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Petter Kjeverud Nyland

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Abstract

In this thesis we study the topological full group of an ample groupoid, with a particular focus on groupoids arising from directed graphs. We mainly address two aspects of the topological full group. The first is to what extent the topological full group is a complete invariant, namely when an ample groupoid can be recovered from the algebraic structure of its topological full group alone. The second is to relate the topological full group to the homology groups of the groupoid, as formulated in Matui's AH conjecture.

Sammendrag

I denne avhandlingen studeres den topologisk fulle gruppen til en ample gruppoide. Det fokuseres spesielt på gruppoider konstruert fra rettede grafer. Vi studerer hovedsakelig to aspekter ved den topologisk fulle gruppen. Det ene er i hvilken grad den topologisk fulle gruppen er en komplett invariant, i den forstand at en ample gruppoide kan rekonstrueres utelukkende fra den algebraiske strukturen til dens topologisk fulle gruppe. Det andre er å relatere den topologisk fulle gruppen til gruppoidens homologigrupper, som formulert i AH-formodningen til Matui.

Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of Philosophiae Doctor (PhD) in Mathematical Sciences at the Norwegian University of Science and Technology (NTNU). The research presented here was conducted at the Department of Mathematical Sciences at NTNU, under the supervision of Associate Professor Eduard Ortega and Professor Toke Meier Carlsen.

The thesis consists of a collection of three research papers and an introductory part that provides background and motivation for the work. The introductory part concludes with a summary of each individual paper, which relates them together and puts them into context. There is a single bibliography at the end of thesis which serves both the introductory part and the research papers.

Acknowledgements

First and foremost I would like to thank my supervisor Eduard Ortega for his steady guidance over the past four years. His office door has been open every day for both longer and shorter mathematical discussions—something I have benefited greatly from. Next, I wish to thank my co-supervisor Toke Meier Carlsen for answering my questions and for suggesting research projects to me. Moreover, I want to thank Toke and his lovely family for hosting me during multiple research stays in the Faroe Islands. I would like to thank Christian Skau for being such a great source of knowledge on C^*-algebras and dynamical systems. Christian also deserves thanks for guiding me through my first semester of plenary teaching—an activity I have enjoyed greatly during my time as a PhD-student. I also wish to thank Franz Luef for many interesting discussions.

I spent the spring semester of 2019 in Copenhagen. I would like to thank the Department of Mathematical Sciences at the University of Copenhagen for their hospitality. Søren Eilers acted as my supervisor while I was there, and I wish to thank him for his guidance. Kevin, Mikala, Clemens, Francesco, and David—

whom I shared office with—helped make it a great stay. I particularly want to thank Kevin for all the great discussions and lunches. The weekly bouldering sessions with Josh, Philipp, Kaif, David, etc., were one of the highlights of my stay. While in Copenhagen I was part of the organizing committee for the annual YMC*A conference. I would like to thank Clemens, Philipp, Henning, Thomas, Johannes and the rest of the organizers—as well as all the participants from all around the globe—for making it such a great event. I am grateful to Josh, Clemens and Calista for allowing me to crash at their place during both the YMC*A and a later research visit to Copenhagen. I sure miss starting the day with listening to Mozart on the piano and ending it with Tool on the stereo.

I wish to thank Thomas Gotfredsen and David Kyed for inviting me over to Odense to give a talk there. Similarly, I want to thank Christian Bönicke for inviting me to Glasgow. Tron Omland deserves thanks for providing both career advice and tex-files during my last year. I want to thank Eduardo Scarparo for insightful discussions on groupoid homology.

During my final year as a PhD-student I spent several weeks in Oslo, and I want to thank the Department of Mathematics at the University of Oslo for allowing me to use their guest offices. Also thanks to the operator algebras group there for providing a cozy environment. My friend and colleague in Oslo, Ulrik Enstad, has been my constant travel companion on numerous conferences and workshops. I want to thank him for all the good times.

My time as a PhD-student at NTNU has been an enjoyable one, greatly due to the camaraderie between all the PhD-students at the department. It has been a pleasure to share an office with Fredrik, Olav and Helge. I want to thank everyone who participated in the department's weekly football games, which provided me with much fun. Special thanks to our player-coach Sølve for doing all the organizing, and to Jon Vegard for providing a competetiveness to match my own. Next, I would like to thank Are, Fredrik, Paul and Magnus for providing extracurricular adventures. Special thanks to Are for all the coffee breaks, chats and proofreading as well.

I am grateful for the support and encouragement of my friends Endre, Håkon and Sigurd. The same goes for my "in-laws" Drude, Tor and Ørjan. I would like to thank Drude and Tor for hosting me a whole month during the outbreak of the coronavirus.

I want to thank my parents Venke and Håvard for raising me to become the man I am today. My father, Håvard, and my brothers, Amund and Lars, have been a fundamental source of love and support over the past four years. I am also thankful for the support from my extended family; including my grandparents, uncles, aunts and cousins.

Then there is my partner, Lisa, without whom these years in Trondheim truly would not have been the same. Thank you for simply being so amazing. And thank you for taking care of literally everything during the last weeks of writing.

Petter Kjeverud Nyland
Trondheim, June 2020

Contents

Part I

Introduction

Chapter 1

Groupoids

Let us begin by introducing the main mathematical object studied in this thesis, namely that of a *groupoid*. We will first discuss (algebraic) groupoids themselves, before introducing topological groupoids. After that, we will introduce étale groupoids and ample groupoids. In short, an *étale* groupoid is a topological groupoid that is locally homeomorphic to its unit space, and an *ample* groupoid is an étale groupoid that is zero-dimensional.

Two short notational remarks before we begin. We denote the positive integers by $\mathbb{N}$ and the non-negative integers by $\mathbb{N}_0$. If two sets A and B are disjoint we will denote their union by $A \sqcup B$ if we wish to emphasize that they are disjoint. When we write $C = A \sqcup B$ we mean that $C = A \cup B$ *and* that A and B are disjoint sets.

1.1 Algebraic groupoids

The typical catchphrase one often encounters is that:

- *"A groupoid is a small category in which all morphisms are isomorphisms"*.

While this is elegant and succinct, we prefer to introduce groupoids in a different way, which we believe gives the "working mathematician" a better feel for them. Namely as an algebraic structure akin to that of a group, but with the very important exception that the binary operation need not be total. In other words, not all elements in a groupoid can be multiplied together. This, of course, makes groupoids quite different from groups. In the words of Alan L. T. Paterson:

- *"A groupoid is a set with a partially defined multiplication for which the usual properties of a group hold whenever they make sense."*

Another advantage of the algebraic definition given below is that it lends itself easily to equipping groupoids with more structure, such as a topology or a differentiable structure. However, we will soon enough return to the categorical picture of a groupoid, and gain some useful intuition from it.

Definition 1.1.1. A *groupoid* is a non-empty set $\mathcal{G}$ together with a distinguished subset $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ equipped with a partial binary operation $\mathcal{G}^{(2)} \to \mathcal{G}$, denoted $(g,h) \mapsto gh$, and a unary operation $\mathcal{G} \to \mathcal{G}$, denoted $g \mapsto g^{-1}$, such that the following axioms are satisfied:

($\mathcal{G}$1): If $(g,h),(h,k) \in \mathcal{G}^{(2)}$, then $(gh,k),(g,hk) \in \mathcal{G}^{(2)}$ and $(gh)k = g(hk)$.

($\mathcal{G}$2): $\left(g^{-1}\right)^{-1} = g$ for all $g \in \mathcal{G}$.

($\mathcal{G}$3): For all $g \in \mathcal{G}$, we have $(g, g^{-1}) \in \mathcal{G}^{(2)}$, and if $(g,h) \in \mathcal{G}^{(2)}$, then $ghh^{-1} = g$ and $g^{-1}gh = h$.

We refer to the set $\mathcal{G}^{(2)}$ as the set of *composable pairs* and to the operation $(g,h) \mapsto gh$ as *multiplication* or *composition* of the elements g and h. The first axiom above says that this multiplication is associative, whenever it is defined. We refer to g^{-1} as the *inverse* of g. We deduce from the third axiom above that the groupoid element $g^{-1}g$ serves as a right idenity for all elements $h \in \mathcal{G}$ such that (h, g^{-1}) is composable. Similarly, gg^{-1} is a left identity for all $k \in \mathcal{G}$ with $(g^{-1}, k) \in \mathcal{G}^{(2)}$. In particular,

$$g(g^{-1}g) = g = (gg^{-1})g.$$

As there are multiple "identities" in $\mathcal{G}$—in contrast to in a group—these elements ($g^{-1}g$ and gg^{-1}) are collectively referred to as *units*. In the words of Aidan Sims:

- *"A groupoid is a group with an identity crisis."*

The set

$$\mathcal{G}^{(0)} := \left\{g^{-1}g \mid g \in \mathcal{G}\right\} = \left\{gg^{-1} \mid g \in \mathcal{G}\right\}$$

is called the *unit space* (or the *set of units* if one wishes to be pedantic). Soon enough we shall exclusively be working with topological groupoids, and then $\mathcal{G}^{(0)}$ will indeed be a topological space in its own right.

The maps $s, r \colon \mathcal{G} \to \mathcal{G}^{(0)}$ given by $s(g) = g^{-1}g$ and $r(g) = gg^{-1}$ are called the *source* and *range* maps, respectively. These maps are occasionally called the "domain" and/or "target" maps. Notation and terminology for groupoids vary somewhat throughout the literature. What is used here aligns with much of the literature on groupoid C^*-algebras.

Let us now consider some examples of groupoids.

Example 1.1.2. Any group Γ becomes a groupoid by declaring all elements to be composable, i.e. $\Gamma^{(2)} = \Gamma \times \Gamma$. The unit space becomes $\Gamma^{(0)} = \{e\}$, where e is the identity element in Γ. Conversely, it can be shown that any groupoid whose unit space is a singleton is a group.

Example 1.1.3. Sitting at the opposite extreme from groups, any set X can be viewed as a groupoid by declaring that $X^{(2)} = X = X^{(0)}$, i.e. nothing is composable, except for an element with itself.

Example 1.1.4. In some sense mixing the former two, let X be a set and let Γ_x be a group for each $x \in X$. The *group bundle* $\mathcal{G} := \sqcup_{x \in X}\{x\} \times \Gamma_x$ becomes a groupoid by only allowing multiplication within each individual fiber (which, a priori, is the only thing that makes sense). The product and inverse are $(x, \gamma)(x, \tau) = (x, \gamma\tau)$ and $(x, \gamma)^{-1} = (x, \gamma^{-1})$ for $\gamma, \tau \in \Gamma_x$. Its unit space is $\mathcal{G}^{(0)} = \sqcup_{x \in X}(x, e_x)$, where e_x is the identity element in Γ_x. By identifying $\mathcal{G}^{(0)}$ with X via $(x, e_x) \leftrightarrow x$ we can write the source and range maps as $s(x, \gamma) = x$ and $r(x, \gamma) = x$.

Example 1.1.5. A less trivial example is that of an *equivalence relation* $\mathcal{R} \subseteq X \times X$ on a set X. By defining $\mathcal{R}^{(2)} = \{((x, y), (y, z)) \mid (x, y), (y, z) \in \mathcal{R}\}$ with product $(x, y)(y, z) = (x, z)$ and inverse $(x, y)^{-1} = (y, x)$. The source and range maps become $s(x, y) = (y, y)$ and $r(x, y) = (x, x)$, so the unit space $\mathcal{R}^{(0)}$ equals the diagonal in X. By identifying $\mathcal{R}^{(0)}$ with X itself via $(x, x) \leftrightarrow x$ we may write $s(x, y) = y$ and $r(x, y) = x$.

As a particular case we have, for each $n \in \mathbb{N}$, the so-called *matrix groupoid* $\mathcal{R}_n := \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$, which is simply the full equivalence relation on a set of n elements. Here $(i, j)(j, k) = (i, k)$ and $(i, j)^{-1} = (j, i)$, and $s(i, j) = j$ and $r(i, j) = i$, after identifying $\mathcal{R}_n^{(0)}$ with $\{1, 2, \ldots, n\}$.

Example 1.1.6. Another important example is that of a group action. Let Γ be a group with identity element e and let X be a Γ-set. Define $\Gamma \ltimes X := \Gamma \times X$ and set $(\Gamma \ltimes X)^{(2)} = \{((\tau, \gamma(x)), (\gamma, x)) \mid \tau, \gamma \in \Gamma, x \in X\}$. So the pairs (τ, y) and (γ, x) are composable if and only if $y = \gamma(x)$, i.e. γ moves x to y. The product and inverse are given by $(\tau, \gamma(x))(\gamma, x) = (\tau\gamma, x)$ and $(\gamma, x)^{-1} = (\gamma^{-1}, \gamma(x))$. We refer to $\Gamma \ltimes X$ as a *transformation groupoid*. Its unit space is $(\Gamma \ltimes X)^{(0)} = \{e\} \times X$, which we will identify with X via $(e, x) \leftrightarrow x$. Then the source and range maps become $s((\gamma, x)) = x$ and $r((\gamma, x)) = \gamma(x)$.

In the case that the action is singly generated (meaning that Γ is a cyclic group) by some bijection $\phi \colon X \to X$, we will denote the transformation groupoid by $\mathcal{G}_\phi$ to emphasize this.

Example 1.1.7. As a final example for now, we explain how one gets an equivalence relation (groupoid) from a group action. If X is a Γ-set, then the *orbit equivalence relation* is $\mathcal{R}_{\Gamma \curvearrowright X} := \{(x, \gamma(x)) \mid x \in X, \gamma \in \Gamma\} \subseteq X \times X$.

The following rudimentary facts follow from Definition 1.1.1.

Proposition 1.1.8. *Let $\mathcal{G}$ be a groupoid. Then:*

1. *$\mathcal{G}$ is cancellative, i.e. if $(g, h), (g, k) \in \mathcal{G}^{(2)}$ and $gh = gk$, then $h = k$, and vice versa.*
2. *If $(g, h) \in \mathcal{G}^{(2)}$, then $(h^{-1}, g^{-1}) \in \mathcal{G}^{(2)}$ and $(gh)^{-1} = h^{-1}g^{-1}$.*
3. *$\mathcal{G}^{(0)} = \left\{g \in \mathcal{G} \mid (g, g) \in \mathcal{G}^{(2)} \text{ and } g^2 = g\right\}$.*
4. *$gs(g) = g = r(g)g$ for all $g \in \mathcal{G}$.*
5. *$s\left(g^{-1}\right) = r(g)$ and $s\left(r^{-1}\right) = s(g)$ for all $g \in \mathcal{G}$.*
6. *$s(x) = x = r(x) = x^{-1}$ for all $x \in \mathcal{G}^{(0)}$.*
7. *If $(g, h) \in \mathcal{G}^{(2)}$, then $s(gh) = s(h)$ and $r(gh) = r(g)$.*
8. *$(g, h) \in \mathcal{G}^{(2)} \iff r(h) = s(g)$.*

Item *8.* above is particularly worth noting. Namely that two groupoid elements g and h are composable (in that order) if and only if the range of h coincides with the source of g. If we now think of a groupoid element g as an arrow (or morphism) from $s(g)$ to $r(g)$, like this:

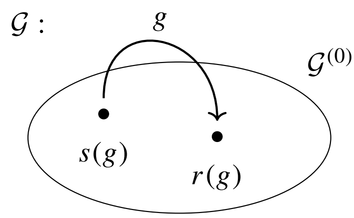
$$\begin{tikzcd} s(g) \arrow[r, bend left, "g"] & r(g) \arrow[l, bend right, dashed, "g^{-1}"'] \end{tikzcd}$$

then it fits our intuiton that h may be followed by g, i.e. composing gh, precisely when g picks up where h left off:

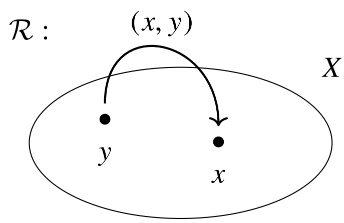
$$\begin{tikzcd} \bullet \arrow[r, bend left, "h"] & \underset{r(h) = s(g)}{\bullet} \arrow[r, bend left, "g"] & \bullet \end{tikzcd}$$

The order of multiplication reflects the order that we compose maps (which is opposite from the way we compose arrows). We now arrive at the categorical picture of a groupoid. Let $\mathcal{C}$ be a small category (the collection of objects and morphisms both form sets) in which every morphism (or arrow) is invertible. Then it is clear that the set of morphisms in $\mathcal{C}$ forms a groupoid under composition and inversion of morphisms. The unit space of $\mathcal{C}$ is the set of identity morphisms. Conversely, if $\mathcal{G}$ is a groupoid, then we may view it as a category by formally identifying the unit space $\mathcal{G}^{(0)}$ with both the set of objects and identity morphisms and for two units (objects) $x, y \in \mathcal{G}^{(0)}$, the set of morphisms (or arrows) from x to y is $\{g \in \mathcal{G} \mid s(g) = x \text{ and } r(g) = y\}$.

Picturing groupoid elements as arrows in this way aids our intuition when working with groupoids arising from various dynamical systems, which is the kind of groupoids studied in this thesis. In many cases (e.g. equivalence relations and transformation groupoids) one constructs a groupoid $\mathcal{G}$ from a set (or space) X with some sort of "dynamics" on it, which is then encoded in the groupoid structure in such a way that the unit space $\mathcal{G}^{(0)}$ may be identified with X. We may then refer to $\mathcal{G}$ as a groupoid *over* X. We can think of $\mathcal{G}$ as a set of arrows over $\mathcal{G}^{(0)}$ as follows:



The equivalence relation groupoid $\mathcal{R}$ and transformation groupoid $\Gamma \ltimes X$ of Examples 1.1.5 and 1.1.6 can be visualized as



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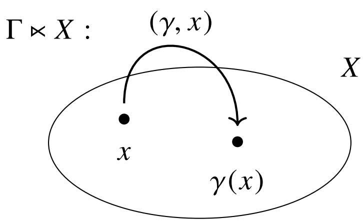

We can think of the transformation groupoid $\Gamma \ltimes X$ as encoding the action by Γ on X in the sense that the groupoid element (γ, x) tells us that γ moves x to $\gamma(x)$.

Groupoids are quite flexible in the sense that they can be combined in many ways to create new groupoids. Some basic constructions are:

1. If $\mathcal{G}$ and $\mathcal{H}$ are groupoids, then their *disjoint union* $\mathcal{G} \sqcup \mathcal{H}$ is a groupoid (with operations within each separate groupoid). For example, a group bundle is a disjoint union of groups.

2. If $\mathcal{G}$ and $\mathcal{H}$ are groupoids, then their *(Cartesian) product* $\mathcal{G} \times \mathcal{H}$ is a groupoid (with coordinate-wise operations).

3. If $\mathcal{G}$ is a groupoid and $A \subseteq \mathcal{G}^{(0)}$ is a subset of the unit space, then the *restriction* $\mathcal{G}|_A := \{g \in \mathcal{G} \mid s(g), r(g) \in A\}$ is a subgroupoid of $\mathcal{G}$ with unit space $\mathcal{G}|_A{}^{(0)} = A$.

Let us now describe homomorphisms and isomorphisms of groupoids.

Definition 1.1.9. Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. A map $\Phi\colon \mathcal{G} \to \mathcal{H}$ is a *groupoid homomorphism* if $(\Phi(g), \Phi(g')) \in \mathcal{H}^{(2)}$ whenever $(g, g') \in \mathcal{G}^{(2)}$, and moreover, $\Phi(gg') = \Phi(g)\Phi(g')$ in this case. If Φ is bijective, then it is an *isomorphism* of groupoids.

Proposition 1.1.10. *Let $\Phi\colon \mathcal{G} \to \mathcal{H}$ be a groupoid homomorphism. Then:*

1. $\Phi(g^{-1}) = \Phi(g)^{-1}$ *for all* $g \in \mathcal{G}$.
2. $\Phi\left(\mathcal{G}^{(0)}\right) \subseteq \mathcal{H}^{(0)}$.
3. $\Phi(s(g)) = s\left(\Phi(g)\right)$ *and* $\Phi(r(g)) = r\left(\Phi(g)\right)$ *for all* $g \in \mathcal{G}$.
4. $\Phi\left(\mathcal{G}\right)$ *is a subgroupoid of* $\mathcal{H}$.

Let us next introduce some more standard terminology for groupoids. For two subsets $U, V \subseteq \mathcal{G}$ of a groupoid $\mathcal{G}$ we define their product and inverse to be

$$UV := \left\{gh \mid g \in U, h \in V \text{ and } (g, h) \in \mathcal{G}^{(2)}\right\}$$

and

$$U^{-1} := \left\{g^{-1} \mid g \in U\right\}.$$

The following terminology is inspired by similar terminology for group actions, as we will see in an example below. Let $\mathcal{G}$ be a groupoid and let $x \in \mathcal{G}^{(0)}$. Define

$$\mathcal{G}_x := \{g \in \mathcal{G} \mid s(g) = x\} \quad \text{and} \quad \mathcal{G}^x := \{g \in \mathcal{G} \mid r(g) = x\}.$$

The *isotropy group* at x is

$$\mathcal{G}_x^x := \mathcal{G}^x \cap \mathcal{G}_x = \{g \in \mathcal{G} \mid s(g) = r(g) = x\}.$$

Note that $\mathcal{G}_x^x$ is indeed a group, whose identity element is x. The *isotropy (bundle)* of $\mathcal{G}$ is

$$\mathcal{G}' \coloneqq \bigsqcup_{x \in \mathcal{G}^{(0)}} \mathcal{G}_x^x = \{g \in \mathcal{G} \mid s(g) = r(g)\}\,.$$

The isotropy bundle $\mathcal{G}'$ is a subgroupoid of $\mathcal{G}$, and it is a group bundle as in Example 1.1.4. We say that $\mathcal{G}$ is *prinicpal* if $\mathcal{G}' = \mathcal{G}^{(0)}$, in other words, each isotropy group $\mathcal{G}_x^x$ is trivial (equalling $\{x\}$). This entails that if $g, h \in \mathcal{G}$ are such that $s(g) = s(h)$ and $r(g) = r(h)$, then we must have $g = h$.

The $\mathcal{G}$*-orbit* of x is

$$\begin{aligned} \mathrm{Orb}_{\mathcal{G}}(x) &\coloneqq s\left(\mathcal{G}^x\right) = r\left(\mathcal{G}_x\right) \\ &= \{y \in X \mid s(g) = x \text{ and } r(g) = y \text{ for some } g \in \mathcal{G}\}\,. \end{aligned}$$

A subset $A \subseteq \mathcal{G}^{(0)}$ is $\mathcal{G}$*-invariant* if for each $g \in \mathcal{G}$, $s(g) \in A$ if and only if $r(g) \in A$. Any invariant subset is a union of orbits and each orbit is an invariant set. All isotropy groups within the same orbit are mutually isomorphic, for if $s(g) = x$ and $r(g) = y$, then the mapping $g' \mapsto gg'g^{-1}$ for $g' \in \mathcal{G}_x^x$ defines a group isomorphism $\mathcal{G}_x^x \cong \mathcal{G}_y^y$. We call $\mathcal{G}$ *transitive* if $\mathrm{Orb}_{\mathcal{G}}(x) = \mathcal{G}^{(0)}$ for some (and hence all) $x \in \mathcal{G}^{(0)}$, i.e. there is only one orbit. Let us illustrate these notions with some examples.

Example 1.1.11. Let Γ be a group and let X be a Γ-set. Then the isotropy group at $x \in X$ of the associated transformation groupoid is

$$(\Gamma \ltimes X)_x^x = \{(x, \gamma) \mid \gamma \in \Gamma \text{ and } \gamma(x) = x\}$$

which can be identified with the usual isotropy (or stabilizer) subgroup

$$\Gamma_x = \{\gamma \in \Gamma \mid \gamma(x) = x\}$$

of Γ. We see that the transformation groupoid is principal if and only if the action is free (meaning that $\gamma(x) = x$ only if $\gamma = e$). The orbit of x in the transformation groupoid is

$$\mathrm{Orb}_{\Gamma \ltimes X}(x) = \{\gamma(x) \in X \mid \gamma \in \Gamma\}$$

which equals the orbit of x under the action, i.e. Γx. The transformation groupoid $\Gamma \ltimes X$ is transitive if and only if the action is transitive (meaning that there is only one orbit).

We also mention that groups (and group bundles), when viewed as groupoids, are as far from being principal as possible, since the isotropy here equals the whole groupoid itself.

Example 1.1.12. An equivalence relation $\mathcal{R} \subseteq X \times X$ is always a principal groupoid. The orbits in $\mathcal{R}$ are precisely the equivalence classes. An equivalence relation is of course a transitive relation, but $\mathcal{R}$ being a transitive groupoid means something else. Namely that there is only one equivalence class, which forces $\mathcal{R} = X \times X$. The matrix groupoids $\mathcal{R}_n$ are such examples.

We also mention that the orbits in the orbit equivalence groupoid $\mathcal{R}_{\Gamma \curvearrowright X}$ associated to a group action $\Gamma \curvearrowright X$ are just the orbits of the action (which are also the equivalence classes in $\mathcal{R}_{\Gamma \curvearrowright X}$).

Example 1.1.13. Generalizing Example 1.1.7 one may also define the *orbit equivalence relation* of a groupoid $\mathcal{G}$ as $\mathcal{R}_{\mathcal{G}} := \{(s(g), r(g)) \mid g \in \mathcal{G}\} \subseteq \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$. It comes with a canonical surjective groupoid homomorphism $\Phi_{\mathcal{R}} \colon \mathcal{G} \to \mathcal{R}_{\mathcal{G}}$ given by $\Phi_{\mathcal{R}}(g) = (s(g), r(g))$. We have that $\mathcal{G}$ is principal if and only if $\Phi_{\mathcal{R}}$ is injective, in which case $\mathcal{G}$ is isomorphic to $\mathcal{R}_{\mathcal{G}}$.

The observant reader will have noticed that algebraically speaking, a principal groupoid is the same as an equivalence relation. However, when we topologize our groupoids in the next section there will be a distinction. A priori, an equivalence relation inherits the subspace topology from $X \times X$, whereas principal groupoids may have all kinds of other (finer) topologies, but now we are getting ahead of ourselves.

Remark 1.1.14. At this point it is worth mentioning that algebraic groupoids are not all that interesting in their own right. Every groupoid is actually (algebraically) isomorphic to a disjoint union of a collection of products between a group and an (full) equivalence relation (see [Put19, Theorem 3.1.11]). However, once we throw topology into the mix in the next section, this changes drastically. This is when groupoids really start to shine.

1.2 Topological groupoids

Let us make our groupoids interesting again, by topologizing them. A topological groupoid generalizes a topological group in the same way that a groupoid generalizes a group.

Definition 1.2.1. A *topological groupoid* is a groupoid $\mathcal{G}$ equipped with a topology under which the multiplication $\mathcal{G}^{(2)} \to \mathcal{G}$ and inversion $\mathcal{G} \to \mathcal{G}$ are continuous when $\mathcal{G}^{(2)}$ is given the subspace topology from $\mathcal{G} \times \mathcal{G}$.

Whenever we deal with a topological groupoid it is understood that the unit space $\mathcal{G}^{(0)}$ is given the subspace topology from $\mathcal{G}$. In order to ensure that topological groupoids behave well, it is common to assume that the topology is locally compact and Hausdorff (or at least that $\mathcal{G}^{(0)}$ is Hausdorff). We will do this eventually, but for now we describe some consequences of the above definition.

Proposition 1.2.2. *Let $\mathcal{G}$ be a topological groupoid. Then*

1. *The source and range maps $s, r\colon \mathcal{G} \to \mathcal{G}^{(0)}$ are continuous.*
2. *The inverse map $g \mapsto g^{-1}$ is a homeomorphism of $\mathcal{G}$.*
3. *The unit space $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}$ if and only if $\mathcal{G}$ is Hausdorff.*
4. *If the unit space $\mathcal{G}^{(0)}$ is Hausdorff, then the set of composable pairs $\mathcal{G}^{(2)}$ is closed in $\mathcal{G} \times \mathcal{G}$.*

The above is one reason why it is desirable for $\mathcal{G}$—or at least $\mathcal{G}^{(0)}$—to be Hausdorff.

Examples 1.2.3. Building on Examples 1.1.2–1.1.6 we have that any topological group and any topological space can be viewed as a topological groupoid (as can any discrete groupoid of course). If X is a topological space and $\mathcal{R} \subseteq X \times X$ is an equivalence relation on X, then $\mathcal{R}$ becomes a topological groupoid when equipped with the subspace topology from $X \times X$. If Γ is a topological group acting continuously on the topological space X, then the transformation groupoid $\Gamma \ltimes X$ equipped with the product topology is a topological groupoid. In all of these examples the identification of the unit space with X is compatible with the groupoid topology (in the sense that their identification is a homeomorphism).

1.3 Étale groupoids

Before introducing étale groupoids, we quickly introduce the larger class of *r-discrete* groupoids and discuss how they are related.

Definition 1.3.1. A topological groupoid $\mathcal{G}$ is called *r-discrete* if $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$.

In an r-discrete groupoid, the range fibers $\mathcal{G}^x = r^{-1}(x)$ (as well as the source fibers $\mathcal{G}_x$) are always discrete subsets of $\mathcal{G}$, which explains the name.

Definition 1.3.2. A topological groupoid $\mathcal{G}$ is *étale* if the range map r is a local homeomorphism, as a map from $\mathcal{G}$ to $\mathcal{G}$.

For convenience we recall that $r\colon \mathcal{G} \to \mathcal{G}$ is a *local homeomorphism* if there for each $g \in \mathcal{G}$, there exists an open set $U \subseteq \mathcal{G}$ containing g such that $r(U)$ is open in $\mathcal{G}$, and so that $r|_U\colon U \to r(U)$ is a homeomorphism. A local homeomorphism is in particular an open map. Note that the source map s is a local homeomorphism (from $\mathcal{G}$ to $\mathcal{G}$) if and only if r is (since they are related through the inversion map, which is a homeomorphism).

Proposition 1.3.3. *Let $\mathcal{G}$ be an étale groupoid. Then*

1. *The source and range maps s and r are open maps.*

2. *The unit space $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$, i.e. $\mathcal{G}$ is r-discrete. In particular, if $\mathcal{G}$ is Hausdorff and étale, then $\mathcal{G}^{(0)}$ is clopen.*

3. *The multiplication map $\mathcal{G}^{(2)} \to \mathcal{G}$ is a local homeomorphism.*

We also note that since an étale groupoid $\mathcal{G}$ is locally homeomorphic to its unit space $\mathcal{G}^{(0)}$, they share all local topological properties. For example, if $\mathcal{G}^{(0)}$ is locally compact Hausdorff, then $\mathcal{G}$ is locally compact and locally Hausdorff.

Remark 1.3.4. We emphasize that in the definition of an étale groupoid, the range map must be a local homeomorphism from $\mathcal{G}$ to $\mathcal{G}$, and not merely to the unit space $\mathcal{G}^{(0)}$. A subtle point is that r being a local homeomorphism into $\mathcal{G}$ is stronger than it being a local homeomorphism into $\mathcal{G}^{(0)}$. However, if $\mathcal{G}$ is an r-discrete groupoid, then $\mathcal{G}$ is étale if and only if $r\colon \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism.

To illustrate the preceding remark we provide an example of a topological groupoid which is not étale, but for which $r\colon \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism.

Example 1.3.5. Let $\mathbb{T}$ denote the unit circle and consider the antipodal equivalence relation

$$\mathcal{R} := \{(z, z), (z, -z) \mid z \in \mathbb{T}\} \subseteq \mathbb{T} \times \mathbb{T}.$$

Equipping $\mathcal{R}$ with the relative topology from $\mathbb{T} \times \mathbb{T}$ turns it into a topological groupoid. Note that $\mathcal{R}$ is not r-discrete, and hence not étale, since the diagonal $\{(z, z) \mid z \in \mathbb{T}\} = \mathcal{R}^{(0)}$ is not open in $\mathbb{T} \times \mathbb{T}$ (compare with Example 1.3.7.4 below). However, the range map *is* a local homeomorphism from $\mathcal{R}$ to $\mathcal{R}^{(0)}$. To see this, let $z \in \mathbb{T}$ be given and let $A \subseteq \mathbb{T}$ be a small open arc containing z. The elements (z, z) and $(z, -z)$ in $\mathcal{R}$ are respectively contained in the open subsets $\mathcal{R} \cap (A \times A)$ and $\mathcal{R} \cap (A \times -A)$ of $\mathcal{R}$, and both of these sets are mapped homeomorphically onto the open subset $\{(z, z) \mid z \in A\} \subseteq \mathcal{R}^{(0)}$ by the range map r.

Let us introduce some important "dynamical" terminology (as in being inspired by terminology for dynamical systems) for étale groupoids. Let $\mathcal{G}$ be an étale groupoid. We say that $\mathcal{G}$ is *minimal* if for each $x \in \mathcal{G}^{(0)}$, the orbit $\mathrm{Orb}_{\mathcal{G}}(x)$ is dense in $\mathcal{G}^{(0)}$. This is equivalent to there being no non-trivial open (or closed) $\mathcal{G}$-invariant subsets in $\mathcal{G}^{(0)}$.

Recall that a groupoid is called principal if $\mathcal{G}' = \mathcal{G}^{(0)}$, i.e. all isotropy groups are trivial. This can be "topologically weakened" in two ways: One, we call $\mathcal{G}$ *effective* if the interior of the isotropy $\mathcal{G}'$ equals $\mathcal{G}^{(0)}$. Note that since $\mathcal{G}^{(0)}$ is open, it is always a subset of the interior of the isotropy, so a groupoid being effective means that the interior of the isotropy is as small as possible. Two, we call $\mathcal{G}$ *topologically principal* if the set of units with trivial isotropy group is dense in $\mathcal{G}^{(0)}$.

For Hausdorff groupoids, being topologically principal is stronger than being effective, but these notions do coincide in many cases (see Proposition 1.3.6). As a result, these definitions are not entirely consistent throughout the literature, so one always has to check which definition is used in a given paper. The term "essentially principal" is also quite common, and is usually used to denote what we here call "effective", but sometimes used for yet another different notion (such as in [Ren80]).

Proposition 1.3.6. *Let $\mathcal{G}$ be an étale groupoid.*

1. *If $\mathcal{G}$ is Hausdorff, then topologically principal implies effective.*
2. *If $\mathcal{G}$ is second countable and $\mathcal{G}^{(0)}$ is locally compact Hausdorff, then effective implies topologically principal.*

In particular, if $\mathcal{G}$ is a locally compact Hausdorff, second countable étale groupoid, then $\mathcal{G}$ is topologically principal if and only if $\mathcal{G}$ is effective.

Proof. Follows from [Ren08, Proposition 3.1]. □

Example 1.3.7. Let us give some examples of étale groupoids.

1. Any topological space is an étale groupoid.
2. A topological group is an étale groupoid if and only if it is discrete.
3. The transformation groupoid $\Gamma \ltimes X$ associated to a group action $\Gamma \curvearrowright X$ is étale if and only if the acting group Γ is discrete. In this case $\Gamma \ltimes X$ is minimal if and only if the action is minimal. The transformation groupoid is effective if and only if for each $\gamma \in \Gamma \setminus \{e\}$, the set $\{x \in X \mid \gamma(x) \neq x\}$ is dense in X. The transformation groupoid is topologically principal if and only if the set $\{x \in X \mid \gamma(x) \neq x \text{ for all } \gamma \in \Gamma \setminus \{e\}\}$ is dense in X.
4. Let $\mathcal{R} \subseteq X \times X$ be an equivalence relation on a topological space X. Equipped with the subspace topology from $X \times X$, $\mathcal{R}$ will never be étale, unless X

is discrete (for that is the only way the diagonal can be open in $X \times X$). However, there are other (finer) topologies that equivalence relations can be equipped with which may result in étale groupoids. See e.g. [GPS04].

Of course, any principal étale groupoid $\mathcal{G}$ can be identified with its orbit equivalence relation $\mathcal{R}_{\mathcal{G}} \subseteq \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ from Example 1.1.13, and letting $\mathcal{R}_{\mathcal{G}}$ inherit the topology from $\mathcal{G}$ makes it étale. Examples include transformation groupoids of free actions by discrete groups, AF-groupoids (see [Ren80, Section III.1]) and groupoids associated to quasicrystals (see [Nek19, Subsection 6.3]).

Motivated by Example 1.3.7.3 above, one may think of an étale groupoid as being part continuous (the "space part") and part discrete (the "acting part"). It can be helpful to think of an étale groupoid as encoding some kind of action by a discrete object on some topological space.

A key notion for étale groupoids is that of a bisection.

Definition 1.3.8. Let $\mathcal{G}$ be an étale groupoid. We call $U \subseteq \mathcal{G}$ a *bisection* if U is open and both s and r are injective when restricted to U.

When $U \subseteq \mathcal{G}$ is a bisection, then the restrictions $s|_U$ and $r|_U$ become homeomorphisms from U onto $s(U)$ and $r(U)$, respectively. If V is another bisection, then UV, U^{-1} and $U \cap V$ are also bisections. An important fact is that an étale groupoid always has a basis of bisections.

Example 1.3.9. Let Γ be a discrete group acting by homeomorphisms on a topological space X. Then a basis of bisections for $\Gamma \ltimes X$ is given by the sets $\{\gamma\} \times A$, where γ ranges over Γ and A ranges over all open subsets of X. For $U = \{\gamma\} \times A$ as above, the homeomorphisms $s|_U : U \to A$ and $r|_U : U \to \gamma(A)$ are given by $(\gamma, x) \mapsto x$ and $(\gamma, x) \mapsto \gamma(x)$, respectively, for $x \in A$.

Remark 1.3.10. We remark that our choice of making openness part of the definition of a bisection is less common in the literature. However, we find it to be convenient here since we never deal with "non-open bisections" in this thesis. This is comparable to neighbourhoods in topology being required to be open by some authors, but not by most. We also mention that the terms *local bisection* or *$\mathcal{G}$-set* are sometimes used instead of bisection.

Remark 1.3.11. If one works with more general topological groupoids, one should change the definition of bisection to being an open set U such that $s(U)$ is open and $s|_U : U \to s(U)$ is a homeomorphism, and similarly for r. With this definition a topological groupoid is étale if and only if it admits a basis of bisections.

Let us now say a few words on subgroupoids and homomorphisms between étale groupoids. If $\mathcal{G}$ is an étale groupoid and $\mathcal{H}$ is an open subgroupoid of $\mathcal{G}$, then $\mathcal{H}$ is also étale. In particular, if $A \subseteq \mathcal{G}^{(0)}$ is open, then the restriction subgroupoid $\mathcal{G}|_A$ is again étale.

Let $\mathcal{G}$ and $\mathcal{H}$ be étale groupoids. We call a groupoid homomorphism $\Phi\colon \mathcal{G} \to \mathcal{H}$ an *étale homomorphism* if it is a local homeomorphism. Then the image $\Phi(\mathcal{G})$ is an open étale subgroupoid of $\mathcal{H}$. In fact, Φ is a local homeomorphism if and only if its restriction to the unit spaces $\Phi^{(0)}\colon \mathcal{G}^{(0)} \to \mathcal{H}^{(0)}$ is. By an *isomorphism* of topogical, or étale, groupoids we mean an algebraic isomorphism which is also a homeomorphism. In other words, a bijective étale homomorphism is the same as an isomorphism of étale groupoids.

1.4 Ample groupoids

We now arrive at the particular kind of topological groupoids that are studied in this thesis. As is done in e.g. [KL16], [Ste19] and even [Sto37], we call a topological space *Boolean* if it is Hausdorff and has a basis of compact open sets.

Definition 1.4.1. An étale groupoid $\mathcal{G}$ is called *ample* if $\mathcal{G}^{(0)}$ is Boolean.

Recall that an étale groupoid is characterized by admitting a basis of bisections. Similarly, an ample groupoid is characterized by admitting a basis of compact bisections. More precisely, we have the following.

Lemma 1.4.2. *Let $\mathcal{G}$ be a locally compact étale groupoid with $\mathcal{G}^{(0)}$ Hausdorff. Then the following are equivalent:*

1. *$\mathcal{G}$ is ample.*
2. *$\mathcal{G}$ admits a basis of compact bisections.*
3. *$\mathcal{G}^{(0)}$ is totally disconnected.*

Proof. We trivially have *2.* $\implies$ *1.* $\implies$ *3.* The implication *3.* $\implies$ *1.* follows from [AT08, Proposition 3.1.7], which says that any totally disconnected locally compact Hausdorff space is Boolean.

As for *1.* $\implies$ *2.*, assume that $\mathcal{G}$ is ample. Recall that any étale groupoid has a basis of bisections. Let $U \subseteq \mathcal{G}$ be a bisection. The set $r(U)$ is open in $\mathcal{G}^{(0)}$, so we may write $r(U) = \cup_i K_i$, where each K_i is compact open. Define $V_i := (r|_U)^{-1}(K_i)$. Since $r|_U$ is a homeomorphism, each V_i is compact open, and since $V_i \subseteq U$ each V_i is a compact bisection. We now see that 2. holds, since $U = \cup_i V_i$. □

Note that if $\mathcal{G}$ is Hausdorff and ample, then $\mathcal{G}$ itself is Boolean as a topological space.

Example 1.4.3. Prominent examples of ample groupoids include:

1. Transformation groupoids associated to Cantor minimal systems. These will appear in both Chapter 2 and 3 as motivating examples. More generally; transformation groupoids associated to discrete groups acting on Boolean spaces.

2. AF-groupoids (or AF-equivalence relations), which are inductive limits of compact principal groupoids with Cantor unit space, see e.g. [GPS04]. See also Subsection A.11.5 and B.2.4.

3. SFT-groupoids, i.e. groupoids associated to one-sided shifts of finite type, see Section 2.3 and Subsection 2.4.3. More generally; graph groupoids, see e.g. [BCW17]. Graph groupoids are presented and studied in detail in Paper A and B. Further generalizations include higher-rank graph groupoids (see e.g. [CR19]) and ultragraph groupoids (see e.g. [dCGvW19]).

4. Spielberg's hybrid 2-graph groupoids from [Spi07].

5. Deaconu–Renault groupoids over Boolean spaces, see e.g. [FKPS18].

6. Groupoids associated to quasicrystals, see e.g. [Kre16].

7. Groupoids of germs associated to self-similar groups, see e.g. [Nek09]. More generally; the tight groupoid of a self-similar graph, see [EP17]. A special case of these, namely Katsura–Exel–Pardo groupoids, are described and studied in Paper C.

There are a number of notions of "equivalence" for étale groupoids in the literature (see [FKPS18, Section 3]). These equivalences are weakened forms of isomorphism which still preserve many structural aspects of the groupoid. A fitting analogy is Morita equivalence (or stable isomorphism) of C^*-algebras. For ample Hausdorff groupoids, many of these notions coincide, see [FKPS18, Theorem 3.12]. We will present two of these equivalences, which make appearences in the papers included in this thesis.

Let $\mathcal{G}$ and $\mathcal{H}$ be two ample Hausdorff groupoids. A subset $A \subseteq \mathcal{G}^{(0)}$ is called *$\mathcal{G}$-full* if $r\left(s^{-1}(A)\right) = \mathcal{G}^{(0)}$, in other words A contains at least one point from each $\mathcal{G}$-orbit. Note in particular that if $\mathcal{G}$ is minimal, then every open subset of $\mathcal{G}^{(0)}$ is full. The two groupoids $\mathcal{G}$ and $\mathcal{H}$ are *Kakutani equivalent* if there exists a $\mathcal{G}$-full clopen subset $A \subseteq \mathcal{G}^{(0)}$ and an $\mathcal{H}$-full clopen subset $B \subseteq \mathcal{H}^{(0)}$ such that $\mathcal{G}|_A \cong \mathcal{H}|_B$ (as topological groupoids). In particular, $\mathcal{G}$ itself is Kakutani equivalent to $\mathcal{G}|_A$ whenever A is full. It takes some work to show that Kakutani equivalence actually is an equivalence relation [Mat12, Lemma 4.5].

Remark 1.4.4. Kakutani equivalence for ample groupoids was introduced by Matui in [Mat12], taking cues from the notion of Kakutani equivalence for Cantor minimal systems introduced in [GPS95]. The concept of Kakutani equivalence has its roots in work of Kakutani in ergodic theory from the 40's [Kak43]. However, Kakutani equivalence for groupoids is slightly weaker in the sense that the transformation groupoids associated with two minimal homeomorphisms ϕ_1, ϕ_2 on a Cantor space are Kakutani equivalent if and only if ϕ_1 is Kakutani equivalent (in the sense of [GPS95]) to either ϕ_2 or ϕ_2^{-1}. This is to be expected as the transformation groupoid construction does not see the difference between ϕ and its inverse ϕ^{-1} (these are flip-conjugate), meaning that we have $\mathcal{G}_\phi \cong \mathcal{G}_{\phi^{-1}}$ as topological groupoids. See Section 2.2 for more on Cantor minimal systems.

The other notion of equivalence that we will introduce is that of *stable isomorphism*. Let $\mathcal{R}_\infty := \mathbb{N} \times \mathbb{N}$ be the full countable equivalence relation, equipped with the discrete topology, which makes it an ample groupoid. Note that $\mathcal{R}_\infty \times \mathcal{R}_\infty \cong \mathcal{R}_\infty$ (as topological groupoids). We refer to the product groupoid $\mathcal{G} \times \mathcal{R}_\infty$ as the *stabilization* of $\mathcal{G}$. We say that $\mathcal{G}$ and $\mathcal{H}$ are *stably isomorphic* if $\mathcal{G} \times \mathcal{R}_\infty \cong \mathcal{H} \times \mathcal{R}_\infty$ (as topological groupoids). This terminology is inspired by the analogous notation for C^*-algebras, as we have $C^*(\mathcal{R}_\infty) \cong \mathbb{K}$ (the compact operators) and $C^*(\mathcal{G} \times \mathcal{R}_\infty) \cong C^*(\mathcal{G}) \otimes \mathbb{K}$.

That Kakutani equivalence is the same as stable isomorphism (for ample groupoids with σ-compact unit spaces) was first observed in [CRS17]. Therein it was also shown to be the same as *groupoid equivalence* in the sense of Renault, which is a bit more involved to define (see [FKPS18, Definition 3.7]). To see how these notions are related, let us explain how stable isomorphism imply Kakutani equivalence (which is the easy direction). The key to this is the following observation.

Lemma 1.4.5. *Let $\mathcal{G}$ be an ample Hausdorff groupoid. Then $\mathcal{G}$ is Kakutani equivalent to its stabilization $\mathcal{G} \times \mathcal{R}_\infty$.*

Proof. The unit space of $\mathcal{R}_\infty$ is identified with $\mathbb{N}$ and so $(\mathcal{G} \times \mathcal{R}_\infty)^{(0)}$ is identified with $\mathcal{G}^{(0)} \times \mathbb{N}$. First observe that $\mathcal{G}^{(0)} \times \{1\}$ is clopen in $\mathcal{G}^{(0)} \times \mathbb{N}$ and that $(\mathcal{G} \times \mathcal{R}_\infty)|_{\mathcal{G}^{(0)} \times \{1\}} \cong \mathcal{G}$. Next, observe that $\mathcal{G}^{(0)} \times \{1\}$ is $\mathcal{G} \times \mathcal{R}_\infty$-full. Indeed, for any $(x, m) \in \mathcal{G}^{(0)} \times \mathbb{N}$, the element $(x, (1, m)) \in \mathcal{G} \times \mathcal{R}_\infty$ has source (x, m) and range $(x, 1) \in \mathcal{G}^{(0)} \times \{1\}$. □

Now, if $\mathcal{G}$ and $\mathcal{H}$ are stably isomorphic, then $\mathcal{G}$ is Kakutani equivalent to $\mathcal{G} \times \mathcal{R}_\infty$, which is isomorphic to $\mathcal{H} \times \mathcal{R}_\infty$, which in turn is Kakutani equivalent to $\mathcal{H}$, so $\mathcal{G}$ and $\mathcal{H}$ are Kakutani equivalent.

Chapter 2

Topological full groups

In this chapter we introduce the *full group* and the *topological full group* associated to a dynamical system, and more generally to an ample groupoid. We focus particularly on topological full groups of Cantor minimal systems and of one-sided shifts of finite type, as these predated and motivated the general definition for groupoids.

