On Orbital Stabilization as an alternative to Reference Tracking Control

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Abstract

With the purpose of highlighting the concept of orbital stabilization as an alternative to the reference tracking control methodology, this paper considers simple, informative examples in relation to motion control of an one-degree-of-freedom, double integrator system. In this regard, the notions of (excessive) transverse coordinates, projections operators and the transverse linearization are introduced, and it is illustrated how these can be used both for the design and analysis of orbitally stabilizing feedback controllers.

Index Terms

Trajectory tracking, orbital stabilization, transverse coordinates, transverse linearization.

I. Introduction

For certain motion control tasks, it is the closeness to the desired motion in the state space of the system which is of paramount importance, whereas the *timing* of the motion, that is, having the system in specific configurations at specific time instants, is of less importance (if important at all). For such tasks, the reference tracking control-methodology, in which the desired motion is represented by a time-varying reference signal which is to be tracked, may not be well suited. Indeed, since the timing-property necessarily will be an inherent part of any such control scheme, it follows that even if just slightly delayed, the system might be momentarily driven away from the nominal motion in order to instead "catch-up in time".

For example, consider a particle $q \in \mathbb{R}$ of unit mass, whose time-evolution is governed by the second-order equation

$$\ddot{q} = u. \tag{1}$$

Here $u \in \mathbb{R}$ is a control input and $\dot{q} = \frac{d}{dt}q$. Suppose we want the particle to move in a sinusoidal fashion with amplitude a and frequency ω, that is,

$$q_*(t) = a\sin(\omega t), \tag{2}$$

where the subscript "$*$" denotes the nominal, desired behaviour. Towards this end, we could define the tracking error $e(t) := q(t) - a\sin(\omega t)$ and utilize some regular reference tracking controller; for example, taking

$$u_{PD} = -a\omega^2 \sin(\omega t) - k_p e - k_d \dot{e} \tag{3}$$

results in the error dynamics $\ddot{e} = -k_p e - k_d \dot{e}$, which are stable for any $k_p, k_d > 0$.

However, the controller (3) has an explicit dependence on time due to the reference signal $r(t) = a\sin(\omega t)$ appearing in the computations of the error, e, its time derivative, as well as in the feedforward term. That is, the closed-loop system is time-varying, not autonomous. As a consequence, if we for instance start the system with, say, the initial conditions $(q_0, \dot{q}_0) = (0, -a\omega)$ the system will experience a "catch-up" effect as the initial velocity error is $\dot{e}_0 = -2a\omega$ even though it starts directly on the desired orbit, defined by

$$\eta_* = \left\{(q, \dot{q}) = (a\sin(\omega t), a\omega\cos(\omega t)),\ t \in [0, 2\pi/\omega)\right\}. \tag{4}$$

Indeed, as the analogy considered in [1] of a car driving on a curvy mountain road illustrates, this effect can potentially have serious consequences, as in its pursuit of catching-up in time from a delay,

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a tracking controller might actually end up cutting a corner and taking the car off the road! Thus, if our main concern is the convergence of the system's states to the orbit η_* rather than to a specific time instant along it, a tracking controller might not necessarily be the best choice.

As an alternative to reference tracking, we can instead try to find a completely state-dependent controller, i.e. $u = u(q, \dot{q})$, resulting in the corresponding autonomous closed-loop system admitting the periodic orbit η_* as an asymptotically stable limit cycle. Such a controller is said to be an *orbitally stabilizing controller* as it stabilizes the desired orbit (the set of all states along the nominal solution) as a whole, not just a particular time-leaf (moving) along it.

Although the concept of orbital stabilization has been well known for several decades,[1] its use in motion control is not wide spread at present. Recently, however, some methods have appeared in the literature which try to make it a more viable option for common robotics tasks. These include immersion and invariance-based approaches [3] and transverse linearization-based approaches [4], [5]. We will in this paper focus on the latter type of approach.

In this regard, orbital stabilization of periodic motions of a fully actuated robot manipulator was considered in [4] by stabilizing an excessive set of so-called *transverse coordinates*. This approach has recently been further extended in [5], [6] to underactuated systems, with the additional relaxation that the so-called *projection operators* – mappings recovering the current "position" along the nominal motion – only have to be state-dependent and not strictly configuration-based as in [4]. Moreover, new insights were there provided in regards to a useful tool for orbitally stabilizing feedback design, as well as for general analysis of orbital (in-)stability in the large: *the transverse linearization*.

In this paper, we consider the approach and analysis tools outlined [5], [6] in relation to the particle system (13). The simplicity of this system allows for constructive- and illustrative examples, both highlighting some advantageous and disadvantageous of orbitally stabilizing controllers in comparison to basic (PD+feedforward-) reference tracking controllers. Furthermore, it facilitates the demonstration of how analysis- and construction of new types orbitally stabilizing feedback controllers can be achieved using the transverse linearization-based tools provided in [5], [6].

A brief outline of the paper follows. In Section II we propose two novel (nonlinear) controllers for orbital stabilization of periodic oscillations of the particle system (see Propositions 1 and 2). A detailed analysis of the transverse linearization of an excessive set of transverse coordinates, eventually resulting in the proof of Proposition 2, is in then given in Section III. We then briefly demonstrate that orbital stabilization is not limited to only periodic motions in Section IV, before we end with some concluding remarks.

II. Orbitally Stabilizing Feedback Design

A. A single transverse coordinate

We begin by noting that the desired trajectory (2) is the solution to the following initial value problem (IVP):

$$\ddot{q} = -\omega^2 q, \quad (q(0), \dot{q}(0)) = (0, a\omega).$$

Thus, using the well-known identity $2\ddot{q} = \partial \dot{q}^2 / \partial q$, we can integrate the above IVP and rearrange it in order to obtain that the function $I : \mathbb{R}^2 \to \mathbb{R}$, defined as

$$I(q, \dot{q}) := \frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2(q^2 - a^2), \tag{5}$$

must be invariant along the solution. In fact, $I(\cdot)$ is a so-called *transverse coordinate*: it it vanishes on the orbit (4), i.e. $I(q_*, \dot{q}_*) \equiv 0$, is non-zero away from it, and its gradient, $DI = [\omega^2 q, \dot{q}]$, evaluated along the orbit can be seen to be orthogonal to the orbit's nominal flow. As a consequence, if we stabilize its origin, we simultaneously stabilize the orbit!

[1]The general notion of orbital (Poincaré) stability we consider here of course dates back to Poincaré [2] over a century ago.

Indeed, the following statement can be used in this regard.

