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The opaque nature of generic examples: The structure of student teachers' arguments in multiplicative reasoning

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ABSTRACT

The study aims to explore the structural aspects of generic examples, to get better insight into what makes them potentially opaque for learners. We have analyzed 27 written arguments, for which student teachers (grades 1–10) were asked to use a generic example to prove a given statement in multiplication. Using Toulmin's framework, we developed five categories of arguments based on their structure: examples, empirical arguments, leap arguments, embedded arguments, and other arguments. Also, we conclude that none of the student teachers provided arguments that we recognize as complete generic examples. The results bring us to a discussion about features of generic examples making them difficult to come to grips with, having implications for how teacher educators can support student teachers' learning to prove. From this, we propose a definition of generic examples that attends to the criteria suggested in previous research, yet, emphasizing their structural nature.

1. Introduction

Reasoning and proving are central aspects of mathematics as a discipline, and many researchers have argued that they should be a central part of school mathematics at all grades and in all topics (e.g., Balacheff, 1988; Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002; Stylianides, 2007, 2008). In exploring ways to teach proof, several studies have shown the crucial role that a teacher plays in helping students identifying the structure of a proof, presenting arguments, and distinguishing between correct and incorrect arguments (see e.g., Stylianides, 2007). However, researchers have found that primary school teachers tend to rely on external authorities, such as textbooks, college instructors or more capable peers, as the basis of their conviction (Lo, Grant, & Flowers, 2018). Primary teachers also seem to believe it is possible to affirm the validity of a mathematical generalization using a few examples (Martin & Harel, 1989). Further, Stylianides, Stylianides, and Philippou (2007) revealed that pre-service teachers had two main types of difficulties with proof: the lack of understanding of the logical mathematical underpinnings of different modes of argumentation, and the inability to use different modes of representations appropriately. Although previous research has identified (student) teachers' challenges when learning to prove in mathematics, there is still a problem finding ways to better support student teachers' learning in terms of communicating clearly what makes an argument a mathematical proof. In this study, we seek to outline challenges that student teachers meet when working with proving in a grade five context. More specifically, we focus on identifying the challenges stemming from the structural nature of proofs, rather than from the student teachers' beliefs or capacities in mathematical reasoning and proving.

We study how student teachers make use of generic examples, which is one way of proving mathematical statements (Stylianides,

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2008). A *generic example* in mathematics is an example illustrating a certain hypothesis or claim, but not used as empiric verification of the claim (i.e., it is true for this example, thus it must generally be true). Quite opposite, the generic example seeks to illuminate the general rather than the particular properties of the example and can therefore be regarded as an argument for the claim's validity, or even as a valid proof (see e.g., Rowland, 1998). Empirical research on generic examples is sparse (Aricha-Metzer & Zaslavsky, 2017) and typically limited to the tertiary level (Dreyfus, Nardi, & Leikin, 2012). Further, previous studies on teachers' work on reasoning and proving show that reasoning by generic example is not necessarily a preferential mode of argumentation. For instance, studies of Kempen (2018) and Kempen and Biehler (2015, 2018) show that pre-service teachers seem to prefer formal proofs to generic examples, both for verification, conviction and explanation. In their research on pre-service teachers' perceptions of generic examples in elementary number theory, the participants typically started with a formal proof when asked to provide an argument. After following a course focusing on proof and proving in mathematics, these same participants viewed generic examples as more valid, but still inferior to formal proof (Kempen & Biehler, 2018). The researchers name possible reasons for this: that the pre-service teachers were not exposed to generic examples during their own schooling while formal proofs were often in focus, that the course during teacher education may have given an impression that formal proofs are shorter (due to use of algebraic notation) and thus more effective, and that the tasks given could be easier for a student on their level to prove by a formal proof than by a generic example. Further, problematizing lacking criteria for what constitutes a generic example, the study of Yopp and Ely (2016) illustrates how prospective elementary teachers in their study used examples in their arguments only to affirm a claim in two-digit subtraction; thus, not appealing to generality.

With the above challenges of reasoning and proving in mind, we study features of generic examples that make them potentially opaque for learners. In other words, we seek better insight into the structural aspects of arguments using generic examples, by stating the following research question:

- What characterizes the structure of student teachers' arguments when asked to give a generic example as a proof for a hypothesis in multiplication?

To answer the research question, we have analyzed 27 student teachers' written arguments, for which they were particularly asked to use a generic example to prove a given statement within multiplicative reasoning. The mathematical content of the statement was taken from a fictional grade five setting, and the arguments provided should be suitable for this audience. To study the structure of the student teachers' arguments (rather than to evaluate the mathematical and logical correctness of their arguments), we have made use of Toulmin's (1958; 2003) model for argumentation. The analysis leads to a discussion on features of generic examples that make them difficult to come to grips with, having implications for how we as teacher educators can support the student teachers' learning to teach reasoning and proving in mathematics. In what follows, we present in more detail Toulmin's framework and our adaptation of it for analyzing student teachers' written arguments. However, we first elaborate on our understanding of generic examples and its basis in previous literature.

2. Theoretical considerations on generic example

As described in Mason and Pimm (1984), "The generic proof, although given in terms of a particular number, nowhere relies on any specific properties of that number" (p. 284). In previous studies (e.g., Balacheff, 1988; Stylianides, 2008), proofs by generic examples are thought of as especially available to mathematics learners, since they take as their basis an accessible example and require no knowledge of certain common aspects of formal mathematical proofs. Accordingly, Hanna (2000) argues that explanatory proofs, a label that usually fits well with generic examples, have great advantages in school. Further, Rowland (1998) claims that generic examples have power to convince, as well as explain, in education at all levels. In a recent study by Aricha-Metzer and Zaslavsky (2017), it was found that over half of the students involved (ranging from middle school students to undergraduates) used examples productively for proving. In addition, Russell, Schifter, and Bastable (2011) show how a similar concept to generic examples (called representation-based proofs) is a tool even for young students to argue for generality, and like Rowland (2002a), they encourage educators to explicitly teach generic examples as an argumentation method. Yet, challenges with generic examples have also been documented. Reid and Vallejo Vargas (2018) point out that generic examples can be viewed as both generic and empiric, depending on the reader's reasoning or what is accepted within a certain community. Hence, there are both psychological and social factors needed to be taken into consideration when communicating the nature of generic examples.

The body of research on generic examples in proving includes several theoretical papers, suggesting frameworks for assessing certain aspects (like validity or choice of examples) of students' arguments. However, the underlying assumptions on what a generic example should be varies in the literature (see e.g., Zaslavsky, 2018; Reid & Vallejo Vargas, 2018; Yopp, Ely, & Johnson-Leung, 2015), and the inconsistencies concern both mathematical and pedagogical aspects. The pedagogical motives for using generic examples in school include providing a tool for argumentation to students who are not yet able to express mathematics in general terms (Harel & Sowder, 1996; Russell et al., 2011), and supplying a means to bridge the gap between informal arguments and formal demonstrations (Dreyfus et al., 2012; Leron & Zaslavsky, 2013). In this paper, we follow Rowland's (1998, 2002a, 2002b) philosophy, that generic examples are useful in their own right, and at all levels, both to explain mathematical claims and to prove them. It follows from this stand that generic examples are valid mathematical proofs (if it fulfils criteria that are discussed below), contrary to the view of e.g., Leron and Zaslavsky (2013).

