practice and academia (Mukhlynina and Nyborg, 2016). Contemporary textbooks in corporate finance (e.g. Berk and DeMarzo, 2019, chapter 18; Brealey and Myers, 2020, chapters 17–19) and firm valuation (e.g. Damodaran, 2006, chapters 5–6; Koller *et al.*, 2010, chapter 6) together with numerous research articles (see Fernández (2007) for an overview) introduce different discounted cash flow (DCF) methods for firm, project and investment valuation. The most prominent are the equity method, the free cash flow (FCF) method (sometimes referred to as WACC method, where WACC stands for weighted average costs of capital), the adjusted present value (APV) method (developed by Myers, 1974) and the capital cash flow (CCF) method (e.g. McConnell and Sandberg, 1975; Nantell and Carlson, 1975; Ruback, 2002). The relationships between these methods are often expressed by the following formulas which are obtained by mathematically transforming one DCF method into another. The first formula arises when the equity method is transformed into the free cash flow method:

$$r_{\mathrm{FCF}} = q \cdot r_{\mathrm{EL}} + (1 - q) \cdot r_{\mathrm{D}} \cdot (1 - \tau) \tag{1}$$

It relates the discount rate r_{FCF} in the FCF method to the required return on levered equity r_{EL} and the cost of debt r_{D}, where q refers to the equity-to-firm-value ratio and τ is the tax rate. This formula is often referred to as the after-tax weighted average costs of capital (e.g. Harris and Pringle, 1985, p. 237; McConnell and Sandberg, 1975, p. 885). The second formula is derived when the equity method is transformed into the CCF method:

$$r_{\mathrm{CCF}} = q \cdot r_{\mathrm{EL}} + (1 - q) \cdot r_{\mathrm{D}} \tag{2}$$

The discount rate r_{CCF} is also referred to as the before-tax weighted average cost of capital. The third formula shows the relationship between the levered (r_{EL}) and unlevered (r_{U}) return on equity. It is obtained by transforming the equity method into the APV method. For example, Modigliani and Miller (1963, p. 439) propose that:

$$r_{\mathrm{EL}} = r_{\mathrm{U}} + (r_{\mathrm{U}} - r_{\mathrm{D}}) \cdot (1 - \tau) \cdot \frac{1 - q}{q} \tag{3}$$

If one assumes the applicability of the Sharpe-Lintner-Mossin capital asset pricing model, formula (3) can be rewritten in terms of beta:

$$\beta_{\mathrm{EL}} = \beta_{\mathrm{U}} + (1-\tau) \cdot (\beta_{\mathrm{U}} - \beta_{\mathrm{D}}) \cdot \frac{1-q}{q} \tag{4}$$

Under the additional assumption of risk-free debt, this formula reduces to (5), which is applied, for example, by Damodaran (2006, pp. 52 and 224).

$$\beta_{\mathrm{EL}} = \beta_{\mathrm{U}} + \left(1 + (1-\tau) \cdot \frac{1-q}{q}\right) \tag{5}$$

While formula (2) is rather general, formulas (3) to (5) are obtained under strict assumptions. One of the most fundamental assumptions is that the cash flows appear in the form of a perpetual and stationary annuity. Using formulas (3) to (5) for cash flows that do not meet this assumption will lead to erroneous and inconsistent valuation results. Among others, Taggart (1991, p. 10) points out that textbook formulas may have limited utility in many practical cases. Nevertheless, several textbooks suggest the use of these formulas for cases where cash flows have a finite lifetime. It could be argued that problems with the precise estimation of future cash flows, risk premia, and interest rates make a mathematically rigorous and consistent valuation less important. In our opinion, however, the estimation problem must never justify an analytically incorrect approach. Moreover, when valuation is carried out by several DCF-methods, inconsistent results reduce the meaningfulness and usefulness of the calculated firm values (or values of investment projects). By “reduced meaningfulness” we mean, that inconsistent results from different valuation approaches are often the consequence of inattentively using opposing assumptions in the different DCF models. The usefulness of such calculations is impaired if these models lead to different recommendations for action or allocation of resources in practice, which they should not if they were based on the same information and assumptions.

Therefore, the primary purpose of this paper is the derivation of a novel and different translation formula between the levered (r_{EL}) and unlevered (r_{U}) return on equity for the valuation case where the cash flows to the firm have a limited lifetime and where debt financing is predetermined, for example, in the form of a constant payment loan, constant amortization loan, bullet loan, or the like. This is practically relevant in the context of the valuation of finite-life projects or in cases where a company's planning period is divided into an explicit planning period and a subsequent infinite continuation period. The formula developed here will be different from formula (3). As we will show, the translation will depend not only on the tax rate, the return on debt (or interest rate) and the financial leverage, but also on the payment structure of debt financing. This effect does not appear in the case of the textbook formulas or alternative formulas presented in the research literature (see section 2).

The secondary purpose of this paper is to promote an understanding of the general relationships between the discount rate in the FCF method and the required returns on levered and unlevered equity (see section 3). This forms the basis for the derivation of the aforementioned translation formula for our special valuation case. More generally, we hope that we can help reduce the confusion that reigns among practitioners about how to take the effect of leverage into account when determining the weighted average costs of capital or the interest tax shield. This confusion has been reported in a survey by Mukhlynina and Nyborg (2016) who looked at valuation practices of consultants, investment bankers, private equity professionals and asset managers.

The remainder of this paper is structured as follows. The next section gives a short overview of the most relevant literature that discusses the relationship between the required return on unlevered and levered equity and the discount rate of the FCF method. In the third section, we lay out the mathematical foundations of our model, and show the general derivation of the discount rate in the FCF method and the translation between the unlevered return and levered return on equity. In section 4, a translation formula is derived for a valuation case where loan financing is predetermined and the cash flow has a limited lifetime. The fifth section is dedicated to a numerical example before the paper ends with a conclusion.

2 Previous literature

There are a considerable number of research articles dealing with the relationships between different DCF methods and the narrower question of what the formal connection between the required return on levered and unlevered equity looks like. In what follows, we will exclusively focus on the literature that is relevant for our analysis and which has the following assumptions in common:

Assumption 1: Cash flows appear at discrete and equidistant points in time.

Assumption 2: The earnings before interest and taxes (EBIT) of the firm are independent of the financing structure of the firm.

Assumption 3: Only corporate taxation is applied and no personal taxation. Among others, Cooper and Nyborg (2008), Miller (1977) Stapleton (1972) and Taggart (1991) included personal taxation in their analyses. The corporate tax rate is furthermore deterministic (non-stochastic), time invariant and does not depend on the size of the earnings before interest and taxes.

Assumption 4: The flow to the debt holders consists of interest payments and changes in the principal of debt only. There do not exist additional fees, discounts, etc.

Assumption 5: There are no transaction or information costs when levering or de-levering the firm.

Assumption 6: The risk-free rate r_{f} is deterministic (non-stochastic).

Assumption 7: There does not exist any event that causes the firm to discontinue before its expected lifetime. The expected lifetime can be finite or infinite. One possible event that could trigger the discontinuation of the firm is the bankruptcy of the firm. Consequently, it is also assumed that there are no bankruptcy costs.

Assumption 8: Debt is risk free. The debt holders receive the negotiated nominal amount of debt and interest. Early research like that by Harris and Pringle (1985), Miles and Ezzell (1980), Modigliani and Miller (1958, 1963) and Myers (1974) explicitly or implicitly assumes risk-free debt financing. Others who applied other than the risk-free rate to debt still treated debt deterministically (for example, Cooper and Nyborg, 2008 and Ruback, 2002). The modelling of risky debt, particularly in finite-life projects, requires additional assumptions and complicates the computations. In case of risky debt, the interest payments may belong to another risk class than the down payments. One can presume that interest payments are lost before down payments if the cash flow of the firm can only partially satisfy the debt holders. The total debt will then have a risk that is composed of both these risk classes, meaning that the required return on debt is a compound. Therefore, the interest tax shield tied to the interest payments cannot be linked to the total flow to the debt holders in a linear fashion, as is often done in existing literature. Throughout this paper it will therefore be convenient to assume debt as risk-free.

Assumption 9: Indirect costs of financial distress, as well as agency benefits and costs (for a detailed discussion, see Berk and DeMarco, 2019, chapter 16), are unaffected by the degree of leverage.

Assumption 10: The value of debt DV_t equals the nominal (contractual) amount of debt DN_t:

$$DV_t = DN_t \text{ for all } t$$

This implies that the nominal (contractual) interest rate equals the risk-free rate. Because the risk-free rate is deterministic and constant over time, the nominal interest rate is also deterministic and time invariant.

Assumption 11: In the case of negative income before taxes ($EBIT_t - I_t < 0$, with I_t representing the interest payment at time t), there is a tax transfer to the firm (reverse taxation). This means, for example, that negative income is not carried forward to another point in time.

