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A converse to the Schwarz lemma for planar harmonic maps [☆]

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ABSTRACT

A sharp version of a recent inequality of Kovalev and Yang on the ratio of the $(H^1)^*$ and H^4 norms for certain polynomials is obtained. The inequality is applied to establish a sharp and tractable sufficient condition for the Wirtinger derivatives at the origin for harmonic self-maps of the unit disc which fix the origin.

1. Introduction

Set $\mathbb{D} := \{z \in \mathbb{C} \,:\, |z| < 1\}$. Suppose that $f\colon \mathbb{D} \to \mathbb{D}$ is harmonic and that $f(0) = 0$. Harmonic functions satisfy Laplace's equation $\Delta f = 0$, and we write $\Delta := \partial\overline{\partial}$ for the Wirtinger derivatives

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \overline{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

In analogy with the classical Schwarz lemma for analytic functions, the quantities $|\partial f(0)|$ and $|\overline{\partial} f(0)|$ are of intrinsic interest.

Set $\mathbb{T}^2 := \{z \in \mathbb{C}^2 \,:\, |z_1| = |z_2| = 1\}$ and let m_2 denote its Haar measure. Every f in $L^p(\mathbb{T}^2)$ can be represented as a Fourier series $f(z) = \sum_{\alpha\in\mathbb{Z}^2} \widehat{f}(\alpha)\, z^\alpha$, where the Fourier coefficients are given by

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$$\widehat{f}(\alpha) = \int_{\mathbb{T}^2} f(z)\, \overline{z^\alpha}\, dm_2(z).$$

The Hardy space $H^p(\mathbb{T}^2)$ is the subspace of $L^p(\mathbb{T}^2)$ comprised of functions f such that $\widehat{f}(\alpha) = 0$ unless both $\alpha_1, \alpha_2 \geq 0$.

A (slightly reformulated) recent result of Kovalev and Yang [5, Thm. 1.1] gives a description of the Wirtinger derivatives of harmonic self-maps of $\mathbb{D}$ fixing the origin in terms of the norm of certain linear functionals on $H^1(\mathbb{T}^2)$.

Theorem 1 *(Kovalev–Yang). Given (α, β) in $\mathbb{C}^2$, the following are equivalent.*

(i) *There is a harmonic $f\colon \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$, $\partial f(0) = \alpha$ and $\overline{\partial} f(0) = \beta$.*
(ii) *$\|\varphi\|_{(H^1(\mathbb{T}^2))^*} \leq 1$ for $\varphi(z) = \alpha z_1 + \beta z_2$.*

Condition (ii) is in general very difficult to check, so more tractable necessary and sufficient conditions in terms of the modulus of α and β are desirable. As explained in [5, Remark 4.2], the following necessary condition follows from Theorem 1. If $f\colon \mathbb{D} \to \mathbb{D}$ is harmonic with $f(0) = 0$, $\partial f(0) = \alpha$ and $\overline{\partial} f(0) = \beta$, then

$$\frac{|\alpha| + |\beta|}{2} \leq \frac{2}{\pi}. \tag{1}$$

The goal of the present note is to obtain a sharp and tractable sufficient condition similar to (1).

Theorem 2. *If $(\alpha, \beta) \in \mathbb{C}^2$ satisfies*

$$\left(\frac{|\alpha|^4 + 4|\alpha\beta|^2 + |\beta|^4}{6}\right)^{\frac{1}{4}} \leq \frac{2}{\pi}, \tag{2}$$

then there is a harmonic $f\colon \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$, $\partial f(0) = \alpha$ and $\overline{\partial} f(0) = \beta$.

Remark. If $\alpha = \beta$, then the necessary and sufficient conditions (1) and (2) coincide, which illustrates that the constant $2/\pi$ cannot be improved in either inequality. This can also be deduced directly by considering the harmonic function

$$f(z) = c\, \mathrm{Arg}\left(\frac{i - z}{i + z}\right)$$

which maps $\mathbb{D}$ to itself if and only if $|c| \leq 2/\pi$ and which satisfies $|\alpha| = |\beta| = |c|$.

The sufficient condition of Theorem 2 with $2/\pi = 0.6366\ldots$ replaced by the smaller constant $5/(3 + 2\sqrt{6}) = 0.6329\ldots$ can be obtained by combining Theorem 1 and [5, Thm. 1.2]. We similarly obtain Theorem 2 after establishing the following sharp version of [5, Thm. 1.2].