Concerning the debates on pedagogical motives, it is not surprising that there is a lack of consensus on mathematical aspects, that is, on what the criteria for a sufficient generic example should be (Zaslavsky, 2018). To ask how "deductive" or "rigorous" the argument should be is a difficult matter, as this is not even agreed upon within the mathematics research community (see Weber, 2008). Yopp et al. (2015) argue that the border between generic examples and abstract proofs is impossible to draw. Instead of going into this debate, we consider criteria suggested by various theoretical frameworks, before outlining the definition of a generic example used in this study.

Several researchers have suggested frameworks to aid analyzing or generating generic examples. The frameworks vary in target audience, focus, and scope. For example, Rowland (Rowland, 2002a) provides guidelines for teachers to apply generic examples in the classroom, within the field of number theory. Buchbinder and Zaslavsky (2009), as well as Yopp and Ely (2016), propose frameworks for assessing the use of examples in argumentation. The former study includes example use for proving as well as disproving (counter-examples) mathematical statements. The status of various types of examples (e.g., confirming or irrelevant) are listed, given the task at hand (e.g., disprove a universal statement). Although Buchbinder and Zaslavsky's (2009) framework has been successfully used to analyze students' arguments (Buchbinder, 2018), we find it impractical in our case, as we focus on the student teachers' argument as a whole, and not particularly on their choice of examples. The latter framework of Yopp and Ely (2016) holds a different character, as the authors "trace" the example throughout the argument and at each step assess whether that part of the argument appeals to a special or a generic feature of the example. Moreover, the definitions used in the hypothesis that is to be proved are considered. The framework is meant for teachers or teacher-researchers in their work of assessing students' arguments. Nevertheless, Yopp and Ely (2016) do not take a holistic perspective on the arguments given. Since we aim to identify characteristics of student teachers' generic arguments, we need to go deeper into the structure of their arguments, beyond tracing the examples used.

Another framework, having the potential to provide insight into the overall nature of generic examples, is suggested by Reid and Vallejo Vargas (2018). They consider written arguments, which is of special interest to us. Two aspects should be included in a written generic example: (1) Evidence of awareness of generality; and (2) Mathematical evidence of reasoning. The first kind of evidence shows that the author is aware that he or she is working with a general argument, valid in all cases involved in the hypothesis. This includes (but is not restricted to) explanations in written natural language, as described by Kempen and Biehler (Kempen & Biehler, 2015). The second "mainly points to the mathematical reasons for why the same structure can be extrapolated for other cases from the example(s) given, and it is based not only on the conditions of the problem given but also on the ground knowledge the community shares at that point" (Reid & Vallejo Vargas, 2018, p. 247). To define the term generic example, we then take as our basis Reid and Vallejo Vargas' (2018) two criteria, which we expand as follows:

(1) Evidence of awareness of generality – the argument concludes with the general claim that was to be proved;
(2) Mathematical evidence of reasoning – the argument contains both a mathematical reasoning concerning the example (establishing the claim using the example), as well as an explicit lifting of this reasoning to the general case.

To illustrate the functionality of the two criteria, we introduce the hypothesis that forms the basis of the task given to the student teachers in our study. The hypothesis, known as "Teddy's hypothesis" (adapted from Skott, Jess, & Hansen, 2008, pp. 223–224), states that whenever you multiply two numbers ending with 5, the result will also end with five. This can be proved by using a generic example and an array model of multiplication, as follows. Given any two numbers both ending with 5, say 125 and 35, one can use the array model for multiplication to represent the multiplication 125×35, as shown in Fig. 1.

Reasoning mathematically on the example given, we can state that every cell in the presented array except the cell with 5×5, is a multiple of 10. Thus, these cells do not contribute to the ones in the product. Only the cell with 5×5 does, and since 5×5 equals 25, we find that the product ends with 5. We can then lift the reasoning to a general case, as there is nothing special about the two numbers (125 and 35) in the example. The product of any two numbers both ending with 5 will have the same structure – the two numbers can be split into tens and a five, with only the number of tens varying, and hence the only cell contributing to the ones in the product is the cell containing $5 \times 5 = 25$. Thus, we conclude the argument with the general claim to be proved: the number resulting from multiplying to numbers ending with 5, will end with 5.

In this argument, the evidence of awareness of generality is found in the last two sentences, in the term "any two numbers" and in the final general claim. The mathematical evidence of reasoning is visible in the explanation of how the structure of the product shown in the example can be generalized. Note that since a proof will depend on the community, this example can be considered by some to include too much information, while others – say, young students not familiar with the array model for multiplication – will not see it as a proof at all. Nevertheless, it is our claim that the argument is suitable for a five-grader, which was a requirement given to the student teachers in the study.

	100	20	5
30	100×30	20×30	5×30
5	100×5	20×5	5×5

Fig. 1. The array model for multiplication used in a proof of Teddy's hypothesis.

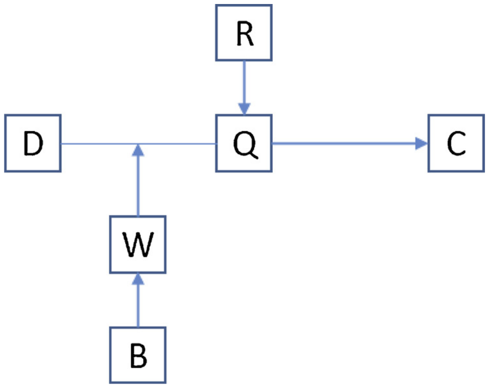


Fig. 2. Toulmin's model of argumentation (Toulmin, 2003).

3. Toulmin’s analytic framework

Analyzing the student teachers' generic arguments, we make use of Toulmin’s (2003) functional model of argumentation. The model is one of the first modern accounts of argumentation theory (Aberdein, 2006a, Aberdein, 2006b; van Eemeren, Grootendoorst, & Kruiger, 1987), and it has gained currency in mathematics education research as a way of analyzing arguments (Simpson, 2015). Rejecting a mathematical logical model for argumentation, Toulmin (1958, 2003) investigates the functional structure of arguments across different fields such as law or medicine. A central theme throughout his studies is the way in which assertion and opinions concerning all sorts of topics, put forward in everyday life or in academic disciplines, can be justified (van Eemeren et al., 1987). Toulmin (1958, 2003) thus assumes that there is a layout of arguments that is *field-invariant*, meaning that components and their functions are similar across disciplines. He begins with the thought that an argument is a claim (*C*) derived from data (*D*) in accordance with a warrant (*W*). More precisely, the claim component is the conclusion of the argument, the data are facts on which the claim is based, and the warrant is the justification for drawing the conclusion on the basis of the data. Warrants thus “act as bridges, and authorize the sort of step to which our particular argument commit us” (Toulmin, 2003, p. 91). The backing (*B*), are “further reasons to believe the warrant” (Toulmin, 1958, p. 101).