2.1 Full groups of measurable transformations

The *full group* of a measurable dynamical system was introduced and studied by Dye in [Dye59] and [Dye63]. Let (X, μ) be a measure space and let $T\colon X \to X$ be an invertible measure preserving transformation. The *full group* of T is

$$[T] := \{S \in \mathrm{Aut}(X) \mid S(x) \in \mathrm{Orb}_T(x) \text{ for a.e. } x \in X\},$$

where $\mathrm{Aut}(X)$ is the group of invertible measure preserving transformations of X. Two measure preserving transformations $T\colon X \to X$ and $T'\colon X' \to X'$ are *orbit equivalent* if there exists a (almost everywhere defined) measure space isomorphism $F\colon X \to X'$ which preserves the orbits, i.e. $F(\mathrm{Orb}_T(x)) = \mathrm{Orb}_{T'}(F(x))$ for almost every $x \in X$. We see that two transformations $T, T' \in \mathrm{Aut}(X)$ on the same measure space have the same orbits, i.e. are orbit equivalent via the identity map, if and only if $T \in [T']$ and $T' \in [T]$.

Dye showed that any two invertible ergodic measure preserving transformations on a non-atomic standard probability space are orbit equivalent. This is a celebrated result within von Neumann algebras and ergodic theory, and it is now referred to as *Dye's Theorem*. Moreover, Dye considered countable group actions and showed that for ergodic measure preserving group actions on non-atomic standard probability spaces, the abstract isomorphism class of the full group completely classifies the orbit equivalence class of the action.

2.2 Topological full groups of Cantor minimal systems

Let us move on to (topological) full groups of topological dynamical systems. In this setting, a fitting analogue of an ergodic measure preserving transformation on a non-atomic standard probability space is that of a *Cantor minimal system* (see Remark 2.2.1). Recall that a *Cantor space* is a (non-empty) totally disconnected compact metric space X with no isolated points, of which there is only one up to homeomorphism. A homeomorphism $\phi\colon X \to X$ is *minimal* if every ϕ-orbit is dense (equivalently, there are no non-trivial open ϕ-invariant subsets). We refer to the pair (X, ϕ) as a *Cantor minimal system*. Two Cantor minimal systems (X_1, ϕ_1) and (X_2, ϕ_2) are

1. *conjugate* if there is a homeomorphism $h\colon X_1 \to X_2$ with $h \circ \phi_1 = \phi_2 \circ h$,
2. *flip-conjugate* if (X_1, ϕ_1) is conjugate to either (X_2, ϕ_2) or (X_2, ϕ_2^{-1}),
3. *(topologically) orbit equivalent* if there is a homeomorphism $h\colon X_1 \to X_2$ with $h(\mathrm{Orb}_{\phi_1}(x)) = \mathrm{Orb}_{\phi_2}(h(x))$ for all $x \in X_1$.

If we widen our scope to group actions, then flip-conjugacy is the same as a conjugacy of $\mathbb{Z}$-actions. A general goal in dynamical systems theory is to classify systems (within a given class) up to various notions of equivalence, like the three notions above.

Remark 2.2.1. That minimality is analogous to ergodicity should be clear. Let us give a few reasons why it is natural to restrict to Cantor spaces in the topological context. Firstly, for minimal homeomorphisms on *connected* compact metric spaces, orbit equivalence actually coincide with flip-conjugacy. This follows from an old theorem of Sierpiński [Sie18], which says that a connected compact Hausdorff space cannot be (non-trivially) partitioned into countably many closed subsets.

Secondly, Cantor minimal systems are “universal” among minimal dynamical systems in the following sense: If Y is a compact metric space and $\psi\colon Y \to Y$ is a minimal homeomorphism, then there exists a Cantor minimal system (X, ϕ) which has (Y, ψ) as a factor. This follows from the Hausdorff–Alexandroff Theorem (see for example [Wil70, Theorem 30.7]), which says any compact metric space is a continuous image of the Cantor set. See [GPS95, page 55] for the construction of (X, ϕ) from (Y, ψ).

Let $\mathrm{Homeo}(X)$ denote the group of self-homeomorphisms of X. The *full group* of a Cantor minimal system (X, ϕ) is

$$[\phi] := \left\{ \psi \in \mathrm{Homeo}(X) \mid \psi(x) \in \mathrm{Orb}_\phi(x) \text{ for all } x \in X \right\}.$$

For each $\psi \in [\phi]$, there is a unique map $n_\psi : X \to \mathbb{Z}$ (since a Cantor minimal system is necessarily a free $\mathbb{Z}$-action), called the *orbit cocycle*, such that $\psi(x) = \phi^{n_\psi(x)}(x)$ for each $x \in X$. The *topological full group* $[\![\phi]\!]$ of (X, ϕ) consists of those homeomorphisms for which this orbit cocycle map n_ψ is continuous, i.e.

$$[\![\phi]\!] \coloneqq \left\{\psi \in [\phi] \mid n_\psi \text{ is continuous}\right\}.$$

We remark that, despite the name, the topological full group is usually not viewed as a topological group. In general the full group is uncountable, whereas the topological full group is countable. To see that $[\![\phi]\!]$ is countable, note that the level sets

$$X_k = n_\psi^{-1}(\{k\}) = \left\{x \in X \mid \psi(x) = \phi^k(x)\right\}$$

form a finite clopen partition of X, which determine ψ, and a Cantor space only has countably many clopen subsets.

The topological full group of a Cantor minimal system appeared already in [Put89] as a quotient of the group of unitary normalizers of $C(X)$ inside the crossed product C^*-algebra $C(X) \rtimes_\phi \mathbb{Z}$. The explicit definition given above appears in [GW95] (in which the topological full group is called the “finite full group”), where the authors prove variants of the results in [GPS95] using purely dynamical arguments. In [Tom96], Tomiyama defines the topological full group for more general topological dynamical systems and uses it to generalize one of the main results from [GPS95].

Then in [GPS99], the full and topological full groups (of Cantor minimal systems) themselves were given a thorough treatment, paralleling that of Dye in the measure theoretic setting. Giordano, Putnam and Skau obtained a topological analogue of Dye’s measure theoretic result; they showed that two Cantor minimal systems (X_1, ϕ_1) and (X_2, ϕ_2) are (topologically) orbit equivalent if and only if their full groups $[\phi_1]$ and $[\phi_2]$ are isomorphic (as abstract groups). Furthermore, they showed that (X_1, ϕ_1) and (X_2, ϕ_2) are flip-conjugate if and only if their topological full groups $[\![\phi_1]\!]$ and $[\![\phi_2]\!]$ are isomorphic.

Suppose that $h \colon X_1 \to X_2$ is an orbit equivalence between two Cantor minimal systems (X_1, ϕ_1) and (X_2, ϕ_2). Then there are unique maps $k_1, k_2 \colon X_1 \to \mathbb{Z}$ satisfying

$$h\left(\phi_1(x)\right) = \phi_2^{k_1(x)}\left(h(x)\right) \quad \text{and} \quad \phi_2\left(h(x)\right) = h\left(\phi_1^{k_2(x)}(x)\right)$$

for all $x \in X_1$. One calls (X_1, ϕ_1) and (X_2, ϕ_2) *continuously orbit equivalent* if there exists an orbit equivalence h for which the orbit cocycles k_1, k_2 are continuous. We see that a conjugacy is the same as a (continuous) orbit equivalence with orbit cocycles constantly equal to 1, while a flip-conjugacy is to have them constantly equal to ± 1.

For more general dynamical systems, continuous orbit equivalence is weaker than (flip-)conjugacy, but for Cantor minimal systems continuous orbit equivalence is actually equivalent to flip-conjugacy. This follows from [Boy83, Theorem 2.6] (see also [BT98, Theorem 3.2]). This was the first example of the phenomenon *continuous orbit equivalence rigidity*, whose systematic study was only recently initiated by Li in [Li18]. As we will see in the next subsection, it is really continuous orbit equivalence that the topological full group can detect in general. See also Section 2.6.

One weakening of continuous orbit equivalence for Cantor minimal systems is to allow the orbit cocycles k_1, k_2 to each have a single discontinuity. This is called *strong orbit equivalence*. This is a natural weakening that accounts for the distinguished maximal path in the Bratteli–Vershik model for Cantor minimal systems [HPS92], [Put18]. In the same spirit as the results mentioned for the full and the topological full group above, it was also shown in [GPS99] that a certain subgroup (which is a locally finite ample group in the sense of Krieger [Kri80]) of the topological full group completely determines the strong orbit equivalence class of the Cantor minimal system.

Remark 2.2.2. The definition of the (topological) full group makes sense also for spaces X which are not Cantor, but it will no longer contain much "dynamical information". Indeed, if the space X is connected, then the (topological) full group of (X, ϕ) reduces to $\{\phi^k \mid k \in \mathbb{Z}\}$ [GPS99, Proposition 1.3], which is isomorphic to $\mathbb{Z}$. So then the (topological) full group can certainly not be used to distinguish any such systems from each other.

2.3 Topological full groups of one-sided SFT's

Let us next look at another type of dynamical systems on Cantor spaces, which is of a different nature from Cantor minimal systems, namely that of *one-sided shifts of finite type* (*SFT* for short) [LM95], [Kit98]. Let $N \in \mathbb{N}$ and let A be an $N \times N$ matrix with entries in $\{0, 1\}$. We call a matrix A *essential* if no row nor column consist entirely of 0's. Assume henceforth that A is essential. Define the one-sided shift space

$$X_A := \left\{ x_1 x_2 x_3 \ldots \in \{1, 2, \ldots, N\}^{\mathbb{N}} \mid A_{x_i, x_{i+1}} = 1 \text{ for all } i \in \mathbb{N} \right\}, \tag{2.3.1}$$

which is equipped with the subspace topology from $\{1, 2, \ldots, N\}^{\mathbb{N}}$. This makes X_A a totally disconnected compact metrizable space. We call a word $\mu = x_1 x_2 \ldots x_n$, where $x_i \in \{1, 2, \ldots, N\}$, *admissible* if $A_{x_i, x_{i+1}} = 1$ for $1 \leq i < n$. The *cylinder set* of μ is

$$Z(\mu) := \{x \in X_A \mid x = \mu z \text{ for some } z \in X_A\}.$$

The collection of cylinder sets forms a countable basis of compact open sets for X_A. In particular, the complement of a cylinder set is a finite disjoint union of cylinder sets.

A matrix A as above is *irreducible* if there for each $1 \leq I, J \leq N$ is some $n \in \mathbb{N}$ for which $(A^n)_{I,J} > 0$. This means that there exists an admissible word $x_1 x_2 \ldots x_n$ with $x_1 = I$ and $x_n = J$. If A is irreducible and not a permutation matrix, then X_A has no isolated points, and is therefore a Cantor space. The *shift map* $\sigma_A \colon X_A \to X_A$ is given by $\sigma_A((x_i)) = (x_{i+1})$, i.e.

$$\sigma_A(x_1 x_2 x_3 \ldots) = x_2 x_3 x_4 \ldots$$

The pair (X_A, σ_A) is called a *one-sided shift of finite type* (or topological Markov shift/chain). One can think of the set $\{1, 2, \ldots, N\}$ as the set of possible states of a system, and the matrix A as specifying the possible transitions between these states. An admissible word is then a possible sequence of states.

Remark 2.3.1. Generally speaking, a shift space (over a finite alphabet) is of *finite type* if there is some finite list of forbidden words such that the shift space consists precisely of all sequences which do not contain any of these forbidden words. In the setting above, the alphabet is $\{1, 2, \ldots, N\}$ and the list of forbidden words are all words IJ where $A_{I,J} = 0$. Up to conjugacy, all shifts of finite type can be described by a matrix A as above. See [LM95] for details.

The shift map is not a homeomorphism, but rather a surjective local homeomorphism. Because of this, the notion of orbits looks a bit different from the case of Cantor minimal systems. The *σ_A-orbit* of a sequence $x \in X_A$ is

$$\mathrm{Orb}_{\sigma_A}(x) := \bigcup_{m=0}^{\infty} \bigcup_{n=0}^{\infty} \sigma_A^{-m}\left(\left\{\sigma_A^n(x)\right\}\right).$$

Two sequences $x, y \in X_A$ are in the same orbit if and only if they have a common (forward) iterate, i.e. $\sigma_A^n(x) = \sigma_A^m(y)$ for some $m, n \in \mathbb{N}_0$. Explicitly, this means that x and y are *tail equivalent*, in the sense that $x = \mu z$ and $y = \nu z$ for some admissible words μ, ν and some sequence $z \in X_A$. The one-sided SFT (X_A, σ_A) is called *minimal* if every σ_A-orbit (as defined above) is dense in X_A. A useful fact is that (X_A, σ_A) is minimal if and only if A is irreducible.

For completeness, we sketch how minimality is related to irreducibility. Given any sequence $x \in X_A$ and admissible word μ, irreducibility of A implies that there exists an admissible word ν such that $\mu\nu x \in X_A$. As $\mu\nu x \in \mathrm{Orb}_{\sigma_A}(x)$ we deduce that $\mathrm{Orb}_{\sigma_A}(x)$ is dense. Conversely, assume that (X_A, σ_A) is minimal and let $I, J \in \{1, 2, \ldots, N\}$ be given. Since A is essential we can find two admissible words $\mu = x_1 x_2 \ldots x_n$ and $\nu = y_1 y_2 \ldots y_m$ such that the words $x_n x_1$, $x_n y_1$ and $y_m J$

are admissible too. By minimality, the orbit of the sequence $\mu\mu\mu\ldots$ intersects the cylinder set $Z(I)$. This results in a sequence of the form $I\tau\mu\mu\ldots \in X_A$, where $I\tau\mu$ is an admissible word. Then $I\tau\mu\nu J$ is admissible, which shows that $(A^t)_{I,J} > 0$ for some positive integer t.

Remark 2.3.2. If one in (2.3.1) instead considers bi-infinite sequences (indexed by $\mathbb{Z}$), then one gets the more common notion of a two-sided shift of finite type. This is usually what is meant by a shift of finite type. In this case, the shift map is in fact a homeomorphism. However, these dynamical systems are in general far from minimal, as the set of periodic points is dense (and a minimal homeomorphism cannot have any periodic points). So two-sided shifts of finite type are not Cantor minimal systems. On the other hand, two-sided subshifts which are not of finite type can of course be minimal, such as e.g. Sturmian shifts [LM95, §13.7].

It was Matsumoto who defined the topological full group of a one-sided shift of finite type in [Mat10]. Here it was used as a tool to study continuous orbit equivalence in relation to diagonal preserving isomorphisms of Cuntz–Krieger algebras. In the same spirit as for Cantor systems, the *full group* of a one-sided shift of finite type (X_A, σ_A) is

$$[\sigma_A] := \{\psi \in \mathrm{Homeo}(X) \mid \psi(x) \in \mathrm{Orb}_{\sigma_A}(x) \text{ for all } x \in X_A\}.$$

Do note the peculiar fact that unlike for Cantor systems, the shift map σ_A itself does *not* belong to $[\sigma_A]$, for it is not a homeomorphism. If $\psi \in [\sigma_A]$, then there exists functions $k, l \colon X_A \to \mathbb{N}_0$ such that

$$\sigma_A^{k(x)}(\psi(x)) = \sigma_A^{l(x)}(x) \text{ for all } x \in X. \tag{2.3.2}$$

In contrast to the case of Cantor minimal systems, these maps are not unique. For instance, the same amount may be added to both k and l. The *topological full group* $[\![\sigma_A]\!]$ of (X_A, σ_A) consists of all $\psi \in [\sigma_A]$ for which there exists continuous maps k, l satisfying (2.3.2). Beware that in [Mat10], the topological full group is denoted $[\sigma_A]_c$, and that $[\![\sigma_A]\!]$ is used to denote another group called the *AF-full group*.

Let B be another square $\{0,1\}$-matrix. As with Cantor systems, two one-sided shifts of finite type (X_A, σ_A) and (X_B, σ_B) are *(topologically) orbit equivalent* if there exists a homeomorphism $h \colon X_A \to X_B$ with $h(\mathrm{Orb}_{\sigma_A}(x)) = \mathrm{Orb}_{\sigma_B}(h(x))$ for all $x \in X_A$. This means that one can find maps $k_A, l_A \colon X_A \to \mathbb{N}_0$ and $k_B, l_B \colon X_B \to \mathbb{N}_0$ satisfying

$$\begin{aligned} \sigma_B^{k_A(x)}(h(\sigma_A(x))) &= \sigma_B^{l_A(x)}(h(x)), \\ \sigma_A^{k_B(y)}\left(h^{-1}(\sigma_B(y))\right) &= \sigma_A^{l_B(y)}\left(h^{-1}(y)\right), \end{aligned} \tag{2.3.3}$$

for all $x \in X_A$ and $y \in X_B$. The shifts (X_A, σ_A) and (X_B, σ_B) are *continuously orbit equivalent* if there exists a homeomorphism h and continuous maps k_A, l_A, k_B, l_B satisfying (2.3.3) [Mat10, Section 5].

In [Mat10], Matsumoto showed that if A and B are irreducible non-permutation matrices, then the one-sided shifts of finite type (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent if and only if the topological full groups $[\![\sigma_A]\!]$ and $[\![\sigma_B]\!]$ are *spatially isomorphic*, meaning that there is some homeomorphism $f\colon X_A \to X_B$ with $f \circ [\![\sigma_A]\!] \circ f^{-1} = [\![\sigma_B]\!]$. Subsequently in [Mat15a], he showed that any (abstract) group isomorphism between such topological full groups must be spatial. (In that paper the topological full group is instead called the "continuous full group" and denoted Γ_A.) This means that the abstract isomorphism class of the topological full group completely determines the continuous orbit equivalence class of the shift. By taking into account the continuous orbit equivalence rigidity mentioned in the previous section, we see that Matsumoto's result is a genuine analogue of Giordano, Putnam and Skau's result for Cantor minimal systems.

Results of the form "every group isomorphism between two topological full groups is spatial" for a given class of dynamical systems—such as [GPS99, Theorem 4.2] and [Mat15a, Theorem 7.2]—can often be deduced from the remarkable reconstruction results of Rubin [Rub89]. See Section A.6 for more details on Rubin's theorems, where they are used to prove such a result for topological full groups of certain ample groupoids.

2.4 Topological full groups of ample groupoids

So far we have seen topological full groups associated to various topological dynamical systems. Although the definition of the topological full group is similar for the different types of dynamical systems, its description still very much depend on the spesific type of dynamical system. Matui's insight in [Mat12] was that by encoding these dynamical systems in ample groupoids, the topological full groups could be described using bisections. This unified the definition of the topological full groups and also many of the results proved about them.

2.4.1 Matui's definition of the topological full group

Before we present Matui's definition of topological full groups for ample groupoids, let us see how bisections give rise to homeomorphisms on the unit space. Let $\mathcal{G}$ be en étale groupoid and let $U \subseteq \mathcal{G}$ be a bisection. Recall that this means that the source and range maps s, r are both injective—and hence homeomorphisms—when restricted to the open set U. From the homeomorphisms $s|_U\colon U \to s(U)$

and $r|_U : U \to r(U)$ we define another homeomorphism

$$\pi_U := r|_U \circ (s|_U)^{-1} : s(U) \to r(U).$$

For each $g \in U$, π_U maps $s(g)$ to $r(g)$. By picturing groupoid elements as arrows from the source to the range, we can think of a bisection as a collection of arrows for which there is at most one arrow starting at a given point and at most one ending at a given point. The homeomorphism π_U is then given by "following the arrows". We call a bisection U *full* if $s(U) = r(U) = \mathcal{G}^{(0)}$. Each full bisection gives rise to a homeomorphism $\pi_U \in \mathrm{Homeo}\left(\mathcal{G}^{(0)}\right)$. Note that if $\mathcal{G}^{(0)}$ is compact, then so is each full bisection, as they are homeomorphic subsets of $\mathcal{G}$. Now we can give Matui's definition of full and topological full groups of étale groupoids (adopting the notation from [Mat15b]).

Definition 2.4.1 ([Mat12, Definition 2.3])**.** Let $\mathcal{G}$ be a Hausdorff étale groupoid with $\mathcal{G}^{(0)}$ compact.

1. The *full group* of $\mathcal{G}$ is
$$[\mathcal{G}] := \left\{\psi \in \mathrm{Homeo}\left(\mathcal{G}^{(0)}\right) \mid \psi(x) \in \mathrm{Orb}_{\mathcal{G}}(x) \text{ for all } x \in \mathcal{G}^{(0)}\right\}.$$

2. The *topological full group* of $\mathcal{G}$ is
$$[\![\mathcal{G}]\!] := \{\pi_U \mid U \subseteq \mathcal{G} \text{ full bisection}\} \subseteq \mathrm{Homeo}\left(\mathcal{G}^{(0)}\right).$$

Note that $[\![\mathcal{G}]\!]$ is still a subgroup of $[\mathcal{G}]$. Indeed, given $x \in \mathcal{G}^{(0)}$ there is a unique $g \in U$ with $x = s(g)$, and we have $\pi_U(x) = r(g) \in \mathrm{Orb}_{\mathcal{G}}(x)$.

Remark 2.4.2. By Remark 2.2.2 one cannot expect the topological full group to be particularly interesting unless the groupoid itself is ample (i.e. the unit space is totally disconnected).

Remark 2.4.3. A special case of Definition 2.4.1 appears in [Mat06] for principal ample groupoids with compact unit space, viewed as étale equivalence relations. Let $\mathcal{G}$ be such a groupoid. The principality of $\mathcal{G}$ means that given two points $x, y \in \mathcal{G}^{(0)}$ in the same orbit, there is only one groupoid element $g \in \mathcal{G}$ with $s(g) = x$ and $r(g) = y$. Denote this groupoid element by g_x^y. The definition given in [Mat06, Definition 3.1] then reads

$$[\![\mathcal{G}]\!] = \left\{\psi \in [\mathcal{G}] \mid \mathcal{G}^{(0)} \ni x \mapsto g_x^{\psi(x)} \in \mathcal{G} \text{ is a continuous map}\right\}$$

This recovers the topological full group of a Cantor minimal system, but not one-sided shifts of finite type.

The group operations in the topological full group $[[\mathcal{G}]]$ (as a group of homeomorphisms) correspond to groupoid operations performed on the defining full bisections. Let $U, V \subseteq \mathcal{G}$ be full bisections. Then UV and U^{-1} are again full bisections since $s(UV) = s(V)$, $r(UV) = r(U)$, $s\left(U^{-1}\right) = r(U)$ and $r\left(U^{-1}\right) = s(U)$. Note that the unit space $\mathcal{G}^{(0)}$ itself is also a full bisection. We have that

$$\pi_U \circ \pi_V = \pi_{UV}, \quad (\pi_U)^{-1} = \pi_{U^{-1}} \quad \text{and} \quad \pi_{\mathcal{G}^{(0)}} = \mathrm{id}_{\mathcal{G}^{(0)}}.$$

Remark 2.4.4. From the above we see that the set of all full bisections, denote it $\mathcal{F}(\mathcal{G})$ for now, forms a group using the groupoid operations. We have a canonical surjective group homomorphism $\Theta\colon \mathcal{F}(\mathcal{G}) \to [[\mathcal{G}]]$ given by $\Theta(U) = \pi_U$. For ample groupoids which are Hausdorff and effective, the map Θ is an isomorphism, because the homeomorphism π_U then determine the bisection U uniquely (see Lemma A.3.1). In general, however, the group $\mathcal{F}(\mathcal{G})$ will be "larger". Some authors take $\mathcal{F}(\mathcal{G})$ as their definition of the topological full group, e.g. [Nek19], [MB18],[BS19], [Sca18].

Let us present two ways of constructing full bisections, i.e. elements of the topological full group, from smaller bisections. These simple constructions occur in all three research papers in this thesis. Let $\mathcal{G}$ be an ample Hausdorff groupoid with $\mathcal{G}^{(0)}$ compact. Suppose that $U \subseteq \mathcal{G}$ is a compact bisection whose source and range coincide, i.e. $s(U) = r(U)$. Then the set $A := s(U) = r(U)$ is compact open and π_U is a homeomorphism from A to itself. Define $\widetilde{U} := U \sqcup \left(\mathcal{G}^{(0)} \setminus A\right)$, which becomes a full bisection. Its associated homeomorphism $\pi_{\widetilde{U}} \in [[\mathcal{G}]]$ then equals π_U on A and is the identity on $\mathcal{G}^{(0)} \setminus A$.

As for the second construction, suppose instead that $U \subseteq \mathcal{G}$ is a compact bisection with disjoint source and range, i.e. $s(U) \cap r(U) = \emptyset$. Then we can define a full bisection by setting

$$\widehat{U} := U \sqcup U^{-1} \sqcup \left(\mathcal{G}^{(0)} \setminus (r(U) \sqcup s(U))\right).$$

Its associated homeomorphism $\pi_{\widehat{U}} \in [[\mathcal{G}]]$ equals π_U from $s(U)$ to $r(U)$, $\pi_{U^{-1}}$ from $r(U)$ to $s(U)$ and is the identity on the rest of $\mathcal{G}^{(0)}$. Such an element of $[[\mathcal{G}]]$ is called a *transposition*, which is an apt name, seeing as it swaps $s(U)$ with $r(U)$ and leaves the rest unchanged. In particular, $\pi_{\widehat{U}}$ is an involution and has order 2 in $[[\mathcal{G}]]$.

2.4.2 Topological full groups of Cantor minimal systems—revisited

Let us now see how the definition of the topological full group of a groupoid recovers the ones we saw in the preceding sections.

Example 2.4.5. Let (X, ϕ) be a Cantor minimal system and consider the associated transformation groupoid $\mathcal{G}_\phi$. A basis of compact bisections for this ample groupoid is given by the sets $\{k\} \times A$ for $k \in \mathbb{Z}$ and $A \subseteq X$ compact open. Recall that $s(\{k\} \times A) = A$ and $r(\{k\} \times A) = \phi^k(A)$. Let $U \subseteq \mathcal{G}_\phi$ be a full bisection and consider $\pi_U \in \llbracket \mathcal{G}_\phi \rrbracket$. Since U is compact and the intersection of two bisections is again a bisection we can write U as a finite disjoint union $U = \sqcup_{i=1}^M \{k_i\} \times A_i$ of basic bisections. Since s and r are injective on U, and U is full, we must have that

$$s(U) = \bigsqcup_{i=1}^{M} A_i = X = r(U) = \bigsqcup_{i=1}^{M} \phi^{k_i}(A_i),$$

i.e. both the A_i's and $\phi^{k_i}(A_i)$'s form finite clopen partitions of X. For $x \in A_i$ we have $x = s(k_i, x)$ with $(k_i, x) \in U$, hence $\pi_U(x) = r(k_i, x) = \phi^{k_i}(x)$. The orbit cocycle is therefore given by $n_{\pi_U}(x) = k_i$ for $x \in A_i$. Its continuity follows from the clopenness of the A_i's. Hence $\pi_U \in \llbracket \phi \rrbracket$.

Conversely, given $\psi \in \llbracket \phi \rrbracket$ with $\psi(x) = \phi^{n(x)}(x)$ for some continuous function $n \colon X \to \mathbb{Z}$, we can write $n(X) = \{k_1, k_2, \ldots, k_M\} \subseteq \mathbb{Z}$ since X is compact. The sets $A_i := n^{-1}(\{k_i\})$ form a clopen partition of X. Hence so do the sets $\phi^{k_i}(A_i)$. This makes $U = \sqcup_{i=1}^M \{k_i\} \times A_i$ a full bisection, and we have $\pi_U = \psi \in \llbracket \mathcal{G}_\phi \rrbracket$.

This shows that $\llbracket \mathcal{G}_\phi \rrbracket = \llbracket \phi \rrbracket$.

Example 2.4.6. Generalizing the previous example, consider the transformation groupoid arising from a discrete group Γ acting on a compact totally disconnected Hausdorff space X. The full bisections in $\Gamma \ltimes X$ are all of the form

$$U = \bigsqcup_{i=1}^{n} \{\gamma_i\} \times A_i,$$

where both the A_i's and $\gamma_i(A_i)$'s form clopen partitions of X. Then the homeomorphism $\pi_U \in \llbracket \Gamma \ltimes X \rrbracket$ equals γ_i on A_i. Hence the topological full group $\llbracket \Gamma \ltimes X \rrbracket$ consists of all homeomorphisms of X which locally look like the action by Γ. We can also describe the topological full group in terms of a continuous orbit cocycle as

$$\llbracket \Gamma \ltimes X \rrbracket = \{\psi \in [\Gamma \ltimes X] \mid \psi(x) = \gamma_x(x) \text{ with } X \ni x \mapsto \gamma_x \in \Gamma \text{ continuous}\}.$$

Remark 2.4.7. In [Kri80], Krieger studied certain locally finite subgroups of $\mathrm{Homeo}(X)$, for a Cantor space X, which he called *ample groups* (see also [GPS99, Definition 2.5]). Recall that a group is *locally finite* if the subgroup generated by any finite set is again finite (this is the same as being isomorphic to an inductive limit of finite groups). Krieger showed that two such ample groups are spatially isomorphic if and only if their associated dimension groups are isomorphic. In hindsight we recognize Krieger's ample groups as being topological full groups of AF-groupoids, the groupoid being the transformation groupoid associated to the action of the ample group on X. The dimension group is the K_0 group of the associated AF-algebra. In fact, it was observed already in Renault's thesis [Ren80] that Krieger's result could be viewed as a classification result for AF-groupoids.

In [GPS99] it was shown that any isomorphism between these ample groups must be spatial, and hence an AF-groupoid is determined up to isomorphism by the abstract isomorphism class of its topological full group. We also mention that the topological full group of an ample groupoid with Cantor unit space satisfies Krieger's definition of an ample group, except for the local finiteness condition. In fact, the topological full group is locally finite if and only if the groupoid is an AF-groupoid [Mat06, Proposition 3.2].

2.4.3 Topological full groups of one-sided SFT's—revisited

Let us describe how a one-sided shifts of finite type naturally gives rise to an ample groupoid. As in Section 2.3, let A be an essential $N \times N$ matrix with entries in $\{0,1\}$ and consider the one-sided shift space (X_A, σ_A). The *SFT-groupoid* $\mathcal{G}_A$ is defined as

$$\mathcal{G}_A := \left\{ (x, m-n, y) \mid m,n \in \mathbb{N}_0,\ x,y \in X_A,\ \sigma_A^m(x) = \sigma_A^n(y) \right\}.$$

Two elements $(x,k,y), (z,l,w) \in \mathcal{G}_A$ are composable if and only if $y = z$ and in this case their product is given by $(x,k,y)(y,l,w) = (x,k+l,w)$. The inverse is defined by $(x,k,y)^{-1} = (y,-k,x)$. Hence the unit space is $\mathcal{G}_A^{(0)} = \{(x,0,x) \mid x \in X_A\}$, which we will identify with X_A via $(x,0,x) \leftrightarrow x$. The source and range maps then become $s(x,k,y) = y$ and $r(x,k,y) = x$. Observe that the $\mathcal{G}_A$-orbits are the same as the σ_A-orbits described in Section 2.3.

The topology on $\mathcal{G}_A$ is specified by the basis consisting of all sets

$$Z(V,m,n,W) := \left\{ (x, m-n, y) \mid x \in V, y \in W, \sigma_A^m(x) = \sigma_A^n(y) \right\},$$

where $m,n \in \mathbb{N}_0$ and $V, W \subseteq X_A$ are compact open subsets such that $\sigma_A^m|_V$ and $\sigma_A^n|_W$ are injective. In fact, it suffices to take the sub-collection

$$Z(\mu,\nu) := Z\left(Z(\mu), |\mu|, |\nu|, Z(\nu)\right)$$

ranging over all admissible words $\mu, \nu \in \{1,2,\ldots,N\}^*$. This topology is compatible with the topology on X_A. It is clear that these basic sets are compact bisections, which makes $\mathcal{G}_A$ a Hausdorff second countable ample groupoid. The SFT-groupoid $\mathcal{G}_A$ is minimal if and only if A is irreducible.

In [CK80], a $\{0,1\}$-matrix A was said to satisfy *Condition (I)* if for each $J \in \{1,2,\ldots N\}$ there are admissible words $\mu = J\mu_2 \ldots \mu_n$, $\nu = \nu_1 \ldots \nu_m$ and

$\nu' = \nu'_1 \dots \nu'_{m'}$ with $\mu_n = \nu_1 = \nu'_1 = \nu_m = \nu'_{m'}$ and where $\nu_j \neq \nu'_j$ for some $j \leq \max\{m, m'\}$. The following are equivalent (see [CK80, page 254] and [BCW17, Proposition 2.3]):

1. A satisfies Condition (I).
2. X_A has no isolated points, and thus is a Cantor space.
3. $\mathcal{G}_A$ is effective.

If A is irreducible, then Condition (I) reduces to A not being a permutation matrix. Note, however, that $\mathcal{G}_A$ is never principal. There is always an admissible word $\mu = \mu_1\mu_2 \dots \mu_n$ with $\mu_1 = \mu_n$ and $n \geq 2$, and then $(x, n-1, x) \in (\mathcal{G}_A)_x^x \setminus \mathcal{G}_A^{(0)}$, where $x := \mu_2 \dots \mu_n\mu_2 \dots \mu_n \dots \in X_A$.

Example 2.4.8. Let us verify that Matui's definition of $[\![\mathcal{G}_A]\!]$ recovers Matsumoto's definition of $[\![\sigma_A]\!]$. The partial homeomorphism associated to a basic bisection $Z(\mu, \nu)$ is given by $\pi_{Z(\mu,\nu)}(\nu z) = \mu z$ for $\nu z \in Z(\nu)$. Let now $U \subseteq \mathcal{G}_A$ be a full bisection. Then we can write $U = \sqcup_{i=1}^M Z(\mu_i, \nu_i)$. It follows that

$$s(U) = \bigsqcup_{i=1}^{M} Z(\nu_i) = X_A = r(U) = \bigsqcup_{i=1}^{M} Z(\mu_i).$$

For $x = \nu_i z \in Z(\nu_i)$, we have $\pi_U(x) = \pi_U(\nu_i z) = \mu_i z$ and so we see that $\sigma_A^{|\mu_i|}(\pi_U(x)) = \sigma_A^{|\nu_i|}(x)$. Now define the continuous maps $k, l \colon X_A \to \mathbb{N}_0$ by $k(x) = |\mu_i|$ for $x \in Z(\mu_i)$ and $l(x) = |\nu_i|$ for $x \in Z(\nu_i)$. These satisfy (2.3.2), hence $\pi_U \in [\![\sigma_A]\!]$.

Conversely, let $\psi \in [\![\sigma_A]\!]$ with $\sigma_A^{k(x)}(\psi(x)) = \sigma_A^{l(x)}(x)$ for some continuous maps $k, l \colon X_A \to \mathbb{N}_0$ be given. Let L be the maximum value of l. By adding $L - l(x)$ to both k and l we have

$$\sigma_A^{n(x)}(\psi(x)) = \sigma_A^L(x)$$

for $n(x) = k(x) + L - l(x)$. Let $\nu_1, \dots, \nu_M$ be all admissible words of length L. Then σ_A^L is injective on each cylinder set $Z(\nu_i)$ and $X_A = \sqcup_{i=1}^M Z(\nu_i)$. The sets $\psi(Z(\nu_i))$ also form a clopen partition of X_A and we may refine it to a clopen partition V_j with $1 \leq j \leq J$ such that the function n is constantly equal to a non-negative integer n_j on V_j. Moreover, since each V_j is a finite disjoint union of cylinder sets $Z(\mu)$, we may by splitting these cylinder sets into cylinder sets of words longer than n_j also assume that $\sigma_A^{n_j}$ is injective on V_j. Now set $W_j = \psi^{-1}(V_j)$, each of which is a subset of some $Z(\nu_i)$, hence σ_A^L is injective on each W_j. Define the full bisection $U = \sqcup_{j=1}^J Z\left(V_j, n_j, L, W_j\right)$. We claim that $\pi_U = \psi \in [\![\sigma_A]\!]$. Indeed, given $x \in X$ we have $x \in W_j$ for some j. Then $\psi(x) \in V_j$ and $\sigma_A^{n_j}(\psi(x)) = \sigma_A^L(x)$. We therefore have $(\psi(x), n_j - L, x) \in Z\left(V_j, n_j, L, W_j\right) \subseteq U$ and the source and range of this element is x and $\psi(x)$, respectively, which means that $\pi_U(x) = \psi(x)$.

This shows that $[\![\mathcal{G}_A]\!] = [\![\sigma_A]\!]$.

Remark 2.4.9. We remark that the definition of SFT-groupoids given in this subsection differs slightly from what is used in the research papers in this thesis, albeit only cosmetically. The use of $\{0, 1\}$-matrices in this chapter is consistent with Matsumoto's papers [Mat10], [Mat15a] and [MM14], while the SFT-groupoids that appear in the research papers allow for matrices with entries in $\mathbb{N}_0$, as in [Mat17, Example 2.5]. However, up to isomorphism, one obtains the same class of groupoids (see e.g. [LM95, Proposition 2.3.9]).

2.5 Topological full groups as novel examples in group theory

We would like to mention a notable application of topological full groups to (geometric) group theory. Until roughly a decade ago, there were no known examples of finitely generated simple groups that were amenable (and infinite). This was a major open problem. Below we will briefly describe how topological full groups provided the first examples of such groups.

Let (X, ϕ) be a Cantor minimal system. Denote the commutator subgroup of the topological full group $[\![\phi]\!]$ by $\mathsf{D}([\![\phi]\!])$. In [Mat06], Matui proved that $\mathsf{D}([\![\phi]\!])$ is a simple group. Moreover, he showed that $\mathsf{D}([\![\phi]\!])$ is finitely generated if and only if (X, ϕ) is conjugate to a minimal subshift. Matui also proved that the commutator subgroup $\mathsf{D}([\![\phi]\!])$ is never finitely presented. Then Grigorchuk and Medynets conjectured the amenability of the topological full group $[\![\phi]\!]$ (for any Cantor minimal system) in the preprint version of [GM14] (see [GM11]). This was subsequently verified by Juschenko and Monod in [JM13]. It follows that $\mathsf{D}([\![\phi]\!])$ is amenable, by virtue of being a subgroup of $[\![\phi]\!]$.

In a similar spirit as [GPS99], Bezuglyi and Medynets proved in [BM08] that also the commutator $\mathsf{D}([\![\phi]\!])$ completely determine the flip-conjugacy class of (X, ϕ). (They also showed that the commutator subgroup of the full group, $\mathsf{D}([\phi])$, determine the orbit equivalence class.) Since (topological) entropy is a flip-conjugacy invariant and minimal subshifts exhaust all possible (finite) entropies, the commutator subgroups of topological full groups of minimal subshifts thus produced uncountably many non-isomorphic infinite, finitely generated, simple, amenable groups.

A similar open problem—posed by Grigorchuk in the 80's—was whether there existed a finitely generated simple group of intermediate growth. Such groups were recently exhibited by Nekrashevych in [Nek18b], in the form of a certain "alternating" subgroup of the topological full group of an ample groupoid, introduced in [Nek19].

Remark 2.5.1. The first (of two) example of a finitely presented simple infinite group was Thompson's group V (see [CFP96]). Later on, Higman [Hig74] constructed a countable family of groups $V_{n,r}$ for $n \geq 2$ and $r \geq 1$ whose commutator subgroups $\mathsf{D}(V_{n,r})$ were all finitely presented and simple (and infinite). We remark that these groups are non-amenable, however. One has that $V \cong V_{2,1} = \mathsf{D}(V_{2,1})$. In [Mat15b, Section 6], Matui describes irreducible non-permutation matrices $A_{n,r}$ such that $V_{n,r} \cong \llbracket \mathcal{G}_{A_{n,r}} \rrbracket$. For example,

$$A_{2,1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$A_{3,5} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

In that same paper, Matui proves that the commutator group $\mathsf{D}(\llbracket \mathcal{G}_A \rrbracket)$ is finitely presented and simple for any irreducible non-permutation matrix A. So one can think of the topological full groups $\llbracket \mathcal{G}_A \rrbracket$ of SFT-groupoids as generalized Higman–Thompson groups.

2.6 Connecting dynamical systems, topological full groups and operator algebras via groupoids

In this section we present a selection of seminal results that illustrate how the following four concepts are interrelated:

- (Continuous) orbit equivalence
- (Topological) full groups
- Operator algebras
- Groupoids

Many of these results have since been generalized further. We refer the reader to the books [Ren80] and [Pat99], or the lecture notes [Sim17] and [Put19], for an introduction to (étale) groupoid C^*-algebras. To put the results below in context we recall the following here:

- For a Cantor (minimal) system (X, ϕ) the C^*-algebra $C_r^*\left(\mathcal{G}_\phi\right)$ is canonically isomorphic to the crossed product $C(X) \rtimes_\phi \mathbb{Z}$ (and this isomorphism maps $C\left(\mathcal{G}_\phi^{(0)}\right)$ onto $C(X)$).
- For a one-sided shift of finite type (X_A, σ_A) the C^*-algebra $C_r^*\left(\mathcal{G}_A\right)$ is canonically isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_A$ (and this isomorphism maps $C\left(\mathcal{G}_A^{(0)}\right)$ onto the diagonal subalgebra $\mathcal{D}_A$).

The first result is measure theoretic, and was the first of this kind.

Theorem 2.6.1 ([Sin55], [Dye63])**.** *Let $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ be essentially free ergodic measure preserving actions by countable groups on non-atomic standard probability spaces. Then the following are equivalent:*

1. *The actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are orbit equivalent.*
2. *The full groups $[\Gamma \curvearrowright X]$ and $[\Lambda \curvearrowright Y]$ are isomorphic.*
3. *There is an isomorphism of the von Neumann algebras $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Lambda$ that maps $L^\infty(X)$ onto $L^\infty(Y)$.*

We emphasize that when we write that two ample groupoids are isomorphic, we mean that they are isomorphic as topological groupoids (see the end of Section 1.3). When we write that two (topological) full groups are isomorphic we mean as abstract groups. The next result is a topological analogue of the former, which was mentioned in Section 2.2.

Theorem 2.6.2 ([GPS95], [GPS99])**.** *Let (X, ϕ) and (Y, ψ) be Cantor minimal systems. Then the following are equivalent:*

1. *The systems (X, ϕ) and (Y, ψ) are flip-conjugate.*
2. *The systems (X, ϕ) and (Y, ψ) are continuously orbit equivalent.*
3. *The topological full groups $[\![\phi]\!]$ and $[\![\psi]\!]$ are isomorphic.*
4. *There is an isomorphism of the C^*-algebras $C(X) \rtimes_\phi \mathbb{Z}$ and $C(Y) \rtimes_\psi \mathbb{Z}$ that maps $C(X)$ onto $C(Y)$.*

The following result of Matsumoto was mentioned in Section 2.3.

Theorem 2.6.3 ([Mat10], [Mat15a]). *Let A and B be irreducible non-permutation $\{0,1\}$-matrices. Then the following are equivalent:*

1. *The one-sided shift spaces (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.*
2. *The topological full groups $[\![\sigma_A]\!]$ and $[\![\sigma_B]\!]$ are isomorphic.*
3. *There is an isomorphism of the C^*-algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ that maps $\mathcal{D}_A$ onto $\mathcal{D}_B$.*

These first three results did not use or include groupoids in any way, but groupoids enter the stage in the remaining results. The following reconstruction theorem due to Renault shows that effective étale groupoids can be recovered from their (reduced) groupoid C^*-algebras together with the position of the diagonal subalgebra. This generalized earlier work of Kumjian for principal groupoids [Kum86]. Renault's theorem holds more generally for twisted groupoids, but we will not go into that here.

Theorem 2.6.4 ([Ren08]). *Let $\mathcal{G}$ and $\mathcal{H}$ be effective locally compact Hausdorff second countable étale groupoids. Then the following are equivalent:*

1. *The topological groupoids $\mathcal{G}$ and $\mathcal{H}$ are isomorphic.*
2. *There is an isomorphism of the C^*-algebras $C_r^*(\mathcal{G})$ and $C_r^*(\mathcal{H})$ that maps $C_0\left(\mathcal{G}^{(0)}\right)$ onto $C_0\left(\mathcal{H}^{(0)}\right)$.*

By Renault's reconstruction theorem, we may add the equivalent condition

- The transformation groupoids $\mathcal{G}_\phi$ and $\mathcal{G}_\psi$ are isomorphic

to Theorem 2.6.2 and

- The SFT-groupoids $\mathcal{G}_A$ and $\mathcal{G}_B$ are isomorphic

to Theorem 2.6.3, respectively.

Remark 2.6.5. There is even a two-sided version of Theorem 2.6.3, which was proved in [MM14]. Let A and B be irreducible non-permutation $\{0,1\}$-matrices and denote their associated two-sided shifts of finite type by $\left(\overline{X}_A, \overline{\sigma}_A\right)$ and $\left(\overline{X}_B, \overline{\sigma}_B\right)$. Then the following are equivalent:

1. The two-sided shift spaces $\left(\overline{X}_A, \overline{\sigma}_A\right)$ and $\left(\overline{X}_B, \overline{\sigma}_B\right)$ are flow equivalent.
2. There is an isomorphism of the C^*-algebras $\mathcal{O}_A \otimes \mathbb{K}$ and $\mathcal{O}_B \otimes \mathbb{K}$ that maps $\mathcal{D}_A \otimes \mathcal{C}$ onto $\mathcal{D}_B \otimes \mathcal{C}$.

The implication "1. $\implies$ 2." appeared almost 35 years earlier in [CK80]. Matsumoto and Matui proves the converse by an elegant use of groupoid techniques, utilizing—among other things—Renault's reconstruction theorem, Matsumoto's theorem from above and the classification of two-sided shifts of finite type up to flow equivalence [Fra84], [Hua94]. Note that we may also add the equivalent conditions

- The stabilized SFT-groupoids $\mathcal{G}_A \times \mathcal{R}_\infty$ and $\mathcal{G}_B \times \mathcal{R}_\infty$ are isomorphic
- The SFT-groupoids $\mathcal{G}_A$ and $\mathcal{G}_B$ are Kakutani equivalent

to the list above.

The next result is Matui's Isomorphism Theorem, which shows that certain ample groupoids can be recovered from the algebraic structure of their topological full group alone.

Theorem 2.6.6 ([Mat15b])**.** *Let $\mathcal{G}$ and $\mathcal{H}$ be effective minimal Hausdorff second countable ample groupoids with $\mathcal{G}^{(0)}$ and $\mathcal{H}^{(0)}$ Cantor spaces. Then the following are equivalent:*

1. *The topological groupoids $\mathcal{G}$ and $\mathcal{H}$ are isomorphic.*
2. *The topological full groups $[\![\mathcal{G}]\!]$ and $[\![\mathcal{H}]\!]$ are isomorphic.*

Remark 2.6.7. In Theorem 2.6.6 one may equivalently replace the topological full group with one of several of its distinguished subgroups, such as the commutator subgroup or the kernel of the index map (the index map is defined in the next section). See [Mat15b, Theorem 3.10] and [Nek19, Theorem 3.11].

The final result of this section cements the fact that a transformation groupoid "remembers" precisely the continuous orbit equivalence class of the underlying action (see [Li18] for the definition of continuous orbit equivalence for general group actions).

Theorem 2.6.8 ([Li18])**.** *Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be topologically free actions by countable groups on locally compact second countable Hausdorff spaces. Then the following are equivalent:*

1. *The actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are continuously orbit equivalent.*
2. *The transformation groupoids $\Gamma \ltimes X$ and $\Lambda \ltimes Y$ are isomorphic.*

Chapter 3

Homology of ample groupoids

In this chapter we will introduce the *homology groups* of an ample groupoid. The homology theory is illustrated by several examples before we go on to describing Matui's two conjectures pertaining groupoid homology.

3.1 The homology theory of Crainic–Moerdijk–Matui

A fairly general homology theory for Hausdorff étale groupoids was developed by Crainic and Moerdijk in [CM00]. In [Mat12], Matui restricted this homology theory to the case of constant coefficients in an abelian group (as opposed to a sheaf) for ample groupoids. This resulted in an elementary and accessible presentation of the theory, which has become somewhat of a standard reference in the literature. We also refer to the paper of Farsi, Kumjian, Pask and Sims [FKPS18] for an excellent exposition, as well as comparisons between [Mat12] and [CM00]. In this section, we will describe groupoid homology with coefficients in a discrete abelian group, largely following [Mat12] and [FKPS18].

Assumption 3.1.1. *Throughout this whole section all groupoids are assumed to be Hausdorff, ample and second countable (cf. [FKPS18, Section 4]).*

3.1.1 The definition of $H_n(\mathcal{G}, A)$

Fix a discrete abelian group A. For a locally compact Hausdorff space X, let $C_c(X, A)$ denote the compactly supported continuous A-valued functions on X. We view $C_c(X, A)$ as an abelian group under pointwise addition. A local homeomorphism $\psi\colon X \to Y$ between locally compact Hausdorff spaces induces a group homomorphism $\psi_*\colon C_c(X,A)\to C_c(Y,A)$ by setting

$$\psi_*(f)(y) = \sum_{x\in\psi^{-1}(y)} f(x) \tag{3.1.1}$$

for $f\in C_c(X,A)$. Note that all but finitely many terms are zero in this sum, as f is compactly supported.

