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Gabor duality theory for Morita equivalent C^*-algebras

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The duality principle for Gabor frames is one of the pillars of Gabor analysis. We establish a far-reaching generalization to Morita equivalence bimodules with some extra properties. For certain twisted group C^*-algebras, the reformulation of the duality principle to the setting of Morita equivalence bimodules reduces to the well-known Gabor duality principle by localizing with respect to a trace. We may lift all results at the module level to matrix algebras and matrix modules, and in doing so, it is natural to introduce (n, d)-matrix Gabor frames, which generalize multi-window super Gabor frames. We are also able to establish density theorems for module frames on equivalence bimodules, and these localize to density theorems for (n, d)-matrix Gabor frames.

Keywords: Gabor frames; twisted group C^*-algebras; Hilbert C^*-modules.

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1. Introduction

Hilbert C^*-modules are well-studied objects in the theory of operator algebras and Rieffel made the crucial observation that they provide the correct framework for the extension of Morita equivalence of rings to C^*-algebras. In his seminal work [25], he noted that his equivalence bimodules between two C^*-algebras are bimodules where the left and right Hilbert C^*-module structures are compatible in a technical sense. We are interested in the features of these equivalence bimodules from the perspective of frame theory. In [13], the notion of standard module frame was introduced for countably generated Hilbert C^*-modules. Already in [27], Rieffel

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observed that finitely generated equivalence bimodules could be described in terms of finite standard module frames. He used this in his study of Heisenberg modules — a class of projective Hilbert C^*-modules over twisted group C^*-algebras. In [22], it was observed that these module frames are closely related to Gabor frames used in time-frequency analysis. Moreover, in [19], the properties of standard module frames for Heisenberg modules were studied from the perspective of the established duality theory of these Gabor frames.

The following central result of Gabor frames is due to the seminal work [10, 20, 28].

Theorem. (Duality Theorem for Gabor systems) *For* $\alpha, \beta > 0$ *and* $g \in L^2(\mathbb{R})$ *the Gabor system* $\{e^{2\pi i\beta l(\,\cdot\,)}g(\,\cdot\, - \alpha k)\}_{k,l\in\mathbb{Z}}$ *is a* frame *for* $L^2(\mathbb{R})$ *if and only if the Gabor system* $\{e^{2\pi i l(\,\cdot\,)/\alpha}g(\,\cdot\, - k/\beta)\}_{k,l\in\mathbb{Z}}$ *is a* Riesz sequence *for the closed span of* $\{e^{2\pi i l(\,\cdot\,)/\alpha}g(\,\cdot\, - k/\beta)\}_{k,l\in\mathbb{Z}}$ *in* $L^2(\mathbb{R})$.

The possible extension of the duality principle from Gabor systems to other types of systems has been investigated in [1, 4, 5, 12] and [16] as well as in the form of the theory of R-duality [7, 9, 30, 31].

Motivated by the link between the duality theory of Gabor frames and the Morita equivalence of noncommutative tori [19, 22], we explore duality theory of module frames for equivalence bimodules between Morita equivalent C^*-algebras and show that this is a true generalization of the duality theory for Gabor frames.

Unlike the treatment of this topic in [19], here, we do not rely on any results from time-frequency analysis. Indeed, we will consider a quite general situation, namely, two Morita equivalent C^*-algebras A and B with Morita equivalence bimodule E, where E is finitely generated and projective as an A-module and B is equipped with a faithful finite trace. We show that module frames for E as an A-module in this situation admit a duality theorem and by localization with respect to a trace, we are able to connect these module frame statements to results on frames in Hilbert spaces. Moreover, we show that some cornerstone results of Gabor analysis can be formulated as C^*-algebraic statements on Morita equivalence bimodules. Also, we establish density results for the existence of module frames, which seemingly have not been explored before.

The application of our duality results to Gabor systems on general locally compact abelian (LCA) groups with time-frequency shifts from closed cocompact subgroups of phase space yields exactly the known key results in duality theory of Gabor systems. Our general approach to duality principles has led us to the introduction of (n, d)-matrix Gabor frames that is a joint generalization of multi-window superframes and Riesz bases and we prove that their Gabor dual systems are (d, n)-matrix Riesz sequences.

Let us summarize the content of this paper. In Sec. 2, we collect some facts about Hilbert C^*-modules which will be of use later, most importantly about localization of Hilbert C^*-modules. We then proceed in Sec. 3, setting up for reformulating central results of Gabor analysis to the setting of Morita equivalence bimodules with some extra conditions. In this section, we also obtain density theorems for existence of module frames. Lastly, in Sec. 4, we localize with respect to a trace to recover the setting of Gabor analysis. Due to the setup of the previous section, we obtain easy proofs for some foundational results of Gabor analysis for a very general type of Gabor frame.

2. Preliminaries

We assume basic knowledge about Banach $*$-algebras, C^*-algebras, Banach modules, and Hilbert C^*-modules. In this section, we collect definitions and basic facts of concepts crucial for this paper.

Suppose ϕ is a positive linear functional on a C^*-algebra B, and that E is a right Hilbert B-module. We define an inner product

$$\langle \cdot, \cdot \rangle_\phi : E \times E \to \mathbb{C}, \quad (f, g) \mapsto \phi(\langle g, f \rangle_B),$$

where $\langle \cdot, \cdot \rangle_B$ is the B-valued inner product. We may have to factor out the subspace $N_\phi := \{f \in E \mid \langle f, f \rangle_B = 0\}$ and complete E/N_ϕ with respect to $\langle \cdot, \cdot \rangle_\phi$. This yields a Hilbert space which we will denote by H_E, and is known as the localization of E in ϕ. There is a natural map $\rho_\phi : E \to H_E$ which induces a map $\rho_\phi : \mathrm{End}_B(E) \to \mathbb{B}(H_E)$. We will focus entirely on the case in which ϕ is a faithful positive linear functional, that is, when $b \in B^+$ with $\phi(b) = 0$ implies $b = 0$. In that case $N_\phi = \{0\}$ and we have the following useful result from [21, pp. 57–58].

Proposition 2.1. *Let B be a C^*-algebra equipped with a faithful positive linear functional $\phi : B \to \mathbb{C}$, and let E be a Hilbert B-module. Then the map $\rho_\phi : \mathrm{End}_B(E) \to \mathbb{B}(H_E)$ is an injective $*$-homomorphism.*

The Hilbert C^*-modules of interest will be A-B-equivalence bimodules for C^*-algebras A and B. We will denote the A-valued inner product by ${}_\bullet\langle \cdot, \cdot \rangle$, and the B-valued inner product by $\langle \cdot, \cdot \rangle_\bullet$.

Definition 2.2. Let A and B be C^*-algebras. A *Morita equivalence bimodule between A and B*, or an *A-B-equivalence bimodule*, is a Hilbert C^*-module E satisfying the following conditions.

(1) $\overline{{}_\bullet\langle E, E \rangle} = A$ and $\overline{\langle E, E \rangle_\bullet} = B$, where ${}_\bullet\langle E, E \rangle = \mathrm{span}_\mathbb{C}\{{}_\bullet\langle f, g \rangle \mid f, g \in E\}$ and likewise for $\langle E, E \rangle_\bullet$.

(2) For all $f, g \in E$, $a \in A$ and $b \in B$,

$$\langle af, g \rangle_\bullet = \langle f, a^* g \rangle_\bullet \quad \text{and} \quad {}_\bullet\langle fb, g \rangle = {}_\bullet\langle f, gb^* \rangle.$$

(3) For all $f, g, h \in E$,

$${}_\bullet\langle f, g \rangle h = f \langle g, h \rangle_\bullet .$$

Now, let $\mathcal{A} \subset A$ and $\mathcal{B} \subset B$ be dense Banach $*$-subalgebras such that the enveloping C^*-algebra of $\mathcal{A}$ is A and the enveloping C^*-algebra of $\mathcal{B}$ is B. Suppose further there is a dense $\mathcal{A}$-$\mathcal{B}$-inner product submodule $\mathcal{E} \subset E$ such that the conditions above hold with $\mathcal{A}, \mathcal{B}, \mathcal{E}$ instead of A, B, E. In that case, we say $\mathcal{E}$ is an *$\mathcal{A}$-$\mathcal{B}$-pre-equivalence bimodule.*

It is a well-known result that if E is an A-B-equivalence bimodule, then $B \cong \mathbb{K}_A(E)$ through the identification $\Theta_{f,g} \mapsto \langle f, g \rangle_{\bullet}$. Here, $\Theta_{f,g}$ is the compact module operator $\Theta_{f,g} : h \mapsto {}_{\bullet}\langle h, f \rangle g$. In particular, $\| {}_{\bullet}\langle f, f \rangle \| = \| \langle f, f \rangle_{\bullet} \|$ for all $f \in E$. We shall only have need for the case when E is a finitely generated Hilbert A-module. For future use, we record the following result.

Proposition 2.3. *Let E be an A-B-equivalence bimodule. Then E is a finitely generated projective A-module if and only if B is unital.*

The result is typically proved by finding sets $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_n\}$ of elements of E for which

$$f = \sum_{I=1}^{n} {}_{\bullet}\langle f, g_i \rangle h_i = \sum_{i=1}^{n} f \langle g_i, h_i \rangle_{\bullet}$$

for all $f \in E$, as done in [26, Proposition 2.1] and later [27, Proposition 1.2]. Note that the systems $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_n\}$ are not necessarily A-linearly independent, but they still provide a reconstruction formula.

The following result concerns frame bounds for module frames consisting of a single element, though we do not formally introduce module frames until Definition 3.6. It will turn out that it is enough to consider module frames consisting of only one element, see Remark 3.8. The results will come into play when we relate module frames and Gabor frames in Sec. 4.

Lemma 2.4. *Let A be any C^*-algebra, and let E be a left Hilbert A-module. Let $T \in \mathrm{End}_A(E)$ be such that there exist $C, D > 0$ such that*

$$C \, {}_{\bullet}\langle f, f \rangle \leq {}_{\bullet}\langle Tf, f \rangle \leq D \, {}_{\bullet}\langle f, f \rangle, \tag{2.1}$$

for all $f \in E$. Then the smallest possible value of D is $\|T\|$, and the largest possible value for C is $\|T^{-1}\|^{-1}$.

Proof. Since T is positive, we have $\|T\| = \sup_{\|f\|=1} \{ \| {}_{\bullet}\langle Tf, f \rangle \| \}$. It follows that the smallest value for D is $\|T\|$. Furthermore, it is easy to see by (2.1) that ${}_{\bullet}\langle T^{-1} f, f \rangle \leq C^{-1} \, {}_{\bullet}\langle f, f \rangle$. Hence, by the same argument applied to T^{-1} the smallest value of C^{-1} is $\|T^{-1}\|$, from which it follows that the largest value of C is $\|T^{-1}\|^{-1}$. □

Since we aim to mimic the situation of Gabor analysis, the positive linear functional that we localize our Morita equivalence bimodule with respect to will have a particular form. In particular, it will be a faithful trace. For unital Morita equivalent C^*-algebras A and B Rieffel showed in [26] that there is a bijection between nonnormalized finite traces on A and nonnormalized finite traces on B under which to a trace tr_B on B, there is an associated trace tr_A on A satisfying

$$\mathrm{tr}_A({}_\bullet\langle f, g\rangle) = \mathrm{tr}_B(\langle g, f\rangle_\bullet) \tag{2.2}$$

for all $f, g \in E$. Here, E is the Morita equivalence bimodule. We will in the sequel almost always consider A or B unital, and so instead, we will suppose the existence of a finite faithful trace on one C^*-algebra (the unital one) and induce a possibly unbounded trace on the other C^*-algebra. The following was proved in [3, Proposition 3.1] and ensures that this procedure works. Note that if both C^*-algebras are unital then the induced trace is also a finite trace as in [26], the result can be deduced from [26, Proposition 2.2].

Proposition 2.5. *Let E be an A-B-equivalence bimodule, and suppose tr_B is a faithful finite trace on B. Then the following hold:*

(i) *There is a unique lower semi-continuous trace on A, denoted tr_A, for which*

$$\mathrm{tr}_A({}_\bullet\langle f, g\rangle) = \mathrm{tr}_B(\langle g, f\rangle_\bullet) \tag{2.3}$$

for all $f, g \in E$. Moreover, tr_A is faithful, and densely defined since it is finite on $\mathrm{span}\{{}_\bullet\langle f, g\rangle : f, g \in E\}$. Setting

$$\langle f, g\rangle_{\mathrm{tr}_A} = \mathrm{tr}_A({}_\bullet\langle f, g\rangle), \quad \langle f, g\rangle_{\mathrm{tr}_B} = \mathrm{tr}_B(\langle g, f\rangle_\bullet),$$

for $f, g \in E$ defines $\mathbb{C}$-valued inner products on E, with $\langle f, g\rangle_{\mathrm{tr}_A} = \langle f, g\rangle_{\mathrm{tr}_B}$ for all $f, g \in E$. Consequently, the Hilbert space obtained by completing E in the norm $\|f\|' = \mathrm{tr}_A({}_\bullet\langle f, f\rangle)^{1/2}$, denoted H_E, is just the localization of E with respect to tr_B.