For mathematical proofs, then, the warrant is backed by various axioms, rules of inference and mathematical techniques, as illustrated by Toulmin, Rieke, and Janik (1979) example of Theaetetus’ proof that there are exactly five polyhedral (conclusion): The data and warrant consist of various facts about the platonic solids, and the warrant is backed by the axioms, postulates and definition of the three-dimensional Euclidean geometry. Further, arguments may have a modal qualifier (*Q*), such as ‘necessarily’ or ‘presumably’, which explicates the force of the warrant (geometrical necessity in the given example of Theaetetus’ proof). If the warrant does not provide necessity, its conditions of exception or rebuttal (*R*) may be noted (for Theaetetus’ proof, there is no rebuttal or exception within the bounds of Euclidean geometry). The overall layout of an argument is often set out graphically, as in Fig. 2. In our study, we follow the tradition of Krummheuer (1995) and others (see e.g., Conner et al., 2008; Conner, 2012; Knipping, 2008; Knipping & Reid, 2015; Nordin & Boistrup, 2018) and focus on “the ‘core’ of an argument” (Krummheuer, 1995), being data and claims, together with their warrants and backings.

While assuming a field-invariant layout of arguments, Toulmin (2003) also proposes his model to display *field-dependency* in arguments, by allowing characterization of arguments being related to their context of use. In our study, field-dependent aspects of the student teachers’ arguments are generally related to mathematical reasoning and the deductive nature of proving in mathematics, and more particularly, to the use of generic examples in multiplicative reasoning in a grade five community. Accordingly, we use Tolumin’s model to elicit the structure of student teachers’ attempts to prove by generic examples, including identifying the warrants and backings explicitly provided in their arguments. Thus, we can outline the “*field* of justifications in which the student teachers operate” (Knipping & Reid, 2015, p. 81), consisting of the statements of multiplication that the student teachers’ take as accepted or available for five-graders.

Although some have criticized Toulmin’s model for having difficulties in precisely defining some of its components (see e.g., van Eemeren et al., 1987), the extent to which it has been used in analyzing arguments suggests it to be a useful tool in research (Simpson, 2015). In the field of mathematics education research, the model was first used by Krummheuer (1995), in his analysis of students’ utterances when taking part in collective argumentation in a primary mathematics classroom. In the wake of Krummheuer’s study, several other researchers have used Toulmin’s model for analyzing collective argumentation in mathematics classrooms (e.g., Conner et al., 2008; Conner, 2012; Knipping, 2008; Knipping & Reid, 2015; Nordin & Boistrup, 2018; Pedemonte & Reid, 2011; Van Ness & Maher, 2019), as well as primary and lower secondary students’ written explanations in mathematics (e.g., (Evens & Houssart, 2004; Hoyles & Küchemann, 2002; Pehkonen, 2000). Further, researchers have used Toulmin’ scheme in studies of mathematics teachers’ arguments, such as in Nardi, Biza, and Zachariades’ (2012) analysis of warrants and backings in secondary mathematics teachers’ written responses and interview utterances. Their study addressed the “range of influences (epistemological, pedagogical, curricular, professional and personal) on the arguments teachers put forward in their scripts and interviews” (Nardi et al., 2012, p. 160). Similarly, Mexatas, Potari, and Zachariades (2016) examined the change of pedagogical argumentation of a teacher participating in a

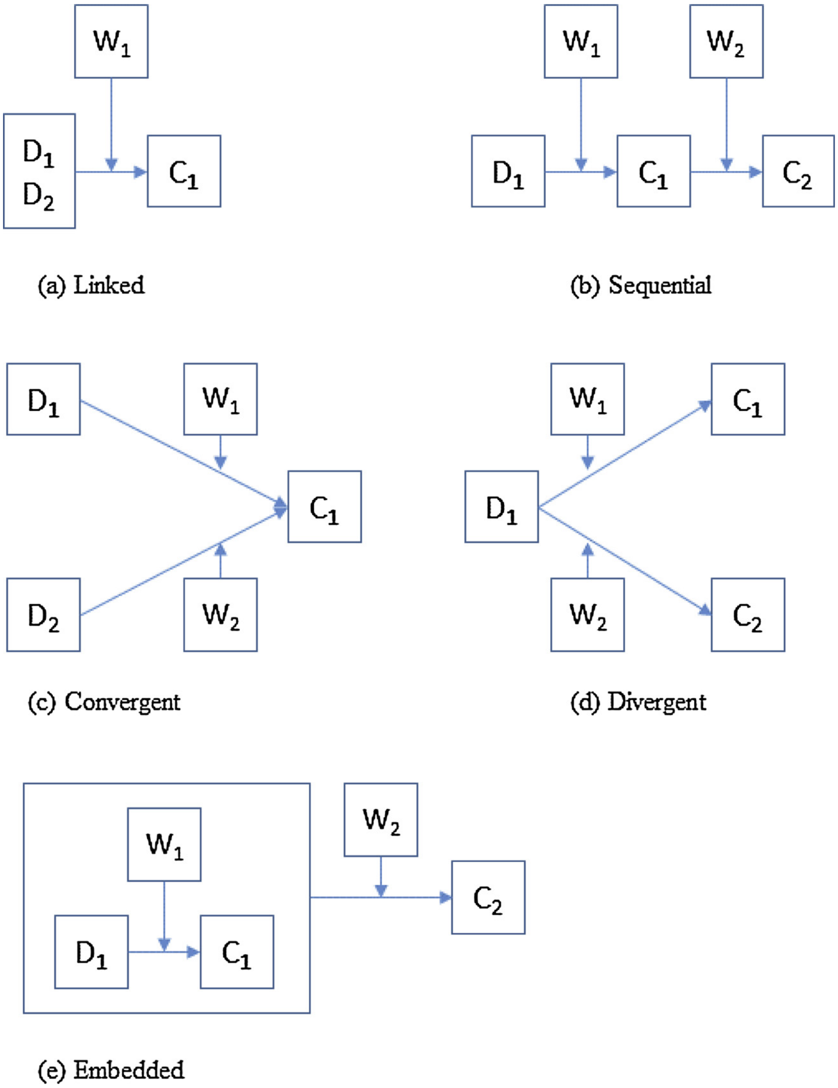


Fig. 3. Five different layouts of arguments.

graduate course. Moreover, Erkek and Bostan (2019) built on the work of Knipping (2008) in their study of global argumentation structures of prospective middle school mathematics teachers in a technology enhanced environment.

Among the aforementioned studies, several adapt Toulmin's model to fit their intentions and their data. In addition to the tendency of omitting the qualifier (*Q*) and rebuttal (*R*) in their frameworks, several researchers consider arguments to be part of complex schemes and therefore make use of a chained model where the conclusion of one step becomes the data of the next (e.g., Conner et al., 2008; Conner, 2012; Erkek & Bostan, 2019; Hoyles & Küchemann, 2002; Knipping, 2008; Knipping & Reid, 2015; Pehkonen, 2000; Simpson, 2015). Following Aberdein (2006b), most mathematical arguments have many steps, and "the number of steps a proof possesses is a function of the detail with which it is presented" (p. 212). Accordingly, we assume that the student teachers' attempts of giving generic examples may show different types of combining layouts, as exemplified by Aberdein (2006b) and illustrated in Fig. 3.