For each $n\geq 1$, let $\mathcal{G}^{(n)}$ denote the set of composable n-tuples in $\mathcal{G}$, that is

$$\mathcal{G}^{(n)} = \{(g_1,g_2,\ldots,g_n)\in\mathcal{G}^n \mid r(g_i)=s(g_{i-1}) \text{ for } 2\leq i\leq n\}.$$

Equip $\mathcal{G}^{(n)}$ with the subspace topology from $\mathcal{G}^n$. This makes each $\mathcal{G}^{(n)}$ a Boolean space. In particular, $\mathcal{G}^{(2)}$ is the set of composable pairs, $\mathcal{G}^{(1)}=\mathcal{G}$ is the groupoid itself, and for $n=0$ we have the unit space $\mathcal{G}^{(0)}$. We will construct a chain complex where the abelian groups involved are the $C_c\left(\mathcal{G}^{(n)},A\right)$'s.

To obtain the differentials, we first define the following maps

$$d_i^{(n)}\colon \mathcal{G}^{(n)}\to\mathcal{G}^{(n-1)}$$

for $n\geq 2$ and $i=0,\ldots,n$ by

$$d_i^{(n)}(g_1,g_2,\ldots,g_n) = \begin{cases} (g_2,g_3,\ldots,g_n) & \text{if } i=0,\\ (g_1,\ldots,g_{i-1},g_ig_{i+1},g_{i+2},\ldots,g_n) & \text{if } 1\leq i\leq n-1,\\ (g_1,g_2,\ldots,g_{n-1}) & \text{if } i=n. \end{cases}$$

Observe that each $d_i^{(n)}$ is a local homeomorphism. This follows from $\mathcal{G}$ having a basis of compact bisections and that multiplication in an étale groupoid is a local homeomorphism (Proposition 1.3.3). We illustrate this in a particular case. For $n=2$ for example, given a composable pair $(g,h)\in\mathcal{G}^{(2)}$ we can find compact bisections $U,V\subseteq\mathcal{G}$ satisfying $g\in U$, $h\in V$ and $s(U)=r(V)$. Restricting $d_i^{(2)}$ (which maps (g,h) to h) to $(U\times V)\cap\mathcal{G}^{(2)}$ produces a homeomorphism onto V.

The differentials

$$\delta_n\colon C_c\left(\mathcal{G}^{(n)},A\right)\to C_c\left(\mathcal{G}^{(n-1)},A\right)$$

are defined, using Equation (3.1.1), as

$$\delta_n \coloneqq \sum_{i=0}^{n}(-1)^i\left(d_i^{(n)}\right)_*$$

for $n\geq 2$. For $n=1$ we define $\delta_1\colon C_c(\mathcal{G},A)\to C_c\left(\mathcal{G}^{(0)},A\right)$, using the source and range maps, as follows:

$$\delta_1 \coloneqq s_* - r_*.$$

Formally, there is also the trival differential $\delta_0\colon C_c\left(\mathcal{G}^{(0)}, A\right) \to 0$. A standard calculation shows that $\delta_n \circ \delta_{n+1} = 0$. Thus we have a chain complex

$$0 \xleftarrow{\delta_0} C_c\left(\mathcal{G}^{(0)}, A\right) \xleftarrow{\delta_1} C_c\left(\mathcal{G}, A\right) \xleftarrow{\delta_2} C_c\left(\mathcal{G}^{(2)}, A\right) \xleftarrow{\delta_3} \cdots$$

and the *homology groups* of $\mathcal{G}$ with coefficients in A are defined as the homology of this complex. To be more precise:

$$H_n(\mathcal{G}, A) := \ker \delta_n / \operatorname{im} \delta_{n+1}.$$

We write $[f] \in H_n(\mathcal{G}, A)$ for the equivalence class of $f \in \ker(\delta_n) \subseteq C_c\left(\mathcal{G}^{(n)}, A\right)$. In the case that $A = \mathbb{Z}$ (the "standard" coefficients), we write $H_n(\mathcal{G}) := H_n(\mathcal{G}, \mathbb{Z})$ for brevity.

Readers familiar with group homology may have noticed the similarity to the so-called non-homogenous description of the chain complex arising from the standard resolution (see [Bro82, Chapter II]). See Example 3.2.1 and 3.2.2 below for comparisons with group homology.

3.1.2 Functoriality and Kakutani equivalence

This homology theory is functorial in the following sense. Suppose $\Phi\colon \mathcal{G} \to \mathcal{H}$ is an étale homomorphism (that is, a groupoid homomorphism which is also local homeomorphism). For $n \geq 0$, let $\Phi^{(n)}\colon \mathcal{G}^{(n)} \to \mathcal{H}^{(n)}$ denote the map given by applying Φ in each coordinate. Each $\Phi^{(n)}$ is a local homeomorphism, so we get the group homomorphisms

$$\Phi_*^{(n)}\colon C_c\left(\mathcal{G}^{(n)}, A\right) \to C_c\left(\mathcal{H}^{(n)}, A\right)$$

from (3.1.1). The $\Phi_*^{(n)}$'s satisfy $\delta_n \circ \Phi_*^{(n)} = \Phi_*^{(n-1)} \circ \delta_n$. Hence they induce group homomorphisms

$$H_n(\Phi)\colon H_n\left(\mathcal{G}, A\right) \to H_n\left(\mathcal{H}, A\right)$$

between the homology groups of $\mathcal{G}$ and $\mathcal{H}$ given by

$$H_n(\Phi)([f]) = \left[\Phi_*^{(n)}(f)\right] \quad \text{for } f \in \ker(\delta_n) \subseteq C_c\left(\mathcal{G}^{(n)}, A\right).$$

It is clear that $H_n(\mathrm{id}_{\mathcal{G}}) = \mathrm{id}_{H_n(\mathcal{G},A)}$ and that $H_n(\Psi \circ \Phi) = H_n(\Psi) \circ H_n(\Phi)$ for two étale homomorphisms $\Phi\colon \mathcal{G} \to \mathcal{H}$ and $\Psi\colon \mathcal{H} \to \mathcal{K}$.

In particular, if $\mathcal{H} \subseteq \mathcal{G}$ is an open subgroupoid, then the inclusion $\iota\colon \mathcal{H} \hookrightarrow \mathcal{G}$ induces group homomorphisms $H_n(\iota)\colon H_n(\mathcal{G}, A) \to H_n(\mathcal{H}, A)$. Moreover, if $\mathcal{H} = \mathcal{G}|_Y$ for some $\mathcal{G}$-full clopen set $Y \subseteq \mathcal{G}^{(0)}$, then the inclusion of the restriction $\mathcal{G}|_Y \subseteq \mathcal{G}$ in fact induces isomorphisms

$$H_n(\iota)\colon H_n(\mathcal{G}|_Y, A) \xrightarrow{\cong} H_n(\mathcal{G}, A)$$

for all $n \geq 0$ [Mat12, Theorem 4.8], [FKPS18, Lemma 4.3]. It follows that Kakutani equivalent groupoids have isomorphic homology groups.

3.1.3 Describing $H_0(\mathcal{G})$ and $H_1(\mathcal{G})$

In this thesis, we will for the most part be working only with the lower homology groups, more precisely, with $H_0(\mathcal{G})$ and $H_1(\mathcal{G})$ (with coefficients in $\mathbb{Z}$). This is because these are the ones that go into Matui's AH conjecure—together with the abelianization of the topological full group (see Section 3.4). Another reason is that in many interesting examples, the higher homology groups vanish (see Section 3.2). Let us therefore describe $H_0(\mathcal{G})$ and $H_1(\mathcal{G})$ in more detail.

Describing $H_0(\mathcal{G})$

The zeroth homology group is $H_0(\mathcal{G}) = C_c\left(\mathcal{G}^{(0)}, \mathbb{Z}\right) / \operatorname{im}(\delta_1)$. Recall that the differential $\delta_1 \colon C_c(\mathcal{G}, \mathbb{Z}) \to C_c\left(\mathcal{G}^{(0)}, \mathbb{Z}\right)$ is given by $\delta_1 = s_* - r_*$. The homomorphisms s_* and r_* are in turn given by

$$s_*(f)(x) = \sum_{g \in \mathcal{G}_x} f(g) \quad \text{and} \quad r_*(f)(x) = \sum_{g \in \mathcal{G}^x} f(g)$$

for $f \in C_c(\mathcal{G}, \mathbb{Z})$ and $x \in \mathcal{G}^{(0)}$. Since $\mathcal{G}^{(0)}$ is Boolean, each function in $C_c\left(\mathcal{G}^{(0)}, \mathbb{Z}\right)$ is a finite sum of indicator functions of compact open subsets of $\mathcal{G}^{(0)}$. It follows that

$$H_0(\mathcal{G}) = \operatorname{span}\left\{[1_A] \mid A \subseteq \mathcal{G}^{(0)} \text{ compact open}\right\},$$

where we by *span* mean linear combinations over $\mathbb{Z}$. A key fact about $H_0(\mathcal{G})$ is that compact bisections implement homological equivalence between indicator functions. Indeed, for any compact bisection $U \subseteq \mathcal{G}$ we have

$$\delta_1(1_U) = s_*(1_U) - r_*(1_U) = 1_{s(U)} - 1_{r(U)}, \tag{3.1.2}$$

which implies that

$$[1_{s(U)}] = [1_{r(U)}] \in H_0(\mathcal{G}).$$

In fact, the image $\operatorname{im}(\delta_1) \subseteq C_c\left(\mathcal{G}^{(0)}, \mathbb{Z}\right)$ is generated by all differences $1_{s(U)} - 1_{r(U)}$ as above, since $\mathcal{G}$ has a basis of compact bisections.

Describing $H_1(\mathcal{G})$

Next up, the first homology group is $H_1(\mathcal{G}) = \ker(\delta_1) / \operatorname{im}(\delta_2)$. Note that each function $f \in C_c(\mathcal{G}, \mathbb{Z})$ can be written as $f = \sum_{i=1}^n k_i 1_{U_i}$ where $k_i \in \mathbb{Z}$ and $U_i \subseteq \mathcal{G}$

are compact bisections. From Equation (3.1.2) we see that

$$f \in \ker(\delta_1) \iff \sum_{i=1}^{n} k_i 1_{s(U_i)} = \sum_{i=1}^{n} k_i 1_{r(U_i)}.$$

In particular, if $U \subseteq \mathcal{G}$ is a compact bisection with $s(U) = r(U)$, then $1_U \in \ker(\delta_1)$ and we obtain an element $[1_U] \in H_1(\mathcal{G})$.

Let us describe the relations imposed by dividing out with $\operatorname{im}(\delta_2)$. The differential $\delta_2 \colon C_c(\mathcal{G}^{(2)}, \mathbb{Z}) \to C_c(\mathcal{G}, \mathbb{Z})$ is given by

$$\delta_2 = \left(d_0^{(2)}\right)_* - \left(d_1^{(2)}\right)_* + \left(d_2^{(2)}\right)_*,$$

where each of these summands are given by

$$\begin{aligned} \left(d_0^{(2)}\right)_* (\psi)(g) &= \sum_{h \in \mathcal{G},\ s(h)=r(g)} \psi(h, g) \\ \left(d_1^{(2)}\right)_* (\psi)(g) &= \sum_{(h_1,h_2) \in \mathcal{G}^{(2)},\ h_1 h_2 = g} \psi(h_1, h_2) \\ \left(d_2^{(2)}\right)_* (\psi)(g) &= \sum_{h \in \mathcal{G},\ r(h)=s(g)} \psi(g, h) \end{aligned}$$

for $\psi \in C_c(\mathcal{G}^{(2)}, \mathbb{Z})$ and $g \in \mathcal{G}$. Suppose that $U, V \subseteq \mathcal{G}$ are compact bisections with $s(U) = r(V)$. Define the set

$$U \circ V := (U \times V) \cap \mathcal{G}^{(2)} = \left\{(g, h) \in \mathcal{G}^{(2)} \mid g \in U,\ h \in V\right\}.$$

Then $U \circ V$ is a compact open subset of $\mathcal{G}^{(2)}$ and the collection of these sets form a basis for $\mathcal{G}^{(2)}$. Observe that

$$\left(d_0^{(2)}\right)_* (1_{U \circ V}) = 1_U, \quad \left(d_1^{(2)}\right)_* (1_{U \circ V}) = 1_{UV}, \quad \left(d_2^{(2)}\right)_* (1_{U \circ V}) = 1_V,$$

hence

$$\delta_2 (1_{U \circ V}) = 1_U - 1_{UV} + 1_V. \tag{3.1.3}$$

In particular, if $r(U) = s(U) = r(V) = s(V)$, then

$$[1_{UV}] = [1_U] + [1_V] \in H_1(\mathcal{G}). \tag{3.1.4}$$

A consequence of (3.1.4) is that for any compact open subset $A \subseteq \mathcal{G}^{(0)}$ of the unit space, setting $U = V = A$ gives $[1_A] = [1_A] + [1_A]$, hence

$$[1_A] = 0 \in H_1(\mathcal{G}). \tag{3.1.5}$$

Next, if $U \subseteq \mathcal{G}$ is any compact bisection, then $1_U + 1_{U^{-1}} \in \ker(\delta_1)$, $s(U) = r\left(U^{-1}\right)$ and $UU^{-1} = r(U) \subseteq \mathcal{G}^{(0)}$, so

$$[1_U + 1_{U^{-1}}] = 0 \in H_1(\mathcal{G}). \tag{3.1.6}$$

The preceding four equations amount to [Mat12, Lemma 7.3].

3.2 Examples

We now list a few examples of groupoids for which the homology groups have been computed (or described). In several of them we will also list the K-groups of the associated groupoid C^*-algebras as reference for Matui's HK conjecture, which is presented in Section 3.4. An introduction to K-theory for C^*-algebras may be found in Rørdam, Larsen and Laustsen's book of the same name [RLL00]. Do note how groupoid homology generalizes group homology in the first two examples.

Example 3.2.1. If Γ is a discrete group and we view $\mathcal{G} = \Gamma$ as an ample groupoid, then

$$H_n(\mathcal{G}) \cong H_n(\Gamma),$$

where the latter is the group homology of Γ (with coefficients in $\mathbb{Z}$) as in [Bro82, Chapter II]. Recall that $H_0(\Gamma) = \mathbb{Z}$ and $H_1(\Gamma) = \Gamma_{\mathrm{ab}}$, in particular.

Example 3.2.2. Next, if the group Γ acts on a locally compact Hausdorff space X, then it is folklore ([Mat12, page 31]) that the homology of the associated transformation groupoid is

$$H_n(\Gamma \ltimes X) \cong H_n(\Gamma, C_c(X, \mathbb{Z})),$$

where the latter is the group homology of Γ with coefficients in $C_c(X, \mathbb{Z})$ as in [Bro82, Chapter III] (where the Γ-module structure on $C_c(X, \mathbb{Z})$ is given by $(\gamma \cdot f)(x) = f(\gamma^{-1}(x))$ for $\gamma \in \Gamma, f \in C_c(X, \mathbb{Z}), x \in X$).

Example 3.2.3. If we view a locally compact Hausdorff space X as a trivial ample groupoid, that is $\mathcal{G} = \mathcal{G}^{(0)} = X$, then its groupoid homology is

$$\begin{aligned} H_0(\mathcal{G}) &= C_c(X, \mathbb{Z}), \\ H_n(\mathcal{G}) &= 0 \quad \text{for } n \geq 1. \end{aligned}$$

Whenever we write $H_n(X)$ in this thesis we will mean the above homology groups (and not singular homology for example, which of course is quite different).

Example 3.2.4. We next consider AF-groupoids (see Subsection A.11.5). Let (V, E) be a Bratteli diagram (see for example [GPS04, Example 2.7.(ii)]). By Theorems 4.10 and 4.11 in [Mat12] the homology of the AF-groupoid $\mathcal{G}_{(V,E)}$ is

$$\begin{aligned} H_0(\mathcal{G}_{(V,E)}) &\cong K_0(V, E) \cong K_0(C^*(V, E)), \\ H_n(\mathcal{G}_{(V,E)}) &= 0 \quad \text{for } n \geq 1. \end{aligned}$$

Here, $K_0(V, E)$ is the dimension group determined by the Bratteli diagram (V, E) (see for example [HPS92, Section 5]) and $C^*(V, E)$ is the AF-algebra determined by (V, E).

In fact, the above isomorphisms for $n = 0$ are order isomorphisms which preserve the distinguished order units. For an AF-groupoid $\mathcal{G}$ the positive cone in $H_0(\mathcal{G})$ is

$$H_0(\mathcal{G})^+ = \left\{[f] \in H_0(\mathcal{G}) \mid f(x) \geq 0 \text{ for all } x \in \mathcal{G}^{(0)}\right\}$$

and the order unit is $[1_{\mathcal{G}^{(0)}}]$ [Mat12]. However, if $\mathcal{G}$ is not an AF-groupoid, then $(H_0(\mathcal{G}), H_0(\mathcal{G})^+)$ need not be an ordered abelian group.

Example 3.2.5. Let A be an $N \times N$ essential $\{0, 1\}$-matrix. Matui [Mat12, Theorem 4.14] computed the homology of the SFT-groupoid $\mathcal{G}_A$ to be

$$\begin{aligned} H_0(\mathcal{G}_A) &\cong \operatorname{coker}(I_N - A) \cong K_0(\mathcal{O}_A), \\ H_1(\mathcal{G}_A) &\cong \ker(I_N - A) \cong K_1(\mathcal{O}_A), \\ H_n(\mathcal{G}_A) &= 0 \quad \text{for } n \geq 2, \end{aligned}$$

where I_N is the $N \times N$ identity matrix and $I_N - A$ is viewed as an endomorphism of $\mathbb{Z}^N$ via left multiplication.

Example 3.2.6. Let $\mathcal{G}_{A,B}$ be the Katsura–Exel–Pardo groupoid associated with two $N \times N$ integer matrices A and B (see Section C.3). Under some mild assumptions on A and B, Ortega [Ort18] computed the homology of $\mathcal{G}_{A,B}$ to be

$$\begin{aligned} H_0(\mathcal{G}_{A,B}) &\cong \operatorname{coker}(I_N - A), \\ H_1(\mathcal{G}_{A,B}) &\cong \ker(I_N - A) \oplus \operatorname{coker}(I_N - B), \\ H_2(\mathcal{G}_{A,B}) &\cong \ker(I_N - B), \\ H_n(\mathcal{G}_{A,B}) &= 0, \quad n \geq 3. \end{aligned}$$

For comparison, the K-theory of the Katsura algebra $\mathcal{O}_{A,B}$ (which is the C^*-algebra of $\mathcal{G}_{A,B}$) was in [Kat08b] found to be given by

$$\begin{aligned} K_0(\mathcal{O}_{A,B}) &\cong \operatorname{coker}(I_N - A) \oplus \ker(I_N - B), \\ K_1(\mathcal{O}_{A,B}) &\cong \ker(I_N - A) \oplus \operatorname{coker}(I_N - B). \end{aligned}$$

Finally, let us spend some time describing the homology groups of transformation groupoids associated to Cantor minimal systems. This will serve several purposes. One, to illustrate Example 3.2.2 further. Two, to compare Matui's general index map to the one introduced by Giordano, Putnam and Skau for Cantor minimal systems, see Subsection 3.3.1. Three, to illustrate how a certain long exact sequence in homology can be used to describe the homology groups, a technique which is employed in both Paper B and C.

Example 3.2.7. Let (X, ϕ) be a Cantor minimal system and consider the associated transformation groupoid $\mathcal{G}_\phi$. Example III.1.1 in [Bro82] describes the group homology of $\Gamma = \mathbb{Z}$ with coefficients. Combining this with Example 3.2.2 we obtain the following descriptions of $H_n\left(\mathcal{G}_\phi\right)$. The zeroth homology group is

$$H_0\left(\mathcal{G}_\phi\right) \cong \frac{C(X,\mathbb{Z})}{\langle f \circ \phi^k - f \mid f \in C(X,\mathbb{Z}), k \in \mathbb{Z}\rangle} = \frac{C(X,\mathbb{Z})}{\{f - f \circ \phi \mid f \in C(X,\mathbb{Z})\}}.$$

This is precisely the K^0-group of the dynamical system, which is denoted by $K^0(X, \phi)$ [GPS95, Definition 1.11]. As for the K_0-group of the crossed product one also has $K_0\left(C(X) \rtimes_\phi \mathbb{Z}\right) \cong K^0(X, \phi)$ [Put89], [HPS92]. (These isomorphisms are even order and order unit preserving.) Next, the first homology group is

$$\begin{aligned} H_1\left(\mathcal{G}_\phi\right) &\cong H_1\left(\mathbb{Z}, C(X,\mathbb{Z}\right) \cong \left\{f \in C(X,\mathbb{Z}) \mid f \circ \phi^k = f \;\forall\, k \in \mathbb{Z}\right\} \\ &= \{f \in C(X,\mathbb{Z}) \mid f \text{ constant}\} \cong \mathbb{Z}, \end{aligned}$$

where the equality is due to the minimality of ϕ. See Proposition 3.3.1 for an explicit isomorphism between $H_1\left(\mathcal{G}_\phi\right)$ and $\mathbb{Z}$. It is well known that $K_1\left(C(X) \rtimes_\phi \mathbb{Z}\right) \cong \mathbb{Z}$ also. Finally, we have $H_n\left(\mathcal{G}_\phi\right) = 0$ for $n \geq 2$.

Example 3.2.8. In both Paper B and C we rely on a long exact sequence in homology (Proposition B.6.1) to get a useful description of the homology groups of the groupoids under study. This long exact sequence is an analogue of the Pimsner–Voiculescu 6-term exact sequence in K-theory for crossed products by $\mathbb{Z}$ (see for example [Bla98, Section 10]), but for ample groupoids equipped with a $\mathbb{Z}$-valued cocycle. Let us, to illustrate this, show how this long exact sequence can be used for an alternative computation of the homology groups of $\mathcal{G}_\phi$ for a Cantor minimal system (X, ϕ), without appealing to group homology with coefficients.

We define a cocycle $c\colon \mathcal{G}_\phi \to \mathbb{Z}$ by simply setting $c(k, x) = k$ for $(k, x) \in \mathcal{G}_\phi$. As we do in Paper B and C, let us denote the kernel of this cocycle by $\mathcal{H}_\phi := \ker(c)$. The kernel is particularly simple in this case since $\mathcal{H}_\phi = \{0\} \times X = \mathcal{G}_\phi^{(0)}$, which we have identified with X. Let $\mathcal{G}_\phi \times_c \mathbb{Z}$ denote the associated skew product groupoid (see Subsection B.2.5) whose unit space is identified with $X \times \mathbb{Z}$ (viewing $\mathbb{Z}$ as a discrete space), which then is only locally compact. We have

$$X \cong \mathcal{H}_\phi \cong \left(\mathcal{G}_\phi \times_c \mathbb{Z}\right)|_{X \times \{0\}}$$

as ample groupoids.

Let us demonstrate that the subset $X \times \{0\} \subseteq \left(\mathcal{G}_\phi \times_c \mathbb{Z}\right)^{(0)}$ is $\left(\mathcal{G}_\phi \times_c \mathbb{Z}\right)$-full. Let $(x, l) \in X \times \mathbb{Z}$ be given. The element $g = ((-l, \phi^l(x)), l) \in \mathcal{G}_\phi \times_c \mathbb{Z}$ satisfies $s(g) = (\phi^l(x), 0) \in X \times \{0\}$ and $r(g) = (x, l)$, and the fullness follows. This makes X—viewed as a trivial ample groupoid—Kakutani equivalent to the skew product $\mathcal{G}_\phi \times_c \mathbb{Z}$. It follows from Example 3.2.3 that

$$H_n\left(\mathcal{G}_\phi \times_c \mathbb{Z}\right) \cong H_n(X) = \begin{cases} C(X,\mathbb{Z}) & \text{for } n = 0, \\ 0 & \text{for } n \geq 1. \end{cases}$$

The isomorphism $C(X,\mathbb{Z}) \cong H_0\left(\mathcal{G}_\phi \times_c \mathbb{Z}\right)$ is given by $f \mapsto [f \times 0]$, where $f \times 0 \in C_c(X \times \mathbb{Z}, \mathbb{Z})$ is given by

$$(f \times 0)(x, l) = \begin{cases} f(x) & \text{if } l = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $H_n\left(\mathcal{G}_\phi \times_c \mathbb{Z}\right)$ vanish for all $n \geq 1$ the long exact sequence from Proposition B.6.1 collapses to

$$0 \longrightarrow H_1(\mathcal{G}_\phi) \longrightarrow H_0(\mathcal{G}_\phi \times_c \mathbb{Z}) \xrightarrow{\mathrm{id} - H_0(\rho)} H_0(\mathcal{G}_\phi \times_c \mathbb{Z}) \longrightarrow H_0(\mathcal{G}_\phi) \longrightarrow 0.$$

and we may immediately conclude that

$$H_n\left(\mathcal{G}_\phi\right) = 0 \quad \text{for } n \geq 2.$$

The homomorphism $H_0(\rho)$ is described in Equation (3.2.1) below (see Section B.6 for a description of the map ρ itself). By identifying $H_0(\mathcal{G}_\phi \times_c \mathbb{Z})$ with $C(X,\mathbb{Z})$ the exact sequence above turns into

$$0 \longrightarrow H_1(\mathcal{G}_\phi) \longrightarrow C(X,\mathbb{Z}) \xrightarrow{\Phi} C(X,\mathbb{Z}) \longrightarrow H_0(\mathcal{G}_\phi) \longrightarrow 0,$$

where Φ is the unique map making the following diagram commute:

$$\begin{tikzcd} C(X,\mathbb{Z}) \arrow[r, "\Phi"] \arrow[d, "\cong"', "{[(\cdot)\times 0]}"] & C(X,\mathbb{Z}) \arrow[d, "\cong"', "{[(\cdot)\times 0]}"] \\ H_0(\mathcal{G}_\phi \times_c \mathbb{Z}) \arrow[r, "\mathrm{id} - H_0(\rho)"] & H_0(\mathcal{G}_\phi \times_c \mathbb{Z}) \end{tikzcd}$$

We claim that $\Phi(f) = f - f \circ \phi^{-1}$ for $f \in C(X,\mathbb{Z})$. Taking this at face value for now, we obtain

$$\begin{aligned} H_1\left(\mathcal{G}_\phi\right) \cong \ker(\Phi) &= \left\{f \in C(X,\mathbb{Z}) \mid f \circ \phi^{-1} = f\right\} \\ &= \{f \in C(X,\mathbb{Z}) \mid f \text{ constant}\} \cong \mathbb{Z} \end{aligned}$$

and

$$H_0\left(\mathcal{G}_\phi\right) \cong \operatorname{coker}(\Phi) = \frac{C(X,\mathbb{Z})}{\{f - f\circ\phi^{-1} \mid f \in C(X,\mathbb{Z})\}} = \frac{C(X,\mathbb{Z})}{\{f - f\circ\phi \mid f \in C(X,\mathbb{Z})\}}$$

as before.

Let us verify the formula for Φ claimed above. The map $H_0(\rho)$ is given by

$$H_0(\rho)\left([1_{A\times\{i\}}]\right) = [1_{A\times\{i+1\}}] \tag{3.2.1}$$

for $A \subseteq X$ clopen and $i \in \mathbb{Z}$. Using this we compute

$$\begin{aligned} C(X,\mathbb{Z}) \ni 1_A \longmapsto [1_A \times 0] = [1_{A\times\{0\}}] &\overset{H_0(\rho)}{\longmapsto} [1_{A\times\{1\}}] \\ &= \left[1_{\phi(A)\times\{0\}}\right] \longmapsto 1_{\phi(A)} = 1_A \circ \phi^{-1}, \end{aligned}$$

where the equality $[1_{A\times\{1\}}] = \left[1_{\phi(A)\times\{0\}}\right] \in H_0(\mathcal{G}_\phi \times_c \mathbb{Z})$ follows from considering the source and range of the bisection $(\{1\}\times A)\times\{0\} \subseteq \mathcal{G}_\phi \times_c \mathbb{Z}$; these being equal to $A\times\{1\}$ and $\phi(A)\times\{0\}$, respectively. The claim follows.

The reader is invited to compare this computation with a computation of the K-groups of the crossed product $C(X) \rtimes_\phi \mathbb{Z}$ using the Pimsner–Voiculescu exact sequence.

3.3 The index map

The index map is a group homomorphism from the topological full group into the first homology group. The index map was introduced in the setting of Cantor minimal systems in [GPS99] and later generalized to étale groupoids over Cantor spaces in [Mat12] (see Subsection 3.3.1).

Let $\mathcal{G}$ be a Hausdorff effective ample groupoid with compact unit space. The *index map*

$$I\colon [\![\mathcal{G}]\!] \to H_1(\mathcal{G})$$

is the homomorphism given by $\pi_U \mapsto [1_U]$, for U a full bisection in $\mathcal{G}$. Recall that since $\mathcal{G}$ is effective, the homeomorphism π_U uniquely determines U. This makes I well-defined (as $1_U \in \ker(\delta_1)$). That the index map is a group homomorphism follows readily from Equation (3.1.4). Indeed, let $U, V \subseteq \mathcal{G}$ be full bisections, then

$$I(\pi_U \circ \pi_V) = I(\pi_{UV}) = [1_{UV}] = [1_U] + [1_V] = I(\pi_U) + I(\pi_V) \in H_1(\mathcal{G}).$$

We denote the induced map on the abelianization $[\![\mathcal{G}]\!]_{\mathrm{ab}}$ by

$$I_{\mathrm{ab}} \colon [\![\mathcal{G}]\!]_{\mathrm{ab}} \to H_1(\mathcal{G}).$$

Note that I is surjective if and only if I_{ab} is surjective. Let us also observe that any transposition in $[\![\mathcal{G}]\!]$ is in the kernel of the index map. Indeed, let $U \subseteq \mathcal{G}$ be a compact bisection with $s(U) \cap r(U) = \emptyset$. Then using (3.1.5) and (3.1.6) we have

$$\begin{aligned} I\left(\pi_{\widehat{U}}\right) &= [1_{\widehat{U}}] = \left[1_{U \sqcup U^{-1} \sqcup (\mathcal{G}^{(0)} \setminus (s(U) \sqcup r(U))}\right] \\ &= [1_U + 1_{U^{-1}}] + \left[1_{\mathcal{G}^{(0)} \setminus (s(U) \sqcup r(U))}\right] = 0 \in H_1(\mathcal{G}). \end{aligned} \tag{3.3.1}$$

In Paper B we explain how to extend the index map to ample groupoids whose unit spaces are not compact.

3.3.1 The index map of a Cantor minimal system

As mentioned above, the index map first appeared in the setting of Cantor minimal systems. There it is the unique surjective homomorphism from the topological full group onto $\mathbb{Z}$. Let us explain this version of the index map and show how Matui's definition generalizes it.

Let (X, ϕ) be a Cantor minimal system. Lemma 5.3 in [GPS99] shows that each element $\psi \in [\![\phi]\!]$ can be written as

$$\psi = \gamma_1 \phi^l \gamma_2, \tag{3.3.2}$$

where $l \in \mathbb{Z}$ and γ_1, γ_2 both have finite order. An immediate consequence is that if $\Theta \colon [\![\phi]\!] \to \mathbb{R}$ is a group homomorphism, then it is determined by its value on ϕ alone, since $\Theta(\psi) = l\Theta(\phi)$ for ψ as above.

Let μ be a ϕ-invariant probability measure on X (which always exists [Wal82, Corollary 6.9.1]). By integrating the orbit cocycle against this measure we obtain a map $I_\mu \colon [\![\phi]\!] \to \mathbb{R}$ given by

$$I_\mu(\psi) = \int_X n_\psi \, \mathrm{d}\mu$$

for $\psi \in [\![\phi]\!]$. Observe that I_μ is a homomorphism, since $n_{\psi_1 \circ \psi_2} = n_{\psi_1} + n_{\psi_2}$. Moreover, we have $I_\mu(\phi) = 1$ as $n_\phi \equiv 1$. It follows from (3.3.2) that $I_\mu([\![\phi]\!]) = \mathbb{Z}$, so $I_\mu \colon [\![\phi]\!] \to \mathbb{Z}$ is a surjective group homomorphism. This is independent of which ϕ-invariant probability measure μ is chosen at the outset. In fact, I_μ is the only homorphism from $[\![\phi]\!]$ to $\mathbb{Z}$ up to scaling. In [GPS99], the integer $I_\mu(\psi)$ is called the *index* of ψ. See the paragraph following [GPS99, Remark 5.6] for a justification of this terminology, in terms of the Fredholm index of a certain Fredholm operator that can be constructed from ψ.

3.3.2 Comparing Matui's index map to that of Giordano, Putnam and Skau

Recall from Example 3.2.7 that $H_1\left(\mathcal{G}_\phi\right) \cong \mathbb{Z}$. To see that Matui's index map $I\colon \left[\!\left[\mathcal{G}_\phi\right]\!\right] \to H_1\left(\mathcal{G}_\phi\right)$ in the case of a transformation groupoid of a Cantor minimal system can be identified with $I_\mu\colon [\![\phi]\!] \to \mathbb{Z}$ we need to verify that I maps $\phi = \pi_{\{1\}\times X}$ to the element in $H_1\left(\mathcal{G}_\phi\right)$ that corresponds to $1 \in \mathbb{Z}$. The following explicit isomorphism between $H_1\left(\mathcal{G}_\phi\right)$ and $\mathbb{Z}$ was found in collaboration with Eduardo Scarparo.

Proposition 3.3.1. *Let (X, ϕ) be a Cantor minimal system and let μ be a ϕ-invariant probability measure on X. Then the map $\Delta\colon H_1\left(\mathcal{G}_\phi\right) \to \mathbb{Z}$ given by*

$$\Delta([f]) = \sum_{k\in\mathbb{Z}} k \int_X f(k,x)\,\mathrm{d}\mu(x)$$

for $f \in \ker(\delta_1)$ is an isomorphism.

Proof. Abusing notation somewhat, we begin by defining a group homomorphism $\Delta\colon C_c\left(\mathcal{G}_\phi, \mathbb{Z}\right) \to \mathbb{R}$ by $f \mapsto \sum_{k\in\mathbb{Z}} k \int_X f(k,x)\,\mathrm{d}\mu(x)$. Note that the outer sum is finite, since $f(k,x) = 0$ for all but finitely many values of k. Recall from Subsection 3.1.3 that $H_1\left(\mathcal{G}_\phi\right) = \ker(\delta_1)/\operatorname{im}(\delta_2)$ and that $\operatorname{im}(\delta_2)$ is spanned by elements of the form $1_U - 1_{UV} + 1_V$, where $U, V \subseteq \mathcal{G}_\phi$ are compact bisections satisfying $s(U) = r(V)$.

We claim that $\operatorname{im}(\delta_2) \subseteq \ker(\Delta)$. Without loss of generality we consider the basic bisections $U = \{k\} \times \phi^l(A)$ and $V = \{l\} \times A$, where $k, l \in \mathbb{Z}$ and $A \subseteq X$ is clopen. For these we have $\Delta(1_U) = k\mu\left(\phi^l(A)\right) = k\mu(A)$ (by ϕ-invariance of μ) and $\Delta(1_V) = l\mu(A)$. Moreover, we have that $UV = \{k+l\} \times A$ and $\Delta(1_{UV}) = (k+l)\mu(A) = \Delta(1_U) + \Delta(1_V)$. This proves the claim, making Δ a well-defined homomorphism from $C_c\left(\mathcal{G}_\phi, \mathbb{Z}\right)/\operatorname{im}(\delta_2)$ to $\mathbb{R}$.

Our next goal is to show that $\Delta([f]) \in \mathbb{Z}$ when $f \in \ker(\delta_1)$. Each function in $C_c\left(\mathcal{G}_\phi, \mathbb{Z}\right)$ is of the form

$$\sum_{j=-N}^{N} \sum_{i=1}^{M_j} m_{i,j} 1_{\{j\}\times A_{i,j}},$$

where $m_{i,j} \in \mathbb{Z}$ and $A_{i,j} \subseteq X$ is clopen. By Equations (3.1.5) and (3.1.6) we have

$$\left[1_{\{0\}\times A}\right] = 0 \quad \text{and} \quad \left[1_{\{-j\}\times A}\right] = \left[1_{\{j\}\times\phi^{-j}(A)}\right],$$

respectively, in $C_c\left(\mathcal{G}_\phi, \mathbb{Z}\right)/\operatorname{im}(\delta_2)$. Similarly, by Equation (3.1.4) we have

$$\left[1_{\{j\}\times A}\right] = \left[1_{\{1\}\times\phi^{j-1}(A)}\right] + \left[1_{\{j-1\}\times A}\right].$$

It follows that the equivalence classes $[f]$ with f of the form

$$f = \sum_{i=1}^{M} m_i 1_{\{1\}\times A_i}$$

exhausts $C_c\left(\mathcal{G}_\phi, \mathbb{Z}\right) / \operatorname{im}(\delta_2)$. For such an f we have $\Delta([f]) = \sum_{i=1}^{M} m_i \mu(A_i)$. Define $g := s_*(f) = \sum_{i=1}^{M} m_i 1_{A_i} \in C(X, \mathbb{Z})$ and observe that $\Delta([f]) = \int_X g \,\mathrm{d}\mu$. Assume now that $f \in \ker(\delta_1)$. This means that $g = s_*(f) = r_*(f) = \sum_{i=1}^{M} m_i 1_{\phi(A_i)}$. Then $g \circ \phi^{-1} = \sum_{i=1}^{M} m_i 1_{\phi(A_i)} = g$, which forces g to be constant (since ϕ, and hence ϕ^{-1}, is minimal). Since μ is a probability measure it readily follows that $\Delta([f]) \in \mathbb{Z}$. This makes $\Delta \colon H_1\left(\mathcal{G}_\phi\right) \to \mathbb{Z}$ well-defined.

Finally, we demonstrate that Δ is surjective as well as injective. For surjectivity it suffices to observe that $\Delta\left([1_{\{1\}\times X}]\right) = 1$. As for injectivity, suppose that $\Delta([f]) = 0$. Since $\{1\} \times A_i$ is the set on which f takes the value m_i we may assume that all the m_i's are distinct and that all the A_i's are disjoint. We have $\Delta([f]) = \int_X g \,\mathrm{d}\mu = 0$ with g constant. This forces $g = \sum_{i=1}^{M} m_i 1_{A_i} = 0$, but then we must have $f = 0$ too. This finishes the proof. □

Returning to the comparison between the two index maps I and I_μ, we have that $I(\phi) = I\left(\pi_{\{1\}\times X}\right) = \left[1_{\{1\}\times X}\right]$ and therefore $\Delta\left(I(\phi)\right) = 1$, as desired. It follows that the following diagram commutes:

$$\begin{tikzcd}
\llbracket \mathcal{G}_\phi \rrbracket \arrow[r, "I"] \arrow[d, equal] & H_1\left(\mathcal{G}_\phi\right) \arrow[d, "\Delta", "\cong"'] \\
\llbracket \phi \rrbracket \arrow[r, "I_\mu"] & \mathbb{Z}.
\end{tikzcd}$$

From this we see that Matui's index map is indeed a generalization.

3.4 Matui's HK and AH conjectures

In this section we describe two conjectures regarding homology of ample groupoids stated by Matui in [Mat16]. We will first state the conjectures and then describe the discoveries that inspired them.

The first conjecture relates the homology of a groupoid to the K-theory of its reduced groupoid C^*-algebra.

Matui's HK Conjecture ([Mat16, Conjecture 2.6]). *Let $\mathcal{G}$ be an effective minimal second countable Hausdorff ample groupoid whose unit space $\mathcal{G}^{(0)}$ is a Cantor space. Then there exists isomorphisms*

$$K_0\left(C_r^*(\mathcal{G})\right) \cong \bigoplus_{n=0}^{\infty} H_{2n}(\mathcal{G}) \quad \textit{and} \quad K_1\left(C_r^*(\mathcal{G})\right) \cong \bigoplus_{n=0}^{\infty} H_{2n+1}(\mathcal{G}).$$

The second conjecture relates the first two homology groups to (the abelianization of) the topological full group.

Matui's AH Conjecture ([Mat16, Conjecture 2.9])**.** *Let $\mathcal{G}$ be an effective minimal second countable Hausdorff ample groupoid whose unit space $\mathcal{G}^{(0)}$ is a Cantor space. Then the following sequence is exact:*

$$H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \xrightarrow{\ j\ } [\![\mathcal{G}]\!]_{ab} \xrightarrow{I_{ab}} H_1(\mathcal{G}) \longrightarrow 0. \tag{3.4.1}$$

In the sequence above, the map I_{ab} is the abelianization of the index map and the map j is described in Section 3.5.

Remark 3.4.1. We remark that there now exists counterexamples to the HK conjecture, see Example 3.4.2. The AH conjecture, however, is still open.

Let us describe some of the results that lead up to the conjectures, beginning with results relevant to the AH conjecture. In [Mat06], Matui essentially showed that

$$[\![\phi]\!]_{\mathrm{ab}} \cong \mathbb{Z} \oplus \left(K^0(X, \phi) \otimes \mathbb{Z}_2\right) \cong H_1\left(\mathcal{G}_\phi\right) \oplus \left(H_0\left(\mathcal{G}_\phi\right) \otimes \mathbb{Z}_2\right)$$

for any Cantor minimal system (X, ϕ). Similarly, it was shown that for any AF-groupoid (for which each orbit contains at least two points) one has

$$[\![\mathcal{G}_{(V,E)}]\!]_{\mathrm{ab}} \cong K_0(V, E) \otimes \mathbb{Z}_2 \cong H_1\left(\mathcal{G}_{(V,E)}\right) \oplus \left(H_0\left(\mathcal{G}_{(V,E)}\right) \otimes \mathbb{Z}_2\right).$$

Then in [Mat15b] it was shown that

$$[\![\mathcal{G}_A]\!]_{\mathrm{ab}} \cong H_1\left(\mathcal{G}_A\right) \oplus \left(H_0\left(\mathcal{G}_A\right) \otimes \mathbb{Z}_2\right)$$

for any minimal effective SFT-groupoid $\mathcal{G}_A$. These results can be interpreted as saying that the sequence (3.4.1) is short exact, i.e. that j is additionally injective. When this is the case, the groupoid is said to have the *strong AH property*. In [Mat16] it was shown that products of such SFT-groupoids also satisfy the AH conjecture, but that the map j may fail to be injective. It was also shown that almost finite groupoids (see [Mat12, Definition 6.2]) that are principal and minimal satisfy the AH conjecture. This class includes transformation groupoids of free minimal $\mathbb{Z}^n$ actions on Cantor spaces.

As for results relevant to the HK conjecture it was observed by Matui in [Mat12] that transformation groupoids of Cantor minimal systems, AF-groupoids and SFT-groupoids all satisfy the HK conjecture (recall Example 3.2.4, 3.2.5 and 3.2.7). In [Mat16], Matui observed that satisfying the HK conjecture is preserved under Kakutani equivalence. It is also clear that satisfying the HK conjecture is preserved under disjoint unions of groupoids. Matui proved a Künneth theorem for groupoid homology, which says that

$$H_n(\mathcal{G}\times\mathcal{H}) \cong \left(\bigoplus_{i+j=n} H_i(\mathcal{G})\otimes H_j(\mathcal{H})\right)\bigoplus\left(\bigoplus_{i+j=n-1}\operatorname{Tor}\left(H_i(\mathcal{G}),H_j(\mathcal{H})\right)\right).$$

As a consequence, if $\mathcal{G}$ and $\mathcal{H}$ are amenable (see e.g. [Sim17, Chapter 4]) ample groupoids with Cantor unit spaces which both satisfy the HK conjecture, then so does $\mathcal{G}\times\mathcal{H}$. This then applies to products of all three kinds of groupoids mentioned above.

We refer to Section B.1 and B.4 for further developments by other authors on Matui's conjectures. We also mention the recent preprint [PY20] that sheds new light on Matui's HK conjecture, and on groupoid homology in general. We end this section by briefly describing Scarparo's counterexample to the HK conjecture.

Example 3.4.2 ([Sca18])**.** There is a topologically free and minimal action by the infinite dihedral group D_∞ on a Cantor space X such that the homology of the associated transformation groupoid is related to the K-theory of the crossed product C^*-algebra as follows:

$$K_0\left(C(X)\rtimes D_\infty\right)\cong H_0\left(D_\infty\ltimes X\right)\oplus\mathbb{Z},\qquad K_1\left(C(X)\rtimes D_\infty\right)=0,$$
$$H_{2n}\left(D_\infty\ltimes X\right)=0 \text{ for } n\geq 1,\qquad H_{2n+1}\left(D_\infty\ltimes X\right)\cong\mathbb{Z}_2 \text{ for } n\geq 0.$$

It is still a possibility that a restricted version of the HK conjecture is true, namely if one restricts to principal groupoids (as opposed to effective ones). For example, for transformation groupoids of free and minimal actions by $\mathbb{Z}^n$ on Cantor spaces it is known that the isomorphisms in the HK conjecture hold modulo torsion (i.e. after tensoring with $\mathbb{Q}$), but it is unknown whether they hold on the nose [Mat12, page 31].

3.5 The map j in the AH conjecture

Let us next describe the second map in the AH conjecture, namely the map

$$j\colon H_0(\mathcal{G})\otimes\mathbb{Z}_2\to[\![\mathcal{G}]\!]_{\mathrm{ab}}.$$

To be clear, by $\mathbb{Z}_2$ we mean the group with two elements and we write $\mathbb{Z}_2=\{\overline{0},\overline{1}\}$. For $f\in C(\mathcal{G}^{(0)},\mathbb{Z})$ we define $O_f := \left\{x\in\mathcal{G}^{(0)}\mid f(x)\text{ is odd}\right\}$. Then we have $f-1_{O_f}\in 2C(\mathcal{G}^{(0)},\mathbb{Z})$. We are going to freely identify $H_0(\mathcal{G})\otimes\mathbb{Z}_2$ with $H_0(\mathcal{G},\mathbb{Z}_2)$ via $[f]\otimes\overline{1}\longmapsto\left[1_{O_f}\right]$ (recall that $H_0(\mathcal{G})=H_0(\mathcal{G},\mathbb{Z})$). Note in particular that each element in $H_0(\mathcal{G},\mathbb{Z}_2)$ is of the form $[1_A]$ for some clopen set $A\subseteq\mathcal{G}^{(0)}$.

Given a compact bisection $U \subseteq \mathcal{G}$ with $s(U) \cap r(U) = \emptyset$ we obtain a transposition $\pi_{\widehat{U}} \in [\![\mathcal{G}]\!]$ (as in Section 2.4). If $\mathcal{G}$ satisfies the assumptions in the AH conjecture and U is as above, then the assignment

$$j\left([1_{s(U)}]\right) := \left[\pi_{\widehat{U}}\right]_{\mathrm{ab}} \in [\![\mathcal{G}]\!]\,\mathrm{ab} \tag{3.5.1}$$

induces a well-defined homomorphism $j\colon H_0(\mathcal{G}, \mathbb{Z}_2) \to [\![\mathcal{G}]\!]_{\mathrm{ab}}$. See [Nek19, Section 7] for a proof of this for effective ample Hausdorff groupoids $\mathcal{G}$ for which every $\mathcal{G}$-orbit has at least 3 elements and where $\mathcal{G}^{(0)}$ is a Cantor space. Alternatively, see the proofs of Theorem 3.6 and 4.4 in [Mat16] for the almost finite case and how that same proof can be adapted to purely infinite groupoids (in the sense of [Mat15b, Definition 4.9]), respectively. It follows readily from Equation (3.3.1) that

$$I_{\mathrm{ab}} \circ j = 0.$$

As long as every $\mathcal{G}$-orbit has at least 2 elements, one can easily show that the elements $[1_{s(U)}]$ as above generate $H_0(\mathcal{G}, \mathbb{Z}_2)$ (using Lemma A.3.9). Furthermore, if $\mathcal{G}$ is purely infinite and minimal, then these elements even exhaust $H_0(\mathcal{G})$, that is

$$H_0(\mathcal{G}) = \left\{[1_{s(U)}] \mid U \subseteq \mathcal{G} \text{ compact bisection with } s(U) \cap r(U) = \emptyset\right\}.$$

Let us briefly explain why. Lemma 5.3 in [Mat15b] shows that every element in $H_0(\mathcal{G})$ is of the form $[1_A]$ for some clopen set $A \subseteq \mathcal{G}^{(0)}$. Now pick any compact bisection V with $s(V) \cap r(V) = \emptyset$ (which exists by e.g. A.3.9). By [Mat15b, Proposition 4.11] there is some bisection W with $s(W) = A$ and $r(W) \subseteq s(V)$. Set $U := (s|_V)^{-1}(r(W)) \subseteq V$. Then U is a compact bisection with $s(U) \cap r(U) = \emptyset$ and $[1_A] = [1_{r(W)}] = [1_{s(U)}]$.

