Chapter 12 Mathematics teaching practices at university level

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Keywords: Teaching university mathematics, theories, characteristics of practices, instruction.

INTRODUCTION

In this chapter the presentations at the first two INDRUM conferences (2016 & 2018) are taken to provide a snapshot of current research into university level mathematics teaching. The papers offer an insight into the variety of theories used to frame research in this specialized field and the issues upon which the research focuses. In recent years, theoretical-methodological issues concerning research in mathematics education at the tertiary level have been considered as a main area of interest (Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2016). Studies presented at INDRUM focus on understanding and describing teaching and on the development of teaching practices aligned with a chosen theoretical perspective. They mostly take qualitative approaches to study cases in depth. They do not adopt positivist approaches that seek evidence of teaching (as cause) and students' learning outcomes (as effect), which might be generalized from sample to population or from experimental or quasi-experimental designs.

Systematic inquiry into mathematics teaching disseminated through INDRUM reflects the international character of the network. The studies of teaching reported at INDRUM conferences are framed within a variety of theories that have roots in traditions of empiricism and pragmatism, discourse, socio-cultural philosophies and institutional perspectives. Sixteen full research papers have been presented, eight each in 2016 and 2018. These papers reveal a significant bias, about one third of the papers, towards theory rooted in institutional, didactic transposition developed by Chevallard (1981, 1998), especially the anthropological theory of didactics. The influence of Chevallard's theoretical development is unsurprising given his background in mathematics and focus on didactics. The composition of this chapter reflects the influence of Chevallard and other, mostly francophone, theorists. Another theory developed in the context of mathematics that has gained some ground in higher mathematics education research is Sfard's (2008) adaptation of discourse theory to mathematics education, referred to as "commognition".

Theories with empirical roots in mathematics classrooms, such as the "Teaching Triad" (Jaworski, 1994), the "Knowledge Quartet" (Rowland, Huckstep, & Thwaites, 2005), and "Mathematical Knowledge for Teaching" (Ball, Thames, & Phelps, 2008) also feature. These theories emerge from careful and sustained observation and interpretation of teachers working in school classroom situations. It can be argued that school and university classrooms are different contexts—size of class, maturity of students, expectations of independent learning activity, pace of instruction, motivation to study, and so on. Therefore, it cannot be taken for granted that a theory developed empirically in one context will easily accommodate to another.

Some research is framed within global socio-cultural theories, which have not been developed with direct roots in mathematics education. Examples of these are cultural historical activity theory (Vygotsky, 1962/1986; Leontiev, 1978; Leplat, 1997) and community of practice theory (Wenger, 1998). These well-developed and rather complex theories are grounded in conceptions of "activity" and "practice" respectively.

This chapter includes brief introductory accounts, sufficient to satisfy the needs for discussion of the range of theories included within the INDRUM presentations. Within a single chapter it is impossible to consider such a range of theories with a depth of treatment that could satisfy even a mildly critical reader seeking to develop an understanding the theories from scratch. The discussion is inevitably brief and incomplete. The intention in outlining the theories is primarily to indicate how research into mathematics teaching in higher education is being framed by scholars included within the INDRUM community.

The chapter moves on to consider the theoretical perspectives of the INDRUM papers, accompanied by the substance of these papers. After that comes a section on characteristics of teaching practices which encompasses teacher learning and teacher knowledge, frameworks of teacher knowledge, issues of communicating mathematics, and teacher-researchers' teaching practices. Then follows a section on methodology for instructional/didactic design which encompasses a comparison of didactic engineering and design research, and a discussion of instructional design for inquiry-based learning. The chapter closes with a conclusion.

THEORIES FRAMING RESEARCH INTO UNIVERSITY MATHEMATICS TEACHING

The anthropological theory of the didactic

According to Chevallard (1998), the Anthropological Theory of the Didactic (ATD) allows an understanding of an individual's "practices" through the notion of *praxeology* which can be broken down into two blocks, *praxis* and *logos*. The *praxis* block comprises the "know-how" made of "types of tasks" (things to accomplish) and "techniques" (methods used to carry out tasks of the given type). The *logos* block comprises the discourse that justifies and produces the praxis at two levels: the "technologies" (justifications of techniques) and "theories" (foundations that underlie the technological discourse). Thus, praxeologies are defined by a quadruple: task, technique, technology and theory. The mathematical organization (MO) describes the mathematical activities of the praxeology, and the didactic organization (DO) describes the activities supporting the teaching or learning of the MO (Bosch & Gascón, 2006). MOs and DOs are influenced by the institutional organization of the teaching and the learning processes. Chevallard (2002) takes into account the different aspects of institutional organization in the didactic analysis of teaching and learning and introduces a scale of levels of didactic co-determination. These levels help distinguish between the conditions and constraints related to the specificities of the discipline taught (discipline, domain, sector, theme, question), and those at a general level of the teaching of any discipline (humanity, civilization, society, school, pedagogy). *Professional praxeologies* were developed by Chevallard for the analysis of the practices of individuals in a given profession such as the training of teachers or of future engineers.

An *institution* is defined by Chevallard (1998) as an evolving social system imposing on its subjects specific praxeological equipment for a given field of activity. Broley (2016) describes the status of some practices within institutions based on how the integration of computer programming in students' learning activities was related to teachers' research. Six levels were identified of interaction of mathematicians or students with programming that vary from "strictly observe the results of a computer program" to "develop a program, including algorithm development, coding, and verification" (Broley, 2016, p. 363).

The exploratory study by Florensa, Bosch, Cuadros and Gascón (2018) deals with the kinds of questions lecturers point out at the beginning of educational courses. The study reveals that the questions asked remained at a pedagogical level and were poorly connected to the mathematical knowledge to be taught. Based on the experience from the courses presented in the paper, Florensa et al. (2018) postulate that didactics can help university teachers better interpret their practice and question it in a more productive way. They highlight elements from ATD that have shown useful in this respect: praxeology, didactic contract, didactic moments, Herbartian schema, media-milieu dialectics, didactic ecology.

A way to characterize teaching activities is through identifying the learning activities they trigger. Chevallard (2013) distinguishes between two paradigms: the "visit of works" where students are required to study according to a firmly pre-established programme; and "questioning the world" where students would be given "questions needing answers" instead of answers without questions. The new paradigm of "questioning the world" defines new professional praxeologies, where teachers would not be transmitters of knowledge, rather facilitators of learning (Bourgade, 2016). Investigation workshops, introduced by Chevallard (2011), can help investigate the professional praxeologies required when a teacher works in the frame of "questioning the world". Seeking possible answers to an open question in an investigation workshop, without pre-defining the direction that the investigation would take, leads to the development of "study and research paths" (SRPs) (Chevallard, 2011). An SRP with its dialectics is the design heuristic of ATD, starting from a problematic question, where mathematical modelling is at the heart of the teaching and learning processes.

According to Winsløw, Matheron, and Mercier (2013), an SRP emphasizes the dialectic between "research" and "study" which characterizes any learning activity. "Research" refers to inquiry, problem solving, etc. while "study" designates the consultation of existing knowledge initiated by the teacher. The term "path" designates the possible open trajectories to be followed in an experimentation of the SRP. An SRP starts by raising a "generating question", an open question that provokes the interest of the community of study. Many questions derived from the initial one arise as the study progresses; the answers to these questions trace out the possible paths to be followed in the experimentation of the SRP.

An SRP can also be applied in ways other than as a heuristic device. For instance, Barquero, Serrano, and Ruiz-Munzón (2016) describe the conditions and constraints influencing the integration of SRP in first year courses of a business and administration degree at university level. The authors analyze the dialectic between inquiry and transmission, emphasizing the role of the different didactic devices mobilized in the survival and evolution of the SRPs. The study highlights possible ways of integrating "Study and Research Activities" (SRAs) (Barquero & Bosch, 2015) into SRP, and subsequently links transmissive teaching devices to inquiry-based ones.