In Aberdein's (2006b) classification, the focus is on types of arguments usually found within the mathematics community, such as proof by induction or proof by contradiction. Nevertheless, we assume that the layouts will have a descriptive power for describing any mathematical argument. For the case of generic examples, the embedded structure can be considered intrinsic to the argument characteristics, in terms of our interpretation of Reid and Vallejo Vargas' (2018) criteria. Using the notation of the embedded argument (*e*) shown in Figure 3, a possible argument layout for a generic example will be to start with data in form of an example (D_1), and then, by an appropriate warrant/backing (W_1), to validate the claim for this example (C_1). This constitutes the first "box" in the embedded structure. Then, the entire argument performed on the example is lifted to the general case, thus leading to the general claim (C_2). The lifted warrant/backing (W_2) is typically a re-use of W_1 or an explanation for why W_1 can be lifted to the general case. Fig. 4 shows the Toulmin model for the generic example of Teddy's hypothesis provided previously. Fig. 5 shows another argument

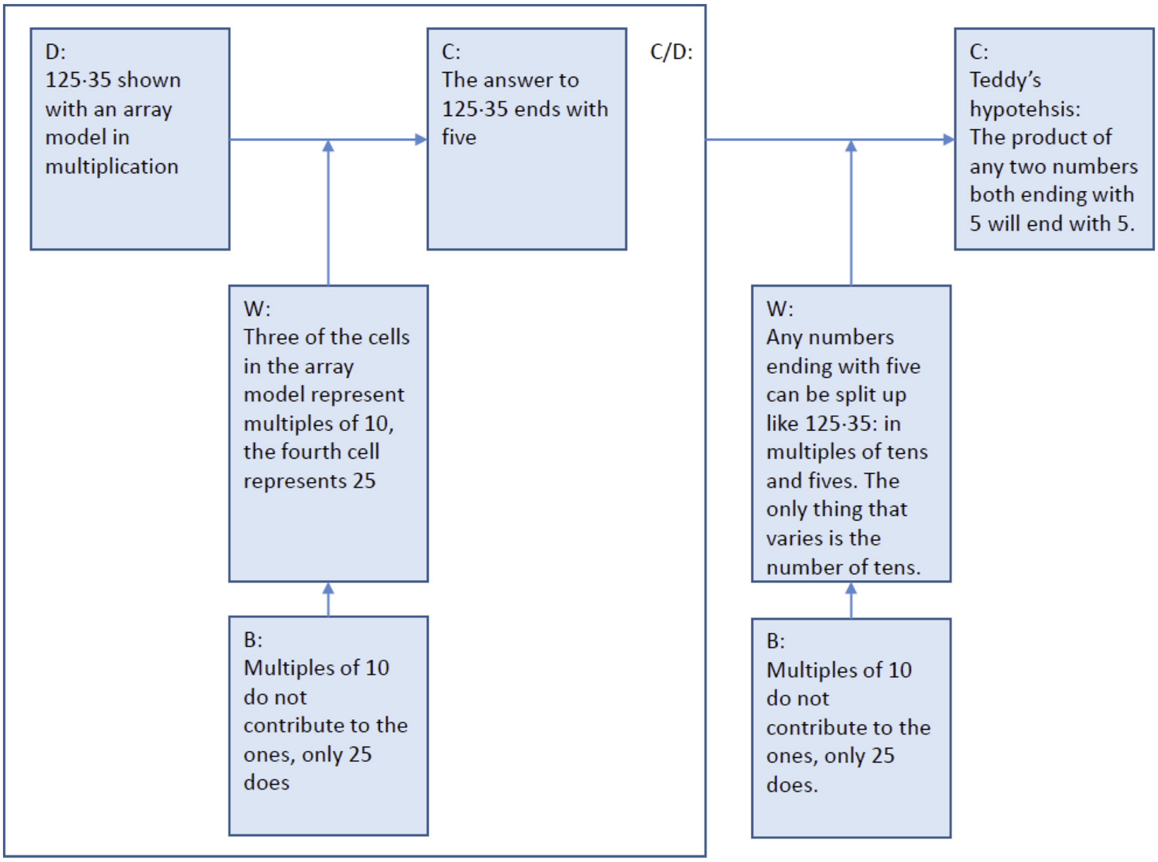


Fig. 4. The embedded structure of a generic example for proving Teddy's hypothesis, alternative 1.

for the same hypothesis, building on other mathematical properties of the products and using another model for multiplication. Both models (array model and equal groups) are well-known to the student teachers in this study.

4. Methods

The study is based on written assignments of 27 student teachers undergoing their first semester of a two-year Master's specialization in mathematics education at one Norwegian university. Of the 27 student teachers, 21 were pre-service teachers taking the Master's specialization in continuation of their undergraduate teacher education program for the grades 1–7 (abbreviated to GLU 1–7) or the grades 5–10 (GLU 5–10), while the remaining 6 student teachers were in-service teachers or professionals taking the Master's specialization as continuing education. All the participants hold at least 60 ECTS credits from the integrated courses in mathematics and mathematics education from the programs GLU 1–7 or GLU 5–10 or similar. However, due to the Norwegian educational context, these credits do not refer to a university degree in mathematics.

The course (15 ECTS credits) in which this investigation took place was organized as five, two-days seminars distributed over one academic semester, in addition to the teachers' individual work on literature and assignments. The topics for the seminars were algebraic thinking (algebra as generalized arithmetic, and algebra as generalization of patterns), group theory and structural aspects of algebra, proof in school mathematics, and learning theories, both from acquisition and participation perspectives. The course was taught by two mathematics teacher educators, one of them being the first author of this paper. Functioning as a homework preparation to the topic "proving in school mathematics" on the second seminar of the course, the student teachers were told to answer what can be denoted a *pedagogy-related mathematics task* (Stylianides & Stylianides, 2014), having both a mathematical focus (multiplicative reasoning), and a substantial pedagogical context (a teaching scenario in grade five). The students were told to give both a generic example and a demonstration for proving Teddy's hypothesis, considering that their proofs should be adapted to grade five students. The complete task is presented in Fig. 6 (translated from Norwegian). The task referred to Styliandes' (2008) analytic framework on reasoning-and-proving and types of proofs and non-proof arguments, and to the work of Stylianides and Ball (2008) and the definition in their article on proof in mathematics classrooms.

In Stylianides' (2008) account of types of proofs, one example of a demonstration is provided, yet, none illustrations of generic examples are given. However, several of the student teachers in this study had experience with generic examples from previous courses in teacher education, especially in the context of representation-based proofs (Schifter, 2009). In addition, the student