Chapter 4

Summary of papers

4.1 Paper A: Topological Full Groups of Ample Groupoids with Applications to Graph Algebras

The first paper revolves around Matui's Isomorphism Theorem (Theorem 2.6.6), or more precisely, around generalizing it. This theorem requires the ample groupoids to be minimal, second countable and have a compact unit space.

In the first part of the paper we provide two generalizations of Matui's Isomorphism Theorem; one that relaxes the minimality assumption and another that relaxes the second countability assumption. Both relax the compactness assumption on the unit space. In the process we have to extend the definition of the topological full group to ample groupoids with non-compact unit spaces.

In the second part of the paper we specialize to graph groupoids. This is a class of ample groupoids that are built from directed graphs, and they generalize SFT-groupoids in the sense that any SFT-groupoid can be viewed as the graph groupoid of a finite graph. By interpreting the general isomorphism theorems from the first part of the paper for the class of graph groupoids we obtain generalizations of Matsumoto's Theorem 2.6.3. In fact, we are able to sharpen one of the isomorphism theorems further for graph groupoids by analyzing its proof separately.

We also obtain a novel embedding result for ample groupoids. We show that a family of ample groupoids, which include graph groupoids and AF-groupoids, all embed into the fixed SFT-groupoid $\mathcal{G}_{A_{2,1}}$, where

$$A_{2,1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

We remark that the groupoid C^*-algebra of $\mathcal{G}_{A_{2,1}}$ is (isomorphic to) the Cuntz algebra $\mathcal{O}_2$. This embedding on the level of the groupoids induces diagonal preserving embeddings of the associated graph C^*-algebras and Leavitt path algebras, which relate to (a special case of) Kirchberg's Embedding Theorem for C^*-algebras and an embedding theorem of Brownlowe and Sørensen for Leavitt path algebras. Another consequence of our embedding result is that all these topological full groups have the Haagerup property.

Finally, we remark that the results in this paper have been utilized in the recent preprint [dCGvW19] to obtain similar isomorphism theorems for ultragraph groupoids.

4.2 Paper B: Matui's AH Conjecture for Graph Groupoids

In the second paper, we continue to study the topological full groups of graph groupoids, but we now shift focus towards Matui's AH conjecture. Since Matui has established the AH conjecture for minimal SFT-groupoids (as mentioned in Section 3.4), the novelty lies in dealing with infinite graphs. In particular, dealing with infinite emitters (i.e. vertices that emit infinitely many edges).

By combining existing results in the literature, we observe that the identification between the homology groups of a graph groupoid with the K-groups of the associated graph C^*-algebra—as in Example 3.2.5—is valid for all (countable) graphs. This means that the AH conjecture relates the topological full group of a graph groupoid with the K-theory of its graph C^*-algebra. Moreover, it follows that the homology groups are easy to compute.

As our main result, we prove that the AH conjecture holds for all graph groupoids that satisfy the assumptions of the conjecture (which we also characterize precisely in terms of the underlying graphs). In fact, the AH conjecture is shown to hold for any ample groupoid that is Kakutani equivalent to such a graph groupoid.

We are not able to decide whether graph groupoids of infinite graphs have the strong AH property or not (recall that SFT-groupoids do). However, we give a partial description of the abelianization of the topological full group and we show that if a graph has an infinite emitter, then the strong AH property rules out simplicity of the topological full group. Furthermore, we show that the topological full group is not finitely generated.

In the proof of the main result we need the fact that general AF-groupoids have cancellation (see [Mat16, Definition 2.11]). We provide a proof of this fact, which may be of independent interest. We also remark that the description of the topological full group from Paper A is used in this paper too, since the skew product groupoid, which appear frequently, has non-compact unit space.

4.3 Paper C: Katsura–Exel–Pardo Groupoids and the AH Conjecture

The third paper also concerns Matui's AH conjecture. Here we consider a different class of groupoids, namely Katsura–Exel–Pardo groupoids. This is a class of ample groupoids that arise from certain self-similar actions by $\mathbb{Z}$ on finite graphs. In the case that the action is trivial, the groupoid is an SFT-groupoid. As our main result, we establish that the AH conjecture is true for (almost) all Katsura–Exel–Pardo groupoids that satisfy the assumptions of the AH conjecture (we additionally assume that the underlying graph has no sources).

We also investigate whether the topological full group of a Katsura–Exel–Pardo groupoid is finitely generated. By extending Nekrashevych's notion of contractivity for self-similar groups to self-similar graphs, we are able to give a sufficient condition for the topological full group to be finitely generated.

Part II

Research Papers

Paper A

Topological Full Groups of Ample Groupoids with Applications to Graph Algebras

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Paper A

Topological Full Groups of Ample Groupoids with Applications to Graph Algebras

Abstract

We study the topological full group of ample groupoids over locally compact spaces. We extend Matui's definition of the topological full group from the compact, to the locally compact case. We provide two general classes of étale groupoids for which the topological full group, as an abstract group, is a complete isomorphism invariant. Hereby extending Matui's Isomorphism Theorem. As an application, we study graph groupoids and their topological full groups, and obtain sharper results for this class. The machinery developed in this process is used to prove an embedding theorem for ample groupoids, akin to Kirchberg's Embedding Theorem for C^*-algebras. Consequences for graph C^*-algebras and Leavitt path algebras are also spelled out. In particular, we improve on a recent embedding theorem of Brownlowe and Sørensen for Leavitt path algebras.

A.1 Introduction

A.1.1 Background

The study of (topological) full groups in the setting of topological dynamics was initiated by Giordano, Putnam and Skau [GPS99]. This was inspired by the work of Dye [Dye63] in the measurable setting, and by Krieger's study of so-called ample groups on the Cantor space [Kri80]. For Cantor minimal systems, Giordano, Putnam and Skau showed that certain distinguished subgroups of the full group determine completely the orbit equivalence class, the strong orbit equivalence class, and the flip conjugacy class, respectively, of the system. The *full group* of a Cantor system (i.e. a $\mathbb{Z}$-action on a Cantor space) consists of all homeomorphisms of the Cantor space which leave the orbits invariant. Roughly speaking, the *topological full group* is the subgroup of the full group consisting of those homeomorphisms which additionally preserve the orbits in a continuous manner. Giordano, Putnam and Skau also connected the dynamics with the theory of C^*-algebras, via the crossed product construction and its K-theory [GPS95]. Thus, they exhibited a strong relationship between these, a priori, quite different mathematical structures.

This is but one example of the rich interplay between the theory of dynamical systems and C^*-algebras. (This interplay essentially goes all the way back to the inception of the field by Murray and von Neumann [MvN43].) Another prominent example of this interplay is the connection between shifts of finite type and Cuntz-Krieger algebras; discovered by Cuntz and Krieger in the early eighties [CK80]. In the setting of irreducible one-sided shifts of finite type, Matsumoto defined the topological full group of such a dynamical system and proved that this group determines the shift up to continuous orbit equivalence, and also the associated Cuntz-Krieger algebra up to diagonal preserving isomorphism [Mat10], [Mat15a]. This parallelled Giordano, Putnam and Skau's results, although the dynamical systems were quite different. For instance, the former has no periodic points whereas the latter has a dense set of periodic points.

Using topological groupoids to model dynamical systems has unified many of these seemingly different connections between dynamics and C^*-algebras. Whenever one has a dynamical system of some sort, one may typically associate to it a topological groupoid, and from the groupoid one can construct its groupoid C^*-algebra. In many cases, isomorphism of such groupoids correspond to some suitable notion of continuous orbit equivalence of the dynamical systems, and also to diagonal preserving isomorphism of the groupoid C^*-algebras [MM14], [BCW17], [Li17], [Li18]. That groupoid isomorphism corresponds to diagonal preserving isomorphism of the C^*-algebras (in the topologically principal case) is due to the pioneering work of Renault [Ren08]. This reconstruction result has recently been generalized in e.g. [CRST17]; wherein it is also shown that by adding more structure on the groupoids, such as gradings, one can recover stronger types of equivalence of the dynamical systems.

In [Mat12], Matui defined the topological full group of an étale groupoid with compact unit space. His definition generalized virtually all the previously given definitions for different kinds of dynamical systems at one fell swoop. Matui realized that homeomorphisms which preserve orbits in a continuous manner are always given by *full bisections* from the associated groupoid. In the subsequent paper [Mat15b], Matui proved (among other things) a remarkable isomorphism theorem. Supressing some assumptions, this theorem says that any two minimal étale groupoids over a Cantor space are isomorphic, as topological groupoids, if and only if their topological full groups are isomorphic, as abstract groups.[1] Matui's Isomorphism Theorem generalized the results of Giordano, Putnam and Skau, and Matsumoto, and others.

The study of topological full groups has also found interesting applications to group theory. Matui's isomorphism theorem means that one can classify the groupoids (and therefore any underlying dynamics, and the C^*-algebras) in terms of the topological full group. However, by going the other direction, one can use étale groupoids to distinguish certain discrete groups. Given two discrete groups, say in terms of their generators and relations, it can be hard to tell whether they are isomorphic or not. But if one can realize these groups as topological full groups (or distinguished subgroups) of some groupoids, then one can use the groupoids (i.e. the dynamics) to tell the groups apart—as one often has much dynamical information about the groupoids. For instance, this was the strategy used by Brin to show that Thompson's group V is not isomorphic to its two-dimensional analog $2V$ [Bri04] (although he did not consider the groupoid explicitly). A more recent application of this form is by Matte Bon [MB18] who showed that the higher dimensional Thompson group nV embeds into mV if and only if $n \leq m$.[2] Matte Bon's paper also includes a novel approach to Matui's Isomorphism Theorem in terms of a certain dichotomy for such groupoids. Another application is that topological full groups have provided new examples of groups with exotic properties. Most notably, topological full groups (or more precisely, their commutator subgroups) of Cantor minimal systems provided the first examples of finitely generated simple groups that are amenable (and infinite) [JM13]. On another note, topological full groups arising from non-amenable groups acting minimally and topologically free on the Cantor space were recently shown to be C^*-simple [BS19].

Topological full groups have also found their way into Lawson's program of non-commutative Stone duality [Law10]. In [Law17], the topological full group of an étale groupoid is shown to coincide with the group of units of the so-called Tarski monoid to which the groupoid corresponds under non-commutative Stone duality.

A.1.2 Our results

The main motivation for the present paper was Matsumoto and Matui's work on irreducible one-sided shifts of finite type mentioned above. If we rephrase their work in terms of (directed) graphs, then they showed that for two strongly connected

[1] Actually, the same is true for several distinguished subgroups of the topological full group as well, such as its commutator subgroup. See [Mat15b] and [Nek19] for details.

[2] It is known that the groups nV are all non-isomorphic [BL10].

finite graphs E and F the following are equivalent:

1. The shifts (E^∞, σ_E) and (F^∞, σ_F) are continuously orbit equivalent.

2. The graph groupoids $\mathcal{G}_E$ and $\mathcal{G}_F$ are isomorphic as topological groupoids.

3. There is an isomorphism of the graph C^*-algebras $C^*(E)$ and $C^*(F)$ which maps the diagonal $\mathcal{D}(E)$ onto $\mathcal{D}(F)$.

4. The topological full groups $[\![\mathcal{G}_E]\!]$ and $[\![\mathcal{G}_F]\!]$ are isomorphic as abstract groups.

The equivalence of (1), (2) and (3) above have since been generalized to more general graphs which need neither be finite nor strongly connected [CEOR19], [BCW17]. Our initial goal was to study the topological full group $[\![\mathcal{G}_E]\!]$ of general graph groupoids $\mathcal{G}_E$ and see if we could also add statement (4) to said equivalence.

Matui's Isomorphism Theorem [Mat15b, Theorem 3.10] gives the equivalence of (2) and (4) above for the general class of ample effective Hausdorff minimal second countable groupoids over (compact) Cantor spaces (see Subsection A.2.3 for definitions). This covers in particular graph groupoids of strongly connected finite graphs. In light of this we attempted to extend Matui's Isomorphism Theorem a little further in order to cover graph groupoids of more general graphs. To do this it is necessary to relax both the compactness assumption of the unit space (which corresponds to the graph having finitely many vertices) and the minimality assumption (which corresponds to strong connectedness of the graph).

As our main findings we first describe two modest extensions of Matui's Isomorphism Theorem that apply to general ample groupoids. Then we describe two (sharper) isomorphism theorems for the class of graph groupoids. Finally, we present a novel embedding theorem for ample groupoids. First of all we have to extend the definition of the topological full group to the locally compact setting. This is done in Definition A.3.2, where we stipulate that the homeomorphisms in the topological full group should be compactly supported (in addition to being induced by bisections). This seems a natural choice, as we then retain the "finitary" nature of the elements in the topological full group, as well as the countability of the topological full group (for second countable groupoids). Additionally, most of the arguments from [Mat15b] still work with suitable modifications. For an ample groupoid $\mathcal{G}$ we denote its unit space by $\mathcal{G}^{(0)}$. The topological full group of $\mathcal{G}$ is denoted by $[\![\mathcal{G}]\!]$. And the commutator subgroup of $[\![\mathcal{G}]\!]$ is denoted by $\mathsf{D}([\![\mathcal{G}]\!])$. The first of these isomorphism theorems is a straightforward extension of Matui's Isomorphism Theorem which relaxes the compactness assumption on $\mathcal{G}^{(0)}$ and the second countability assumption on $\mathcal{G}$.

Theorem A.1.1 (see Theorem A.7.2, [Mat15b, Theorem 3.10])**.** *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are effective ample minimal Hausdorff groupoids whose unit spaces have no isolated points. Then following are equivalent:*

1. *$\mathcal{G}_1 \cong \mathcal{G}_2$ as topological groupoids.*
2. *$[\![\mathcal{G}_1]\!] \cong [\![\mathcal{G}_2]\!]$ as abstract groups.*
3. *$\mathsf{D}([\![\mathcal{G}_1]\!]) \cong \mathsf{D}([\![\mathcal{G}_2]\!])$ as abstract groups.*

We mention that when restricting to the class of graph groupoids we are also able to relax the minimality assumption in Theorem A.1.1 substantially (see Theorem A.1.3 below). The second isomorphism theorem replaces the minimality assumption with a significantly weaker "mixing property" that we call *non-wandering* (see Definition A.7.8). However, the result does not apply to the commutator subgroups. And we also require the unit spaces to be second countable. (By a *locally compact Cantor space* we mean either the *compact* Cantor space or the locally compact *non-compact* Cantor space (up to homeomorphism), see Subsection A.2.1.)

Theorem A.1.2 (see Theorem A.7.10)**.** *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be effective ample Hausdorff groupoids over locally compact Cantor spaces. If, for $i = 1, 2$, $\mathcal{G}_i$ is non-wandering and each $\mathcal{G}_i$-orbit has length at least 3, then the following are equivalent:*

1. *$\mathcal{G}_1 \cong \mathcal{G}_2$ as topological groupoids.*
2. *$[\![\mathcal{G}_1]\!] \cong [\![\mathcal{G}_2]\!]$ as abstract groups.*

Let us say a few words about the proofs. As the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) in Theorem A.1.1 and (1) $\Rightarrow$ (2) in Theorem A.1.2 are trivial, there is only one direction to prove. The proof strategy is similar in both cases and is summarized in Figure A.1. The first step is showing that for certain classes of homeomorphism groups, any (abstract) group isomorphism is induced by a homeomorphism of the underlying spaces.[3] We call this a *spatial realization result*. In [Mat15b], Matui proves a spatial realization result that applies to any Γ with $\mathsf{D}([\![\mathcal{G}]\!]) \leq \Gamma \leq [\![\mathcal{G}]\!]$ (for minimal $\mathcal{G}$). And from a spatial isomorphism he directly constructs an isomorphism of the groupoids and obtains his Isomorphism Theorem. In this paper we have chosen to break this direct step into two more parts in order to also study when the groupoid can be recovered from the action of (subgroups of) the topological full group on the unit space, as the groupoid of germs of this action. We find that such a groupoid of germs always embed into the groupoid we started with, and that

[3]If $\Gamma \leq \mathrm{Homeo}(X)$ and $\Lambda \leq \mathrm{Homeo}(Y)$ are groups of homeomorphisms, then a *spatial isomorphism* between them is a homeomorphism $\phi\colon X \to Y$ such that $\gamma \mapsto \phi \circ \gamma \circ \phi^{-1}$ for $\gamma \in \Gamma$ is a group isomorphism.

$$\begin{array}{cc}
\begin{array}{c}
\Gamma_1 \cong \Gamma_2 \\
\Big\Downarrow \\
\left(\Gamma_1, \mathcal{G}_1^{(0)}\right) \cong \left(\Gamma_2, \mathcal{G}_2^{(0)}\right) \\
\Big\Downarrow \text{Functoriality} \\
\mathrm{Germ}\left(\Gamma_1, \mathcal{G}_1^{(0)}\right) \cong \mathrm{Germ}\left(\Gamma_2, \mathcal{G}_2^{(0)}\right) \\
\Big\Downarrow \Gamma_i \text{ covers } \mathcal{G}_i \\
\mathcal{G}_1 \cong \mathrm{Germ}\left(\Gamma_1, \mathcal{G}_1^{(0)}\right) \cong \mathrm{Germ}\left(\Gamma_2, \mathcal{G}_2^{(0)}\right) \cong \mathcal{G}_2
\end{array}
&
\begin{array}{l}
\text{(abstract isomorphism)} \\
\\
\text{(spatial isomorphism)}
\end{array}
\end{array}$$

Figure A.1: Proof strategy for Theorem A.1.1 and A.1.2. Here Γ_i is a subgroup of $\mathrm{Homeo}\left(\mathcal{G}_i^{(0)}\right)$.

they are isomorphic if and only if the subgroup in question is generated by enough bisections to cover the groupoid (Proposition A.4.10, Corollary A.4.13). We also show that for a natural choice of maps, the assignment of the groupoid of germs is functorial (Proposition A.5.4).

Having this machinery in place, proving Theorem A.1.1 is then just a matter of checking that Matui's spatial realization result also holds in the locally compact setting (Theorem A.6.6). Although this is but a small extension of Matui's result we have chosen to include it as a theorem since it is applicable to a larger class of groupoids. Regarding our initial motivation, namely the graph groupoids, we are able to characterize exactly when the aforementioned spatial realization result applies, and it turns out that we can get away with much weaker mixing properties than minimality when we restrict to graph groupoids—see Theorem A.1.3 below.

For the proof of Theorem A.1.2 we employ a spatial realization result (Theorem A.6.19) based on Rubin's work in [Rub89] in the first step. We mention that Medynets has previously obtained a similar spatial realization result [Med11, Remark 3] for (topological) full groups arising from group actions on the Cantor space, building on Fremlins work in [Fre04, Section 384]. After some modifications, Theorem A.1.2 could also be deduced from this result. However, Theorem A.6.19 is more general as it can potentially be applied to other groups than topological full groups, e.g. homeomorphism groups of zero-dimensional linearly ordered spaces. See Remark A.6.20 for a more detailed discussion on the differences and similarities of these approaches. Although Theorems A.1.1 and A.1.2 can be deduced by employing arguments along the lines of [Mat15b] and [Med11], we believe that the way we trisect the proofs does add some new insight. In particular, this was how we discovered the embedding result given below in Theorem A.1.5.

Let us now describe the isomorphism theorem we obtain for graph groupoids, when starting with the spatial reconstruction result à la Matui. As mentioned above, it turns out that we can replace minimality (strong connectedness of the graphs) with some weaker "exit and return"-conditions. Each of these three conditions (see Definition A.10.1) can be considered strengthenings of the three conditions that characterize when the boundary path space ∂E has no isolated points (Proposition A.8.1). Condition (K) means that every cycle can be exited, and then returned to. Condition (W) means that every wandering path can be exited, and then returned to. And Condition (∞) means that every singular vertex can be exited (i.e. is an infinite emitter), and then returned to (along infinitely many of the emitted edges).

Theorem A.1.3 (see Theorem A.10.10)**.** *Let E and F be graphs with no sinks, and suppose they both satisfy Condition (K), (W) and (∞). Then the following are equivalent:*

1. *$\mathcal{G}_E \cong \mathcal{G}_F$ as topological groupoids.*
2. *$[\![\mathcal{G}_E]\!] \cong [\![\mathcal{G}_F]\!]$ as abstract groups.*
3. *$\mathsf{D}([\![\mathcal{G}_E]\!]) \cong \mathsf{D}([\![\mathcal{G}_F]\!])$ as abstract groups.*

By interpreting the assumptions in Theorem A.1.2 for graph groupoids we obtain Theorem A.1.4 below. Therein, Condition (L) is the well known exit condition of Kumjian, Pask and Raeburn [KPR98], namely, that every cycle should have an exit. Condition (T) (see Definition A.10.5) essentially means that the graph does not have a component which is a tree. Finally, what we call *degenerate vertices* (see Definition A.10.6) are the ones giving $\mathcal{G}_E$-orbits of length 1 or 2. This theorem may be considered a generalization of Matsumoto's result in the case of irreducible one-sided shifts of finite type [Mat15a] (which correspond to finite strongly connected graphs).

Theorem A.1.4 (see Theorem A.10.11)**.** *Let E and F be countable graphs satisfying Condition (L) and (T), and having no degenerate vertices. Then the following are equivalent:*

1. *$\mathcal{G}_E \cong \mathcal{G}_F$ as topological groupoids.*
2. *$[\![\mathcal{G}_E]\!] \cong [\![\mathcal{G}_F]\!]$ as abstract groups.*

Hence we establish the equivalence of (1)–(4) mentioned in the beginning of this subsection for graphs satisfying the assumptions of Theorem A.1.4. In Corollary A.10.13, we spell out this rigidity result for the associated graph algebras.

Our final main result is an embedding theorem for ample groupoids—inspired by embedding theorems for C^*-algebras and Leavitt path algebras. The seminal embedding theorem of Kirchberg [KP00] states that any separable exact (unital) C^*-algebra embeds (unitally) into the Cuntz algebra $\mathcal{O}_2$. In particular, this means that any graph C^*-algebra $C^*(E)$, where E is a countable graph, embeds into $\mathcal{O}_2$. The latter, being the universal C^*-algebra generated by two orthogonal isometries, can be canonically identified with a graph C^*-algebra. Namely, the graph C^*-algebra of the graph E_2 which consists of a single vertex with two loops. In [BS16], Brownlowe and Sørensen show that the Leavitt path algebra $L_R(E)$, where E is any countable graph and R any commutative unital ring, embeds into $L_R(E_2)$—the algebraic analog of $\mathcal{O}_2$. An inspection of their proof reveals that this embedding also maps the canonical diagonal subalgebra $D_R(E)$ into $D_R(E_2)$. As a consequence, Kirchberg's embedding for the graph C^*-algebras may then also be taken to be diagonal preserving—with respect to the diagonal[4] in $\mathcal{O}_2$ coming from its identification with $C^*(E_2)$. At this point, it starts smelling a bit like groupoids might be lurking about. Indeed, using the properties of the *Germ-functor* (see Section A.5), we are able to prove that the underlying graph groupoid $\mathcal{G}_E$ embeds into the Cuntz groupoid $\mathcal{G}_{E_2}$ (modulo topological obstructions in the sense of isolated points). Thus, the known embeddings of the graph algebras actually occur at the level of the underlying groupoid models. We were also able to extend this embedding result to all groupoids which are groupoid equivalent (or stably isomorphic) to a graph groupoid. To the best of the authors' knowledge, this is the first embedding result of its kind for ample groupoids.

Theorem A.1.5 (see Theorem A.11.16)**.** *Let $\mathcal{H}$ be an effective ample second countable Hausdorff groupoid with $\mathcal{H}^{(0)}$ a locally compact Cantor space. If $\mathcal{H}$ is groupoid equivalent to $\mathcal{G}_E$, for some countable graph E satisfying Condition (L) and having no sinks nor semi-tails, then $\mathcal{H}$ embeds into $\mathcal{G}_{E_2}$. Moreover, if $\mathcal{H}^{(0)}$ is compact, then the embedding maps $\mathcal{H}^{(0)}$ onto E_2^∞.*

In particular, any graph groupoid $\mathcal{G}_E$, with E as above, embeds into $\mathcal{G}_{E_2}$, and any AF-groupoid (with perfect unit space) embeds into $\mathcal{G}_{E_2}$.

The main ingredient in the proof is constructing an injective local homeomorphism $\phi\colon \partial E \to E_2^\infty$ which induces a spatial embedding of the associated topological full groups. This construction is entirely explicit. As a consequence we also obtain explicit embeddings of any graph C^*-algebra $C^*(E)$ (or Leavitt path algebra $L_R(E)$), in terms of their canonical generators, into $\mathcal{O}_2$ (or $L_R(E_2)$. This embedding is *diagonal preserving*, and when $C^*(E)$ is unital (i.e. E^0 is finite) this embedding is unital and maps the diagonal *onto* the diagonal. These embeddings

[4]Technically, this is a Cartan subalgebra in the sense of Renault, not a C^*-diagonal in the sense of Kumjian. But it is common to refer to it as "the diagonal" in a graph C^*-algebra.

are described in Corollary A.11.5 and Remark A.11.6. We also record a result on diagonal embeddings of AF-algebras in Corollary A.11.27.

Another consequence of Theorem A.1.5 is that each topological full group $[\![\mathcal{G}_E]\!]$, for E as above, embeds into Thompson's group V—since V is isomorphic to $[\![\mathcal{G}_{E_2}]\!]$. The Higman-Thompson groups $V_{n,r}$ (where $nV = V_{n,1}$) can be realized as topological full groups of graph groupoids of certain strongly connected finite graphs (see Subsection 11.3). Hence, our embedding theorem may be considered a generalization of the well known embedding of $V_{n,r}$ into V. The embedding entails that the topological full groups $[\![\mathcal{H}]\!]$, of groupoids $\mathcal{H}$ as in Theorem A.1.5, has the Haagerup property (but they are generally not amenable). In terms of groups, our embedding also includes all the so-called *LDA-groups* (see Remark A.11.24).

In [Mat16], Matui introduced two conjectures for minimal ample groupoids over the Cantor space. The *HK-conjecture* relates the groupoid homology to the K-theory of the groupoid C^*-algebra. And the *AH-conjecture* relates the topological full group to the groupoid homology. These conjectures have been verified in several cases [Mat17], in particular for (products of) graph groupoids arising from strongly connected finite graphs. For the more general graph groupoids studied in the present paper, the second named author will, together with Toke Meier Carlsen, attack these conjectures in a forthcoming paper. (In the recent preprint [Ort18], the second named author verifies the HK-conjecture for a class of groupoids which includes the graph groupoids of row-finite graphs.)

A.1.3 Précis

The structure of the paper is as follows. We recall some basic notions regarding étale groupoids and (classical) Stone duality in Section A.2. This section also serves the purpose of establishing notation and conventions. The rest of the paper is divided into two parts. The first, sections A.3–A.7, deals with ample groupoids in general, while the second, sections A.8–A.11, deals with graph groupoids.

In Section A.3 we give the definition of the topological full group $[\![\mathcal{G}]\!]$ of an ample groupoid $\mathcal{G}$ with locally compact unit space $\mathcal{G}^{(0)}$. We also prove some elementary results on the existence of elements in the topological full group with certain properties. Then we move on to study the groupoid of germs $\mathrm{Germ}\left(\Gamma, \mathcal{G}^{(0)}\right)$ associated to a subgroup $\Gamma \leq [\![\mathcal{G}]\!]$ of the topological full group, in Section A.4. We establish that $\mathrm{Germ}\left(\Gamma, \mathcal{G}^{(0)}\right)$ always embeds into $\mathcal{G}$, and that this embedding is an isomorphism as long as Γ contains "enough elements". In Section A.5 we introduce two categories; **SpatG** and **Gpoid**. The former consists of pairs (Γ, X) where X is a space and Γ is a subgroup of $\mathrm{Homeo}(X)$. The latter consists of certain ample groupoids. By defining suitable morphisms in these categories and what the germ of a morphism in **SpatG** should be, we establish that the assigment $(\Gamma, X) \mapsto \mathrm{Germ}(\Gamma, X)$ is functorial. We also show that monomorphisms in **SpatG** induce étale embeddings of the associated groupoids of germs.

The spatial realization results needed to deduce that an abstract isomorphism of two topological full groups always is spatially implement are provided in Section A.6. In Section A.7 we prove the two general isomorphism theorems, Theorem A.1.1 and Theorem A.1.2. This is now mostly a matter of interpreting the spatial realization results from Section A.6 in terms of the groupoid and its topological full group, and then combine this with the results of Section A.4 and Section A.5.

In Section A.8 we begin our in-depth study of graph groupoids $\mathcal{G}_E$ of general graphs E. This section is devoted to a thorough introduction of graph terminology and the dynamics that give rise to the graph groupoids. For several of the generic properties a topological groupoid can have, we list their characterizations for graph groupoids in terms of the graphs. We continue in Section A.9 with describing explicitly all elements in the topological full group $[\![\mathcal{G}_E]\!]$ of any graph groupoid. To do this we need to specify a new (yet equivalent) basis for the topology on $\mathcal{G}_E$. We then pursue specialized isomorphism theorems for the class of graph groupoids in Section A.10. This yields Theorem A.1.3 and Theorem A.1.4. At the end of this section we spell out the induced rigidity result for the associated graph algebras.

In the final section of the paper we employ the machinery from Sections A.4, A.5 and A.9 to obtain our groupoid embedding result; Theorem A.1.5. We also describe the explicit diagonal embeddings of the graph algebras that follow from the embedding of the groupoids. Examples of these embeddings for graph algebras are provided for several infinite graphs. At the end of Section A.11 we show that any AF-groupoid is groupoid equivalent to a graph groupoid, going via Bratteli diagrams, hence $\mathcal{G}_{E_2}$-embeddable. We then spell out consequences for diagonal embeddings of AF-algebras. Additionally, we remark that transformation groupoids arising from locally compact (non-compact) Cantor minimal systems are AF-groupoids, and hence $\mathcal{G}_{E_2}$-embeddable as well.

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A.2 Preliminaries

We will now recall the basic notions needed throughout the paper, as well as establish notation and conventions. We denote the positive integers by $\mathbb{N}$ and the non-negative integers by $\mathbb{N}_0$. If two sets A and B are disjoint we will denote their union by $A \sqcup B$ if we wish to emphasize that they are disjoint. When we write that $C = A \sqcup B$ we mean that $C = A \cup B$ *and* that A and B are disjoint sets.

A.2.1 Topological notions

Following [KL16], [Ste19] we say that a topological space is *Boolean* if it is Hausdorff and has a basis of compact open sets. (This is also the terminology orginally used by Stone [Sto37].) A *Stone space* is then a compact Boolean space. We say that a topological space is *perfect* if it has no isolated points. By a *locally compact Cantor space* we mean a (non-empty) second countable perfect Boolean space. Up to homeomorphism there are two such spaces; one compact (the Cantor set) and one non-compact (the Cantor set with a point removed). The latter may also be realized as any non-closed open subset of the Cantor set, or as the product of the Cantor set and a countably infinite discrete space.

For a topological space X we denote the group of self-homeomorphisms of X by $\operatorname{Homeo}(X)$. We will occasionally denote id_X simply by 1 for brevity. By an *involution* we mean a homeomorphism (or more generally, a group element) ϕ with $\phi^2 = 1$. For a homeomorphism $\phi \in \operatorname{Homeo}(X)$, we define the *support of* ϕ to be the (regular) closed set $\overline{\{x \in X \mid \phi(x) \neq x\}}$, and denote it by $\operatorname{supp}(\phi)$. We also define

$$\operatorname{Homeo}_c(X) := \{\phi \in \operatorname{Homeo}(X) \mid \operatorname{supp}(\phi) \text{ compact open}\}.$$

When Γ is a subgroup of a group Γ' we write $\Gamma \leq \Gamma'$. Beware that we will abuse this notation when we write $\Gamma \leq \operatorname{Homeo}_c(X)$ to mean that Γ is a subgroup of $\operatorname{Homeo}(X)$ *and* that $\Gamma \subseteq \operatorname{Homeo}_c(X)$. (It is not clear whether $\operatorname{Homeo}_c(X)$ itself is a group.)

A.2.2 Stone duality

We will now briefly recall the basics of (classical) Stone duality needed for Section A.6. For more details the reader may consult [Kop89], [Fre04, Chapter 31] (or even the fountainhead [Sto37], [Doc64]). By a *Boolean algebra* we mean a complemented distributive lattice with a top and bottom element. And by a *generalized Boolean algebra* we mean a relatively complemented distributive lattice with a bottom element. For a topological space X, we denote the set of clopen subsets of X by $\mathrm{CO}(X)$. The set of compact open subsets of X are denoted by $\mathrm{CK}(X)$. Finally, the set of regular open subsets of X are denoted by $\mathcal{R}(X)$.

Example A.2.1. Let X be a topological space.

1. $\mathrm{CO}(X)$ is a Boolean algebra under the operations of set-theorietic union, intersection and complement by X.

2. $\mathrm{CK}(X)$ is a generalized Boolean algebra in the same way as $\mathrm{CO}(X)$, except for admitting only relative (set-theoretic) complements.

3. $\mathcal{R}(X)$ is a Boolean algebra with the following operations. Let $A, B \in \mathcal{R}(X)$. The join of A and B is $\left(\overline{A \cup B}\right)^{\circ}$, where $\circ$ denotes the interior. The meet of A and B is $A \cap B$. And the complement of A is $\sim A \coloneqq (X \setminus A)^{\circ}$.

A crude way of stating Stone duality is to say that every Boolean algebra arises as $\mathrm{CO}(X)$ for some Stone space X, and that every generalized Boolean algebra arises as $\mathrm{CK}(Y)$ for some Boolean space Y. Hence, Stone spaces correspond to Boolean algebas and Boolean spaces correspond to generalized Boolean algebras.

More precisely, it is a duality in the following sense. Recall that a continuous map $f \colon X \to Y$ between topological spaces X and Y is *proper* if $f^{-1}(K)$ is compact in X whenever K is a compact subset of Y. A map $\psi \colon \mathcal{A} \to \mathcal{B}$ between generalized Boolean algebras $\mathcal{A}$ and $\mathcal{B}$ is a *Boolean homomorphism* if it preserves joins, meets and relative complements. We say that ψ is *proper* if for each $b \in \mathcal{B}$, there exists $a \in \mathcal{A}$ such that $\psi(a) \geq b$. Boolean spaces with proper continuous maps form a category. So does generalized Boolean algebras with proper Boolean homomorphisms. For a proper continuous map $f \colon X \to Y$, define

$$\mathrm{CK}(f)(A) \coloneqq f^{-1}(A) \quad \text{for } A \in \mathrm{CK}(Y).$$

This makes $\mathrm{CK}(-)$ a contravariant functor from the category of Boolean spaces to the category of generalized Boolean algebras (with maps as above).

For a generalized Boolean algebra $\mathcal{A}$, let $\mathbb{S}(\mathcal{A})$ denote the set of ultrafilters in $\mathcal{A}$. For each $a \in \mathcal{A}$, let $\mathbb{S}(a) \coloneqq \{\alpha \in \mathbb{S}(\mathcal{A}) \mid a \in \alpha\}$. Equipping $\mathbb{S}(\mathcal{A})$ with the topology generated by the (compact open) cylinder sets $\mathbb{S}(a)$ turns it into a Boolean space. For a proper Boolean homomorphism $\psi \colon \mathcal{A} \to \mathcal{B}$ and an ultrafilter $\beta \in \mathbb{S}(\mathcal{B})$, let $\mathbb{S}(\psi)(\beta) \coloneqq \{\psi^{-1}(b) \mid b \in \beta\}$. This makes $\mathbb{S}(-)$ a contravariant functor in the other direction, and we refer to it as the *Stone functor*. *Stone duality* asserts that the contravariant functors $\mathrm{CK}(-)$ and $\mathbb{S}(-)$ implement a dual equivalence. In other words, the category of Boolean spaces is dually equivalent to the category of generalized Boolean algebras. It is more common to state Stone duality in terms of Stone spaces and Boolean algebras. This is just the restriction of the duality above to the aforementioned sub-categories.

For a generalized Boolean algebra $\mathcal{A}$, we let $\mathrm{Aut}(\mathcal{A})$ denote the group of Boolean isomorphisms from $\mathcal{A}$ to $\mathcal{A}$.

A.2.3 Étale groupoids

Standard references for étale groupoids (and their C^*-algebras) are Renault's thesis [Ren80] and Paterson's book [Pat99]. See also the excellent lecture notes by Sims [Sim17]. A *groupoid* is a small category of isomorphisms, that is, a set $\mathcal{G}$ (the morphisms, or arrows in the category) equipped with a partially defined multiplication $(g_1, g_2) \mapsto g_1 \cdot g_2$ for a distinguished subset $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$, and everywhere defined involution $g \mapsto g^{-1}$ satisfying the following axioms:

1. If $g_1 g_2$ and $(g_1 g_2) g_3$ are defined, then $g_2 g_3$ and $g_1(g_2 g_3)$ are defined and $(g_1 g_2) g_3 = g_1(g_2 g_3)$,
2. The products gg^{-1} and $g^{-1}g$ are always defined. If $g_1 g_2$ is defined, then $g_1 = g_1 g_2 g_2^{-1}$ and $g_2 = g_1^{-1} g_1 g_2$.

A *topological groupoid* is a groupoid equipped with a topology making the operations of multiplication and taking inverse continuous. The elements of the form gg^{-1} are called *units*. We denote the set of units of a groupoid $\mathcal{G}$ by $\mathcal{G}^{(0)}$, and refer to this as the *unit space*. We think of the unit space as a topological space equipped with the relative topology from $\mathcal{G}$. The *source* and *range* maps are

$$s(g) := g^{-1}g \qquad \text{and} \qquad r(g) := gg^{-1}$$

for $g \in \mathcal{G}$. These maps are necessarily continuous when $\mathcal{G}$ is a topological groupoid. We implicitly assume that all unit spaces appearing are of infinite cardinality (in order to avoid some degenerate cases). An *étale* groupoid is a topological groupoid where the range map (and necessarily also the source map) is a local homeomorphism (as a map from $\mathcal{G}$ to $\mathcal{G}$). The unit space $\mathcal{G}^{(0)}$ of an étale groupoid is always an open subset of $\mathcal{G}$. An *ample* groupoid is an étale groupoid whose unit space is a Boolean space.

It is quite common for operator algebraists to restrict to Hausdorff groupoids. One reason for this is that a topological groupoid is Hausdorff if and only if the unit space is a closed subset of the groupoid. In the end our main results will only apply to groupoids that are Hausdorff, but some of the theory applies when $\mathcal{G}$ is merely ample (and effective). For as long as the unit space $\mathcal{G}^{(0)}$ is Hausdorff the groupoid will be locally Hausdorff. We shall therefore clearly indicate whenever we actually need the groupoid to be Hausdorff for some result to hold.

Two units $x, y \in \mathcal{G}^{(0)}$ belong to the same *$\mathcal{G}$-orbit* if there exists $g \in \mathcal{G}$ such that $s(g) = x$ and $r(g) = y$. We denote by $\mathrm{Orb}_{\mathcal{G}}(x)$ the $\mathcal{G}$-orbit of x. When every $\mathcal{G}$-orbit is dense in $\mathcal{G}^{(0)}$, $\mathcal{G}$ is called *minimal*. In the special case that there is just one orbit, we call $\mathcal{G}$ *transitive*. A subset $A \subseteq \mathcal{G}^{(0)}$ is called *$\mathcal{G}$-full* if $r(s^{-1}(A)) = \mathcal{G}^{(0)}$, in other words if A meets every $\mathcal{G}$-orbit. For an open subset

$A \subseteq \mathcal{G}^{(0)}$ the subgroupoid $\mathcal{G}_{|A} := \{g \in \mathcal{G} \mid s(g), r(g) \in A\}$ is called the *restriction* of $\mathcal{G}$ to A. When $\mathcal{G}$ is étale, the restriction $\mathcal{G}_{|A}$ is an open étale subgroupoid. The *isotropy group* of a unit $x \in \mathcal{G}^{(0)}$ is the group $\mathcal{G}_x^x := \{g \in \mathcal{G} \mid s(g) = r(g) = x\}$, and the *isotropy bundle* is

$$\mathcal{G}' := \{g \in \mathcal{G} \mid s(g) = r(g)\} = \bigsqcup_{x \in \mathcal{G}^{(0)}} \mathcal{G}_x^x.$$

A groupoid $\mathcal{G}$ is said to be *principal* if $\mathcal{G}' = \mathcal{G}^{(0)}$, i.e. if all isotropy groups are trivial. Any principal groupoid can be identified with an equivalence relation on its unit space $\mathcal{G}^{(0)}$, but the topology need not be the relative topology from $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$. We say that $\mathcal{G}$ is *effective* if the interior of $\mathcal{G}'$ equals $\mathcal{G}^{(0)}$. We call $\mathcal{G}$ *topologically principal* if the set of points in $\mathcal{G}^{(0)}$ with trivial isotropy group are dense in $\mathcal{G}^{(0)}$.

Remark A.2.2. We should point out that the condition we are calling *effective* often goes under the name *essentially principal* (or even topologically principal) elsewhere in the literature. In general, topologically principal implies effective. However, for most groupoids considered by operator algebraists the two notions are in fact equivalent (see [Ren08, Proposition 3.1]), so often these names all mean the same thing. In particular, this is the case for second countable locally compact Hausdorff étale groupoids.

Definition A.2.3. Let $\mathcal{G}$ be an étale groupoid. A *bisection* is an open subset $U \subseteq \mathcal{G}$ such that s and r are both injective when restricted to U. A bisection U is called *full* if we have $s(U) = r(U) = \mathcal{G}^{(0)}$.

When U is a bisection in $\mathcal{G}$, then $s_{|U} \colon U \to s(U)$ is a homeomorphism, and similarly for the range map. An étale groupoid can thus be characterized by admitting a topological basis consisting of bisections, and an ample groupoid as one with a basis of compact bisections. In particular, ample groupoids are locally compact, and if $\mathcal{G}$ is Hausdorff and ample, then $\mathcal{G}$ is also a Boolean space. One of the most basic class of examples of étale groupoids are the following, which arise from group actions.

Example A.2.4. Let Γ be a discrete group acting by homeomorphisms on a topological space X. The associated *transformation groupoid* is

$$\Gamma \ltimes X := \Gamma \times X$$

with product according to $(\tau, \gamma(x)) \cdot (\gamma, x) = (\tau\gamma, x)$ (and undefined otherwise), and inverse $(\gamma, x)^{-1} = (\gamma^{-1}, \gamma(x))$. Identifying the unit space $(\Gamma \ltimes X)^{(0)} = \{1\} \times X$ with X in the obvious way we have $s((\gamma, x)) = x$ and $r((\gamma, x)) = \gamma(x)$. Equipping

$\Gamma \ltimes X$ with the product topology makes it an étale groupoid (essentially because Γ is discrete), and a basis of bisections is given by the cylinder sets

$$Z(\gamma, U) \coloneqq \{(\gamma, x) \mid x \in U\}$$

indexed over $\gamma \in \Gamma$ and open subsets $U \subseteq X$. The identification of X with the unit space as above is compatible with this topology. In particular $\Gamma \ltimes X$ is Hausdorff and ample exactly when X is Boolean, and second countable when Γ is countable and X is second countable. The transformation groupoid is effective if and only if every non-trivial group element has support equal to X. In the second countable setting, this coincides with the action being topologically principal (meaning that the set of points that are fixed only by the identity element of the group form a dense subset of X). The groupoid orbit $\mathrm{Orb}_{\Gamma \ltimes X}(x)$ of a point $x \in X$ coincide with the orbit under the action, i.e. $\mathrm{Orb}_{\Gamma \ltimes X}(x) = \{\gamma(x) \mid \gamma \in \Gamma\} = \mathrm{Orb}_{\Gamma \curvearrowright X}(x)$.

A *groupoid homomorphism* is a map $\Phi \colon \mathcal{G} \to \mathcal{H}$ such that $(\Phi(g), \Phi(g')) \in \mathcal{H}^{(2)}$ whenever $(g, g') \in \mathcal{G}^{(2)}$, and moreover $\Phi(g) \cdot \Phi(g') = \Phi(g \cdot g')$. It follows that $\Phi(g^{-1}) = \Phi(g)^{-1}$ for all $g \in \mathcal{G}$, Φ commutes with the source and range maps and $\Phi\left(\mathcal{G}^{(0)}\right) \subseteq \mathcal{H}^{(0)}$. If Φ is a bijection, then Φ^{-1} is a groupoid homomorphism and we call Φ an *algebraic isomorphism*. For étale groupoids $\mathcal{G}$ and $\mathcal{H}$ an *étale homomorphism* is a groupoid homomorphism $\Phi \colon \mathcal{G} \to \mathcal{H}$ which is also a local homeomorphism. It is a fact that a groupoid homomorphism $\Phi \colon \mathcal{G} \to \mathcal{H}$ between étale groupoids is a local homeomorphism if and only if the restriction $\Phi^{(0)} \colon \mathcal{G}^{(0)} \to \mathcal{H}^{(0)}$ to the unit spaces is a local homemorphism. By an *isomorphism* of topogical (or étale) groupoids we mean an algebraic isomorphism which is also a homeomorphism. So a bijective étale homomorphism is an isomorphism of étale groupoids. Note that if $\Phi \colon \mathcal{G} \to \mathcal{H}$ is an étale homomorphism, then the image $\Phi(\mathcal{G})$ is an open étale subgroupoid of $\mathcal{H}$.

A.3 The topological full group

In this section we will expand Matui's definition of the topological full group of an ample groupoid from the compact to the locally compact case, and establish some elementary properties. To each bisection $U \subseteq \mathcal{G}$ in an étale groupoid we associate a homeomorphism

$$\pi_U \colon s(U) \to r(U)$$

given by $r_{|U} \circ (s_{|U})^{-1}$. This means that for each $g \in U$, π_U maps $s(g)$ to $r(g)$. Whenever U is a full bisection, π_U is a homeomorphism of $\mathcal{G}^{(0)}$. We now show that the (partial) homeomorphism π_U determines the bisection U, when the groupoid is effective and Hausdorff.

Lemma A.3.1. *Let $\mathcal{G}$ be an effective ample Hausdorff groupoid and let $U, V \subseteq \mathcal{G}$ be bisections with $s(U) = s(V)$ and $r(U) = r(V)$. If $\pi_U = \pi_V$, then $U = V$.*

Proof. To have $\pi_U = \pi_V$ means that for each $x \in s(U)$, the unique elements $g \in U, h \in V$ with $s(g) = x = s(h)$ also satisfies $r(g) = r(h)$. This in turn implies that $V^{-1}U \subseteq \mathcal{G}'$. As $\mathcal{G}$ is Hausdorff, $\mathcal{G}^{(0)}$ is closed, and therefore $V^{-1}U \cap \left(\mathcal{G} \setminus \mathcal{G}^{(0)}\right)$ is an open subset of $\mathcal{G}' \setminus \mathcal{G}^{(0)}$. But since $\mathcal{G}$ is effective this set must be empty. This entails that $V^{-1}U \subseteq \mathcal{G}^{(0)}$, and hence $U = V$. □

Definition A.3.2. Let $\mathcal{G}$ be an effective ample groupoid. The *topological full group* of $\mathcal{G}$, denoted $[\![\mathcal{G}]\!]$, is the subgroup of Homeo $\left(\mathcal{G}^{(0)}\right)$ consisting of all homeomorphisms of the form π_U, where U is a full bisection in $\mathcal{G}$ such that $\operatorname{supp}(\pi_U)$ is compact. We will denote by $\mathsf{D}([\![\mathcal{G}]\!])$ its commutator subgroup.

In the topological full group, composition and inversion of the homeomorphisms correspond to multiplication and inversion of the bisections, viz.:

- $\pi_{\mathcal{G}^{(0)}} = \mathrm{id}_{\mathcal{G}^{(0)}} = 1$
- $\pi_U \circ \pi_V = \pi_{UV}$
- $(\pi_U)^{-1} = \pi_{U^{-1}}$

Remark A.3.3. It is clear that when the unit space is compact, this definition coincides with Matui's [Mat12, Definition 2.3]—which again generalizes the definitions given in [GPS99] and [Mat10], for Cantor dynamical systems and one-sided shifts of finite type, respectively, to étale groupoids. Moreover, in [Mat02] Matui defined six different full groups associated with a minimal homeomorphism ϕ of a locally compact Cantor space. The smallest one of these, denoted $\tau[\phi]_c$ in [Mat02], equals the topological full group (as in Definition A.3.2) of the associated transformation groupoid.