Theorem 3. *Suppose that $\varphi(z) = c_1 z_1 + c_2 z_2$ for $(c_1, c_2) \neq (0, 0)$. Then*

$$1 \leq \frac{\|\varphi\|_{(H^1(\mathbb{T}^2))^*}}{\|\varphi\|_{H^4(\mathbb{T}^2)}} \leq \frac{\pi}{2\sqrt[4]{6}} = 1.0035\ldots. \tag{3}$$

Moreover,

(a) *the upper bound is sharp in the sense that (3) no longer holds if $\frac{\pi}{2\sqrt[4]{6}}$ is replaced by any smaller number.*

(b) *the lower bound is sharp in the sense that* (3) *no longer holds if* $\|\varphi\|_{H^4(\mathbb{T}^2)}$ *is replaced by* $\|\varphi\|_{H^p(\mathbb{T}^2)}$ *for any* $p > 4$.

Comparing Theorem 3 and [5, Thm. 1.2], one finds that the novelty of our result is the sharp upper bound in (3) and the statements (a) and (b). For the sake of completeness (and since it does not require much additional effort), we will also include a proof of the lower bound in (3) in our exposition.

The sharp upper bound in (3) is obtained after replacing an estimate due to Ramanujan on the complete elliptic integral (see [5, pp. 6–7]) with certain explicit expressions obtained using the Hahn–Banach theorem. As in [5], some polynomial estimates are required as well. Part (b) of Theorem 3 is also a consequence of the Hahn–Banach theorem along with a counter-example to a related problem from [6].

Organization. In Section 2, some preliminary results pertaining to $H^p(\mathbb{T}^2)$ are compiled. Section 3 is devoted to the proof of Theorem 3. Some related work is also discussed.

2. Preliminaries

We require certain basic properties of $H^p(\mathbb{T}^2)$. Our aim is that our note be self-contained, so we refer to broadly to the monographs [4,7]. Suppose that φ is an analytic polynomial. The bounded linear functional generated by φ on $H^p(\mathbb{T}^2)$ is

$$L_\varphi(f) := \langle f, \varphi\rangle. \tag{4}$$

In (4) and in what follows, the inner product will always denote that of $L^2(\mathbb{T}^2)$. Suppose that $1 \le p < \infty$. We view φ in (4) as an element in $(H^p(\mathbb{T}^2))^*$. Hence

$$\|\varphi\|_{(H^p(\mathbb{T}^2))^*} := \sup_{f\in H^p(\mathbb{T}^2)} \frac{|\langle f,\varphi\rangle|}{\|f\|_{H^p(\mathbb{T}^2)}} = \frac{\langle g,\varphi\rangle}{\|g\|_{H^p(\mathbb{T}^2)}} \tag{5}$$

for some g in $H^p(\mathbb{T}^2)$ with $\langle g, \varphi\rangle \ge 0$.

By the Hahn–Banach theorem, L_φ extends to a bounded linear functional on $L^p(\mathbb{T}^2)$ with the same norm. Every functional on $L^p(\mathbb{T}^2)$ is of the form

$$L_\psi(f) := \langle f, \psi\rangle$$

where ψ is in $L^q(\mathbb{T}^2)$ for $1/p+1/q = 1$. Since the bounded linear functional L_ψ extends the bounded linear functional L_φ, we must have $\|\varphi\|_{(H^p(\mathbb{T}^2))^*} = \|\psi\|_{L^q(\mathbb{T}^2)}$ and $P\psi = \varphi$, where P is the orthogonal projection (Riesz projection) from $L^2(\mathbb{T}^2)$ to $H^2(\mathbb{T}^2)$.

In particular, we get from (5) that

$$\|\varphi\|_{(H^p(\mathbb{T}^2))^*} = \frac{\langle g,\varphi\rangle}{\|g\|_{H^p(\mathbb{T}^2)}} = \frac{\langle g,\psi\rangle}{\|g\|_{L^p(\mathbb{T}^2)}} = \|\psi\|_{L^q(\mathbb{T}^2)}.$$

From the rightmost equality and Hölder's inequality, we see that $|g|^{p-2}g = C\psi$ for some constant $C > 0$. Taking the Riesz projection, we conclude that

$$P\big(|g|^{p-2}g\big) = C\varphi \tag{6}$$

if and only if φ and g are related as in (5).