(ii) *If E and F are equivalence A-B-bimodules, then every adjointable A-linear operator $E \to F$ has a unique extension to a bounded linear operator $H_E \to H_F$. Furthermore, the map $\mathrm{End}_A(E, F) \to \mathrm{End}(H_E, H_F)$ given by sending T to its unique extension is a norm-decreasing linear map of Banach spaces. Finally, if $E = F$, the map $\mathrm{End}_A(E) \to \mathbb{B}(H_E)$ is an isometric $*$-homomorphism of C^*-algebras.*

3. Duality for Equivalence Bimodules

3.1. *The equivalence bimodule picture*

In Gabor analysis one considers not just Gabor frames, but multi-window super Gabor frames. Indeed, we will in Sec. 4 introduce matrix Gabor frames, which will turn out to generalize multi-window super Gabor frames. To aid in our approach to these types of frames, we shall want to lift an A-B-equivalence bimodule E to an equivalence module between matrix algebras over A and B. We will soon make this precise. For $k \in \mathbb{N}$, let $\mathbb{Z}_k$ denote the group $\mathbb{Z}/(k\mathbb{Z})$. To simplify formulas in the sequel, we will zero-index matrices, i.e. the top left corner will correspond to $(0,0)$. For $n, d \in \mathbb{N}$, we then consider the space of $n \times d$-matrices with entries from E, denoted $M_{n,d}(E)$. $M_{n,d}(E)$ naturally becomes an $M_n(A)$-$M_d(B)$-equivalence bimodule with actions and inner products defined below. Here, $M_n(A)$ is the usual C^*-algebra consisting of $n \times n$-matrices over A, and likewise for $M_d(B)$. We will still let the A-valued inner product on E be denoted by ${}_\bullet\langle -, - \rangle$, and the B-valued inner product on E be denoted $\langle -, - \rangle_\bullet$. Define an $M_n(A)$-valued inner product on $M_{n,d}(E)$ by

$$
\begin{aligned}
&{}_\bullet[-,-] : M_{n,d}(E) \times M_{n,d}(E) \to M_n(A) \\
&(f,g) \mapsto \sum_{k \in \mathbb{Z}_d} \begin{pmatrix} {}_\bullet\langle f_{0,k}, g_{0,k} \rangle & {}_\bullet\langle f_{0,k}, g_{1,k} \rangle & \cdots & {}_\bullet\langle f_{0,k}, g_{n-1,k} \rangle \\ {}_\bullet\langle f_{1,k}, g_{0,k} \rangle & {}_\bullet\langle f_{1,k}, g_{1,k} \rangle & \cdots & {}_\bullet\langle f_{1,k}, g_{n-1,k} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ {}_\bullet\langle f_{n-1,k}, g_{0,k} \rangle & {}_\bullet\langle f_{n-1,k}, g_{1,k} \rangle & \cdots & {}_\bullet\langle f_{n-1,k}, g_{n-1,k} \rangle \end{pmatrix}.
\end{aligned}
$$

The action of $M_n(A)$ on $M_{n,d}(E)$ is defined in the natural way, that is

$$
(af)_{i,j} = \sum_{k \in \mathbb{Z}_n} a_{i,k} f_{k,j},
$$

for $a \in M_n(A)$ and $f \in M_{n,d}(E)$. Likewise, we define an $M_d(B)$-valued inner product on $M_{n,d}(E)$ in the following way

$$
\begin{aligned}
&[-,-]_\bullet : M_{n,d}(E) \times M_{n,d}(E) \to M_d(B) \\
&(f,g) \mapsto \sum_{k \in \mathbb{Z}_n} \begin{pmatrix} \langle f_{k,0}, g_{k,0} \rangle_\bullet & \langle f_{k,0}, g_{k,1} \rangle_\bullet & \cdots & \langle f_{k,0}, g_{k,d-1} \rangle_\bullet \\ \langle f_{k,1}, g_{k,0} \rangle_\bullet & \langle f_{k,1}, g_{k,1} \rangle_\bullet & \cdots & \langle f_{k,1}, g_{k,d-1} \rangle_\bullet \\ \vdots & \vdots & \ddots & \vdots \\ \langle f_{k,d-1}, g_{k,0} \rangle_\bullet & \langle f_{k,d-1}, g_{k,1} \rangle_\bullet & \cdots & \langle f_{k,d-1}, g_{k,d-1} \rangle_\bullet \end{pmatrix}.
\end{aligned}
$$

The right action of $M_d(B)$ on $M_{n,d}(E)$ is defined by

$$
(fb)_{i,j} = \sum_{k \in \mathbb{Z}_d} f_{i,k} b_{k,j}
$$

for $f \in M_{n,d}(E)$ and $b \in M_d(B)$.

With this setup, $M_{n,d}(E)$ becomes an $M_n(A)$-$M_d(B)$-equivalence bimodule. It is not hard to verify the three conditions of Definition 2.2. Indeed, the setup above is just the one induced by the usual Morita equivalence of $\mathbb{C}$ with $M_k(\mathbb{C})$, $k \in \mathbb{N}$. In particular, we have for $f, g, h \in M_{n,d}(E)$ that

$$
{}_\bullet[f,g]h = f[g,h]_\bullet,
$$

and also

$$
\begin{aligned}
M_n(A) &= \mathbb{K}_{M_d(B)}(M_{n,d}(E)), \\
M_d(B) &= \mathbb{K}_{M_n(A)}(M_{n,d}(E)).
\end{aligned}
$$

Moreover, since the new inner products are defined using the inner products ${}_{\bullet}\langle -, -\rangle$ and $\langle -, -\rangle_{\bullet}$, we see that in case we have Banach $*$-subalgebras $\mathcal{A} \subset A$ and $\mathcal{B} \subset B$, as well as an $\mathcal{A}$-$\mathcal{B}$-subbimodule $\mathcal{E} \subset E$ as above, we get

$$ {}_{\bullet}[M_{n,d}(\mathcal{E}), M_{n,d}(\mathcal{E})] \subset M_n(\mathcal{A}), \quad [M_{n,d}(\mathcal{E}), M_{n,d}(\mathcal{E})]_{\bullet} \subset M_d(\mathcal{B}), \tag{3.1} $$

as well as

$$ M_n(\mathcal{A})M_{n,d}(\mathcal{E}) \subset M_{n,d}(\mathcal{E}), \quad M_{n,d}(\mathcal{E})M_d(\mathcal{B}) \subset M_d(\mathcal{E}). \tag{3.2} $$

Remark 3.1. While it is far from surprising that $M_{n,d}(E)$ becomes an $M_n(A)$-$M_d(B)$-equivalence bimodule, the resulting actions and inner products above will in Sec. 4 make natural the construction of a new type of Gabor frame which generalizes the n-multi-window d-super Gabor frames considered in [19], see Definition 4.7 and Proposition 4.29.

Definition 3.2. Let E be an A-B-equivalence bimodule and let $n, d \in \mathbb{N}$. For $g \in M_{n,d}(E)$, we define the *analysis operator* by

$$ \Phi_g : M_{n,d}(E) \to M_n(A) $$
$$ f \mapsto {}_{\bullet}[f, g], $$

and the *synthesis operator*:

$$ \Psi_g : M_n(A) \to M_{n,d}(E) $$
$$ a \mapsto a \cdot g. $$

An elementary computation shows that $\Phi_g^* = \Psi_g$. We will not make n, and later d, explicit in the notation for the analysis and synthesis operators. It will be implicit from the atom g being in $M_{n,d}(E)$.

Remark 3.3. As $M_{n,d}(E)$ is an $M_n(A)$-$M_d(B)$-bimodule, we could just as well have defined the analysis operator and the synthesis operator with respect to the $M_d(B)$-valued inner product. Indeed, we will need this later, but it will then be indicated by writing Φ_g^B. Unless otherwise indicated, the analysis operator and synthesis operator will be with respect to the left inner product module structure.

Definition 3.4. Let E be an A-B-equivalence bimodule and let $n, d \in \mathbb{N}$. For $g, h \in M_{n,d}(E)$, we define the *frame-like operator* $\Theta_{g,h}$ to be

$$ \Theta_{g,h} : E \to E $$
$$ f \mapsto {}_{\bullet}[f, g] \cdot h. $$

In other words, $\Theta_{g,h} = \Psi_h \Phi_g = \Phi_h^* \Phi_g$. The *frame operator of* g is the operator

$$ \Theta_g := \Theta_{g,g} = \Phi_g^* \Phi_g : E \to E $$
$$ f \mapsto {}_{\bullet}\langle f, g\rangle g. $$

Remark 3.5. The frame operator Θ_g is a positive operator since $\Theta_g = (\Phi_g)^*\Phi_g$.

Definition 3.6. Let E be an A-B-equivalence bimodule and let $n, d \in \mathbb{N}$. We say $g \in M_{n,d}(E)$ generates a *(single)* $M_n(A)$*-module frame for* $M_{n,d}(E)$ if Θ_g is an invertible operator $M_{n,d}(E) \to M_{n,d}(E)$. Equivalently, there exist constants $C, D > 0$ such that

$$C\,{}_\bullet[f, f] \leq {}_\bullet[f, g]\,{}_\bullet[g, f] \leq D\,{}_\bullet[f, f],$$

holds for all $f \in M_{n,d}(E)$.

Remark 3.7. When g generates a module frame for E, Θ_g is a positive invertible operator on E.

Remark 3.8. If we are willing to change the integer n in the above setup, we can show that it is really always sufficient to consider a single generator. Indeed, suppose we have $g_1, \ldots, g_k \in M_{n,d}(E)$, $k \in \mathbb{N}$, such that $\sum_{i=1}^{k} \Theta_{g_i}$ is invertible as a map $M_{n,d}(E) \to M_{n,d}(E)$. This is equivalent to existence of constants $C, D > 0$ such that

$$C\,{}_\bullet[f, f] \leq \sum_{i=1}^{k} {}_\bullet[f, g_i]\,{}_\bullet[g_i, f] \leq D\,{}_\bullet[f, f]$$

for all $f \in M_{n,d}(E)$. In other words, $(g_i)_{i=1}^{k}$ is what is typically known as an $M_n(A)$-module frame for $M_{n,d}(E)$. This is then equivalent to existence constants $C', D' > 0$ such that

$$C'\,{}_\bullet[f', f'] \leq {}_\bullet[f', g]\,{}_\bullet[g, f'] \leq D'\,{}_\bullet[f', f']$$

for all $f' \in M_{kn,d}(E)$ and where $g = (g_1, \ldots, g_k)^T \in M_{kn,d}(E)$. In the last equation, the inner products are $M_{kn}(A)$-valued.

We will now begin to formulate the Morita equivalence bimodule versions of central results of Gabor analysis, and we will show in Sec. 4 that the results localize to the well-known Gabor analysis results, but for the very general type of Gabor frame introduced in Definition 4.7.

The following result, while quite obvious in this context, will localize to one of the cornerstones of Gabor analysis, namely the Wexler–Raz biorthogonality relations, see Proposition 4.30.

Proposition 3.9 (Wexler–Raz for equivalence bimodules). *Let E be an A-B-equivalence bimodule and let $n, d \in \mathbb{N}$. Let $g, h \in M_{n,d}(E)$. Then $f = \Theta_{g,h} f = \Theta_{h,g} f$ for all $f \in M_{n,d}(E)$ if and only if $M_d(B)$ is unital and $\langle g, h\rangle_\bullet = \langle h, g\rangle_\bullet = 1_{M_d(B)}$.*

Proof. In the standard isomorphism $\mathbb{K}_{M_n(A)}(M_{n,d}(E)) \cong M_d(B)$, we send $\Theta_{g,h}$ to the element $[g, h]_\bullet \in M_d(B)$, from which the result follows immediately. □

We also record the following result for use in Sec. 4.

Proposition 3.10. *Let E be an A-B-equivalence bimodule and let $n, d \in \mathbb{N}$. Let $g, h \in M_{n,d}(E)$ be so that ${}_\bullet[f, h]g = f$ for all $f \in M_{n,d}(E)$. Then*

$$f = h[g, f]_\bullet \quad \textit{for all } f \in \overline{h \cdot M_d(B)}.$$

Proof. By assumption $M_{n,d}(E)$ is finitely generated and projective as an $M_n(A)$-module, hence $M_d(B) \cong \mathbb{K}_{M_n(A)}(M_{n,d}(E)) = \mathrm{End}_{M_n(A)}(M_{n,d}(E))$ and $M_d(B)$ is unital. We may rewrite the equality to $f = f[h, g]_\bullet$ for all $f \in M_{n,d}(E)$, which implies $[h, g]_\bullet = 1_{M_d(B)}$ as $M_d(B)$ acts faithfully on $M_{n,d}(E)$. But then

$$[g, h]_\bullet = [h, g]_\bullet{}^* = 1^*_{M_d(B)} = 1_{M_d(B)}$$

as well. Then if we let $f \in h \cdot M_d(B)$, we may write $f = hb$ for some $b \in M_d(B)$, and we get

$$h[g, f]_\bullet = h[g, hb]_\bullet = h[g, h]_\bullet b = h1_{M_d(B)}b = hb = f.$$

We extend the reconstruction formula to $\overline{h \cdot M_d(B)}$ by continuity. □

We shall have use for the following definition, which can be formulated on more general modules than equivalence bimodules, but we shall not need the more general setting.

Definition 3.11. Let E be an A-B-equivalence bimodule and let $n, d \in \mathbb{N}$. If $g \in M_{n,d}(E)$ is such that Θ_g is invertible on $M_{n,d}(E)$, then $h = (\Theta_g)^{-1}g$ is called the *canonical dual atom of g*.

Remark 3.12. Note that if g is such that $\Theta_g : M_{n,d}(E) \to M_{n,d}(E)$ is invertible, then $M_n(A) \cdot g = M_{n,d}(E)$. To see this, let $f \in M_{n,d}(E)$. Then

$$f = \Theta_g(\Theta_g)^{-1}f = {}_\bullet[\Theta_g^{-1}f, g]g \in M_n(A) \cdot g.$$

There is a correspondence between projections in Morita equivalent C^*-algebras, see for example [27, Proposition 1.2]. We formulate the following variant.

Proposition 3.13. *Let E be an A-B-equivalence bimodule between a C^*-algebra A and a unital C^*-algebra B, and let $n, d \in \mathbb{N}$. If $g, h \in M_{n,d}(E)$ are such that $[g, h]_\bullet = 1_{M_d(B)}$, then ${}_\bullet[g, h]$ is an idempotent in $M_n(A)$. If $h = \Theta_g^{-1}g$, then ${}_\bullet[g, h]$ yields a projection in $M_n(A)$.*

Proof. From $[g, h]_\bullet = 1_{M_d(B)} = 1^*_{M_d(B)} = [h, g]_\bullet$, we get

$${}_\bullet[g, h]\,{}_\bullet[g, h] = {}_\bullet[\,{}_\bullet[g, h]g, h] = {}_\bullet[g[h, g]_\bullet, h] = {}_\bullet[g \cdot 1_{M_d(B)}, h] = {}_\bullet[g, h],$$

so ${}_\bullet[g, h]$ is an idempotent in $M_n(A)$. If $h = (\Theta_g)^{-1}g$, we also have

$${}_\bullet[g, h] = {}_\bullet[g, \Theta_g^{-1}g] = {}_\bullet[\Theta_g^{-1}g, g] = {}_\bullet[h, g] = {}_\bullet[g, h]^*,$$

so ${}_\bullet[g, h]$ is a projection in $M_n(A)$. □

The duality principle is a cornerstone of the field of Gabor analysis, see for example [10, 20, 28]. One of the main intentions of this investigation is a reformulation of this duality principle in our module framework. Indeed, the duality principle in the Hilbert C^*-module picture turns out to be quite an elementary statement. To this end, we introduce the following operator. As before, let E be an A-B-equivalence bimodule and let $n, d \in \mathbb{N}$. For an element $g \in M_{n,d}(E)$, we define the $M_d(B)$-coefficient operator by

$$\Phi_g^B : M_{n,d}(E) \to M_d(B)$$
$$f \mapsto [g, f]_{\bullet}.$$

Note that this operator is B-adjointable with adjoint

$$(\Phi_g^B)^* b \mapsto g \cdot b.$$

We are now in the position to state and prove the module version of the duality principle which will localize to the duality principle of Gabor analysis in Theorem 4.31.