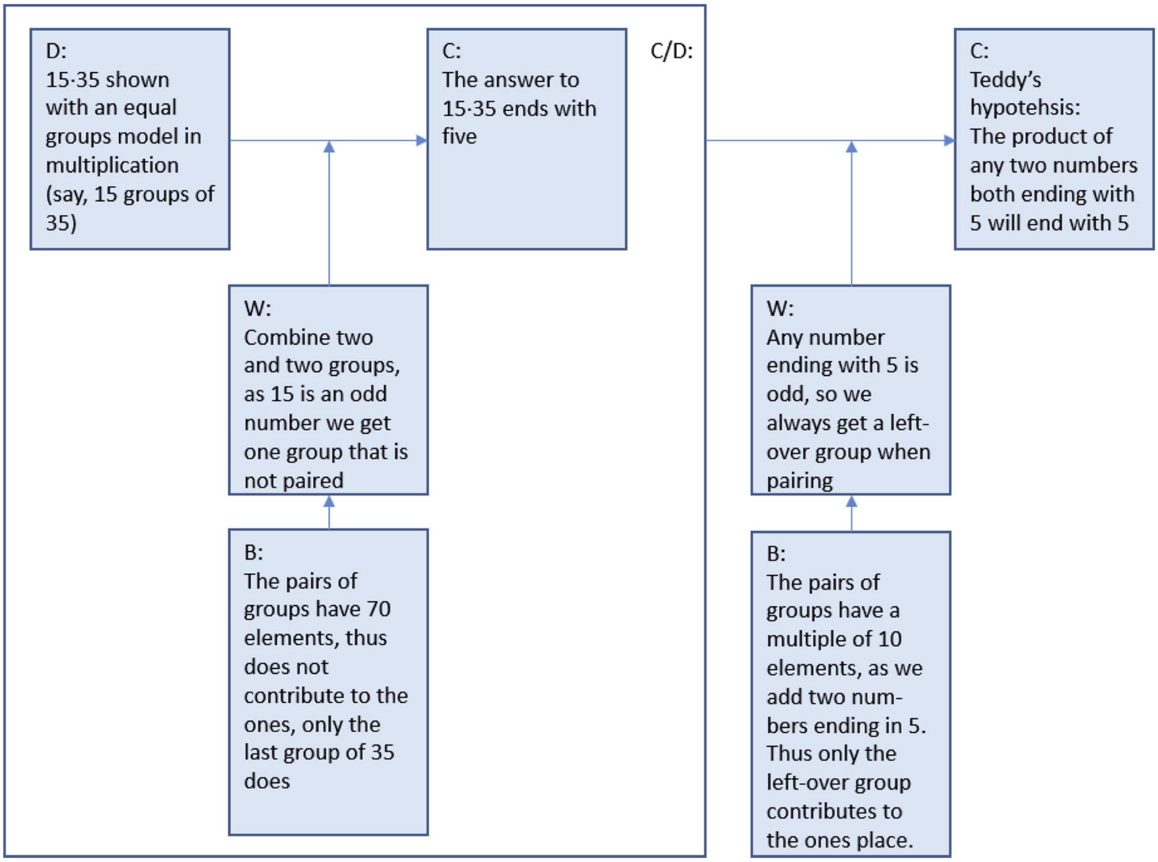


Fig. 5. The embedded structure of a generic example for proving Teddy's hypothesis, alternative 2.

teachers had in the previous seminar worked on the article of Lannin (2005), where generic examples are given in the context of figural numbers and several examples from students' justifications are provided. We thus assumed that the participants in our study had some experience from working on generic examples in advance of being given the homework preparation. Note that in the task, both the terms "generic example" and "generic argument" were used. This was not intentionally done by the teacher educators, and the analysis shows no indications that the terms were interpreted by the student teachers as bearing two different meanings.

Our analysis is based on the student teachers' written responses to task a. The analysis was performed in several steps. First, the authors of this paper collaborated on analyzing five out of the 27 written assignments, using Toulmin's model for argumentation to make visible the structures of the student teachers' generic examples. The analysis of an argument using Toulmin's model does not result in a unique structure, and a statement's status within an argument as data, conclusion, warrant is not always well defined. Based on the collaborative analysis of five written assignments, we therefore continued the analytic work individually, and then met to compare, contrast and adjust our interpretations of the remaining 22 written arguments.

We reconstructed the student teachers' argument using Toulmin's scheme and the variety of layouts shown in Fig. 3. Hence, we identified whether a given argument made use of one or several data, whether there was one or several claims stated in argument and whether Teddy's hypothesis was explicitly stated as a claim, and whether there were warrants or backings provided. From this, we could identify linked, sequential, convergent, divergent, and embedded layouts of the student teacher arguments. Researchers have discussed implicit aspects in arguments (e.g., (Freeman, 2005); Conner, Singletary, Smith, Wagner, & Francisco, 2014); however, we have only displayed components that were explicitly stated in the written arguments. That is, possible implicit warrants remained implicit in our reconstructions.

Based on this structural analysis, we further compared the 27 Toulmin schemes, to categorize those arguments holding similar structural properties. This was done using the following questions:

1 Does the argument contain a claim for the given example (e.g., "The answer to 15·25 is 375, which ends with five")?
2 Does the argument contain a general claim (i.e., is Teddy's hypothesis explicitly mentioned)?
3 Are there any warrants or backings supporting the claims stated from the example (if any)?
4 Are there any warrants or backings supporting the general claim(s) (if any)?
5 Are the warrants or backings supporting the claim from the example (if any) re-stated in some more general form to support the general claim?

Task 4:
In Stylianides and Ball (2008), a mathematical proof is defined as a logical argument building on accepted statements and leading towards a conclusion on whether a general hypothesis is true or not. By accepted statements, we mean here definitions and relations that are assumed to be known or previously proved within a given community (a classroom, among mathematicians, and the like) and that do not need any further justification. It therefore depends on the community what one builds on in a proof. The language and forms of expression will also be different depending on the community. In a community of mathematicians, a proof will often make use of algebraic symbols, while a proof in an elementary school classroom often will be expressed through natural language and drawings, or real-life contexts. The starting point of an argument in mathematics is different definitions, axioms, previously established results and mathematical relations that are accepted by the class and that do not need further justification. Both the starting point, ways of reasoning and ways of expressing reasoning depend on the students' previous knowledge and experience and will be different at different grades and classes. An argument must not only be mathematically valid, but also adapted to the student group.

Teddy is a grade 5 student. He and his classmates are working on square and cubic numbers. After completing some tasks, Teddy says to the teacher: "Look here, if you multiply … take two numbers and multiply… and both numbers end with 5… then the result also will end with 5".

Stylianides (2008) outlines four types of argumentation, where two of them are mathematically invalid (non-proof): empirical argument and rationale; two of them are mathematically valid (proof): generic example and demonstration.

a. Prove Teddy's hypothesis by using a generic argument.
b. Prove Teddy's hypothesis by using a demonstration.

Note: The proofs in a. and b. must be adapted to grade five students and the standard algorithm in multiplication shall not be taken as an accepted statement. A hint: Try to make sense of/represent multiplication using the model of equal groups or an area.

How would you prove the hypothesis yourself, if you did not have to adapt the proof to a grade five student?

Fig. 6. The task given to the student teachers, on proving Teddy's hypothesis.

6 Are there traces of any embedded structure in the argument, that is, is the argument related to the given example lifted to a general form, concluding with Teddy's hypothesis?

From answering these questions for each of the 27 responses, and by comparing and contrasting the layouts of the arguments, we developed five categories of the student teachers' written arguments, named by their characteristics: *examples*, *empirical arguments*, *leap arguments*, *embedded arguments* and *other arguments*. They are further explained in the following section.

5. Results

Based on the analysis of the 27 arguments, we conclude that none of the 27 written arguments completely satisfy our definition of generic examples. Yet, some are close, and we are aware that the details missing could have been provided by the student teachers had their arguments been presented orally and challenged by a discussion (Stylianides, 2019). We are also aware that for all the five categories, the associated arguments provide to some degree a structure of the example(s) provided, that is, a structure of the numbers due to the positional system, as well as an interpretation of multiplication by either referring to the array model or the equal groups model. This is probably due to the hint given in the task (see Fig. 6): "Try to make sense of/represent multiplication using the model of equal groups or an area". In what follows, we present the categories emerging from the analysis. For each category, we show the number of arguments identified, we provide a description of the category's characteristic features and give an example from the student teachers' arguments, both in its written form (translated from Norwegian) and as a Toulmin scheme.