Further, Bourgade (2016) aims to answer the question: "To what extent is the elaboration of a professional praxeological equipment dependent on professional difficulties on the one hand, and on the dissemination of this equipment on the other hand?" (p. 351). He initiated and supported the implementation of an investigation workshop at *la prépa des INP*[1] for two years. The paper focuses on the task of "formulating a generating question" from a real-life situation and tries to identify the different didactic moments in the realization of this task (first encounter, exploration, building of the technological-theoretical block). Analyzing the task in terms of didactic moments helps identify professional difficulties and dissemination issues as two catalyzers of the *logos* production in the building of a professional praxeology, which usually does not require elaborate *logos*.

A contribution by Romo, Barquero and Bosch (2016) is an adaptation of the methodology of "Study and Research Path" for teacher education (SRP-TE) developed by Sierra (2006) and Ruiz-Olarría (2015). SRP-TE methodology takes as a starting point of the teacher education process a problematic teaching question and approaches it in many phases: searching for available answers in mathematics didactics literature; experiencing one of the findings; analyzing the experience from a didactics perspective; and finally, when possible, designing, implementing and developing a class activity adapted to a given institutional context. The study considers a course on mathematical modelling implemented in a Master's Programme for in-service secondary mathematics teachers using an online modality. The design of the course is based on four phases associated with the four stages of didactic analysis: 1) creating a teaching proposal based on the epistemological analysis; 2) elaborating a lesson plan; 3) putting the plan into practice with secondary school students; and 4) redesigning the activity after experimentation taking into account the institutional constraints revealed. Results show that when teachers design modelling activities and implement them in their classrooms, they become aware of the impact institutional constraints have on the implementation of the pedagogical and mathematical forms of these activities.

Instrumental and Documentational approach

The *instrumental approach* focuses on studying the activity of one or more subjects, involved in a goal-directed activity, and mediated by *artefacts*. An artefact is defined by Rabardel (1995/2002) as being a product of human activity, whose design is oriented towards a given goal. An artefact associated with a scheme of use of this artefact in a given situation generates an instrument. The process of development of an instrument is called *instrumental genesis* and has two intricate aspects: *instrumentation* (shaping of the activity to accommodate the tool) and *instrumentalisation* (re-shaping of the tool by the user to adapt to the activity's goals).

A scheme is a relation between knowledge and actions. Vergnaud (1998) defines it as an invariant organization of conduct for a given set of situations comprising four interacting components (Gueudet, Buteau, Mesa, & Misfeldt, 2014): the *aim* of the activity (intentionality in the organization of the activity); the *rules of action* (ways of acting to reach a given aim); *operational invariants* (theorems-in-action or concepts-in-action that influence rules of action); and *inferences* (adaptations that can be brought to answer the specificities of a situation). *Instrumental orchestration* (Trouche, 2004) encompasses the intentional and systematic organization of both the mathematical situation and the artefacts, in order to facilitate the students' instrumental geneses.

The *documentational approach* draws on the instrumental approach, taking into account the broader range of available *resources*. It retains the general notion of resource described by Adler (2000, p. 207): "It is possible to think about resource as the verb re-source, to source again or differently". For a given subject, a resource is thus anything that can possibly intervene in his/her activity. Associating resources with a "scheme of use" of these resources generates a "document". Teachers may select, combine, and design their own resources. They use these resources in class, modify them on the spot or afterwards, share them with colleagues, etc. All this constitutes the teacher's *documentation work*.

The instrumental and the documentational approaches can both be used to describe learning (including teacher learning) and teaching processes analytically, and to design teaching or teacher training devices (Gueudet et al., 2014). A poster contribution by Syrdalen (2018) draws on the instrumental orchestration to explore the role of a lecturer in a flipped classroom. Three elements of orchestration, introduced by Drijvers, Doorman, Boon, Reed and Gravemeijer (2010), are planned used to explore the lecturers' role in a flipped classroom: the didactic configuration (arrangement of the artifacts and the teaching setting); the exploitation mode (how the didactic configuration is used); and the didactic performance (spontaneous decisions).

From a different perspective, a poster contribution by Hadjerrouit and Gautestad (2016) uses instrumental orchestration in an empirical study on the consequences of integrating an interactive visualization tool "SimReal+" in mathematics education at university level. The study mobilizes instrumental orchestration as defined in the work of Drijvers et al. (2010) and defines new types of orchestrations that emerge in the digital era: 13 different types of orchestration are identified. Results show that SimReal+ can motivate students and enhance their learning of some mathematical topics such as differentiation and integration.

The documentational approach (Gueudet, Pepin, & Trouche, 2012) is often used to study the interactions between a teacher, or a group of teachers, and resources. Tabchi (2018) employs the documentational approach—as developed in (Gueudet, 2017)—to characterize the practices of university mathematics teacher-researchers, particularly the impact of their research activity on their teaching practices, through the lens of their interactions with resources. This exploratory study draws on interviews with two teacher-researchers, whose domain of research is discrete mathematics. They both insisted on the importance of reasoning and proofs, and therefore select contents that contribute to develop students' ability to write proofs. Further, they considered that, despite software is important in research, it is not essential for teaching purposes. Moreover, Tabchi (2018) highlights the impact of the institutional context (level of education such as Bachelor and Master levels, progressivity in teaching in certain fields, etc.) on teaching practices at university level. Furthermore, in what concerns the relationship between the research activity and the teaching practices of university teacher-researchers, the study shows that the choice to consider the interactions with resources is revealing. But that step should be developed in further studies.

The theory of didactic situations in mathematics

The theory of didactic situations (TDS) offers a systemic framework for studying teaching and learning processes and for carrying out didactic design in mathematics, where the nature of the knowledge at stake is crucial (Brousseau, 1997). The aim of TDS is to investigate the relationship between the design of a teaching situation and the knowledge it generates. This involves studying the conditions for the functioning of the knowledge at stake—that is, how a generic and epistemic subject would be induced to *use* the particular mathematics to make decisions.

A *didactic situation* in mathematics is a specific plan organized so as to cause students to appropriate some piece of mathematical reference knowledge. TDS postulates that every mathematical concept is the solution of at least one specific system of mathematical conditions. This "system" can be interpreted by at least one situation, for example, a game, whose solution (decision, message, argument) is one of the typical manifestations of the concept (Brousseau & Warfield, 2014).

An *adidactic situation* is a situation in which the students take a mathematical task as their own and try to solve it without the teacher's interventions and without trying to figure out the teacher's didactic intention. The *milieu* models the physical and intellectual reality with which the students interact in an adidactic situation. An appropriate milieu acts as a "piece of nature" for the students, that is, something giving objective feedback that tells them whether their responses are adequate or not with respect to the target knowledge. The milieu may be composed of: material or symbolic tools provided (artefacts, informative texts, data, etc.), students' prior knowledge, other students, and arrangement of the classroom and rules for operating in the situation (determinative of who is supposed to interact with whom).

Every didactic process consists of three situations where the teacher's role changes (Brousseau & Warfield, 2014): 1) A *situation of devolution* in which the teacher: a) Engages students in a challenging mathematical situation that can be undertaken without requiring the teacher's help; and b) informs students about the conditions, rules, goal, and the criterion for success. 2) An *adidactic* situation that supports the students in autonomous mathematical activities, both individual and collective, through engaging them in: a) producing "new" statements and discussing their validity; b) making decisions, formulating hypotheses, predicting and judging their consequences, attempting to communicate information, producing and organizing models, arguments and proofs, etc.; and c) evaluating and correcting by themselves the consequences of their choices. Students' autonomy depends on the adidactic potential of the milieu. The teacher might need to *regulate* the situation to make the milieu evolve to ensure progression of the knowledge; this can be done by informational jumps or by making references to the didactic contract. 3) A *situation of institutionalisation* in which the teacher informs students about the place, importance and future of the mathematical knowledge reached in the adidactic situation by: a) taking note of the progress of the adidactic situation, of the questions and answers that have been obtained or studied from it and placing them within the perspective of the curriculum; b) distinguishing among the pieces of knowledge that have appeared, and presenting in conventional form those that will serve as reference (the correct ones).