Let $H^p_1(\mathbb{T}^2)$ be the two-dimensional subspace of $H^p(\mathbb{T}^2)$ consisting of functions $f(z) = az_1 + bz_2$ for (a, b) in $\mathbb{C}^2$. The orthogonal projection $P_1\colon H^2(\mathbb{T}^2) \to H^2_1(\mathbb{T}^2)$ extends to a contraction on $H^p(\mathbb{T}^2)$ for every $1 \le p \le \infty$. This claim can be easily deduced from the integral representation

$$P_1 f(z) = \int_0^{2\pi} f(e^{i\theta} z_1, e^{i\theta} z_2)\, e^{-i\theta}\, \frac{d\theta}{2\pi}. \tag{7}$$

Suppose that $\varphi(z) = c_1 z_1 + c_2 z_2$ for $(c_1, c_2) \neq (0, 0)$ and that f is in $H^p(\mathbb{T}^2)$. Then

$$\frac{|\langle f, \varphi\rangle|}{\|f\|_{H^p(\mathbb{T}^2)}} = \frac{|\langle P_1 f, \varphi\rangle|}{\|f\|_{H^p(\mathbb{T}^2)}} \le \frac{|\langle P_1 f, \varphi\rangle|}{\|P_1 f\|_{H^p(\mathbb{T}^2)}}.$$

Hence, it is clear that $\|\varphi\|_{(H^p(\mathbb{T}^2))^*} = \|\varphi\|_{(H^p_1(\mathbb{T}^2))^*}$. Moreover, the optimal g in (5), and equivalently any solution of (6), is of the form $g(z) = az_1 + bz_2$.

We will next establish three results needed in the proof of Theorem 3. The first lemma shows that we may swap 1 and 4 in the ratio appearing in Theorem 3 when considering optimal lower and upper bounds. Here and elsewhere, we use the notation $\varphi_y(z) := z_1 + yz_2$, with the presumption that $0 \le y \le 1$.

Lemma 4. *Suppose that $\varphi(z) = c_1 z_1 + c_2 z_2$ for $(c_1, c_2) \neq (0, 0)$. The estimates*

$$\inf_{0\le y\le 1} \frac{\|\varphi_y\|_{(H^4(\mathbb{T}^2))^*}}{\|\varphi_y\|_{H^1(\mathbb{T}^2)}} \le \frac{\|\varphi\|_{(H^1(\mathbb{T}^2))^*}}{\|\varphi\|_{H^4(\mathbb{T}^2)}} \le \sup_{0\le y\le 1} \frac{\|\varphi_y\|_{(H^4(\mathbb{T}^2))^*}}{\|\varphi_y\|_{H^1(\mathbb{T}^2)}}$$

are both attained.

Proof. Recall that if $\varphi(z) = c_1 z_1 + c_2 z_2$, then

$$\frac{\|\varphi\|_{(H^1(\mathbb{T}^2))^*}}{\|\varphi\|_{H^4(\mathbb{T}^2)}} = \frac{\|\varphi\|_{(H^1_1(\mathbb{T}^2))^*}}{\|\varphi\|_{H^4_1(\mathbb{T}^2)}}.$$

The point of this reformulation is that the two-dimensional space $H^1_1(\mathbb{T}^2)$ is reflexive (while $H^1(\mathbb{T}^2)$ is not). Considering the identity operator $I\colon H^4_1(\mathbb{T}^2) \to (H^1_1(\mathbb{T}^2))^*$ and using duality, we find that

$$\sup_{(c_1,c_2)\neq(0,0)} \frac{\|\varphi\|_{(H^1_1(\mathbb{T}^2))^*}}{\|\varphi\|_{H^4_1(\mathbb{T}^2)}} = \sup_{(c_1,c_2)\neq(0,0)} \frac{\|\varphi\|_{(H^4_1(\mathbb{T}^2))^*}}{\|\varphi\|_{H^1_1(\mathbb{T}^2)}}.$$

To see that the same statement holds with sup replaced by inf, consider instead $I\colon (H^1_1(\mathbb{T}^2))^* \to H^4_1(\mathbb{T}^2)$. Hence, we may equivalently investigate sharp upper and lower bounds for the ratio

$$\frac{\|\varphi\|_{(H^4_1(\mathbb{T}^2))^*}}{\|\varphi\|_{H^1_1(\mathbb{T}^2)}} = \frac{\|\varphi\|_{(H^4(\mathbb{T}^2))^*}}{\|\varphi\|_{H^1(\mathbb{T}^2)}}.$$