Proposition 1: Given some Lipschitz continuous mapping $f : \mathbb{R} \to \mathbb{R}$, let $w(t) = w(t+T)$, $T > 0$, be the (bounded) periodic solution of the initial value problem

$$\ddot{w} = f(w), \quad (w(0), \dot{w}(0)) = (q_0, \dot{q}_0), \tag{6}$$

satisfying $|f(w(t))|^2 + |\dot{w}(t)|^2 > 0$ for all $t \in [0, T)$. Then for all $(q, \dot{q})$ within the orbit

$$\eta_* := \{(q, \dot{q}) = (w(t), \dot{w}(t)),\ t \in [0, T)\},$$

the scalar function $I = I(q, \dot{q})$, defined by

$$I := \frac{1}{2}\dot{q}^2 - \frac{1}{2}\dot{q}_0^2 - \int_{q_0}^{q} f(\sigma)d\sigma, \tag{7}$$

vanishes. Moreover, taking in (1) $u = u_I$, with

$$u_I := f(q) - k_I \mathrm{sgn}_+(\dot{q}) I,\ \mathrm{sgn}_+(\dot{q}) = \begin{cases} 1 & \text{if } \dot{q} \geq 0 \\ -1 & \text{if } \dot{q} < 0 \end{cases}, \tag{8}$$

for some $k_I > 0$, renders $q_*(t) = w(t)$ an asymptotically stable solution of the particle system (1) in the orbital sense.

Proof: If the controller is taken according to (8), straightforward computations show that $\dot{I} = -k_I |\dot{q}| I$. Defining the $\mathcal{C}^1$ Lyapunov function candidate $V_I := \frac{1}{2} I^2$, we therefore obtain $\dot{V}_I = -k_I |\dot{q}| I^2 \leq 0$, which is negative definite as long as $|\dot{q}| \neq 0$. From the relation $|f(w(t))|^2 + |\dot{w}(t)|^2 > 0$ and the continuity of $f(\cdot)$, it follows that there exists some nonzero neighbourhood of the orbit η_* in which $|\ddot{q}| > 0$ whenever $\dot{q} \equiv 0$. Indeed, if $\dot{q} \equiv 0$ and q is in η_*, then $\ddot{q} = f(q) + k_i(\frac{1}{2}\dot{q}_0^2 + \int_{q_0}^{q} f(\sigma)d\sigma) = \ddot{q}_*(t) + \frac{k_I}{2}\dot{q}_*(t)$. Hence, by LaSalle's invariance principle, we can conclude that the origin of I is locally asymptotically stable, which again implies that the orbit η_* is asymptotically stable. ■

Whilst this approach of using the coordinate I and the controller (8) will indeed orbitally stabilize the periodic orbit within some non-zero neighbourhood, it is very case specific. That is to say, the system, in addition to being scalar, is both fully actuated and has trivial dynamics. We therefore present a more general approach utilizing an *excessive* number of transverse coordinates next.

B. Excessive transverse coordinates

Let us begin by observing that the nominal trajectory (2) can be reparameterized on the following form:

$$\begin{aligned} q_*(s) &= a\sin(s) &&=: \phi(s), \\ \dot{q}_*(s) &= a\cos(s)\omega &&=: \phi'(s)\dot{s}_*, \\ \dot{s}_*(s) &= \omega &&=: \zeta(s). \end{aligned} \tag{9}$$

The scalar variable, $s \in \mathcal{S} := [0, 2\pi)$, is referred to as the *motion generator* (MG), and is simply a re-scaling of time along η_*, i.e. $s = \omega t$. More importantly, as $s - \mathrm{atan2}(\omega q_*, \dot{q}_*) \equiv 0$, it can be recovered by a projection of the system's states down upon the orbit using the *projection operator*

$$s = P(q, \dot{q}) := \mathrm{atan2}(\omega q, \dot{q}), \tag{10}$$

with atan2$(\cdot)$ denoting the four-quadrant arctangent function.

With this parameterization, one has several natural candidates for transverse coordinates of which to stabilize:

$$y = q - \phi(s), \tag{11a}$$

$$\dot{y} = \dot{q} - \phi'(s)\dot{s}, \tag{11b}$$

$$z = \dot{q} - \phi'(s)\zeta(s), \tag{11c}$$

$$\xi = \dot{s} - \zeta(s). \tag{11d}$$

So which should we pick? Or just as importantly, how many of them do we need to consider? That is, if $x_\perp$ is the vector of transverse coordinates, whose origin we want to stabilize, what should its dimension be? By simple reasoning, as the system has two states, $(q, \dot{q})$, and since we have one none-vanishing coordinate uniquely parameterizing the orbit, i.e. the MG s, it is implied that $x_\perp$ has to be of unit dimension for there to exist a well defined (local) diffeomorphism $(q, \dot{q}) \mapsto (s, x_\perp)$. Hence, there exists at most one independent transverse coordinate at any given time.

The only candidate always satisfying this among the possible coordinates in (11) is ξ, which takes the place of the coordinate I defined in the previous section; that is $\hat{I} = \frac{1}{2}(\dot{s} + \zeta(s))\xi$. However, note that it is dependent upon the velocity, $\dot{s}$, of the MG, which can be non-trivial to find as it will be dependent on $\ddot{q}$; indeed, $\dot{s} = \frac{\partial P}{\partial q}\dot{q} + \frac{\partial P}{\partial \dot{q}}\ddot{q}$.

Thus, suppose we instead take[2] $x_\perp = (y, z)^\mathsf{T}$, that is, an *excessive* number of coordinates, which then only depends on the states and the MG, not its velocity $\dot{s}$. It is clear that, even though the triplet (s, y, z) can have at most two independent elements at any given time, if we stabilize the origin of the excessive coordinates $x_\perp$, we simultaneously stabilize the nominal trajectory in the orbital sense.

The following statement demonstrates this fact.

Proposition 2: For any two constants $k_y, k_z \in \mathbb{R}$ with $k_z > 0$, the controller

$$u = -\omega^2 a \sin(s) - k_y y - k_z z, \tag{12}$$

exponentially orbitally stabilizes the desired trajectory (2). □

The proof of Proposition 2, as well as further analysis in regards to excessive transverse coordinates for this example, is given in Section III. The analysis there is based on a *transverse linearization*, that is, the linearization of the dynamics of the coordinates (y, z) along the solution. Thus the above stability result is of course only local. Nevertheless, it is quite surprising that it is true regardless of the value of k_y. Taking $k_y > 0$ is however likely advantageous in terms of the convergence rate.