The *didactic contract* is an interpretation of the rules of the interaction between the teacher and the students in a didactic situation. Some of these rules are explicit and related to the mathematics at stake; others are implicit and of more general character related to mathematics teaching and learning. These rules form a set of reciprocal obligations between the teacher and the students. The didactic contract is the source of several didactic phenomena that constrain the meaning of the knowledge taught and learned. Brousseau (1997) has described six such phenomena: the Topaze effect, the Jourdain effect, the improper use of analogy, the metacognitive shift, the metamathematical shift, and the Dienes effect.

In TDS, students' learning is seen as a combination of processes of *adaptation* and *acculturation* (Brousseau, 2000). Independent adaptation to a milieu takes place in an adidactic situation; acculturation into an educational system takes place in a didactic situation with help of a didactic contract. In this model, the devolution ensures the conditions for adaptation, and the institutionalisation ensures the conditions for acculturation.

Didactic engineering is the primary research methodology of TDS. It is based on qualitative analyses and is structured into four phases: preliminary analyses; design and *a priori* analysis; realisation, observation and data collection; *a posteriori* analysis and validation (Artigue, 2015). The design heuristic of TDS consists of the *adidactic situation* with its *milieu*: This is designed with a problem (or problems) that has the target knowledge as an (in some sense) optimal solution, where the milieu gives feedback to the students on their responses, with respect to this knowledge.

Although TDS originates experimentally in the context of primary school, numerous aspects of the theory have been used by researchers at secondary and tertiary level. For instance, González-Martín, Bloch, Durand-Guerrier and Maschietto (2014) discuss the use of TDS at university level. They present three research cases where TDS is applied—two on calculus and one on proof. Further, Strømskag Måsøval (2013) have used TDS tools in teacher education to identify relationships between the design of an algebraic task and the outcome of student teachers' engagement with the task.

Strømskag (2018) discusses instructional design in teacher education with data from three student teachers' experiments in secondary school. A didactic engineering model developed by Strømskag (2017) was used by student teachers to design, implement and analyze didactic situations. The results show how the student teachers, on the basis of comparison between the a priori and a posteriori analyses, were able to identify relationships between the milieus of the adidactic situations and the knowledge generated by the pupils in those situations. The utility at university level of the applied instructional model involves considering the experimented situations as material milieus for instructional design in teacher education.

Theory of Commognition

The basic principle of Sfard's (2008) theory of *commognition*—an amalgam of *communication* and *cognition*—is that thinking can be regarded as a communicative act. Mathematics can be seen as a discourse and doing mathematics can be seen as engaging in a mathematical discourse. This viewpoint simplifies things, as *thinking* and *thoughts* are traditionally conceived as something less accessible to analysis, than *communicating* and *discourses*.

There is an increasing number of studies in mathematics education at university level that use commognition theory (Nardi, Ryve, Stadler, & Viirman, 2014; Viirman, 2014, 2015; Thoma & Nardi, 2016). Thoma and Nardi's contribution uses commognition to provide insight into closed-book assessment practices. They identify the routines of the assessment discourse of the examination task of a year one compulsory mathematics course in the United Kingdom. We will take a closer look at this contribution in the section devoted to the issues of communicating mathematics.

Cultural-Historical Activity Theory

The cultural-historical *Activity Theory* of Vygotsky (1962/1986) and others (e.g. Leontiev, 1978; Leplat, 1997) is a global theory of cognition that has been used to frame research into university level mathematics teaching. In this framework, it is hypothesized that students' (possible) learning emerges from their activities as they work on mathematical tasks, where the *activity* is "what a subject engages in during the completion of the task" whereas the task is "the goal to be attained under certain circumstances" (Rogalski, 2013, p. 4). It is interesting to note that, in the context of mathematics education, activity theory has developed with divergent interpretations, as illustrated by Abboud et al. (2018). Two contributions at INDRUM 2016 align with these different interpretations, each of which has specific theoretical-methodological hypotheses related to teachers' practices.

In Grenier-Boley, Bridoux, and Hache (2016), teachers' practices are analyzed by using the "Double Approach" framework (Robert & Hache, 2013) which is a didactic and ergonomic approach to practices. Within the Double Approach, mathematics teachers' goals and decisions are described in terms of five components: cognitive, mediative, institutional, personal, social. The authors study discursive activities of the teacher that reflect a *proximity* with students' activities, knowledge or remarks in analogy with the Vygtoskian notion of zone of proximal development (ZPD) (Rogalski, 2013). In Petropoulou, Jaworski, Potari, and Zachariades (2016), teachers' practices during lectures are identified by using the Teaching Triad framework (Jaworski, 1994), an analytic framework oriented by teachers' goals and actions. Its purpose is to describe the complexity of teachers' activities in terms of *management of learning*, *sensitivity to students* and *mathematical challenge*.

Community of practice theory

Community of practice theory (CPT) has been used as a framework for developmental research that seeks to establish inquiry communities (at university level, e.g., Biza, Jaworski, & Hemmi, 2014). An INDRUM study framed within CPT is (Viirman, 2018), which reports the collaboration between a mathematician and a mathematics education researcher.

CHARACTERISTICS OF TEACHING PRACTICES

Teacher learning and teacher knowledge

Some ongoing studies have focused on the design and implementation of initiatives dedicated to teachers' training at the university. Hamann and Schmidt-Thieme (2018) study the possibility of enriching links between curriculum courses of a pre-service training spiral curriculum by attending to basic notions, history and language of mathematics through teaching episodes referred to as "keynotes". The study described in the poster contribution by Viirman, Pettersson, Björklund and Boustedt (2018) aims to help Swedish student teachers integrate programming in their mathematics classes (as required by a revision of the national curriculum). To do so, the authors propose to implement a training course focusing on programming in a mathematical context with a collaborative teaching approach, and studying the potential transformation of knowledge acquired during teachers' training into teaching practices *in situ*. Huget (2018) reports on the creation of a tutor training program ("BiMathTutor") conceptualized and implemented at the University of Bielefeld, Germany. These contributions make it possible to raise important issues related to the implementation of such initiatives at the university. We may wonder about the extent to which theoretical frameworks can help evaluate the "efficacy" of such an initiative (also pointed out by Hamann & Schmidt-Thieme, 2018).

The question of transition from being a university student to being a teacher in a certain institution has also been addressed. Huget (2018) questions the effects of “BiMathTutor”, both on the possible change in tutors’ practice during the training sessions and on the expectations and results of students who attend tutoring classes. Mathieu-Soucy, Corriveau and Hardy (2018) address the main issue of the transition from being a university student to becoming a new teacher in the *cegep*[2] institution in Quebec, Canada. To do so, they plan to investigate *significant experiences* (to be understood from the perspective of Dewey’s philosophy (1938)) within these teachers’ *relationship with mathematics and its teaching and learning* (RWMTL). The main result of the pilot study is the classification of these teachers’ RWMTL into several categories, including personal issues, issues linked to the institution, the relation between teachers and students, and teachers’ expectations towards students’ mathematical level.