Proposition 3.14 (Module Duality Principle). *Let E be an A-B-equivalence bimodule, let $n, d \in \mathbb{N}$, and let $g \in M_{n,d}(E)$. The following are equivalent.*

(1) $\Theta_g : M_{n,d}(E) \to M_{n,d}(E)$ *is invertible.*
(2) $\Phi_g^B(\Phi_g^B)^* : M_d(B) \to M_d(B)$ *is an isomorphism.*

Proof. We show that both statements are equivalent to $[g, g]_{\bullet}$ being invertible in $M_d(B)$. Suppose Θ_g is invertible. Then $M_{n,d}(E)$ is finitely generated and projective as an $M_n(A)$-module, so $M_d(B) \cong \mathbb{K}_{M_n(A)}(M_{n,d}(E))$ is unital. As

$$\Theta_g f = f[g, g]_{\bullet},$$

statement (1) is equivalent to $[g, g]_{\bullet}$ being invertible in $M_d(B)$. On the other hand,

$$\Phi_g^B(\Phi_g^B)^* b = \Phi_g^B(g \cdot b) = [g, g \cdot b]_{\bullet} = [g, g]_{\bullet}\, b.$$

Since $\Phi_g^B(\Phi_g^B)^* \in \mathrm{End}_{M_d(B)}(M_d(B))$ and $M_d(B)$ is an ideal in $\mathrm{End}_{M_d(B)}(M_d(B))$, statement (2) implies that $M_d(B)$ is unital and the statement is equivalent to $[g, g]_{\bullet}$ being invertible in $M_d(B)$. □

In Gabor analysis, one is often concerned with the regularity of the atoms generating a Gabor frame, as these often have desirable properties. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{E}$ be as in the setup in (3.1) and (3.2). In case g is so that Θ_g is invertible on all of $M_{n,d}(E)$ with $g \in M_{n,d}(\mathcal{E})$, and $\mathcal{B} \subset B$ is a spectrally invariant Banach $*$-subalgebra with the same unit as B, the canonical dual atom has the following important property.

Proposition 3.15. *Let E be an A-B-equivalence bimodule, with an $\mathcal{A}$-$\mathcal{B}$-pre-equivalence bimodule $\mathcal{E} \subset E$, and let $n, d \in \mathbb{N}$. Suppose $\mathcal{B} \subset B$ is spectrally invariant with the same unit. If $g \in M_{n,d}(\mathcal{E})$ is such that $\Theta_g : M_{n,d}(E) \to M_{n,d}(E)$ is invertible, then the canonical dual $(\Theta_g)^{-1} g \in M_{n,d}(\mathcal{E})$ as well.*

Proof. For $f \in M_{n,d}(E)$, we have

$$\Theta_g f = {}_\bullet[f, g]g = f[g, g]_\bullet.$$

We deduce that $[g, g]_\bullet$ is invertible in $M_d(B)$ and $(\Theta_g)^{-1}g = g[g, g]_\bullet{}^{-1}$. But as $g \in M_{n,d}(\mathcal{E})$, we have $[g, g]_\bullet \in M_d(\mathcal{B})$. By spectral invariance of $\mathcal{B}$ in B, it follows that $[g, g]_\bullet{}^{-1} \in M_d(\mathcal{B})$, see [29, Theorem 2.1]. Then, since $M_{n,d}(\mathcal{E}) \cdot M_d(\mathcal{B}) \subset M_{n,d}(\mathcal{E})$, it follows that

$$(\Theta_g)^{-1}g = g[g, g]_\bullet{}^{-1} \in M_{n,d}(\mathcal{E}),$$

which is the desired assertion. □

There are well-known theorems in Gabor analysis known as density theorems. Postponing the precise formulations and technicalities, they give restrictions on existence of certain spanning sets, e.g. Gabor frames, in terms of the volume of cocompact subgroups of phase space, see Propositions 4.33 and 4.34. We proceed to establish analogous statements for module frames on certain equivalence bimodules.

More precisely, let E be an A-B-equivalence bimodule, and let B be unital with faithful finite trace tr_B. We induce a trace tr_A on A by ways of Proposition 2.5. Now let $n, d \in \mathbb{N}$, and consider $M_{n,d}(E)$ as an $M_n(A)$-$M_d(B)$ equivalence bimodule. Then there are traces on $M_n(A)$ and $M_d(B)$ satisfying

$$\mathrm{tr}_{M_n(A)}({}_\bullet[f, g]) = \mathrm{tr}_{M_d(B)}([g, f]_\bullet)$$

for all $f, g \in M_{n,d}(E)$. They can be written as

$$\mathrm{tr}_{M_n(A)}({}_\bullet[f, g]) = \frac{1}{n}\sum_{i\in\mathbb{Z}_n} \mathrm{tr}_A({}_\bullet[f, g]_{i,i}), \quad \mathrm{tr}_{M_d(B)}([f, g]_\bullet) = \frac{1}{n}\sum_{i\in\mathbb{Z}_d} \mathrm{tr}_B([f, g]_{\bullet\, i,i}). \tag{3.3}$$

The trace on $M_d(B)$ extends to a finite trace on the whole algebra, but the same might not be true for the densely defined trace on $M_n(A)$. It is however true if A, and hence, also $M_n(A)$, is unital. It is easy to show that the lifting process on the traces preserves both finiteness and faithfulness. We may then present our density theorems for equivalence bimodules. These appear to be new in the literature, and we will in Sec. 4, use them to deduce density theorems statements for matrix Gabor frames, which will also be introduced in the same section.

Theorem 3.16. *Let E be an A-B-equivalence bimodule where both A and B are unital and equipped with faithful finite traces tr_A and tr_B related by* (2.3), *and let $n, d \in \mathbb{N}$. If $g \in M_{n,d}(E)$ is such that $\Theta_g : M_{n,d}(E) \to M_{n,d}(E)$ is invertible, then*

$$d\,\mathrm{tr}_B(1_B) \le n\,\mathrm{tr}_A(1_A). \tag{3.4}$$

Proof. The assumption that Θ_g is invertible implies $[g, g]_\bullet$ is invertible. Then

$$u = \Theta_g^{-1}g = g[g, g]_\bullet{}^{-1}$$

is the canonical dual frame for $M_{n,d}(E)$. We have $[g, u]_\bullet = [u, g]_\bullet = 1_{M_d(B)}$, and by Proposition 3.13 ${}_\bullet[g, u]$ is a projection in $M_n(A)$. Observing that ${}_\bullet[g, u] \le 1_{M_n(A)}$

in $M_n(A)$ and using (3.3), we get

$$d\,\mathrm{tr}_B(1_B) = n\cdot\frac{1}{n}\sum_{i=1}^{d}\mathrm{tr}_B(1_B) = n\,\mathrm{tr}_{M_d(B)}(1_{M_d(B)}) = n\,\mathrm{tr}_{M_d(B)}([u,g]_\bullet)$$

$$= n\,\mathrm{tr}_{M_n(A)}({}_\bullet[g,u]) \leq n\,\mathrm{tr}_{M_n(A)}(1_{M_n(A)}) = n\cdot\frac{1}{n}\sum_{i=1}^{n}\mathrm{tr}_A(1_A) = n\,\mathrm{tr}_A(1_A).$$

□

Theorem 3.17. *Let E be an A-B-equivalence bimodule, where both A and B are unital and equipped with faithful finite traces* tr_A *and* tr_B *related by Eq.* (2.3), *and let $n, d \in \mathbb{N}$. If $g \in M_{n,d}(E)$ is such that $\Phi_g\Phi_g^*: M_n(A) \to M_n(A)$ is an isomorphism, then*

$$d\,\mathrm{tr}_B(1_B) \geq n\,\mathrm{tr}_A(1_A). \tag{3.5}$$

Proof. The assumptions imply ${}_\bullet[g,g]^{-1} \in M_n(A)$, so it follows that

$$1_{M_n(A)} = {}_\bullet[g,g]^{-1}\,{}_\bullet[g,g] = {}_\bullet[{}_\bullet[g,g]^{-1}g, g],$$

and $[{}_\bullet[g,g]^{-1}g, g]_\bullet$ is a projection in $M_d(B)$ by Proposition 3.13. Since B is unital, then by observing that $[{}_\bullet[g,g]^{-1}g, g]_\bullet \leq 1_{M_d(B)}$ in $M_d(B)$ and using (3.3), we get

$$n\,\mathrm{tr}_A(1_A) = n\cdot\frac{1}{n}\sum_{i=1}^{n}\mathrm{tr}_A(1_A) = n\,\mathrm{tr}_{M_n(A)}(1_{M_n(A)}) = n\,\mathrm{tr}_{M_n(A)}({}_\bullet[{}_\bullet[g,g]^{-1}g, g])$$

$$= n\,\mathrm{tr}_{M_d(B)}([g, {}_\bullet[g,g]^{-1}g]_\bullet) \leq n\,\mathrm{tr}_{M_d(B)}(1_{M_d(B)})$$

$$= n\cdot\frac{1}{n}\sum_{i=1}^{d}\mathrm{tr}_B(1_B) = d\,\mathrm{tr}_B(1_B).$$

□

3.2. *Passing to the localization*

In [22], the existence of multi-window Gabor frames for $L^2(\mathbb{R}^d)$ with windows in Feichtinger's algebra was proved through considerations on a related Hilbert C^*-module. Furthermore, in [23], projections in noncommutative tori were constructed from Gabor frames with sufficiently regular windows. Thus being able to pass from an equivalence bimodule E to a localization H_E and back is quite important, and we dedicate this section to results on this procedure. We will interpret this in terms of Gabor analysis in Sec. 4, and we will explain how $L^2(G)$, for G a second countable LCA group, relates to H_E for specific modules E which arise in the study of twisted group C^*-algebras.

In the following, let E be an A-B-equivalence bimodule. We will make the presence of traces precise in the individual results. To ease notation, we will not formulate the below results in the setting of $M_{n,d}(E)$ being an $M_n(A)$-$M_d(B)$-equivalence bimodule, $n, d \in \mathbb{N}$, as such a reformulation is easy but notationally tedious.

Proposition 3.18. *Let E be an A-B-equivalence bimodule, where B is unital and equipped with a faithful finite trace tr_B. We induce a trace tr_A on A by (2.3) and denote by H_E the localization of E in tr_A, and by $(-,-)_E$ the inner product on the localization of E in tr_A, i.e. $(f_1, f_2)_E = \mathrm{tr}_A({}_\bullet\langle f_1, f_2\rangle)$ for all $f_1, f_2 \in E$. Now suppose $g \in E$. Then there exists an $h \in E$ such that we have ${}_\bullet\langle f, g\rangle h = f$ for all $f \in E$ if and only if there exist constants $C, D > 0$ such that*

$$C(f, f)_E \leq (f\langle g, g\rangle_\bullet, f)_E \leq D(f, f)_E \tag{3.6}$$

for all $f \in H_E$. In other words, g generates a module frame for E as an A-module if and only if the inequalities in (3.6) are satisfied for some $C, D > 0$.

Remark 3.19. We should note that in the setting of Proposition 3.18 it is possible to say that $\langle g, g\rangle_\bullet$ is invertible in B if and only if there is h such that $\langle g, h\rangle_\bullet = 1_B$. Indeed, one may obtain this by [13, Theorem 5.9], in the case A is unital, and by [3, Proposition 2.6], in the case that A is not unital. One could use this to deduce Proposition 3.18. However, our proof gives frame bounds which are of independent interest, see Proposition 4.36. Since we want to focus on the link between module frames and Hilbert space frames, we therefore offer a more direct argument.

Proof. Suppose first that there is an $h \in E$ such that ${}_\bullet\langle f, g\rangle h = f$ for all $f \in E$. By Morita equivalence this implies

$$f = {}_\bullet\langle f, g\rangle h = f\langle g, h\rangle_\bullet$$

for all $f \in E$. As before, this implies $1_B = \langle g, h\rangle_\bullet = \langle h, g\rangle_\bullet$. Since tr_B is a positive linear functional, we obtain

$$\begin{aligned}
(f, f)_E &= \mathrm{tr}_A({}_\bullet\langle f, f\rangle) \\
&= \mathrm{tr}_A({}_\bullet\langle f\langle g, h\rangle_\bullet\langle h, g\rangle_\bullet, f\rangle) \\
&= \mathrm{tr}_A({}_\bullet\langle f\langle g, h\langle h, g\rangle_\bullet\rangle_\bullet, f) \\
&= \mathrm{tr}_A({}_\bullet\langle f\langle g, {}_\bullet\langle h, h\rangle g\rangle_\bullet, f\rangle) \\
&= \mathrm{tr}_B(\langle f, f\langle g, {}_\bullet\langle h, h\rangle g\rangle_\bullet\rangle_\bullet) \\
&\leq \mathrm{tr}_B(\langle f, f\langle g, g\rangle_\bullet \|{}_\bullet\langle h, h\rangle\|\rangle_\bullet) \\
&= \|{}_\bullet\langle h, h\rangle\| \,\mathrm{tr}_B(\langle f, f\langle g, g\rangle_\bullet\rangle_\bullet) \\
&= \|{}_\bullet\langle h, h\rangle\| \,\mathrm{tr}_A({}_\bullet\langle f\langle g, g\rangle_\bullet, f\rangle) \\
&= \|{}_\bullet\langle h, h\rangle\|(f\langle g, g\rangle_\bullet, f)_E,
\end{aligned}$$

for all $f \in E$, where we have used

$$\langle g, {}_\bullet\langle h, h\rangle g\rangle_\bullet \leq \|{}_\bullet\langle h, h\rangle\| \langle g, g\rangle_\bullet,$$

see e.g. [24, Corollary 2.22]. We then get the lower frame bound with $C = \|{}_\bullet\langle h, h\rangle\|^{-1}$, that is

$$\frac{1}{\|{}_\bullet\langle h, h\rangle\|}(f, f)_E \leq (f\langle g, g\rangle_\bullet, f)_E$$

for all $f \in E$. By Proposition 2.1, all intermediate steps involve operators that extend to bounded operators on H_E, so we may extend by continuity. We get the upper frame bound by use of [24, Corollary 2.22], in the following manner

$$\begin{aligned}(f\langle g, g\rangle_\bullet, f)_E &= \mathrm{tr}_A({}_\bullet\langle f\langle g, g\rangle_\bullet, f\rangle) \\ &= \mathrm{tr}_A({}_\bullet\langle f\langle g, g\rangle_\bullet{}^{1/2}, f\langle g, g\rangle_\bullet{}^{1/2}\rangle) \\ &\leq \|\langle g, g\rangle_\bullet{}^{1/2}\|^2 \,\mathrm{tr}_A({}_\bullet\langle f, f\rangle) \\ &= \|\langle g, g\rangle_\bullet\| \,\mathrm{tr}_A({}_\bullet\langle f, f\rangle) \\ &= \|{}_\bullet\langle g, g\rangle\|(f, f)_E,\end{aligned}$$

for all $f \in E$. Once again, all intermediate steps involve operators that extend to bounded operators on H_E by Proposition 2.1, so we may extend the result to all of H_E. Thus, we have shown that

$$\frac{1}{\|{}_\bullet\langle h, h\rangle\|}(f, f)_E \leq (f\langle g, g\rangle_\bullet, f)_E \leq \|{}_\bullet\langle g, g\rangle\|(f, f)_E$$

for all $f \in H_E$.