It is also important to note that even though the structure of the controller (12) is reminiscent of the reference tracking controller (3), it is an inherently nonlinear controller due to the definition of the projection operator (10).

C. Simulation: Reference tracking vs. orbital stabilization

Consider again the scalar particle example (1), but suppose its true mass is in fact $1.5\,\mathrm{kg}$ and that it is also subject to an unknown dry friction term:

$$\frac{3}{2}\ddot{q} = u - 2\mathrm{sgn}(\dot{q}). \tag{13}$$

The task is as before to converge to the periodic orbit (4). Towards this end, we compared the following three controllers: the u_{PD}-controller (3) with $(k_p, k_d) = (4, 4)$; the controller (8) taken as $u_I = -\omega^2 q - \mathrm{sgn}_+(\dot{q})I$ with I given by (5); and the u_{yz}-controller (12) with $(k_y, k_z) = (4, 4)$.[3]

Figure 1 shows the obtained phase portraits and control inputs from numerically simulating the system (13) when starting at $(q_0, \dot{q}_0) = (0, -3a\omega/2)$ with $a = \pi/2$, $\omega = \pi$ and with white noise added to the measurements. It is clear that, whereas the orbitally stabilizing controllers u_I and u_{yz} quite quickly converge close to the orbit in a quite natural way, the reference tracking (feedforward+PD)-controller takes a much more aggressive action and a significant detour corresponding to the previously mentioned "catch-up"-effect, before eventually converging to a somewhat perturbed orbit.

Although this simple example provides some experimental evidence for the u_{yz}-controller based on the excessive transverse coordinates indeed being orbitally stabilizing for the system here considered,

[2]Taking $\dot{y}$ as one of the coordinates instead of z would also be a valid choice but again requires knowledge of $\dot{s}$, which, as stated, can be nontrivial to find and which can be highly sensitive to model uncertainty.

[3]The controller gains are here picked rather arbitrary in order to best illustrate the concept of orbital stabilization compared to reference tracking, and are therefore not chosen towards the goal of optimizing performance.

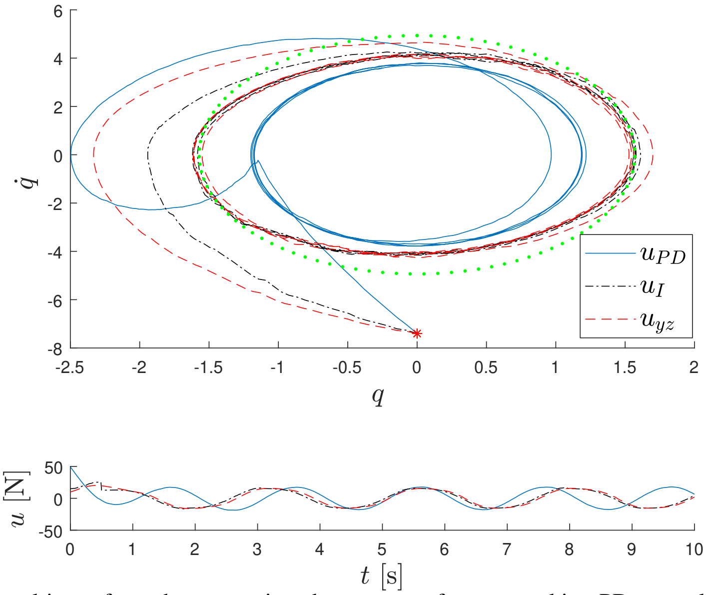

Fig. 1. Phase portrait and control input from the comparison between a reference tracking PD-controller (–) and two orbitally stabilizing controllers using, respectively, one (-·-) and two (- -) transverse coordinates. The green, dotted line (···) corresponds to the nominal orbit.

we have yet to prove Proposition 2. Therefore, in the next section, we will give such such a proof by first deriving the linearization of the dynamics of the excessive transverse coordinates along the trajectory, a so-called *transverse linearization*, and in doing so, provide further insight into this form of linearization.

III. A Useful Tool: The Transverse Linearization

A. Deriving the Linearized Transverse Dynamics

Let $x = [q, \dot{q}]^{\mathsf{T}} \in \mathbb{R}^2$ denote the states of (1), and suppose we know a feasible trajectory parameterized on the form

$$x_s(s) = \begin{bmatrix} \phi(s) \\ \phi'(s)\zeta(s) \end{bmatrix}, \quad s \in \mathcal{S}, \tag{14}$$

given some known scalar, $\mathcal{C}^2$-smooth function ϕ, with $\phi'(s) = \frac{d}{ds}\phi(s)$, and where the $\mathcal{C}^1$-function $\zeta : \mathcal{S} \to \mathbb{R}_{>0}$ recovers the nominal velocity of s along the orbit, i.e. $\dot{s}_*(t) = \zeta(s_*(t))$. Further suppose that we know a smooth projection operator $P : \mathbb{R}^2 \to \mathcal{S}$, that is $s = P(x)$, which is well defined in a tubular neighbourhood of the trajectory. We will denote by $DP(x) = [\frac{\partial P}{\partial q}(x), \frac{\partial P}{\partial \dot{q}}(x)]$ its gradient and by $DDP(x) = \frac{\partial^2 P}{\partial x^2}(x)$ its symmetric Hessian matrix.

As $\ddot{q}_*(s) = u_*(s)$, it follows from the parameterization (14) that the nominal control input along η_* is given by

$$u_*(s) = [\phi''(s)\zeta(s) + \phi'(s)\zeta'(s)]\,\zeta(s). \tag{15}$$

Thus suppose we take[4] $u = u_*(s) + v$ for some $v \in \mathbb{R}$ to be defined. Recalling the definitions of (y, z) in (11), it is not difficult to show that the dynamics of the system then can be rewritten on the following equivalent form:

$$\dot{x} = x_s'(s)\zeta(s) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v. \tag{16}$$

[4]Rather than taking $u_*(s)$ in (15) directly, on can instead take any smooth function $v : \mathbb{R}^2 \times \mathcal{S} \to \mathbb{R}$ satisfying $v(x_s(s), s) \equiv u_*(s)$ for all $s \in \mathcal{S}$; see [6]. Although note that this could alter the subsequent analysis somewhat.

Now, let $x_\perp := x - x_s(s)$ be a vector of the transverse coordinates corresponding to the pair (y, z) as defined in (11). Defining the Jacobian matrix function

$$\Omega(x) := I_2 - x_s'(s) DP(x) \tag{17}$$

such that $\dot{x}_\perp = \Omega(x)\dot{x}$, we thus find, using (16), that

$$\dot{x}_\perp = \Omega(x) \left(x_s'(s)\zeta(s) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \right) \tag{18}$$

which are the dynamics of the excessive transverse coordinates, also referred to simply as the *transverse dynamics*.