Pre-service and in-service teachers’ competences to diagnose mathematics teaching situations have also been studied (Biza & Nardi, 2016, Biza, Nardi, & Zachariades, 2018). Two issues related to teacher learning emerge from these studies. An important matter of interest seems to be the possibility to integrate competences to diagnose teaching situations or to address teaching issues in mathematics teachers’ pre- and in-service development programmes (Biza & Nardi, 2016; Biza et al., 2018). Another is the question of the role of mathematics education theories in teachers’ professional development.

Frameworks of teacher knowledge

Shulman (1987) proposed three components that together constitute the professional knowledge base of teaching: subject matter knowledge (SMK); pedagogical content knowledge (PCK); and curriculum knowledge. Ball et al. (2008) build on Shulman’s work and focus simultaneously on defining distinct forms of mathematical knowledge for teaching (MKT) and on developing related measures. They distinguish between forms of SMK consisting of common, specialized, and horizon content knowledge, and forms of PCK comprising knowledge of mathematics and students, mathematics and teaching, and mathematics and curriculum. They describe specialized content knowledge (SCK) as mathematical knowledge that is unique to the work of teaching and distinct from the common content knowledge (CCK) needed and used by teachers as well as others. The works of Shulman (1987) and Ball et al. (2008) form the basis for the Mathematics Teacher’s Specialized Knowledge model (MTSK) developed by Carrillo, Climent, Contreras and Muñoz-Catalán (2013).

In Germany, the project Professional Competence of Teachers, Cognitively Activating Instruction, and Development of Students’ Mathematical Literacy (COACTIV) developed open response measures of content knowledge (CK, which aligns with SMK) and PCK at the secondary level and used these together with data from a large sample of grade 10 teachers and their students’ progress (Baumert et al., 2010). In COACTIV, content knowledge is defined as deep understanding of the mathematics taught in the secondary school—it is associated with the idea of “elementary mathematics from a higher standpoint" in Klein's (1933/2016) textbook, and with Ma's (1999) notion of "profound understanding of mathematics". Results from COACTIV studies showed that while PCK was inconceivable without sufficient CK, the latter could not substitute for PCK.

The above-mentioned frameworks for mathematics teachers' knowledge are developed—and mainly used—in the context of primary or secondary school. Delgado and Zakaryan (2018) build on MTSK (Carrillo et al., 2013) in their attempt to characterize a university teacher's knowledge of the practice of mathematics in the content area of mathematical analysis. They found indicators of the lecturer's knowledge of the practice of mathematics related to ways of *reasoning*, *validating*, and *proceeding* in mathematics.

Another categorization of teachers' knowledge is that created by Rowland et al. (2005). On the basis of 24 video-recorded mathematics lessons in primary school, they developed the "Knowledge Quartet" (KQ), comprising foundation, transformation, connection, and contingency knowledge. Foundation knowledge aligns with SMK, and the other three with PCK. The first dimension of the KQ, *foundation*, is composed of teachers' mathematics-related knowledge, beliefs and understanding. The second dimension, *transformation,* is composed of knowledge-in-action as manifested in the planning of teaching and in the act of teaching itself. The third dimension, *connection*, is composed of ways in which the teacher obtains coherence within and between lessons. The fourth dimension, *contingency*, is about the teacher's ability to "think on his/her feet" and deals with unexpected contributions from students.

Breen, Meehan, O'Shea and Rowland (2018) used the Knowledge Quartet framework to analyse the features of university teaching that are highlighted in a set of three university lecturers' (the first three authors of the paper) brief-but-vivid accounts of memorable incidents that occurred during their teaching over the course of two years. The accounts were written as part of a professional development project using the Discipline of Noticing (Mason, 2002), and focused on troublesome aspects of classes as perceived by the lecturers.

Within the frame of ATD and the paradigm of "questioning the world" (Chevallard, 2015), Bourgade's (2016) study shows how teacher knowledge is developed as a consequence of elaboration and dissemination of a professional praxeological equipment. Bourgade explored how to design a *generating question* (see above) for an investigation workshop in engineering education. He describes the difficulties that a (generic) teacher faces while disseminating a praxeology under construction (initiated by a new type of task) to other subjects of the institution. Furthermore, he explains how these difficulties can contribute towards and are instrumental for the construction of a high-level *logos* of the new type of tasks.

In Syrdalen's (2018) on-going study, the intention is to apply the Systematic Classroom Analysis Notation (Beeby, Burkhardt, & Fraser, 1979) to describe teacher knowledge through evaluation of the depth of demand on, and level of guidance given by, the lecturer during in-class interactions.

Issues of communicating mathematics

The contributions that address the issue of communicating mathematical ideas can be divided in two main directions: those focusing on teaching as a discursive practice and those interested in teachers' discourse in order to address the way students' needs are taken into account.

In the first direction, one possibility is to see teaching "as a highly-situated discursive practice" like in (Biza & Nardi, 2016) and (Viirman et al., 2018). Another possibility is to see language as a medium and an aim of mathematics lessons (Schmidt-Thieme, 2018). In this work, Schmidt-Thieme proposes to implement a "language curriculum" for pre-service teachers in order to focus on mathematics teachers' language competencies as well as transitions between the registers of mathematical representation (Duval, 2006).

In the second direction, two contributions that adopt an Activity Theory approach to cognition have studied university teachers' discourse in Calculus lectures. First, Grenier-Boley et al. (2016) compare two courses devoted to the introduction of the formal definition of limit of function at the beginning of university[3]: the course presented in a textbook and the course given by a teacher during a lecture. Several proximities (upward, downward, horizontal)[4] are spotted in the textbook and the teacher's discourses: they correspond to different types of occasions to induce and/or operate a proximity between students' previous knowledge and the new knowledge to be learnt. A finding from the study is that the textbook does not contain occasions of proximities whereas the teacher's discourse contains several proximities that are supposed to address several students' needs. These needs include the use of a progressive formalism, reformulation of part of a definition (words, graph, absolute value, distance, neighborhood), and recall of logical elements. Second, Petropoulou et al. (2016) study the way in which a lecturer conceptualizes students' learning needs and enacts these conceptualizations in his actual teaching. They propose two main categories of this lecturer's goals with respect to students' needs within his practice. The first encompasses introducing students to aspects of advanced mathematical thinking, such as refining a problem, using alternative methods, representing, and justifying. The second is about supporting students' content-related understanding, including providing steps and methods for solving tasks, and highlighting subtle aspects of the mathematics.

The issue of communicating mathematics to non-mathematics students has also been addressed at INDRUM conferences. Martìn-Molina (2016) studies her own adaptation of content and teaching of statistics in several undergraduate programmes. Two outcomes of the study were the importance of disciplinary knowledge that may help contextualize the statistics taught and the need to reduce the level of formalism in instruction. The study by Viirman (2018) focuses on the collaboration between mathematicians and mathematics educators within a project in which modelling activities are proposed to Biology students. The study highlights, in the frame of Community of Practice Theory, the contrasting stances towards students and mathematics by the mathematician and the mathematics education researcher. The former making mathematical content his didactic priority, the latter's endeavor being to try to set himself in the position of a student and see the mathematics from the students' perspective.

In connection with the above studies, one could ask: how can an instructor's teaching practice be influenced by his/her conceptualization of students' learning needs? The studies by Grenier-Boley et al. (2016) and Petropoulou et al. (2016) present many similarities[5]: the former formulates hypotheses about students' needs as a consequence of three types of studies (epistemological, curricular and didactical) whereas the starting point of the latter is students' needs as perceived by teachers (cognitive, affective, social).

One of the aims of inquiry based learning (IBL) is to teach students to construct proofs. Accordingly, one purpose of examinations would be to assess students' ability to generate such a discourse. Thoma and Nardi (2016) show, in their study of a closed-book examination, that students are explicitly guided to justify and construct their responses. This not only fosters a somewhat limited image of mathematics as a predominantly step by step, highly directed activity, as the authors observe, but also reveals a reluctance to ask students to develop a complete and correct reasoning by themselves, the reasoning required in the exams being imitative and not creative.