5.1. Examples ($n = 5$)

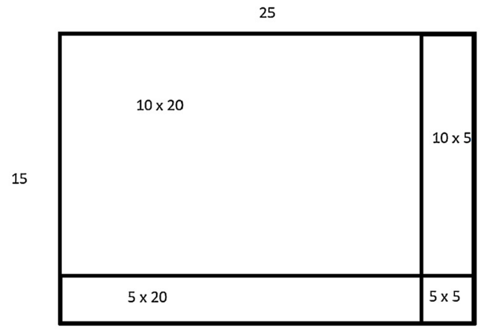
The arguments in this category do not include a general claim, that is, Teddy's hypothesis is not explicitly mentioned. Hence, the argument contains one or more examples; however, no general statements are given. The student teachers state an example, the warrants given (if any) are concerned with the example, and then the argument stops. Five arguments were identified within this category, and Fig. 7 and 8 show the argument and the Toulmin scheme for Student 15:

5.2. Empirical arguments ($n = 6$)

This category resembles empirical arguments previously discussed in the literature (e.g., Balacheff, 1988; Stylianides, 2008): the argument states one or more examples followed by a general conclusion (Teddy's hypothesis in this case). Any warrants given are for this category only concerned with the example and are not of a general nature. One case is Student 22′s written argument (Fig. 9 and 10), where there are no traces of warranting the final conclusion. Instead, the conclusion immediately follows from the example.

A generic argument is according to Stylianides (2008) to argue by using a concrete example to prove something. In this task I will thus use a concrete example in which two numbers that both end with 5 are to be multiplied and the product is supposed to end with 5. For five-graders, I would have used the area model to easier make an overview of the two numbers. So that I, as a teacher, should not decide upon two numbers where the product ends with 5, I can let the students choose two numbers ending with 5.

But here I choose the numbers 15 multiplied by 25. These two numbers I consider to be too difficult for 5th graders to calculate mentally. By placing the numbers around a rectangular area, it makes the following area model:



(it is assumed that they have worked on area and its meaning in practice)

By making the four calculations, the students can find the total product:

$10 \cdot 20 = 200$
$5 \cdot 20 = 100$
$10 \cdot 5 = 50$
$5 \cdot 5 = 25$

Sum: 200 + 100 + 50 + 25 = 375, the number thus ends with 5.

Fig. 7. Student 15′s written argument.

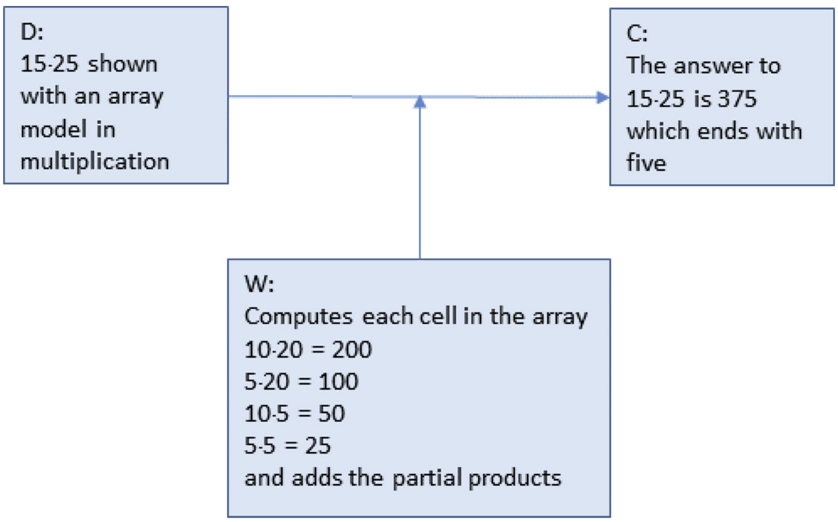
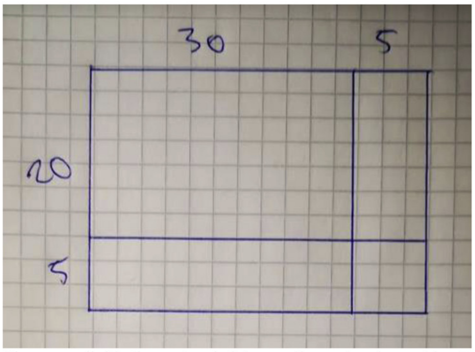


Fig. 8. Student 15′s argument.

5.3. Leap arguments (n = 8)

Our analysis revealed several arguments (eight in total) that state one or more examples, draw a general conclusion, and give a general warrant for this conclusion – none of which would be considered unsuitable elements of a generic example. However, the eight arguments moreover provide no warrants or claims explicitly related to the given example(s). Hence, one can say that the

For Stylianides, a "Generic example" would be that you show the validity of your hypothesis by a random, yet, concrete example. If you start with the area model, you can explain the arithmetic problem 25x35 like this:



Thus, you divide the rectangle into smaller rectangles and has 20+5 as the length of the one side, and 30+5 as the length of the other side. By calculating the four different rectangles that have appeared, you see that you get the answers 600, 150, 100 and 25. If you add this together for the total area, you get 875 and you have shown that by multiplying two numbers ending with five, you will also get a number ending with 5.

Fig. 9. Student 22′s written argument.

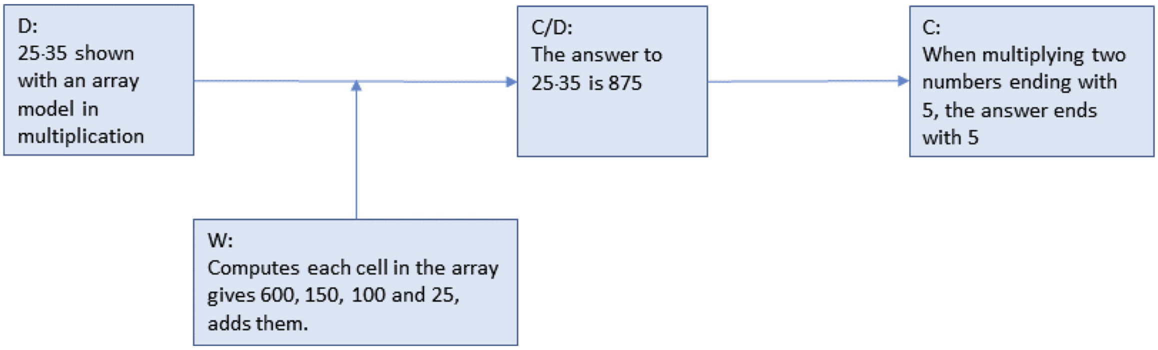


Fig. 10. Student 22′s argument.

arguments in this category do not "use" the example(s). We have named such arguments *leap arguments*, as they make a leap directly from stating the example to treating the general hypothesis, without explicitly linking these two parts of the argument. Fig. 11 and 12 shows the argument of Student 10:

In the given argument, we note that Student 10 provides two examples of a slightly different nature: two different models for multiplication are used, thus two different warrants and backings are needed. The (translated) wording of the first warrant is:

Then we see that when we multiply two odd numbers, one [group] is always left alone.

For the second example, the warrant and backing are written as:

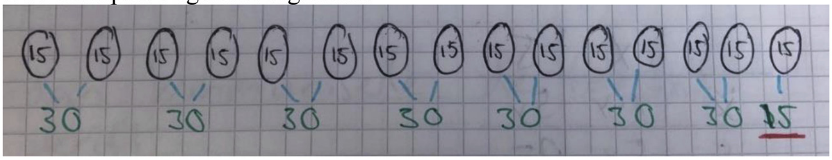
Here you use the principle of distributivity, one splits tens and ones and multiply the various parts. By using a table like this we see that you will always end with 5*5 on the one's place, the other calculations times'es with tens so they won't influence the one's place.