Conversely, suppose there are $C, D > 0$ such that

$$C(f, f)_E \leq (f\langle g, g\rangle_\bullet, f)_E \leq D(f, f)_E$$

for all $f \in H_E$. We wish to show that this implies there exists $h \in E$ such that ${}_\bullet\langle f, g\rangle h = f$ for all $f \in E$. The assumption implies that $f \mapsto f\langle g, g\rangle_\bullet$ is a positive, invertible operator on H_E. As C^*-algeras are inverse closed it follows that $\langle g, g\rangle_\bullet$ is invertible in B. Thus, $f \mapsto f\langle g, g\rangle_\bullet$ is a positive, invertible operator on E as well. Hence the operator

$$\begin{aligned}\Theta_g : E &\to E \\ f &\mapsto {}_\bullet\langle f, g\rangle g = f\langle g, g\rangle_\bullet\end{aligned}$$

is invertible with inverse

$$\Theta_g^{-1} f = f\langle g, g\rangle_\bullet{}^{-1}.$$

Define $h := \Theta_g^{-1} g$, and let $f \in E$ be arbitrary. Then we have

$${}_\bullet\langle f, g\rangle h = {}_\bullet\langle f, g\rangle \Theta_g^{-1} g = \Theta_g^{-1}({}_\bullet\langle f, g\rangle g) = \Theta_g^{-1}\Theta_g f = f,$$

from which the result follows. □

We are interested in module frames and module Riesz sequences, and their relationship to frames and Riesz sequences in Gabor analysis for LCA groups. To get results on Riesz sequences in Sec. 4, we need a module version of Riesz sequences which, when localized, yields the Riesz sequences we know from Gabor analysis. To make the transition to Gabor frames in Sec. 4 easier, we will in the following result let A be unital with a faithful trace tr_A, and we will localize A as a Hilbert A-module in the trace tr_A, i.e. we let $(a_1, a_2)_A := \mathrm{tr}_A(a_1 a_2^*)$. The completion of A in this inner product will be denoted H_A, and the action of A on H_A is the continuous extension of the multiplication action of A on itself.

Proposition 3.20. *Let E be an A-B-equivalence bimodule where A is unital and equipped with a faithful finite trace tr_A. We localize E as in the setting of Proposition* 3.18 *and localize A as described above. Now suppose $g \in E$. Then $\Phi_g \Phi_g^* : A \to A$ is an isomorphism if and only if there exist $C, D > 0$ such that for all $a \in A$ it holds that*

$$C(a,a)_A \leq (ag, ag)_E \leq D(a,a)_A. \tag{3.7}$$

Proof. First, suppose $\Phi_g \Phi_g^* : A \to A$ is an isomorphism. Then, as ${}_\bullet\langle g, g\rangle \geq 0$ in A, we have

$${}_\bullet\langle ag, ag\rangle = a\, {}_\bullet\langle g, g\rangle a^* \leq \| {}_\bullet\langle g, g\rangle \| aa^*,$$

and we may deduce

$$(ag, ag)_A = \mathrm{tr}_A({}_\bullet\langle ag, ag\rangle) \leq \| {}_\bullet\langle g, g\rangle \| \, \mathrm{tr}_A(aa^*) = \| {}_\bullet\langle g, g\rangle \| (a,a)_A.$$

Hence, in (3.7), we may set $D = \| {}_\bullet\langle g, g\rangle \|$. Since $\Phi_g \Phi_g^* : A \to A$ is an isomorphism and $\Phi_g \Phi_g^* a = a\, {}_\bullet\langle g, g\rangle$, it follows that there is ${}_\bullet\langle g, g\rangle^{-1} \in A$. Then

$$\begin{aligned}
(a,a)_A &= \mathrm{tr}_A(aa^*)_A \\
&= \mathrm{tr}_A(a\, {}_\bullet\langle g, g\rangle^{1/2}\, {}_\bullet\langle g, g\rangle^{-1}\, {}_\bullet\langle g, g\rangle^{1/2} a^*) \\
&\leq \| {}_\bullet\langle g, g\rangle^{-1} \| \, \mathrm{tr}_A(a\, {}_\bullet\langle g, g\rangle a^*) \\
&= \| {}_\bullet\langle g, g\rangle^{-1} \| \, \mathrm{tr}_A({}_\bullet\langle ag, ag\rangle) \\
&= \| {}_\bullet\langle g, g\rangle^{-1} \| (ag, ag)_E,
\end{aligned}$$

which implies that we may set $C = \| {}_\bullet\langle g, g\rangle^{-1} \|^{-1}$ in (3.7). All intermediate steps extend to H_A by Proposition 2.1.

Suppose now that (3.7) is satisfied. The lower inequality in (3.7) tells us that for all $a \in A$,

$$\begin{aligned}
(a({}_\bullet\langle g, g\rangle - C), a)_A &= \mathrm{tr}_A(a({}_\bullet\langle g, g\rangle - C)a^*) \\
&= \mathrm{tr}_A(a\, {}_\bullet\langle g, g\rangle a^*) - C\, \mathrm{tr}_A(aa^*) \\
&= \mathrm{tr}_A({}_\bullet\langle ag, ag\rangle) - C\, \mathrm{tr}_A(aa^*) \\
&= (ag, ag)_E - C(a,a)_A \geq 0.
\end{aligned}$$

Note that we need the upper inequality of (3.7) to extend all intermediate steps to H_A via Proposition 2.1. It follows that ${}_\bullet\langle g, g\rangle$ is a positive invertible operator on $H_A \supset A$. As C^*-algebras are inverse closed it follows that ${}_\bullet\langle g, g\rangle$ is invertible in A. Then, since

$$\Phi_g \Phi_g^* a = a \, {}_\bullet\langle g, g\rangle,$$

it follows that $\Phi_g \Phi_g^* : A \to A$ is an isomorphism. □

Remark 3.21. Note that in the proofs of the two preceding results the upper bounds in (3.6) and (3.7) were both satisfied with $D = \| {}_\bullet\langle g, g\rangle\|$. We will see in Sec. 4 that in the Gabor analysis setting, this means that all atoms coming from the Hilbert C^*-module are Bessel vectors for the localized frame system.

For use in Sec. 4, we introduce the following notion.

Definition 3.22. Let E be an A-B-equivalence bimodule, let $n, d \in \mathbb{N}$, and let $g \in M_{n,d}(E)$. If $\Phi_g \Phi_g^* : M_n(A) \to M_n(A)$ is an isomorphism, we say g generates a *module Riesz sequence for $M_{n,d}(E)$ with respect to $M_n(A)$.*

4. The Link to Gabor Analysis

In this section, we show how the above results reproduce some of the core results of Gabor analysis for LCA groups. We will find that some of the cornerstones of Gabor analysis on LCA groups are trivial consequences of the above framework.

To present the results, we will need to explain how time-frequency analysis on LCA groups relates to Morita equivalence of twisted group C^*-algebras. In the interest of brevity, we refer the reader to [19] for a more in-depth treatment of time frequency analysis and its relation to twisted group C^*-algebras, and to [17] for a survey on the Feichtinger algebra. We can also not omit to mention [27], which is a major inspiration for a lot of work done in the intersection of Gabor analysis and operator algebras.

Throughout this section, we fix a second-countable LCA group G and let $\widehat{G}$ be its dual group. We fix a Haar measure μ_G on G and normalize the Haar measure $\mu_{\widehat{G}}$ on $\widehat{G}$ such that the Plancherel theorem holds. By Λ, we denote a closed subgroup of the time-frequency plane $G \times \widehat{G}$. The induced topologies and group multiplications on Λ and $(G \times \widehat{G})/\Lambda$ turn them into LCA groups as well, and we may equip them with their respective Haar measures. Having fixed the Haar measures on $G, \widehat{G}$, and Λ, we will assume $(G \times \widehat{G})/\Lambda$ is equipped with the unique Haar measure such that Weil's formula holds, see e.g. [11, Theorem 1.5.3]. In this setting, we can define the *size of* Λ by

$$s(\Lambda) := \int_{(G\times\widehat{G})/\Lambda} 1 d\mu_{(G\times\widehat{G})/\Lambda}. \tag{4.1}$$

Note that $s(\Lambda)$ is finite if and only if Λ is cocompact in $G \times \widehat{G}$.

For any $\xi = (x, \omega) \in G \times \widehat{G}$, we may then define the time-frequency shift operator

$$\begin{aligned}\pi(\xi)\colon L^2(G) &\to L^2(G)\\ \pi(\xi)f(t) &= \omega(t)f(t-x)\end{aligned}$$

for $t \in G$ and $f \in L^2(G)$. We also define the 2-cocycle

$$\begin{aligned}c\colon (G \times \widehat{G}) \times (G \times \widehat{G}) &\to \mathbb{T}\\ (\xi_1, \xi_2) &\mapsto \overline{\omega_2(x_1)}\end{aligned}$$

for $\xi_1 = (x_1, \omega_1), \xi_2 = (x_2, \omega_2) \in G \times \widehat{G}$. Note then that

$$\pi(\xi_1)\pi(\xi_2) = c(\xi_1, \xi_2)\,\pi(\xi_1 + \xi_2).$$

For the reader's convenience, we also note that

$$\pi(\xi)^* = c(\xi, \xi)\pi(-\xi)$$

for all $\xi \in G \times \widehat{G}$.

For the given closed subgroup $\Lambda \subseteq G \times \widehat{G}$, we define its *adjoint subgroup* Λ° by

$$\Lambda^\circ := \{\xi \in G \times \widehat{G} \mid \pi(\xi)\,\pi(\lambda) = \pi(\lambda)\,\pi(\xi) \text{ for all } \lambda \in \Lambda\}.$$

Note that $(\Lambda^\circ)^\circ = \Lambda$ and $\widehat{\Lambda^\circ} \cong (G \times \widehat{G})/\Lambda$, see for example [18]. Moreover, Λ is cocompact if and only if Λ° is discrete. With these identifications, we put on Λ° the Haar measure such that the Plancherel theorem holds with respect to Λ° and $(G \times \widehat{G})/\Lambda$.

We want to reframe time-frequency analysis in terms of Morita equivalence bimodules for certain twisted group C^*-algebras. To do this, we use the Feichtinger algebra. In order to introduce this, we first define the *short-time Fourier transform* with respect to a function $g \in L^2(G)$ as the operator

$$V_g \colon L^2(G) \to L^2(G \times \widehat{G}), \quad V_g f(\xi) = \langle f, \pi(\xi)g\rangle,$$

for $\xi \in G \times \widehat{G}$. The *Feichtinger algebra* $S_0(G)$ can be defined by

$$S_0(G) = \{f \in L^2(G) \mid V_f f \in L^1(G \times \widehat{G})\}.$$

A norm on $S_0(G)$ is given by

$$\|f\|_{S_0(G)} = \|V_g f\|_{L^1(G\times\widehat{G})} \quad \text{for some } g \in S_0(G)\backslash\{0\}.$$

It is a nontrivial fact that all elements of $S_0(G)\backslash\{0\}$ determine equivalent norms on $S_0(G)$. In case G is discrete one has $S_0(G) = \ell^1(G)$ with equivalent norms. Furthermore, $S_0(G)$ consists of continuous functions and is dense in both $L^1(G)$ and $L^2(G)$.

With the above norm, $S_0(\Lambda)$ becomes a Banach $*$-algebra when equipped with the twisted convolution and involution given by

$$F_1 \natural F_2(\lambda) := \int_\Lambda F_1(\lambda')F_2(\lambda-\lambda')c(\lambda',\lambda-\lambda')\mathrm{d}\lambda',$$
$$F_1^*(\lambda) := c(\lambda,\lambda)\overline{F_1(-\lambda)},$$

for $F_1, F_2 \in S_0(\Lambda)$ and $\lambda \in \Lambda$. We denote the resulting Banach $*$-algebra by $S_0(\Lambda, c)$.