Remark A.3.4. After the completion of this work, we were made aware of Matte Bon's preprint [MB18] where he defines the topological full group of an arbitrary étale groupoid $\mathcal{G}$ as the group of all full bisections $U \subseteq \mathcal{G}$ such that $U \setminus \mathcal{G}^{(0)}$ is compact. For effective groupoids, this agrees with Definition A.3.2, modulo identifying a full bisection with its associated homeomorphism. For not necessarily effective groupoids it is arguably better to define the topological full group in terms of the bisections themselves, for then one does not "lose" the information contained in the (non-trivial) isotropy (but also to separate the group from its canonical—no longer faithful—action on the unit space). This is done in e.g. [Nek19] and [BS19] as well. However, the approach taken in this paper—in particular in Section A.6—is based on working with subgroups of the homeomorphism group of a space (i.e. faithful group actions), which is why we have defined $[\![\mathcal{G}]\!]$ as we have.

Remark A.3.5. We emphasize that the topological full group $[[\mathcal{G}]]$ is viewed as a *discrete* group. The term *topological* is historical, and refers to the fact that the homeomorphisms in the topological full group preserves orbits in a "continuous way", as opposed to the full groups, which appeared first—in the measurable setting—see [GPS99, page 2].

For descriptions of the topological full group in certain classes of examples, see Proposition A.9.4, Remark A.11.22 and Remark A.11.28. See also [Mat17] for a survey of about topological full groups of étale groupoids with compact unit space.

By virtue of the groupoid being effective, the support of a homeomorphism in the topological full group is in fact open as well. Matui's proof of this fact for compact unit spaces carries over verbatim to our setting.

Lemma A.3.6 (cf. [Mat15b, Lemma 2.2]). *Let $\mathcal{G}$ be an effective ample Hausdorff groupoid. Then* $\mathrm{supp}(\pi_U) = s(U \setminus \mathcal{G}^{(0)})$ *for each* $\pi_U \in [[\mathcal{G}]]$. *In particular,* $\mathrm{supp}(\pi_U)$ *is a compact open subset of* $\mathcal{G}^{(0)}$.

We now present a few basic results on the existence of elements in the topological full group. They will be used in later sections to construct elements in the topological full group with localized support.

Lemma A.3.7. *Let $\mathcal{G}$ be an effective ample groupoid, and let $\pi_U \in [[\mathcal{G}]]$. Then we have a decomposition*

$$U = U^{\perp} \bigsqcup \left(\mathcal{G}^{(0)} \setminus \mathrm{supp}(\pi_U)\right),$$

where $U^{\perp}$ is a compact bisection with $s(U^{\perp}) = r(U^{\perp}) = \mathrm{supp}(\pi_U)$.

Conversely, any compact bisection $V \subseteq \mathcal{G}$ with $s(V) = r(V)$ defines an element $\pi_{\tilde{V}} \in [[\mathcal{G}]]$ with $\mathrm{supp}(\pi_{\tilde{V}}) \subseteq s(V)$ *by setting* $\tilde{V} = V \sqcup \left(\mathcal{G}^{(0)} \setminus s(V)\right)$.

Proof. It is clear that $\mathrm{supp}(\pi_U)$ is invariant under π_U. Therefore we may simply put $U^{\perp} = s_{|U}^{-1}(\mathrm{supp}(\pi_U))$. The second statement is obvious. □

Lemma A.3.8. *Let $\mathcal{G}$ be an effective ample groupoid. Any compact bisection $V \subseteq \mathcal{G}$ satisfying $s(V) \cap r(V) = \emptyset$ defines an involutive element $\pi_{\hat{V}} \in [[\mathcal{G}]]$ with* $\mathrm{supp}(\pi_{\hat{V}}) \subseteq s(V) \cup r(V)$ *by setting* $\hat{V} = V \sqcup V^{-1} \sqcup \left(\mathcal{G}^{(0)} \setminus (s(V) \cup r(V))\right)$.

Proof. Immediate. □

Lemma A.3.9. *Let $\mathcal{G}$ be an effective ample groupoid. If $g \in \mathcal{G} \setminus \mathcal{G}'$, that is $s(g) \neq r(g)$, then there is a (nontrivial) bisection $U \subseteq \mathcal{G}$ containing g with $\pi_U \in [[\mathcal{G}]]$. Furthermore, for any open set $A \subseteq \mathcal{G}^{(0)}$ containing both $s(g)$ and $r(g)$, U can be chosen so that* $\mathrm{supp}(\pi_U) \subseteq A$. *We may also choose π_U to be an involution.*

Proof. As $\mathcal{G}$ is ample there is a compact bisection W containing g. Let B_1, B_2 be disjoint open neighbourhoods of $s(g)$, $r(g)$ respectively in $\mathcal{G}^{(0)}$. By intersecting we may take $B_1 \subseteq s(W) \cap A$ and $B_2 \subseteq r(W) \cap A$. By continuity of s and r there are compact open sets $W_1, W_2 \subseteq W$, both containing g, such that $s(W_1) \subseteq B_1$ and $r(W_2) \subseteq B_2$. And then $V = W_1 \cap W_2$ is a compact bisection containing g with $s(V) \cap r(V) = \emptyset$ and $s(V) \cup r(V) \subseteq A$. Hence $U = \hat{V}$ (as in Lemma A.3.8) is the desired full bisection. □

Remark A.3.10. In the non-compact case we may view the topological full group as a direct limit of topological full groups of groupoids over *compact spaces* as follows. Consider $\mathrm{CK}\left(\mathcal{G}^{(0)}\right)$ as a directed set (ordered by inclusion). Given two sets $A, B \in \mathrm{CK}\left(\mathcal{G}^{(0)}\right)$ with $A \subseteq B$ we define a homomorphism $\iota_{A,B} \colon [\![\mathcal{G}_A]\!] \to [\![\mathcal{G}_B]\!]$ by $\pi_U \mapsto \pi_{\tilde{U}}$, where $\tilde{U} = U \sqcup (B \setminus A)$. Then we have that

$$[\![\mathcal{G}]\!] \cong \varinjlim([\![\mathcal{G}_A]\!], \iota).$$

A.4 The groupoid of germs

We are now going to adapt the notions of [Ren08, Section 3] to the (special) case of groups, rather than inverse semigroups, to fit the framework of the topological full group and its subgroups, rather than the *pseudogroup* studied in [Ren08]. Our goal is to reconstruct an ample groupoid $\mathcal{G}$ from subgroups of the topological full group $[\![\mathcal{G}]\!]$ as a so-called *groupoid of germs*—which is a quotient of a transformation groupoid.

Remark A.4.1. In the following three sections we will be working with subgroups of $\mathrm{Homeo}(X)$, where X is a topological space. Thus we are essentially studying faithful actions by discrete groups on X. In the end we will have $X = \mathcal{G}^{(0)}$ for some ample groupoid $\mathcal{G}$, and we will be looking at subgroups of $[\![\mathcal{G}]\!]$. Yet it will be convenient to state most results for general subgroups $\Gamma \leq \mathrm{Homeo}(X)$ without reference to groupoids. Also, beware that the term *faithful* will be used differently in Section A.6 (see Definition A.6.1).

Recall that two homeomorphisms $\gamma, \tau \colon X \to X$ have the same *germ* at a point $x \in X$ if there is a neighbourhood U of x such that $\gamma_{|U} = \tau_{|U}$.

Definition A.4.2. Let X be a locally compact Hausdorff space and let Γ be a subgroup of $\mathrm{Homeo}(X)$. The *groupoid of germs* of (Γ, X) is

$$\mathrm{Germ}(\Gamma, X) \coloneqq (\Gamma \ltimes X) \,/\, \sim$$

where $(\gamma, x) \sim (\tau, y)$ iff $x = y$ and γ, τ have the same germ at x.

Denote the equivalence class of $(\gamma, x) \in \Gamma \ltimes X$ under $\sim$ by $[\gamma, x]$. It is straightforward to check that the groupoid operations of the transformation groupoid are well-defined on representatives of the equivalence classes in the groupoid of germs (and that they are continuous). The bisections

$$Z[\gamma, A] := \{[\gamma, x] \mid x \in A\},$$

for $\gamma \in \Gamma$ and $A \subseteq X$ open, form a basis for the quotient topology. The unit space of $\mathrm{Germ}(\Gamma, X)$ is also identified with X in the obvious way. Hence the groupoid $\mathrm{Germ}(\Gamma, X)$ is étale (and ample when X is Boolean), and it is furthermore always effective (as any group element acting identically on an open set is identified with the identity at each point of this open set). Hausdorffness of the groupoid however, is no longer guaranteed, but it can be characterized as follows.

Lemma A.4.3. *Let X be a locally compact Hausdorff space and let $\Gamma \leq \mathrm{Homeo}(X)$. Then the groupoid of germs $\mathrm{Germ}(\Gamma, X)$ is Hausdorff if and only if $\mathrm{supp}(\gamma)$ is clopen in X for every $\gamma \in \Gamma$.*

Proof. Since X is Hausdorff, any two groupoid elements $[\gamma, x], [\tau, y] \in \mathrm{Germ}(\Gamma, X)$ with distinct sources (i.e. $x \neq y$) can always be separated by open sets. We only have to worry about separating elements in the same isotropy group, and it suffices to be able to separate the unit from any other element. Also note that $[\gamma, x] \neq [1, x]$ if and only if $x \in \mathrm{supp}(\gamma)$.

First, assume that all the supports are clopen. If $[\gamma, x] \neq [1, x]$, then by the observation above, $Z[\gamma, \mathrm{supp}(\gamma)]$ and $Z[1, \mathrm{supp}(\gamma)]$ are disjoint open neighbourhoods of these elements. To separate $[\gamma, x]$ from $[\tau, x]$ (when these are distinct), we first note that $[\gamma, x][\tau, x]^{-1} = [\gamma\tau^{-1}, \tau(x)] \neq [1, \tau(x)]$. Hence $\tau(x) \in \mathrm{supp}(\gamma\tau^{-1})$, so by the argument above $Z[\gamma\tau^{-1}, A]$ and $Z[1, A]$, with $A = \mathrm{supp}(\gamma\tau^{-1})$, separates $[\gamma\tau^{-1}, \tau(x)]$ from $[1, \tau(x)]$. It follows that $Z[\gamma, \tau^{-1}(A)]$ and $Z[\tau, \tau^{-1}(A)]$ separates $[\gamma, x]$ and $[\tau, x]$.

Conversely, suppose there is a $\gamma \in \Gamma$ such that $\mathrm{supp}(\gamma)$ is not open. Let x be any point on the boundary of $\mathrm{supp}(\gamma)$. Then $\gamma(x) = x$, but $[\gamma, x] \neq [1, x]$, and these two groupoid elements cannot be separated by open sets. To see this take any two basic neighbourhoods $Z[\gamma, A], Z[1, B]$ where A, B are open neighbourhoods of x in X. They both contain the basic set $Z[1, C]$ where $C = (A \cap B) \setminus \mathrm{supp}(\gamma)$, since γ acts identically on C. □

In the sequel we shall restrict our attention to groups of homeomorphisms which have open, as well as compact, support. Topological full groups are determined by the "local behaviour" of its elements. This is made precise in the following definition.

Definition A.4.4. Let X be a locally compact Hausdorff space and let Γ be a subgroup of $\mathrm{Homeo}_c(X)$. We say that a homeomorphism $\varphi \in \mathrm{Homeo}_c(X)$ *locally belongs to* Γ if for every $x \in X$, there exists an open neighborhood U of x and an element $\gamma \in \Gamma$ such that $\varphi_{|U} = \gamma_{|U}$. The group Γ is called *locally closed* if whenever $\varphi \in \mathrm{Homeo}_c(X)$ locally belongs to Γ, then $\varphi \in \Gamma$.

Proposition A.4.5. *Let $\mathcal{G}$ be an effective ample Hausdorff groupoid. Then the topological full group* $[\![\mathcal{G}]\!] \leq \mathrm{Homeo}_c\left(\mathcal{G}^{(0)}\right)$ *is locally closed.*

Proof. Let $\varphi \in \mathrm{Homeo}_c(\mathcal{G}^{(0)})$ locally belong to $[\![\mathcal{G}]\!]$. Then, since $\mathrm{supp}(\varphi)$ is compact open, we can find finitely many open sets $A_i \subseteq \mathrm{supp}(\varphi)$, covering $\mathrm{supp}(\varphi)$, such that $\varphi_{|A_i} = (\pi_{U_i})_{|A_i}$, where $\pi_{U_i} \in [\![\mathcal{G}]\!]$. Since $\mathcal{G}^{(0)}$ is Boolean we may assume that the A_i's are clopen and disjoint. We then have a clopen partition $\mathrm{supp}(\varphi) = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$, and φ restricts to a self-homeomorphism of $\mathrm{supp}(\varphi)$ which on each region A_i equals π_{U_i}. It follows that the set $V = \cup_{i=1}^n V_i$, where $V_i = \left(s_{|U_i}\right)^{-1}(A_i)$, is a compact bisection in $\mathcal{G}$ with $s(V) = \mathrm{supp}(\varphi) = r(V)$. And then $\varphi = \pi_{\tilde{V}} \in [\![\mathcal{G}]\!]$, where $\tilde{V}$ is as in Lemma A.3.7. □

Given a group $\Gamma \leq \mathrm{Homeo}_c(X)$ we denote by $\langle\Gamma\rangle$ the set of $\varphi \in \mathrm{Homeo}_c(X)$ which locally belong to Γ. Clearly $\langle\Gamma\rangle$ is a locally closed group in $\mathrm{Homeo}_c(X)$ and $\Gamma \leq \langle\Gamma\rangle$. As the groupoid of germs is defined in the same local terms as the local closure we have a canonical isomorphism $\mathrm{Germ}(\langle\Gamma\rangle, X) \cong \mathrm{Germ}(\Gamma, X)$. From this we obtain the analog of [Ren08, Proposition 3.2], namely that the topological full group of a groupoid of germs equals the local closure of the group we started with.

Proposition A.4.6. *Let X be a Boolean space and let $\Gamma \leq \mathrm{Homeo}_c(X)$. Then we have* $[\![\mathrm{Germ}(\Gamma, X)]\!] \cong \langle\Gamma\rangle$.

Proof. Since $\mathrm{Germ}(\Gamma, X)$ can be identified with $\mathrm{Germ}(\langle\Gamma\rangle, X)$, it suffices to show that $[\![\mathrm{Germ}(\langle\Gamma\rangle, X)]\!] = \langle\Gamma\rangle$. For each $\varphi \in \langle\Gamma\rangle$ the full bisection $Z[\varphi, X] = U_\varphi$ in $\mathrm{Germ}(\langle\Gamma\rangle, X)$ satisfies $\pi_{U_\varphi} = \varphi$. And since φ has compact support it belongs to $[\![\mathrm{Germ}(\langle\Gamma\rangle, X)]\!]$.

For the reverse inclusion, take any $\pi_U \in [\![\mathrm{Germ}(\langle\Gamma\rangle, X)]\!]$. Recall that the support of π_U is open, as well as compact, since any groupoid of germs is effective (Lemma A.3.6). To see that π_U locally belongs to Γ take any $x \in X$, and let $[\varphi, x]$ be the unique element in U whose source is x. Since U is open there is a basic set $Z[\varphi, A] \subseteq U$, where A is an open neighbourhood of x in X. As $\varphi \in \langle\Gamma\rangle$ there is an open neighbourhood B of x and an element $\gamma \in \Gamma$ with $\varphi_{|B} = \gamma_{|B}$. By intersecting with A we may assume that $B \subseteq A$. Now observe that $(\pi_U)_{|B} = \varphi_{|B} = \gamma_{|B}$, and we are done. □

As topological full groups are locally closed (Proposition A.4.5) we obtain the following immediate corollary.

Corollary A.4.7. *Let $\mathcal{G}$ be an effective ample Hausdorff groupoid. Then*

$$[\![\mathrm{Germ}([\![\mathcal{G}]\!], \mathcal{G}^{(0)})]\!] \cong [\![\mathcal{G}]\!].$$

The preceding results show that a locally closed group $\Gamma \leq \mathrm{Homeo}_c(X)$ can be reconstructed from its associated groupoid of germs $\mathrm{Germ}\left(\Gamma, \mathcal{G}^{(0)}\right)$, namely as the topological full group of this groupoid. We now turn to the question of how an ample groupoid $\mathcal{G}$ relates to the groupoid of germs, $\mathrm{Germ}\left([\![\mathcal{G}]\!], \mathcal{G}^{(0)}\right)$, determined by its topological full group. We will see that these will also be isomorphic under some mild condition on the groupoid—namely that the groupoid can be covered by bisections as in the following definition.

Definition A.4.8. Let $\mathcal{G}$ be an effective ample groupoid. We say that a subgroup $\Gamma \leq [\![\mathcal{G}]\!]$ *covers* $\mathcal{G}$ if there for each $g \in \mathcal{G}$ exists a $\pi_U \in \Gamma$ such that $g \in U$.

Note that if $\Gamma \leq [\![\mathcal{G}]\!]$ covers $\mathcal{G}$, then so does any group Γ' in between, i.e. $\Gamma \leq \Gamma' \leq [\![\mathcal{G}]\!]$, and in particular $[\![\mathcal{G}]\!]$ itself covers $\mathcal{G}$. Sufficient conditions on the orbits of $\mathcal{G}$ for $[\![\mathcal{G}]\!]$, or the commutator $\mathsf{D}([\![\mathcal{G}]\!])$, to cover $\mathcal{G}$ is given by the following result (which is the analog of [Mat15b, Lemma 3.7]).

Lemma A.4.9. *Let $\mathcal{G}$ be an effective ample groupoid.*

1. *If $|\mathrm{Orb}_{\mathcal{G}}(x)| \geq 2$ for every $x \in \mathcal{G}^{(0)}$, then $[\![\mathcal{G}]\!]$ covers $\mathcal{G}$.*
2. *If $|\mathrm{Orb}_{\mathcal{G}}(x)| \geq 3$ for every $x \in \mathcal{G}^{(0)}$, then $\mathsf{D}([\![\mathcal{G}]\!])$ covers $\mathcal{G}$.*

Proof. (1) First consider $g \in \mathcal{G} \setminus \mathcal{G}'$. Then Lemma A.3.9 immediately gives a $\pi_U \in [\![\mathcal{G}]\!]$ with $g \in U$. Next, suppose $s(g) = r(g) = x$. By assumption there is a point y different from x in $\mathrm{Orb}_{\mathcal{G}}(x)$. This means that there is some $h \in \mathcal{G}$ with $s(h) = x \neq y = r(h)$. And then h^{-1} is composable with g and $gh^{-1} \in \mathcal{G} \setminus \mathcal{G}'$. Applying Lemma A.3.9 to both gh^{-1} and h we get $\pi_{U_1}, \pi_{U_2} \in [\![\mathcal{G}]\!]$ with $gh^{-1} \in U_1$ and $h \in U_2$. Since $\pi_{U_1 U_2} \in [\![\mathcal{G}]\!]$ and $g \in U_1 U_2$ we see that $[\![\mathcal{G}]\!]$ covers $\mathcal{G}$.

(2) As in the previous part we first consider $g \in \mathcal{G} \setminus \mathcal{G}'$. By assumption there is a third (distinct) point y in the same orbit as $s(g)$ and $r(g)$. Therefore there is an element $h \in \mathcal{G}$ with $s(h) = y$ and $r(h) = s(g)$. Lemma A.3.9 gives involutions $\pi_U, \pi_V \in [\![\mathcal{G}]\!]$ such that $g \in U$ and $h \in V$. We may also arrange so that $y \notin \mathrm{supp}(\pi_U)$ by the second part of Lemma A.3.9. Then

$$[\pi_U, \pi_V] = \pi_U \pi_V (\pi_U)^{-1} (\pi_V)^{-1} = \pi_{(UV)^2} \in \mathsf{D}([\![\mathcal{G}]\!]),$$

and we claim that g belongs to the associated full bisection $(UV)^2$. Indeed, note that $y \in U$ since $y \notin \mathrm{supp}(\pi_U)$. Thus we have $g = g \cdot h \cdot y \cdot h^{-1} \in UVUV$ since $s(h) = y$.

Finally, for the case $s(g) = r(g)$ we proceed similar as in part (1). We take $h \in \mathcal{G}$ with $s(h) = s(g)$ and $r(h) \neq s(g)$ and apply the above part to gh^{-1} and h, which both belong to $\mathcal{G} \setminus \mathcal{G}'$. Multiplying the bisections we get gives the desired bisection containing g. □

The conditions in Lemma A.4.9 are not necessary (see Example A.9.6), but they are typically easy to check in specific examples. Note that for minimal groupoids all orbits are in particular infinite, so the covering as above is automatic. We are now ready to give the main result on how a groupoid $\mathcal{G}$ can be reconstructed from the germs of $[\![\mathcal{G}]\!]$. It is the analog of [Ren08, Proposition 3.2].

Proposition A.4.10. *Let $\mathcal{G}$ be an effective ample Hausdorff groupoid and let Γ be a subgroup of $[\![\mathcal{G}]\!]$. Then there is an injective étale homomorphism*

$$\iota\colon \operatorname{Germ}\left(\Gamma, \mathcal{G}^{(0)}\right) \hookrightarrow \mathcal{G}$$

given by $\iota([\pi_U, x]) = (s_{|U})^{-1}(x)$ for $[\pi_U, x] \in \operatorname{Germ}\left(\Gamma, \mathcal{G}^{(0)}\right)$. Furthermore, ι is surjective, and hence an isomorphism, if and only if Γ covers $\mathcal{G}$.

Proof. We first have to verify that ι is well-defined. Let $x \in \mathcal{G}^{(0)}$ and suppose that $\pi_U, \pi_V \in \Gamma$ have the same germ over x. Let A be an open neighbourhood of x on which π_U and π_V agree. Then

$$\pi_{UA} = (\pi_U)_{|A} = (\pi_V)_{|A} = \pi_{VA},$$

so by Lemma A.3.1 we have $UA = VA$. This means that the unique groupoid elements in U and V that have source equal to x coincide, so ι is well-defined.

To see that ι is a groupoid homomorphism recall that $([\pi_V, y], [\pi_U, x])$ is a composable pair iff $\pi_U(x) = y$. Suppose this is the case and let $g \in U$ be the element with $s(g) = x$, and let $h \in V$ be the element with $s(h) = y$. As $r(g) = \pi_U(x) = y = s(h)$ we have $(h, g) \in \mathcal{G}^{(2)}$ and

$$\iota([\pi_V, y] \cdot [\pi_U, x]) = \iota([\pi_{VU}, x]) = hg,$$

since $hg \in VU$ and $s(hg) = x$.

Now note that $\iota(x) = x$ for $x \in \mathcal{G}^{(0)}$ (under the identification of the unit space of the groupoid of germs). So $\iota^{(0)} = \mathrm{id}_{\mathcal{G}^{(0)}}$ is a (local) homeomorphism, hence ι is an étale homomorphism.

To see that ι is injective note first that $\iota([\pi_U, x]) \neq \iota([\pi_V, y])$ if $x \neq y$ since $\iota^{(0)}$ is the identity. Suppose now that $\iota([\pi_U, x]) = \iota([\pi_V, x])$ for some $\pi_U, \pi_V \in \Gamma$. This means that there is a groupoid element $g \in U \cap V$ with $s(g) = x$. Thus $B = s(U \cap V)$ is an open neighbourhood of x in $\mathcal{G}^{(0)}$ and clearly $(\pi_U)_{|B} = (\pi_V)_{|B}$, which means that $[\pi_U, x] = [\pi_V, x]$.

Finally, that ι is surjective is easily seen to be the same as Γ covering $\mathcal{G}$. □

Remark A.4.11. When the map ι in the previous proposition is an isomorphism the inverse is given by $\iota^{-1}(g) = [\pi_U, s(g)]$, where U is any full bisection such that $\pi_U \in \Gamma$ and $g \in U$.

Remark A.4.12. Let $\mathcal{G}$ be an effective ample Hausdorff groupoid. Combining Propositions A.4.10 and A.4.6 we see that for each locally closed subgroup Γ of $[\![\mathcal{G}]\!]$, there is an open étale subgroupoid $\mathcal{H}_\Gamma \subseteq \mathcal{G}$ such that $[\![\mathcal{H}_\Gamma]\!] \cong \Gamma$, namely $\mathcal{H}_\Gamma = \mathrm{Germ}(\Gamma, \mathcal{G}^{(0)})$.

As we really are interested in knowing when $\mathcal{G}$ is isomorphic to $\mathrm{Germ}\left(\Gamma, \mathcal{G}^{(0)}\right)$ (particularly for the case $\Gamma = [\![\mathcal{G}]\!]$) it is natural to ask whether they could be isomorphic even if the canonical map ι fails to be an isomorphism. We will see shortly that this is not possible. For $\Gamma \leq \mathrm{Homeo}_c(X)$ with X Boolean we have seen that $\Gamma \leq \langle\Gamma\rangle \cong [\![\mathrm{Germ}(\Gamma, X)]\!]$. Identifying the latter two we see that Γ covers $\mathrm{Germ}(\Gamma, X)$ since $[\gamma, x] \in Z[\gamma, X]$ and $\pi_{Z[\gamma,X]} = \gamma \in \Gamma$ for each $[\gamma, x] \in \mathrm{Germ}(\Gamma, X)$.

Corollary A.4.13. *Let $\mathcal{G}$ be an effective ample Hausdorff groupoid. Then $\mathcal{G}$ and* $\mathrm{Germ}\left([\![\mathcal{G}]\!], \mathcal{G}^{(0)}\right)$ *are isomorphic as étale groupoids if and only if $[\![\mathcal{G}]\!]$ covers $\mathcal{G}$.*

Proof. Suppose $\Phi\colon \mathcal{G} \to \mathrm{Germ}\left([\![\mathcal{G}]\!], \mathcal{G}^{(0)}\right)$ is an isomorphism. Then Φ induces an isomorphism between the topological full groups by $\pi_U \mapsto \pi_{\Phi(U)}$ for $\pi_U \in [\![\mathcal{G}]\!]$. Let $g \in \mathcal{G}$ be given. As $[\![\mathcal{G}]\!]$ covers $\mathrm{Germ}\left([\![\mathcal{G}]\!], \mathcal{G}^{(0)}\right)$ there is a full bisection V containing $\Phi(g)$ such that $\pi_V \in [\![\mathrm{Germ}([\![\mathcal{G}]\!], \mathcal{G}^{(0)})]\!] = [\![\mathcal{G}]\!]$. And then $\Phi^{-1}(V)$ is a full bisection in $\mathcal{G}$ containing g with $\pi_{\Phi^{-1}(V)} \in [\![\mathcal{G}]\!]$. Hence $[\![\mathcal{G}]\!]$ covers $\mathcal{G}$. $\square$

A.5 The category of spatial groups

In this section we will study the groupoid of germs from a categorical point of view. By introducing suitable categories we will see that the assigment

$$(\Gamma, X) \mapsto \mathrm{Germ}(\Gamma, X)$$

is functorial. We will also see that certain equivariant maps between the spaces induce embeddings of the groupoids of germs.

Definition A.5.1. The category of *spatial groups*, denoted **SpatG**, consists of pairs (Γ, X), where X is a Boolean space and $\Gamma \leq \mathrm{Homeo}_c(X)$. A morphism in **SpatG** from (Γ_1, X_1) to (Γ_2, X_2) is a local homeomorphism $\phi\colon X_1 \to X_2$ satisfying $\phi \circ \Gamma_1 \subseteq \Gamma_2 \circ \phi$.

We shall sometimes refer to a pair (Γ, X) as a *space-group pair*. Observe that an isomorphism in the category **SpatG** is a homeomorphism ϕ such that $\phi \circ \Gamma_1 \circ \phi^{-1} = \Gamma_2$. We call such an isomorphism a *spatial isomorphism* (as it is a group isomorphism implemented by a homeomorphism).

Definition A.5.2. The category **Gpoid** consists of ample effective Hausdorff groupoids, and the morphisms are étale homomorphisms.

Remark A.5.3. The choice of morphisms in **SpatG** is done so that they induce étale homomorphisms between the groupoid of germs in a natural way. As for the morphisms in **Gpoid**, there are several reasons for stipulating that they should be étale homomorphisms (rather than merely continuous groupoid homomorphisms). First of all, since all the structure maps in an étale groupoid are local homeomorphisms, it is reasonable to prescribe that maps between étale groupoids should be as well. Moreover, the image under an étale homomorphism is always an open étale subgroupoid in the codomain. An important consequence of this is that an injective étale homomorphism induce (diagonal preserving) injective $*$-homomorphisms between both the full and reduced groupoid C^*-algebras, respectively (and also between the Steinberg algebras), see [BNR+16, page 113] and [Phi05, Proposition 1.9]. In general, however, the groupoid C^*-algebra construction is not functorial.

It is straightforward to check that **SpatG** and **Gpoid** indeed are categories. We will now define a functor from **SpatG** to **Gpoid**, which on objects is the groupoid of germs. Let ϕ be a spatial morphism between two space-group pairs (Γ_1, X_1) and (Γ_2, X_2) in **SpatG**. Given $[\gamma, x] \in \mathrm{Germ}(\Gamma_1, X_1)$, there is a $\gamma' \in \Gamma_2$ with $\phi \circ \gamma = \gamma' \circ \phi$. We then propose to define an étale homomorphism

$$\mathrm{Germ}(\phi)\colon \mathrm{Germ}(\Gamma_1, X_1) \to \mathrm{Germ}(\Gamma_2, X_2)$$

by setting $\mathrm{Germ}(\phi)([\gamma, x]) = [\gamma', \phi(x)]$.

Proposition A.5.4. *The mapping* $\mathrm{Germ}(\phi)$ *described above is a well-defined étale homomorphism, and* $\mathrm{Germ}(-)\colon$ **SpatG** $\to$ **Gpoid** *is a (covariant) functor.*

Proof. Let $\phi\colon (\Gamma_1, X_1) \to (\Gamma_2, X_2)$ be a spatial morphism. We first verify that $\mathrm{Germ}(\phi)$ is well-defined. Given $[\gamma, x] \in \mathrm{Germ}(\Gamma_1, X_1)$, suppose $\gamma', \gamma'' \in \Gamma_2$ satisfy

$$\phi \circ \gamma = \gamma' \circ \phi = \gamma'' \circ \phi.$$

Then γ' and γ'' agree on $\phi(X_1)$, which is an open neighbourhood of $\phi(x)$, hence we have $[\gamma', \phi(x)] = [\gamma'', \phi(x)]$. So the choice of γ' doesn't matter. As for the choice of γ, suppose $\tau \in \Gamma_1$ has the same germ over x as γ, i.e. $\gamma_{|A} = \tau_{|A}$ for some open neighbourhood A of x in X_1. Let $\tau' \in \Gamma_2$ satisfy $\phi \circ \tau = \tau' \circ \phi$. Then

$$\gamma' \circ \phi_{|A} = \phi \circ \gamma_{|A} = \phi \circ \tau_{|A} = \tau' \circ \phi_{|A}.$$

This means that $\gamma'_{|\phi(A)} = \tau'_{|\phi(A)}$, hence $[\gamma', \phi(x)] = [\tau', \phi(x)]$. This shows that $\mathrm{Germ}(\phi)$ is well-defined.

Observe that the restriction to the unit spaces is just $\mathrm{Germ}(\phi)^{(0)} = \phi\colon X_1 \to X_2$. From this we obtain

$$s\left(\mathrm{Germ}(\phi)([\gamma, x])\right) = \phi(x) = \mathrm{Germ}(\phi)\left(s([\gamma, x])\right),$$

and

$$r\left(\mathrm{Germ}(\phi)([\gamma, x])\right) = \gamma' \circ \phi(x) = \phi \circ \gamma(x) = \mathrm{Germ}(\phi)\left(r([\gamma, x])\right).$$

This means that $\mathrm{Germ}(\phi)$ takes composable pairs to composable pairs. As for preserving the product itself, we verify that

$$\begin{aligned}\mathrm{Germ}(\phi)([\tau, \gamma(x)]) \cdot \mathrm{Germ}(\phi)([\gamma, x]) &= [\tau', \phi\gamma(x)] \cdot [\gamma', \phi(x)] = [\tau'\gamma', \phi(x)] \\ &= \mathrm{Germ}(\phi)([\tau\gamma, x]), \text{ since } \phi\tau\gamma = \phi\tau'\gamma'.\end{aligned}$$

As $\mathrm{Germ}(\phi)^{(0)} = \phi$ is a local homeomorphism, we have shown that $\mathrm{Germ}(\phi)$ is an étale homomorphism. Similar computations as above shows that $\mathrm{Germ}(-)$ sends identity morphisms to identity morphisms and preserves composition of morphisms. □

We record some consequences of this functoriality.

Corollary A.5.5. *Let* $\phi\colon (X_1, \Gamma_1) \to (X_2, \Gamma_2)$ *be a morphism in* **SpatG** *and consider the induced étale homomorphism* $\mathrm{Germ}(\phi)\colon \mathrm{Germ}(\Gamma_1, X_1) \to \mathrm{Germ}(\Gamma_2, X_2)$.

1. *If* ϕ *is a spatial isomorphism, then* $\mathrm{Germ}(\phi)$ *is an isomorphism of étale groupoids.*

2. *We have* $\mathrm{Germ}(\phi)^{(0)} = \phi$*. In particular,* $\mathrm{Germ}(\phi)$ *maps* X_1 *onto* X_2 *if and only if* ϕ *is surjective.*

3. *If* $\phi\colon X_1 \to X_2$ *is injective, then* $\mathrm{Germ}(\phi)$ *is also injective.*

4. *If* $\phi\colon X_1 \to X_2$ *is surjective and* $\phi \circ \Gamma_1 = \Gamma_2 \circ \phi$*, then* $\mathrm{Germ}(\phi)$ *is surjective.*

Proof. Statement (1) follows immediately from functoriality, and statement (2) was observed in the proof of Proposition A.5.4.

(3) Assume that $\phi\colon X_1 \to X_2$ is injective. Then clearly $\mathrm{Germ}(\phi)$ maps elements with distinct sources to distinct elements. So suppose

$$[\gamma', \phi(x)] = \mathrm{Germ}(\phi)([\gamma, x]) = \mathrm{Germ}(\phi)([\tau, x]) = [\tau', \phi(x)].$$

Then $\gamma'_{|A} = \tau'_{|A}$ for some open neighbourhood A of $\phi(x)$ in X_2. As $\phi \circ \gamma = \gamma' \circ \phi$ and $\phi \circ \tau = \tau' \circ \phi$ we have that $\phi \circ \gamma$ and $\phi \circ \tau$ agree on $\phi^{-1}(A)$. The injectivity of ϕ now implies that γ and τ agree on $\phi^{-1}(A)$, which is an open neighbourhood of x, hence $[\gamma, x] = [\tau, x]$ and $\mathrm{Germ}(\phi)$ is injective.

(4) Suppose $\phi\colon X_1 \to X_2$ is surjective and that $\phi \circ \Gamma_1 = \Gamma_2 \circ \phi$. Given an element $[\tau, y]$ in $\mathrm{Germ}(\Gamma_2, X_2)$, pick any $x \in X_1$ with $\phi(x) = y$. By assumption there is some $\gamma \in \Gamma_1$ such that $\phi \circ \gamma = \tau \circ \phi$, and then $\mathrm{Germ}(\phi)([\gamma, x]) = [\tau, y]$. □

Remark A.5.6. One might ask whether a spatial morphism $\phi\colon (X_1, \Gamma_1) \to (X_2, \Gamma_2)$ induces a (algebraic) group homomorphism from Γ_1 to Γ_2. This is not so clear. But at least if $\phi\colon X_1 \to X_2$ is injective and Γ_2 is locally closed, then one can define an injective group homomorphism $f_\phi\colon \Gamma_1 \to \Gamma_2$ in the following way. First observe that given $\gamma \in \Gamma_1$, there is a $\gamma_2 \in \Gamma_2$ with $\phi \circ \gamma = \gamma_2 \circ \phi$, and then $\gamma_2(\phi(X_1)) = \phi(X_1)$ and $\mathrm{supp}((\gamma_2)_{|\phi(X_1)}) = \phi(\mathrm{supp}(\gamma))$. Given another $\gamma_3 \in \Gamma_2$ with $\phi \circ \gamma = \gamma_3 \circ \phi$ we have

$$(\gamma_2)_{|\phi(X_1)} = (\gamma_3)_{|\phi(X_1)} \in \mathrm{Homeo}_c(\phi(X_1)).$$

So we can define $f_\phi(\gamma) = \gamma'$ to be the homeomorphism γ' on X_2 given by

$$(\gamma')_{|\phi(X_1)} = (\gamma_2)_{|\phi(X_1)} \quad \text{and} \quad (\gamma')_{|X_2 \setminus \phi(X_1)} = \mathrm{id}_{X_2 \setminus \phi(X_1)}.$$

The homeomorphism γ' belongs to Γ_2 because Γ_2 is locally closed. It is straightforward to check that f_ϕ is an injective group homomorphism, and also that $\mathrm{supp}(f_\phi(\gamma)) = \phi(\mathrm{supp}(\gamma))$ for every $\gamma \in \Gamma_1$. If ϕ is a spatial isomorphism, then f_ϕ is a group isomorphism and f_ϕ satisfies $f_\phi(\gamma) = \phi \circ \gamma \circ \phi^{-1}$ for each $\gamma \in \Gamma_1$.

Remark A.5.7. Viewing the functor Germ as a “free” functor turning a space-group pair into an effective ample Hausdorff groupoid (in the “most efficient” way), one could ask for a “forgetful” functor in the opposite direction. Proposition A.4.6 suggests that this functor should be

$$[\![-]\!]\colon \mathbf{Gpoid} \to \mathbf{SpatG} \quad \text{assigning} \quad \mathcal{G} \mapsto \left([\![\mathcal{G}]\!], \mathcal{G}^{(0)}\right).$$

The natural choice of mapping on the morphisms is for an étale homomorphism $\Phi\colon \mathcal{G} \to \mathcal{H}$ to let

$$[\![\Phi]\!] \coloneqq \Phi^{(0)}\colon \left([\![\mathcal{G}]\!], \mathcal{G}^{(0)}\right) \to \left([\![\mathcal{H}]\!], \mathcal{H}^{(0)}\right),$$

i.e. restriction to the unit space. Unfortunately, this *fails* to be a morphism in **SpatG** in general. For injective étale homomorphisms though, the restriction to the unit spaces does yield an injective spatial morphism.

A.6 Spatial realization theorems

In this section we shall study reconstruction of topological spaces from subgroups of their homeomorphism group in the sense of the following definition.

Definition A.6.1. A class K of space-group pairs is called *faithful* if every group isomorphism $\Phi\colon \Gamma_1 \to \Gamma_2$, where $(\Gamma_1, X_1), (\Gamma_2, X_2) \in K$, is *spatially implemented*, that is, there is a homeomorphism $\phi\colon X_1 \to X_2$ such that $\Phi(\gamma) = \phi \circ \gamma \circ \phi^{-1}$ for every $\gamma \in \Gamma_1$.

We stress the fact that the isomorphisms Φ considered in the preceding definition are, a priori, *abstract* group isomorphisms. They only "see" the algebraic structure of the Γ_i's, not the actions on the underlying spaces. We may rephrase faithfulness to saying that "every group isomorphism is a spatial isomorphism". In relation to the previous section we obtain the following from Corollary A.5.5.

Proposition A.6.2. *Suppose K is a faithful class of space-group pairs from* **SpatG**. *If (Γ_1, X_1) and (Γ_2, X_2) belong to K and Γ_1 is isomorphic to Γ_2 as abstract groups, then the groupoids of germs* $\mathrm{Germ}(\Gamma_1, X_1)$ *and* $\mathrm{Germ}(\Gamma_2, X_2)$ *are isomorphic as topological groupoids.*

In conjunction with Proposition A.4.10 this will allow us to deduce that in many cases, the topological full group of an ample groupoid, considered as an abstract group, is a complete invariant for the isomorphism class of the groupoid. This will be done in the next section. The rest of this section will be devoted to proving two faithfulness results. The first one is a straightforward extension of Matui's spatial realization result [Mat15b, Theorem 3.5] to our locally compact setting (Theorem A.6.6). This result will not only apply to the topological full group, but also to any subgroup containing the commutator. The second result we present (Theorem A.6.19) has more relaxed assumptions on the "mixing properties" of the action, but we were not able to apply it to the commutator subgroup of the topological full group.

A.6.1 The class K^F

We now present the main definition from [Mat15b, Section 3], adapted to our setting.

Definition A.6.3. We define the class K^F to consist of all space-group pairs $(\Gamma, X) \in$ **SpatG** which satisfy the following conditions:

(F1) For any $x \in X$ and any clopen neighbourhood $A \subset X$ of x, there exists an involution $\alpha \in \Gamma$ such that $x \in \operatorname{supp}(\alpha)$ and $\operatorname{supp}(\alpha) \subseteq A$.

(F2) For any involution $\alpha \in \Gamma \setminus \{1\}$, and any non-empty clopen set $A \subseteq \operatorname{supp}(\alpha)$, there exists a $\beta \in \Gamma \setminus \{1\}$ such that $\operatorname{supp}(\beta) \subseteq A \cup \alpha(A)$ and $\alpha(x) = \beta(x)$ for every $x \in \operatorname{supp}(\beta)$.

(F3) For any non-empty clopen set $A \subseteq X$, there exists an $\alpha \in \Gamma$ such that $\operatorname{supp}(\alpha) \subseteq A$ and $\alpha^2 \neq 1$.

Remark A.6.4. In [Mat15b, Definition 3.1] there is also a condition (F0), stipulating that the support of any involution should be clopen. This is already implicit in the definition above, since all supports of elements in Γ are assumed to be compact and open. We also remark that Definition A.6.3 does not impose any countability restrictions on the space X. However, condition (F1) (and also (F3)) implies that the space X cannot have isolated points.

Remark A.6.5. The notation K^F to denote a class of space-group pairs is in the same style as Rubin uses in his paper [Rub89]. Elsewhere in the literature, in particular [Mat15b] and [GPS99], groups Γ with $(\Gamma, X) \in K^F$ are called groups of *class F* (and X is assumed to be a (compact) Cantor space).

We now state a simple extension of Matui's Spatial Realization Theorem.

Theorem A.6.6 (cf. [Mat15b, Theorem 3.5])**.** *The class K^F is faithful.*

Proof. By closely inspecting the proof of [Mat15b, Theorem 3.5] and the three lemmas preceding it, one finds that the compactness of the spaces is not needed until the proof of [Mat15b, Theorem 3.5] itself. The lemmas preceding it are completely algebraic. Furthermore, the compactness is used only to guarantee that a certain intersection of supports become non-empty—by appealing to the finite intersection property. However, since all supports in our setting are already compact (by assumption) the conclusion that the intersection is non-empty still holds. The second countability is never needed. Therefore, Matui's proof remains valid. □

Remark A.6.7. We remark that Matui's proof of [Mat15b, Theorem 3.5] is similar to the approach used by Bezuglyi and Medynets in [BM08, Section 5], wherein the authors prove a precursor of Matui's Isomorphism Theorem for Cantor minimal systems. Both of these build on Fremlin's book [Fre04, Section 384].

A.6.2 The class K^{LCC}

We now turn to obtaining the second spatial realization result, by providing another faithful class of space group-pairs. In comparison with K^F, we'll impose more restrictions on the spaces (second countability—resulting in locally compact Cantor spaces), but the conditions on the actions will be less "localized" in some sense. We will of course still need the groups Γ to be very "rich" in order to recover the action on the space X, but we do not focus solely on involutive group elements, as was the case for K^F.

Some of the (many) results from Rubins remarkable paper [Rub89] will form the backbone of this spatial realization result. In that paper, Rubin exhibits the faithfulness of several general classes of space-group pairs. However, many of the classes considered there required quite different proofs. Arguably, the most commonly cited result from [Rub89] in our context is [Rub89, Corollary 3.5], but this spatial realization result is not strong enough to prove Theorem A.1.2. We essentially end up reprove Rubin's result on zero-dimensional spaces, but we obtain a slightly different statement. Also, our proof is a bit more straightforward (since we aim for a less general setting; namely perfect unit spaces of ample groupoids).

Reconstructing the Boolean algebra $\mathcal{R}(X)$

The main theorem from Section 2 of Rubin's paper (given below in Theorem A.6.11) gives general conditions for when the abstract isomorphism class of a group $\Gamma \leq \mathrm{Homeo}(X)$ determines the Boolean algebra $\mathcal{R}(X)$, and the induced action by Γ on it. We may view Γ as a subgroup of $\mathrm{Aut}(\mathcal{R}(X))$ by taking images of regular open sets in $\mathcal{R}(X)$ under the homeomorphisms in Γ. In [Rub89, Section 3], Rubin defines several classes of space-group pairs and proves, in a case-by-case manner, that the space X and the action by Γ on it, can be recovered from the induced action of Γ on $\mathcal{R}(X)$. Let us begin with some terminology (adapted from [Rub89]).

Definition A.6.8. Let (Γ, X) be a space-group pair.

1. We say that (Γ, X) is *locally moving* if for every non-empty open subset $A \subseteq X$ there exists $\gamma \in \Gamma \setminus \{1\}$ with $\mathrm{supp}(\gamma) \subseteq A$.

2. An open set $B \subseteq X$ is called *flexible* if for every pair of open subsets $C_1, C_2 \subseteq B$, if there exists $\gamma \in \Gamma$ such that $\gamma(C_1) \cap C_2 \neq \emptyset$, then there exists $\tau \in \Gamma$ such that $\tau(C_1) \cap C_2 \neq \emptyset$ and $\mathrm{supp}(\tau) \subseteq B$.

3. We say that (Γ, X) is *locally flexible* if every non-empty open subset A contains a non-empty open flexible subset $B \subseteq A$.

Remark A.6.9. Note that if (Γ, X) is locally moving, then the space X has no isolated points.

Remark A.6.10. In [Rub89], "locally moving" goes by the name "regionally disrigid", whilst the former terminology is from a later paper of Rubin [Rub96].

We now state a special case of the main result from [Rub89, Section 2].

Theorem A.6.11 (cf. [Rub89, Theorem 0.2, Theorem 2.14(a)])**.** *Let* (Γ_1, X_1) *and* (Γ_2, X_2) *be in* **SpatG**. *Assume that they are both locally moving and locally flexible. If* $\Phi\colon \Gamma_1 \to \Gamma_2$ *is an isomorphism of groups, then there exists a Boolean isomorphism* $\psi\colon \mathcal{R}(X_1) \to \mathcal{R}(X_2)$ *such that* $\psi(g(A)) = \Phi(g)(\psi(A))$ *for each* $A \in \mathcal{R}(X_1)$ *and* $g \in \Gamma_1$.

If we think of g and $\Phi(g)$ as elements in $\mathrm{Aut}(\mathcal{R}(X_1))$ and $\mathrm{Aut}(\mathcal{R}(X_2))$ respectively, then we can rewrite the conclusion in the preceding theorem as

$$\Phi(g) = \psi \circ g \circ \psi^{-1}.$$

Thus, Theorem A.6.11 says that any group isomorphism between Γ_1 and Γ_2 is actually induced by an isomorphism of the Boolean algebras of regular open sets of the underlying spaces.

Remark A.6.12. We remark that what Rubin proves in [Rub89, Theorem 2.14(a)] is a somewhat stronger statement than the one we gave above. First of all, the spaces need really only be Hausdorff (and perfect). Rubin shows that if (Γ, X) is locally moving and locally flexible, then starting with Γ alone, one can canonically reconstruct the Boolean algebra $\mathcal{R}(X)$ (up to isomorphism) using only group theoretic constructions. Moreover, one obtains a natural action by Γ on this Boolean algebra which is conjugate to the action by Γ on $\mathcal{R}(X)$. The strategy of the proof is to model a regular set $A \in \mathcal{R}(X)$ by its rigid stabilizer $Q(A) \coloneqq \{\gamma \in \Gamma \mid \mathrm{supp}(\gamma) \subseteq A\}$, and then to describe the Boolean operations in $\mathcal{R}(X)$ in group theoretic terms, in terms of the subgroups $Q(A)$. Finally one shows that there are enough regular sets A for which subgroups of the form $Q(A)$ can be detected inside Γ in order to generate the whole of $\mathcal{R}(X)$.

Reconstructing the space X

We now turn to reconstructing X (and the original action by Γ) from its Boolean algebra of regular sets. The strategy is to first impose conditions making it possible to detect clopenness. And then characterize the compact open sets among the clopen sets, which in turn allow us to recover X from Stone duality.