Set $\varphi(z) = c_1 z_1 + c_2 z_2$ for some $(c_1, c_2) \neq (0, 0)$. By the rotational invariance of the Haar measure m_2, we may assume that $c_1, c_2 \ge 0$. By symmetry, we may also assume that $c_1 \ge c_2$ so $c_1 > 0$. Dividing φ by a non-zero constant does not change the ratio, so with $y = c_2/c_1$, which satisfies $0 \le y \le 1$, we obtain

$$\frac{\|\varphi\|_{(H^4(\mathbb{T}^2))^*}}{\|\varphi\|_{H^1(\mathbb{T}^2)}} = \frac{\|\varphi_y\|_{(H^4(\mathbb{T}^2))^*}}{\|\varphi_y\|_{H^1(\mathbb{T}^2)}}.$$

Sharp upper and lower bounds are therefore obtained by taking the supremum and infimum, respectively, over $0 \le y \le 1$. $\square$

The second result readily demonstrates the virtue of the first lemma, since equation (6) is easy to solve explicitly for φ_y when $p = 4$.

Lemma 5. *Given $0 \leq y \leq 1$, let $0 \leq x \leq 1$ be the unique real number such that*

$$y = \sqrt{x}\frac{2+x}{1+2x},$$

and set $g(z) = z_1 + \sqrt{x}z_2$. Then $P\big(|g|^2 g\big) = (1+2x)\varphi_y$.

Proof. Since $|g(z)|^2 = 1 + x + \sqrt{x}(z_1\overline{z_2} + \overline{z_1}z_2)$, we find that

$$P(|g|^2 g)(z) = (1+x)g(z) + xz_1 + \sqrt{x}z_2 = (1+2x)\left(z_1 + \sqrt{x}\frac{2+x}{1+2x}z_2\right).$$

The proof is completed by checking that $y(x) = \sqrt{x}(2+x)/(1+2x)$ is an increasing function on $0 \leq x \leq 1$. $\square$

We require the third lemma only for $p = 1$, but we state and prove it in the general case since it requires no additional effort. Note that $\binom{1}{1/2} = 4/\pi$, which explains the appearance of π in Theorem 3.

Lemma 6. *For $1 \leq p < \infty$, we have the identities*

$$\binom{p}{p/2} = \|z_1 + z_2\|_{H^p(\mathbb{T}^2)}^p = \sum_{j=0}^{\infty} \binom{p/2}{j}^2 = \frac{4}{p}\sum_{j=0}^{\infty}\binom{p/2}{j}^2 j. \tag{8}$$

Proof. The proof relies on expressing $\|z_1 + xz_2\|_{H^p(\mathbb{T}^2)}^p$, for $0 \leq x \leq 1$ in two different ways. First, we note that

$$\|z_1 + xz_2\|_{H^p(\mathbb{T}^2)}^p = \|(1 + x\overline{z_1}z_2)^2\|_{L^{p/2}(\mathbb{T}^2)}^{2/p} = \int_0^{2\pi} \big(1 + 2x\cos(\theta) + x^2\big)^{p/2}\,\frac{d\theta}{2\pi}. \tag{9}$$

Setting $x = 1$, we obtain the first equality in (8) from a well-known integral formula for the beta function (see e.g. [3, Sec. 9.3]),

$$\int_0^{2\pi} (2 + 2\cos(\theta))^{p/2}\,\frac{d\theta}{2\pi} = \frac{2^{p+1}}{\pi}\int_0^{\pi/2} (\cos\theta)^p\,d\theta = \frac{2^p}{\pi}\,\mathrm{B}\left(\frac{p+1}{2}, \frac{1}{2}\right) = \binom{p}{p/2}.$$

Second, we expand

$$(1 + x\overline{z_1}z_2)^{p/2} = \sum_{j=0}^{\infty}\binom{p/2}{j}\left(\overline{z_1}z_2\right)^j x^j.$$

Consequently, Parseval's identity shows that

$$\|z_1 + xz_2\|_{H^p(\mathbb{T}^2)}^p = \|(1 + x\overline{z_1}z_2)^{p/2}\|_{L^2(\mathbb{T}^2)}^2 = \sum_{j=0}^{\infty}\binom{p/2}{j}^2 x^{2j}. \tag{10}$$

Setting $x = 1$ in (10), we obtain the second equality in (8). For the third equality in (8), we differentiate the expressions (9) and (10) with respect to x to obtain

$$\sum_{j=0}^{\infty} \binom{p/2}{j}^2 2jx^{2j-1} = \frac{p}{2} \int_0^{2\pi} \left(1 + 2x\cos(\theta) + x^2\right)^{p/2-1} \left(2\cos(\theta) + 2x\right) \frac{d\theta}{2\pi}.$$

Setting $x = 1$ and using (9) yet again, we obtain the third equality in (8). $\square$

We close the present section by explaining the connection between Theorem 2 and Theorem 3.