The first relevant analysis is that of Modigliani and Miller (1963). In addition to the above-mentioned assumptions, they assume that cash flows appear as an infinite and stationary annuity. This means that the relationship between the required return on levered equity and unlevered equity can be described by formula (3) (see Modigliani and Miller, 1963, p. 439). It can also be shown (as in Myers, 1974, pp. 11, 12) that the discount rate in the FCF method in the case of Modigliani and Miller (1963) is related to the required return on unlevered equity according to the following formula.

$$r_{\mathrm{FCF}} = [1 - \tau \cdot (1 - q)] \cdot r_{\mathrm{U}} \tag{6}$$

Contrary to this analysis, Miles and Ezzell (1980, 1985) assume that the cash flow follows a Martingale process. Their analysis therefore suggests a different formula for the relationship between the required return on unlevered equity and the discount rate in the FCF method (Miles and Ezzell, 1980, p. 726):

$$r_{\mathrm{FCF}} = r_{\mathrm{U}} - \tau \cdot (1 - q) \cdot r_{\mathrm{f}} \cdot \frac{1 + r_{\mathrm{U}}}{1 + r_{\mathrm{f}}} \tag{7}$$

Since Miles and Ezzell (1980, p. 727), like Modigliani and Miller (1963), conclude the validity of formula (1), the relationship between the required return on unlevered equity and levered equity takes the following form:

$$r_{\mathrm{EL}} = r_{\mathrm{U}} + (r_{\mathrm{U}} - r_{\mathrm{f}}) \cdot \frac{1 + r_{\mathrm{f}} \cdot (1 - \tau)}{1 + r_{\mathrm{f}}} \cdot \frac{1 - q}{q} \tag{8}$$

Formulas (3), (6), (7) and (8) emerge directly from the consistency of the FCF method, equity method and APV method. The latter method involves the valuation of the interest tax shield. Therefore, the composition of formulas (3), (6), (7) and (8) is inextricably linked to the discount rate of the interest tax shield. Both Miller and Modigliani (1963) and Myers (1974) conclude that the interest tax shield can be discounted by the cost of debt, which in their analysis equals the risk-free rate: $r_{\mathrm{TS}} = r_{\mathrm{f}}$. Contrary to this, in the analysis by Miles and Ezzell (1980, p. 724), the amount of debt needs to be adjusted to firm-value fluctuations across time. This is necessary to maintain a given target capital structure. Therefore, the future tax benefits are not deterministic and can therefore not be discounted with the required return on debt. Their required return on the tax shield is therefore given by $r_{\mathrm{TS}} = r_{\mathrm{U}} \cdot \frac{1+r_{\mathrm{f}}}{1+r_{\mathrm{U}}}$ (see Arzac and Glosten, (2005, equation 13 with growth rate $g = 0$; Barbi, 2012, equation 15). Contrary to the two approaches above, Harris and Pringle (1985, pp. 240, 241) claim that the tax benefit has to be discounted with the required return on unlevered equity at all times: $r_{\mathrm{TS}} = r_{\mathrm{U}}$. This implies the following translation for a perpetual stream of cash flows:

$$r_{\mathrm{FCF}} = r_{\mathrm{U}} - (1 - q) \cdot r_{\mathrm{f}} \cdot \tau \tag{9}$$

For the past 50 years, many researchers have suggested that these discount rates may not be applicable to cash flows that do not represent a perpetual annuity. Miles and Ezzell (1980, p. 725, formula 12) and Myers (1974, pp. 12, 13) show that this relationship takes the following form in the case of a single-period cash flow (lifetime of one period):

$$r_{\mathrm{FCF}} = r_{\mathrm{U}} - \tau \cdot r_{\mathrm{D}} \cdot (1 - q) \cdot \frac{1 + r_{\mathrm{U}}}{1 + r_{\mathrm{D}}} \tag{10}$$

This is the same formula as (7), which implies that the translation is independent of the lifetime of the cash flows as long as one assumes a Miles-Ezell type of cash flow. However, for a Modigliani-Miller type of cash flow, it can be concluded that the formula $r_{\mathrm{FCF}} = f(r_{\mathrm{U}})$ will depend on the lifetime of the cash flow. This has the consequence that both formula (6) (infinite stationary annuity) and formula (10) (single-period cash flow) cannot be generally used for cash flows of arbitrary structure.

In the translation formulas above, we clearly see that the relevant parameters are the required return on debt, the tax rate and the equity-to-firm value ratio. An indication that these parameters are generally not sufficient comes from Arzac and Glosten (2005), who discuss the valuation of the tax shield. During their considerations, and more particularly with their formula (12), they represent a relationship between the value of the tax shield VTS, the value of debt DV, and the value of the principal payments on debt VPP as follows:

$$VTS = \tau \cdot (DV - VPP)$$

The difference $DV - VPP$ represents the value of the interest payments. Although Arzac and Glosten (2005) pursue another purpose with their formula, it offers an interesting implication which is confirmed in this paper: namely, for cash flows with a finite lifetime, the tax shield does depend on the value of the interest payments. However, the value of interest payments depends on the particular type of loan. Hence, if the tax shield value VTS depends on the characteristics of debt, then so will the levered firm value FV_L, i. e. the translation formulas between r_{FCF} (or r_{EL}) and r_{U} will depend on the loan structure. This implies that formulas (3), (6), (7) and (8) cannot be generally applied.

Before we show the specific translation formula for the case of finite-life cash flows with prespecified debt payments (section 4), the following section lays the necessary foundation by describing the general context of valuation and the general relationship between the required return on unlevered and levered equity.

3 Prerequisites and mathematical approach

3.1 Valuation by means of backward induction

According to assumption 1, cash flows occur at discrete and equidistant points in time $t = 1, .., T$ (T can be infinite). Furthermore, cash flows can be considered as stochastic, and their evolution can be visualized by means of a scenario tree as indicated in Figure 1. A scenario tree assumes a discrete number of transitions from one state s (node) at time t to a new state (node) at time $t + 1$. This model is used here to support the analysis illustratively, but not ruling out that stochastic cash flows can have a continuous (infinite) support of values.

In Figure 1, the nodes are enumerated by $s = 1, \dots, S$. For all states, the free cash flows FCF_s, the continuation values $V_{\mathrm{L},s}$ and the one-period required return (or discount rate) r_s are given.

One can generally imagine that all these variables are path-dependent. In such a case, the valuation has to follow a backward-iterative process with the following steps:

Step 0: Initialization: start with the next-to-last period $t = T - 1$

Step 1: Let S_t be the set of all nodes at time t.

Let $F(s)$ be the set of all the offspring nodes that evolve from node s.

For each node $s \in S_t$, determine the value of the stochastic cash flow $CF_{j \in F(s)}$ and continuing value $V_{j \in F(s)}$ as follows:

$$V_s = \frac{\mathbb{E}_{j \in F(s)}\left[CF_j + V_j\right]}{1 + r_s} \text{ for all } s \in S_t.$$

Here, $\mathbb{E}_{j \in F(s)}$ represents the expectation operator.

Step 2: If $t = 0$, the valuation is complete.

Otherwise, go back one time period, i.e. $t - 1 \rightarrow t$, and continue with step 1.

With respect to Figure 1 (part A), one would have to first calculate the four separate values $V_{s=4}$ to $V_{s=7}$. In a next step, the two separate values $V_{s=2}$ and $V_{s=3}$ would be determined. Finally, the present value $V_{s=1}$ can be computed.

This procedure can be simplified if the required returns (discount rates) are path-independent, i.e. if all discount rates r_s in a given time period are the same. Looking at Figure 1 (part A), if all the discount rates $r_{s \in S_2}$ (i. e. $s = 4, \ldots, 7$) have the same value, and if all the discount rates $r_{s \in S_1}$ (i. e. $s = 2$ and 3) have the same value, we can apply a deterministic backward- iterative process described by:

$$V_t = \frac{\mathbb{E}_{s \in S_{t+1}}[CF_s + V_s]}{1 + r_t} \text{ for all } t = 0, \ldots, T - 1 \tag{11}$$

This process is exemplified in part B of Figure 1. In this paper, we assume the applicability of this scheme. Hence, all the following considerations will be based on the following additional assumption:

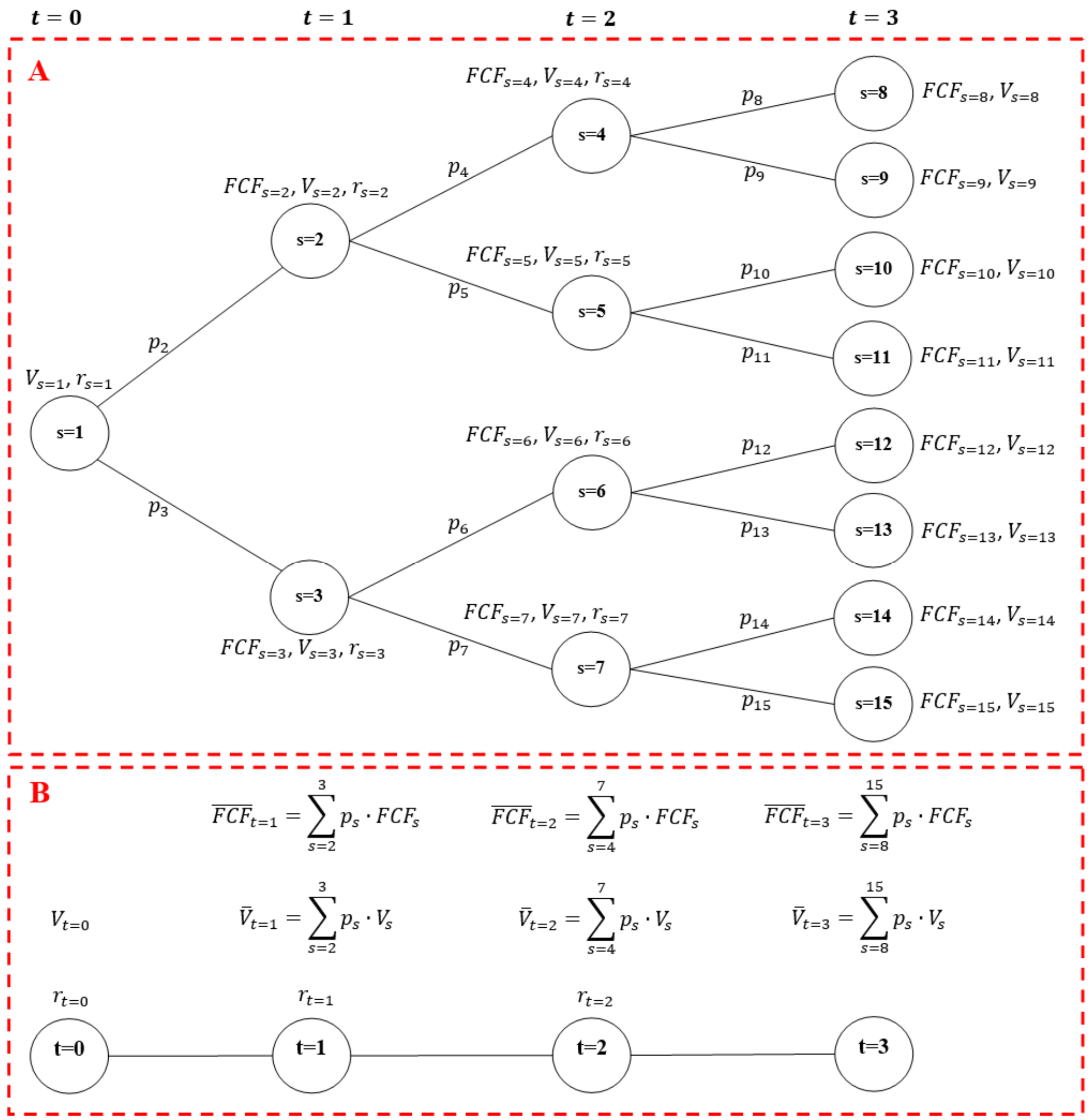

Figure 1: Scenario tree with evolving cash flows, values and discount rates over time

Assumption 12: We assume that the discount rates used in the equity method, FCF method, CCF method and APV method are path-independent. Assumption 8 already forces the required return on debt to be path-independent.

At first sight, this assumption seems to be unproblematic. However, it is important to be aware that the path-dependency of the cash flows and continuing values can affect the path-dependency of the discount rates. For example, if one assumes the Sharpe-Lintner-Mossin capital asset pricing model, the required return r_s is defined as follows:

$$r_{\mathrm{s}} = r_{\mathrm{f,s}} + \frac{\operatorname{cov}\left[\frac{CF_{j\in F(s)} + V_{j\in F(s)}}{V_s} - 1\ ,\ r_{\mathrm{M},j\in F(s)}\right]}{\operatorname{var}\left[r_{\mathrm{M},j\in F(s)}\right]} \cdot \left(\mathbb{E}\left[r_{\mathrm{M},j\in F(s)}\right] - r_{\mathrm{f,s}}\right) \qquad (12)$$

There are many possibilities that allow the return r_s to be the same in all $s \in S_t$. One of these possibilities is that the risk-free rate $r_{\mathrm{f,s}}$, the variance and expectation of the stochastic market return $r_{\mathrm{M},j\in F(s)}$ and the covariance between the market returns and the asset returns (firm or equity) are the same in all states $s \in S_t$. Particularly the covariance requires a critical eye because the values V_s, and the distributions of the cash flows $CF_{j\in F(s)}$, continuing values $V_{j\in F(s)}$ and market returns $r_{\mathrm{M},j\in F(s)}$ are not necessarily path-independent (i. e. the same for different s). For example, in the analysis of Miles and Ezzell (1980) the free cash flows and all values are path dependent, whereas they are not in Modigliani and Miller (1963). Therefore, we additionally require that the term $a_s = \frac{\operatorname{cov}\left[CF_{j\in F(s)}+V_{j\in F(s)}\ ,\ r_{M,j\in F(s)}\right]}{\mathbb{E}\left[CF_{j\in F(s)}+V_{j\in F(s)}\right]}$ is the same for each $s \in S_t$. Both Miles and Ezzell (1980) and Modigliani and Miller (1963) fulfil this requirement, and all the required returns (discount rates) are path-independent in their analyses. Having assumption 12 in place, we can now turn to the relevant DCF methods, which will be described by the deterministic backward-iterative process (11).

3.2 Discounted cash flow methods

In order to put the different DCF methods in context, we will depart from the following cash flow statement:

$$\sum_{i=1}^{N} CF_{i,t} = 0 \text{ for all } t = 1, \ldots, T \qquad (13)$$

where $CF_{i,t}$ is a stochastic cash flow of category i at time t. As relevant categories, one may consider payments from/to customers, suppliers, employees, equity holders, debt holders, tax authorities and other interest groups. For our analysis, we shall narrow this statement to the following:

$$CF_{\mathrm{Res},t} + CF_{\mathrm{D},t} + CF_{\mathrm{Tax},t} + CF_{\mathrm{E},t} = 0 \text{ for all } t = 1, \ldots, T \qquad (14)$$

where $CF_{\mathrm{E},t}$ ($CF_{\mathrm{D},t}$, $CF_{\mathrm{Tax},t}$) represents the cash flow to or from the equity holders (debt holders, tax authorities) and $CF_{\mathrm{Rest},t}$ is the cash flow to or from other stakeholders (suppliers of goods and services, customers, etc.).

One of the principles used in financial theory is the value additivity, i.e. $\sum_{i=1}^{N} V(\mathbf{CF}_i) = V(\sum_{i=1}^{N} \mathbf{CF}_i)$, where $\mathbf{CF}_i$ is an uncertain (stochastic) multi-period stream of cash flows and V represents its corresponding value. In our case, we therefore have:

$$V_{\mathrm{Rest},t-1} + V_{\mathrm{D},t-1} + V_{\mathrm{Tax},t-1} + V_{\mathrm{E},t-1} = 0 \text{ for all } t = 1, \dots, T \tag{15}$$

where the value $V_{i,t-1}$ of cash flow $CF_{i,t}$ is defined as $V_{i,t-1} = \frac{\mathbb{E}[CF_{i,t}]}{1+r_{i,t-1}}$, with $\mathbb{E}$ being the expectation operator and $r_{i,t-1}$ being the corresponding one-period required return (discount rate). Value additivity in financial models is usually justified by the requirement of arbitrage freeness. Since we are interested in the relationship between the levered and unlevered values of the firm and equity, we base our assumption on equation (14) and define the cash flows to all interest groups involved in the unlevered firm on the one hand and the levered firm on the other:

Levered: $$CF_{\mathrm{Rest_L},t} + CF_{\mathrm{D},t} + CF_{\mathrm{Tax_L},t} + CF_{\mathrm{EL},t} = 0 \text{ for all } t = 1, \dots, T \tag{16}$$

Unlevered: $$CF_{\mathrm{Rest_U},t} + CF_{\mathrm{Tax_U},t} + CF_{\mathrm{U},t} = 0 \text{ for all } t = 1, \dots, T \tag{17}$$

Based on these two equations, we define:

$$CF_{\mathrm{EL},t} = CF_{\mathrm{U},t} + CF_{\mathrm{Rest_L},t} - CF_{\mathrm{Rest_U},t} + CF_{\mathrm{Tax_L},t} - CF_{\mathrm{Tax_U},t} - CF_{\mathrm{D},t} \tag{18}$$

With $TS_t = CF_{\mathrm{Tax_L},t} - CF_{\mathrm{Tax_U},t}$ (tax shield due to leverage) and $FD_t = CF_{\mathrm{Rest_L},t} - CF_{\mathrm{Rest_U},t}$ (change in the remaining cash flow due to leverage), we can write:

$$CF_{\mathrm{EL},t} = CF_{\mathrm{U},t} + TS_t + FD_t - CF_{\mathrm{D},t} \tag{19}$$

In the literature, TS_t is commonly restricted to the tax benefits coming from the tax deductibility of interest payments. The most common reason for the existence of FD_t is costs related to financial distress, as well as agency costs and benefits (see Berk and DeMarco, 2019, chapter

16). Note that we will later neglect the existence of FD_t (because of assumptions 7 and 9 in section 2).

In what follows, we will state all the formulas for calculating the values of the cash flows that are relevant in our analysis. These formulas are defined according to the deterministic backward-iterative process given by (11), which requires assumption 12, discussed in the previous subsection.

The value of equity is determined by means of the equity method as follows:

$$V_{\mathrm{EL},t} = \frac{\overline{CF}_{\mathrm{EL},t+1} + \bar{V}_{\mathrm{EL},t+1}}{1 + r_{\mathrm{EL},t}} \tag{20}$$

where $\overline{CF}_{\mathrm{EL},t+1}$ are the expected payments to the equity holders, $\bar{V}_{\mathrm{EL},t+1}$ is the expected continuation value and $r_{\mathrm{EL},t}$ is the required return on levered equity.