Definition A.6.13. Let (Γ, X) be a space-group pair. A clopen set $A \subseteq X$ is said to be *recognizable by* Γ if it satisfies:

1. For every $\gamma \in \Gamma$ with $\gamma(A) = A$ the homeomorphism τ given by

$$\tau(x) = \begin{cases} \gamma(x) & x \in A, \\ x & \text{otherwise}, \end{cases}$$

belongs to Γ.

2. For every $\gamma \in \Gamma$ with $\gamma(A) \cap A = \emptyset$ the involution α given by

$$\alpha(x) = \begin{cases} \gamma(x) & x \in A, \\ \gamma^{-1}(x) & x \in \gamma(A), \\ x & \text{otherwise}, \end{cases}$$

belongs to Γ.

We shall see later that in our setting of topological full groups, all clopen subsets of the unit space are recognizable. And whenever this is the case, it is possible to characterize when a regular set is closed (i.e. clopen) using the following Boolean algebra notion.

Definition A.6.14. Let (Γ, X) be a space-group pair, and let $A \in \mathcal{R}(X)$ be a regular open set. We say that A is *weakly clopen* if for every $\gamma \in \Gamma$ satisfying $\gamma(A \cap \gamma(A)) = A \cap \gamma(A)$, there exists an element $\rho \in \Gamma$ such that

1. $\rho(B) = \gamma(B)$ for each $B \in \mathcal{R}(X)$ with $B \subseteq A \cap \gamma(A)$,
2. $\rho(B) = B$ for each $B \in \mathcal{R}(X)$ with $B \subseteq\, \sim (A \cap \gamma(A))$.

Note that the notion of being weakly clopen is formulated solely in terms of the action by Γ on the Boolean algebra $\mathcal{R}(X)$. And as the next result shows—under suitable hypotheses—being weakly clopen is the same as being clopen.

Lemma A.6.15. *Let* $(\Gamma, X) \in$ **SpatG**. *Assume that every clopen subset of X is recognizable by Γ, and that the Γ-orbit of each point contains at least 3 points. Then a regular open set $A \in \mathcal{R}(X)$ is clopen if and only if both A and $\sim A$ are weakly clopen.*

Proof. This is a special case of [Rub89, Lemma 3.45], where the dense subset R is taken to be all of $\mathcal{R}(X)$. The assumptions 3.V.1 (a), (b), (c) and 3.V.2 (a), (b) preceding [Rub89, Lemma 3.45] follow from those above. In particular, what Rubin calls "recognizably clopen" coincides with (2) in Definition A.6.13, and "strongly recognizably clopen" is slightly weaker than (1) in Definition A.6.13 (together with (2)). □

In order to invoke Stone duality for Boolean spaces we need to recover the generalized Boolean algebra of compact open sets. The previous lemma gives us the clopen sets, and from these we obtain the compact open ones as follows.

Lemma A.6.16. *Let X be a second countable Boolean space. Then X is compact if and only if* $\mathrm{CO}(X)$ *is countable.*

Proof. If X is compact, then $\mathrm{CO}(X) = \mathrm{CK}(X)$, and any second countable space has countably many compact open subsets.

Suppose X is non-compact. Let $\{K_n\}_{n=1}^{\infty}$ be a countable basis for X consisting of compact open sets. Now form the compact open sets $C_k = \cup_{n=1}^{k} K_n$. As X is not compact, we must have $C_k \neq X$ for each k. Also, $C_k \subseteq C_{k+1}$ and they cover X. By passing to a subsequence, if necessary, we may assume that $C_k \subsetneq C_{k+1}$ for each k. Finally, let $D_k = C_{k+1} \setminus C_k$. Then the D_k's are pairwise disjoint non-empty compact open sets. We claim that for each subset S of the natural numbers, the set $\cup_{k \in S} D_k$ is clopen. And then we have produced uncountably many distinct clopen sets. The claim follows from the fact that for each C_m, the intersection $C_m \cap (\cup_{k \in S} D_k)$ is a finite intersection, hence closed, and that the C_m's cover X. □

Corollary A.6.17. *Let X be a second countable Boolean space, and let $A \in \mathrm{CO}(X)$ be a clopen set. Then A is compact if and only if the set $\{B \in \mathrm{CO}(X) \mid B \subseteq A\}$ is countable.*

Proof. The set $\{B \in \mathrm{CO}(X) \mid B \subseteq A\}$ coincides with $\mathrm{CO}(A)$ when viewing A as a subspace of X. The result now follows from Lemma A.6.16. □

This shows that in the generalized Boolean algebra $\mathrm{CO}(X)$ compactness is characterized by having only countably many elements below. We are now ready to define the class K^{LCC} and give the second spatial realization result of this section.

Definition A.6.18. We define the class K^{LCC} to consist of all space-group pairs (Γ, X) in **SpatG** which satisfy the following conditions:

(K1) X is a locally compact Cantor space.

(K2) (Γ, X) is locally moving.

(K3) (Γ, X) is locally flexible.

(K4) Every clopen subset of X is recognizable by Γ.

(K5) The Γ-orbit of each point contains at least 3 points.

Theorem A.6.19 (cf. [Rub89, Theorem 3.50(a)])**.** *The class K^{LCC} is faithful.*

Proof. Suppose we have two space-group pairs (Γ_1, X_1), $(\Gamma_2, X_2) \in K^{LCC}$ and a group isomorphism $\Phi\colon \Gamma_1 \to \Gamma_2$. Invoking Theorem A.6.11 yields an isomorphism of Boolean algebras $\psi\colon \mathcal{R}(X_1) \to \mathcal{R}(X_2)$ such that $\psi(g(A)) = \Phi(g)(\psi(U))$ for each $A \in \mathcal{R}(X_1)$ and $g \in \Gamma_1$. We first argue that $\psi(\mathrm{CO}(X_1)) = \mathrm{CO}(X_2)$, and then that $\psi(\mathrm{CK}(X_1)) = \mathrm{CK}(X_2)$.

First of all, note that both $\mathrm{CO}(X_i)$ and $\mathrm{CK}(X_i)$ are invariant under Γ_i $(i = 1, 2)$. Lemma A.6.15 characterizes clopenness of regular sets in X_i solely in terms of the (induced) actions by Γ_i on $\mathcal{R}(X_i)$. Since ψ is an equivariant Boolean algebra isomorphism, it follows that $\psi(\mathrm{CO}(X_1)) = \mathrm{CO}(X_2)$. Next, Corollary A.6.17 characterizes compactness of a clopen set in terms of a countability condition in the generalized Boolean algebra $\mathrm{CO}(X_i)$. Clearly, this is then also preserved by ψ. Consequently, ψ restricts to an equivariant isomorphism of the generalized Boolean algebras $\mathrm{CK}(X_1)$ and $\mathrm{CK}(X_2)$.

By applying the Stone functor to the generalized Boolean algebra isomorphism

$$\psi\colon \mathrm{CK}(X_1) \to \mathrm{CK}(X_2)$$

we obtain a homeomorphism

$$\mathbb{S}(\psi)\colon \mathbb{S}(\mathrm{CK}(X_2)) \to \mathbb{S}(\mathrm{CK}(X_1))$$

of the spaces of ultrafilters. The induced actions by the groups Γ_i on $\mathbb{S}(\mathrm{CK}(X_i))$ is given by $g \cdot \alpha = \{g(K) \mid K \in \alpha\}$ for an ultrafilter $\alpha \in \mathbb{S}(\mathrm{CK}(X_i))$. Finally, let $\phi\colon X_1 \to X_2$ be the homeomorphism given by the composition

$$X_1 \xrightarrow{\Omega_{X_1}} \mathbb{S}(\mathrm{CK}(X_1)) \xrightarrow{\mathbb{S}(\psi)^{-1}} \mathbb{S}(\mathrm{CK}(X_2)) \xrightarrow{\Omega_{X_2}^{-1}} X_2$$

where Ω_{X_i} is the canonical homeomorphism mapping a point to its compact open neighbourhood ultrafilter. It is now easy to check that the original group isomorphism Φ is spatially implemented by ϕ, i.e. that $\Phi(g) = \phi \circ g \circ \phi^{-1}$ for each $g \in \Gamma_1$. $\square$

Remark A.6.20. As mentioned in the introduction, Medynets has obtained a spatial realization result for full groups of group actions on the Cantor space [Med11]. The arguments therein also apply to the topological full group, and could be adapted to the topological full group of the ample groupoids over locally compact Cantor spaces considered here. And then in turned be used to prove Theorem A.1.2 instead of using Theorem A.6.19. Medynets' starting point is a Boolean algebra reconstruction result of Fremlin [Fre04, Theorem 384D]. This result is very similar to Rubin's Boolean algebra reconstruction result; Theorem A.6.11. Rubin requires the space-group pair to be locally moving and locally flexible, whereas Fremlin requires it to be locally moving in terms of involutions. Yet they both apply to the topological full group, since it is both (globally) flexible and has enough involutions to witness locally moving. Medynets then goes on to characterize the clopen sets among the regular open sets in an algebraic way and use this to show that the Boolean algebra isomorphism must preserve the Boolean subalgebra of clopen subsets and in turn give rise to a spatial isomorphism via Stone duality. This is exactly the same approach as we use here, via Rubin, but Medynets' characterization of the clopens [Med11, Lemma 2.5] looks (at least on the surface) a bit different from the one we give here in Lemma A.6.15. Finally, we remark that Medynets' arguments does not seem to apply to the commutator subgroup either (see Remark A.7.11).

A.7 Isomorphism theorems for ample groupoids

In this section we shall apply the spatial realization results of the previous section to (subgroups of) the topological full group. As corollaries we are able to reconstruct certain ample groupoids from their topological full group. The two faithful classes considered in the previous section allows us to lift an abstract group isomorphism of (subgroups of) the topological full groups to a spatial one. This in turn yields an isomorphism of the associated groupoids of germs (Corollary A.5.5). In order to conclude that the groupoids themselves are isomorphic we need, by Proposition A.4.10 and Corollary A.4.13, to assume that the subgroups in question cover the groupoids. As we saw in Lemma A.4.9, if every $\mathcal{G}$-orbit has length at least 2, or respectively 3, then $[\![\mathcal{G}]\!]$, or respectively any Γ with $D([\![\mathcal{G}]\!]) \leq \Gamma \leq [\![\mathcal{G}]\!]$, covers $\mathcal{G}$.

We first extract an isomorphism theorem from the faithfulness of the class K^F. For a general ample groupoid the only general condition we know to imply that $\left([\![\mathcal{G}]\!], \mathcal{G}^{(0)}\right)$ belong to K^F is minimality. So for general groupoids we obtain only a straightforward minor extension of Theorems 3.9 and 3.10 from [Mat15b] in Theorem A.7.2 below. However, for the class of graph groupoids we will see in Section A.10 that we can weaken minimality quite a lot and still have the topological full group (and its commutator) in K^F, and thereby obtain a significantly more general result within the class of graph groupoids. It would therefore be interesting to find general conditions on a general ample groupoid $\mathcal{G}$, weaker than minimality, ensuring that $\left([\![\mathcal{G}]\!], \mathcal{G}^{(0)}\right)$ and $\left(\mathsf{D}([\![\mathcal{G}]\!]), \mathcal{G}^{(0)}\right)$ belong to K^F.

Proposition A.7.1 (cf. [Mat15b, Proposition 3.6])**.** *Let $\mathcal{G}$ be an effective ample Hausdorff groupoid whose unit space has no isolated points. If $\mathcal{G}$ is minimal and Γ is any subgroup of $[\![\mathcal{G}]\!]$ containing $\mathsf{D}([\![\mathcal{G}]\!])$, then $\left(\Gamma, \mathcal{G}^{(0)}\right) \in K^F$.*

Proof. The proof of [Mat15b, Proposition 3.6] goes through verbatim in this slightly more general setting. The proof makes heavy use of the minimality of $\mathcal{G}$ and combine this with Lemma A.3.8 to find the desired elements in $\mathsf{D}([\![\mathcal{G}]\!])$. □

Theorem A.7.2. *Let $\mathcal{G}_1, \mathcal{G}_2$ be effective ample minimal Hausdorff groupoids whose unit spaces have no isolated points. Suppose Γ_1, Γ_2 are subgroups satisfying $\mathsf{D}([\![\mathcal{G}_i]\!]) \leq \Gamma_i \leq [\![\mathcal{G}_i]\!]$. If $\Gamma_1 \cong \Gamma_2$ as abstract groups, then $\mathcal{G}_1 \cong \mathcal{G}_2$ as topological groupoids. In particular, the following are equivalent:*

1. *$\mathcal{G}_1 \cong \mathcal{G}_2$ as topological groupoids.*
2. *$[\![\mathcal{G}_1]\!] \cong [\![\mathcal{G}_2]\!]$ as abstract groups.*
3. *$\mathsf{D}([\![\mathcal{G}_1]\!]) \cong \mathsf{D}([\![\mathcal{G}_2]\!])$ as abstract groups.*

Proof. Clearly every $\mathcal{G}_i$-orbit is infinite, for $i = 1, 2$. Thus the result follows from combining Proposition A.7.1, Theorem A.6.6, Proposition A.6.2, Lemma A.4.9 and Proposition A.4.10. □

Remark A.7.3. For transformation groupoids arising from minimal $\mathbb{Z}$-actions on locally compact Cantor spaces, a variant of this result appears in [Mat02, Theorem 4.13 (vi)]. See also Remark A.3.3.

Remark A.7.4. In [Mat15b, Theorem 3.10] the kernel of the so-called *index map* also appears (as $[\![\mathcal{G}]\!]_0$). We could equally well have included it in Theorem A.7.2 since it is a distinguished subgroup lying between $[\![\mathcal{G}]\!]$ and $\mathsf{D}([\![\mathcal{G}]\!])$.

Our next goal is to analyze the conditions in the definition of the class K^{LCC}, when the space-group pair under consideration is the topological full group and the unit space of an ample groupoid. Unfortunately, the commutator subgroup $\mathsf{D}([\![\mathcal{G}]\!])$ does not seem to belong to K^{LCC}, which is why we only consider $[\![\mathcal{G}]\!]$ itself (see Remark A.7.11 below). We begin by showing that the groupoid-orbits coincide with the orbits of the action by the topological full group on the unit space.

Lemma A.7.5. *Let $\mathcal{G}$ be an effective ample groupoid and let $x \in \mathcal{G}^{(0)}$. Then*

$$\mathrm{Orb}_{\mathcal{G}}(x) = \mathrm{Orb}_{[\![\mathcal{G}]\!] \curvearrowright \mathcal{G}^{(0)}}(x).$$

Proof. From the definition of the topological full group it is obvious that the groupoid orbit $\mathrm{Orb}_{\mathcal{G}}(x)$ contains the orbit of the action $\mathrm{Orb}_{[\![\mathcal{G}]\!] \curvearrowright \mathcal{G}^{(0)}}(x)$. For the reverse inclusion, suppose $y \in \mathrm{Orb}_{\mathcal{G}}(x)$ is distinct from x, and let $\gamma \in \mathcal{G}$ be an arrow from x to y. Applying Lemma A.3.9 to γ we obtain an element $\pi_U \in [\![\mathcal{G}]\!]$ with $\pi_U(x) = y$. Thus $y \in \mathrm{Orb}_{[\![\mathcal{G}]\!] \curvearrowright \mathcal{G}^{(0)}}(x)$. □

In other words, when the space group pair is $([\![\mathcal{G}]\!], \mathcal{G}^{(0)})$ condition (K5) of Definition A.6.18 is equivalent to saying that every $\mathcal{G}$-orbit has length at least 3 (which, incidentally, implies that $[\![\mathcal{G}]\!]$ covers $\mathcal{G}$). Next we show that conditions (K3) and (K4) of Definition A.6.18 are always satisfied for topological full groups. In fact, $([\![\mathcal{G}]\!], \mathcal{G}^{(0)})$ is even "globally flexible".

Lemma A.7.6. *Let $\mathcal{G}$ be an effective ample groupoid. Then every open subset of $\mathcal{G}^{(0)}$ is flexible with respect to $[\![\mathcal{G}]\!]$. In particular, $([\![\mathcal{G}]\!], \mathcal{G}^{(0)})$ is locally flexible.*

Proof. Let A be a non-empty open subset of $\mathcal{G}^{(0)}$, and let B_1, B_2 be two open subsets of A. We may assume that these are disjoint, for otherwise the identity homeomorphism trivially witnesses flexibility. Suppose $\pi_U \in [\![\mathcal{G}]\!]$ satisfies $\pi_U(B_1) \cap B_2 \neq \emptyset$. Then there is a $g \in U$ with $s(g) \in B_1$ and $r(g) \in B_2$. Lemma A.3.9 applied to g and $B_1 \sqcup B_2$ produces an element $\pi_V \in [\![\mathcal{G}]\!]$ with $\operatorname{supp}(\pi_V) \subseteq B_1 \sqcup B_2 \subseteq A$ and $\pi_V(B_1) \cap B_2 \neq \emptyset$. This shows that A is flexible. □

Lemma A.7.7. *Let $\mathcal{G}$ be an effective ample groupoid. Then every clopen subset of $\mathcal{G}^{(0)}$ is recognizable by $[\![\mathcal{G}]\!]$.*

Proof. Let $A \subseteq \mathcal{G}^{(0)}$ be clopen.

(1) Suppose $\pi_U \in [\![\mathcal{G}]\!]$ satisfies $\pi_U(A) = A$. Then $V = s_{|U}^{-1}(A) \subseteq U$ is a clopen bisection with $s(V) = r(V) = A$. Then $\tilde{V}$ as in Lemma A.3.7 is a full bisection with $\operatorname{supp}(\pi_{\tilde{V}}) \subseteq \operatorname{supp}(\pi_U)$, hence $\pi_{\tilde{V}} \in [\![\mathcal{G}]\!]$. The homeomorphism $\pi_{\tilde{V}}$ is the one from condition (1) of Definition A.6.13.

(2) Suppose now that $\pi_U \in [\![\mathcal{G}]\!]$ satisfies $\pi_U(A) \cap A = \emptyset$. Again we set $V = s_{|U}^{-1}(A)$. Then $s(V) \cap r(V) = A \cap \pi_U(A) = \emptyset$. The full bisection $\hat{V}$ as in Lemma A.3.8 also has compact support since $\operatorname{supp}(\pi_{\hat{V}}) \subseteq \operatorname{supp}(\pi_U)$, and so $\pi_{\hat{V}} \in [\![\mathcal{G}]\!]$. The involution $\pi_{\hat{V}}$ is the one from condition (2) of Definition A.6.13. □

It remains to consider condition (K2) of Definition A.6.18. Inspired by [Med11, Proposition 2.2] we introduce the the notion of a *non-wandering* groupoid, in order to characterize when $\left([\![\mathcal{G}]\!], \mathcal{G}^{(0)}\right)$ is locally moving in terms of the groupoid $\mathcal{G}$.

Definition A.7.8. Let $\mathcal{G}$ be an ample groupoid. A subset $A \subseteq \mathcal{G}^{(0)}$ is called *wandering* if $|A \cap \operatorname{Orb}_{\mathcal{G}}(x)| = 1$ for all $x \in A$. We say that $\mathcal{G}$ is *non-wandering* if $\mathcal{G}^{(0)}$ has no non-empty clopen wandering subsets.

In words, a non-wandering groupoid is one in which every clopen subset of the unit space meets some orbit at least twice. This may be viewed as a "mixing condition" which is far weaker than minimality. For if $\mathcal{G}$ is minimal, then in particular $|A \cap \operatorname{Orb}_{\mathcal{G}}(x)|$ is infinite (from being dense) for each clopen neighbourhood A of x.

Proposition A.7.9. *Let $\mathcal{G}$ be an effective ample Hausdorff groupoid. Then the following are equivalent:*

1. *The space-group pair $\left([\![\mathcal{G}]\!], \mathcal{G}^{(0)}\right)$ is locally moving.*
2. *The groupoid $\mathcal{G}$ is non-wandering.*

Proof. Let A be a non-empty clopen subset of $\mathcal{G}^{(0)}$. We will prove that A meets some $\mathcal{G}$-orbit twice (i.e. A is not wandering) if and only if there is an element $\pi_U \in [\![\mathcal{G}]\!] \setminus \{1\}$ with $\mathrm{supp}(\pi_U) \subseteq A$. If $\emptyset \neq \mathrm{supp}(\pi_U) \subseteq A$, then, since both sets are clopen, there is an $x \in A$ with $x \neq \pi_U(x) \in A$. In other words, $|A \cap \mathrm{Orb}_{\mathcal{G}}(x)| \geq 2$. Conversely, if $|A \cap \mathrm{Orb}_{\mathcal{G}}(x)| \geq 2$ holds for some $x \in A$, then there is a $g \in \mathcal{G} \setminus \mathcal{G}'$ such that $s(g)$ and $r(g)$ both belong to A. Now Lemma A.3.9 gives us a nontrivial group element in $[\![\mathcal{G}]\!]$ supported on A. As the clopens form a base for the topology on $\mathcal{G}^{(0)}$ we are done. □

Putting it all together, we arrive at the second main result of this section.

Theorem A.7.10. *Let $\mathcal{G}_1, \mathcal{G}_2$ be effective ample Hausdorff groupoids over locally compact Cantor spaces. Suppose that, for $i = 1, 2$, $\mathcal{G}_i$ is non-wandering and that each $\mathcal{G}_i$-orbit has length at least* 3*. Then any isomorphism between $[\![\mathcal{G}_1]\!]$ and $[\![\mathcal{G}_2]\!]$ is spatial. In particular, the following are equivalent:*

1. *$\mathcal{G}_1 \cong \mathcal{G}_2$ as topological groupoids.*
2. *$[\![\mathcal{G}_1]\!] \cong [\![\mathcal{G}_2]\!]$ as abstract groups.*

Remark A.7.11. It would be desirable to also obtain a spatial realization result for the commutator subgroup $\mathsf{D}([\![\mathcal{G}]\!])$ in terms of the class K^{LCC}. Unfortunately we were not able to show that $\mathsf{D}([\![\mathcal{G}]\!])$ satisfies condition (K4). This is also the reason why the arguments of [Med11] do not apply to the commutator subgroup either. However, it might be that Theorem A.7.10 holds for the commutator subgroups as well.

As mentioned above, non-wandering is a much weaker "mixing property" than minimality. Below we include two other "mixing properties" that lie between non-wandering and minimality.

Definition A.7.12 (see [Nek19, page 8])**.** An ample groupoid $\mathcal{G}$ is called *locally minimal* if there exists a basis for $\mathcal{G}^{(0)}$ consisting of clopen sets A such that $\mathcal{G}_A$ is minimal.

Definition A.7.13. An ample groupoid $\mathcal{G}$ is called *densely minimal* if for every non-empty open subset A of $\mathcal{G}^{(0)}$ there exists a non-empty clopen subset $B \subseteq A$ such that $\mathcal{G}_B$ is minimal.

We clearly have the following implications for an ample groupoid:

$$\text{minimal} \implies \text{locally minimal} \implies \text{densely minimal} \implies \text{non-wandering}.$$

We will give examples of densely minimal groupoids which are not minimal in the next section (Examples A.9.6 and A.9.7), as well as non-wandering groupoids which are not densely minimal (Remark A.10.8).

A.8 Graph groupoids

The rest of the paper will be focused on graph groupoids. This section recalls the relevant terminology for graphs and their associated groupoids (as they appear in the literature on graph algebras). We also record the characterizations of many properties of a graph groupoid in terms of the graph. This is fairly standard and may also be found in many other papers, e.g. [BCW17], [KPRR97].

A.8.1 Graph terminology

By a *graph* we shall always mean a directed graph, i.e. a quadruple $E = (E^0, E^1, r, s)$, where E^0, E^1 are (non-empty) sets and $r, s\colon E^1 \to E^0$ are maps. The elements in E^0 and E^1 are called *vertices* and *edges*, respectively, while the maps r and s are called the *range* and *source* map, respectively.[5] We say that E is *finite* if E^0 and E^1 both are finite sets, and similarly that E is *countable* if E^0 and E^1 are countable.

A *path* in E is a sequence of edges $\mu = e_1 e_2 \ldots e_n$ such that $r(e_i) = s(e_{i+1})$ for $1 \leq i \leq n-1$. The *length* of μ is $|\mu| := n$. The set of paths of length n is denoted E^n. The vertices, E^0, are considered trivial paths of length 0. The set of all finite paths is denoted $E^* := \bigcup_{n=0}^{\infty} E^n$. The range and source maps extend to E^* by setting $r(\mu) := r(e_n)$ and $s(\mu) := s(e_1)$. For $v \in E^0$, we set $s(v) = r(v) = v$. Given another path $\lambda = f_1 \ldots f_m$ with $s(\lambda) = r(\mu)$ we denote the concatenated path $e_1 \ldots e_n f_1 \ldots f_m$ by $\mu\lambda$. In particular, we set $s(\mu)\mu = \mu = \mu\, r(\mu)$ for each $\mu \in E^*$. Given two paths $\mu, \mu' \in E^*$ we write $\mu < \mu'$ if there exists a path λ with $|\lambda| \geq 1$ such that $\mu' = \mu\lambda$. Writing $\mu \leq \mu'$ allows for $\mu = \mu'$. We say that μ and μ' are *disjoint* if $\mu \nleq \mu'$ and $\mu' \nleq \mu$, i.e. neither is a subpath of the other.

A *cycle* is a nontrivial path μ (i.e. $|\mu| \geq 1$) with $r(\mu) = s(\mu)$, and we say that μ is *based* at $s(\mu)$. We also say that the vertex $s(\mu)$ *supports* the cycle μ. By a *loop* we mean a cycle of length 1. Beware that some authors use the term loop to denote what we here call cycles. When μ is a cycle and $k \in \mathbb{N}$, μ^k denotes the cycle $\mu\mu \ldots \mu$, where μ is repeated k times. A cycle $\mu = e_1 \ldots e_n$ is called a *return path* if $r(e_i) \neq r(\mu)$ for all $i < n$. This simply means that μ does not pass through $s(\mu)$ multiple times. An *exit* for a path $\mu = e_1 \ldots e_n$ is an edge e such that $s(e) = s(e_i)$ and $e \neq e_i$ for some $1 \leq i \leq n$.

[5]Although the notation collides with the range and source maps in a groupoid, both conventions are well established. In the sequel it will always be clear from context whether we mean the source/range of an edge in a graph or of an element in a groupoid.

For $v, w \in E^0$ we set

$$\begin{aligned} vE^n &:= \{\mu \in E^n \mid s(\mu) = v\}, \\ E^n w &:= \{\mu \in E^n \mid r(\mu) = w\}, \\ vE^n w &:= vE^n \cap E^n w. \end{aligned}$$

A vertex $v \in E^0$ is called a *sink* if $vE^1 = \emptyset$, and a *source* if $E^1 v = \emptyset$. Further, v is called an *infinite emitter* if vE^1 is an infinite set. The set of *regular* vertices is $E^0_{\mathrm{reg}} := \{v \in E^0 \mid 0 < |vE^1| < \infty\}$, and the set of *singular* vertices is $E^0_{\mathrm{sing}} := E^0 \setminus E^0_{\mathrm{reg}}$. In other words, sinks and infinite emitters are singular vertices, while all other vertices are regular. We equip the vertex set E^0 with a preorder $\geq$ by definining $v \geq w$ iff $vE^* w \neq \emptyset$, i.e. there is a path from v to w. The graph E is called *strongly connected* if for each pair of vertices $v, w \in E^0$ we have $v \geq w$.

To close this subsection we describe three exit conditions on graphs that appear frequently in the graph algebra literature. They will play a central role in what follows. A graph E is said to satisfy *Condition (L)* if every cycle in E has an exit. The graph E satisfies *Condition (K)* if for every vertex $v \in E^0$, either there is no return path based at v or there are at least two distinct return paths based at v. We say that E satisfies *Condition (I)* if for every vertex $v \in E^0$, there exists a vertex $w \in E^0$ supporting at least two distinct return paths and $v \geq w$. These conditions first appeared in [KPR98], [KPRR97] and [CK80], respectively. In general, Condition (K) and (I) both imply (L), while (K) and (I) are not comparable. For graphs with finitely many vertices and no sinks, Condition (I) is equivalent to Condition (L).

A.8.2 The boundary path space

An *infinite path* in a graph E is an infinite sequence of edges $x = e_1 e_2 e_3 \ldots$ such that $r(e_i) = s(e_{i+1})$ for all $i \in \mathbb{N}$. We define $s(x) := s(e_1)$ and $|x| := \infty$. The set of all infinite paths in E is denoted E^∞. Given a finite path $\mu = f_1 \ldots f_n$ and an infinite path $x = e_1 e_2 e_3 \ldots \in E^\infty$ such that $r(\mu) = s(x)$ we denote the infinite path $f_1 \ldots f_n e_1 e_2 e_3 \ldots$ by μx. For natural numbers $m < n$, we set $x_{[m,n]} := e_m e_{m+1} \ldots e_n$, and we denote the infinite path $e_m e_{m+1} e_{m+2} \ldots$ by $x_{[m,\infty)}$. Given a cycle $\lambda \in E^*$ we denote the infinite path $\lambda\lambda\lambda \ldots$ by λ^∞. An infinite path of the form $\mu\lambda^\infty$, where λ is a cycle with $s(\lambda) = r(\mu)$, is called *eventually periodic*. An infinite path $e_1 e_2 \ldots \in E^\infty$ is *wandering* if the set $\{i \in \mathbb{N} \mid s(e_i) = v\}$ is finite for each $v \in E^0$. Note that there are no wandering infinite paths in a graph with finitely many vertices. We call a wandering infinite path $e_1 e_2 \ldots \in E^\infty$ a *semi-tail*[6] if $s(e_i)E^1 = \{e_i\}$ for each $i \in \mathbb{N}$. The graph E is called *cofinal* if for every vertex

[6]By comparison, a *tail* is a wandering path with $s(e_i)E^1 = \{e_i\} = E^1 r(e_i)$ for all i, see [BPRS00].

$v \in E^0$ and for every infinite path $e_1 e_2 \ldots \in E^\infty$, there exists $n \in \mathbb{N}$ such that $v \geq s(e_n)$.

The *boundary path space* of E is

$$\partial E := E^\infty \cup \{\mu \in E^* \mid r(\mu) \in E^0_{\text{sing}}\},$$

whose topology will be specified shortly. Note that if $v \in E^0$ is a singular vertex, then v belongs to ∂E. For any vertex $v \in E^0$ we define $v\partial E := \{x \in \partial E \mid s(x) = v\}$ and similarly $vE^\infty := \{x \in E^\infty \mid s(x) = v\}$. The *cylinder set* of a finite path $\mu \in E^*$ is $Z(\mu) := \{\mu x \mid x \in r(\mu)\partial E\}$. Given a finite subset $F \subseteq r(\mu)E^1$, we define the "punctured" cylinder set $Z(\mu \setminus F) := Z(\mu) \setminus \left(\bigcup_{e \in F} Z(\mu e)\right)$. Note that two finite paths are disjoint if and only if their cylinder sets are disjoint sets. A basis for the topology on the boundary path space ∂E is given by $\left\{Z(\mu \setminus F) \mid \mu \in E^*, F \subseteq_{\text{finite}} r(\mu)E^1\right\}$ ([Web14]). Each basic set $Z(\mu \setminus F)$ is compact open and these separate points, so ∂E is a Boolean space. Moreover, each open set in ∂E is a disjoint union of basic sets $Z(\mu \setminus F)$ ([BCW17, Lemma 2.1]). The boundary path space ∂E is second countable exactly when E is countable, and it is compact if and only if E^0 is finite. When it comes to (topologically) isolated points, these are classified as follows.

Proposition A.8.1 ([CW18, Proposition 3.1])**.** *Let E be a graph.*

1. *If $v \in E^0$ is a sink, then any finite path $\mu \in E^*$ with $r(\mu) = v$ is an isolated point in ∂E.*
2. *If $x = \mu\lambda^\infty \in E^\infty$ is eventually periodic, then x is an isolated point if and only if the cycle λ has no exit.*
3. *If $x = e_1 e_2 \ldots \in E^\infty$ is wandering, then x is an isolated point if and only if for some $n \in \mathbb{N}$, $e_n e_{n+1} \ldots$ is a semi-tail.*

These are the only isolated points in ∂E.

For each $n \in \mathbb{N}$ we set

$$\partial E^{\geq n} := \{x \in \partial E \mid |x| \geq n\} \quad \text{and} \quad \partial E^n := \{x \in \partial E \mid |x| = n\}.$$

Each of the sets $\partial E^{\geq n}$ is an open subset of ∂E. The *shift map* on E is the map $\sigma_E \colon \partial E^{\geq 1} \to \partial E$ given by $\sigma_E(e_1 e_2 e_3 \ldots) = e_2 e_3 e_4 \ldots$ for $e_1 e_2 e_3 \ldots \in \partial E^{\geq 2}$ and $\sigma_E(e) = r(e)$ for $e \in \partial E^1$. In other words, $\sigma_E(x) = x_{[2,\infty)}$. We have that

$$\sigma_E\left(\partial E^{\geq 1}\right) = \{x \in \partial E \mid E^1 s(x) \neq \emptyset\} = \partial E \setminus \left(\cup_{E^1 v \neq \emptyset} Z(v)\right),$$

which is an open set, and we see that σ_E is surjective if and only if E has no sources. We let $\sigma_E^n \colon \partial E^{\geq n} \to \partial E$ be the n-fold composition of σ_E with itself, and we set $\sigma_E^0 = \mathrm{id}_{\partial E}$. Each σ_E^n is then a local homeomorphism between open subsets of ∂E. Note that an infinite path $x \in E^\infty$ is eventually periodic if and only if there are distinct numbers $m, n \in \mathbb{N}_0$ such that $\sigma_E^m(x) = \sigma_E^n(x)$.

A.8.3 Graph groupoids and their properties

The *graph groupoid* of a graph E is the (generalized) Renault-Deaconu groupoid ([Dea95], [Ren00]) of the dynamical system $(\partial E, \sigma_E)$, that is

$$\mathcal{G}_E := \{(x, m-n, y) \mid m, n \in \mathbb{N}_0, x \in \partial E^{\geq m}, y \in \partial E^{\geq n}, \sigma_E^m(x) = \sigma_E^n(y)\}$$

as a set. The groupoid structure is given by $(x, k, y) \cdot (y, l, z) := (x, k+l, z)$ (and undefined otherwise) and $(x, k, y)^{-1} := (y, -k, x)$. The unit space is

$$\mathcal{G}_E^{(0)} = \{(x, 0, x) \mid x \in \partial E\},$$

which we will identify with ∂E via $(x, 0, x) \leftrightarrow x$. Then $s(x, k, y) = y$ and $r(x, k, y) = x$. We equip $\mathcal{G}_E$ with the topology generated by the basic sets

$$Z(U, m, n, V) := \{(x, m-n, y) \mid x \in U, y \in V, \sigma_E^m(x) = \sigma_E^n(y)\},$$

where $U \subseteq \partial E^{\geq m}$ and $V \subseteq \partial E^{\geq n}$ are open sets such that $\left(\sigma_E^m\right)_{|U}$ and $\left(\sigma_E^n\right)_{|V}$ are injective, and $\sigma_E^m(U) = \sigma_E^n(V)$. This makes $\mathcal{G}_E$ an étale groupoid, and the identification of the unit space with ∂E is compatible with the topology on ∂E. Note however, that this topology on $\mathcal{G}_E$ is finer than the relative topology induced from $\partial E \times \mathbb{Z} \times \partial E$. According to [BCW17, page 394] the family

$$\left\{Z(U, |\mu|, |\lambda|, V) \mid \sigma_E^{|\mu|}(U) = \sigma_E^{|\lambda|}(V)\right\}, \tag{A.8.1}$$

parametrized over all $\mu, \lambda \in E^*$ with $r(\mu) = r(\lambda)$, $U \subseteq Z(\mu)$ and $V \subseteq Z(\lambda)$ compact open, is also a basis for the same topology. Each set $Z(U, |\mu|, |\lambda|, V)$ is a compact open bisection, and they separate the elements of $\mathcal{G}_E$, so $\mathcal{G}_E$ is an ample Hausdorff groupoid. The family in (A.8.1) is countable precisely when E is countable, and so the graph groupoid $\mathcal{G}_E$ is second countable exactly when E is countable.

For a boundary path $x \in \partial E$, the isotropy group of $(x, 0, x) \in \mathcal{G}_E^{(0)}$ is nontrivial if and only if x is eventually periodic (and infinite). For graph groupoids, effectiveness coincides with topological principality (even without assuming second countability), which in turn is well known to coincide with the graph satisfying Condition (L).

Proposition A.8.2 (cf. [BCW17, Proposition 2.3]). *Let E be a graph. The following are equivalent:*

1. *The groupoid $\mathcal{G}_E$ is effective.*
2. *The groupoid $\mathcal{G}_E$ is topologically principal.*

3. *The set of infinite paths which are not eventually periodic form a dense subset of the boundary path space ∂E.*

4. *The graph E satisfies Condition (L).*

Proof. The equivalence of (2), (3) and (4) is proved in [BCW17, Proposition 2.3] for countable graphs, but the proof does not rely on the countability of the graph. As it is always the case that (2) implies (1) (Remark A.2.2), we only have to show that (1) implies (4). To that end, assume that E does not satisfy Condition (L). Then there is a cycle $\lambda \in E^*$ with no exit, and λ^∞ is an isolated point in ∂E. But then the bisection

$$Z\left(Z\left(\lambda^2\right), |\lambda|^2, |\lambda|, Z(\lambda)\right) = \{(\lambda^\infty, |\lambda|, \lambda^\infty)\}$$

is an open subset of $\mathcal{G}_E \setminus \mathcal{G}_E^{(0)}$, and hence $\mathcal{G}_E$ is not effective. □

We end this subsection by giving a characterization of minimality for graph groupoids. Let E be a graph. Two infinite paths $x, y \in E^\infty$ are called *tail equivalent* if there are natural numbers k, l such that $x_{[k,\infty)} = y_{[l,\infty)}$. Similarly, two finite paths $\mu, \lambda \in E^*$ are *tail equivalent* if $r(\mu) = r(\lambda)$. From the definition of $\mathcal{G}_E$ one sees that two boundary paths belong to the same $\mathcal{G}_E$-orbit if and only if they are tail equivalent. By combining [BCFS14, Theorem 5.1] with [DT05, Corollary 2.15] we arrive at the following result—of which we provide a self-contained proof.

Proposition A.8.3. *Let E be a graph. Then the following are equivalent:*

1. *The groupoid $\mathcal{G}_E$ is minimal.*

2. *The graph E is cofinal, and for each $v \in E^0$ and $w \in E^0_{sing}$, we have $v \geq w$.*

Proof. If E has a sink $w \in E^0_{\mathrm{sing}}$, then one immediately deduces from both statements that E cannot have any other singular vertices, nor any infinite paths. Consequently

$$\partial E = \mathrm{Orb}_{\mathcal{G}_E}(w) = \{\mu \in E^* \mid r(\mu) = w\},$$

and this entails that $\mathcal{G}_E$ is a discrete transitive groupoid. Now, *(1)* and *(2)* are clearly equivalent in this case.

For the remainder of the proof we assume that E has no sinks. Assume that *(2)* holds. Let $x \in E^\infty$ and let $\lambda \in E^*$. By cofinality, there is a path λ' from $r(\lambda)$ to $s(x_n)$ for some $n \in \mathbb{N}$. The infinite path $\lambda\lambda' x_n x_{n+1} \dots$ then belongs to both $Z(\lambda)$ and $\mathrm{Orb}_{\mathcal{G}_E}(x)$. Hence the latter is dense in ∂E (since every open set contains a cylinder set when there are no sinks). Next, suppose $\mu \in \partial E \cap E^*$ with $r(\mu)$ an infinite emitter. By assumption there is a path λ'' from $r(\lambda)$ to $r(\mu)$, and then $\lambda\lambda'' \in Z(\lambda) \cap \mathrm{Orb}_{\mathcal{G}_E}(\mu)$. This shows that $\mathcal{G}_E$ is minimal.

Assume now that $\mathcal{G}_E$ is minimal. To see that E is cofinal, let $x \in E^\infty$ and $v \in E^0$ be given. By minimality there is a $y \in E^\infty$ tail equivalent to x such that $y \in Z(v)$. This implies that v can reach x. As for the second part of *(2)*, let $v \in E^0$ and $w \in E^0_{\text{sing}}$ be given. Again by minimality there is a $\lambda \in E^* \cap Z(v)$ tail equivalent to w, but this is just a path from v to w, so $v \geq w$. □

Remark A.8.4. The notion of cofinality is slightly weaker than strong connectedness. But for finite graphs with no sinks and no sources, cofinality coincides with strong connectedness. In fact, this is also true for infinite graphs which additionaly have no *semi-heads* (the direction-reversed notion of a semi-tail). We also remark that for cofinal graphs, Condition (L) is equivalent to Condition (K).

A.9 Topological full groups of graph groupoids

We are now going to describe the elements in the topological full group of a graph groupoid. Some examples will be given at the end of the section. We begin by specifying yet another (equivalent) basis for $\mathcal{G}_E$, which in turn will allow us to describe bisections combinatorially in terms of the graph.

For two finite paths $\mu, \lambda \in E^*$ with $r(\mu) = r(\lambda) = v$ we define

$$Z(\mu, \lambda) \coloneqq Z\left(Z(\mu), |\mu|, |\lambda|, Z(\lambda)\right).$$

More generally, given a finite subset $F \subseteq vE^1$ as well, we define

$$Z(\mu, F, \lambda) \coloneqq Z\left(Z(\mu \setminus F), |\mu|, |\lambda|, Z(\lambda \setminus F)\right).$$

Each $Z(\mu, F, \lambda)$ is a compact open bisection in $\mathcal{G}_E$, and we will see shortly that they also form a basis. Observe that if $v \in E^0_{\text{reg}}$, then $Z(\mu, F, \lambda) = \bigsqcup_{e \in vE^1 \setminus F} Z(\mu e, \lambda e)$, and that this is a finite union.

Lemma A.9.1. *Let E be a graph. Let $\mu, \mu', \lambda, \lambda' \in E^*$ with $r(\mu) = r(\lambda) = v$, $r(\mu') = r(\lambda') = v'$ and let $F \subseteq_{\text{finite}} vE^1$, $F' \subseteq_{\text{finite}} v'E^1$. Then the intersection $Z(\mu, F, \lambda) \bigcap Z(\mu', F', \lambda')$ equals either*

1. *$\emptyset$, or*
2. *$Z(\mu, F, \lambda)$, or*
3. *$Z(\mu', F', \lambda')$, or*
4. *$Z(\mu, F \cup F', \lambda)$, in which case $\mu = \mu'$, $\lambda = \lambda'$ and*

$$Z(\mu, F, \lambda) \cup Z(\mu', F', \lambda') = Z(\mu, F \cap F', \lambda).$$

Proof. Suppose that $Z(\mu, F, \lambda) \cap Z(\mu', F', \lambda') \neq \emptyset$. Then we must have that $|\mu| - |\lambda| = |\mu'| - |\lambda'|$, $Z(\mu \setminus F) \cap Z(\mu' \setminus F') \neq \emptyset$ and $Z(\lambda \setminus F) \cap Z(\lambda' \setminus F') \neq \emptyset$. Since

$$Z(\mu \setminus F) \bigcap Z(\mu' \setminus F') = \begin{cases} Z(\mu \setminus (F \cup F')) & \text{if } \mu = \mu', \\ Z(\mu \setminus F) & \text{if } \mu' < \mu \text{ and } \mu_{|\mu'|+1} \notin F', \\ Z(\mu' \setminus F') & \text{if } \mu < \mu' \text{ and } \mu'_{|\mu|+1} \notin F, \\ \emptyset & \text{otherwise,} \end{cases}$$

we may suppose without loss of generality that $\mu \leq \mu'$. The equality $|\mu| - |\lambda| = |\mu'| - |\lambda'|$ then forces $\lambda \leq \lambda'$ as well. If $\mu = \mu'$, then we must also have $\lambda = \lambda'$ and it is easy to see that *(4)* holds in this case.

Next, suppose $\mu < \mu'$, which forces $\lambda < \lambda'$. As the intersections above are non-empty we have $Z(\mu' \setminus F') \subseteq Z(\mu \setminus F)$ and $Z(\lambda' \setminus F') \subseteq Z(\lambda \setminus F)$. It follows from this that $Z(\mu', F', \lambda') \subseteq Z(\mu, F, \lambda)$, and we are done. □

Lemma A.9.2. *The family*

$$\left\{ Z(\mu, F, \lambda) \mid \mu, \lambda \in E^*, r(\mu) = r(\lambda), F \subseteq_{\mathit{finite}} r(\mu)E^1 \right\}$$

forms a basis for the topology on $\mathcal{G}_E$.

Proof. It suffices to write each basic set $Z(U, |\mu|, |\lambda|, V)$, where $\mu, \lambda \in E^*$ with $r(\mu) = r(\lambda)$, $U \subseteq Z(\mu)$, $V \subseteq Z(\lambda)$ compact open and $\sigma_E^{|\mu|}(U) = \sigma_E^{|\lambda|}(V)$, as a union of $Z(\mu', F', \lambda')$'s. Given such a basic set $Z(U, |\mu|, |\lambda|, V)$, we can then write

$$\sigma_E^{|\mu|}(U) = \sigma_E^{|\lambda|}(V) = \bigsqcup_{i=1}^{k} Z(\eta_i \setminus F_i),$$

for some $\eta_i \in E^*$, $F_i \subseteq_{\text{finite}} r(\eta_i)E^1$, since the former two are compact open subsets of ∂E. It follows that

$$U = \bigsqcup_{i=1}^{k} Z(\mu\eta_i \setminus F_i) \quad \text{and} \quad V = \bigsqcup_{i=1}^{k} Z(\lambda\eta_i \setminus F_i).$$

Hence

$$Z(U, |\mu|, |\lambda|, V) = \bigsqcup_{i=1}^{k} Z(\mu\eta_i, F_i, \lambda\eta_i).$$

□

Using the basis above, we may concretely describe the bisections in $\mathcal{G}_E$ as follows.

Lemma A.9.3. *Let E be graph, and let $U \subseteq \mathcal{G}_E$ be a compact open bisection with $s(U) = r(U)$. Then U is of the form*

$$U = \bigsqcup_{i=1}^{k} Z(\mu_i, F_i, \lambda_i),$$

where $\mu_i, \lambda_i \in E^*$ with $r(\mu_i) = r(\lambda_i)$, $F_i \subseteq_{\text{finite}} r(\mu_i)E^1$ and

$$s(U) = \bigsqcup_{i=1}^{k} Z(\lambda_i \setminus F_i) = \bigsqcup_{i=1}^{k} Z(\mu_i \setminus F_i).$$

Proof. Since U is a compact open subset of $\mathcal{G}_E$ we may, by the preceding two lemmas, write U as a finite disjoint union of basic sets as $U = \bigsqcup_{i=1}^{k} Z(\mu_i, F_i, \lambda_i)$. As r and s are injective on U they preserve disjoint unions, so we have

$$\begin{aligned} s(U) = s\left(\bigsqcup_{i=1}^{k} Z(\mu_i, F_i, \lambda_i)\right) &= \bigsqcup_{i=1}^{k} s\left(Z(\mu_i, F_i, \lambda_i)\right) = \bigsqcup_{i=1}^{k} Z(\lambda_i \setminus F_i) \\ = r(U) = r\left(\bigsqcup_{i=1}^{k} Z(\mu_i, F_i, \lambda_i)\right) &= \bigsqcup_{i=1}^{k} r\left(Z(\mu_i, F_i, \lambda_i)\right) = \bigsqcup_{i=1}^{k} Z(\mu_i \setminus F_i). \end{aligned}$$

□

In conjunction with Lemma A.3.7 we get that the elements in $[\![\mathcal{G}_E]\!]$ for an effective graph groupoid (i.e. the graph E satisfying Condition (L)) may be described as follows, in terms of E.

Proposition A.9.4. *Let E be a graph satisfying Condition (L). If $\pi_U \in [\![\mathcal{G}_E]\!]$, then the full bisection U can be written as*

$$U = \left(\bigsqcup_{i=1}^{k} Z(\mu_i, F_i, \lambda_i)\right) \bigsqcup \left(\partial E \setminus \operatorname{supp}(\pi_U)\right),$$

where $\mu_i, \lambda_i \in E^*$ with $r(\mu_i) = r(\lambda_i)$, $F_i \subsetneq_{\text{finite}} r(\mu_i)E^1$ and

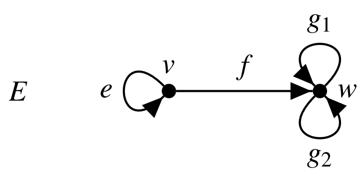
$$\operatorname{supp}(\pi_U) = \bigsqcup_{i=1}^{k} Z(\lambda_i \setminus F_i) = \bigsqcup_{i=1}^{k} Z(\mu_i \setminus F_i).$$

Moreover, $\mu_1, \dots \mu_k$ are pairwise disjoint, $\lambda_1, \dots \lambda_k$ are pairwise disjoint, and $\mu_i \neq \lambda_i$ for each i. The associated homeomorphism $\pi_U \colon \partial E \to \partial E$ is given by $x = \lambda_i z \longmapsto \mu_i z$ for $x \in Z(\lambda_i \setminus F_i)$ and $x \longmapsto x$ otherwise.