Proof of Theorem 2. Suppose that $\varphi(z) = \alpha z_1 + \beta z_2$. By using the upper bound of Theorem 3, we see that

$$\frac{\|\varphi\|_{H^4(\mathbb{T}^2)}}{\sqrt[4]{6}} \leq \frac{2}{\pi} \qquad \Longrightarrow \qquad \|\varphi\|_{(H^1(\mathbb{T}^2))^*} \leq 1.$$

The proof is completed by computing $\|\varphi\|_{H^4(\mathbb{T}^2)} = \left(|\alpha|^4 + 4|\alpha\beta|^2 + |\beta|^4\right)^{1/4}$ and appealing to Theorem 1. $\square$

3. Proof of Theorem 3

We will start from Lemma 4. Let F be a polynomial that is strictly positive for $0 \leq y \leq 1$, and write

$$\frac{\|\varphi_y\|_{(H^4(\mathbb{T}^2))^*}}{\|\varphi_y\|_{H^1(\mathbb{T}^2)}} = \frac{\|\varphi_y\|_{(H^4(\mathbb{T}^2))^*}}{F(y)} \frac{F(y)}{\|\varphi_y\|_{H^1(\mathbb{T}^2)}}. \tag{11}$$

Our idea is to choose F in such a way that we can treat the two fractions on the right hand side of (11) independently. We begin with the most technical part of the proof, which pertains to the second fraction.

Lemma 7. *The function*

$$\frac{1 + \frac{y^2}{4} + \frac{y^4}{64} + \frac{y^6}{256} + cy^8}{\|\varphi_y\|_{H^1(\mathbb{T}^2)}} \tag{12}$$

is increasing on $0 \leq y \leq 1$ *if and only if* $c \geq \frac{5}{768}$.

Proof. Set $\xi = y^2$. Recall from the proof of Lemma 6 that

$$\|\varphi_y\|_{H^1(\mathbb{T}^2)} = \sum_{j=0}^{\infty} \binom{1/2}{j}^2 \xi^j = 1 + \frac{\xi}{4} + \frac{\xi^2}{64} + \frac{\xi^3}{256} + \sum_{j=4}^{\infty} \binom{1/2}{j}^2 \xi^j.$$

We will divide by ξ^4 upstairs and downstairs in (12) and equivalently investigate

$$R(\xi) := \frac{\Sigma_1(\xi) + c}{\Sigma_1(\xi) + \Sigma_2(\xi)}$$

where

$$\Sigma_1(\xi) := \sum_{j=0}^{3} \binom{1/2}{j}^2 \xi^{j-4} \qquad \text{and} \qquad \Sigma_2(\xi) := \sum_{j=4}^{\infty} \binom{1/2}{j}^2 \xi^{j-4}.$$

We begin by computing

$$\Sigma_1(1) = \frac{325}{256}, \qquad \Sigma_1'(1) = -\frac{1225}{256}, \qquad \Sigma_2(1) = \frac{4}{\pi} - \frac{325}{256}. \tag{13}$$

The two first are direct computations, while the last uses Lemma 6 to obtain that

$$\Sigma_1(1)+\Sigma_2(1)=\sum_{j=0}^{\infty}\binom{1/2}{j}^2=\binom{1}{1/2}=\frac{4}{\pi}.$$

It is clear that Σ_1 is positive and decreasing on $0\le y\le 1$ and that Σ_2 is positive and increasing on $0\le y\le 1$. Differentiating term by term, we find that each summand is maximized when $y=1$. Hence

$$\big(\Sigma_1(\xi)+\Sigma_2(\xi)\big)'\le \Sigma_1'(1)+\Sigma_2'(1)=\sum_{j=0}^{\infty}\binom{1/2}{j}^2(j-4)=-\frac{15}{\pi}.$$