It was shown in [19] that when Λ is a closed subgroup of $G \times \widehat{G}$ the map $\lambda \mapsto \pi(\lambda)$ is a faithful c-projective unitary representation of Λ, and the integrated representation becomes a nondegenerate $*$-representation of $S_0(\Lambda, c)$ as bounded operators on $L^2(G)$. In other words, given $\mathbf{a} \in S_0(\Lambda, c)$, we have the representation given by

$$\pi(\mathbf{a})f = \int_\Lambda \mathbf{a}(\lambda)\pi(\lambda)f\mathrm{d}\lambda,$$

for $f \in L^2(G)$, and where we interpret the integral weakly. It is well known that this $*$-representation is faithful. Indeed, it was shown in [27] for the case of the Schwartz–Bruhat space, and the arguments easily carry over to the Feichtinger algebra. By completing $S_0(\Lambda, c)$ in the C^*-algebra norm coming from the integrated representation, we obtain a C^*-algebra which we denote by $C^*(\Lambda, c)$. It is well known that this coincides with the enveloping C^*-algebra of $S_0(\Lambda, c)$. We do the same for $S_0(\Lambda^\circ, \overline{c})$, and denote its universal enveloping C^*-algebra by $C^*(\Lambda^\circ, \overline{c})$.

Now, $S_0(G)$ becomes a pre-equivalence $S_0(\Lambda, c)$-$S_0(\Lambda^\circ, \overline{c})$-equivalence bimodule as in Definition 2.2 when equipped with the actions

$$\mathbf{a} \cdot f = \int_\Lambda a(\lambda)\pi(\lambda)f d\lambda, \quad f \cdot \mathbf{b} = \int_{\Lambda^\circ} b(\lambda^\circ)\pi(\lambda^\circ)^* f d\lambda^\circ \tag{4.2}$$

for $\mathbf{a} \in S_0(\Lambda, c)$, $\mathbf{b} \in S_0(\Lambda^\circ, \overline{c})$, and $f, g \in S_0(G)$, and with algebra-valued inner products given by

$${}_\bullet\langle f, g\rangle(\lambda) = \langle f, \pi(\lambda)g\rangle, \quad \langle f, g\rangle_\bullet(\lambda^\circ) = \langle g, \pi(\lambda^\circ)^* f\rangle \tag{4.3}$$

for $f, g \in S_0(G)$, $\lambda \in \Lambda$, $\lambda^\circ \in \Lambda^\circ$. The inner products on the right-hand sides of the equality signs are those of $L^2(G)$. That these are well-defined was noted in [19, Sec. 3]. As is typical, we pass to the C^*-completions. The resulting completion of $S_0(G)$ will be denoted $E_{G,\Lambda}$. As done in the Schwartz–Bruhat case in [27], we note that $E_{G,\Lambda}$ is a $C^*(\Lambda, c)$-$C^*(\Lambda^\circ, \overline{c})$-equivalence bimodule.

Remark 4.1. The fact that we get the same twisted group C^*-algebras by using $S_0(\Lambda, c)$ as we get when using the more traditional approach with $L^1(\Lambda, c)$ was noted in [3].

An important consequence of working with S_0 instead of L^1 is that we have well-defined traces on dense Banach $*$-subalgebras of $C^*(\Lambda, c)$ and $C^*(\Lambda^\circ, \overline{c})$. Indeed, since S_0-functions are continuous, there are well-defined canonical faithful traces on $S_0(\Lambda, c)$ and $S_0(\Lambda^\circ, \overline{c})$ given by evaluation in 0. We will denote the trace on $S_0(\Lambda, c)$ by tr_Λ and the trace on $S_0(\Lambda^\circ, \overline{c})$ by $\mathrm{tr}_{\Lambda^\circ}$ in the sequel.

Remark 4.2. Although the traces tr_Λ and $\mathrm{tr}_{\Lambda^\circ}$ do not in general extend to the C^*-algebras $C^*(\Lambda, c)$ and $C^*(\Lambda^\circ, \overline{c})$, we can guarantee they extend in one case. Namely, tr_Λ extends to all of $C^*(\Lambda, c)$ if $C^*(\Lambda, c)$ is unital, which is equivalent to Λ being discrete. The same is of course true for $C^*(\Lambda^\circ, \overline{c})$ and $\mathrm{tr}_{\Lambda^\circ}$, with the discreteness condition on Λ°. This is due to the fact that the trace given by evaluation in the identity extends to twisted group C^*-algebras when the underlying group is discrete [6, p. 951].

The following result is straightforward to prove and explains why we in the sequel will focus mostly on the case where Λ is closed and cocompact.

Proposition 4.3. *Let $\Lambda \subset G \times \widehat{G}$ be a closed subgroup. Then $E_{G,\Lambda}$ is a finitely generated projective $C^*(\Lambda, c)$-module if and only if $\Lambda \subset G \times \widehat{G}$ is a cocompact subgroup.*

As a very last preparation before starting to connect our results of Sec. 3 to Gabor analysis, we note the following important result. It was shown in [15], in the case of lattices in $\mathbb{R}^{2d}$ and in the same paper it was claimed to hold for more general lattices in phase spaces of arbitrary LCA groups. It was shown for arbitrary discrete subgroups of phase spaces of LCA groups in [2].

Proposition 4.4. *For a discrete subgroup Λ in $G \times \widehat{G}$ the involutive Banach algebra $S_0(\Lambda, c)$ is spectrally invariant in $C^*(\Lambda, c)$.*

To get results on Gabor frames for $L^2(G)$ with windows in E from the above setup, we will need to localize certain subsets of the C^*-algebras $C^*(\Lambda, c)$ and $C^*(\Lambda^\circ, \overline{c})$, as well as the Morita equivalence bimodule $E_{G,\Lambda}$, just as explained in Sec. 2. For simplicity, let Λ be cocompact in $G \times \widehat{G}$ from now on, unless otherwise specified. Then Λ° is discrete and $\mathrm{tr}_{\Lambda^\circ}$ is defined on all of $C^*(\Lambda^\circ, \overline{c})$. The localization of $C^*(\Lambda^\circ, \overline{c})$ in $\mathrm{tr}_{\Lambda^\circ}$ is induced by the inner product $(-,-)_{\Lambda^\circ}$ given by

$$(b_1, b_2)_{\Lambda^\circ} := \mathrm{tr}_{\Lambda^\circ}(b_1^* b_2).$$

Since $S_0(\Lambda^\circ, \overline{c})$ is dense in $C^*(\Lambda^\circ, \overline{c})$ and $\mathrm{tr}_{\Lambda^\circ}$ is continuous with respect to the C^*-norm, it follows that their localizations in $\mathrm{tr}_{\Lambda^\circ}$ are the same. For $b_1, b_2 \in S_0(\Lambda^\circ, \overline{c})$, we then have

$$\begin{aligned}(b_1, b_2)_{\Lambda^\circ} &= \mathrm{tr}_{\Lambda^\circ}(b_1^* b_2) \\ &= \mathrm{tr}_{\Lambda^\circ}\left(\sum_{\lambda^\circ \in \Lambda^\circ} \overline{b_1(\lambda^\circ)} \pi(\lambda^\circ)^* \sum_{\xi \in \Lambda^\circ} b_2(\xi) \pi(\xi) \right) \\ &= \mathrm{tr}_{\Lambda^\circ}\left(\sum_{\lambda^\circ \in \Lambda^\circ} \sum_{\xi \in \Lambda^\circ} \overline{b_1(\lambda^\circ)} b_2(\xi) c(\lambda^\circ, \lambda^\circ) \pi(-\lambda^\circ) \pi(\xi) \right)\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{tr}_{\Lambda^\circ}\left(\sum_{\lambda^\circ\in\Lambda^\circ}\sum_{\xi\in\Lambda^\circ}\overline{b_1(\lambda^\circ)}b_2(\xi)c(\lambda^\circ,\lambda^\circ)c(-\lambda^\circ,\xi)\pi(-\lambda^\circ+\xi)\right)\\
&= \mathrm{tr}_{\Lambda^\circ}\left(\sum_{\lambda^\circ\in\Lambda^\circ}\sum_{\xi\in\Lambda^\circ}\overline{b_1(\lambda^\circ+\xi)}b_2(\xi)c(\lambda^\circ+\xi,\lambda^\circ+\xi)c(-\lambda^\circ-\xi,\xi)\pi(-\lambda^\circ)\right)\\
&= \sum_{\xi\in\Lambda^\circ}\overline{b_1(\xi)}b_2(\xi)c(\xi,\xi)c(-\xi,\xi)\\
&= \sum_{\xi\in\Lambda^\circ}\overline{b_1(\xi)}b_2(\xi)\\
&= \langle b_1,b_2\rangle_{\ell^2(\Lambda^\circ)}.
\end{aligned}
$$

As $S_0(\Lambda^\circ,\overline{c}) = \ell^1(\Lambda^\circ,\overline{c})$ is dense in $\ell^2(\Lambda^\circ)$, we may identify the localization $H_{C^*(\Lambda^\circ,\overline{c})}$ of $C^*(\Lambda^\circ,\overline{c})$ with $\ell^2(\Lambda^\circ)$. By [3, Proposition 3.3], we also obtain that the localization of $E_{G,\Lambda}$ in $\mathrm{tr}_{\Lambda^\circ}$ is $L^2(G)$. Note that this is the same as the localization of E in tr_Λ by construction, and that there is an action of $C^*(\Lambda,c)$ on $L^2(G)$ by extending the action of $C^*(\Lambda,c)$ on $E_{G,\Lambda}$.

It is slightly more tricky to localize subsets of $C^*(\Lambda,c)$. Indeed, it is not in general possible as the trace might not be defined everywhere. However, even if $C^*(\Lambda,c)$ is not unital, we may localize the algebraic ideal ${}_\bullet\langle E_{G,\Lambda},E_{G,\Lambda}\rangle\subset C^*(\Lambda,c)$ in the trace tr_Λ. Indeed, by [3, Theorem 3.10], elements of $E_{G,\Lambda}$ are such that whenever $g\in E_{G,\Lambda}$ and $f\in L^2(G)$, then $\{\langle f,\pi(\lambda)g\rangle\}_{\lambda\in\Lambda}\in L^2(\Lambda)$. This is the property of being a Bessel vector, which we will discuss in more detail below. Hence, for any $f,g\in E_{G,\Lambda}$, we may identify ${}_\bullet\langle f,g\rangle\in C^*(\Lambda,c)$ with $(\langle f,\pi(\lambda)g\rangle)_{\lambda\in\Lambda}$ in $L^2(\Lambda)$ by doing the analogous procedure with tr_Λ as for $\mathrm{tr}_{\Lambda^\circ}$ above.

We may do the same for the matrix algebras and matrix modules considered in Sec. 3. Note that ${}_\bullet[M_{n,d}(E_{G,\Lambda}),M_{n,d}(E_{G,\Lambda})] = M_{n,d}({}_\bullet\langle E_{G,\Lambda},E_{G,\Lambda}\rangle)$ in the setup of Sec. 3. Adapting the setting of twisted group C^*-algebras and Heisenberg modules above to the matrix algebra setting of Sec. 3, we see that we obtain the following identifications

$$
\begin{aligned}
H_{M_d(C^*(\Lambda^\circ,\overline{c}))} &= \ell^2(\Lambda^\circ\times\mathbb{Z}_d\times\mathbb{Z}_d),\\
H_{M_n({}_\bullet\langle E_{G,\Lambda},E_{G,\Lambda}\rangle)} &= L^2(\Lambda\times\mathbb{Z}_n\times\mathbb{Z}_n),\\
H_{M_{n,d}(E_{G,\Lambda})} &= L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d).
\end{aligned}\tag{4.4}
$$

Remark 4.5. Should Λ° be cocompact and therefore Λ discrete, we do the obvious changes. Also if both Λ and Λ° are discrete, that is, they are both lattices, then we may localize all of $M_n(C^*(\Lambda,c))$ and all of $M_d(C^*(\Lambda^\circ,\overline{c}))$.

Remark 4.6. Note that when we do the above lifting process to obtain the identifications of (4.4), we may still identify Λ as being in $G\times\widehat{G}$. That is, even though after the lifting process Λ is technically inside $G \times \mathbb{Z}_n \times \mathbb{Z}_d \times \widehat{G} \times \widehat{\mathbb{Z}_n} \times \widehat{\mathbb{Z}_d}$, Λ will be identified as embedded along the units of $\mathbb{Z}_n$, $\mathbb{Z}_d$ and their duals in this product space. This will be a standing assumption throughout the rest of the paper.

We are finally ready to present the material and constructions which constitute the main results and novelty of this paper in terms of time-frequency analysis. As a first step towards this, we will consider a novel type of Gabor frames. To ease notation, we will for $f \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ write $f_{i,j}$ instead of $f(\cdot, i, j)$, and the same for elements of $L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$ and $L^2(\Lambda^\circ \times \mathbb{Z}_d \times \mathbb{Z}_d)$.

Definition 4.7. Let Λ be a closed subgroup of $G \times \widehat{G}$. For $g \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$, we define the coefficient operator C_g by

$$C_g : L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \to L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$$

$$C_g(f) = \left\{ \sum_{m \in \mathbb{Z}_d} \langle f_{k,m}, \pi(\lambda) g_{l,m} \rangle \right\}_{\lambda \in \Lambda, k,l \in \mathbb{Z}_n}$$

and the synthesis operator D_g by

$$D_g : L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) \to L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$$

$$D_g a = \left\{ \sum_{m \in \mathbb{Z}_n} \int_\Lambda a_{k,m}(\lambda) \pi(\lambda) g_{m,l} \mathrm{d}\lambda \right\}_{k \in \mathbb{Z}_n, l \in \mathbb{Z}_d}.$$

Furthermore, we define the *frame-like operator* $S_{g,h} = D_h C_g$, and for brevity, we write S_g for $D_g C_g$. We say S_g is the *frame operator associated to* g.

We say g generates an (n,d)*-matrix Gabor frame for* $L^2(G)$ with respect to Λ if $S_g : L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \to L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ is an isomorphism. Equivalently, the collection of time-frequency shifts

$$\mathcal{G}(g; \Lambda) := \{\pi(\lambda) g_{i,j} \mid \lambda \in \Lambda\}_{i \in \mathbb{Z}_n, j \in \mathbb{Z}_d}$$

is a frame for $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$. We then say that $\mathcal{G}(g; \Lambda)$ is an (n,d)*-matrix Gabor frame for* $L^2(G)$. Equivalently, there exists $h \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ such that for all $f \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ we have

$$f_{r,s} = \sum_{k \in \mathbb{Z}_d} \sum_{l \in \mathbb{Z}_n} \int_\Lambda \langle f_{r,k}, \pi(\lambda) g_{l,k} \rangle \pi(\lambda) h_{l,s} d\lambda, \tag{4.5}$$

for all $r \in \mathbb{Z}_n$ and $s \in \mathbb{Z}_d$. When g and h satisfy (4.5), we say $\mathcal{G}(g; \Lambda)$ and $\mathcal{G}(h; \Lambda)$ are a *dual pair of* (n,d)*-matrix Gabor frames*. If Λ is implicit, we may also say h is a *dual* (n,d)*-matrix Gabor atom for* g, or just a *dual atom of* g.