Remark A.9.5. The elements in $[\![\mathcal{G}_E]\!]$ may alternatively be described in more dynamical terms via the orbits by the shift map. From [BCW17, Proposition 3.3] one deduces that a homeomorphism $\alpha \in \mathrm{Homeo}(\partial E)$ belongs to $[\![\mathcal{G}_E]\!]$ if and only if there are compactly supported continuous functions $m, n \colon \partial E \to \mathbb{N}_0$ such that $\sigma_E^{m(x)}(\alpha(x)) = \sigma_E^{n(x)}(x)$. This parallels Matui's definition for locally compact Cantor minimal systems mentioned in Remark A.3.3, and Matsumoto's definition for one-sided shifts of finite type in [Mat10].

Having completely described the topological full group of a graph groupoid, we provide an example to show that the assumption on the orbits in Lemma A.4.9 is not a necessary condition. On the other hand, we also give an example to show that the statement is generally false without said assumption. These examples also provide examples of densely minimal groupoids which are not minimal.

Example A.9.6. Consider the following graph:

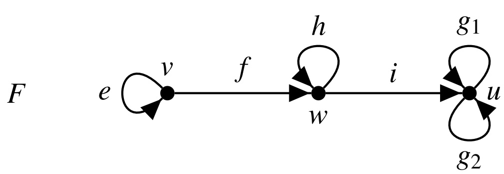


The graph E satisfies condition (L), but is not cofinal, so $\mathcal{G}_E$ is effective, but not minimal. We claim that $\mathcal{G}_E$ is densely minimal. To see this, note that any non-empty open subset of E^∞ must contain a cylinder set $Z(\mu)$ where $r(\mu) = w$. And the restriction of $\mathcal{G}_E$ to $Z(\mu)$ is minimal. As for covering, observe that the orbit of $e^\infty \in \partial E$ has length 1, i.e. $\mathrm{Orb}_{\mathcal{G}_E}(e^\infty) = \{e^\infty\}$. However, the topological full group $[\![\mathcal{G}_E]\!]$ still covers $\mathcal{G}_E$. For instance, the isotropy element $(e^\infty, 1, e^\infty)$ belongs to the full bisection

$$U = Z(e^2, e) \bigsqcup Z(ef, g_1 g_2) \bigsqcup Z(g_1, g_1 g_1) \bigsqcup Z(f, f) \bigsqcup Z(g_2, g_2).$$

Similar full bisections can be found for (e^∞, k, e^∞) where k is any integer.

Example A.9.7. Consider the following graph:



As in the previous example, $e^\infty \in \partial F$ has a singleton orbit. However, in contrast to the previous example, $[\![\mathcal{G}_F]\!]$ does *not* cover $\mathcal{G}_F$. For there is no full bisection containing the element $(e^\infty, 1, e^\infty)$. Indeed, if U is a bisection containing

$(e^\infty, 1, e^\infty)$, then U must contain a bisection of the form $Z(e^{k+1}, e^k)$. Now since $Z(e^k) = Z(e^{k+1}) \sqcup Z(e^k f)$, it will be impossible to enlarge U to a full bisection. By adding disjoint $Z(\mu, \lambda)$'s to write U as in Proposition A.9.4 one will always have one more μ ending in w than λ's. See also [BS19, Example 3.5] for the same phenomenon in a restricted transformation groupoid.

A.10 Isomorphism theorems for graph groupoids

In this section we will pursue specialized isomorphism theorems for the class of graph groupoids. We will determine exactly when the topological full group of a graph groupoid belongs to K^F, and the conditions for this turn out to be weaker than minimality. We will also determine, in terms of the graph, exactly when it belongs to K^{LCC}. From this we obtain two isomorphism theorems for graph groupoids.

A.10.1 The class K^F

We are now going to give necessary and sufficient conditions for when $(\Gamma, \partial E)$ belongs to K^F—for a graph E, and a subgroup $\Gamma \leq [\![\mathcal{G}_E]\!]$ containing $\mathsf{D}([\![\mathcal{G}_E]\!])$. Of the three conditions (F1), (F2) and (F3) in Definition A.6.3, (F1) is the "hardest" one to satisfy. This is essentially because we need to produce elements in the topological full group with support containing a given point $x \in \partial E$, but also contained in a given neighbourhood of x. In the other two conditions we can get away with simply choosing a "small enough" support. As both conditions (F1) and (F3) fails in the presence of isolated points, we will only consider graphs that have no sinks, no semi-tails, and satisfy Condition (L). We will see that Condition (K) will be necessary for (F1) to hold for periodic[7] points. The two conditions in Definition A.10.1 below are needed to ensure that (F1) holds for wandering infinite paths, and for finite boundary paths, respectively. For notational convenience we make the following ad-hoc definitions.

Definition A.10.1. Let E be a graph.

1. We say that E satisfies *Condition (W)* if for every wandering infinite path $x \in E^\infty$, we have $|s(x)E^* r(x_n)| \geq 2$ for some $n \in \mathbb{N}$.

2. We say that E satisfies *Condition (∞)* if for every infinite emitter $v \in E^0$, the set $\{e \in vE^1 \mid r(e) \geq v\}$ is infinite.

[7] That is, $x = \lambda^\infty$ for some cycle $\lambda \in E^*$.

The three conditions (K), (W) and (∞) can be thought of as strengthenings of each of the three criteria for the boundary path space ∂E being perfect (Proposition A.8.1). The latter three criteria can informally be described as "can exit", whereas the former three can be described as "can exit *and* return". More specifically, Condition (L) means that one can exit every cycle, whereas Condition (K) means that one can also return back to the same cycle. That E has no semi-tails means that every wandering infinite path has an exit, and Condition (W) means that one can return to the same infinite path again. That E has no sinks can be reformulated as saying that every singular vertex has an exit (and hence infinitely many), whereas Condition (∞) says that one can also return to the same vertex (from infinitely many of these exits). Note that Condition (∞) holds in particular if every infinite emitter supports infinitely many loops. Also note that if $|s(x)E^*r(x_n)| \geq 2$ for some $n \in \mathbb{N}$, then the same is true for each $m \geq n$. We now make two elementary observations needed in the proof of the next proposition.

Lemma A.10.2. *Let E be a graph.*

1. *If $\mu \in E^*$ is a cycle and E satisfies Condtion (K), then there are infinitely many cycles $\lambda_1, \lambda_2, \ldots$ based at $s(\mu)$ such that $\mu, \lambda_1, \lambda_2, \ldots$ are mutually disjoint.*

2. *If $x = x_1 x_2 \ldots \in E^\infty$ is a wandering infinite path and E satisfies Condition (W), then for each $N \in \mathbb{N}$ there is an $n \in \mathbb{N}$ and paths $\mu_1, \ldots, \mu_N$ from $s(x)$ to $r(x_n)$ such that $x_{[1,n]}, \mu_1, \ldots, \mu_N$ are mutually disjoint.*

Proof. For the first part, let τ_1 and τ_2 be two distinct return paths based at $s(\mu)$. As distinct return paths are disjoint we must have that μ is disjoint from one of them, say τ_1. And then the cycles $\mu, \tau_1\mu, \tau_1^2\mu, \tau_1^3\mu, \ldots$ are all disjoint.

We argue inductively for the second part. From Condition (W) we can find a number $n_1 \in \mathbb{N}$ with $|s(x)E^*r(x_{n_1})| \geq 2$. Set $v := r(x_{n_1})$. Since x is wandering we can let $m_1 \geq n_1$ be the largest index such that $r(x_{m_1}) = v$. So that x never returns to v after the m_1'th edge. Let μ be a path in $s(x)E^*r(x_{m_1})$ distinct from $x_{[1,m_1]}$. If $x_{[1,m_1]}$ and μ are disjoint, then we are done with the base case. If not, then either $x_{[1,m_1]} < \mu$ or $x_{[1,m_1]} > \mu$. In the former case we have that $\mu = x_{[1,m_1]}\rho$, where ρ is a cycle based at v. As x does not return to v again we must have that $x_{[m_1+1,m_1+|\rho|]} \neq \rho$, and then $x_{[1,m_1+|\rho|]}$ is disjoint from the path

$$\mu_1 := \mu x_{[m_1+1,m_1+|\rho|]} = x_{[1,m_1]}\rho x_{[m_1+1,m_1+|\rho|]}.$$

If the latter is the case, then $\mu = x_{[1,k]}$ for some $k < m_1$ and $x_{[k+1,m_1]}$ is a cycle. And then the previous argument applied to $x_{[1,m_1]}$ and $\mu' = x_{[1,k]}x_{[k+1,m_1]}x_{[k+1,m_1]}$ shows that the statement holds for $N = 1$.

Applying the above to the tail $x_{[m_1+1,\infty)}$, which is again a wandering infinite path, we get an index $m_2 > m_1$ and a path μ_2 from $r(x_{m_1})$ to $r(x_{m_2})$ disjoint from $x_{[m_1+1,m_2]}$. By concatenating $x_{[1,m_1]}$ and μ_1 with $x_{[m_1+1,m_2]}$ and μ_2 we obtain three paths from $s(x)$ to $r(x_{m_2})$ that are mutually disjoint, as well as disjoint from $x_{[1,m_2]}$. By continuing in this manner one sees that the result is true for all $N \in \mathbb{N}$. $\square$

Proposition A.10.3. *Let E be a graph with no sinks and let $\Gamma \leq [\![\mathcal{G}_E]\!]$ be a subgroup containing $\mathsf{D}([\![\mathcal{G}_E]\!])$. Then $(\Gamma, \partial E)$ belongs to K^F if and only if E satisfies Condition (K), (W) and (∞).*

Proof. This proof is inspired by Matui's proof of [Mat15b, Proposition 3.6]. We employ similar tricks in this more concrete, yet non-minimal context. We will first show that (F2) and (F3) holds when E satisfies Condition (K) and (W). And then we will show, in turn, that all three conditions are necessary and sufficient for (F1) to hold at certain boundary paths.

Suppose E satisfies Condition (K) and (W) (in addition to having no sinks). We verify (F3) first. Let A be any non-empty clopen subset of ∂E. There is then a path η such that $Z(\eta) \subseteq A$. Now there are two possibilities. Either $r(\eta)$ connects to a cycle, or $r(\eta)E^\infty$ consists only of wandering paths. In the first case we may assume, by extending η, that $r(\eta)$ supports a cycle. By Lemma A.10.2 we can find three disjoint cycles $\lambda_1, \lambda_2, \lambda_3$ based at $r(\eta)$. Define $V = Z(\eta\lambda_1, \eta\lambda_2)$, $W = Z(\eta\lambda_2, \eta\lambda_3)$ and $\alpha = [\pi_{\hat{V}}, \pi_{\hat{W}}]$ (as in Lemma A.3.8). Then $\alpha \in \Gamma \setminus \{1\}$ has order 3 and $\mathrm{supp}(\alpha) \subseteq Z(\eta) \subseteq A$. In the case that $r(\eta)E^\infty$ consists only of wandering paths we may find, again by Lemma A.10.2, three disjoint paths $\lambda_1, \lambda_2, \lambda_3$ starting at $r(\eta)$, and such that $r(\lambda_1) = r(\lambda_2) = r(\lambda_3)$. Defining α as above shows that (F3) holds in this case as well.

Next we verify (F2). To that end, let $\alpha \in \Gamma \setminus \{1\}$ with $\alpha^2 = 1$ and $\emptyset \subsetneq A \subseteq \mathrm{supp}(\alpha)$ a clopen be given. We have $\alpha = \pi_U$ with

$$U = \left(\bigsqcup_{i=1}^{k} Z(\mu_i, F_i, \lambda_i) \right) \bigsqcup \left(\mathcal{G}_E^{(0)} \setminus \mathrm{supp}(\pi_U) \right)$$

as in Proposition A.9.4. Arguing as above, we can find a finite path η and an index $1 \leq j \leq k$ such that $Z(\eta) \subseteq A \cap Z(\lambda_j \setminus F_j)$, as well as two disjoint paths τ_1, τ_2 satisfying $s(\tau_1) = s(\tau_2) = r(\eta)$ and $r(\tau_1) = r(\tau_2)$. As $\lambda_j \leq \eta$ we can write $\eta = \lambda_j \rho$ for some path ρ whose first edge does not belong to F_j. Define the bisections

$$V = Z(\lambda_j\rho\tau_1, \lambda_j\rho\tau_2) \bigsqcup Z(\mu_j\rho\tau_1, \mu_j\rho\tau_2)$$

and

$$W = Z(\mu_j\rho\tau_1, \lambda_j\rho\tau_1).$$

Put $\beta = [\pi_{\hat{V}}, \pi_{\hat{W}}]$. As α is an involution we have that

$$\alpha(\lambda_j z) = \mu_j z \quad \text{for} \quad \lambda_j z \in Z(\lambda_j \setminus F_j)$$

and vice versa. Now observe that $\beta \in \Gamma$,

$$\begin{aligned}\operatorname{supp}(\beta) &= Z(\lambda_j \rho \tau_1) \sqcup Z(\lambda_j \rho \tau_2) \sqcup Z(\mu_j \rho \tau_1) \sqcup Z(\mu_j \rho \tau_2) \\ &\subseteq Z(\eta) \cup \alpha(Z(\eta)) \subseteq A \cup \alpha(A),\end{aligned}$$

and that α and β agree on $\operatorname{supp}(\beta)$ (as they both swap the inital paths λ_j and μ_j).

Assume now that E merely has no sinks, no semi-tails and satisfies Condition (L). We will show that (F1) holds if and only if E satisfies Condition (K), (W) and (∞). Let $x \in \partial E$ and A a clopen neighbourhood of x be given. We further divide this part into three cases, each one yielding the necessity of one of the three conditions.

Condtion (K): Assume that the graph E satisfies Condtion (K), and suppose that $x = x_1 x_2 \ldots \in E^\infty$ is an infinite non-wandering path. For $m \in \mathbb{N}$ large enough, we have that $Z(x_{[1,m]}) \subseteq A$. As x contains infinitely many cycles we can, by possibly choosing m larger, assume that $x_{[m+1,n]}$ is a return path at $r(x_m)$ for some $n > m$. Using Lemma A.10.2 we can find three mutually disjoint cycles $\lambda_1, \lambda_2, \lambda_3$ based at $r(x_m)$ which are also disjoint from $x_{[m+1,n]}$. Let $\mu_i = x_{[1,m]}\lambda_i$ for $i = 1, 2, 3$ and let $\mu_4 = x_{[1,n]}$. Define

$$V = Z(\mu_1, \mu_2) \bigsqcup Z(\mu_3, \mu_4)$$

and

$$W = Z(\mu_1, \mu_3).$$

Then $\alpha = [\pi_{\hat{V}}, \pi_{\hat{W}}] \in \Gamma$ satisfies $\operatorname{supp}(\alpha) = \bigsqcup_{i=1}^4 Z(\mu_i) \subseteq Z(x_{[1,m]}) \subseteq A$, $\alpha^2 = 1$ and $x \in Z(\mu_4) \subseteq \operatorname{supp}(\alpha)$ as desired.

To see that Condition (K) is necessary, suppose that E does not satisfy it. Then there is a vertex $v \in E^0$ supporting a unique return path, say τ. We may assume that τ has an exit f with $s(f) = v$. Consider $x = \tau^\infty$ and its neighbourhood $A = Z(\tau)$. We claim that (F1) fails for this pair. To see this, suppose $\pi_U \in [\![\mathcal{G}_E]\!]$ satisfies $\tau^\infty \in \operatorname{supp}(\pi_U) \subseteq Z(\tau)$. By Proposition A.9.4 we can find $Z(\mu, \lambda) \subseteq U$ with $r(\mu) = r(\lambda)$, $\mu \neq \lambda$ and $\tau^\infty \in Z(\lambda)$, which means that $\lambda \leq \tau^k$ for some $k \geq 1$. By possibly extending μ and λ we may assume that $\lambda = \tau^k$. We also have $Z(\mu) \subseteq Z(\tau)$, i.e. $\tau \leq \mu$, and $r(\mu) = r(\lambda) = v$. But since τ is the only return path based at v we must have $\mu = \tau^l$ for some $l \neq k$ as $\mu \neq \lambda$. Let $z \in r(f)\partial E$. Then $(\pi_U)^2 \left(\tau^{2k} f z\right) = \tau^{2l} f z \neq \tau^{2k} f z$, hence π_U is not an involution, and therefore $(\Gamma, \partial E)$ does not satisfy (F1).

Condtion (W): Assume E satisfies Condtion (W), and let $x = x_1 x_2 \ldots \in E^\infty$ be an infinite wandering path. Choose m large enough so that $Z(x_{[1,m]}) \subseteq A$. By

Lemma A.10.2 there is an $n \geq m$ and three paths $\lambda_1, \lambda_2, \lambda_3$ from $s(x)$ to $r(x_n)$ such that $\lambda_1, \lambda_2, \lambda_3, x_{[1,n]}$ are mutually disjoint. Setting $\mu_i = x_{[1,m]}\lambda_i$ for $i = 1, 2, 3$ and $\mu_4 = x_{[1,n]}$, and defining α in the same way as in the case of Condition (K) above gives the desired element in Γ.

To see that Condition (W) is necessary, suppose that there is an infinite wandering path $x = x_1x_2 \ldots \in E^\infty$ such that $|s(x)E^*s(x_n)| = 1$ for all $n \in \mathbb{N}$. We claim that (F1) fails for $A = Z(x_1)$. Indeed, suppose $\pi_U \in [\![\mathcal{G}_E]\!]$ satisfies $x \in \operatorname{supp}(\pi_U) \subseteq Z(x_1)$. By Proposition A.9.4 we can find $Z(\mu, \lambda) \subseteq U$ with $r(\mu) = r(\lambda)$, $\mu \neq \lambda$ and $x \in Z(\lambda)$, which implies that $\lambda = x_{[1,m]}$ for some $m \geq 1$. But as $Z(\mu) \subseteq Z(x_1)$ we have that $s(\mu) = s(x)$ and $r(\mu) = r(x_m)$. It now follows that $\mu = \lambda$ since $|s(x)E^*s(x_m)| = 1$. This contradiction shows that there is not even an element $\pi_U \in [\![\mathcal{G}_E]\!]$ such that $x \in \operatorname{supp}(\pi_U) \subseteq Z(x_1)$.

Condtion (∞): Assume E satisfies Condtion (∞), and suppose $x = x_1 \ldots x_m \in E^*$ is a finite boundary path. Then for some $F \subseteq_{\text{finite}} r(x)E^1$ we have $Z(x \setminus F) \subseteq A$. By Condition ($\infty$) we can find three distinct edges $e_1, e_2, e_3 \in r(x)E^1 \setminus F$, and three (necessarily disjoint) cycles τ_1, τ_2, τ_3 based at $r(x)$ such that $e_i \leq \tau_i$ for $i = 1, 2, 3$. Let $F' = F \sqcup \{e_1, e_2, e_3\}$. Now define

$$V = Z(x\tau_1, F', x) \bigsqcup Z(x\tau_2, F', x\tau_3)$$

and

$$W = Z(x\tau_1, F', x\tau_2).$$

Then $\alpha = [\pi_{\hat{V}}, \pi_{\hat{W}}] \in \Gamma$ satisfies

$$\operatorname{supp}(\alpha) = Z(x \setminus F') \bigsqcup_{i=1}^{3} Z(x\tau_i \setminus F') \subseteq Z(x \setminus F) \subseteq A,$$

$\alpha^2 = 1$ and $x \in Z(x \setminus F') \subseteq \operatorname{supp}(\alpha)$.

Finally, if E does not satisfy Condition (∞), then there is an infinite emitter $v \in E^0$ such that the set $F = \{e \in vE^1 \mid r(e) \geq v\}$ is finite. And then (F1) fails for $x = v$ and $A = Z(v \setminus F)$ as there is no element $\pi_U \in [\![\mathcal{G}_E]\!]$ whose support is contained in $Z(v \setminus F)$ and contains v. The argument for this is essentially the same as in the necessity of Condition (W) above. □

Remark A.10.4. From Proposition A.10.3 we see that for a graph groupoid $\mathcal{G}_E$, the topological full group $[\![\mathcal{G}_E]\!]$ (on the boundary path space ∂E) belongs to the class K^F if and only if its commutator subgroup $\mathsf{D}([\![\mathcal{G}_E]\!])$ does. This is not something one would expect in general from the definition of K^F. It is clear that (F1) and (F3) in Definition A.6.3 passes to supergroups, but (F2) need not do so. It is even more peculiar that the properties (F1), (F2) and (F3) pass down to the commutator from $[\![\mathcal{G}_E]\!]$. This phenomenon might be an artifact of the combinatorial nature of the topological full group of a graph groupoid, and so it might also hold for other concrete classes of groupoids.

A.10.2 The class K^{LCC}

Our next objective is to perform a similar analysis of when the space-group pair $([\![\mathcal{G}_E]\!], \partial E)$ for a graph E belongs to K^{LCC}. In this case the "mixing conditions" will be weaker than for K^F (see Proposition A.10.3), but we are only able to prove membership for the topological full group itself—no proper subgroups. As in the case of K^F we need to stipulate that the boundary path space ∂E has no isolated points (by condition (K1) in Definition A.6.18), but also that the graphs are countable (this also for condition (K1)). By the results in Section A.7 we only have to determine when $\mathcal{G}_E$ is non-wandering, and when all orbits have length at least 3. We shall soon see that the former property is characterized by excluding certain "tree-like" components in the graph E, which we make precise in the following defintion.

Definition A.10.5. We say that a graph E satisfies *Condition (T)* if for every vertex $v \in E^0$, there exists a vertex $w \in E^0$ such that $|vE^*w| \geq 2$.

Note that Condition (T) implies that there are no sinks and no semi-tails. It does not, however, imply Condition (L) as one can traverse a cycle twice to get two different paths. As long as there are no sinks, Condition (W) implies Condition (T). Condition (T) is a fairly weak condition; it is in fact satisfied by all graphs that have finitely many vertices and no sinks, and more generally by any graph in which every vertex connects to a cycle. The archetypical example of graphs not satisfying Condition (T) are trees, or more generally graphs containing such components.

As for when $\mathcal{G}_E$ can have orbits of length 1 or 2 one finds, by merely exhausting all possibilites, that this happens exactly if one or more of the following kinds of vertices are present in the graph E.

Definition A.10.6. Let E be a graph. We say that a vertex $v \in E^0$ is *degenerate* if it is one of the following types:

1. **"1-loop-source"**: $E^1 v = \{e\}$ where e is a loop.
2. **"1 source to 1-loop-source"**: $E^1 v = \{e, f\}$ where e is a loop and $s(f)$ is a source.
3. **"2-loop-source"**: There is another vertex $w \in E^0$ distinct from v such that $E^1 v = \{e\} = wE^1 v$ and $E^1 w = \{f\} = vE^1 w$.
4. **"Infinite source"**: vE^1 is infinite and $E^1 v$ is empty.

5. **"1 source to singular"**: v is singular and $E^1 v = \{f\}$ where $s(f)$ is a source.

6. **"Stranded"**: vE^1 and $E^1 v$ are both empty.

Proposition A.10.7. *Let E be a graph.*

1. $\mathcal{G}_E$ is non-wandering if and only if E satisfies Condition (L) and (T).

2. $|\mathrm{Orb}_{\mathcal{G}_E}(x)| \geq 3$ for all $x \in \partial E$ if and only if E has no degenerate vertices.

Proof. We prove part *(1)* first. We may assume that E has no sinks, as this is implied by both of the statements in *(1)*. Suppose E satisfies Condition (L) and (T). Let A be a non-empty clopen subset of ∂E. Then there is a path $\mu \in E^*$ such that $Z(\mu) \subseteq A$. Suppose first that $r(\mu)$ connects to a cycle. Let λ be such a cycle and let ρ be a path from $r(\mu)$ to $s(\lambda)$. We may assume that λ has an exit f with $s(f) = s(\lambda)$. Let $x \in r(f)E^\infty$. Then $\mu\rho f x$ and $\mu\rho\lambda f x$ are two distinct tail-equivalent boundary paths in A. If, on the other hand, $r(\mu)$ does not connect to a cycle, then $r(\mu)E^\infty$ consists only of wandering paths that visit each vertex at most once. Let $w \in E^0$ be a vertex such that there are two distinct paths ρ_1, ρ_2 from $r(\mu)$ to w. Again letting $x \in wE^\infty$ be arbitrary we have that $\mu\rho_1 x$ and $\mu\rho_2 x$ are two distinct tail-equivalent boundary paths in A. Hence A is not wandering.

To see that Condition (L) and (T) are both necessary, note first that if E does not satisfy Condition (L), then ∂E has an isolated point, and a clopen singleton is surely wandering. Assume instead that E fails to satisfy Condition (T), and let $v \in E^0$ be a vertex such that there is either no path or a unique path from v to any other vertex in E. We claim that the cylinder set $Z(v)$ is wandering. We first consider a finite boundary path μ beginning in v (if such a path exists). Then $r(\mu)$ is a singular vertex and

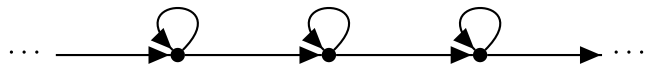
$$\mathrm{Orb}_{\mathcal{G}_E}(\mu) \cap Z(v) = \{\lambda \in E^* \mid s(\lambda) = v,\ r(\lambda) = r(\mu)\} = vE^* r(\mu) = \{\mu\},$$

as desired. Similarly, if $x \in vE^\infty$ and $y \in \mathrm{Orb}_{\mathcal{G}_E}(x) \cap Z(v)$, then there are $k, l \in \mathbb{N}$ such that $x_{[k,\infty)} = y_{[l,\infty)}$. In particular $x_{[1,k-1]}$ and $y_{[1,l-1]}$ are finite paths from v to $s(x_k) = s(y_l)$, hence these are equal and it follows then that $x = y$. Thus $\mathrm{Orb}_{\mathcal{G}_E}(x) \cap Z(v) = \{x\}$. This proves the first part of the proposition.

For part *(2)*, simply note that an orbit of length 1 can only occur if there are degenerate vertices of type (1), (4), or (6) as in Definition A.10.6 (the corresponding orbits of length 1 being $\{e^\infty\}$, $\{v\}$, $\{v\}$, respectively). And that an orbit of length 2 can only occur if there are degenerate vertices of type (2), (3), or (5) (the corresponding orbits of length 2 being $\{e^\infty, fe^\infty\}$, $\{(ef)^\infty, (fe)^\infty\}$, $\{v, f\}$, respectively). ◻

Remark A.10.8. By an argument as in Example A.9.6 one deduces that if a graph E satisfies Condition (I), then the graph groupoid $\mathcal{G}_E$ is densely minimal. However, statement *(1)* in Proposition A.10.7 is strictly weaker than $\mathcal{G}_E$ being densely minimal. It is easy to cook up examples of infinite graphs satisfying Condition (L) and (T), but whose graph groupoids are not densely minimal. One such example is:

E



A.10.3 Isomorphism theorems

Recall that all orbits having length at least 3 is sufficient for the commutator subgroup of the topological full group to cover the groupoid (Lemma A.4.9). This in turn means that the groupoid can be recovered as the groupoid of germs of any subgroup between the topological full group and its commutator. Combined with Propositions A.10.3 and A.10.7 we will obtain the two isomorphism results for graph groupoids. We begin by first observing that the conditions on the graph for membership in K^F actually implies that all orbits are infinite.

Lemma A.10.9. *Let E be a graph with no sinks and suppose E satisfies Condition (K) and (∞). Then $\mathrm{Orb}_{\mathcal{G}_E}(x)$ is infinite for each $x \in \partial E$. In particular, E has no degenerate vertices.*

Proof. We first consider the $\mathcal{G}_E$-orbits of finite boundary paths. Suppose $v \in E^0$ is an infinite emitter. Condition (∞) implies that there are infinitely many distinct return paths at v, hence $\mathrm{Orb}_{\mathcal{G}_E}(\mu)$ is infinite for each $\mu \in \partial E \cap E^*$.

Next, let $x \in E^\infty$ be an infinite path. If x is eventually periodic, then $x = \mu\lambda^\infty$ for some finite path μ and some cycle λ. Lemma A.10.2 gives a sequence of mutually disjoint cycles $\tau_1, \tau_2, \ldots$ based at $s(\lambda)$. And then $\{\tau_1\lambda^\infty, \tau_2\lambda^\infty, \ldots\}$ is an infinite subset of $\mathrm{Orb}_{\mathcal{G}_E}(x)$. If x is not eventually periodic, then $\{x, x_{[2,\infty]}, x_{[3,\infty]}, \ldots\}$ is an infinite subset of $\mathrm{Orb}_{\mathcal{G}_E}(x)$. $\square$

In terms of the class K^F we obtain the following isomorphism result, which relaxes the assumptions in Theorem A.7.2 considerably for graph groupoids.

Theorem A.10.10. *Let E and F be graphs with no sinks, and suppose they both satisfy Condition (K), (W) and (∞). Suppose Γ, Λ are subgroups that satisfy $\mathsf{D}([\![\mathcal{G}_E]\!]) \leq \Gamma \leq [\![\mathcal{G}_E]\!]$ and $\mathsf{D}([\![\mathcal{G}_F]\!]) \leq \Lambda \leq [\![\mathcal{G}_F]\!]$. If $\Gamma \cong \Lambda$ as abstract groups, then $\mathcal{G}_E \cong \mathcal{G}_F$ as topological groupoids. In particular, the following are equivalent:*

1. *$\mathcal{G}_E \cong \mathcal{G}_F$ as topological groupoids.*

2. $[\![\mathcal{G}_E]\!] \cong [\![\mathcal{G}_F]\!]$ *as abstract groups.*

3. $\mathsf{D}([\![\mathcal{G}_E]\!]) \cong \mathsf{D}([\![\mathcal{G}_F]\!])$ *as abstract groups.*

Proof. This follows from combining Proposition A.10.3, Theorem A.6.6, Proposition A.6.2, Lemma A.10.9, Lemma A.4.9 and Proposition A.4.10. □

The preceding result covers—in particular—all finite graphs that have no sinks and satisfy Condition (K). As for an isomorphism result in terms of K^{LCC}, we combine Proposition A.10.7 with Theorem A.7.10 to get the following result.

Theorem A.10.11. *Let E and F be countable graphs satisfying Condition (L) and (T), and having no degenerate vertices. Then the following are equivalent:*

1. $\mathcal{G}_E \cong \mathcal{G}_F$ *as topological groupoids.*

2. $[\![\mathcal{G}_E]\!] \cong [\![\mathcal{G}_F]\!]$ *as abstract groups.*

This result covers—in particular—all finite graphs that have no degenerate vertices nor sinks, and which satisfy Condition (L).

Remark A.10.12. Matsumoto established a version of Theorem A.10.11 for finite graphs which are strongly connected (and satisfy Condition (L), or equivalently Condition (K)) in [Mat15a]. At about the same time, Matui announced [Mat15b], and his Isomorphism Theorem therein applies to the enlarged class of graphs which have finitely many vertices, countably many edges, no sinks, are cofinal, satisfy Condition (L) and for which every vertex can reach every infinite emitter.

Combining Theorem A.10.11 with [BCW17, Theorem 5.1] and [CR18, Corollary 4.2] we obtain the rigidity result in Corollary A.10.13 below, which ties in many of the mathematical structures associated to (directed) graphs. For background on graph C^*-algebras, see [Rae05],[8] and for Leavitt path algebras, see [AASM17].

Corollary A.10.13. *Let E and F be countable graphs satisfying Condition (L) and (T), and having no degenerate vertices. Let R be an integral domain. Then the following are equivalent:*

1. *The graph groupoids $\mathcal{G}_E$ and $\mathcal{G}_F$ are isomorphic as topological groupoids.*

2. *There is an isomorphism of the graph C^*-algebras $C^*(E)$ and $C^*(F)$ which maps the diagonal $\mathcal{D}(E)$ onto $\mathcal{D}(F)$.*

[8]Beware that the convention for paths in graphs in Raeburn's book is opposite of the one used in this paper.

3. *There is an isomorphism of the Leavitt path algebras $L_R(E)$ and $L_R(F)$ which maps the diagonal $D_R(E)$ onto $D_R(F)$.*

4. *The pseudogroups $\mathcal{P}_E$ and $\mathcal{P}_F$ are spatially isomorphic.*

5. *The graphs E and F are (continuously) orbit equivalent.*

6. *The topological full groups $[\![\mathcal{G}_E]\!]$ and $[\![\mathcal{G}_F]\!]$ are isomorphic as abstract groups.*

Remark A.10.14. Statement (5) in Corollary A.10.13 coincides with Li's notion of *continuous orbit equivalence* for the partial dynamical systems associated to the graphs, see [Li17].

Remark A.10.15. We remark that in Corollary A.10.13, statements (1), (2) and (3) are always equivalent, statements (4) and (5) are always equivalent and they are implied by (1), (2) and (3). Furthermore, if the graphs satisfy Condition (L), then statements (1)–(5) are equivalent. Additionally, the equivalence of (1) and (2) has recently been shown in greater generality [CRST17]. The same is true for (1) and (3) by recent work of Steinberg [Ste19], even with weaker assumptions on the ring R.

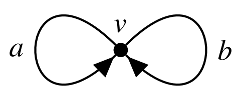
A.11 Embedding theorems

In this final section we will show that several classes of groupoids embed into a certain fixed graph groupoid—namely the groupoid of the graph that consists of a single vertex and two edges. This class includes graph groupoids and AF-groupoids. We will also discuss the induced embeddings of the associated graph algebras and the topological full groups.

A.11.1 Embedding graph groupoids

Let E_2 denote the graph with a single vertex v, and two edges a and b:

E_2



In [BS16], Brownlowe and Sørensen proved an algebraic analog of Kirchberg's Embedding Theorem (see [KP00]) for Leavitt path algebras. They showed that for any countable graph E, and for any commutative unital ring R, the Leavitt path algebra $L_R(E)$ embeds (unitally, whenever it makes sense) into $L_R(E_2)$. By inspecting their proof one finds that this embedding is also *diagonal-preserving*, i.e. that the canonical diagonal $D_R(E)$ is mapped into $D_R(E_2)$. A special case of Kirchberg's Embedding Theorem is that any graph C^*-algebra, $C^*(E)$, embeds into the Cuntz algebra $\mathcal{O}_2$, which is canonically isomorphic to the graph C^*-algebra $C^*(E_2)$ (and the groupoid C^*-algebra $C_r^*(\mathcal{G}_{E_2})$). We denote the canonical diagonal subalgebra in $\mathcal{O}_2$ by $\mathcal{D}_2$. A priori, Kirchberg's embedding is of an analytic nature, but Brownlowe and Sørensen's results shows that in the case of graph C^*-algebras, algebraic embeddings exist. Both graph C^*-algebras and Leavitt path algebras have the same underlying groupoid models (being canonically isomorphic to the groupoid C^*-algebra, and the Steinberg R-algebra ($A_R(\mathcal{G}_E)$) of $\mathcal{G}_E$, respectively). Generally, isomorphisms of the graph groupoids correspond to diagonal preserving isomorphisms of the algebras. Thus, one could wonder whether there is an embedding of the underlying graph groupoids. We will show that this is indeed the case, modulo topological obstructions. Our proof is inspired by [BS16, Proposition 5.1] (and the examples following it).

Lemma A.11.1. *Let E be a countable graph with no sinks, no semi-tails, and suppose that E satisfies Condition (L). Then there exists an injective local homeomorphism $\phi\colon \partial E \to E_2^\infty$ such that*

$$\phi \circ [\![\mathcal{G}_E]\!] \subseteq [\![\mathcal{G}_{E_2}]\!] \circ \phi.$$

If E^0 is finite, then ϕ is surjective (hence a homeomorphism), and if E^0 is infinite, then $\phi(\partial E) = E_2^\infty \setminus \{a^\infty\}$. In particular, there exists an injective étale homomorphism

$$\Phi\colon \mathrm{Germ}([\![\mathcal{G}_E]\!], \partial E) \to \mathcal{G}_{E_2}.$$

Proof. For transparency we first treat the case when E^0 is finite. The infinite case requires only a minor tweak. Let $n = |E^0|$. Label the vertices and edges of E (arbitrarily) as

$$E^0 = \{w_1, w_2, \ldots w_n\} \quad \text{and} \quad w_i E^1 = \{e_{i,j} \mid 1 \le j \le k(i)\} \text{ for each } 1 \le i \le n,$$

where $k(i) = |s^{-1}(w_i)|$. When w_i is an infinite emitter, $k(i) = \infty$, and we let j range over $\mathbb{N}$. For each pair j, i with $j \in \mathbb{N}$, $i \in \mathbb{N} \cup \{\infty\}$ and $j \le i$ we define a finite path $\alpha_{j,i} \in E_2^*$ as follows: $\alpha_{1,1} := v$ and for $j \ge 2$

$$\alpha_{j,i} := \begin{cases} b & \text{if } j = 1, \\ a^{j-1}b & \text{if } 1 < j < i, \\ a^{j-1} & \text{if } j = i. \end{cases}$$

Observe that for each fixed $i \in \mathbb{N}$, the set $\{Z(\alpha_{j,i}) \mid 1 \leq j \leq i\}$ forms a partition of E_2^∞. And for $i = \infty$, $\{Z(\alpha_{j,i}) \mid 1 \leq j < \infty\}$ forms a partition of $E_2^\infty \setminus \{a^\infty\}$.

We now define the map $\phi \colon \partial E \to E_2^\infty$ as follows. For $x = e_{i_1,j_1} e_{i_2,j_2} \ldots \in E^\infty$ we set

$$\phi(x) = \alpha_{i_1,n} \alpha_{j_1,k(i_1)} \alpha_{j_2,k(i_2)} \ldots .$$

If $w_i \in E^0$ is an infinite emitter, then

$$\phi(w_i) = \alpha_{i,n} a^\infty.$$

For notational convenience, we define

$$\phi^*(\mu) \coloneqq \alpha_{i_1,n} \alpha_{j_1,k(i_1)} \alpha_{j_2,k(i_2)} \ldots \alpha_{j_m,k(i_m)} \in E_2^*$$

for each finite path $\mu = e_{i_1,j_1} e_{i_2,j_2} \ldots e_{i_m,j_m} \in E^*$. Finally, if μ is a finite boundary path, then

$$\phi(\mu) = \phi^*(\mu) a^\infty.$$

Recall that $v\alpha = \alpha = \alpha v$ for each $\alpha \in E_2^*$. A priori, $\phi(x)$ could be a finite path in E_2. We argue that this is not the case. For a finite path $\mu \in E^*$, $\phi(\mu)$ is clearly infinite. For an infinite path $x = e_{i_1,j_1} e_{i_2,j_2} \ldots$, $\phi(x)$ is finite if and only if for some $M \in \mathbb{N}$, $\alpha_{j_m,k(i_m)} = v$ for all $m > M$, that is $k(i_m) = 1$ and $j_m = 1$. This means that $e_{i_{M+1},j_{M+1}} e_{i_{M+2},j_{M+2}} \ldots$ is either a semi-tail, or an eventually periodic point whose cycle has no exit. But there are by assumption no such paths in E. So we conclude that ϕ is well-defined.

Using the fact that $\{Z(\alpha_{j,i})\}$ for each fixed i forms a partition of E_2^∞, or $E_2^\infty \setminus \{a^\infty\}$, one easily sees that ϕ is a bijection. As for continuity, define the sets

$$F_{i,l} \coloneqq \{e_{i,1}, e_{i,2}, \ldots, e_{i,l}\} \subseteq w_i E^1 \quad \text{for } 1 \leq l < k(w_i) + 1.$$

Let $\mu = e_{i_1,j_1} e_{i_2,j_2} \ldots e_{i_m,j_m} \in E^*$ and suppose $r(\mu) = w_i$. Observe that

$$\phi(Z(\mu)) = Z(\phi^*(\mu))$$

and

$$\phi(Z(\mu \setminus F_{i,l})) = Z(\phi^*(\mu) a^l).$$

For arbitrary $F = \{e_{i,j_1}, \ldots, e_{i,j_m}\}$ we have

$$Z(\mu \setminus F) = Z(\mu \setminus F_{i,j_m+1}) \bigsqcup \bigsqcup_{j \in J_F} Z(\mu e_{i,j}), \tag{A.11.1}$$

where J_F is the set of j's with $1 \leq j \leq j_m$ and $e_{i,j} \notin F$. Thus ϕ is an open map. Conversely, we have that for $\beta \in E_2^*$

$$\phi^{-1}(Z(\beta)) = \left(\bigcup_{\beta \leq \phi^*(\mu)} Z(\mu) \right) \bigcup \left(\bigcup_{l=1}^{\infty} \bigcup_{\beta \leq \phi^*(\lambda) a^l} Z(\lambda \setminus F_{r(\lambda),l}) \right),$$

(and these unions may actually be taken to be finite). Hence ϕ is a homeomorphism.

To see that $\phi \circ [\![\mathcal{G}_E]\!] \circ \phi^{-1} \subseteq [\![\mathcal{G}_{E_2}]\!]$, let $\mu, \lambda \in E^*$ with $r(\mu) = r(\lambda) = w_i$ be given, and let $1 \leq l < k(w_i) + 1$. Observe that

$$\phi \circ \pi_{Z(\mu,\lambda)} \circ \phi^{-1} = \pi_{Z(\phi^*(\mu),\phi^*(\lambda))} \colon Z(\phi^*(\lambda) \to Z(\phi^*(\mu)),$$

and

$$\phi \circ \pi_{Z(\mu,F_l,\lambda)} \circ \phi^{-1} = \pi_{Z(\phi^*(\mu)a^l,\phi^*(\lambda)a^l)} \colon Z(\phi^*(\lambda)a^l \to Z(\phi^*(\mu)a^l),$$

as partial homeomorphisms. By utilizing a similar decomposition as in Equation (A.11.1) for the basic set $Z(\mu, F, \lambda)$ for arbitrary F, together with the description of elements in $[\![\mathcal{G}_E]\!]$ from Proposition A.9.4, we see that for each $\pi_U \in [\![\mathcal{G}_E]\!]$, the homeomorphism $\phi \circ \pi_U \circ \phi^{-1}$ belongs to $[\![\mathcal{G}_{E_2}]\!]$.

In the case that E^0 is infinite, all the arguments above still go through, with the minor adjustment that the first word in $\phi(x)$ is $\alpha_{i_1,\infty}$. This word always ends with a b, so we see that ϕ becomes a homeomorphism from ∂E onto $E_2^\infty \setminus \{a^\infty\}$.

The final statement follows from Corollary A.5.5 and Proposition A.4.10 ($[\![\mathcal{G}_{E_2}]\!]$ covers $\mathcal{G}_{E_2}$ since the groupoid is minimal). □

Remark A.11.2. The local homeomorphism ϕ constructed in the preceding proof depends on the choice of labeling of the graph. And there are of course many ways to label a graph, but each one gives a local homeomorphism ϕ with the desired properties.

In order to conclude that $\mathcal{G}_E$ embeds into $\mathcal{G}_{E_2}$ it seems like we have to assume that $[\![\mathcal{G}_E]\!]$ covers $\mathcal{G}_E$ (as this is not always the case). However, in the proof of Lemma A.11.1 we are really showing that $\phi \circ \mathcal{P}_c(\mathcal{G}_E) \subseteq \mathcal{P}_c(\mathcal{G}_{E_2}) \circ \phi$, where $\mathcal{P}_c(\mathcal{G})$ denotes the inverse semigroup of partial homeomorphisms $\pi_U \colon s(U) \to r(U)$ coming from compact bisections $U \subseteq \mathcal{G}$. It is a sub-inverse semigroup of Renault's pseudogroup as in [Ren08], [BCW17] (when $\mathcal{G}$ is effective). The constructions in Sections A.4 and A.5 apply more or less verbatim to $\mathcal{P}_c(\mathcal{G})$. The crucial difference is that $\mathcal{P}_c(\mathcal{G})$ always covers $\mathcal{G}$, when $\mathcal{G}$ is ample. Thus, the analogs of Corollary A.5.5 and Proposition A.4.10 for $\mathcal{P}_c(\mathcal{G})$ applied to ϕ induces the desired embedding of the graph groupoids—which we record in the following theorem.

Theorem A.11.3. *Let E be a countable graph satisfying Condition (L) and having no sinks nor semi-tails. Then there is an embedding $\Phi \colon \mathcal{G}_E \hookrightarrow \mathcal{G}_{E_2}$ of étale groupoids. If E^0 is finite, then Φ maps ∂E onto E_2^∞.*

Remark A.11.4. Theorem A.11.3 is optimal in the sense there is no embedding if one relaxes the assumptions on E. For if ∂E has isolated points, then there is no local homeomorphism from ∂E to E_2^∞, as the latter has no isolated points. And if E is uncountable, then there is no embedding either, for then ∂E is not second countable, while E_2^∞ is. Similarly, ∂E cannot map onto E_2^∞ if E^0 is infinite, for then the former is not compact.

A.11.2 Diagonal embeddings of graph algebras

From Theorem A.11.3 we recover Brownlowe and Sørensen's embedding theorem for Leavitt path algebras (albeit for the slightly smaller class of graphs E with ∂E having no isolated points). However, we get the additional conclusion that when E^0 is finite (i.e. the algebras are unital), the embedding can be chosen to not only be unital, but also to map the diagonal *onto* the diagonal.

Corollary A.11.5. *Let E be a countable graph with no sinks, no semi-tails, and satisyfing Condition (L).*

1. *There is an injective $*$-homomorphism $\psi\colon C^*(E) \to \mathcal{O}_2$ such that $\psi(\mathcal{D}(E))$ is contained in $\mathcal{D}_2$. If E^0 is finite, then ψ is unital and $\psi(\mathcal{D}(E)) = \mathcal{D}_2$.*
2. *For any commutative unital ring R, there is an injective $*$-algebra homomorphism $\rho\colon L_R(E) \to L_R(E_2)$ such that $\rho(D_R(E))$ is contained in $D_R(E_2)$. If E^0 is finite, then ρ is unital and $\rho(D_R(E)) = D_R(E_2)$.*

Remark A.11.6. For each labeling of a graph E as in the proof of Lemma A.11.1, one obtains explicit embeddings of both the graph C^*-algebras and the Leavitt path algebras into $\mathcal{O}_2$ and $L_R(E_2)$, respectively, in terms of their canonical generators. This is done by expanding the scheme in [BS16, Proposition 5.1]. The canonical isomorphism between both $C^*(E)$ and $C^*(\mathcal{G}_E)$, and $L_R(E)$ and $A_R(\mathcal{G}_E)$ is given by $p_v \leftrightarrow 1_{Z(v)}$ for $v \in E^0$ (vertex projections) and $s_e \leftrightarrow 1_{Z(e,r(e))}$ for $e \in E^1$ (edge partial isometries). Denote the generators in $\mathcal{O}_2$ and $L_R(E_2)$ by s_a and s_b. Given a labeling $E^0 = \{w_1, w_2, w_3 \ldots\}$ and $E^1 = \{e_{i,j} \mid 1 \le i \le n,\ 1 \le j \le k(i)\}$, the embedding of the algebras induced by ϕ as in Lemma A.11.1 is given on the generators by

$$p_{w_i} \longmapsto s_{\phi^*(w_i)}\left(s_{\phi^*(w_i)}\right)^*, \qquad s_{e_{i,j}} \longmapsto s_{\phi^*(e_{i,j})}\left(s_{\phi^*(r(e_{i,j}))}\right)^*,$$

where $\phi^*(\mu) \in \{a,b\}^*$ is as in the proof of Lemma A.11.1 (recall that for a finite path $\mu = e_1, \ldots e_n \in E^*$, we define $s_\mu := s_{e_1} \cdots s_{e_2}$).

Remark A.11.7. In the case that E has infinitely many vertices, the image of the diagonals in Corollary A.11.5 can be described as follows:

$$\psi(\mathcal{D}(E)) = \overline{\operatorname{span}\{s_\alpha s_\alpha^* \mid \alpha \in E_2^* \setminus \{a, a^2, a^3, \ldots\}\}},$$

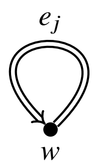
and

$$\rho(D_R(E)) = \operatorname{span}_R\{s_\alpha s_\alpha^* \mid \alpha \in E_2^* \setminus \{a, a^2, a^3, \ldots\}\}.$$

For examples of explicit embeddings for finite graphs satisfying Condition (L) (possibly even having sinks), see Section 5 of [BS16]. As for infinite graphs, we provide a few examples below.