In the final equality, we used Lemma 6 twice. Hence we find that $\Sigma_1'+\Sigma_2'$ is negative for $0<\xi\le 1$ and that

$$\Sigma_2'(1)=\frac{1225}{256}-\frac{15}{\pi}. \tag{14}$$

We want to find a requirement on c such that $R'(\xi)\ge 0$ for $0\le\xi\le 1$. Note that

$$0\le R'=\frac{\Sigma_1'(\Sigma_1+\Sigma_2)-(\Sigma_1+c)(\Sigma_1'+\Sigma_2')}{(\Sigma_1+\Sigma_2)^2}\qquad\Longleftrightarrow\qquad c\ge\frac{\Sigma_1'\Sigma_2-\Sigma_1\Sigma_2'}{\Sigma_1'+\Sigma_2'}, \tag{15}$$

where we used that $\Sigma_1'(\xi)+\Sigma_2'(\xi)<0$ for $0<\xi\le 1$. If we could prove that the right-hand side of (15) is increasing on the interval $0<\xi\le 1$, then we would get the stated requirement on c by (13) and (14), since

$$\frac{\Sigma_1'(1)\Sigma_2(1)-\Sigma_1(1)\Sigma_2'(1)}{\Sigma_1'(1)+\Sigma_2'(1)}=\frac{-\frac{1225}{256}\left(\frac{4}{\pi}-\frac{325}{256}\right)-\frac{325}{256}\left(\frac{1225}{256}-\frac{15}{\pi}\right)}{-\frac{15}{\pi}}=\frac{5}{768}.$$

To prove that the right-hand side of (15) is increasing on $0<\xi\le 1$, we begin by rewriting it as

$$\frac{\Sigma_1'\Sigma_2-\Sigma_1\Sigma_2'}{\Sigma_1'+\Sigma_2'}=\frac{\Sigma_2-\Sigma_2'\frac{\Sigma_1}{\Sigma_1'}}{1+\frac{\Sigma_2'}{\Sigma_1'}}. \tag{16}$$

Note that

$$0\le -\frac{d}{d\xi}\frac{\Sigma_1(\xi)}{\Sigma_1'(\xi)}\qquad\Longleftrightarrow\qquad 0\le \Sigma_1''(\xi)\Sigma_1(\xi)-(\Sigma_1'(\xi))^2.$$

The second statement can be checked directly because

$$\Sigma_1''(\xi)\Sigma_1(\xi)-(\Sigma_1'(\xi))^2=\frac{4}{\xi^{10}}+\frac{2}{\xi^9}+\frac{11}{32\xi^8}+\frac{5}{64\xi^7}+\frac{17}{2048\xi^6}+\frac{1}{4096\xi^5}+\frac{1}{65536\xi^4}.$$

Hence $-\Sigma_1/\Sigma_1'$ is positive and increasing. Since both Σ_2 and Σ_2' are positive and increasing, we find that the numerator on the right-hand side of (16) is increasing. Since Σ_1 is positive and decreasing and $-\Sigma_2'\Sigma_1/\Sigma_1'$ is positive and increasing, we conclude that $-\Sigma_2'/\Sigma_1'$ is positive and increasing. Consequently, the denominator on the right-hand side of (16) is decreasing. $\square$

We will use the polynomial

$$F(y):=1+\frac{y^2}{4}+\frac{y^4}{64}+\frac{y^6}{256}+\frac{5}{768}y^8$$

in (11).

Remark. We can establish that $\|\varphi_y\|_{H^1(\mathbb{T}^2)} \leq F(y)$ for $0 \leq y \leq 1$, similarly to how the inequality

$$\|\varphi_y\|_{H^1(\mathbb{T}^2)} \leq 1 + \frac{y^2}{4} + \frac{y^4}{64} + \frac{y^6}{128} =: \widetilde{F}(y)$$

is proved in [5, p. 7]. The latter estimate is sharper for $3/5 \leq y^2 \leq 1$, but the statement of Lemma 7 does not hold if F is replaced by $\widetilde{F}$.

By Lemma 7, we know that

$$\frac{F(0)}{\|\varphi_0\|_{H^2(\mathbb{T}^2)}} \leq \frac{F(y)}{\|\varphi_y\|_{H^1(\mathbb{T}^2)}} \leq \frac{F(1)}{\|\varphi_1\|_{H^2(\mathbb{T}^2)}}, \tag{17}$$

so it remains to verify that the analogous estimates hold for

$$\frac{\|\varphi_y\|_{(H^4(\mathbb{T}^2))^*}}{F(y)}. \tag{18}$$

Recalling that (5) and (6) are equivalent, we invoke Lemma 5 to see that

$$\|\varphi_y\|_{(H^4(\mathbb{T}^2))^*} = \frac{\langle g, \varphi_y\rangle}{\|g\|_{H^4(\mathbb{T}^2)}} = \frac{1+\sqrt{x}y}{(1+4x+x^2)^{1/4}} = \frac{\left(1+4x+x^2\right)^{3/4}}{1+2x}, \tag{19}$$

where $y = \sqrt{x}(2+x)/(1+2x)$. By (18) and (19), it is equivalent to consider

$$\left(\frac{(1+2x)^7\|\varphi_y\|_{(H^4(\mathbb{T}^2))^*}}{(1+2x)^7 F(y)}\right)^4 = \frac{(1+2x)^{28}(1+4x+x^2)^3}{(1+2x)^{32}(F(y))^4} =: \frac{P(x)}{Q(x)}. \tag{20}$$