Our goal will now be to linearize these transverse dynamics along the orbit, that is, perform a so-called *transverse linearization*. Towards this end, using the subscript notation $DP_s(s) := DP(x_s(s))$, it follows from the definitions of $P(\cdot)$ and $\zeta(\cdot)$ that $P(x_s(s)) = s$, or equivalently $\zeta(s) = DP_s(s)x_s'(s)\zeta(s)$, which again implies the relation

$$DP_s(s) x_s'(s) \equiv 1 \tag{19}$$

holding for all $s \in \mathcal{S}$. As a consequence, the matrix function $\Omega_s(s) := \Omega(x_s(s))$ has $x_s'(s)$ as a right annihilator, while $DP_s(s)$ is its left annihilator (see also [5, Lemma 3]).

Consider now the first-order Taylor expansion of $DP(\cdot)$:

$$DP(x) = DP_s(s) + x_\perp^\mathsf{T} DDP_s(s) + O(\|x_\perp\|^2).$$

By defining $B := [0, 1]^\mathsf{T}$, as well as the matrix function

$$A_\perp(s) := \ \Omega_s(s) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - x_s'(s) x_s'^\mathsf{T}(s) DDP_s(s)\zeta(s),$$

it follows that we can rewrite the transverse dynamics (18) on the following equivalent form:

$$\dot{x}_\perp = A_\perp(s) x_\perp + \Omega(x) B v + O(\|x_\perp\|^2).$$

Similarly, by defining $\psi := \int_0^t (\dot{s}(\tau) - \zeta(s(\tau)) d\tau$ and

$$\mathcal{A}_\parallel(s) := DP_s(s) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + x_s'^\mathsf{T}(s) DDP_s(s)\zeta(s),$$

as well as using the fact that $\dot{s} = DP(x)\dot{x}$, we obtain

$$\dot{\psi} = \mathcal{A}_\parallel(s) x_\perp + DP(x) B v + O(\|x_\perp\|^2). \tag{20}$$

Letting $B_\perp(s) := \Omega_s(s)B$, it therefore follows that the first approximation (variational system) of the transverse dynamics (18) along the solution (14) is given by the constrained (differential-algebraic) linear periodic system

$$\frac{d}{dt}\delta x_\perp = A_\perp(s)\delta x_\perp + B_\perp(s) v, \quad DP_s(s)\delta x_\perp = 0; \tag{21}$$

whereas the first approximation of (20) is

$$\frac{d}{dt}\delta\psi = \mathcal{A}_\parallel(s)\delta x_\perp + DP_s(s) B v, \ DP_s(s)\delta x_\perp = 0. \tag{22}$$

We can here infer from (22) that if the origin of (21) is asymptotically (exponentially) stable, then $\delta\psi$ approaches some constant number; hence $\dot{s} \to \zeta(s)$ if $x_\perp \to 0$. Indeed, this observation just corresponds to the well-known fact that a shift in the phase along a periodic orbit does not influence its stability [7], letting us simply discard (22) in the sequel.

It is here also important to note that the latter condition in (21), that is $DP_s(s)\delta x_\perp \equiv 0$, is necessary in order to restrict the solutions to the transverse plane. Indeed, it is not difficult to show that $\Omega_s^2(s) = \Omega_s(s)$, and as $\delta x_\perp = \Omega_s(s)\delta x$, we must therefore have $\delta x_\perp = \Omega_s(s)\delta x_\perp$, from which the condition follows. We further demonstrate its necessity next.

B. Necessity of the transversality condition

From the two properties $DP_s(s)\Omega_s(s) \equiv 0$ and $\delta x_\perp = \Omega_s(s)\delta x_\perp$, it follows that we can readily take

$$\hat{A}_\perp(s) := \Omega_s(s)A(s)\Omega_s(s) - x_s'(s)x_s'(s)^\mathsf{T} DDP_s(s)\zeta(s)$$

instead of $A_\perp$ in (21). Here the matrix $A(\cdot)$, given by

$$A(s) := \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ u_*'(s) \end{bmatrix} DP_s(s) \right),$$

corresponds to the variational system of (16) evaluated along the orbit, that is $\frac{d}{dt}\delta x = A(s)\delta x + Bv$. Moreover, from the definition of the matrix function $\Omega_s(\cdot)$ in (17), we can also rewrite $A(s)$ as $A(s) = A(s)\Omega_s(s) + A(s)x_s'(s)DP_s(s)$.

Thus, let us for a moment consider the system

$$\dot{w} = \left[\hat{A}_\perp(s) + \Omega_s(s)A(s)x_s'(s)DP_s(s) \right] w + B_\perp(s)v, \tag{23}$$

which if we added the constraint $DP_s(s)w \equiv 0$ would be equivalent to (21). We will demonstrate that without this condition, however, the origin of (23) can never be asymptotically stable, regardless of the control law v. Moreover, we will show that this also implies that (21) without the transversality condition cannot be stabilizable either.

Towards this end, let $p_\perp : \mathcal{S} \to \mathbb{R}^2$ denote a smooth unitary basis of the kernel of $DP_s(s)$, i.e. $DP_s(s)p_\perp(s) \equiv 0$ and $\|p_\perp\| = \|p_\perp\|_2 = 1$. It is not difficult to show that

$$\dot{p}_\perp(s) = \left(I_2 - p_\perp(s)p_\perp^\mathsf{T}(s)\right) A_\perp(s)p_\perp(s).$$

Similarly, denote by $\hat{p}_\perp : \mathcal{S} \to \mathbb{R}^2$ a smooth normalized basis of the kernel of $(x_s'(s))^\mathsf{T}$.