Teacher-researchers' teaching practices

On an international level, there is an emerging interest in studying university teacher-researchers' practices (Annoot & Fave-Bonnet, 2004). Becher (1994) and Poteaux (2013) affirm the need to take into consideration the nature of the content taught and the possible impact of the research activity of teacher-researchers on their teaching when studying their practices. However, existing research on university teaching practices seldom takes them into account (Henkel, 2004; Neumann, 2001). In this regard, several contributions to INDRUM show an increasing interest in the practices of university teacher-researchers in mathematics based on didactical approaches. Examples are these studies referred in the previous section: Broley (2016), Florensa et al. (2018), Strømskag (2018), and Tabchi (2018).

METHODOLOGIES FOR INSTRUCTIONAL/DIDACTIC DESIGN

Didactic engineering versus design research: four theoretical approaches

According to Artigue (2009), didactic design includes controlled intervention research into the processes of *planning*, *delivering* and *evaluating* mathematics teaching and learning, and further, it includes the problem of *reproducibility* of results from such interventions. Here, we examine the methodologies for instructional design of four frequently used theoretical approaches to mathematics teaching and learning: ATD, TDS, Realistic Mathematics Education (RME) and Lesson Study. Whereas the methodology of ATD and TDS is *didactic engineering*, the methodology of RME and Lesson Study is *design research* (even if the research component may not be central in Lesson Study). The design heuristics of ATD and TDS are presented above, in the sections on ATD and TDS respectively.

The design heuristic of RME consists of either *didactic phenomenology* (Freudenthal, 1983) or *emergent modelling* (Gravemeijer, 2004). Didactic phenomenology identifies the phenomena that the target mathematical concept helps organize and understand, whereas emergent modelling starts with identifying the initial problems that refer to a paradigmatic context situation for the target mathematical concept.

The design heuristic of Lesson Study is the *research lesson* (Fernandez & Yoshida, 2004). This is a lesson in which observational data is collected from the test of a teaching-learning experiment. An overview of the four approaches and their methodologies is given in Table 1.

THEORY	RESEARCH METHODOLOGY	CONSTITUENTS of the methodology
ATD	Didactic engineering SRP (*study and research path*)	- Preliminary analyses - Design and a priori analysis of an SRP - Implementation, observation, data collection, *in vivo* analysis - A posteriori analysis and validation (Barquero & Bosch, 2015)
TDS	Didactic engineering (*adidactic situation*)	- Preliminary analyses - Design and a priori analysis of a didactic situation - Implementation, observation, data collection - A posteriori analysis and validation (Artigue, 2015)
RME	Design research (*emergent modelling*)	Cyclic process of: - Thought experiment - Design of a teaching sequence - Test of the teaching sequence, observation, data collection - A posteriori analysis (Gravemeijer & Cobb, 2013)
Lesson Study	"Open approach method" (*research lesson*)	- Planning of a research lesson - Implementation, observation, data collection - Reflection on the research lesson - Possibly, revision and teaching of the revised research lesson (Murata, 2011)

Table 1. Research methodologies of the theories (*design heuristics* in parentheses)

We note that all four approaches involve the design of some educational tool, and they are all informed by, and contribute to the development of, educational theory.

Furthermore, they all reject standardised validation processes based on comparison of experiment groups and control groups.

When it comes to vision of mathematics education and characteristics of the two methodologies, they are different. The vision of didactic engineering is mathematics education as a fundamental science, where the principal aim is understanding of didactic systems and didactic phenomena (an applied component is relevant, but secondary). Didactic engineering is a kind of "phenomenotechnics" that cares for the necessity to *produce* didactic phenomena in order to gain control over them (Bachelard, cited in Chevallard, 1981).

The vision of design research is mathematics education as a design science, where the principal aim is to develop robust teaching sequences through controlled production of educational tools (to understand didactic phenomena is relevant, but secondary). Design research is interventionist and iterative, cyclic in nature.

Studies based on didactic design that have been presented at INDRUM and included in the foregoing, are Barquero et al. (2016), Bourgade (2016), Romo et al. (2016), Florensa et al. (2018), and Strømskag (2018).

Instructional design for inquiry-based learning

As noted above, there are several approaches to instructional design that have been theorized. Despite the theorization, there appears to be a broad common ground when considering the intended learning activities of the students. This common ground is devoted to achieving so-called *inquiry-based learning*, or IBL in short.

From the perspective of studies conducted within ATD, the idea of inquiry is implicit in the notion of *study and research path (SRP)* (Chevallard, 2015). Several contributions to INDRUM 2016 and 2018 are related to this, for instance the papers: Barquero et al. (2016); Romo et al. (2016); and Gascón and Nicolás (2018). SRP was introduced earlier in the chapter, we briefly recall that a SRP consists in a process that starts with a question of genuine interest for the community of study and entails looking for satisfactory answers to this question. Along this quest, as it would be the case for real mathematicians, students are allowed to "research" (inquiry, problem solving, problem posing, etc.) and "study" (consulting existing knowledge, attending lectures where the teacher acts as the main means to provide mathematical knowledge, etc.). In this way, students progressively examine different possible answers until they finally get what the study community (formed by teacher plus students) regards as a suitable answer to the initial question.

A priori one can consider two types of IBL. On the one hand, we can consider inquiries where there is no predetermined knowledge to be rebuilt by students. The only important point would be to find an "admissible" answer to an initial question (Chevallard, 2015). To adopt this kind of teaching entails a radical change in the conception of education, as the emphasis in this case would be on considering questions and starting with inquiries rather than learning a piece of knowledge explicitly anticipated by the teacher.

On the other hand, inquiries can be considered in which the students are expected to rediscover a certain piece of knowledge previously chosen by the teacher. This kind of IBL is still groundbreaking and has to face many challenges of different nature: political, cultural, epistemological, concerning teacher's training and teacher's view of their own profession, etc. Gascón and Nicolás (2018) consider some of these challenges by applying Hintikka's conceptualization of inquiry (Hintikka, Halonen, & Mutanen, 2002) as a useful tool not only to describe but also to plan inquiries. Among the challenges they address, they consider the problem of finding criteria in order to attach meanings to the objects involved in the expected inquiries. If, in a proposed inquiry, some the meanings are dismissed, then students are most probably deprived of relevant information. Furthermore, one can raise a question about the possible ways in which an inquiry in which things have several meanings can be planned. This is the first challenge of IBL considered by Gascón and Nicolás (2018): the conundrum of implementing semantics in mathematics syntax. The second challenge these researchers consider is concerned with how to plan inquiries. Every inquiry is materialized in the quest of an answer to an initial question. If it is desired that students discover a certain piece of knowledge through an inquiry process, the question to initiate the inquiry must be chosen with great care.

The questions considered in (Gascón & Nicolás, 2018) are still open problems in the study of IBL, but, of course, they are not the only ones. Many issues were not addressed in this contribution, among them the problem of student's assessment in an IBL context. We are used to assess student's answers to our questions, or student's observance of our requests or command. But how should we assess an inquiry? Should examinations be inquiries? Should examinations disappear?

CONCLUSION

Teaching mathematics at university is an interesting and compelling area of study for mathematics education researchers that is only just beginning to receive close attention. The special abstract nature of mathematical truths and the big ideas such as function, variability, change, limit and structure create a unique context for the study of mathematical meaning making, communication and instruction. The studies presented at INDRUM conferences by posters and papers bear witness to the truth of the assertion: the more we learn, the more there is to find out and to investigate further and deeper; the closer we look, the more there is to see.