Lemma 8. *Let P and Q be as in (20). Then*

$$\frac{P(0)}{Q(0)} \leq \frac{P(x)}{Q(x)} \leq \frac{P(1)}{Q(1)}.$$

Proof. To prove the upper bound, it is equivalent to verify that

$$R_1(x) := \frac{P(1)Q(x) - Q(1)P(x)}{(1-x)^2}$$

is non-negative for $0 \leq x \leq 1$. We claim that R_1 is a polynomial of degree 46 with positive coefficients. Hence $R_1(x) \geq 0$.

To prove the lower bound, we note that since $P(0) = Q(0) = 1$, it is equivalent to verify that $R_2(x) := P(x) - Q(x)$ is non-negative for $0 \leq x \leq 1$. Here we claim that R_2 is a polynomial of degree 48 for which the first 28 coefficients are positive and the rest are negative. Moreover, we claim that $R_2(1) > 0$. By comparing coefficients and using that $x^j \geq x^k$ for $0 \leq x \leq 1$ and $0 \leq j \leq k$, we deduce from this that $R_2(x) \geq 0$ for $0 \leq x \leq 1$.

The claims on R_1 and R_2 can be easily verified using a computer algebra system. We checked them using `Maple` and `Mathematica`. □

Returning to (18) and recalling that $y = y(x)$ is increasing from $y(0) = 0$ to $y(1) = 1$, we get from (19), (20) and Lemma 8 that

$$\frac{\|\varphi_0\|_{H^4(\mathbb{T}^2)}}{F(0)} \leq \frac{\|\varphi_y\|_{(H^4(\mathbb{T}^2))^*}}{F(y)} \leq \frac{\|\varphi_1\|_{H^4(\mathbb{T}^2)}}{F(1)}. \tag{21}$$

Final part in the proof of Theorem 3. We begin with the proof of the estimates (3). By Lemma 4, (11), (17), and (21), we obtain

$$\frac{\|\varphi_0\|_{(H^4(\mathbb{T}^2))^*}}{\|\varphi_0\|_{H^1(\mathbb{T}^2)}} \le \frac{\|\varphi\|_{(H^1(\mathbb{T}^2))^*}}{\|\varphi\|_{H^4(\mathbb{T}^2)}} \le \frac{\|\varphi_1\|_{(H^4(\mathbb{T}^2))^*}}{\|\varphi_1\|_{H^1(\mathbb{T}^2)}},$$

where as before $\varphi_y(z) = z_1 + yz_2$ and $\varphi(z) = c_1z_1 + c_2z_2$ for arbitrary $(c_1, c_2) \neq (0, 0)$. These estimates are evidently sharp and the lower bound is equal to 1. To obtain a numerical value for the upper bound, we first get $\|\varphi_1\|_{H^1(\mathbb{T}^2)} = 4/\pi$ from Lemma 6. Next we use (19), recalling that $y = 1$ corresponds to $x = 1$, to establish that $\|\varphi_1\|_{(H^4(\mathbb{T}^2))^*} = 6^{3/4}/3$. Hence

$$\frac{\|\varphi_1\|_{(H^1(\mathbb{T}^2))^*}}{\|\varphi_1\|_{H^4(\mathbb{T}^2)}} = \frac{\pi}{2\sqrt[4]{6}},$$

and so the proof of (3) and part (a) is complete.