Although the Student 10′s drawing illustrates the examples, the language used in the argument is general, including the word "always", which indicates that the student teacher is not only concerned with the associated example. One can question whether arguments of this form can be valid mathematical arguments, and if considered valid, whether they are generic examples. The first part of student 10′s argument lacks some detail in the warrant (i.e., explaining why the pairs of groups always ends with 0), but the second part includes a warrant and a backing that could be seen as mathematically sufficient depending on the community. Nevertheless, it is our claim that this argument is neither a sufficient formal deduction nor a generic example. An example does not belong in a formal deduction, and in a generic example, the argument should be based on the example(s) given.

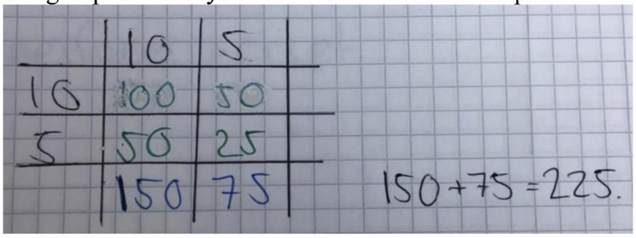
5.4. Embedded arguments ($n = 5$)

In Section 3, we provided two generic examples proving Teddy's hypothesis that held an embedded structure (Aberdein, 2006b). The category named *embedded arguments* thus consists of those arguments that were analysed with an embedded Toulmin scheme. That is, arguments that first consider an example with an explicit conclusion, followed by an "embedding" where the entire argument so far is lifted to a general form, concluding with Teddy's hypothesis and possibly repeating the warrants and backings used within the example in a more general form. We identified five arguments of this type in our data material. As previously mentioned, none of

Two examples of generic argument:



Here, the arithmetic problem 15 × 15 is illustrated by the equal groups model. Then we see that when we multiply two odd numbers, one group is always left alone. When multiplying two numbers ending with 5, one group will always be left alone and hence the product will have a 5 in the one's place.



Here you use the principle of distributivity, one splits tens and ones and multiply the various parts. By using a table like this we see that you will always end with 5*5 on the one's place, the other calculations times'es with tens so they won't influence the one's place. One can therefore see that when multiplying two numbers ending with the digit 5, the product will always end with 5.

Fig. 11. Student 10's written argument.

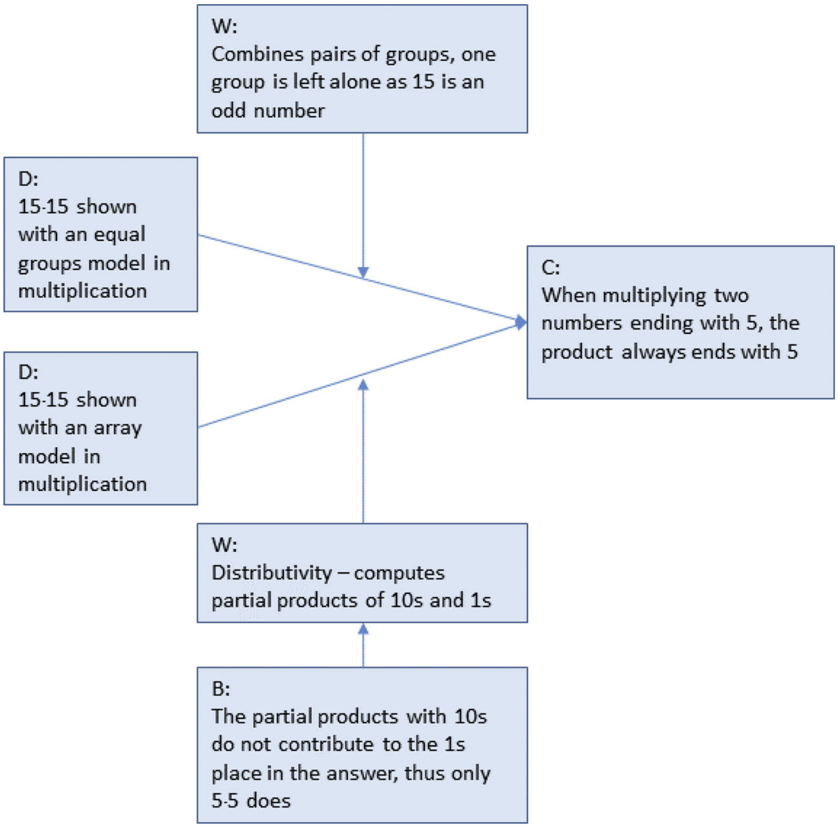
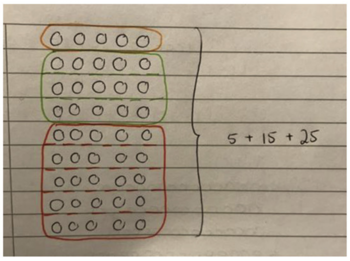


Fig. 12. Student 10's argument.

When you add two or more numbers that all end with 5 you will either get a number ending with 0 or with 5, depending on the number of numbers you add. If you add an even number of times it ends with 0 and if you add an odd number of times it ends with 5. This is because a number ending with 5 can always be distributed as equal groups of 5, as shown in figure 1. The same concept holds when multiplying two numbers ending with 5. If we for example consider the case *5 · 15*, it can be interpreted as 5 added 15 times. Because 15 is an odd number, we will end up with a product ending with 5. This is because two and two five's can be put together into a ten, that is an even number, until we added 14 times. Then we are left with one five that stands alone, and therefore the answer will end with 5. See figure 2. Since all numbers that end with 5 is an odd number, we will always get a product ending with 5 if we multiply two numbers ending with 5, because there is always a group of 5 that is standing alone in the end.



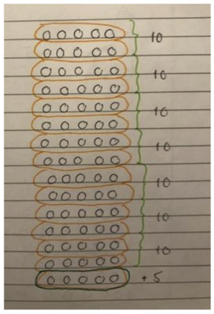


Fig. 13. Student 8′s written argument.

them were found by the authors to be valid proofs of Teddy's hypothesis. In fact, some of the embedded arguments are separated from those categorized as *no general warrant* only because they use terms like "the same argument is true for the general case" (i.e., explicit reference to embedding) instead of simply "thus, the general case is true" (no traces of embedding). Such arguments have no explicitly stated general warrants, or they have only partly lifted the argument to the general case. Other arguments in the category include generality, as well as an explicit reference to the embedding, exemplified by Student 8′s argument (Fig. 13 and 14):

In this argument, Student 8 first gives a careful description of why the product in the example ends with 5, proceeding with the general case, where, essentially, the same warrant/backing is repeated and leading to the statement of Teddy's hypothesis. Yet, Student 8 seems to rely on the fact that one of the factors in the chosen example is 5. Thus, the backing for the general conclusion that you always get a single group of 5 is not true in general, meaning that this argument is not a valid proof.