The value of debt is derived by discounting the expected flow to the debt holders $\overline{CF}_{\mathrm{D},t+1}$ (interest payments, down payments, issues of new debt) and the continuation value $\bar{V}_{\mathrm{D},t+1}$ by means of the required return on debt $r_{\mathrm{D},t}$:

$$V_{\mathrm{D},t} = \frac{\overline{CF}_{\mathrm{D},t+1} + \bar{V}_{\mathrm{D},t+1}}{1 + r_{\mathrm{D},t}} \tag{21}$$

The value of the unlevered firm will be computed by discounting the expected free cash flow $\overline{FCF}_{t+1}$ (the flow to the equity holders as if there were no debt financing) and the expected unlevered continuation value $\bar{V}_{\mathrm{U},t+1}$ by means of the required return on unlevered equity $r_{\mathrm{U},t}$.

$$V_{\mathrm{U},t} = \frac{\overline{FCF}_{t+1} + \bar{V}_{\mathrm{U},t+1}}{1 + r_{\mathrm{U},t}} \tag{22}$$

The value of the tax shield is found by discounting the expected interest tax shield $\overline{TS}_{t+1}$ and the continuation value of future interest tax shields $\bar{V}_{\mathrm{TS},t+1}$ by means of the required return on the tax shield $r_{\mathrm{TS},t}$:

$$V_{\mathrm{TS},t} = \frac{\overline{TS}_{t+1} + \bar{V}_{\mathrm{TS},t+1}}{1 + r_{\mathrm{TS},t}} \tag{23}$$

The value of financial distress is determined by the following expression, where $\overline{FD}_{t+1}$ denotes the expected financial distress costs, $\bar{V}_{\text{FD},t+1}$ is the expected continuation value and $r_{\text{FD},t}$ is the corresponding required return.

$$V_{\text{FD},t} = \frac{\overline{FD}_{t+1} + \bar{V}_{\text{FD},t+1}}{1 + r_{\text{FD},t}} \tag{24}$$

The calculated values from (21) to (24) are the ingredients of the adjusted present value method that calculates the value of the levered firm as $V_{\text{FL},t} = V_{\text{U},t} + V_{\text{TS},t} + V_{\text{FD},t}$. From this, the value of the levered equity can be calculated as: $V_{\text{EL},t} = V_{\text{FL},t} - V_{\text{D},t}$.

The value of the levered firm by means of the FCF method is determined by discounting the expected free cash flow $\overline{FCF}_{t+1}$ (note that this is the hypothetical cash flow to the equity holders as if there were no debt financing) and the expected continuation value of the levered firm $\bar{V}_{\text{FL},t+1}$ with some 'modified' discount rate $r_{\text{FCF},t}$:

$$V_{\text{FL},t} = \frac{\overline{FCF}_{t+1} + \bar{V}_{\text{FL},t+1}}{1 + r_{\text{FCF},t}} \tag{25}$$

By 'modified', we want to stress the fact that this discount rate is used to derive the value of the levered firm from a cash flow in the unlevered firm, which seems odd at first sight. In the next subsection, we will show in more detail how the discount rate $r_{\text{FCF},t}$ is calculated so that the results from different DCF approaches remain consistent.

3.3 General deviation of the FCF method and $r_{FCF,t}$

In this section, we will look briefly at the general derivation of the free cash flow method as it will be used later. In what follows, we will temporarily neglect assumptions 8 and 9. This method can be derived with the following steps.

1. Depart from the equity method given by (20), where we define the cash flow to the equity holders based on the cash flow relationship given in equation (19):

$$V_{\text{EL},t} = \frac{\overline{FCF}_{t+1} + \overline{TS}_{t+1} + \overline{FD}_{t+1} - \overline{CF}_{\text{D},t+1} + \bar{V}_{\text{EL},t+1}}{1 + r_{\text{EL},t}} \tag{26}$$

2. Based on equation (21), we replace the cash flow to the debt holders with $V_{\mathrm{D},t} \cdot \left(1 + r_{\mathrm{D},t}\right) + \bar{V}_{\mathrm{D},t+1}$. We can do this because of assumptions 4 and 10. Furthermore, we replace $V_{\mathrm{EL},t}$ with $q_t \cdot V_{\mathrm{FL},t}$ and $V_{\mathrm{D},t}$ with $(1 - q_t) \cdot V_{\mathrm{FL},t}$. We obtain:

$$V_{\mathrm{FL},t} = \frac{\overline{FCF}_{t+1} + \overline{TS}_{t+1} + FD_{t+1} - V_{\mathrm{FL},t} \cdot \left(1 + r_{\mathrm{D},t}\right) \cdot (1 - q_t) + \bar{V}_{\mathrm{FL},t+1}}{\left(1 + r_{\mathrm{EL},t}\right) \cdot q_t} \tag{27}$$

This can be rearranged to:

$$\begin{aligned} &V_{\mathrm{FL},t} \cdot \left[\left(1 + r_{\mathrm{EL},t}\right) \cdot q_t + \left(1 + r_{\mathrm{D},t}\right) \cdot (1 - q_t)\right] - \overline{TS}_{t+1} - \overline{FD}_{t+1} \\ &= \overline{FCF}_{t+1} + \bar{V}_{\mathrm{FL},t+1} \end{aligned} \tag{28}$$

On the left-hand side of this expression, we recognize the familiar before-tax weighted average costs of capital r_{CCF} from formula (2).

3. In order to obtain an expression where the value of the levered firm can be determined by discounting the free cash flow, we need to relate $\overline{TS}_{t+1}$ and $\overline{FD}_{t+1}$ in some way to the value of the levered firm $V_{\mathrm{FL},t}$. Let us do this here by means of the ratios $\varphi_t = \frac{\overline{TS}_{t+1}}{V_{\mathrm{FL},t}}$ and $\gamma_t = \frac{\overline{FD}_{t+1}}{V_{\mathrm{FL},t}}$, as we lack additional specific assumptions regarding tax benefit and bankruptcy-related payments:

$$V_{\mathrm{FL},t} = \frac{\overline{CF}_{\mathrm{U},t+1} + \bar{V}_{\mathrm{FL},t+1}}{1 + q_t \cdot r_{\mathrm{EL},t} + (1 - q_t) \cdot r_{\mathrm{D},t} - \gamma_t - \varphi_t} \tag{29}$$

If there are no bankruptcy costs or interest tax shields, then the discount rate equals the before-tax weighted average costs of capital $q_t \cdot r_{\mathrm{EL},t} + (1 - q_t) \cdot r_{\mathrm{D},t}$. In the case of Modigliani and Miller (1963) or Miles and Ezzell (1980), we would have $\gamma_t = 0$ and $\varphi_t = (1 - q_t) \cdot r_{\mathrm{D},t} \cdot \tau_t$.

3.4 General relationship between $r_{EL,t}$ and $r_{EU,t}$

While the last subsection focused on the relationship between the required return on levered equity and the discount rate in the free cash flow approach, this section analyses the relationship between the levered and the unlevered return on equity. For the derivation of the translation formula between $r_{\mathrm{EL},t}$ and $r_{\mathrm{U},t}$, we look at the following relationship:

$$V_{\mathrm{EL},t} + V_{\mathrm{D},t} = V_{\mathrm{U},t} + V_{\mathrm{TS},t} + V_{\mathrm{FD},t} \tag{30}$$

More precisely, this is:

$$\begin{aligned} &\frac{\overline{CF}_{\mathrm{EL},t+1} + \bar{V}_{\mathrm{EL},t+1}}{1 + r_{\mathrm{EL},t}} + \frac{\overline{CF}_{\mathrm{D},t+1} + \bar{V}_{\mathrm{D},t+1}}{1 + r_{\mathrm{D},t}} \\ &\qquad = \frac{\overline{FCF}_{t+1} + \bar{V}_{\mathrm{U},t+1}}{1 + r_{\mathrm{U},t}} + \frac{\overline{TS}_{t+1} + \bar{V}_{\mathrm{TS},t+1}}{1 + r_{\mathrm{TS},t}} + \frac{\overline{FD}_{t+1} + \bar{V}_{\mathrm{FD},t+1}}{1 + r_{\mathrm{FD},t}} \end{aligned} \tag{31}$$

When solving this expression for the required return on equity $r_{\mathrm{EL},t}$, we notice that without further assumptions, this return depends on all the other discount rates, cash flows and continuation values. Furthermore, we face the dilemma that all but a few variables in this equation will generally depend on the capital structure $q_t = \frac{\bar{V}_{\mathrm{EL},t}}{\bar{V}_{\mathrm{EL},t} + \bar{V}_{\mathrm{D},t}}$, except for $r_{\mathrm{U},t}$ and $\overline{FCF}_{t+1}$ (assumption 2 in section 2). Various researchers have therefore made different assumptions to solve this dilemma and to be able to derive formulas with relatively few parameters, like the ones presented above by Miles and Ezzell (1980) or Modigliani and Miller (1963). In what follows, we will use Modigliani and Miller (1963) as an example to show how to derive the relationship between $r_{\mathrm{EL},t}$ and $r_{\mathrm{U},t}$. We start with expression (30). Because of assumptions 7 and 9, the value of financial distress is zero $V_{\mathrm{FD},t} = V_{\mathrm{FD},t+1} = 0$. Furthermore, Modigliani and Miller (1963) assume a stationary constant perpetuity. This implies that the value of debt is constant and deterministic across time. Therefore, both the interest tax shield and its value are also constant and deterministic and can be discounted with the risk-free rate. The tax shield value is therefore $V_{\mathrm{TS},t} = \frac{V_{\mathrm{D},t} \cdot \tau \cdot r_{\mathrm{f}} + V_{\mathrm{TS},t+1}}{1 + r_{\mathrm{f}}} = \frac{V_{\mathrm{D},t} \cdot \tau \cdot r_{\mathrm{f}}}{r_{\mathrm{f}}} = V_{\mathrm{D},t} \cdot \tau$. This leads immediately to (we neglect the time index t because of stationary perpetuity):

$$V_{\mathrm{EL}} + V_{\mathrm{D}} \cdot (1 - \tau) = V_{\mathrm{U}} = \frac{\overline{FCF}}{r_{\mathrm{U}}} \tag{32}$$

Now we multiply this expression by r_{U}, replace $\overline{FCF}$ with $V_{\mathrm{FL}} \cdot [q \cdot r_{\mathrm{EL}} + (1-q) \cdot (1-\tau) \cdot r_{\mathrm{D}}]$,and use $V_{\mathrm{EL}} = q \cdot V_{\mathrm{FL}}$ and $V_{\mathrm{D}} = (1-q) \cdot V_{\mathrm{FL}}$. Then we solve for r_{EL} and obtain:

$$r_{\mathrm{EL}} = r_{\mathrm{U}} + (r_{\mathrm{U}} - r_{\mathrm{f}}) \cdot (1-\tau) \cdot \frac{(1-q)}{q}$$

which corresponds to formula (3). This is a relatively simple formula that, compared to (31), only contains four relevant parameters for calculating the required return on levered equity. In the following section, we apply the same approach to derive a translation formula that will not match formulas (3) or (8).