It remains to settle (b). Suppose that the estimate

$$\|\varphi\|_{H^p(\mathbb{T}^2)} \le \|\varphi\|_{(H^1(\mathbb{T}^2))^*} \tag{22}$$

holds for every $\varphi(z) = c_1z_1 + c_2z_2$ for some $p \ge 4$. By the Hahn–Banach theorem and the fact that $(L^1(\mathbb{T}^2))^* = L^\infty(\mathbb{T}^2)$, we get that

$$\|\varphi\|_{(H^1(\mathbb{T}^2))^*} = \inf_{P\psi=\varphi} \|\psi\|_{L^\infty(\mathbb{T}^2)}. \tag{23}$$

For $0 \le \varepsilon < 1$, consider

$$\psi(z) := z_2\frac{(1-\varepsilon z_1\overline{z_2})^2}{|1-\varepsilon z_1\overline{z_2}|^2} = z_2\left(-\varepsilon z_1\overline{z_2} + \left(1-\varepsilon^2\right)\sum_{j=0}^{\infty}\varepsilon^j(\overline{z_1}z_2)^j\right).$$

Clearly $\|\psi\|_{L^\infty(\mathbb{T}^2)} = 1$. Moreover, the Riesz projection of ψ is

$$\varphi(z) := P\psi(z) = -\varepsilon z_1 + \left(1-\varepsilon^2\right)z_2.$$

If $\varepsilon > 0$ is so small that $\varepsilon \le 1-\varepsilon^2$, then

$$\|\varphi\|_{H^p(\mathbb{T}^2)} = \left(1-\varepsilon^2\right)\left(\sum_{j=0}^{\infty}\binom{p/2}{j}^2\left(\frac{\varepsilon}{1-\varepsilon^2}\right)^{2j}\right)^{\frac{1}{p}} = 1 + \left(\frac{p}{4}-1\right)\varepsilon^2 + O(\varepsilon^4).$$

Hence we can obtain a contradiction to (22) from (23) whenever $p > 4$ by choosing $\varepsilon > 0$ sufficiently small. We conclude that part (b) also is true. □

The proof of Theorem 3 (b) is adapted from the proof of a result of Marzo and Seip [6, Thm. 1], which we shall now recall. Let $P\colon L^2(\mathbb{T}) \to H^2(\mathbb{T})$ denote the Riesz projection on $\mathbb{T}$. The inequality

$$\|Pf\|_{H^p(\mathbb{T})} \le \|f\|_{L^\infty(\mathbb{T})} \tag{24}$$

holds for every f in $L^\infty(\mathbb{T})$ if and only if $p \le 4$. It is also demonstrated in [6] that (24) does not hold if $\mathbb{T}$ is replaced by $\mathbb{T}^2$ and $p = 4$.

Let P_1 denote the operator defined by (7). The space $P_1 L^\infty(\mathbb{T}^2)$ is comprised of essentially bounded functions on $\mathbb{T}^2$ whose Fourier coefficients are supported on the straight line $\alpha_1 + \alpha_2 = 1$ in $\mathbb{Z}^2$. The lower bound in Theorem 3 and its optimality can be restated as follows. The inequality

$$\|Pf\|_{H^p(\mathbb{T}^2)} \leq \|f\|_{L^\infty(\mathbb{T}^2)} \tag{25}$$

holds for every f in $P_1 L^\infty(\mathbb{T}^2)$ if and only if $p \leq 4$. Hence one can think of (25) as a partial analogue of (24) on $\mathbb{T}^2$.

Fix $1 \leq q \leq 2$. What is the largest $2 \leq p \leq 4$ such that

$$\|\varphi\|_{H^p(\mathbb{T}^2)} \leq \|\varphi\|_{(H^q(\mathbb{T}^2))^*}$$

holds for every $\varphi(z) = c_1 z_1 + c_2 z_2$?

By adapting the counter-example from [2, Thm. 9] similarly to how we adapted the counter-example from [6, Thm. 1] in the proof of Theorem 3 (b), the necessary condition $p \geq 4/q$ can be established. By the lower bound in Theorem 3, we know that this is sharp for $q = 1$. It is also trivially sharp for $q = 2$. Similarly, the answer to the following question is affirmative in the endpoint cases $q = 1, 2$.

Question. Fix $1 < q < 2$. Is it true that

$$1 \leq \frac{\|\varphi\|_{(H^q(\mathbb{T}^2))^*}}{\|\varphi\|_{H^{4/q}(\mathbb{T}^2)}} \leq 2\binom{q}{q/2}^{-1/q}\binom{4/q}{2/q}^{-q/4}$$

for every $\varphi(z) = c_1 z_1 + c_2 z_2$ with $(c_1, c_2) \neq (0, 0)$?

The upper bound in the question is obtained by setting $c_1 = c_2 = 1$. To compute the ratio in this case, we first use [1, Lem. 5] to see that

$$\|z_1 + z_2\|_{(H^q(\mathbb{T}^2))^*} = 2\|z_1 + z_2\|_{H^q(\mathbb{T}^2)}^{-1}$$

and then Lemma 6 twice.

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