If the data material had included arguments of similar forms as those provided by the authors in Fig. 3 or 4, they would have been classified into the category of embedded arguments, thus demonstrating that this category can include generic examples that prove

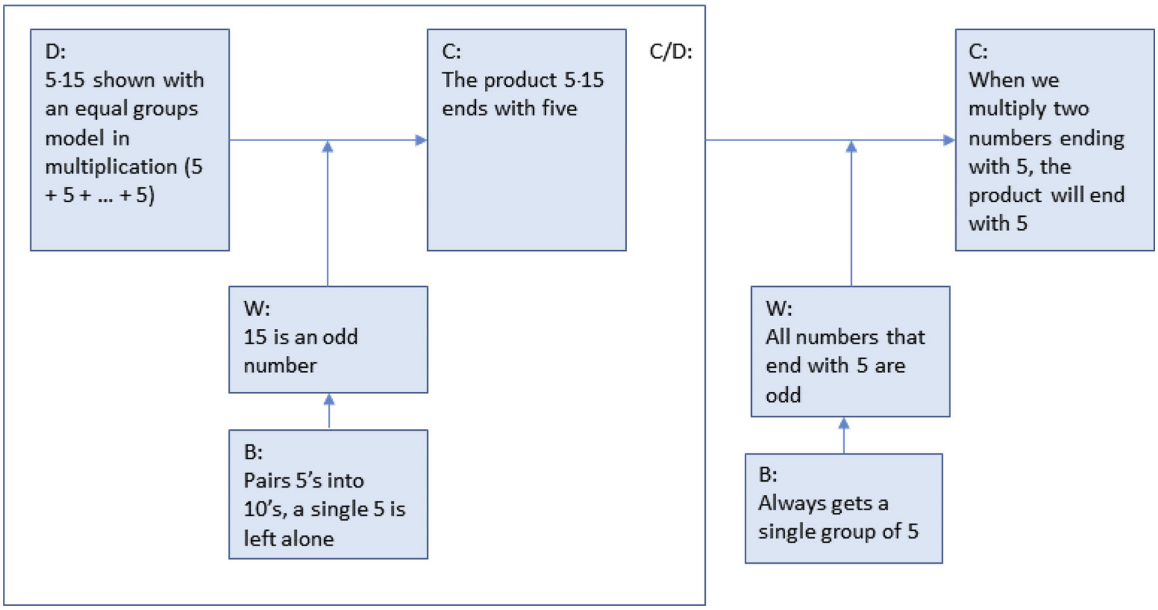


Fig. 14. Student 8′s argument.

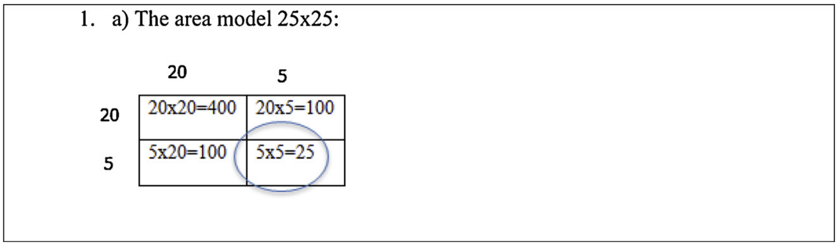


Fig. 15. Example of other arguments.

the given hypothesis. In the discussion, we argue that an embedded structure is a necessary (but not sufficient, as shown above) condition for an argument to be a generic example.

5.5. Other arguments ($n = 2$)

Two students provided arguments with a picture/example only, without any explanatory text. These arguments cannot be analyzed with the Toulmin model and are therefore categorized as other arguments in our study. A typical argument in this category is shown in Fig. 15.

6. Discussion

It is remarkable that none of the student teachers provided arguments that we recognized as complete generic examples. However, in this study, we have claimed that the problem is not necessarily the student teachers' lack of knowledge about proof in general (although many of our findings confirm earlier findings on this matter), but instead the opaque nature of generic examples. In the following, we discuss our findings considering previous research and discussing how structural aspects of generic examples can be emphasized to learners.

We have through the five presented categories identified the characteristics of student teachers' attempts to construct generic examples. In line with previous research, the results show evidence of student teachers' challenges of making the transition from empirical to generic arguments (e.g., Martin & Harel, 1989; Stylianides, Stylianides, & Philippou, 2007). This is apparent from the number of arguments being categorized as either *examples* or *empirical arguments,* as the associated arguments provide warrants only concerned with the example given and not having a general nature. However, the analysis also reveals categories of arguments with characteristics not previously discussed in the literature. The *leap argument* appears in our study as such a category: there is a tendency in the student teachers' arguments of approaching generic examples via formal demonstrations (Stylianides, 2008), which contrasts struggling with leaving empirical arguments. In other words, some characteristics of arguments labelled leap arguments (they draw a general conclusion and give a general warrant) are overlapping with those of proofs not relying on the representativeness of a particular example. The leap arguments also bear resemblance to Yopp and Ely's (2016) category named "examples not appealed to", in their study of prospective primary school teachers' arguments. Their category contains arguments where the respondents' warrants do not appeal to the examples in such a way that they become generic. Instead, the examples are additional empirical support for the claim. However, we argue that this category is best described as demonstrations further validated by examples, rather than generic examples "gone wrong", since the respondents in Yopp and Ely's study were asked to prove a claim by any preferred method. In our study, the leap arguments constitute the largest category of the student teachers' arguments ($n = 8$). It is worth noting that our respondents were told to provide proofs both by generic example and by demonstration, thus leaving them no choice of "proof method", but at the same time giving the student teachers the opportunity to reflect on the differences between the two types of proof. However, the number of leap arguments may relate to Kempen and Biehler's (Kempen & Biehler, 2015, 2018) findings, that student teachers prefer formal proofs to generic examples, both for verification, conviction and explanation.

By means of Toulmin's framework, we identified in our analysis not only what was lacking, but just as much what the student teachers managed to do when attempting to provide generic examples. The number of leap arguments indicates that several of the student teachers in our study were successful in some important parts of mathematical argumentation. Following Reid and Vallejo Vargas (2018), what makes a written argument a generic example is evidence of the author's awareness of generality, typically phrased as "the same reasoning can be used for other cases". In the student teachers' arguments providing general warrants, there is evidence of awareness of generality, as well as mathematical evidence of reasoning. However, these two criteria do not uncover the cases where the student teachers do not use an initial reasoning on the example as a basis for warranting the general claim. From this, it seems that an embedded structure, as described for generic examples in Chapter 2, is necessary when providing a generic example. At the same time, the embedded structure of generic examples is what possibly makes them opaque for learners, perhaps because structural aspects of generic examples are rarely highlighted in the literature. To meet this deficiency, we suggest that a definition focusing on structure will be helpful. Such a definition could be:

A *generic example* is a mathematical argument that takes as its basis an example of the claim to be proved, continues with a mathematical reasoning on why the claim holds for the example, and winds up with a lifting of this mathematical reasoning to the general claim.

We claim that this definition attends to the two criteria of Reid and Vallejo Vargas (2018). Further, it is implicit in the definition (but should be emphasized in teaching) that the example chosen must be appropriate, and that it should be explicitly explained why the reasoning is not dependent on the chosen example. With the suggested definition as point of departure in teacher education, we further claim that the work on generic examples must not only focus on the difference between empirical and generic argumentation, but also on the difference between generic and demonstrative arguments. Learning and learning to teach this difference should thus be supported by focusing on structural aspects in addition to mathematical aspects (as rigor and validity) of arguments.

Declarations of interest

See a separate Declaration of Interest form.

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