Remark 4.8. The equivalence of the definitions of (n,d)-matrix Gabor frames given in Definition 4.7 follows by [8, Lemma 6.3.2] and Proposition 4.12.

Remark 4.9. When $\mathcal{G}(g;\Lambda)$ is an (n,d)-matrix Gabor frame for $L^2(G)$, there is always a dual (n,d)-matrix Gabor atom for g, namely $h = S_g^{-1}g$. This is known as the *canonical dual of* g.

Remark 4.10. One can verify that $C_g = D_g^*$. Thus, S_g is always a positive operator between Hilbert spaces, just as for the module frame operator in Sec. 3.

For general, $g \in L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ the operator C_g will not be bounded. Functions g such that C_g is bounded are of interest on their own.

Definition 4.11. If $g \in L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ is so that $C_g : L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d) \to L^2(\Lambda\times\mathbb{Z}_n\times\mathbb{Z}_n)$ is a bounded operator, we say g is an (n,d)*-matrix Gabor Bessel vector for* $L^2(G)$ *with respect to* Λ, or that $\mathcal{G}(g;\Lambda)$ is an (n,d)*-matrix Gabor Bessel system for* $L^2(G)$. Equivalently, there is $D > 0$ such that for all $f \in L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$, we have

$$\langle f, f\rangle \le D\langle C_g f, C_g f\rangle, \tag{4.6}$$

which may also be written as

$$\sum_{i\in\mathbb{Z}_n}\sum_{j\in\mathbb{Z}_d}\int_G |f_{i,j}(\xi)|^2 d\xi \le D\sum_{k,l\in\mathbb{Z}_n}\int_\Lambda \left|\sum_{m\in\mathbb{Z}_d}\langle f_{k,m}, \pi(\lambda)g_{l,m}\rangle\right|^2 d\lambda.$$

The smallest $D > 0$ such that the condition of (4.6) holds is called the *optimal Bessel bound of* $\mathcal{G}(g;\Lambda)$.

The Gabor frames of Definition 4.7 seemingly generalize the n-multi-window d-super Gabor frames of [19]. Indeed, we obtain n-multi-window d-super Gabor frames if we only require reconstruction of $f \in L^2(G\times\mathbb{Z}_d)$ and we identify $L^2(G\times\mathbb{Z}_d) \subset L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ by embedding along a single element of $\mathbb{Z}_n$. Hence, (4.5) generalizes both multi-window Gabor frames and super Gabor frames as well, setting $d = 1$ or $n = 1$, respectively. However, we will in Proposition 4.29 show that any n-multi-window d-super Gabor frame for $L^2(G)$ with respect to Λ is an (n,d)-matrix Gabor frame for $L^2(G)$ with respect to Λ. In spite of this, we continue to call them by separate names, since, as mentioned above, they are used for reconstruction in different Hilbert spaces.

The following proposition was noted in the $(n,1)$-matrix case in [3, Theorem 3.10], and its proof in the (n,d)-matrix Gabor case goes through the same except with more bookkeeping.

Proposition 4.12. *Let* $\Lambda \subset G\times\widehat{G}$ *be closed and cocompact. For every* $g \in M_{n,d}(E_{G,\Lambda})$, $C_g : L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d) \to L^2(\Lambda\times\mathbb{Z}_n\times\mathbb{Z}_n)$ *is a bounded operator. In other words, every* $g \in M_{n,d}(E_{G,\Lambda})$ *is a Bessel vector.*

For ease of notation, the localization map in $M_n(C^*(\Lambda,c))$ will be denoted by ρ_Λ, though note that we might not be able to localize all of $M_n(C^*(\Lambda,c))$. With the above definitions, the following calculation is justified for $f, g \in M_{n,d}(E_{G,\Lambda}) \subset L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ by Proposition 4.12

$$\begin{aligned}\rho_\Lambda \Phi_g(f) &= \rho_\Lambda({}_\bullet[\, f, g]) \\ &= \rho_\Lambda\left(\left\{\sum_{m\in\mathbb{Z}_d}\int_\Lambda \langle f_{k,m}, \pi(\lambda)g_{l,m}\rangle \pi(\lambda)\right\}_{k,l\in\mathbb{Z}_n}\right) \\ &= \left\{\sum_{m\in\mathbb{Z}_d}\langle f_{k,m}, \pi(\lambda)g_{l,m}\rangle\right\}_{\lambda\in\Lambda,k,l\in\mathbb{Z}_n} = C_g \rho_{M_{n,d}(E_{G,\Lambda})}(f).\end{aligned}$$

Hence, we obtain the following result.

Lemma 4.13. *Let $\Lambda \subset G \times \widehat{G}$ be closed and cocompact. For every $g \in M_{n,d}(E_{G,\Lambda})$, the module coefficient operator Φ_g localizes to give the coefficient operator C_g. Equivalently, the diagram*

$$\begin{array}{ccc} M_{n,d}(E_{G,\Lambda}) & \xrightarrow{\quad\Phi_g\quad} & M_n(C^*(\Lambda, c)) \\ \downarrow{\scriptstyle \rho_{M_{n,d}(E_{G,\Lambda})}} & & \downarrow{\scriptstyle \rho_\Lambda} \\ L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) & \xrightarrow{\quad C_g\quad} & L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) \end{array}$$

commutes for all $g \in M_{n,d}(E_{G,\Lambda})$.

Likewise, one may obtain $C_g^*\rho_\Lambda = \rho_{M_{n,d}(E_{G,\Lambda})}\Phi_g^* : M_n({}_\bullet\langle E_{G,\Lambda}, E_{G,\Lambda}\rangle) \to L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ for all $g \in M_{n,d}(E_{G,\Lambda})$. Note that the domain might be larger, but we cannot guarantee this unless $C^*(\Lambda, c)$ is unital, that is, when Λ is discrete.

Lemma 4.14. *Let $\Lambda \subset G \times \widehat{G}$ be closed and cocompact. For every $g \in M_{n,d}(E_{G,\Lambda})$, the module synthesis operator Φ_g^* localizes to the Gabor synthesis operator C_g^*. Equivalently, the diagram*

$$\begin{array}{ccc} M_n({}_\bullet\langle E_{G,\Lambda}, E_{G,\Lambda}\rangle) & \xrightarrow{\quad\Phi_g^*\quad} & M_{n,d}(E_{G,\Lambda}) \\ \downarrow{\scriptstyle \rho_\Lambda} & & \downarrow{\scriptstyle \rho_{M_{n,d}(E_{G,\Lambda})}} \\ L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) & \xrightarrow{\quad C_g^*\quad} & L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \end{array}$$

commutes for every $g \in M_{n,d}(E_{G,\Lambda})$.

Combining Lemmas 4.13 and 4.14, we then obtain

Proposition 4.15. *Let $\Lambda \subset G \times \widehat{G}$ be closed and cocompact. For all $g, h \in M_{n,d}(E_{G,\Lambda})$, $S_{g,h}\rho_{M_{n,d}(E_{G,\Lambda})} = \rho_{M_{n,d}(E_{G,\Lambda})}\Theta_{g,h}$, meaning the module frame-like operator $\Theta_{g,h}$ localizes to the frame-like operator $S_{g,h}$. Equivalently,* *the diagram*

$$\begin{array}{ccc} M_{n,d}(E_{G,\Lambda}) & \xrightarrow{\Theta_{g,h}} & M_{n,d}(E_{G,\Lambda}) \\ \Big\downarrow \rho_{M_{n,d}(E_{G,\Lambda})} & & \Big\downarrow \rho_{M_{n,d}(E_{G,\Lambda})} \\ L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) & \xrightarrow{S_{g,h}} & L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \end{array}$$

commutes for all $g \in M_{n,d}(E_{G,\Lambda})$.

As $\rho_{M_{n,d}(E_{G,\Lambda})} : M_{n,d}(E_{G,\Lambda}) \to \rho_{M_{n,d}(E_{G,\Lambda})}(M_{n,d}(E_{G,\Lambda}))$ is a linear bijection intertwining both the $C^*(\Lambda, c)$-actions and the $C^*(\Lambda^\circ, \overline{c})$-actions, we see by Proposition 4.15 that for $g \in M_{n,d}(E_{G,\Lambda})$, Θ_g is invertible if and only if $S_g|_{\rho_{M_{n,d}(E_{G,\Lambda})}(M_{n,d}(E_{G,\Lambda}))}$ is invertible. But we also have the following result.

Lemma 4.16. *Let* $\Lambda \subset G \times \widehat{G}$ *be closed and cocompact, and let* $g \in M_{n,d}(E_{G,\Lambda})$. *Then*

$$S_g|_{\rho_{M_{n,d}(E_{G,\Lambda})}(M_{n,d}(E_{G,\Lambda}))} : \rho_{M_{n,d}(E_{G,\Lambda})}(M_{n,d}(E_{G,\Lambda})) \to \rho_{M_{n,d}(E_{G,\Lambda})}(M_{n,d}(E_{G,\Lambda}))$$

is invertible if and only if

$$S_g : L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \to L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$$

is invertible.

Proof. Suppose first

$$S_g|_{\rho_{M_{n,d}(E_{G,\Lambda})}(M_{n,d}(E_{G,\Lambda}))} : \rho_{M_{n,d}(E_{G,\Lambda})}(M_{n,d}(E_{G,\Lambda})) \to \rho_{M_{n,d}(E_{G,\Lambda})}(M_{n,d}(E_{G,\Lambda}))$$

is invertible. Since any $g \in M_{n,d}(E_{G,\Lambda})$ is a Bessel vector by Proposition 4.12, we may extend the operator by continuity to obtain that $S_g : L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \to L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ is invertible as well.

Conversely, suppose $S_g : L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \to L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ is invertible. Since S_g is the continuous extension of Θ_g, it then follows by Proposition 2.1 and inverse closedness of C^*-algebras that Θ_g is invertible, which implies $S_g|_{\rho_{M_{n,d}(E_{G,\Lambda})}(M_{n,d}(E_{G,\Lambda}))}$ is invertible. ◻

Remark 4.17. From now on, we will identify $M_{n,d}(E)$ and its image in the localization, and we will do this without mention.

Combining Proposition 4.15 and Lemma 4.16, we obtain the following important result.

Proposition 4.18. *Let* $\Lambda \subset G \times \widehat{G}$ *be closed and cocompact. For* $g \in M_{n,d}(E_{G,\Lambda})$, *we have that* Θ_g *is invertible if and only if* S_g *is invertible. In other words,* g *generates a module frame for* $M_{n,d}(E_{G,\Lambda})$ *as an* $M_n(C^*(\Lambda, c))$*-module if and only* $\mathcal{G}(g; \Lambda)$ *is an* (n, d)*-matrix Gabor frame for* $L^2(G)$.

We also have the following important corollary.

Corollary 4.19. *Let $\Lambda \subset G \times \widehat{G}$ be closed and cocompact, and let $g, h \in M_{n,d}(E_{G,\Lambda})$. Then g and h generate dual (n,d)-matrix Gabor frames for $L^2(G)$ with respect to Λ if and only if $[g,h]_\bullet$ extends to the identity operator on $L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$.*

Proof. Suppose first $g, h \in M_{n,d}(E_{G,\Lambda})$ generate dual (n,d)-matrix Gabor frames for $L^2(G)$ with respect to Λ. Then we know that for all $f \in M_{n,d}(E_{G,\Lambda})$, we have

$$f = {}_\bullet[f,g]h = f[g,h]_\bullet,$$

from which we as before deduce that $[g,h]_\bullet = 1_{M_d(C^*(\Lambda^\circ,\overline{c}))}$. This extends by continuity to the identity operator on all of $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$.

Conversely, if $[g,h]_\bullet$ extends to the identity operator on $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$, then $[g,h]_\bullet$ acts as the identity on $M_{n,d}(E_{G,\Lambda})$. For any $f \in M_{n,d}(E_{G,\Lambda})$, we then have

$$f = f[g,h]_\bullet = {}_\bullet[f,g]h,$$

hence, (4.5) holds for all $f \in M_{n,d}(E_{G,\Lambda})$. But this extends to $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ by continuity, which implies that g and h generate dual (n,d)-matrix Gabor frames. □

Amongst other results, we wish to establish a duality principle for (n,d)-matrix Gabor frames. For this, we also need to treat (n,d)-matrix Gabor Riesz sequences and relate them to Definition 3.22.

Definition 4.20. Let $g \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$. We say g generates an *(n,d)-matrix Gabor Riesz sequence for $L^2(G)$ with respect to* Λ, or that $\mathcal{G}(g;\Lambda)$ is an *(n,d)-matrix Gabor Riesz sequence for $L^2(G)$*, if

$$C_g C_g^* : L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) \to L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$$

is an isomorphism. Equivalently, there exists $h \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ such that for all $a \in L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$, we have

$$a_{r,s}(\mu) = \sum_{i\in\mathbb{Z}_d}\sum_{j\in\mathbb{Z}_n}\left\langle \int_\Lambda a_{r,j}(\lambda)\pi(\lambda)g_{j,i}\,d\lambda, \pi(\mu)h_{s,i}\right\rangle \tag{4.7}$$

for all $r,s \in \mathbb{Z}_n$ and all $\mu \in \Lambda$. If (4.7) is satisfied, we will say h generates a *dual (n,d)-matrix Gabor Riesz sequence of g.*

Remark 4.21. Note that the equivalence of the definitions of (n,d)-matrix Gabor Riesz sequences in Definition 4.20 follows by [8, Theorem 3.6.6] and Proposition 4.12.

Remark 4.22. Equation (4.7) can be seen to be equivalent to $C_g C_h^* = C_h C_g^* = \mathrm{Id}_{L^2(\Lambda\times\mathbb{Z}_n\times\mathbb{Z}_n)}$.