4 Translation formula for explicit cash flows and fixed risk-free debt

In this section, we derive the translation formula between the required return on levered equity and unlevered equity for a situation where the cash flow can have a finite lifetime. However, the expected cash flow does not need to come in the form of an annuity. Both the expectation and standard deviation of this cash flow can vary across different time periods, and the cash flow is allowed to be path-dependent or path-independent. In addition to the assumptions presented in sections 2 and 3, the following assumption is required.

Assumption 13: Debt financing is prespecified. It can take the form of a constant payment loan, constant amortization loan, bullet loan, or another type of loan for which payments are given in advance. This assumption does not exclude the Modigliani-Miller type of constant leverage where both cash flows and debt are path-independent. However, it excludes the Miles-Ezzell type of constant leverage, because here the flow to the debt holders is path-dependent.

Let us start with expression (30), where we neglect financial distress or bankruptcy (assumptions 7 and 9) and use the risk-free rate for the valuation of debt (assumption 6):

$$V_{\mathrm{FL},t} = V_{\mathrm{U},t} + V_{\mathrm{TS},t} \tag{33}$$

Similar to (31), we render this expression more precisely as follows:

$$V_{\mathrm{FL},t} = \frac{\overline{FCF}_{t+1} + \bar{V}_{\mathrm{U},t+1}}{1 + r_{\mathrm{U},t}} + \frac{\overline{TS}_{t+1} + \bar{V}_{\mathrm{TS},t+1}}{1 + r_{\mathrm{TS},t}} \tag{34}$$

If the debt is risk-free and the payment process of debt is given path-independently in advance, the tax benefit will also be risk-free. Hence, we can define the value of the interest tax shield as follows:

$$V_{\mathrm{TS},t} = \frac{\overline{TS}_{t+1} + \bar{V}_{\mathrm{TS},t+1}}{1 + r_{\mathrm{TS},t}} = \frac{r_{\mathrm{f},t} \cdot V_{\mathrm{D},t} \cdot \tau_t + \bar{V}_{\mathrm{TS},t+1}}{1 + r_{\mathrm{f},t}} = \frac{r_{\mathrm{f},t} \cdot (1-q) \cdot V_{\mathrm{FL},t} \cdot \tau_t + \bar{V}_{\mathrm{TS},t+1}}{1 + r_{\mathrm{f},t}} \quad (35)$$

Note that we are working with the assumption that the future risk-free interest rate is deterministic (it may not be constant, but for simplicity's sake, we still use a time-invariant risk-free interest rate in the further discourse): $r_{\mathrm{f},t} = r_{\mathrm{f}}$. The interest payments are based on the nominal amount of debt that is assumed to be equal to the value of the debt: $I_{t+1} = V_{\mathrm{D},t} \cdot r_f$. The tax shield then becomes $TS_{t+1} = \tau_{t+1} \cdot I_{t+1}$. Therefore, the value of the tax saving at time t is equal to:

$$V_{\mathrm{TS},t} = \sum_{n=t+1}^{T} \frac{TS_n}{(1 + r_{\mathrm{f}})^n} = \sum_{n=t+1}^{T} \frac{\tau_t \cdot I_n}{(1 + r_{\mathrm{f}})^n} \quad (36)$$

Now we notice that the time-invariance of the tax rate according to assumption 3 is a critical requirement for further simplification of expression (36), which can now be written as:

$$V_{\mathrm{TS},t} = \tau \cdot \sum_{n=t+1}^{T} \frac{I_n}{(1 + r_{\mathrm{f}})^n} \quad (37)$$

The tax rate is thus multiplied by the value of the interest payments $V_{\mathrm{Int},t}$. Let $v_t = \frac{V_{\mathrm{Int},t}}{V_{\mathrm{D},t}}$ be the ratio that describes the value of the interest payments in relation to the value of debt. Then the value of the interest tax shield at time t becomes:

$$V_{\mathrm{TS},t} = v_t \cdot \tau \cdot V_{\mathrm{D},t} = \frac{r_f \cdot V_{\mathrm{D},t} \cdot \tau + V_{\mathrm{TS},t+1}}{1 + r_{\mathrm{f}}} \quad (38)$$

It follows immediately that the value of the interest tax shield at time $t + 1$ is:

$$V_{\mathrm{TS},t+1} = v_t \cdot \tau \cdot V_{\mathrm{D},t} \cdot (1 + r_{\mathrm{f}}) - r_{\mathrm{f}} \cdot V_{\mathrm{D},t} \cdot \tau \quad (39)$$

With this definition, we can return to formula (34), which now becomes:

$$V_{\mathrm{FL},t} = \frac{\overline{FCF}_{t+1} + \bar{V}_{\mathrm{U},t+1}}{1 + r_{\mathrm{U},t}} + v_t \cdot \tau \cdot V_{\mathrm{D},t} \tag{40}$$

Since $\bar{V}_{\mathrm{U},t+1} = \bar{V}_{\mathrm{FL},t+1} - \bar{V}_{\mathrm{TS},t+1}$, we have:

$$V_{\mathrm{FL},t} = \frac{\overline{FCF}_{t+1} + \bar{V}_{\mathrm{FL},t+1} - \bar{V}_{\mathrm{TS},t+1}}{1 + r_{\mathrm{U},t}} + v_t \cdot \tau \cdot V_{\mathrm{D},t} \tag{41}$$

The value of the tax shield $V_{\mathrm{TS},t+1}$ has been described by (39). This brings us to:

$$V_{\mathrm{FL},t} = \frac{\overline{FCF}_{t+1} + \bar{V}_{\mathrm{FL},t+1}}{1 + r_{\mathrm{U},t}} - \frac{v_t \cdot \tau \cdot V_{\mathrm{D},t} \cdot (1 + r_{\mathrm{f}}) - r_{\mathrm{f}} \cdot V_{\mathrm{D},t} \cdot \tau}{1 + r_{\mathrm{U},t}} + v_t \cdot \tau \cdot V_{\mathrm{D},t} \tag{42}$$

We recognize the term $\overline{FCF}_{t+1} + \bar{V}_{\mathrm{L},t+1}$ from the FCF method, which can be rearranged to $V_{\mathrm{FL},t} \cdot \left(1 + r_{\mathrm{FCF},t}\right) = \overline{FCF}_{t+1} + \bar{V}_{\mathrm{L},t+1}$. We add this term together with $V_{\mathrm{D},t} = (1 - q_t) \cdot V_{\mathrm{FL},t}$ into expression (42). This gives:

$$\begin{aligned} V_{\mathrm{FL},t} = {} & \frac{V_{\mathrm{FL},t} \cdot \left(1 + r_{\mathrm{FCF},t}\right)}{1 + r_{\mathrm{U},t}} \\ & - \frac{v_t \cdot \tau \cdot (1 - q_t) \cdot V_{\mathrm{FL},t} \cdot (1 + r_{\mathrm{f}}) - r_{\mathrm{f}} \cdot (1 - q_t) \cdot V_{\mathrm{FL},t} \cdot \tau}{1 + r_{\mathrm{U},t}} + v_t \cdot \tau \cdot (1 - q_t) \cdot V_{\mathrm{FL},t} \end{aligned} \tag{43}$$

After dividing by $V_{\mathrm{FL},t}$, we obtain the following result:

$$\begin{aligned} 1 = {} & \frac{1 + r_{\mathrm{FCF},t}}{1 + r_{\mathrm{U},t}} \\ & - \frac{v_t \cdot \tau \cdot (1 - q_t) \cdot (1 + r_{\mathrm{f}}) - r_{\mathrm{f}} \cdot (1 - q_t) \cdot \tau}{1 + r_{\mathrm{U},t}} + v_t \cdot \tau \cdot (1 - q_t) \end{aligned} \tag{44}$$

Solving for $r_{\mathrm{FCF},t}$ brings us to:

$$r_{\mathrm{FCF},t} = r_{\mathrm{U},t} - \tau \cdot (1 - q_t) \cdot \left([1 - v_t] \cdot r_{\mathrm{f}} + v_t \cdot r_{\mathrm{U},t}\right) \tag{45}$$

The discount rate in the FCF method has been shown in section 3. We can therefore immediately conclude the relationship between $r_{\mathrm{E_L}}$ and $r_{\mathrm{E_U}}$ as follows:

$$r_{\mathrm{EL},t} = r_{\mathrm{U},t} + \left(r_{\mathrm{U},t} - r_{\mathrm{f}}\right) \cdot (1 - \tau \cdot v_t) \cdot \frac{1 - q_t}{q_t} \tag{46}$$

We now see that this formula has an extra parameter compared to formulas (3) or (8). It is the parameter v that describes a specific property of different types of loan financing, i. e. this parameter will be different for constant payment, constant amortization, bullet and other types of loans. We also notice that formula (3) proposed by Modigliani and Miller (1963) is a special case of formula (46). Whenever $v_t = 1$ then formula (46) becomes the same as (3). The meaning of $v_t = 1$ is that the debt value consists of 100 % interest payments and that there never happen any down payments of debt. This is exactly what happens in the perpetual case of Modigliani and Miller (1963).