Consider now the following change of coordinates:

$$w = U(s)\xi, \quad U(s) := \begin{bmatrix} \frac{x_s'(s)}{\|x_s'(s)\|} & p_\perp(s) \end{bmatrix}, \tag{24}$$

in which one can note that by defining $p_\perp^\dagger := (\hat{p}_\perp^\mathsf{T} p_\perp)^{-1}\hat{p}_\perp^\mathsf{T}$, one can take $U^{-1} = [DP_s^\mathsf{T}\|x_s'\|, (p_\perp^\dagger)^\mathsf{T}]^\mathsf{T}$. Moreover, as

$$\frac{d}{dt}\left(\frac{x_s'}{\|x_s'\|}\right) = \left(I_2 - \frac{x_s' {x_s'}^\mathsf{T}}{\|x_s'\|^2}\right) A \frac{x_s'}{\|x_s'\|},$$

where we have dropped the s-argument for readability, it follows that

$$\dot{U} = \left[\left(I_2 - \frac{x_s' {x_s'}^\mathsf{T}}{\|x_s'\|^2}\right) A x_s' DP_s + \left(I_2 - p_\perp p_\perp^\mathsf{T}\right) A_\perp \Omega_s \right] U.$$

Straightforward computations then show that the dynamics of $\xi = [\xi_\parallel, \xi_\perp]^\mathsf{T}$ with $\xi_\parallel, \xi_\perp \in \mathbb{R}$ are given by

$$\dot{\xi}_\parallel = \left(\frac{{x_s'}^\mathsf{T}}{\|x_s'\|} A \frac{x_s'}{\|x_s'\|} - \zeta' \right) \xi_\parallel, \tag{25a}$$

$$\dot{\xi}_\perp = p_\perp^\mathsf{T} A_\perp p_\perp \xi_\perp + p_\perp^\dagger B v. \tag{25b}$$

As here $\int_0^t \left(\frac{{x_s'}^\mathsf{T}}{\|x_s'\|} A \frac{x_s'}{\|x_s'\|} - \zeta' \right) dt = \ln\|x_s'(s(t))\| - \ln\|x_s'(s_0)\|$, it is clear that $\xi_\parallel(t) = c\|x_s'(s(t))\|$ for some $c \in \mathbb{R}$. Therefore, from (24), we can conclude that $x_s'(s)$ must always be a solution of (23) regardless of the control input; while, moreover, it follows that the stability of (21) corresponds to that of the one-dimensional subsystem (25b).

With this in mind, suppose there exists a smooth stabilizing feedback controller for (21) of the form $v = K(s)\delta x_\perp$. We can then write the state transition matrix of (25) as

$$\Xi(t,t_0) = \begin{bmatrix} \frac{\|x_s'(s(t))\|}{\|x_s'(s_0)\|} & 0 \\ 0 & \Xi_\perp(t,t_0) \end{bmatrix} \tag{26}$$

where $\Xi_\perp(t,t_0)$ is the state transition matrix of (25b) with $v = K(s)p_\perp(s)\xi_\perp$, that is, $\xi_\perp(t) = \Xi_\perp(t,t_0)\xi_\perp(t_0)$.

So what happens if we remove the $(\Omega_s A x_s' DP_s)$-part in (23) such that it would correspond to (21) if we added transversality condition $DP_s(s)w \equiv 0$? It turns out that the dynamics of $\xi_\parallel$ then stay the same, whereas (25b) becomes

$$\frac{d}{dt}\hat{\xi}_\perp = p_\perp^\top A_\perp p_\perp \hat{\xi}_\perp + p_\perp^\dagger(Bv - Ax_s'\hat{\xi}_\parallel(t_0)). \tag{27}$$

So for $\hat{\xi}_\parallel(t_0) \equiv 0$ it has the same solution $\xi_\perp^*(t)$ as (25b), but this is not true whenever $\hat{\xi}_\parallel(t_0) \neq 0$. Indeed, the state transition matrix is then instead given by

$$\hat{\Xi}(t,t_0) = \begin{bmatrix} \frac{\|x_s'(s(t))\|}{\|x_s'(s_0)\|} & 0 \\ \int_{t_0}^t \Xi_\perp(t_0,\tau)p_\perp^\dagger\left(BK - A\right)x_s' dt & \Xi_\perp(t,t_0) \end{bmatrix}.$$

Nevertheless, as clearly

$$\det \hat{\Xi}(t,t_0) = \frac{\|x_s'(s(t))\|}{\|x_s'(s_0)\|}\det \Xi_\perp(t,t_0) = \det \Xi(t,t_0),$$

it is implied by Liouville's formula that the sum of their characteristic exponents are the same (see e.g. [8, Def. 10]). Moreover, as (25b) and (27) share the solution $\xi_\perp^*(t)$, we can conclude that the condition $DP_s(s)\delta x_\perp = 0$ in (21) is vital, as without it the system would always have a non-vanishing (zero characteristic exponent) solution and thus would never be stabilizable. We will use this fact in the following section.

C. Proof of Proposition 2

Let us again consider the task of stabilizing sinusoidal oscillations of the particle system. That is, we consider $\phi(s) = a\sin(s)$ in (14), but rather than just taking $\zeta = \omega$ will we for now consider the more general case of some arbitrary smooth, periodic function $\zeta : \mathcal{S} \to \mathbb{R}_{>0}$.

In order to find the motion generator s in this case, we consider the implicit equation $h(q,\dot{q},s) = s - \text{atan2}\left(\frac{q\zeta(s)}{\dot{q}}\right)$ which clearly vanishes on the nominal orbit, letting us utilize Proposition 2 in [6]. That is, we can use the implicit function theorem in order to find the projection operator in a neighbourhood of the orbit, as well as to obtain that

$$DP(x) = \frac{\zeta(s)}{\zeta^2(s)q^2 + \dot{q}^2 - q\dot{q}\zeta'(s)}\begin{bmatrix}\dot{q} & -q\end{bmatrix};$$

which, when evaluated along the orbit, becomes

$$DP_s(s) = \frac{\begin{bmatrix}\cos(s)\zeta(s) & -\sin(s)\end{bmatrix}}{a\zeta(s) - a\zeta'(s)\sin(s)\cos(s)}. \tag{28}$$

Consider now the controller

$$u = u_*(s) + Kx_\perp \tag{29}$$

for some constant vector $K = [k_y, k_z]$. The linearized transverse dynamics (21) thus become

$$\frac{d}{dt}\delta x_\perp = A_\perp^{cl}(s)\delta x_\perp, \quad DP_s(s)\delta x_\perp = 0,$$

in which, by utilizing that $\Omega_s(s)\delta x_\perp = \delta x_\perp$, we have

$$A_\perp^{cl} := \Omega_s \left(A + BK\right) \Omega_s - x_s' {x_s'}^\mathsf{T} DDP_s \zeta.$$

Our aim will now be to derive a condition for the exponential stability of the origin of this system. For this purpose, we recall that for a regular linear system $\dot{\chi} = M(t)\chi$ (e.g. $M(\cdot)$ is constant or periodic) the sum of its characteristic exponents is given by the formula

$$\sigma(M) := \liminf_{t\to\infty} \frac{1}{t}\int_0^t \text{Tr } M(\tau)d\tau$$

where Tr M denotes the trace of M [8]. Thus, as the system here considered is of dimension two, and we have already demonstrated that it has one solution with zero characteristic exponent, we can conclude the following statement, which has strong ties to the classical Poincaré test [9, Thm. 5.5].

