Discussion of 'Virtual age, is it real?'

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Abstract

I congratulate the authors on this very interesting article discussing various aspects of the virtual age concept, which has been in active use in reliability and lifetime analyses for at least three decades. At the same time, the authors should be acknowledged for their own long time research on the subject, some of which is reviewed in their present article. In my discussion I will elaborate on some of the main issues in the article, mostly with a view towards my own interests.

Keywords: Virtual age, imperfect repair, trend-renewal process, heterogeneity

1 Introduction

The authors have chosen a title of the article which at once settles the question to be discussed. Their main idea at the outset is that real maintenance actions do not usually conform with a pure age reduction. It might not be difficult to agree in such a view, but still virtual age models have proven useful in modeling and understanding of many complicated maintenance processes.

My focus in the present discussion paper will mainly be on topics from my own research. This typically applies to the study of repairable systems, where recurrent events is a keyword. The seminal papers on virtual ages

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of Kijima (1989) and Doyen and Gaudoin (2004) indeed consider repairable and maintainable systems where event processes are behind the models. The authors of the article under discussion have, on the other hand, been able to concentrate on the imperfect *repair* part of the issue, which essentially plays the key role in their presentation.

In order not to confuse equation numbers from the article with equation numbers from the present discussion contribution, I let the former be marked as (FC1) etc.

2 Age correspondence for recurrent events

In Section 3, the authors discuss what they call the age-correspondence problem, where a concept of virtual age appears as a means of comparing working condition of an item to an idealized baseline condition. The idea is to compare an item that operates in a baseline regime, having survival function $\bar{F}_b(t)$, to an identical item operating in a severer environment, having survival function $\bar{F}_s(t)$. Assuming a stochastically larger time to failure under the baseline conditions than under the severer conditions, there exists an increasing function $W(t) \geq t$ such that

$$\bar{F}_s(t) = \bar{F}_b(W(t))$$

for all $t \geq 0$. The function $W(t)$ is then interpreted as the *virtual age* function of an item in the milder regime that would correspond to the real age of an item that was operating for time t in the severer regime. Thus, in some sense, a more severe environment corresponds to time running faster in the baseline conditions. As noted by the authors, such a relation is commonly the basis of models for accelerated life testing.

It follows that if the item operating in the severer environment fails at time $T_s \sim \bar{F}_s$, then

$$W(T_s) \sim \bar{F}_b \tag{1}$$

(indeed, $P(W(T_s) > u) = P(T_s > W^{-1}(u)) = \bar{F}_s(W^{-1}(u)) = \bar{F}_b(u)$).

Suppose then that the item operating in the severer environment fails at time T_s and is then repaired. How should one model the time to the next failure? This problem has not been discussed in the article, where the focus is on modeling the time to first failure. Obviously, one might think of going back to the classical virtual age models, and define some direct reduction of the virtual age resulting from the repair. Instead, the authors, at the end of Section 3, indicate an interesting connection between the age correspondence principle and the failure modeling by degradation processes, e.g., the gamma process.

In the spirit of the authors' consideration of a baseline condition as compared to a "real life" condition, leading to the connection (1), it is tempting to extend the age correspondence approach of the article to a consideration of recurrent events modeled by the *trend-renewal process* (Lindqvist, 2006).

This would here mean that for the observed failure times in the severer environment, $T_{s1}, T_{s2}, \ldots$, we assume that $W(T_{s1}), W(T_{s2}), \ldots$ is a renewal process with inter-arrival distribution given by $\bar{F}_b$. Letting

$$T_{bj} = W(T_{sj}) \text{ for } j = 1, 2, \ldots,$$

this implies that the times between failures in the baseline regime, $T_{b1}, T_{b2} - T_{b1}, T_{b3} - T_{b2} \ldots$, are i.i.d. with survival function $\bar{F}_b$, and hence that repairs in the baseline regime are always perfect.

It should be noted (Lindqvist, 2006) that by appropriate choices of the distribution F_b and the function $W(t)$, the failure process $T_{s1}, T_{s2}, \ldots$ may itself be a renewal process (i.e., have perfect repairs), but could also be a minimal repair process. The latter property is obtained by letting $\bar{F}_b$ correspond to an exponential distribution.

The ability to model both perfect and minimal repairs, as well as situations "between" the two extreme repair regimes, makes the trend-renewal process as a way of modeling imperfect repair (Gámiz et al., 2011, Ch. 4).

3 Component dependent virtual ages

In Section 5 in the article, the authors consider state or component dependent virtual ages. They show in particular that the virtual age defined for a system coincides with the component-wise virtual ages when the components have i.i.d. exponential lifetimes, while such a property is not generally valid even in the i.i.d. case when the component distribution is not exponential.