5 Numerical example

In this section, we will underpin the preceding analysis by means of a numerical example. We will use the following agenda for walking through this example. In the *first step*, we will introduce the numerical data that is given for a stream of cash flows with a finite lifetime. In the *second step* we will choose the most appropriate DCF method for determining the value of the firm. In the *third step*, we will confirm the calculated firm value by applying alternative DCF methods. Here the new formula (46) will be applied. In the *fourth step*, we will look at where and how inconsistencies are generated by M&M's formula (3) or M&E's formula (8). In the *fifth step*, we will look at how the discount rates and the values of the levered firm and the levered equity depend on leverage. This allows us to conceptually compare our calculations with the results that are found in contemporary textbooks. In the *sixth and final step*, we will look at how different types of loans affect the values of the firm and equity.

Step 1 – Numerical information: Assume that an investment project generates earnings before interest and taxes (EBIT) as shown in Table 1. For simplicity, depreciations are the only difference between EBIT and the unlevered cash flow to the firm. These are also shown in Table 1.

Table 1. Input data for earnings before interest and taxes (EBIT) and depreciation

Point in time	$t = 0$	$t = 1$	$t = 2$	$t = 3$
$EBIT_t$		30 000.00	40 000.00	50 000.00
Depreciation (Dep_t)		20 000.00	15 000.00	10 000.00

Furthermore, the required return on unlevered equity is assumed to be constant as $r_{\mathrm{U},t} = 0.20$ for all $t = 0, \ldots, 2$, the tax rate is $\tau = 0.30$, and the risk-free interest rate is $r_\mathrm{f} = 0.05$. A constant amortization loan with an initial amount (at $t = 0$) of 45,000 will be paid down during the project period (here, three periods). This loan is assumed to be risk-free, and the interest rate is equal to the risk-free interest rate. The outstanding principal is assumed to be equal to the value of the loan at any point in time.

Step 2 – Choice of valuation approach and determination of all values: Since the cash flows to the debt holders are known and deterministic, we can determine both the value of debt and the value of the tax shield. Furthermore, the required return on unlevered equity is given. We can therefore calculate the value of the unlevered firm. Consequently, the values of the levered firm and levered equity can be computed by means of the adjusted-present-value method, i.e. $V_{\mathrm{FL},t} = V_{\mathrm{U},t} + V_{\mathrm{TS},t}$ and $V_{\mathrm{EL},t} = V_{\mathrm{FL},t} - V_{\mathrm{D},t}$ for all $t = 0, .., 2$.

Let us begin with the calculations concerning the debt. Table 2 shows the cash flows to the debt holders and the development of the value of the loan. The value of the loan at time t can be calculated by formula (21). For example, at $t = 1$, the value of debt can be computed as $V_{\mathrm{D},1} = \frac{CF_{\mathrm{D},2} + V_{\mathrm{D},2}}{1 + r_\mathrm{f}} = \frac{16\,500 + 15\,000}{1 + 0.05} = 30\,000$.

Table 2. Payments related to fixed debt financing

Point in time	$t = 0$	$t = 1$	$t = 2$	$t = 3$
Outstanding principal of debt	45 000.00	30 000.00	15 000.00	0.00
Down payment of debt ($\Delta V_{\mathrm{D},t}$)		15 000.00	15 000.00	15 000.00
Interest payment (I_t)		2 250.00	1 500.00	750.00
Flow to debt holders ($CF_{\mathrm{D},t}$)		17 250.00	16 500.00	15 750.00
Value of debt ($V_{\mathrm{D},t}$)	45 000.00	30 000.00	15 000.00	
Value of interest payments	4 151.28	2 108.84	714.29	
Parameter v_t	0.0923	0.0703	0.0476	

Table 3. Value of the interest tax shield

Point in time	$t = 0$	$t = 1$	$t = 2$	$t = 3$
Interest payment (I_t)		2 250.00	1 500.00	750.00
Interest tax shield ($TS_t = I_t \cdot \tau$)		675.00	450.00	225.00
Value of interest tax shield ($V_{\mathrm{TS},t}$)	1 245.38	632.65	214.29	

In Table 3 the interest tax shield is calculated on the basis of the tax rate τ and the interest payments I_t, i.e. $TS_t = I_t \cdot \tau$. Since the interest tax shield is deterministic, its value can be calculated using the risk-free interest rate. Exemplified for time $t = 1$, the calculation is as follows: $V_{\mathrm{TS},1} = \frac{TS_2 + V_{\mathrm{TS},2}}{1 + r_{\mathrm{f}}} = \frac{450 + 214.29}{1 + 0.05} = 632.65$.

Table 4 shows the remaining calculations of the APV method. The value of the unlevered firm is calculated by discounting the free cash flow $\overline{FCF}_{t+1} = \overline{EBIT}_{t+1} \cdot (1 - \tau) + Dep_{t+1}$. For example, at point of time $t + 1 = 2$ we obtain: $\overline{FCF}_2 = \overline{EBIT}_2 \cdot (1 - \tau) + Dep_2 = 40\,000 \cdot (1 - 0.30) + 15\,000$. The value of the unlevered firm at time $t = 1$ can now be computed as $V_{\mathrm{U},1} = \frac{\overline{FCF}_2 + \bar{V}_{\mathrm{U},2}}{1 + r_{\mathrm{U},1}} = \frac{43\,000 + 37\,500}{1 + 0.20} = 67\,083.33$.

In the end, we can merge the value of the tax shield with the value of the unlevered firm to the value of the levered firm. Table 4 also shows the calculation of the value of levered equity as the difference between the value of the levered firm and the value of debt $V_{\mathrm{EL},t} = V_{\mathrm{FL},t} - V_{\mathrm{D},t}$.

Table 4. Value of levered firm and levered equity by means of APV method

Point in time	$t = 0$	$t = 1$	$t = 2$	$t = 3$
Free cash flow (unlevered) ($\overline{FCF}_t$)		41 000.00	43 000.00	45 000.00
Value of unlevered firm ($V_{\mathrm{U},t}$)	90 069.44	67 083.33	37 500.00	
Value of interest tax shield ($V_{\mathrm{TS},t}$)	1 245.38	632.65	214.29	
Value of the levered firm ($V_{\mathrm{FL},t}$)	91 314.83	67 715.99	37 714.29	
Value of debt ($V_{\mathrm{D},t}$)	45 000.00	30 000.00	15 000.00	
Value of levered equity ($V_{\mathrm{EL},t}$)	46 314.83	37 715.99	22 714.29	
Equity-to-firm-value ratio (q_t)	0.5072	0.5570	0.6023	

Step 3 – Confirmation of the calculated values by means of the equity method and FCF method: We now turn to the calculation of the value of the levered equity and levered firm by means of the equity method according to formula (20). The cash flow to the equity holders is $\overline{CF}_{\mathrm{EL},t+1} = (\overline{EBIT}_{t+1} - I_{t+1}) \cdot (1 - \tau) + Dep_{t+1} - \Delta V_{\mathrm{D},t+1}$. For example, at time $t + 1 = 2$, we can calculate the following flow to the equity holders: $\overline{CF}_{\mathrm{EL},2} = (40\,000 - 1\,500) \cdot$

$(1-0.30)+15\,000-15\,000=26\,950$. In order to discount this cash flow, the corresponding required return needs to be calculated. Here, the newly derived translation formula (46) will be used. This formula assumes that we know the equity-to-firm-value ratio q_t and the parameter v_t, which indicates the value of the interest payments in relation to the debt value. For example at time $t=1$, we can calculate: $r_{\mathrm{EL},1}=r_{\mathrm{U},1}+\left(r_{\mathrm{U},1}-r_{\mathrm{f}}\right)\cdot(1-\tau\cdot v_1)\cdot\frac{1-q_1}{q_1}=0.20+(0.20-0.05)\cdot(1-0.30\cdot 0.0703)\cdot\frac{1-0.5570}{0.5570}=0.3168$. Table 5 shows all calculated figures in connection with the equity method. Note that the equity-to-firm-value ratio cannot be calculated unless the value of the levered firm and levered equity value are known. We thus encounter a circularity problem in both the equity method and FCF method. In the valuation frameworks of Miles and Ezzell (1980, 1985) or Modigliani and Miller (1963), the circularity problem would occur in the APV method, where the equity-to-firm-value ratio was given, and the flow to the debtholders depended on this ratio.

Table 5. Value of levered equity determined by means of equity method

Point in time	$t=0$	$t=1$	$t=2$	$t=3$
Cash flow to levered equity ($\overline{CF}_{\mathrm{EL},t}$)		24 425.00	26 950.00	29 475.00
Required return on levered equity ($r_{\mathrm{EL},t}$)	0.3417	0.3168	0.2976	
Value of levered equity ($V_{\mathrm{EL},t}$)	46 314.83	37 715.99	22 714.29	

Note that our calculations satisfy the claim for consistency in the sense that the value of levered equity is the same for both the equity method and the adjusted present value method. Table 6 shows the calculations with respect to the FCF method. Here, we apply the after-tax weighted average costs of capital. For time $t=1$, this is: $r_{\mathrm{FCF},1}=q_1\cdot r_{\mathrm{EL},1}+(1-q_1)\cdot(1-\tau)\cdot r_{\mathrm{D},1}=0.5570\cdot 0.3168+(1-0.5570)\cdot(1-0.30)\cdot 0.05=0.1920$. Again, the calculations are consistent with the adjusted present value method.