Before treating localization of module matrix Riesz sequences and how they relate to matrix Gabor Riesz sequences, we do a necessary but justified simplification. Recall that existence of finite module matrix Riesz sequences for $M_{n,d}(E_{G,\Lambda})$ with respect to $M_n(C^*(\Lambda,c))$ requires $C^*(\Lambda,c)$ to be unital by Proposition 3.14. In the following, we therefore let Λ be discrete, but not necessarily cocompact. Hence, $C^*(\Lambda,c)$ is unital with a faithful trace, but $C^*(\Lambda^\circ,\overline{c})$ might not have that property. By [18, p. 251], we know that $\mathcal{G}(g;\Lambda)$ is a Bessel system with Bessel bound D if and only if $\mathcal{G}(g;\Lambda^\circ)$ is a Bessel system with Bessel bound D. Applying Proposition 4.12, we immediately get the following from Lemmas 4.13 and 4.14.

Proposition 4.23. *Let $\Lambda \subset G\times\widehat{G}$ be discrete. For all $g,h\in M_{n,d}(E_{G,\Lambda})$, we have $(C_hC_g^*)\circ\rho_{M_n(C^*(\Lambda,c))} = \rho_\Lambda\circ(\Phi_h\Phi_g^*)$. Equivalently, the diagram*

$$\begin{array}{ccc}
M_n(C^*(\Lambda,c)) & \xrightarrow{\quad\Phi_h\Phi_g^*\quad} & M_n(C^*(\Lambda,c)) \\
\Big\downarrow{\scriptstyle\rho_\Lambda} & & \Big\downarrow{\scriptstyle\rho_\Lambda} \\
L^2(\Lambda\times\mathbb{Z}_n\times\mathbb{Z}_n) & \xrightarrow{\quad C_hC_g^*\quad} & L^2(\Lambda\times\mathbb{Z}_n\times\mathbb{Z}_n)
\end{array}$$

commutes.

As $\rho_\Lambda\colon M_n(C^*(\Lambda,c)) \to \rho_\Lambda(M_n(C^*(\Lambda,c))$ is a linear bijection respecting the actions of $C^*(\Lambda,c)$, we see by Proposition 4.23 that for $g\in M_{n,d}(E_{G,\Lambda})$, $\Phi_g\Phi_g^*$ is an isomorphism if and only if $(C_gC_g^*)|_{\rho_\Lambda(M_n(C^*(\Lambda,c))}$ is an isomorphism. In analogy with Lemma 4.16, we have the following result.

Lemma 4.24. *Let $\Lambda\subset G\times\widehat{G}$ be discrete. For $g\in M_{n,d}(E)$, we have that*

$$(C_gC_g^*)|_{\rho_\Lambda(M_n(C^*(\Lambda,c)))}\colon \rho_\Lambda(M_n(C^*(\Lambda,c)))\to\rho_\Lambda(M_n(C^*(\Lambda,c)))$$

is invertible if and only if

$$C_gC_g^*\colon L^2(\Lambda\times\mathbb{Z}_n\times\mathbb{Z}_n)\to L^2(\Lambda\times\mathbb{Z}_n\times\mathbb{Z}_n)$$

is invertible.

Proof. Suppose first that $(C_gC_g^*)|_{\rho_\Lambda(M_n(C^*(\Lambda,c)))} : \rho_\Lambda(M_n(C^*(\Lambda,c))) \to \rho_\Lambda(M_n(C^*(\Lambda,c)))$ is invertible. Since any $g\in M_{n,d}(E_{G,\Lambda})$ is a Bessel vector by Proposition 4.12, we may extend the operator by continuity to obtain that $C_gC_g^*\colon L^2(\Lambda\times\mathbb{Z}_n\times\mathbb{Z}_n)\to L^2(\Lambda\times\mathbb{Z}_n\times\mathbb{Z}_n)$ is invertible as well.

Conversely, suppose $C_gC_g^*\colon L^2(\Lambda\times\mathbb{Z}_n\times\mathbb{Z}_n)\to L^2(\Lambda\times\mathbb{Z}_n\times\mathbb{Z}_n)$ is invertible. Since $C_gC_g^*$ is the continuous extension of $\Phi_g\Phi_g^*$, it then follows by Proposition 2.1 and inverse closedness of C^*-algebras that $\Phi_g\Phi_g^*$ is invertible as well, which implies $(C_gC_g^*)|_{\rho_\Lambda(M_n(C^*(\Lambda,c)))} : \rho_\Lambda(M_n(C^*(\Lambda,c)))\to\rho_\Lambda(M_n(C^*(\Lambda,c)))$ is invertible. $\square$

Remark 4.25. From now on, we will identify $M_n({}_\bullet\langle E_{G,\Lambda},E_{G,\Lambda}\rangle)$ (and potentially a larger domain) and its localization. The same goes for $M_d(C^*(\Lambda^\circ,\overline{c}))$.

Now the following is an immediate consequence.

Proposition 4.26. *Let $\Lambda \subset G \times \widehat{G}$ be discrete. For $g \in M_{n,d}(E_{G,\Lambda})$, we have that $\Phi_g \Phi_g^* \colon M_n(C^*(\Lambda, c)) \to M_n(C^*(\Lambda, c))$ is invertible if and only if $C_g C_g^* \colon L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_n) \to L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_n)$ is invertible. In other words, g generates a module Riesz sequence for $M_{n,d}(E_{G,\Lambda})$ as an $M_n(C^*(\Lambda, c))$-module if and only if $\mathcal{G}(g; \Lambda)$ is an (n, d)-matrix Gabor Riesz sequence for $L^2(G)$.*

By the proof of Lemma 4.24, we then have the following statement.

Corollary 4.27. *Let $\Lambda \subset G \times \widehat{G}$ be discrete. Suppose $g, h \in M_{n,d}(E_{G,\Lambda})$. Then g and h generate dual (n, d)-matrix Gabor Riesz sequences for $L^2(G)$ with respect to Λ if and only if ${}_\bullet[g, h]$ extends to the identity operator on $L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$.*

Proof. Suppose first that g and h generate dual (n, d)-matrix Gabor Riesz sequences for $L^2(G)$ with respect to Λ. Then for all $a \in M_n(C^*(\Lambda, c))$, we have

$$(a_{r,s}) = \left\{ \sum_{i\in\mathbb{Z}_d} \sum_{j\in\mathbb{Z}_n} \left\langle \int_\Lambda a_{r,j}(\lambda)\pi(\lambda)g_{j,i}\,d\lambda, \pi(\mu)h_{s,i} \right\rangle \right\}_{\mu\in\Lambda, r,s\in\mathbb{Z}_n},$$

which is equivalent to $a = a\,{}_\bullet[g, h]$ for all $a \in M_n(C^*(\Lambda, c))$. But the first expression extends by continuity to $L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$, so ${}_\bullet[g, h]$ extends to the identity on $L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$.

Conversely, suppose ${}_\bullet[g, h]$ extends to the identity on $L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$. Once again, for all $a \in M_n(C^*(\Lambda, c))$, we then have

$$(a_{r,s}) = \left\{ \sum_{i\in\mathbb{Z}_d} \sum_{j\in\mathbb{Z}_n} \left\langle \int_\Lambda a_{r,j}(\lambda)\pi(\lambda)g_{j,i}\,d\lambda, \pi(\mu)h_{s,i} \right\rangle \right\}_{\mu\in\Lambda, r,s\in\mathbb{Z}_n},$$

which again extends to $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_n)$. Hence, g and h are dual (n, d)-matrix Gabor Riesz sequences for $L^2(G)$ with respect to Λ. □

Note how the above results guarantee that when $\Lambda \subset G \times \widehat{G}$ is closed and cocompact and $g \in M_{n,d}(E_{G,\Lambda})$ is such that $\mathcal{G}(g; \Lambda)$ is an (n, d)-matrix Gabor frame for $L^2(G)$, the canonical dual frame $S_g^{-1} g \in M_{n,d}(E_{G,\Lambda})$. Indeed,

$$S_g^{-1} g = \Theta_g^{-1} g = g[g, g]_\bullet^{-1} \in M_{n,d}(E_{G,\Lambda}).$$

Likewise, for Riesz sequences, there is the notion of *canonical biorthogonal atom*, see for example [8, p. 160]. Restricting to Λ discrete, it is given by $(S_g^{\Lambda^\circ})^{-1} g$, where $S_g^{\Lambda^\circ}$ is the frame operator with respect to the right-hand side, that is, with respect to Λ°. We see that for all $f \in M_{n,d}(E_{G,\Lambda})$

$$S_g^{\Lambda^\circ} f = (\Phi_g^{C^*(\Lambda^\circ, \overline{c})})^* \Phi_g^{C^*(\Lambda^\circ, \overline{c})} f = (\Phi_g^{C^*(\Lambda^\circ, \overline{c})})^*([g, f]_\bullet) = g[g, f]_\bullet = {}_\bullet[g, g] f.$$

Thus, it follows that

$$(S_g^{\Lambda^\circ})^{-1} g = (\Theta_g^{C^*(\Lambda^\circ, \overline{c})})^{-1} g = {}_\bullet[g, g]^{-1} g \in M_{n,d}(E).$$

Hence, for both matrix Gabor frames and matrix Gabor Riesz sequences with generating atom in $M_{n,d}(E_{G,\Lambda})$, the canonically associated dual atoms are also in $M_{n,d}(E_{G,\Lambda})$. We have the following result which shows that in the cases we are interested in, if the generating atom is regular, the canonical dual atom has the same regularity.

Proposition 4.28. *Let $g \in M_{n,d}(S_0(G))$.*

(i) *If $\mathcal{G}(g;\Lambda)$ is an (n,d)-matrix Gabor frame for $L^2(G)$ and Λ is closed and cocompact in $G \times \widehat{G}$, then the canonical dual atom is in $M_{n,d}(S_0(G))$.*
(ii) *If $\mathcal{G}(g;\Lambda)$ is an (n,d)-matrix Gabor Riesz sequence for $L^2(G)$ and Λ is discrete, then the canonical biorthogonal atom is also in $M_{n,d}(S_0(G))$.*

Proof. For the proof of (i), note that the assumption that Λ is cocompact implies that Λ° is discrete, so $M_d(C^*(\Lambda^\circ,\overline{c}))$ is unital. Also $M_d(C^*(\Lambda^\circ,\overline{c}))$ is a C^*-subalgebra of $\mathbb{B}(H_{M_{n,d}(E_{G,\Lambda})})$ by Proposition 2.1. That $\mathcal{G}(g;\Lambda)$ is an (n,d)-matrix Gabor frame for $L^2(G)$, then means that (3.6) is satisfied for our current setting. We deduce, as in the proof of Proposition 3.18, that $[g,g]_\bullet$ is invertible in $M_d(B)$. Since $[g,g]_\bullet \in M_d(S_0(\Lambda^\circ,\overline{c}))$ and $M_d(S_0(\Lambda^\circ,\overline{c}))$ is spectrally invariant in $M_d(C^*(\Lambda^\circ,\overline{c}))$ by Proposition 4.4 and [29, Theorem 2.1], the canonical dual atom is $g[g,g]_\bullet{}^{-1} \in M_{n,d}(S_0(G))$.

For the proof of (ii), note that the assumption that Λ is discrete implies $M_n(C^*(\Lambda,c))$ is unital. Also, $M_n(C^*(\Lambda,c))$ is a C^*-subalgebra of $\mathbb{B}(H_{M_n(C^*(\Lambda,c))})$ by Proposition 2.1. That $\mathcal{G}(g;\Lambda)$ determines an (n,d)-matrix Gabor Riesz sequence for $L^2(G)$, then means that (3.7) is satisfied for our current setting. The middle term of (3.7) can be written as $(a\,{}_\bullet[g,g],a)_{C^*(\Lambda,c)}$, so ${}_\bullet[g,g]$ extends to a positive, invertible operator on $L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_n)$. We deduce as in the proof of Proposition 3.20 that ${}_\bullet[g,g]$ is invertible in $M_n(C^*(\Lambda,c))$. Since $g \in M_{n,d}(S_0(G))$, we have ${}_\bullet[g,g] \in M_n(S_0(\Lambda,c))$, and again $M_n(S_0(\Lambda,c))$ is spectrally invariant in $M_n(C^*(\Lambda,c))$. It follows that the canonical dual atom $h := {}_\bullet[g,g]^{-1}g$ is in $M_{n,d}(S_0(G))$. □

When applying the module setup of Sec. 3 to Gabor analysis, we take as a pre-equivalence bimodule $\mathcal{E} = S_0(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$, which is a proper subspace of $L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ unless G is a finite group. Even the Hilbert C^*-module completion $E_{G,\Lambda}$ is properly contained in $L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ general Λ. As such, we cannot hope to treat general atoms in $L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ by applying just this method. But indeed the module reformulation is made exactly to guarantee some regularity of the atoms generating frames.

From Definition 4.7, we see that (n,d)-matrix Gabor frames generalize n-multi-window d-super Gabor frames considered in [19]. However, we now make clear how they fit into the module framework. As mentioned earlier, we obtain n-multi-window d-super Gabor frames if we only require reconstruction of $f \in L^2(G\times\mathbb{Z}_d)$ and we identify $L^2(G\times\mathbb{Z}_d) \subset L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$ by embedding it along a single element in

$\mathbb{Z}_n$. The module reformulation of this is that $g, h \in M_{n,d}(E_{G,\Lambda})$ are dual n-multi-window d-super Gabor frames if for all $f \in M_{n,d}(E_{G,\Lambda})$ supported only one row, we have

$$f = {}_\bullet[\, f, g]h = f[g, h\,]_\bullet\,.$$

Likewise, it is clear that the (n, d)-matrix Gabor Riesz sequences of Definition 4.20 generalize the n-multi-window d-super Gabor Riesz sequences also considered in [19]. Indeed, we obtain n-multi-window d-super Gabor Riesz sequences if we only require reconstruction of $a \in L^2(\Lambda \times \mathbb{Z}_n)$ and we identify $L^2(\Lambda \times \mathbb{Z}_n) \subset L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$ by embedding it along a single element in the middle copy of $\mathbb{Z}_n$. The module reformulation of this is that $g, h \in M_{n,d}(E_{G,\Lambda})$ are dual n-multi-window d-super Gabor Riesz sequences if for all $a \in M_n(C^*(\Lambda, c))$ supported only one row, we have

$$a = {}_\bullet[\, ag, h] = a\,{}_\bullet[\, g, h].$$

We proceed to prove that all n-multi-window d-super Gabor frames for $L^2(G)$ with respect to Λ are (n, d)-matrix Gabor frames for $L^2(G)$ with respect to Λ, as well as the analogous statement for Riesz sequences. The converse statements are true as well.