In higher education, the majority of students who study mathematics are required to do so because it is part of another programme such as engineering or economics. In these programmes, mathematics is often blamed for poor performance and retention, high failure and drop-out; this is a global problem which mathematics education research must address. Presentations at INDRUM reflect both the attraction of research into teaching mathematics and the societal value of the research.

Researching university mathematics education presents a great opportunity for mathematics education researchers and mathematicians to collaborate and thus bridge the divide between the disciplines of mathematics and mathematics education. Research evidence can challenge the, often naïve, epistemological assumptions of teaching by introducing theoretical lenses and frameworks to provide means for analyzing, interpreting and making sense of classroom practices. It is essential that mathematics education researchers communicate with mathematics teachers the results of research and the possibilities for improving practice. The presentations at INDRUM are framed in a plethora of theories: ATD, Instrumental and Documentational approaches, TDS, Commognition, CHAT and CPT—it is possible to get a glimpse of how mathematicians react to this situation in the work of Nardi (2008) and in a scathing review by the mathematician Steen (1999) of an ICMI study of mathematics education research. Do the theories drive a wedge between mathematics education researchers and mathematicians/mathematics teachers? For example, in Viirman (2018), there is no mention of "theories" of mathematical instruction entering the discourse of the collaboration between mathematics education researchers and mathematics teachers reported on. The papers that refer to the "teaching triad" (Petropoulou et al., 2016) and the "knowledge quartet" (Breen et al., 2018) reveal the value of less complex models that have empirical roots in mathematics classes, albeit at school level.

The use of models of teaching and learning practices developed in school-based research presents a challenge to INDRUM researchers: are these models transferable to learning in higher education? The differences between the school and higher education contexts are significant. Students study mathematics in higher education because it is a part of a programme of study they have chosen, rather than a compulsory part of the curriculum. In higher education, students are expected to have greater independence and agency, and we assume maturity. The teaching and learning contexts differ radically in terms of class size, intensity and pace of content covered in lectures and sometimes use of minimally 'trained' student teaching/learning assistants. The teachers or lecturers in higher education are likely to have a rather different relationship with mathematics than teachers in school, and the workloads and working days of schoolteachers and lecturers are likely to be quite different. Thus, when taking models developed from school-based empirical studies and applying them in the context of higher education, researchers need to consider the utility and applicability of the model, this may not be taken for granted.

It has to be recognized that the presentations at conferences such as INDRUM cover only part of the research field at large. Consequently, the discussion of the theories included in this chapter can only reflect the research presented at INDRUM. What might be concluded from the proportion of papers framed within ATD? Does it reveal something of the utility of the theories, or perhaps the nature of the issues that interest researchers of didactics of university mathematics, or is it about national characteristics and the maturity of research in this field in different national communities (or communities united by sharing a common language)? INDRUM conferences serve a purpose by bringing together these research communities using these theories.

Contrasting theoretical perspectives can offer alternative approaches for exploring and developing mathematics teaching at university. For example: TDS and the Double Approach offer developmental and analytic tools for teachers/researchers. TDS (with its concepts of adidactic situation and milieu) can be used as a tool to design and analyze the implementation of mathematics teaching sequences with respect to (generic) students' appropriation of some particular knowledge. The Double Approach (with its concept of proximities) can be used to analyze the distance between what students do and know and the teacher's goals for the students, and how students' responses influence the actions of the teacher in trying to reduce this distance. Also, there is a difference in the nature of the didactic devices: TDS aims at adidactic functioning of the knowledge, and its evolution, by designing and managing an appropriate milieu; the Double Approach's concept of proximities aims at didactic actions that the teacher can use to bridge the gap between students' existing knowledge and the new knowledge aimed at (Mangiante-Orsola, Perrin-Glorian, & Strømskag, 2018).

More can be developed about teacher-researchers' teaching practices. What is the relationship between teacher practice and researcher practice? To what extent does the researcher's practice influence the teacher and vice versa? In other words, what balance exists between mathematical practice and research in the field? In Bridoux et al. (2019), the purpose of a comparative and interdisciplinary research based on three academic disciplines (Mathematics, Chemistry, Physics) is to question the research discipline's imprint on teaching practices at tertiary level. More generally, the influence of research practices on teacher-researchers learning practices (already stressed by Biza et al., 2016) or the study of teacher-researchers' (possible) training should be a main point of interest. Attention could be focused on the practices of teacher-researchers from the point of view of their interaction with resources and with other disciplines.

Another area calling for research is to relate teaching practices to student performance. However, valid and reliable research that is likely to expose correlations and more importantly cause-effect relationships is at the moment not covered by the qualitative studies mostly reported at INDRUM conferences. Without large-scale research using experimental and quasi-experimental approaches an important area of knowledge will remain unexplored. Is this a challenge the INDRUM community should confront?

INDRUM serves a purpose for mathematics education researchers to negotiate the value of theoretical frameworks in structuring and making sense of mathematics teaching at university level. The network provides an opportunity for critical reflection on the application of theory to teaching practice, and the development of teaching. It is difficult, and perhaps foolish, to try to predict the direction of research in this area. One thing that can be asserted by these contributions though is the enthusiasm of INDRUM researchers for sustaining and continuing to create knowledge of, and for, teaching mathematics at university.

NOTES

1. “A preparatory class that leads to engineering schools after two years of intense training in sciences and humanities.” (Bourgade, 2016)

2. Cegep institutions refer to general or vocational colleges and constitute the first step in post-secondary education.

3. See also Bridoux, Grenier-Boley, Hache, and Robert (2016).

4. Roughly speaking: *upward proximities* are inductive and used during a generalization or decontextualization process, *downward proximities* are deductive and are used between the “general” (course) and the “particular” (examples, exercises). *Horizontal proximities* do not change the level of discourse and correspond to analogies or translations.

5. The general framework and the study of Calculus notions related to the learning of advanced mathematics (abstraction, formalism…) accentuated by the context of the secondary-tertiary transition, the type of issue.

REFERENCES

Abboud, M., Goodchild, S., Jaworski, B., Potari, D., Robert, A., & Rogalski, J. (2018). Use of activity theory to make sense of mathematics teaching: A dialogue between perspectives. *Annales de Didactique et de Sciences Cognitives*, *Volume Spécial English-French*, 61–92.

Adler, J. (2000). Conceptualising resources as a theme for teacher education. *Journal of Mathematics Teacher Education, 3*, 205–224.

Annoot, E., & Fave-Bonnet, M. F. (2004). *Pratiques pédagogiques dans l’enseignement supérieur: enseigner, apprendre, évaluer [Pedagogical practices in higher education: teach, learn, evaluate]*. Paris: L’Harmattan.

Artigue, M. (2009). Didactical design in mathematics education. In C. Winsløw (Ed.), *Nordic research in mathematics education* (pp. 7–16). Rotterdam, The Netherlands: Sense.

Artigue, M. (2015). Perspectives on design research: The case of didactical engineering. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education* (pp. 467–496). Dordrecht, The Netherlands: Springer.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*, 389–407.

Barquero, B., & Bosch, M. (2015). Didactic engineering as a research methodology: From fundamental situations to study and research paths. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education* (pp. 249–272). Dordrecht, The Netherlands: Springer.

Barquero, B., Serrano, L., & Ruiz-Munzón, N. (2016). A bridge between inquiry and transmission: the study and research paths at university level. INDRUM 2016 Proceedings, pp. 340–349.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., . . . Tsai, Y.-M. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Education Research Journal*, *47*(1), 133–180.

Becher, T. (1994). The signifiance of disciplinary differences. *Studies in Higher Education, 19*(2), 151–161.

Beeby, T., Burkhardt, H., & Fraser, R. (1979). *Systematic classroom analysis notation for mathematics lessons*. Nottingham, UK: Shell Centre for Mathematical Education.