Table 6. Value of the levered firm determined by means of the FCF method

Point in time	$t=0$	$t=1$	$t=2$	$t=3$
Free cash flow ($\overline{FCF}_t$)		41,000.00	43 000.00	45 000.00
Discount rate in FCF method ($r_{\mathrm{FCF},t}$)	0.1906	0.1920	0.1932	
Value of levered firm ($V_{\mathrm{FL},t}$)	91 314.83	67 715.99	37 714.29	

Step 4 – Inconsistencies with the translation formulas by M&M and M&E: Table 7 and Table 8 show the erroneous and inconsistent calculations if the translation formulas (3) or (6) according to Modigliani and Miller (1963) are applied. More particularly, Table 7 shows the calculations with respect to the equity method. Note that the translation formula (3) requires the equity-to-firm value ratio q_t. We assume here the same values as calculated in Table 4. We notice that the equity values calculated in Table 7 deviate from the correct equity values calculated in Table 5. Table 8 shows the calculations according to the FCF method, where the discount rate corresponds to translation formula (6). We notice here, too, that the values of both levered equity and the levered firm are incorrect compared to Table 6. In addition, the equity method is inconsistent with respect to the FCF method, since the values of levered equity are different in Table 7 and Table 8. The same effects occur if translation formulas (7) or (8), according to Miles and Ezzell (1980), are applied. The corresponding results are shown in Table 9 and Table 10.

Table 7. Inconsistent calculation of levered equity value based on translation formula (3)

Point in time	$t = 0$	$t = 1$	$t = 2$	$t = 3$
Flow to levered equity ($\overline{CF}_{\mathrm{EL},t}$)		24 425.00	26 950.00	29 475.00
Required return on levered equity ($r_{\mathrm{EL},t}$)	0.3020	0.2835	0.2693	
Value of levered equity ($V_{\mathrm{EL},t}$)	48 780.72	39 088.43	23 220.74	

Table 8. Inconsistent calculation of the values of levered equity and firm based on translation formula (6)

Point in time	$t = 0$	$t = 1$	$t = 2$	$t = 3$
Free cash flow ($\overline{FCF}_t$)		41 000.00	43 000.00	45 000.00
Discount rate in FCF method ($r_{\mathrm{FCF},t}$)	0.1704	0.1734	0.1761	
Value of levered firm ($V_{\mathrm{FL},t}$)	94 197.19	69 251.40	38 260.87	
Value of levered equity ($V_{\mathrm{EL},t}$)	49 197.19	39 251.40	23 260.87	

Table 9. Inconsistent calculation of levered equity value based on translation formula (8)

Point in time	$t = 0$	$t = 1$	$t = 2$	$t = 3$
Flow to levered equity ($\overline{CF}_{\mathrm{EL},t}$)		24 425.00	26 950.00	29 475.00
Required return on levered equity ($r_{\mathrm{EL},t}$)	0.3437	0.3176	0.2976	
Value of levered equity ($V_{\mathrm{EL},t}$)	46 230.27	37 692.75	22 714.29	

Table 10. Inconsistent calculation of the values of levered equity and firm based on translation formula (7)

Point in time	$t = 0$	$t = 1$	$t = 2$	$t = 3$
Free cash flow ($\overline{FCF}_t$)		41 000.00	43 000.00	45 000.00
Discount rate in FCF method ($r_{\mathrm{FCF},t}$)	0.1916	0.1924	0.1932	
Value of levered firm ($V_{\mathrm{FL},t}$)	91 217.43	67 690.31	37 714.29	
Value of levered equity ($V_{\mathrm{EL},t}$)	46 217.43	37 690.31	22 714.29	

By means of Figure 2, we will look more closely into the inconsistencies that arise from using the wrong translation formula. More precisely, Figure 2 shows the calculation steps of the APV method on the left-hand side and the FCF method on the right-hand side. These two methods are connected by translation formulas like (3), (8) or (46) which is indicated by T1 in Figure 2. In case of a project cash flows with a limited lifetime (satisfying the assumptions stated above) formulas (3) and (8) are incorrect. If these formulas are used anyway for the computation of the required return on levered equity, then the following inconsistencies occur.

(a) Since, the required return on levered equity is erroneously calculated in T1, the discount rate in the free-cash-flow method (F1) is also incorrect and deviates from the true discount rate in A8.

(b) Since the discount rate in F1 is incorrect, also the values of the levered firm (F2) and levered equity (F3) are incorrect and deviate from the true values (A4 and A5).

(c) Since the values in F2 and F3 are incorrect, also the debt-to-firm-value ratio in F4 is incorrect and different from the true ratio in A6.

(d) Finally, recalculating the required return on equity in F5 yields an erroneous value, which not only is different from the true value in A7, but also different from the value computed in T1 (if the incorrect translation formula is applied).

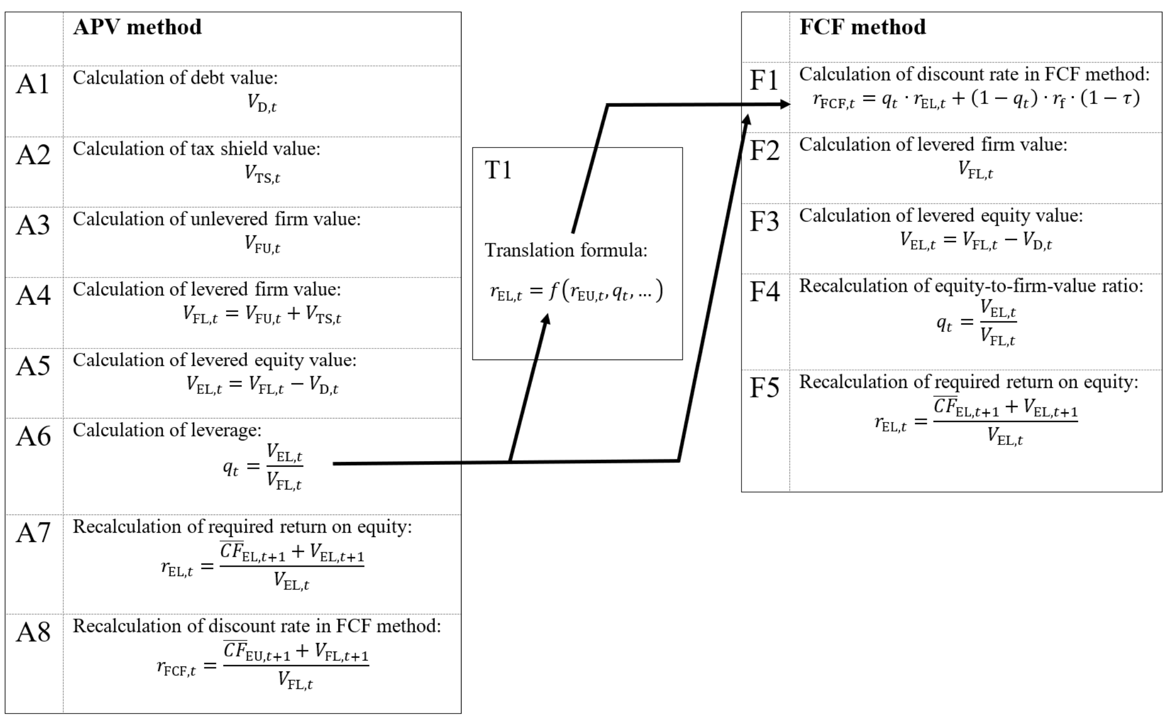

Figure 2: The connecting formula between the APV and the FCF method

Step 5 – Dependency of discount rates and values on leverage: We will now have a look at how the discount rate in the free-cash-flow method ($r_{\text{FCF},t}$, also referred to as weighted average costs of capital after tax) and the required return on levered equity $r_{\text{EL},t}$ depend on leverage. This is done by varying the amount of the outstanding loan in point of time $t = 0$. Table 11 shows the values of the firm and equity, the leverage and the discount rates for point in time $t = 0$. It is important to note that the firm and equity values, the discount rates ($r_{\text{FCF},t}$ and $r_{\text{EL},t}$) and the equity-to-firm-value-ratio (q_t) change with the remaining maturity $T - t$ of project.

In Figure 3, the relation between leverage (both in terms of debt-to-firm value ratio and debt-to-equity-value ratio) are plotted. We recognize the same shape of the functions as depicted in contemporary textbooks (compare for example with Brealey et al. (2020, chapter 19) and Berk and DeMarzo (2017, chapter 15). This means that with increasing leverage the discount rate in the FCF method (weighted average cost of capital after tax) decreases and the required return on levered equity increases. Hence, Modigliani and Miller's (1963) famous propositions I and II are not rejected by formula (46). Only the numerical change in the risk premium due to financial leverage is different. Looking at formulas (3), (8) and (46), we notice that they all have the same functional form $r_{\text{EL}} = r_{\text{EU}} + s \cdot \frac{1-q}{q}$, where s denotes the change of the required return due to financial leverage. For the three aforementioned formulas the slopes are:

M&E's formula (8): $$s_{\text{M\&E}} = (r_{\text{U}} - r_{\text{f}}) \cdot \frac{1 + r_{\text{f}} \cdot (1 - \tau)}{1 + r_{\text{f}}}$$

M&M's formula (3): $$s_{\text{M\&M}} = (r_{\text{U}} - r_{\text{f}}) \cdot (1 - \tau)$$

This paper's formula (46): $$s_{45} = (r_{\text{U}} - r_{\text{f}}) \cdot (1 - \tau \cdot v)$$

For $r_{\text{U}} > r_{\text{f}} > 0$, we can easily see that $s_{\text{M\&E}} < s_{\text{M\&M}} < s_{45}$, which means that the increase of the risk premium due to financial leverage is highest in the case of finite-life projects with fixed debt financing compared to the constellations of M&M or M&E.