Lemma 1: A periodic, non-vanishing solution of the particle system (1) under the controller (29) is exponentially orbitally stable if and only if $\sigma(A_\perp^{cl}) < 0$. $\square$

We will now find conditions on K such that Lemma 1 is satisfied. In this regard, we have

$$\begin{aligned}\sigma(A_\perp^{cl}) &= \liminf_{t\to\infty} \frac{1}{t}\int_{s_0}^s \frac{1}{\zeta}\text{Tr } \Omega_s \left(A + BK\right)\Omega_s d\tau \\ &\quad - \liminf_{t\to\infty} \frac{1}{t}\int_{s_0}^s \text{Tr } x_s' {x_s'}^\mathsf{T} DDP_s d\tau.\end{aligned}$$

Utilizing the properties of the trace and the fact that the system is 2π-periodic, the latter part simplifies as follows:

$$\liminf_{t\to\infty} \frac{1}{t}\int_{s_0}^s \text{Tr } x_s' {x_s'}^\mathsf{T} DDP_s d\tau = \frac{1}{2\pi}\int_0^{2\pi} {x_s'}^\mathsf{T} DDP_s x_s' d\tau.$$

Thus we can utilize that the following relation here holds:

$$\int_0^{2\pi} x_s'(\tau)^\mathsf{T} DDP_s(\tau) x_s'(\tau) d\tau \equiv 0. \tag{30}$$

Indeed, differentiating $DP_s(s)x_s'(s) = 1$ from (19) we obtain ${x_s'}^\mathsf{T} DDP_s x_s' = -DP_s x_s''$. Thus by (28) and (9),

$$\begin{aligned}DP_s(s)x_s''(s) &= \frac{2a\zeta'(s)\sin(s)^2 - a\zeta''(s)\sin(s)\cos(s)}{a\zeta(s) - a\zeta'(s)\sin(s)\cos(s)} \\ &= \frac{\frac{d}{ds}\left(\zeta(s) - \zeta'(s)\sin(s)\cos(s)\right)}{\zeta(s) - \zeta'(s)\sin(s)\cos(s)}.\end{aligned}$$

Hence, due to the 2π-periodicity of $\zeta(\cdot)$, (30) evaluates to $[\ln(\zeta(s) - \zeta'(s)\sin(s)\cos(s))^{-1}]_0^{2\pi} \equiv 0$.

Coming back to computing $\sigma(A_\perp^{cl})$, by the above relation and using the cyclic property of the trace, it follows that

$$\begin{aligned}&\int_0^{2\pi} \frac{1}{\zeta(\tau)}\text{Tr } \Omega_s(\tau)\left(A(\tau) + BK\Omega_s(\tau)\right)d\tau \\ &= \int_0^{2\pi} \frac{1}{\zeta}\text{Tr } \left(A + BK\Omega_s\right)d\tau - \int_0^{2\pi} \frac{1}{\zeta} DP_s A x_s' d\tau.\end{aligned}$$

Using that $x_s''\zeta + x_s'\zeta' = Ax_s'$ together with (30), we have $DP_s A x_s' = DP_s x_s''\zeta + \zeta' = -{x_s'}^\mathsf{T} DDP_s x_s' + \zeta'$; hence

$$\sigma(A_\perp^{cl}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\zeta(\tau)}\text{Tr } \left(A(\tau) + BK\Omega_s(\tau)\right)d\tau. \tag{31}$$

It is here interesting to note that this is simply equivalent to the sum of the characteristic exponents of the original first approximation system under the controller (29), that is $\frac{d}{dt}\delta x = (A(s) + BK\Omega_s(s))\delta x$, thus further solidifying the connection between the (planar) Poincaré test [9, Thm. 5.5] and Lemma 1, which then also readily follows from the Andronov–Vitt theorem [4, Thm. 1].

With the above in place, let us finally compute $\sigma(A_\perp^{cl})$ for the case considered in Sec. II-B, that is $\zeta(s) \equiv \omega$. As then $DP_s(s) = [\omega\cos(s), -\sin(s)]/(a\omega)$ and $x_s'(s) = [a\cos(s), -a\omega\sin(s)]^\mathsf{T}$, it is straightforward to show that

$$\text{Tr}\left[(A(s) + BK)\Omega_s(s)\right] = k_z\cos^2(s) + \frac{\sin(2s)}{2\omega}\left(\omega^2 + k_y\right).$$

Hence by (31) we obtain $\sigma(A_\perp^{cl}) = k_z/(2\omega)$, such that we can conclude by Lemma 1 that Proposition 2 indeed holds.

IV. Orbital stabilization of non-periodic motion

In the example we have considered so far, we assumed the motion we were to stabilize was periodic. However, orbital stabilization is not restricted just to periodic motions, but rather to any motion defined over a finite time interval. Indeed, (asymptotic) orbital stability of some trajectory $x_*(t) = x_*(t, x_0)$, $x_*(t_0) = x_0$, defined over the finite interval $[t_0, t_f]$, just means (asymptotic) stability of the orbit

$$\eta_* := \{x = x_*(t),\ t \in [t_0, t_f]\}. \tag{32}$$

As an example, consider a smooth dynamical system

$$\dot{x} = f(x) + g(x)u \tag{33}$$

with state $x \in \mathbb{R}^n$ and control input $u \in \mathbb{R}^m$. Let $x_*(t)$ be a known, bounded trajectory of the unforced system, i.e. $\dot{x}_*(t) = f(x_*(t))$, beginning and terminating at two equilibrium states, denoted x_0 and x_f, respectively. Suppose we know a smooth, regular reparametrization $x_s(s)$ with $s \in \mathcal{S} := [s_0, s_f]$ such that $x_s(s_0) = x_0$ and $x_s(s_f) = x_f$. Moreover, assume some projection operator $P : \mathbb{R}^n \to \mathcal{S}$ is known, which is well defined within some region $\mathcal{X} \subset \mathbb{R}^n$ containing η_*, and which, by defining the hypersurface $\Gamma_{\hat{s}} := \{x \in \mathcal{X} : P(x) = \hat{s}\}$, is smooth within $\{\Gamma_{\hat{s}},\ \hat{s} \in (s_0, s_f)\}$.