Biza, I., Jaworski, B., & Hemmi, K. (2014). Communities in university mathematics. *Research in Mathematics Education, 16*, 161–172.

Biza, I., Giraldo, V., Hochmuth, R., Khakbaz, A., & Rasmussen, C. (2016). *Research on teaching and learning mathematics at the tertiary level: State-of-the-art and looking ahead. ICME-13 Topical Surveys*. Cham: Switzerland: Springer.

Biza, I., & Nardi, E. (2016). Consistency, reification of pedagogical and mathematical discourses in student teachers' narratives on the challenges of their school placement experience. INDRUM 2016 Proceedings, pp. 425–426

Biza, I., Nardi, E., & Zachariades, T. (2018). Competences of mathematics teachers in diagnosing teaching situations and offering feedback to students: Specificity, consistency and reification of pedagogical and mathematical discourses. In T. Leuders, K. Philipp, & J. Leuders (Eds.), *Diagnostic competence of mathematics teachers* (pp. 55–78). New York, NY: Springer.

Bosch, M., & Gascón, J. (2006). Twenty-five years of didactic transposition. ICMI Bulletin, 58, 51–64.

Bourgade, J.-P. (2016). On the diffusion of professional praxeologies at university level: The case of investigation workshops. INDRUM 2016 Proceedings, pp. 350–359.

Breen, S., Meehan, M., O'Shea, A., & Rowland, T. (2018). An analysis of university teaching using the Knowledge Quartet. INDRUM 2018 Proceedings, pp. 383–392.

Bridoux, S., de Vleeschouwer, M., Grenier-Boley, N., Khanfour-Armalé, R., Lebrun, N., Mesnil, Z., & Nihoul C. (2019). L'identité professionnelle des enseignants-chercheurs en mathématiques, chimie et physique [The professional identity of teacher-researchers in Mathematics, Chemistry, Physics]. In M. Abboud (Ed.), Actes du colloque EMF 2018 (540-547). Paris : IREM de Paris.

Bridoux, S., Grenier-Boley, N., Hache, C., & Robert, A. (2016). Les moments d'exposition des connaissances, analyses et exemples [Presenting knowledge and teaching moments, analyses and examples]. *Annales de Didactique et de Sciences Cognitives*, *21*, 187–234.

Broley, L. (2016). The place of computer programming in (undergraduate) mathematical practices. INDRUM 2016 Proceedings, pp. 360–369.

Brousseau, G. (1997). *The theory of didactical situations in mathematics*: *Didactique des mathématiques, 1970–1990* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Eds. & Trans.). Dordrecht, The Netherlands: Kluwer.

Brousseau, G. (2000). Educacion y didactica de las matematicas. *Educacion Matematica, 12*(1), 5–38. Available in French: https://hal.archives-ouvertes.fr/hal-00466260

Brousseau, G., & Warfield, V. (2014). Didactic situations in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 163–170). Dordrecht, The Netherlands: Springer.

Carrillo, J., Climent, N., Contreras, L. C., & Muñoz-Catalán, M. C. (2013). Determining specialised knowledge for mathematics teaching. In B. Ubuz, Ç. Haser, & M. A. Mariotti (Eds.), *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education* (pp. 2985–2994). Ankara, Turkey: Middle East Technical University & ERME.

Chevallard, Y. (1981). The didactics of mathematics: Its problematic and related research. *Recherches en Didactique des Mathématiques*, *2*(1), 146–158.

Chevallard, Y. (1998). Analyse des pratiques enseignantes et didactique des mathématiques: L'approche anthropologique [Analysis of teaching practices and didactics of mathematics: The anthropological approach]. Retrieved from yves.chevallard.free.fr/spip/spip/IMG/pdf/Analyse_des_pratiques_enseignantes.pdf

Chevallard, Y. (2002). Organiser l’étude. Écologie & régulation [Organising the study. Ecology & regulation]. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot, & R. Floris (Eds.), *Actes de la XIe École d’Été de Didactique des Mathématiques* (pp. 41–56). Grenoble: La Pensée sauvage.

Chevallard, Y. (2011). La notion d’ingénierie didactique, un concept à refonder. Questionnement et éléments de réponse à partir de la TAD [The notion of didactic engineering, a concept that needs to be redesigned. Questioning and answers from the TAD. In C. Margolinas, M. Abboud-Blanchard, L. Bueno-Ravel, N. Douek, A. Flückiger, P. Gibel, . . . F. Wozniak (Eds.) , *En amont et en aval des ingénieries didactiques, Actes de la XVe école d’été de didactique des mathématiques* (pp. 81–108). Grenoble: La pensée sauvage.

Chevallard, Y. (2013). Enseñar Matemáticas en la Sociedad de Mañana: Alegato a Favor de un Contraparadigma Emergente [Teaching Mathematics in tomorrow's society: Plea for an emerging counter paradigm]. *Journal of Research in Mathematics Education, 2*, 161–182.

Chevallard, Y. (2015). Teaching mathematics in tomorrow’s society: A case for an oncoming counter paradigm. In S. J. Cho (Ed.), *Proceedings of the 12th International Congress on Mathematical Education* (pp. 173–187). Berlin, Germany: Springer.

Delgado, R., & Zakaryan, D. (2018). Knowledge of the practice in mathematics in university teachers. INDRUM 2018 Proceedings, pp. 393–402.

Dewey, J. (1938). *Experience and education*. New York: Collier Books.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics, 75*, 213–234.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*, 103–131.

Fernandez, C., & Yoshida, M. (2004). *Lesson study: A Japanese approach to improving mathematics teaching and learning*. New York: Routledge, Taylor & Francis Group.

Florensa, I., Bosch, M., Cuadros, J., & Gascón, J. (2018). Helping lecturers address and *formulate* teaching challenges: An exploratory study. INDRUM 2018 Proceedings, pp. 373–382.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht, The Netherlands: Reidel.

Gascón, J., & Nicolás, P. (2018). Inquiry-based learning and pre-service teachers education. INDRUM 2018 Proceedings, pp. 412–421.

González-Martín, A. S., Bloch, I., Durand-Guerrier, V., & Maschietto, M. (2014). Didactic Situations and Didactical Engineering in university mathematics: Cases from the study of Calculus and proof. *Research in Mathematics Education, 16*, 117–134.

Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning*, *6*(2), 105–128.

Gravemeijer, K., & Cobb, P. (2013). Design research from the learning design perspective. In T. Plomp & N. Nieveen (Eds.), *Educational design research Part A: An introduction* (pp. 72–113). Enschede, The Netherlands: SLO.

Grenier-Boley, N., Bridoux, S., & Hache, C. (2016). Moments d'exposition des connaissances à l'université : Le cas de la notion de limite [Presenting knowledge at the university: The case of the notion of limit]. INDRUM 2016 Proceedings, pp. 380–389.

Gueudet, G. (2017). University teachers' resources systems and documents. *International Journal of Research in Undergraduate Mathematics Education 3*(1), 198–224.

Gueudet, G., Buteau, C., Mesa, V., & Misfeldt, M. (2014). Instrumental and documentational approaches: From technology use to documentation systems in university mathematics education. *Research in Mathematics Education, 16*, 139–155.

Gueudet, G., Pepin, B., & Trouche, L. (2012). *From text to 'Lived resources': Mathematics curriculum material and teacher development*. New York: Springer.

Hadjerrouit, S., & Gautestad, H. (2016). Using the interactive visualization tool SimReal+ to teach mathematics at the university level: An instrumental approach. INDRUM 2016 Proceedings, pp. 429–430.

Hamann, T., & Schmidt-Thieme, B. (2018). Cross-linking maths – using keynotes to structure a curriculum for future teachers. INDRUM 2018 Proceedings, pp. 456–457.