Table 11. Values and discount rates dependent on leverage

Debt value $V_{D,0}$	Levered firm value $V_{\mathrm{FL},0}$	Levered equity value $V_{\mathrm{EL},0}$	Debt-to-firm-value ratio $(1-q_0)$	Debt-to-equity-value ratio $(1-q_0)/q_0$	Discount rate in FCF-method $r_{\mathrm{FCF},0}$	Required return of levered equity $r_{\mathrm{EL},0}$
0	90 069	90 069	0 %	0 %	20.0 %	20.0 %
10 000	90 346	80 346	11 %	12 %	19.8 %	21.8 %
20 000	90 623	70 623	22 %	28 %	19.6 %	24.1 %
30 000	90 900	60 900	33 %	49 %	19.4 %	27.2 %
40 000	91 176	51 176	44 %	78 %	19.2 %	31.4 %
45 000*	91 315	46 315	49 %	97 %	19.1 %	34.2 %
50 000	91 453	41 453	55 %	121 %	19.0 %	37.6 %
60 000	91 730	31 730	65 %	189 %	18.7 %	47.6 %
70 000	92 007	22 007	76 %	318 %	18.5 %	66.4 %
80 000	92 283	12 283	87 %	651 %	18.3 %	115.0 %
90 000	92 560	2 560	97 %	3515 %	18.1 %	532.7 %

*These numbers correspond to the numerical example above.

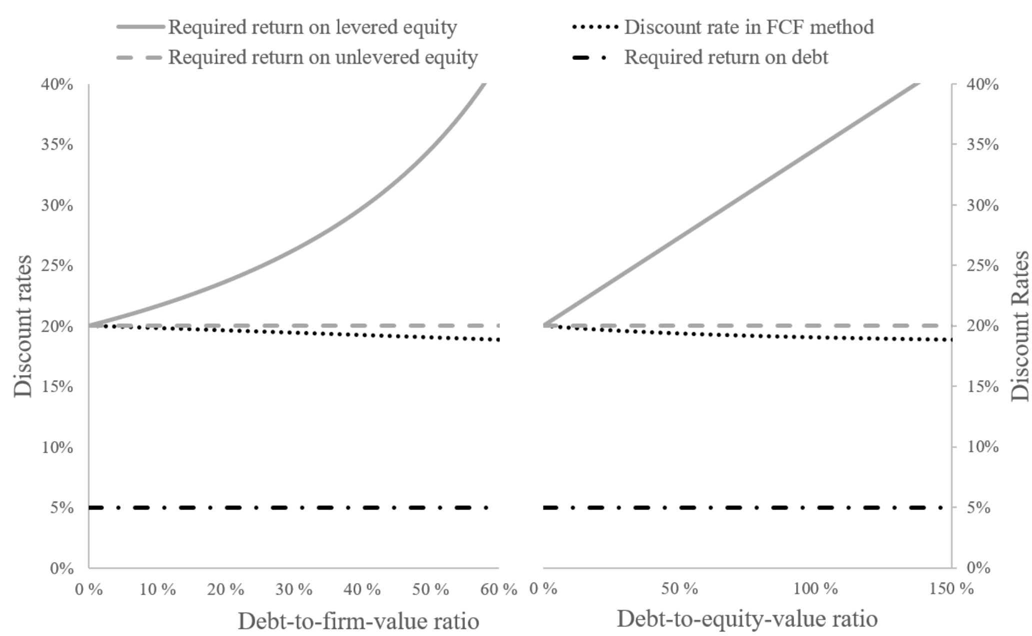

Figure 3: Discount rates for different levels of leverage

Step 6 – The effect of different loan types on discount rates and values: Finally, we will look at how the values of the levered firm and levered equity and the interest-to-debt-value ratio v_t behave for different types of loans. For all loans we assume the same initial amount of 45,000 and the same interest rate of 5 %. The results are shown in Table 13. To avoid any misunderstandings about the structure of these loans, Table 12 summarizes the corresponding payments (point of time $t = 0$ contains the initial amount of debt, and $t = 1$ to $t = 3$ contain the debt service that consists of interest and down payments). In order to compare the results with the constant leverage policies of M&M and M&E, we have also added a loan that maintains a constant leverage throughout the maturity of the project (in this case $q_0 = q_1 = q_2 = 50.77$ %) In Table 14, we calculate the errors that occurs if the translation formulas of M&M and M&E are applied. Even for constant leverage the formula of M&M cannot be applied because it requires a perpetual annuity. The reason, why the formula of M&E cannot be applied, is different. The formula of M&E is applicable to finite-life projects with constant leverage. It requires, however, stochastic (firm-value adjusted) debt payments.

Table 12. Payment structure of different types of loans

Type of loan	Amount of loan (debt value) in $t = 0$	Debt service (interest and down payment)		
		in $t = 1$	in $t = 2$	in $t = 3$
Bullet Loan	45 000	2 250.00	2 250.00	47 250.00
Constant Payment Loan	45 000	16 524.39	16 524.39	16 524.39
Constant Amortization Loan*	45 000	17 250.00	16 500.00	15 750.00
Constant Leverage	45 000	13 865.47	16 461.71	19 521.64

*These numbers correspond to the numerical example above.

Table 13. Levered firm and equity values and interest-to-debt-value ratios for different types of loans

Type of loan	Levered firm value $V_{\mathrm{FL},0}$	Levered equity value $V_{\mathrm{EL},0}$	Interest-to-debt-value ratio		
			v_0	v_1	v_2
Bullet Loan	91 907.64	46 907.64	13.62 %	9.30 %	4.76 %
Constant Payment Loan	91 334.26	46 334.26	9.37 %	7.08 %	4.76 %
Constant Amortization Loan*	91 314.83	46 314.83	9.23 %	7.03 %	4.76 %
Constant Leverage	91 407.42	46 407.42	9.91 %	7.29 %	4.76 %

*These numbers correspond to the numerical example above.

Table 14. Errors that occurs if the translation formulas of M&M or M&E are applied to finite-life projects

Type of loan	M&M - FCF Method		M&M - Equity Method	
	$V_{\mathrm{EL},0}$	Error	$V_{\mathrm{EL},0}$	Error
Bullet Loan	51 011.17	8.7 %	51 328.49	9.4 %
Constant Payment Loan	49 256.37	6.3 %	48 826.07	4.1 %
Constant Amortization Loan*	49 197.19	6.2 %	48 780.72	4.0 %
Constant Leverage	49 479.62	6.6 %	48 991.79	4.4 %

Type of loan	M&E - FCF Method		M&E - Equity Method	
	$V_{\mathrm{EL},0}$	Error	$V_{\mathrm{EL},0}$	Error
Bullet Loan	46 692.33	-0.5 %	46 727.93	-0.4 %
Constant Payment Loan	46 233.35	-0.2 %	46 246.75	-1.4 %
Constant Amortization Loan*	46 217.43	-0.2 %	46 230.27	-1.4 %
Constant Leverage	46 293.19	-0.2 %	46 308.78	-1.3 %

*These numbers correspond to the numerical example above.

6 Conclusion

In this paper, we have shown how to consistently compute the values of the firm and equity when streams of cash flows have a finite life. By "consistency" we mean that the same value of the firm and the same value of equity are obtained when applying both the adjusted-present-value method, the free-cash-flow method and the equity method. We have achieved this consistency by deriving a new formula for translating between the required return on unlevered and levered equity.

As outlined earlier, the consistent valuation of a firm is necessary for computing values that are meaningful and useful for decision making, may it be the acquisition or sale of a firm, the acceptance or rejection of an investment project or the adjustment of the leverage of a firm or investment project.

We have stated that the translation formulas in contemporary textbooks are based on strict assumptions. If these assumptions are not met, the valuation will be incorrect or inconsistent if these formulas are applied anyway. More particularly, these formulas are invalid in cases where both debt financing is known in advance and the stream of cash flows has a limited lifetime. This applies also to cases where the stream of cash flows is divided into an explicit planning period and a subsequent perpetual continuation period, which is the most common approach in practice (Mukhlynina and Nyborg, 2016).

Contrary to the conventional formulas, we observe that the translation formula between the unlevered return and levered return on equity depends on an additional parameter that reflects the loan payment structure. This parameter is different for constant-amortization, constant-payment, bullet, and other types of loans. In other words, the required return on equity and the discount rate in the FCF method do not only depend on the required return on debt, the tax rate and the leverage (for example expressed by the debt-to-firm-value ratio) in some given point of time, but also on how debt will be paid back.

The formula that we have developed in this paper contains the well-known translation formula by Modigliani and Miller (1963; see formula (3) above) as a special case. More particularly, if the lifetime of a stationary cash flow approaches infinity, and if constant leverage is enforced, our formula (46) will coincide with Modigliani and Miller's formula.

Both the case outlined in our paper and the models presented in previous research have specific assumptions that allow the derivation of specific translation formulas. Hence, these models are not generally valid. Therefore, our discussion does not end the long-lasting debate and research on DCF-methods. However, it is a departing point for the development of a consistent valuation theory for more advanced cash flows with a finite lifetime or when cash flows are divided into different planning periods. In our opinion, the most urgent improvements concern stochastic debt payments and personal taxation. Since formula (46) does not deal with these issues, we have explained the general procedure to depart from when deriving more advanced translation formulas.

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