Thus, if we can design some feedback $u = K(s)x_\perp$ which along the orbit ensures contraction of the transverse coordinates, $x_\perp = x - x_s(s)$, towards their origin, then consequently there will be some tubular subset of $\Gamma_\mathcal{S} := \bigcup_{s\in\mathcal{S}}\Gamma_s$, containing η_*, in which we also have contraction to the orbit. If, in addition, the controller is such that for $s_e \in \{s_0, s_f\}$, the constant matrix $\left[\frac{\partial f}{\partial x}(x_s(s)) + g(x_s(s))K(s)\right]\big|_{s=s_e}$ is Hurwitz, then the controller renders the orbit (32) asymptotically stable. A geometrical illustration of this orbital stabilization scheme for point-to-point motions can be seen in Figure 2.

As a practical example, let us again consider the particle system (1) with the objective of restating the following motion control problem as an orbital stabilization problem.

Problem 1: Find a control input $u : [t_0, t_f] \to \mathbb{R}$ with $u(t_0) = u(t_f) = 0$, such that by starting the system at rest from $q(t_0) = q_0$, the system is driven to $(q(t_f), \dot{q}(t_f)) = (q_f, 0)$ for some $q_f > q_0$. □

Taking for simplicity $t_0 = 0$, and using the procedure outlined in Sec. 5.5.1 in [10], we can easily find one possible solution to this problem on the form of a quintic polynomial:

$$q_*(t) = q_0 + \sum_{i=1}^{5} a_i t^i \tag{34}$$

for some constant coefficient $a_i \in \mathbb{R}$. Moreover, as $u_*(t) = \ddot{q}_*(t)$, a simple reference tracking (feedforward+PD) controller similar to (3) considered in the introduction, that is $u = u_*(t) - k_p e - k_d\dot{e}$, can of course then be utilized, but would again result in a non-autonomous closed-loop system.

While there are many possible way of generating an orbitally stabilizing controller for solving Problem 1 without the explicit timing aspect, we will here utilize the fact that $\dot{q}_* \geq 0$ in order to allow us to use (34) for this purpose. More specifically, we take $s = \text{sat}_{q_0}^{q_f}(q)$, where $\text{sat}_{q_0}^{q_f}(\cdot)$ is the saturation function

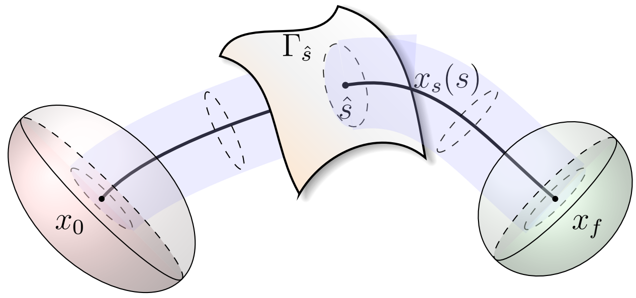

Fig. 2. Illustration of the hypersurface $\Gamma_{\hat{s}}$ travelling along the nominal trajectory $x_s(s)$. Suppose the space corresponding to the union of the blue shaded tube that encloses the orbit and the two ellipsoids is made an invariant (and contracting) set by some control action. If in addition the Jacobian linearization evaluated at both the initial and final equilibrium states, x_0 and x_f, have regions of attractions containing the red- and blue shaded ellipsoids, respectively, then an estimate of the region of attraction of the orbit under this orbitally stabilizing control action is contained within the union of the blue-shaded tube and the darkly shaded semi-ellipsoids.

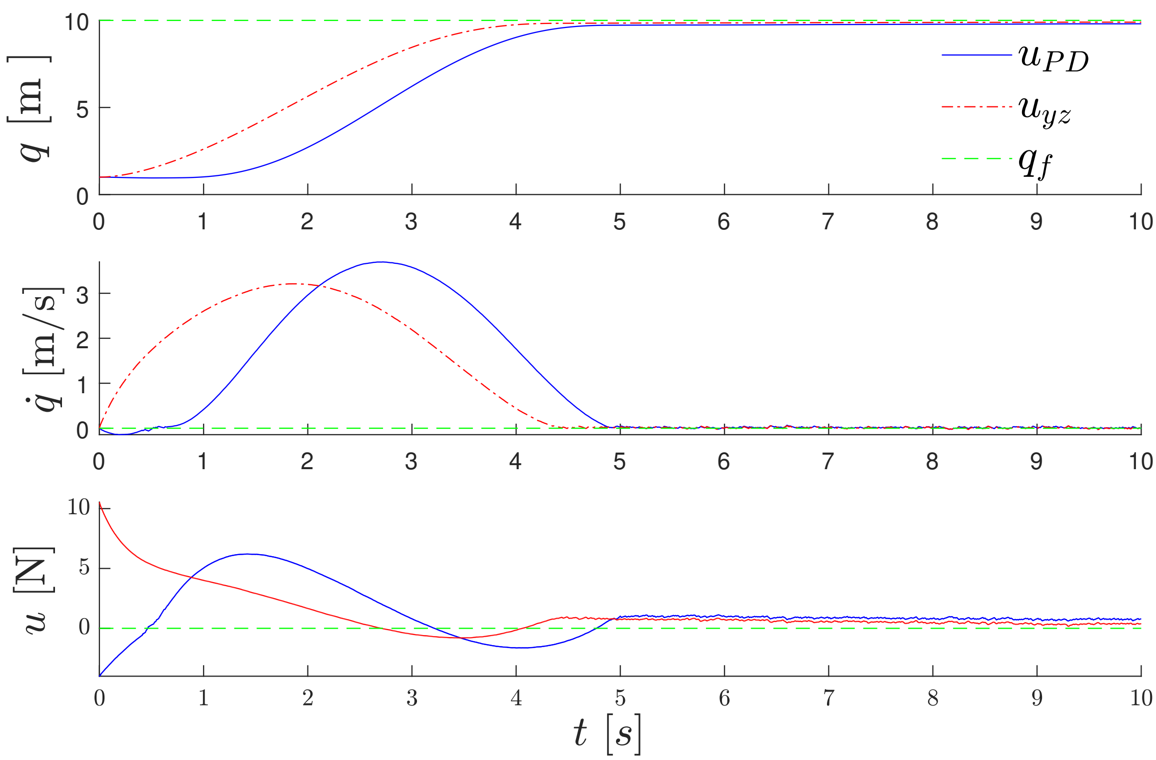

Fig. 3. Reference tracking PD-control (**blue**) vs the orbitally stabilizing controller (**red**) when starting at $q(0) = 1$.

with lower- and upper bound q_0 and q_f, respectively, such that $s \equiv q$ for $q \in [q_0, q_f]$. This allows us to readily use (34), together with its first- and second derivatives, to find the nominal control input as a function of s, that is $u_*^s : [q_0, q_f] \to \mathbb{R}$, as well as the function $\zeta : [q_0, q_f] \to \mathbb{R}_{\geq 0}$ such that $\dot{q}_*(t) \equiv \zeta(q_*(t))$ and with the reparametrization regular: $\|x_s'(s)\| \neq 0$ for all $s \in [q_0, q_f]$.