Henkel, M. (2004). La relation enseignement-recherche [The relationship between teaching and research]. *Politiques et gestion de l'enseignement supérieur, 16*(2), 21–36.

Hintikka J., Halonen, I., & Mutanen, A. (2002). Interrogative logic as a general theory of reasoning. In D. M. Gabbay, R. H. Johnson, H. J. Ohlbach, & J. Woods (Eds.), *Handbook of the logic of argument and inference: The turn towards the practical* (Vol. 1, pp. 295–337). Amsterdam, The Netherlands: Elsevier.

Huget, J. (2018). BiMathTutor – The program for further training of new tutors in University Mathematics. INDRUM 2018 Proceedings, pp. 458–459.

Jaworski, B. (1994). *Investigating mathematics teaching: A constructivist enquiry.* London: Falmer Press.

Klein, F. (2016). *Elementary mathematics from a higher standpoint. Volume I: Arithmetic, Algebra, Analysis*. Berlin/Heidelberg, Germany: Springer-Verlag (Original work published 1933)

Leontiev, A. N. (1978). *Activity, consciousness and personality*. Englewood Cliffs, NJ: Prentice Hall.

Leplat, J. (1997). *Regards sur l'activité en situation de travail* [*About the activity in a job situation*]. Paris: PUF.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understandings of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Mangiante-Orsola, C., Perrin-Glorian, M.-J., & Strømskag, H. (2018). Theory of didactical situations as a tool to understand and develop mathematics teaching practices. *Annales de Didactique et de Sciences Cognitives, Volume Spécial English-French*, 145–174.

Martìn-Molina, V. (2016). The challenge of being a mathematics teacher. INDRUM 2016 Proceedings, pp. 431–432.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London: Routledge.

Mathieu-Soucy, S., Corriveau, C., & Hardy, N. (2018). Exploration of new post-secondary mathematics teachers' experiences: Preliminary results of a narrative inquiry. INDRUM 2018 Proceedings, pp. 403–411.

Murata, A. (2011). Introduction: Conceptual overview of Lesson Study. In L. Hart, A. Alston, & A. Murata (Eds.), *Lesson Study research and practice in mathematics education* (pp. 1-12). Dordrecht, The Netherlands: Springer.

Nardi, E. (2008). *Amongst mathematicians: Teaching and learning mathematics at university level*. New York: Springer.

Nardi, E., Ryve, A., Stadler, E., & Viirman, O. (2014). Commognitive analyses of the learning and teaching of mathematics at university level: The case of discursive shifts in the study of Calculus. *Research in Mathematics Education, 16,* 182–198.

Neumann, R. (2001). Disciplinary Differences and University Teaching. *Studies in Higher Education*, *26*(2), 135–146.

Petropoulou, G., Jaworski, B., Potari, D., & Zachariades, T. (2016). Addressing large cohorts of first year mathematics students in lectures. INDRUM 2016 Proceedings, pp. 390–399.

Poteaux, N. (2013). Pédagogie de l'enseignement supérieur en France: état de la question [Pedagogy of higher education in France: state of the art]. *Distances et médiations des savoirs. Distance and Mediation of Knowledge*, *1*(4).

Rabardel, P. (2002). Les hommes et les technologies: Approche cognitive des instruments contemporains [People and technology, a cognitive approach to contemporary instruments]. Paris: Armand Colin. Retrieved from http://ergoserv.psy.univ-paris8.fr/Site/default.asp?Act_group=1 (Original work published 1995)

Robert, A., & Hache, C. (2013). Why and how to understand what is at stake in a mathematics class. In F. Vandebrouck (Ed.). *Mathematics classroom: students' activities and teacher's practices* (pp. 23–73). Rotterdam, The Netherlands: Sense Publishers.

Rogalski, J. (2013). Theory of Activity and developmental frameworks for an analysis of teachers' practices and students' learning. In F. Vandebrouck (Ed.). *Mathematics classroom: students' activities and teacher's practices* (pp. 3–22). Rotterdam, The Netherlands: Sense Publishers.

Romo, A., Barquero, B., & Bosch, M. (2016). Study and research paths in online teacher professional development. INDRUM 2016 Proceedings, pp. 400–410.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, *8*(3), 255–281.

Ruiz-Olarría, A. (2015). *La formación matemático-didáctica del profesorado de secundaria: De las matemáticas por enseñar a las matemáticas para la enseñanza [Mathematical-didactical training of secondary teachers: From mathematics to teach to mathematics for teaching]*. (PhD dissertation). Universidad Autónoma de Madrid.

Schmidt-Thieme, B. (2018). Getting language awareness: a curriculum for language and language teaching for pre-service studies for teachers of mathematics. INDRUM 2018 Proceedings, pp. 460–461.

Sfard, A. (2008). *Thinking as communicating: Human development, development of discourses, and mathematizing.* New York, NY: Cambridge University Press.

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1–23.

Sierra, T. A. (2006). *Lo matemático en el diseño y análisis de organizaciones didácticas. Los sistemas de numeración y la medida de magnitudes [Mathematics in the design and analysis of didactic organizations. Numbering systems and the measurement of magnitudes]*. (Doctoral dissertation, Universidad Complutense de Madrid, Spain).

Steen, L. A. (1999). Theories that gyre and gimble in the wabe. *Journal for Research in Mathematics Education, 30*(2), 235–241.

Strømskag Måsøval, H. (2013). Shortcomings in the *milieu* for algebraic generalisation arising from task design and vagueness in mathematical discourse. In C. Margolinas (Ed.), *Task design in mathematics education. Proceedings of ICMI Study 22* (pp. 231–239). Oxford, UK: The International Commission on Mathematical Instruction.

Strømskag, H. (2017). A methodology for instructional design in mathematics—with the generic and epistemic student at the centre. *ZDM Mathematics Education*, *49*, 909–921.

Strømskag, H. (2018). A model for instructional design in mathematics implemented in teacher education. INDRUM 2018 Proceedings, pp. 422–431.

Syrdalen, E. (2018). A university lecturer's role and interactions in a flipped classroom. INDRUM 2018 Proceedings, pp. 462–463.

Tabchi, T. (2018). University teachers-researchers' practices: The case of teaching discrete mathematics. INDRUM 2018 Proceedings, pp. 432–441.

Thoma, A., & Nardi, E. (2016). A commognitive analysis of closed-book examination task and lecturer's perspectives. INDRUM 2016 Proceedings, pp. 411–420.

Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning*, *9*(3), 281–307.

Vergnaud, G. (1998). Toward a cognitive theory of practice. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 227–240). Dordrecht, The Netherlands: Kluwer Academic.

Viirman, O. (2014). The functions of function discourse-university mathematics teaching from a commognitive standpoint. *International Journal of Mathematical Education in Science and Technology, 45*(4), 512–527.

Viirman, O. (2015). Explanation, motivation and question posing routines in university mathematics teachers' pedagogical discourse: a commognitive analysis. *International Journal of Mathematical Education in Science and Technology, 46*(8), 1165–1181.

Viirman, O. (2018). Characteristics of participation – A mathematician and a mathematics educator collaborating on a developmental research project. INDRUM 2018 Proceedings, pp. 442–451.

Viirman, O., Pettersson, I., Björklund, J., & Boustedt, J. (2018). Programming in mathematics teacher education – a collaborative teaching approach. INDRUM 2018 Proceedings, pp. 464–465.

Vygotsky, L. (1986). *Thought and language*. Cambridge MA: MIT Press. (Original work published 1962)

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge, UK: Cambridge University Press.

Winsløw, C., Matheron, Y., & Mercier, A. (2013). Study and research courses as an epistemological model for didactics. *Educational Studies in Mathematics, 83,* 267–284.