As noted by the authors in the Introduction of their article, both minimal and perfect repairs have clear "physical" meanings in imperfect repair modeling involving virtual ages. Bedford and Lindqvist (2004) studied a series system with n components, where at each failure, the failing component is replaced by a new one while the other components are left as they are. Thus one can consider that, at each failure, one component (the failed one) is perfectly repaired and hence its individual virtual age is set to 0, while the virtual ages of the other components are unchanged. This results in an n-dimensional vector $\mathbf{V}(t) = (V_1(t), \ldots, V_n(t))$ of virtual ages, where at each failure of the series system, one of the component ages is set to 0, while the other are kept unchanged. Between failures, each component ages according to calendar time. $\mathbf{V}(t)$ is hence a vector of "justifiable" virtual ages according to the above.

The main result of Bedford and Lindqvist (2004) is that the components' virtual ages in the long run will mix in such a manner that the individual failure rates for each component can be estimated for any given vector of virtual ages for all components. This model is a special case of a more general model for multivariate virtual ages considered by Lindqvist (2006).

4 Virtual age and heterogeneity

The authors finally discuss the virtual age concept assuming heterogeneous populations of items, demonstrating an apparent ambiguity in the treatment of heterogeneity in connection with virtual age. It seems that this ambiguity comes from the somewhat understated assumption of (FC1), that there is a conditioning on $T > t^*$ behind this formula. A precise description of the situation leading to (FC1) might be as follows.

Consider an item with life distribution function $F(t)$ and corresponding failure rate $\lambda(t)$, monitored from time 0 and until failure at time T. Let $t^* > 0$ be an apriori given time such that, if the item is still working at time t^*, then a maintenance is performed, reducing the age of the item to

$\tau \in (0, t^*)$. The (unconditional) survival function of the resulting lifetime T is then found to be

$$P(T > t) = \begin{cases} \exp\{-\int_0^t \lambda(u)du\} & ; \ t \le t^* \\ \exp\{-[\int_0^{t^*} \lambda(u)du + \int_\tau^{\tau+t-t^*} \lambda(u)du)]\} & ; \ t > t^* \end{cases} . \tag{2}$$

Equation (FC1) now follows from

$$\begin{aligned} P(T > t|T > t^*) &= \frac{P(T > t)}{P(T > t^*)} = \exp\left\{-\int_\tau^{\tau+t-t^*} \lambda(u)du)\right\} \\ &= \frac{F(\tau + t - t^*)}{F(\tau)} \qquad \text{for } t > t^*. \end{aligned} \tag{3}$$

Assume now that, conditional on a positive frailty variable Z, the failure rate of the item is $Z\lambda(t)$. Then $P(T > t) = E[P(T > t|Z)]$, so we get from (2),

$$P(T > t) = \begin{cases} E[\exp\{-Z \int_0^t \lambda(u)du\}] & ; \ t \le t^* \\ E[\exp\{-Z[\int_0^{t^*} \lambda(u)du + \int_\tau^{\tau+t-t^*} \lambda(u)du]\}] & ; \ t > t^* \end{cases} . \tag{4}$$

Consequently, for $t > t^*$,

$$P(T > t|T > t^*) = \frac{E[\exp\{-Z[\int_0^{t^*} \lambda(u)du + \int_\tau^{\tau+t-t^*} \lambda(u)du]\}]}{E[\exp\{-Z \int_0^{t^*} \lambda(u)du\}]} \tag{5}$$

which is exactly formula (FC26).

The erroneous formula (FC25) is based on (3), which involves a cancellation of the term $\exp\{-\int_0^{t^*} \lambda(u)du\}$ at the second equality sign. Such a cancellation is, however, not valid under the above frailty calculations. This is clearly seen from (5), where we cannot cancel the term $\exp\{-Z \int_0^{t^*} \lambda(u)du\}$ in the numerator and denominator unless Z is a constant.

I agree with the authors that the proper consideration of unobserved heterogeneities among otherwise identical items, is important. Still, as noted in the article, they are often ignored in practical imperfect repair analyses and so far they are not much studied in connection with virtual age models. Thus it is interesting to note a very recent paper, where Liu et al. (2020) study the effect of ignoring an unobserved heterogeneity in the failure process of a repairable system that undergoes imperfect repairs. They consider in particular the so called ARA$_\infty$ (Doyen and Gaudoin, 2004) process and consider a limiting state of this process.

5 Final remarks

The topic of imperfect repair and maintenance is of great importance and will certainly be so also in the future. The authors are therefore encouraged to continue their research in this field. While their article is concerned with various notions of virtual age, there are clearly many related unresolved issues to be considered. In my discussion, I have not been able to cover all aspects that are treated in the article. I think though that representations of virtual age-like measures connected with degradation processes and shock models may prove to be useful in applications. Connecting such approaches to recurrent events models will then be of particular interest.

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