We again compared the reference tracking controller $u_{PD} = u_*(t) - k_p e - k_d \dot{e}$ and the orbitally stabilizing controller $u_{yz} = u_*^s(s) - k_y y - k_z z$ in relation to Problem 1 on the particle system with uncertainties given by (13), for $(q_0, q_f) = (0, 10)$ and $(t_0, t_f) = (0, 5)$. We used the same gains as in Sec. II-C and added white noise to the controller measurements. Notice here that $y \equiv 0$ whenever $q \in [q_0, q_f]$.

The results from starting the system at $q(0) = 1$ and $q(0) = -1$ can be seen in Figures 3 and 4, respectively. It is clear from Figure 3 that when starting inside the interval $[q_0, q_f]$, the orbitally

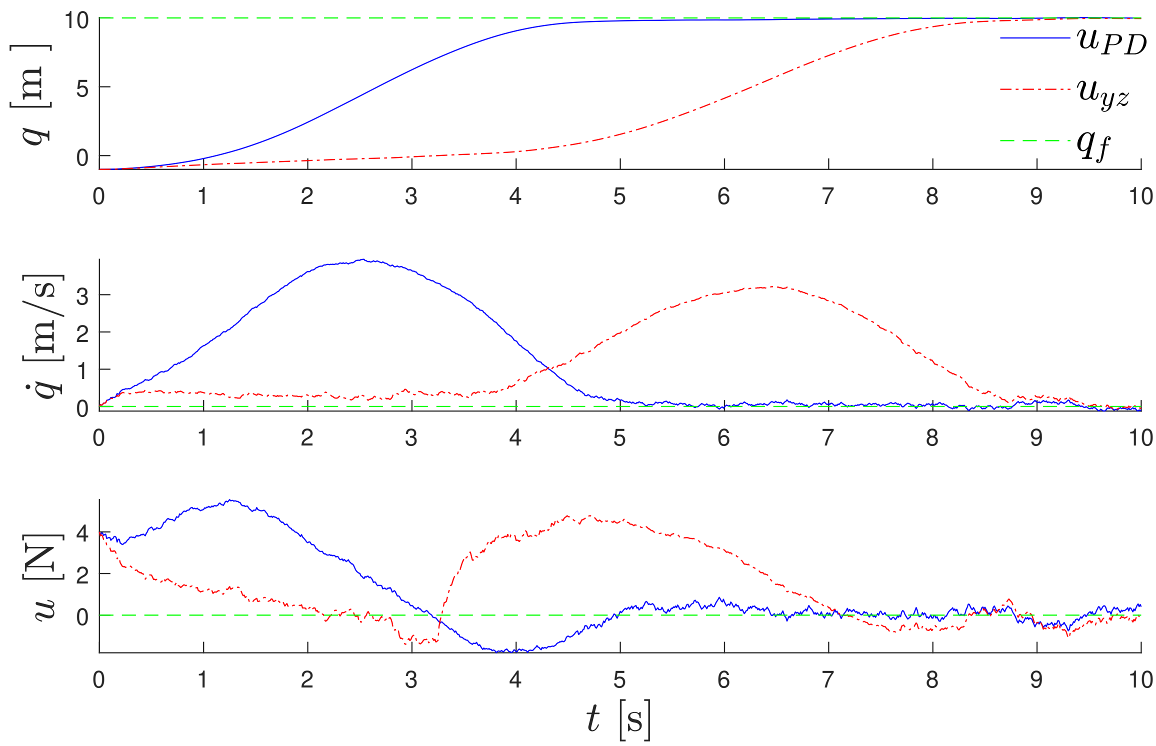

Fig. 4. Reference tracking PD-control (**blue**) vs the orbitally stabilizing controller (**red**) when starting at $q(0) = -1$.

stabilizing controller, u_{yz}, drives the system quickly to the goal states, whereas the reference tracking controller initially reverses the motion in order to maintain the "timing" given by time-varying reference signal. On the other hand, when starting outside this interval, as seen in Figure 4, the system is driven by the orbitally stabilizing controller to the goal state with quite a significant delay.

V. Discussion and Concluding Remarks

Through the simple example of an one-degree-of-freedom particle system, we have demonstrated that orbital stabilization can be a feasible alternative to reference tracking controllers, especially for motion control tasks in which the importance of the timing of the motion is secondary to the closeness to the nominal orbit in state space.

Some comments and observations on the presented results and examples which are of particular interest are stated next.

Implications of Proposition 2: The fact that the controller proposed in Proposition 2 is stabilizing even if it only depends on the coordinate z is quite surprising, as the map $x \mapsto (s, z)$ is not everywhere a well-defined diffeomorphism. Checking the possibility of generalizing this property to more complex (periodic) trajectories and to higher dimensional systems is therefore an interesting topic for further study.

Transverse linearization and the Poincaré test: As was previously stated, the analysis carried out in Section III, in particular Lemma 1, can in many ways be seen as equivalent (or at least as an alternative) to the well-known Poincaré test [9, Theorem 5.5] in relation to testing orbital stability of planar systems. The aforementioned connection with the Andronov–Vitt theorem is also quite interesting in this regard. Indeed, this type of (transverse) linearization is evidently a useful tool not only for control design, but also for (in-)stability analysis in the large (see also [5], [9]).

Orbital stabilization of non-period motion: It was demonstrated how orbitally stabilizing controllers also could be used for stabilization of certain non-periodic trajectories. Indeed, the presented approach can be extended to orbital stabilization of point-to-point motions of higher dimensional underactuated systems, in which a single controller then can be used to solve the well-known “swing-up and balance” problem without the need for any control switching.

Yet in the example considered, some limitations were also revealed. More specifically, it was demonstrated that when starting sufficiently far away from the orbit, an orbitally stabilizing controller might lead to unsatisfactory performance and delays if it is not complemented by some additional method correcting for this deviation. Possible candidates includes replanning the path from the initial state, or some higher level heuristic which modifies the velocity of the motion generator along the path on-line, e.g. in a similar manner to the “timing-law” in certain types of path following schemes; see e.g. [11] and the references therein.

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