Proposition 4.29. *Let g be in $M_{n,d}(E_{G,\Lambda})$.*

(i) *If $\mathcal{G}(g; \Lambda)$ is an n-multi-window d-super Gabor frame for $L^2(G)$ with a dual window $h \in M_{n,d}(E_{G,\Lambda})$, then $\mathcal{G}(g; \Lambda)$ is an (n, d)-matrix Gabor frame for $L^2(G)$ with dual window h.*

(ii) *If $\mathcal{G}(g; \Lambda)$ is an n-multi-window d-super Gabor Riesz sequence for $L^2(G)$ with a dual Gabor Riesz sequence $\mathcal{G}(h; \Lambda)$ with $h \in M_{n,d}(E_{G,\Lambda})$, then $\mathcal{G}(g; \Lambda)$ is an (n, d)-matrix Gabor Riesz sequence for $L^2(G)$ with dual Gabor Riesz sequence $\mathcal{G}(h; \Lambda)$.*

Proof. If $\mathcal{G}(g; \Lambda)$ is an n-multi-window d-super Gabor frame for $L^2(G)$ with respect to Λ with a dual window $h \in M_{n,d}(E_{G,\Lambda})$, we can, as noted above, reconstruct any $f \in M_{n,d}(E_{G,\Lambda})$ supported on a single row, say the kth row. In other words, $f = f[g, h\,]_\bullet$ for all $f \in M_{n,d}(E_{G,\Lambda})$ supported on the kth row. Writing out this expression, we find that

$$f_{k,i} = \sum_{j \in \mathbb{Z}_d} f_{k,j} \cdot [g, h\,]_{\bullet\,i,j},$$

for all $i \in \mathbb{Z}_d$. Here, $[g, h\,]_{\bullet\,i,j} \in C^*(\Lambda^\circ, \overline{c})$ denotes the entry (i, j) in $[g, h\,]_\bullet$. Since this holds for all f supported on the kth, row, we deduce that $b_{i,j} = \delta_{i,j} 1_{C^*(\Lambda^\circ, \overline{c})}$, meaning $[g, h\,]_\bullet = 1_{M_d(C^*(\Lambda^\circ, \overline{c}))}$. The actions extend to $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ and we deduce that $\mathcal{G}(g; \Lambda)$ is an (n, d)-matrix Gabor frame for $L^2(G)$ with dual window h.

The proof of (ii) is completely analogous. □

At last, we may present core results of time-frequency analysis for (n, d)-matrix Gabor frames. Due to all the work we have just put in to properly establishing the link between module frame theory on Morita equivalence bimodules and Gabor frame theory, we will see that the statements below more or less follow from the analogous statements in Sec. 3.

Proposition 4.30 (Wexler–Raz biorthogonality relations). *Let $\Lambda \subset G \times \widehat{G}$ be a closed and cocompact subgroup, and let $g, h \in M_{n,d}(E_{G,\Lambda})$. Then the following are equivalent*:

(i) *$\mathcal{G}(g;\Lambda)$ and $\mathcal{G}(h;\Lambda)$ are dual (n, d)-matrix Gabor frames for $L^2(G)$.*
(ii) *For all $i, j \in \mathbb{Z}_d$, we have $\sum_{k\in\mathbb{Z}_n} \langle g_{k,i}, \pi(\lambda^\circ)h_{k,j}\rangle_{\ell^2(\Lambda^\circ)} = \delta_{0,\lambda^\circ}\delta_{i,j}s(\Lambda)$.*

Proof. As Λ is cocompact, we know Λ° is discrete, so $M_d(C * (\Lambda^\circ, \overline{c}))$ is unital. Knowing this, we can see that both the above statements are equivalent to the statement $[g, h]_\bullet = [h, g]_\bullet = 1_{M_d(B)}$. □

In the previous paragraphs we did quite a lot of work to establish a connection between module Riesz sequences and Riesz sequences in Gabor analysis, a connection, we have yet to use for anything significant. However, as a result, we now obtain the following statement of the duality principle in Gabor analysis and a very short proof.

Theorem 4.31 (Duality principle). *Let $\Lambda \subset G \times \widehat{G}$ be a closed cocompact subgroup, and let $g \in M_{n,d}(E_{G,\Lambda})$. Then the following are equivalent.*

(i) *$\mathcal{G}(g;\Lambda)$ is an (n, d)-matrix Gabor frame for $L^2(G)$.*
(ii) *$\mathcal{G}(g;\Lambda^\circ)$ is a (d, n)-matrix Gabor Riesz sequence for $L^2(G)$.*

Proof. Statement (i) can be seen to be equivalent to $[g, g]_\bullet$ being invertible in $M_d(C^*(\Lambda^\circ, \overline{c}))$ by Proposition 4.18. But statement (ii) is also equivalent to $[g, g]_\bullet$ being invertible in $M_d(C^*(\Lambda^\circ, \overline{c}))$ by Proposition 4.26. This finishes the proof. □

For completeness, we also include the following result related to the duality principle. This is a strengthening of the corresponding result in [19].

Proposition 4.32. *Let $\Lambda \subset G \times \widehat{G}$ be closed and cocompact, and let $g, h \in M_{n,d}(E_{G,\Lambda})$ be such that $[g, h]_\bullet$ extends to the identity operator on $L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d)$. Then ${}_\bullet[g, h]$ extends to an idempotent operator from $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ onto $\overline{\text{span}}\{\bigoplus_{i\in\mathbb{Z}_n} \bigoplus_{j\in\mathbb{Z}_d} \pi(\lambda^\circ)g_{i,j}\}$.*

Proof. As $[g, h]_\bullet$ extends to the identity operator, we have $[g, h]_\bullet = [h, g]_\bullet = 1_{M_d(C^*(\Lambda^\circ,\overline{c}))}$. That ${}_\bullet[g, h]$ is an idempotent, then follows by Proposition 3.13.

By Proposition 3.10 $\bullet[g,h]$ is then an idempotent from $M_{n,d}(E_{G,\Lambda})$ onto $\overline{gM_d(C^*(\Lambda^\circ,\overline{c}))}$. But this passes to the localization, and the localization of $\overline{gM_d(C^*(\Lambda^\circ,\overline{c}))}$ is

$$\overline{\operatorname{span}}\left\{\bigoplus_{i\in\mathbb{Z}_n}\bigoplus_{j\in\mathbb{Z}_d}\pi(\lambda^\circ)g_{i,j}\right\}\subset L^2(G\times\mathbb{Z}_n\times\mathbb{Z}_d).$$ □

Given a closed and cocompact subgroup Λ, we may ask if there are restrictions on $n,d\in\mathbb{N}$ for there to possibly exist (n,d)-matrix Gabor frames for $L^2(G)$ with respect to Λ. Conversely, if we fix n and d, we may ask if there are restrictions on the size of the subgroup Λ, see (4.1), for there to possibly exist (n,d)-matrix Gabor frames for $L^2(G)$ with respect to Λ. When Λ is a lattice, we have the following proposition.

Proposition 4.33. *Let $\Lambda\subset G\times\widehat{G}$ be a lattice. If there is $g\in M_{n,d}(E_{G,\Lambda})$ such that $\mathcal{G}(g;\Lambda)$ is an (n,d)-matrix Gabor frame for $L^2(G)$, then*

$$s(\Lambda)\leq\frac{n}{d},$$

where $s(\Lambda)$ is defined as in (4.1).

Proof. Since Λ is discrete and cocompact, both $C^*(\Lambda,c)$ and $C^*(\Lambda^\circ,\overline{c})$ are unital. We also know by Proposition 4.28 that the canonical dual of g is in $M_{n,d}(E_{G,\Lambda})$. Hence, we are in the setting of Theorem 3.16. Since module (n,d)-matrix frames localize to (n,d)-matrix Gabor frames for the localization, and we have $\operatorname{tr}_\Lambda(1_{C^*(\Lambda,c)})=1$, and $\operatorname{tr}_{\Lambda^\circ}(1_{C^*(\Lambda^\circ,\overline{c})})=s(\Lambda)$ (since the identity on $C^*(\Lambda^\circ,\overline{c})$ is $s(\Lambda)\delta_0$, where δ_0 is the indicator function in the group identity, see for example [27]), the result is immediate by Theorem 3.16. □

Likewise, given a lattice Λ, we may ask if there is a relationship between the size of Λ (4.1) and the integers n and d such that there can possibly exist (n,d)-matrix Gabor Riesz sequences for $L^2(G)$ with respect to Λ. This is the content of the following proposition.

Proposition 4.34. *Let $\Lambda\subset G\times\widehat{G}$ be a lattice. If $g\in M_{n,d}(E_{G,\Lambda})$ is such that $\mathcal{G}(g;\Lambda)$ is an (n,d)-matrix Gabor Riesz sequence for $L^2(G)$, then*

$$s(\Lambda)\geq\frac{n}{d},$$

where $s(\Lambda)$ is defined as in (4.1).

Proof. As before, we know by the conditions on Λ that both $C^*(\Lambda,c)$ and $C^*(\Lambda^\circ,\overline{c})$ are unital, and by Proposition 4.28, the canonical dual of g is in $M_{n,d}(E_{G,\Lambda})$. Thus, we are in the setting of Theorem 3.17. Since module (n,d)-matrix Riesz sequences localize to (n,d)-matrix Gabor Riesz sequences for the localization, and

$\mathrm{tr}_\Lambda(1_{C^*(\Lambda,c)}) = 1$ and $\mathrm{tr}_{\Lambda^\circ}(1_{C^*(\Lambda^\circ,\overline{c})}) = s(\Lambda)$ (once again since the identity on B is $s(\Lambda)\delta_0$), the result is immediate by Theorem 3.17. ☐

Remark 4.35. The two preceding propositions contain statements known as density theorems in Gabor analysis. This is due to the fact that they give conditions on the density of a lattice for there to possibly exist Gabor frames and Riesz sequences.

Lastly, in this paper, we prove that whenever Λ is cocompact, there is a close relationship between the module frame bounds and the Gabor frame bounds in the localization.

Proposition 4.36. *Let $\Lambda \subset G \times \widehat{G}$ be a closed cocompact subgroup. Then $g \in M_{n,d}(E_{G,\Lambda})$ generates a module (n,d)-matrix frame for $E_{G,\Lambda}$ as a $C^*(\Lambda,c)$-module with lower frame bound C and upper frame bound D if and only if $\mathcal{G}(g;\Lambda)$ is an (n,d)-matrix Gabor frame for $L^2(G)$ with lower frame bound C and upper frame bound D.*

Proof. By Lemma 2.4, it suffices to prove that the optimal frame bounds are equal for both the module frame and the Gabor frame. We know that the localization of a module frame for $M_{n,d}(E_{G,\Lambda})$ as an $M_n(C^*(\Lambda,c))$-module becomes an (n,d)-matrix Gabor frame for $L^2(G)$ with respect to Λ. Since Λ is cocompact, we also know that if $g \in M_{n,d}(E)$ is such that $\mathcal{G}(g;\Lambda)$ is an (n,d)-matrix Gabor frame for $L^2(G)$, then the canonical dual $S_g^{-1}g \in M_{n,d}(E_{G,\Lambda})$ also. By Proposition 4.18, we have $\rho(\Theta_g) = S_g$. From standard Hilbert space frame theory, we know that the optimal upper frame bound for S_g is $\|S_g\|$, and the optimal lower frame bound for S_g is $\|S_g^{-1}\|^{-1}$, see for example [14, Sec. 5.1]. We know by Proposition 2.1 that $\|\Theta_g\| = \|\rho(\Theta_g)\| = \|S_g\|$ and $\|\Theta_g^{-1}\| = \|\rho(\Theta_g^{-1})\| = \|S_g^{-1}\|$. The result then follows by Lemma 2.4. ☐

Remark 4.37. A straightforward calculation will show that $\|\Theta_g\| = \|{}_\bullet[g,g]\|$. Indeed, for an A-B-equivalence bimodule E this follows by the usual isomorphism $B \cong \mathbb{K}_A(E)$.

Corollary 4.38. *Let $g \in M_{n,d}(E_{G,\Lambda})$ and let $\Lambda \subset G \times \widehat{G}$ be a lattice. If D_Λ denotes the optimal Bessel bound for $\mathcal{G}(g;\Lambda)$ and D_{Λ° denotes the optimal Bessel bound for $\mathcal{G}(g;\Lambda^\circ)$, then $D_{\Lambda^\circ} = s(\Lambda)^{-1}D_\Lambda$.*

Proof. By Proposition 4.36 and Remark 4.37, it follows that $D_\Lambda = \|S_g\| = \|{}_\bullet[g,g]\|$. But the analogous argument can be made to work with Λ° instead of Λ, since the important part for the setup with localization as done in this paper is that Λ or Λ° is cocompact. Hence, we may obtain D_{Λ° by similar considerations. We show how to do this. First, we make $S_0(G)$ into a $S_0(\Lambda^\circ,c)$-$S_0(\Lambda,\overline{c})$-pre-equivalence bimodule similarly to what we did in (4.2) and (4.3), and then complete it to obtain a $C^*(\Lambda^\circ,c)$-$C^*(\Lambda,\overline{c})$-equivalence bimodule E_{G,Λ°. By [3, Proposition 3.17], we have $E_{G,\Lambda} = E_{G,\Lambda^\circ}$ as function spaces (hence, the same for the matrix cases). Denote by ${}_\bullet[\cdot,\cdot]'$, the $M_d(C^*(\Lambda^\circ, c))$-valued inner product on $M_{d,n}(E_{G,\Lambda^\circ})$, and by $\|\cdot\|_{\Lambda^\circ}$ the resulting norm on E_{G,Λ°. Then [3, Proposition 3.17] tells us that for any $f \in M_{d,n}(E_{G,\Lambda})$, we have $\|{}_\bullet[f,f]'\|_{\Lambda^\circ} = s(\Lambda)^{-1}\|{}_\bullet[f,f]\|$. Following the first line of this proof for D_{Λ° instead, we then obtain $D_{\Lambda^\circ} = \|{}_\bullet[g,g]'\|_{\Lambda^\circ} = s(\Lambda)^{-1}\|{}_\bullet[g,g]\| = s(\Lambda)^{-1}D_\Lambda$, which is what we wanted to show. □

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