Example A.11.8. Consider the following graph, whose graph C^*-algebra is the Cuntz algebra $\mathcal{O}_\infty$:

E_∞

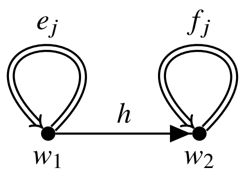


The double arrow indicates infinitely many edges, i.e. $E^1 = \{e_1, e_2, e_3, \ldots\}$. For simplicity, we denote the edge isometries by s_j for $j \in \mathbb{N}$. We label $w = w_1$ and $e_j = e_{1,j}$. Following the recipe in Remark A.11.6 we obtain a unital embedding of $\mathcal{O}_\infty$ into $\mathcal{O}_2$ (and similarly of $L_R(E_\infty)$ into $L_R(E_2)$) which maps the diagonal onto the diagonal, in terms of generators as follows:

$$p_w = 1_{\mathcal{O}_\infty} \longmapsto 1_{\mathcal{O}_2} = p_v, \qquad s_j \longmapsto s_{a^{j-1}b}.$$

Example A.11.9. Next, consider the following graph:

E

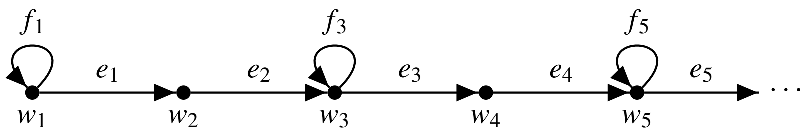


By labeling the edges as $h = e_{1,1},\ e_j = e_{1,j+1},\ f_j = e_{2,j}$ we get the following unital diagonal preserving embedding of $C^*(E)$ into $\mathcal{O}_2$:

$$\begin{aligned} &p_{w_1} \longmapsto s_b s_b^*, && p_{w_2} \longmapsto s_a s_a^*, && \\ &s_h \longmapsto s_{bb} s_a^*, && s_{e_j} \longmapsto s_{ba^j b} s_b^*, && s_{f_j} \longmapsto s_{ba^j b} s_a^*. \end{aligned}$$

Example A.11.10. Finally, let us look at a graph with infinitely many vertices:

F



We label the edges as $e_j = e_{j,1}$ for $j \in \mathbb{N}$, and $f_j = e_{j,2}$ for j odd. The induced diagonal preserving embedding of $C^*(F)$ into $\mathcal{O}_2$ is then given on the generators as follows:

$$p_{w_i} \longmapsto s_{a^{i-1}b}\left(s_{a^{i-1}b}\right)^*, \qquad s_{f_j} \longmapsto s_{a^{j-1}ba}\left(s_{a^{j-1}b}\right)^* \ (j \text{ odd}),$$
$$s_{e_j} \longmapsto \begin{cases} s_{a^{j-1}b}\left(s_{a^{j}b}\right)^* & j \text{ even}, \\ s_{a^{j-1}b^2}\left(s_{a^{j}b}\right)^* & j \text{ odd}. \end{cases}$$

A.11.3 Analytic properties of $[\![\mathcal{G}_E]\!]$

Before generalizing the groupoid embedding theorem to a larger class of groupoids in the next subsection we take brief pause to discuss some analytic properties of the topological full groups $[\![\mathcal{G}_E]\!]$ for graphs E as in Lemma A.11.1. First of all, $[\![\mathcal{G}_E]\!]$ is generally not amenable, as it often contains free products [Mat15b, Proposition 4.10].

Let E_n for $n \geq 2$ denote the graph consisting of a single vertex and n edges. And more generally, for $r \in \mathbb{N}$, let $E_{n,r}$ be the graph with r vertices $w_1, w_2, \ldots, w_r$ and $n + r - 1$ edges $e_1, \ldots, e_n, f_1, \ldots, f_{r-1}$ such that $s(e_i) = w_1, r(e_i) = w_r$ for each $1 \leq i \leq n$ and $s(f_i) = w_{i+1}, r(f_i) = w_i$ for each $1 \leq i \leq r - 1$. According to [Mat15b, Section 6], the topological full group $[\![\mathcal{G}_{E_{n,r}}]\!]$ is isomorphic to the *Higman-Thompson group* $V_{n,r}$. In particular, $[\![\mathcal{G}_{E_2}]\!] \cong V_{2,1} = V$ (Thompson's group V). As Lemma A.11.1 in particular induces an algebraic embedding of the topological full groups, we have that $[\![\mathcal{G}_E]\!]$ embeds into V for each graph E as in Lemma A.11.1. Thus, Lemma A.11.1 may be considered a generalization of the well known embedding of $V_{n,r}$ into V. As V has the *Haagerup property* [Far03], we deduce that $[\![\mathcal{G}_E]\!]$ does as well.

Corollary A.11.11. *Let E be a countable graph with no sinks, no semi-tails, and suppose E satisfies Condition (L). Then the topological full group $[\![\mathcal{G}_E]\!]$ has the Haagerup property.*

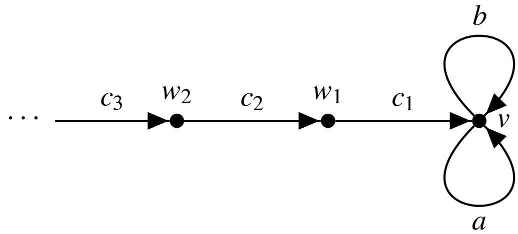
Remark A.11.12. For finite, strongly connected graphs, this was proved directly, using so-called *zipper actions*, by Matui in [Mat15b]. Later, in [Mat16], Matui proved that for any finite, strongly connected graph E, $[\![\mathcal{G}_E]\!]$ embeds into $[\![\mathcal{G}_{E_2}]\!]$. In fact, he proved even more, namely that $\mathcal{G}_{E_2}$ could be replaced by any groupoid with similar properties (see [Mat16, Proposition 5.14] for the details). By our results, one may relax the conditions on E considerably in Matui's embedding result.

A.11.4 Embedding equivalent groupoids

We are now going to expand on the embedding theorem for graph groupoids to include all groupoids that are merely groupoid equivalent to a graph groupoid. To accomplish this we will make us of the fundamental results by Carlsen, Ruiz and Sims in [CRS17]. Following their notation, let $\mathcal{R}$ denote the countably infinite discrete full equivalence relation, that is $\mathcal{R} = \mathbb{N} \times \mathbb{N}$ equipped with the discrete topology, whose product and inverse are given by $(k, m) \cdot (m, n) := (k, n)$ and $(m, n)^{-1} := (n, m)$. We refer to the product groupoid $\mathcal{G} \times \mathcal{R}$ as the *stabilization* of the groupoid $\mathcal{G}$. For a graph E, let SE denote the graph obtained from E by adding a *head* at every vertex—see the example below (see also [Tom04]). It is shown in [CRS17] that $\mathcal{G}_E \times \mathcal{R} \cong \mathcal{G}_{SE}$ as topological groupoids for any graph E.

Example A.11.13. The stabilized graph of E_2 is the following graph:

SE_2



Let us first just say a few words on necessary conditions for an étale groupoid $\mathcal{H}$ to be embeddable into $\mathcal{G}_{E_2}$. First of all, it is clearly necessary that $\mathcal{H}$ is ample, Hausdorff and second countable, since $\mathcal{G}_{E_2}$ is. As we observed for the graph groupoids, it is also necessary that $\mathcal{H}^{(0)}$ has no isolated points, and hence that $\mathcal{H}^{(0)}$ is a locally compact Cantor space. Furthermore, since subgroupoids of effective groupoids are effective, it is also necessary that $\mathcal{H}$ be effective. As a final observation in this regard, any embedding $\Phi \colon \mathcal{H} \hookrightarrow \mathcal{G}_{E_2}$ induces an embedding of the isotropy bundles $\mathcal{H}' \hookrightarrow (\mathcal{G}_{E_2})'$, meaning that Φ restricts to an embedding of the isotropy group $\mathcal{H}_y^y$ into $(\mathcal{G}_{E_2})_{\Phi(y)}^{\Phi(y)}$ for each $y \in \mathcal{H}^{(0)}$. Now recall that for any graph groupoid $\mathcal{G}_E$ the isotropy groups are

$$(\mathcal{G}_E)_x^x \cong \begin{cases} \mathbb{Z} & \text{if } x \text{ is eventually periodic,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, a final necessary condition for embeddability is that the istropy bundle of $\mathcal{H}$ consists only of the groups 0 and $\mathbb{Z}$. This rules out for instance (most) products of graph groupoids, since they typically have isotropy groups that are free abelian of rank up to the number of factors in the product. Note however, that taking the product with a principal groupoid does no harm in this regard. As we'll see imminently, taking the product with $\mathcal{R}$ (i.e. stabilizing) does not affect embeddability into $\mathcal{G}_{E_2}$.

Proposition A.11.14. *Let $\mathcal{H}$ be an effective ample second countable Hausdorff groupoid with $\mathcal{H}^{(0)}$ a locally compact Cantor space. Then $\mathcal{H}$ embeds into $\mathcal{G}_{E_2}$ if and only if the stabilized groupoid $\mathcal{H} \times \mathcal{R}$ embeds into $\mathcal{G}_{E_2}$.*

Proof. The "if statement" is trivial as a groupoid always embeds into its stabilization. Suppose $\Phi\colon \mathcal{H} \to \mathcal{G}_{E_2}$ is an injective étale homomorphism. Then $\phi \times \mathrm{id}\colon \mathcal{H} \times \mathcal{R} \to \mathcal{G}_{E_2} \times \mathcal{R}$ is an injective étale homomorphism as well. By [CRS17, Lemma 4.1] we have $\mathcal{G}_{E_2} \times \mathcal{R} \cong \mathcal{G}_{SE_2}$, and SE_2 is a countable graph satisfying Condition (L) with no sinks nor semi-tails. So by Theorem A.11.3, $\mathcal{G}_{SE_2}$ embeds into $\mathcal{G}_{E_2}$. Thus $\mathcal{H} \times \mathcal{R}$ embeds into $\mathcal{G}_{E_2}$. □

The next lemma shows that any étale embedding of a groupoid $\mathcal{H}$, with compact unit space, into $\mathcal{G}_{E_2}$ can be "twisted" into an embedding that hits the whole unit space of $\mathcal{G}_{E_2}$.

Lemma A.11.15. *Let $\mathcal{H}$ be an effective ample second countable Hausdorff groupoid with $\mathcal{H}^{(0)}$ a Cantor space. If $\mathcal{H}$ embeds into $\mathcal{G}_{E_2}$, then there exists an embedding $\Phi\colon \mathcal{H} \hookrightarrow \mathcal{G}_{E_2}$ such that $\Phi\left(\mathcal{H}^{(0)}\right) = E_2^\infty$.*

Proof. Let $\Psi\colon \mathcal{H} \to \mathcal{G}_{E_2}$ be an injective étale homomorphism and define the set $Y = \Psi\left(\mathcal{H}^{(0)}\right)$. Then Y is a compact open (hence clopen) subset of E_2^∞. We claim that there exists a compact open bisection $U \subseteq \mathcal{G}_{E_2}$ such that $s(U) = Y$ and $r(U) = E_2^\infty$. The claim follows from [Mat15b, Theorem 6.4] and [Mat17, Example 3.3 (3)] by identifying $\mathcal{G}_{E_2}$ with the *SFT-groupoid* of the 1×1 matrix $A = [2]$ (see [Mat17, Example 2.5]). Now define $\Phi(h) = U \cdot \Psi(h) \cdot U^{-1}$ for $h \in \mathcal{H}$. Then Φ is an injective étale homomorphism and

$$\Phi\left(\mathcal{H}^{(0)}\right) = UYU^{-1} = UU^{-1} = r(U) = E_2^\infty.$$

□

We now state the most general version of our embedding theorem.

Theorem A.11.16. *Let $\mathcal{H}$ be an effective ample second countable Hausdorff groupoid whose unit space $\mathcal{H}^{(0)}$ is a locally compact Cantor space. If $\mathcal{H}$ is groupoid equivalent to $\mathcal{G}_E$, for some countable graph E satisfying Condition (L) and having no sinks nor semi-tails, then $\mathcal{H}$ embeds into $\mathcal{G}_{E_2}$. Moreover, if $\mathcal{H}^{(0)}$ is compact, then the embedding maps $\mathcal{H}^{(0)}$ onto E_2^∞.*

Proof. Suppose $\mathcal{H}$ is groupoid equivalent to $\mathcal{G}_E$ as above. Then by [CRS17, Theorem 3.2] we have $\mathcal{H} \times \mathcal{R} \cong \mathcal{G}_E \times \mathcal{R}$. By Theorem A.11.3 and Proposition A.11.14, $\mathcal{G}_E \times \mathcal{R}$ embeds into $\mathcal{G}_{E_2}$, hence so does $\mathcal{H} \times \mathcal{R}$ and $\mathcal{H}$. The second statement follows from Lemma A.11.15. □

Remark A.11.17. We note that for any groupoid $\mathcal{H}$ as in the above theorem, its topological full group $[\![\mathcal{H}]\!]$ also has the Haagerup property.

A.11.5 Embedding AF-groupoids

A well-studied class of groupoids satisfying the hypothesis of Theorem A.11.16, yet conceptually different from graph groupoids, are the *AF-groupoids*. See [GPS04] (wherein they are dubbed *AF-equivalence relations*). Let $\mathcal{G}$ be an ample Hausdorff second countable groupoid with $\mathcal{G}^{(0)}$ a locally compact Cantor space. Then $\mathcal{G}$ is called an *AF-groupoid* if there exists an increasing sequence $\mathcal{K}_1 \subseteq \mathcal{K}_2 \subseteq \ldots \subseteq \mathcal{G}$ of clopen subgroupoids such that

- $\mathcal{K}_n$ is principal for each $n \in \mathbb{N}$.
- $\mathcal{K}_n^{(0)} = \mathcal{G}^{(0)}$ for each $n \in \mathbb{N}$.
- $\mathcal{K}_n \setminus \mathcal{G}^{(0)}$ is compact for each $n \in \mathbb{N}$.
- $\bigcup_{n=1}^{\infty} \mathcal{K}_n = \mathcal{G}$.

This entails that $\mathcal{G}$ is principal.

Remark A.11.18. The terminology AF-groupoid is due to Renault [Ren80], and is also used by Matui in [Mat12] and [Mat17]. Note however, that Matui only considered the case of a compact unit space therein.

In the following example we explain how *Bratteli diagrams* give rise to AF-groupoids.

Example A.11.19 (cf. [GPS04, Example 2.7(ii)])**.** A *Bratteli diagram* B is a directed graph whose vertex set V and edge set E can be written as countable disjoint unions of non-empty finite sets

$$V = V_0 \sqcup V_1 \sqcup V_2 \sqcup \ldots \quad \text{and} \quad E = E_1 \sqcup E_2 \sqcup E_3 \sqcup \ldots \tag{A.11.2}$$

such that the source and range maps satisfy $s(E_n) = V_{n-1}$ and $r(E_n) \subseteq V_n$.[9] In particular, there are no sinks in B. Let $S_B \subseteq V$ denote the set of sources in B. Then $V_0 \subseteq S_B$. We call B a *standard* Bratteli diagram if there is only one source in B, i.e. $S_B = \{v_0\} = V_0$. We say that B is *simple* if for every vertex $v \in V_n$, there is an $m > n$ such that there is a path from v to every vertex in V_m. The partitions of the vertices and edges (into *levels* as in Equation (A.11.2)) is considered part of the data of the Bratteli diagram B. We let E_B denote the underlying graph where we "forget" about the partitions.

[9]This notation is inconsistent with what we have been using for directed graphs so far. But since Bratteli diagrams are very special kinds of graphs we have chosen to use the well-established notation from the literature. In this way we can, albeit somewhat artificially, distinguish a Bratteli diagram from its underlying graph.

For a source $v \in S_B \cap V_n$ on level n we let X_v denote the set of infinite paths starting in v, that is

$$X_v := \{e_{n+1}e_{n+2}e_{n+3} \dots \mid s(e_{n+1}) = v, e_{n+k} \in E_{n+k}, s(e_{n+k}) = r(e_{n+k-1}), k > 1\}$$

The *path space* of B is

$$X_B := \bigsqcup_{v \in S_B} X_v$$

whose topology is given by the basis of *cylinder sets*

$$C(\mu) := \{e_{n+1}e_{n+2} \dots \in X_{s(\mu)} \mid e_{n+1} \dots e_{n+|\mu|} = \mu\}$$

where μ is a finite path such that $s(\mu) = v$ for some source $v \in S_B \cap V_n$. The path space X_B is Boolean, and it is compact if and only if S_B is finite. Further, X_B is perfect if and only if E_B has no semi-tails. Two infinite paths in X_B are *tail-equivalent* if they agree from some level on. With this equivalence relation as the starting point, let for each $N \in \mathbb{N}$

$$\mathcal{P}_N := \{(x, y) \in X_B \times X_B \mid s(x) \in V_m, s(y) \in V_n, m, n \leq N, x_k = y_k \text{ for all } k > N\}.$$

That is, $\mathcal{P}_N$ consists of all pairs of infinite paths which start before the N'th level and agrees from the N'th level and onwards. Equipping $\mathcal{P}_N$ with the relative topology from $X_B \times X_B$ makes $\mathcal{P}_N$ a compact principal ample Hausdorff groupoid whose unit space is identified with $\bigsqcup_{n=1}^{N} \bigsqcup_{v \in S_B \cap V_n} Z(v)$.

We define the *groupoid of the Bratteli diagram B* as the increasing union

$$\mathcal{G}_B := \bigcup_{N=1}^{\infty} \mathcal{P}_N$$

equipped with the inductive limit topology. For any two finite paths μ, λ with $s(\mu), s(\lambda) \in S_B$ and $r(\mu) = r(\lambda)$ we define

$$C(\mu, \lambda) := \left\{(x, y) \in C(\mu) \times C(\lambda) \mid x_{[|\mu|+1, \infty)} = y_{[|\lambda|+1, \infty)}\right\}.$$

A straightforward computation shows that the family of $C(\mu, \lambda)$'s form a compact open basis for the inductive limit topology on $\mathcal{G}_B$. We identify $\mathcal{G}_B^{(0)}$ with X_B. By setting $\mathcal{K}_n = \mathcal{P}_n \cup \mathcal{G}_B^{(0)}$ one sees that $\mathcal{G}_B$ is an AF-groupoid. The groupoid $\mathcal{G}_B$ is minimal if and only if B is a simple Bratteli diagram.

Remark A.11.20. Although the AF-groupoid $\mathcal{G}_B$ is defined in terms of a very special graph, namely the Bratteli diagram B, it is generally not isomorphic to a graph groupoid. To see this, recall that $\mathcal{G}_B$ is always principal, while a graph groupoid $\mathcal{G}_E$ is principal if and only if the graph E has no cycles. If X_B is compact, perfect and infinite (this is essentially stipulating that the Bratteli diagram is standard and “non-degenerate”), then $\mathcal{G}_B$ cannot be isomorphic to any graph groupoid. For any such $\mathcal{G}_E$ would have a compact unit space, i.e. E has finitely many vertices, and E would have no cycles and no sinks. There are clearly no such graphs.

Giordano, Putnam and Skau showed that, just as with AF-algebras [Bra72], every AF-groupoid can be realized by a Bratteli diagram as in Example A.11.19.

Theorem A.11.21 ([GPS04, Theorem 3.9])**.** *Let $\mathcal{H}$ be an AF-groupoid. Then there exists a Bratteli diagram B such that $\mathcal{H} \cong \mathcal{G}_B$. If $\mathcal{H}^{(0)}$ is compact, then B can be chosen to be standard.*

Remark A.11.22. As another example of a concrete description of the topological full group of an ample groupoid, we remark that Matui described the topological full group of an AF-groupoid with compact unit space in terms of a definining Bratteli diagram in [Mat06, Proposition 3.3]. The topological full group $[\![\mathcal{G}_B]\!]$, where B is a Bratteli diagram, is the direct limit of the finite groups Γ_N for $N \in \mathbb{N}$, where $\Gamma_N \leq \operatorname{Homeo}(X_B)$ consists of all permutations of the finite set of paths from level V_0 to V_N such that the permutation preserves the range of these paths (and the action on X_B is by permuting the intial segment of an infinite path). We should also mention that these groups were originally studied by Krieger in [Kri80], without emphasis on the underlying groupoids.

By the preceding remark it is clear that the topological full group of any AF-groupoid is a locally finite group. And actually, this characterizes the AF-groupoids. This is somewhat of a folklore result, but a proof is published by Matui in the compact case, and it is not hard to see that his proof extends to locally compact unit spaces as well.

Proposition A.11.23 (cf. [Mat06, Proposition 3.2])**.** *Let $\mathcal{G}$ be an ample principal Hausdorff second countable groupoid with $\mathcal{G}^{(0)}$ a locally compact Cantor space. Then the topological full group $[\![\mathcal{G}]\!]$ is locally finite if and only if $\mathcal{G}$ is an AF-groupoid.*

Remark A.11.24. The commutator subgroups $\mathsf{D}(\mathcal{G}) \leq [\![\mathcal{G}]\!]$ for AF-groupoids $\mathcal{G}$ are quite interesting in their own right. In fact, these exhaust[10] the class of so-called *strongly diagonal limits of products of alternating groups* (also called *LDA-groups*, see [LN07] where these are classified using the dimension groups of their Bratteli diagrams). These form a subclass of the locally finite simple groups. By Corollary A.11.26 below, all the LDA-groups embed into Thompson’s group V.

[10]With the single exception of the infinite finitary alternating group.

We now demonstrate that every AF-groupoid is groupoid equivalent to a graph groupoid. This is essentially just a reformulation of the main theorem from [Dri00], wherein it is shown that any AF-algebra can be recovered as a certain *pointed* graph C^*-algebra of a defining Bratteli diagram. In contrast, in Proposition A.11.25 below we emphasize the groupoids, rather than their C^*-algebras. Also, since we use "unlabeled" Bratteli diagrams here, as opposed to *labeled Bratteli diagrams* (as in [Dri00, Section 2]), the computations are easier.

Proposition A.11.25. *Let B be a Bratteli diagram. Then the AF-groupoid $\mathcal{G}_B$ is isomorphic to the restriction of the graph groupoid $\mathcal{G}_{E_B}$ to the open subset $\bigsqcup_{v\in S_B} Z(v) \subseteq E_B^\infty$. It follows that every AF-groupoid is groupoid equivalent to a graph groupoid.*

Proof. Let $A = \bigsqcup_{v\in S_B} Z(v)$. Then

$$(\mathcal{G}_{E_B})_{|A} = \{(x,k,y) \mid s(x), s(y) \in S_B, \sigma_{E_B}(x)^m = \sigma_{E_B}(y)^n, k = m-n\}.$$

Due to the special structure of the graph E_B, the lag k in $(x,k,y) \in (\mathcal{G}_{E_B})_{|A}$ is uniquely determined by x and y. In fact, k is determined by the levels on which x and y start in the Bratteli diagram. Indeed, let $m, n \in \mathbb{N}$ be such that $s(x) \in V_m$ and $s(y) \in V_n$, then $k = n - m$. This means that the map $\Phi\colon (\mathcal{G}_{E_B})_{|A} \to \mathcal{G}_B$ defined by $\Phi((x,k,y)) = (x,y)$ is a bijection. It is easy to see that Φ is also a groupoid homomorphism. Finally, to see that Φ is a homeomorphism simply note that the family of $Z(\mu,\lambda)$'s where μ,λ are finite paths with $s(\mu), s(\lambda) \in S_B$ and $r(\mu) = r(\lambda)$ form a basis for $(\mathcal{G}_{E_B})_{|A}$, and that $\Phi(Z(\mu,\lambda)) = C(\mu,\lambda)$. Thus $(\mathcal{G}_{E_B})_{|A} \cong \mathcal{G}_B$ as étale groupoids.

We claim that A is a $\mathcal{G}_{E_B}$-full subset of E_B^∞, and then the second statement follows from [CRS17, Theorem 3.2]. To see this, let $z \in E_B^\infty$ be an infinite path starting anywhere in the Bratteli diagram and simply note that by following $s(z)$ upwards in the Bratteli diagram, one eventually reaches a source $v \in S_B$ such that v connects to $s(z)$. Letting μ be any path from v to $s(z)$ we have that z belongs to the $\mathcal{G}_{E_B}$-orbit of $\mu z \in A$. □

As a special case of Theorem A.11.16 we obtain the following.

Corollary A.11.26. *Let $\mathcal{G}$ be an AF-groupoid with $\mathcal{G}^{(0)}$ perfect. Then there exists an embedding of étale groupoids $\mathcal{G} \hookrightarrow \mathcal{G}_{E_2}$. If $\mathcal{G}^{(0)}$ is compact, then $\mathcal{G}^{(0)}$ maps onto E_2^∞.*

From this we obtain an analogue of Corollary A.11.5 for AF-algebras and their diagonals. Let A be an AF-algebra. By an *AF Cartan subalgebra* $D \subseteq A$ we mean a Cartan subalgebra arising from the diagonalization method of Strătilă and Voiculescu [SV75]. See [Dri00, Section 4] for a description of these diagonals for non-unital AF-algebras. Note that they are also C^*-diagonals in the sense of Kumjian [Kum86]. According to [Ren08, Subsection 6.2] these are precisely the Cartain pairs arising as $(C_r^*(\mathcal{G}_B), C_0(X_B))$ for a Bratteli diagram B.

Corollary A.11.27. *Let A be an infinite-dimensional AF-algebra and let $D \subseteq A$ be any AF Cartan subalgebra in A whose spectrum is perfect. Then there is an injective $*$-homomorphism $\psi\colon A \hookrightarrow \mathcal{O}_2$ such that $\psi(D) \subseteq \mathcal{D}_2$. If A is unital, then so is ψ, and $\psi(D) = \mathcal{D}_2$.*

Remark A.11.28. As a final remark, we note that certain transformation groupoids (by virtue of actually being AF-groupoids) also embed into $\mathcal{G}_{E_2}$. Let X be a *non-compact* locally compact Cantor space and let T be a minimal homeomorphism on X. It follows from [GPS04, Theorem 4.3] that the transformation groupoid $\mathbb{Z} \ltimes_T X$ is an AF-groupoid, and consequently $\mathbb{Z} \ltimes_T X$ embeds into $\mathcal{G}_{E_2}$.

An indirect way of seeing that $\mathbb{Z} \ltimes_T X$ is an AF-groupoid is via Proposition A.11.23. By realizing the dynamical system (X, T) as a *Bratteli-Vershik system* on a (standard) *almost simple orderered Bratteli diagram* $B = (V, E, \geq)$ (see [Dan01]), one easily observes (as Matui did in [Mat02]) that $[\![\mathbb{Z} \ltimes_T X]\!]$ is locally finite. This is because each element of $[\![\mathbb{Z} \ltimes_T X]\!]$ only depends on the inital edges down to level N for some fixed N (determined by the group element), for each infinite path in X_B. This actually allows one to describe the topological full group $[\![\mathbb{Z} \ltimes_T X]\!]$ explicitly in terms of a conjugate Bratteli-Vershik system.

A third way of demonstrating that $\mathbb{Z} \ltimes_T X$ is an AF-groupoid is to go from a conjugate Bratteli-Vershik system on an ordered Bratteli diagram $B = (V, E, \geq)$ to an "unordered" Bratteli diagram B' such that $\mathbb{Z} \ltimes_T X \cong \mathcal{G}_{B'}$ as étale groupoids. Indeed, let $e_1 e_2 e_3 \ldots \in X_B$ denote the unique maximal and minimal path in X_B (see [Dan01]). By "forgetting" the ordering and removing each of the edges e_n for all $n \in \mathbb{N}$, and thereby introducing a source at each of the vertices $s(e_n)$, one obtains the modified Bratteli diagram B', and it is not hard to see that the AF-groupoid $\mathcal{G}_{B'}$ is isomorphic to $\mathbb{Z} \ltimes_T X$.

Paper B

Matui's AH Conjecture for Graph Groupoids

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Submitted for publication.

Paper B

Matui's AH Conjecture for Graph Groupoids

Abstract

We prove that Matui's AH conjecture holds for graph groupoids of infinite graphs. This is a conjecture that relates the topological full group of an ample groupoid with the homology of the groupoid. Our main result complements Matui's result in the finite case, which makes the AH conjecture true for all graph groupoids covered by the assumptions of said conjecture. Furthermore, we observe that for arbitrary graphs, the homology of a graph groupoid coincides with the K-theory of its groupoid C^*-algebra.

B.1 Introduction

B.1.1 Background

Building on the discoveries in the series of papers [Mat06], [Mat12] and [Mat15b] Hiroki Matui stated two conjectures concerning effective minimal étale groupoids over Cantor spaces in [Mat16]. The *HK conjecture* predicts that the K-theory of a reduced groupoid C^*-algebra is determined by the groupoid's homology as follows:

$$K_0\left(C_r^*(\mathcal{G})\right) \cong \bigoplus_{n=0}^{\infty} H_{2n}(\mathcal{G}) \quad \text{and} \quad K_1\left(C_r^*(\mathcal{G})\right) \cong \bigoplus_{n=0}^{\infty} H_{2n+1}(\mathcal{G}).$$

The *AH conjecture* predicts that the abelianization of the topological full group of a groupoid together with its first two homology groups fit together in an exact sequence as follows:

$$H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \xrightarrow{\ j\ } [\![\mathcal{G}]\!]_{\mathrm{ab}} \xrightarrow{\ I\ } H_1(\mathcal{G}) \longrightarrow 0.$$

In several cases (including graph groupoids) the K-groups actually coincide with the first two homology groups, which means that the AH conjecture in these cases relates the K-theory of the groupoid C^*-algebra with the topological full group.

Topological full groups associated to dynamical systems (and more generally to étale groupoids) are perhaps best known for being complete invariants for continuous orbit equivalence (and groupoid isomorphism), in addition to diagonal preserving isomorphisms of associated C^*-algebras. Roughly speaking, the topological full group consists of all homeomorphisms which preserve the orbits of the dynamical system in a continuous manner. Consult [GPS99], [Med11], [Mat15a], [Mat15b], [NO19] and [dCGvW19] for some of these rigidity results. Topological full groups also provide means of constructing new groups with interesting properties, most notably by providing the first examples of finitely generated simple groups that are amenable (and infinite) [JM13].

In the works of Matui mentioned above, both conjectures were verified for key classes of groupoids, such as AF-groupoids, transformation groupoids of minimal $\mathbb{Z}$-actions and groupoids associated to shifts of finite type (*SFT-groupoids*). Subsequently, other authors have expanded upon this. The HK conjecture has been shown to hold for Katsura–Exel–Pardo groupoids [Ort18], Deaconu–Renault groupoids of rank 1 and 2 [FKPS18] and groupoids of unstable equivalence relations on one-dimensional solenoids [Yi20].

Alas, the HK conjecture is now known to be false in general. It fails to hold for transformation groupoids associated to odometers on the infinite dihedral group, as demonstrated in [Sca18]. Nevertheless, it is still interesting to investigate for which groupoids the conclusion of the HK conjecture holds. We will say that a groupoid has the *HK property* when this is the case. In spite of them providing counterexamples to the HK conjecture, the AH conjecture was shown, also in [Sca18], to hold for transformation groupoids arising from odometers. Hence the AH conjecture remains open. A notable difference between the two conjectures is that in the AH conjecture the maps involved are specified, whereas in the HK conjecture it is only predicted that some isomorphisms exist.

B.1.2 Our results

The purpose of this paper is to investigate the AH conjecture for the class of graph groupoids. As the SFT-groupoids prominently studied by Matui can be realized as graph groupoids of finite graphs, the novelty lies in dealing with infinite (directed) graphs, and in particular with the presence of infinite emitters, that is, vertices that emit infinitely many edges.

Our main motivating example has been the graph E_∞ which has one vertex and infinitely many loops. The graph groupoid $\mathcal{G}_{E_\infty}$ is the canonical groupoid model for the (infinitely generated) Cuntz algebra $\mathcal{O}_\infty$. This was a natural example to explore as E_∞ is the simplest possible graph having an infinite emitter. On the other hand, its graph C^*-algebra $\mathcal{O}_\infty$ has played—and continues to play—an important role in the theory of C^*-algebras. Seeing as the topological full groups of the canonical graph groupoid models of the other Cuntz algebras $\mathcal{O}_n$ are isomorphic to the highly interesting Higman–Thompson groups $V_{n,1}$, we believe it worthwhile to also investigate the topological full group $[\![\mathcal{G}_{E_\infty}]\!]$.

One of the assumptions in the AH conjecture is that the unit space of the groupoid is compact, and this translates into the underlying graph having finitely many vertices. We were indeed able to show that the AH conjecture holds for these graph groupoids as well, so that our main result is the following.

Theorem B.1.1 (see Corollary B.9.5)**.** *Let E be a strongly connected graph with finitely many vertices which is not a cycle graph. Then the AH conjecture holds for the graph groupoid $\mathcal{G}_E$.*

Let us remark that Corollary B.9.5 applies to a slightly more general family of graphs than in the preceding theorem, as well as to all restrictions of these graph groupoids. The conclusion is that the AH conjecture holds for all graph groupoids covered by the assumptions in said conjecture. Additionally, it holds for any groupoid which is Kakutani equivalent to such a graph groupoid.

It should be mentioned that Matui in [Mat15b] not only proved that the AH conjecture is true for restrictions of SFT-groupoids, but that these also have the *strong AH property*. This means that the map j is injective, so that one has a short exact sequence. This was done by constructing a suitable finite presentation of the topological full group. We investigate this subject in Section B.10, but we find that when the graph has an infinite emitter, then the topological full group is not even finitely generated.

We also observe that all graph groupoids have the HK property. The following theorem is an extension of already existing similar results (see the paragraph following Theorem B.4.6).

Theorem B.1.2 (see Theorem B.4.6)**.** *Let E be any graph. Then*

$$\begin{aligned} H_0(\mathcal{G}_E) &\cong K_0(C^*(E)), \\ H_1(\mathcal{G}_E) &\cong K_1(C^*(E)), \\ H_n(\mathcal{G}_E) &= 0, \quad n \geq 2. \end{aligned}$$

Here $C^*(E)$ denotes the graph C^*-algebra of E, which is canonically isomorphic to the groupoid C^*-algebra $C_r^*(\mathcal{G}_E)$. Since the K-groups of a graph C^*-algebra are relatively easy to compute, Theorem B.1.2 allows us to give a partial description of the abelianization of the topological full group $[\![\mathcal{G}_E]\!]_{\mathrm{ab}}$ via the AH conjecture.

Our proof of the AH conjecture for graph groupoids of infinite graphs will in broad strokes follow a similar strategy as Matui's proof for finite graphs in [Mat15b]. However, we emphasize that there are several major differences which make this a nontrivial generalization. There are steps and techniques in Matui's proof that no longer work—or even make sense—in the infinite setting. A couple of significant differences are described below.

If E is a graph with infinite emitters (or sinks), then the unit space of its graph groupoid is no longer full in the associated skew product (compare [FKPS18, Lemma 6.1] and Remark B.7.2). This means that we cannot deduce that the kernel of the canonical graph cocycle is Kakutani equivalent to the skew product, and in turn we cannot identify their homologies as is done in Matui's proof.

A key component in Matui's proof is the reduction to *mixing* shifts of finite type. This is equivalent to the adjacency matrix of the associated finite graph being *primitive*. In this case, the kernel of the cocycle is a minimal AF-groupoid admitting a unique invariant probability measure arising from the Perron eigenvalue of the adjacency matrix. This measure can then be used to compare clopen subsets of the unit space and produce certain bisections connecting them. When passing to the infinite setting we lose all of this. We no longer have a shift of finite type (nor any shift space for that matter) and no Perron–Frobenius theory. Furthermore, the kernel of the cocycle is not minimal anymore.

We also wish to remark that even though certain parts of the paper are quite similar to parts of [Mat15b, Section 6], such as Section B.8 and the second half of the proof of Theorem B.9.4, we have chosen to keep the exposition mostly self-contained. We have done this in the best interest of the reader. There are several subtle differences, such as indices being shifted or reversed, and some steps being done in the opposite order. This is in part due to us having to consider the inverse of a certain map from Matui's proof, see Remarks B.7.6 and B.8.8. We supply several remarks along the way which compare our approach to Matui's to signify where they differ.

The work laid down in this paper is not done with graph groupoids alone in mind. It is our belief that these techniques can also be applied to other groupoids which have an underlying "graph skeleton", such as groupoids arising from self-similar actions by groups on graphs, as studied by Nekrashevych [Nek09] and by Exel and Pardo [EP17]. The authors plan to explore this avenue in future work. Groupoids associated to k-graphs and ultragraphs are also obvious candidates.

B.1.3 Summary

We begin in Section B.2 by giving the necessary background regarding étale groupoids. This includes the topological full group, homology and skew products by cocycles. More background is given in Section B.3, regarding graphs and their associated groupoids. The graph groupoid $\mathcal{G}_E$ associated to a graph E has a canonical $\mathbb{Z}$-valued cocycle denoted c_E. Both the skew product groupoid $\mathcal{G}_E \times_{c_E} \mathbb{Z}$ and the kernel subgroupoid $\mathcal{H}_E \coloneqq \ker(c_E) \subseteq \mathcal{G}_E$ play important roles in the rest of the paper. We show that the graph groupoids of acyclic graphs are AF-groupoids. From this we deduce that both $\mathcal{G}_E \times_{c_E} \mathbb{Z}$ and $\mathcal{H}_E$ are AF-groupoids.

In Section B.4 we describe the AH conjecture in more detail. One of the maps appearing in the AH conjecture is the *index map* $I \colon [\![\mathcal{G}]\!] \to H_1(\mathcal{G})$. We extend its definition to groupoids with non-compact unit space. Then the assumptions in the AH conjecture for graph groupoids are translated into properties of the underlying graphs. These turn out to be equivalent to the graph C^*-algebra being a unital Kirchberg algebra. We also note that all graph groupoids have the HK property by combining known results in the row-finite case with the concept of desingularization. This yields Theorem B.1.2. The graph groupoids satisfying the assumptions in the AH conjecture are shown to be purely infinite. It then follows from a result of Matui (see Remark B.4.12) that the AH conjecture is equivalent to *Property TR*. Property TR means that the kernel of the index map is generated by transpositions. Hence the rest of the paper, except for the final section, is devoted to establishing Property TR for these graph groupoids.

Section B.5 is devoted to showing that all AF-groupoids have cancellation, something which is needed several times in the proof of the main result. We point out that this cancellation result may be of independent interest. Then in Section B.6 we present two long exact sequences in ample groupoid homology. One of them relates the homology of a groupoid equipped with a cocycle with that of the associated skew product. The other relates the homology of restrictions to nested invariant subsets.

Both of these long exact sequences are applied to graph groupoids in Section B.7. This allows us to relate the homology of a graph groupoid $\mathcal{G}_E$ with both the skew product $\mathcal{G}_E \times_{c_E} \mathbb{Z}$ and the kernel $\mathcal{H}_E$. As the latter two are AF-groupoids, this truncates the long exact sequences to finite exact sequences. After some work, we obtain the embeddings $H_1(\mathcal{G}_E) \hookrightarrow H_0(\mathcal{H}_E) \hookrightarrow H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$. In particular, we identify $H_1(\mathcal{G}_E)$ with $\ker(\mathrm{id} - \varphi)$, where φ is an endomorphism of $H_0(\mathcal{H}_E)$ given by "extending paths backwards". We have to do some extra work here because we cannot deduce that $H_0(\mathcal{H}_E) \cong H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$, as one can for finite graphs. In Section B.8 we associate each element α in the topological full group $[\![\mathcal{G}_E]\!]$ with a finite clopen partition of the unit space $\mathcal{G}_E^{(0)}$. This partition is then used to give a description of the value $I(\alpha)$ of the index map under the correspondence $H_1(\mathcal{G}_E) \cong \ker(\mathrm{id} - \varphi)$ from the previous section.

The proof of our main result, Theorem B.1.1, is given in Section B.9. We begin the section by proving a technical lemma which plays a similar role as mixing of the shift space does in Matui's proof for SFT-groupoids. The way it is used in our proof, however, is quite different from the way mixing is used. Next we show that the assumptions in said lemma can always be arranged, by appealing to the geometric moves on graphs from the classification program of unital graph C^*-algebras [ERRS16]. After that we prove that strongly connected graphs with infinite emitters have Property TR. The proof is quite long and draws upon all of the preceding sections. By combining Matui's result for strongly connected finite graphs with our result for infinite graphs, together with another geometric move on graphs, we deduce that the AH conjecture holds for all graph groupoids satisfying the assumptions in the AH conjecture.

We end the paper with Section B.10 where we give a couple of examples and obtain some consequences of the AH conjecture. In particular, we consider the canonical graph groupoid model of $\mathcal{O}_\infty$ and observe that either the topological full group $[\![\mathcal{G}_{E_\infty}]\!]$ is simple or $\mathcal{G}_{E_\infty}$ has the strong AH property, but not both. In fact, these two properties are shown to be mutually exclusive whenever the graph has an infinite emitter. This is in contrast to the case of finite graphs, where one can have both. We also observe that when E has an infinite emitter, then $[\![\mathcal{G}_E]\!]$ is not finitely generated. A partial description of the abelianization $[\![\mathcal{G}_E]\!]_{\mathrm{ab}}$ is also given in terms of the first two homology groups.

B.2 Étale groupoids

In this section we will collect the basic notions regarding étale groupoids that we will need, as well as establish notation and conventions. Two standard references for étale groupoids (and their C^*-algebras) are Renault's thesis [Ren80] and Paterson's book [Pat99]. More recent accounts are found in e.g. [Exe08] and [Sim17].

If two sets A and B are disjoint we will denote their union by $A \sqcup B$ when we wish to emphasize that they are disjoint. When we write $C = A \sqcup B$ we mean that $C = A \cup B$ and that A and B are disjoint sets.

B.2.1 Topological groupoids

A *groupoid* is a set $\mathcal{G}$ equipped with a partially defined product $\mathcal{G}^{(2)} \to \mathcal{G}$ denoted $(g, h) \mapsto gh$, where $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ is the set of *composable pairs*, and an everywhere defined involutive inverse $g \mapsto g^{-1}$ satisfying the following axioms:

1. If $(g_1, g_2), (g_2, g_3) \in \mathcal{G}^{(2)}$, then we have $(g_1 g_2, g_3), (g_1, g_2 g_3) \in \mathcal{G}^{(2)}$ and $(g_1 g_2) g_3 = g_1 (g_2 g_3)$.

2. For all $g \in \mathcal{G}$, we have $(g, g^{-1}), (g^{-1}, g) \in \mathcal{G}^{(2)}$.

3. If $(g,h) \in \mathcal{G}^{(2)}$, then $ghh^{-1} = g$ and $g^{-1}gh = h$.

The set $\mathcal{G}^{(0)} := \{gg^{-1} \mid g \in \mathcal{G}\}$ is called the *unit space*, and the maps $r, s \colon \mathcal{G} \to \mathcal{G}^{(0)}$ given by $r(g) = gg^{-1}$ and $s(g) = g^{-1}g$ are called the *range* and *source* maps, respectively.

If $\mathcal{G}$ is given a topology in which the product and inverse map are continuous we call $\mathcal{G}$ a topological groupoid. A topological groupoid is *étale* if it has a locally compact topology in which the unit space is open and Hausdorff, and the range and source maps are local homeomorphisms. For the most part we will be dealing with étale groupoids which are (globally) Hausdorff, and then $\mathcal{G}^{(0)}$ is clopen in $\mathcal{G}$. We say that an étale groupoid $\mathcal{G}$ is *ample* if $\mathcal{G}^{(0)}$ is zero-dimensional, i.e. admits a basis of compact open sets. Étale groupoids are characterized by admitting a basis of *bisections* (defined below), and ample groupoids by admitting a basis of *compact bisections*.

For a subset $A \subseteq \mathcal{G}^{(0)}$ we set

$$\mathcal{G}^A := \{g \in \mathcal{G} \mid r(g) \in A\} \quad \text{and} \quad \mathcal{G}_A := \{g \in \mathcal{G} \mid s(g) \in A\}.$$

For singleton sets $A = \{x\}$ we drop the braces and write $\mathcal{G}^x$ and $\mathcal{G}_x$, respectively. The *isotropy group* of $x \in \mathcal{G}^{(0)}$ is $\mathcal{G}_x^x := \mathcal{G}^x \cap \mathcal{G}_x$, and the *isotropy* of $\mathcal{G}$ is

$$\mathcal{G}' := \bigsqcup_{x \in \mathcal{G}^{(0)}} \mathcal{G}_x^x.$$

We say that $\mathcal{G}$ is *principal* if $\mathcal{G}' = \mathcal{G}^{(0)}$, and *effective* if the interior of $\mathcal{G}'$ equals $\mathcal{G}^{(0)}$. We remark that the literature is not entirely consistent regarding this notion. For example in [Mat15b] the term "essentially principal" is used. The term *topologically principal* also appear in the literature, but this usually refers to a slightly stronger notion.

The $\mathcal{G}$*-orbit* of a unit x is the set $\mathrm{Orb}_{\mathcal{G}}(x) := s(\mathcal{G}^x) = r(\mathcal{G}_x)$. We call $\mathcal{G}$ *minimal* when every $\mathcal{G}$-orbit is dense in $\mathcal{G}^{(0)}$. This is equivalent to there being no nontrivial open (or closed) $\mathcal{G}$*-invariant* subsets $A \subseteq \mathcal{G}^{(0)}$, meaning that $\mathcal{G}^A = \mathcal{G}_A$. The *restriction* of $\mathcal{G}$ to A is $\mathcal{G}|_A := \mathcal{G}^A \cap \mathcal{G}_A$, and this is a subgroupoid of $\mathcal{G}$ with unit space A. If A is open and $\mathcal{G}$ is étale, then $\mathcal{G}_{|A}$ is an open étale subgroupoid of $\mathcal{G}$. We say that A is $\mathcal{G}$*-full* if $r(\mathcal{G}_A) = \mathcal{G}^{(0)}$, in other words if A intersects every $\mathcal{G}$-orbit. Two étale groupoids $\mathcal{G}$ and $\mathcal{H}$ are *Kakutani equivalent* if there exists a $\mathcal{G}$-full clopen subset $A \subseteq \mathcal{G}^{(0)}$ and an $\mathcal{H}$-full clopen subset $B \subseteq \mathcal{H}^{(0)}$ such that $\mathcal{G}|_A \cong \mathcal{H}|_B$ (as topological groupoids). This notion of groupoid equivalence admits many different descriptions, see [FKPS18, Theorem 3.12].

B.2.2 The topological full group

An open subset $U \subseteq \mathcal{G}$ of an étale groupoid $\mathcal{G}$ is called a *bisection* if both r and s are injective on U. It follows then that $r|_U \colon U \to r(U)$ is a homeomorphism, and similarly for s. Thus we get a homeomorphism $\pi_U := r_{|U} \circ (s_{|U})^{-1}$ from $s(U)$ to $r(U)$ which maps $s(g)$ to $r(g)$ for each $g \in U$. We say that the bisection U is *full* if $r(U) = s(U) = \mathcal{G}^{(0)}$, and in this case π_U is a homeomorphism of $\mathcal{G}^{(0)}$. For a homeomorphism $\alpha \colon X \to X$ of a topological space X we define the *support* of α to be the set $\operatorname{supp}(\alpha) := \overline{\{x \in X \mid \alpha(x) \neq x\}}$.

The *topological full group* of an effective étale groupoid $\mathcal{G}$ is

$$[\![\mathcal{G}]\!] := \{\pi_U \mid U \subseteq \mathcal{G} \text{ full bisection with } \operatorname{supp}(\pi_U) \text{ is compact}\},$$

which is a subgroup of the homeomorphism group of $\mathcal{G}^{(0)}$. The commutator subgroup of $[\![\mathcal{G}]\!]$ is denoted by $\mathsf{D}([\![\mathcal{G}]\!])$. We remark that when $\mathcal{G}$ is effective and Hausdorff, then $\operatorname{supp}(\pi_U)$ is also open for any full bisection U. If $V \neq U$ are different bisections, then $\pi_U \neq \pi_V$. As a notational remark, if we are given an element $\alpha \in [\![\mathcal{G}]\!]$ we let U_α denote the unique full bisection which gives rise to α, i.e. the one with $\alpha = \pi_{U_\alpha}$.

The following construction will be used several times. Suppose $U \subseteq \mathcal{G}$ is a compact bisection with $r(U) \cap s(U) = \emptyset$. Define

$$\widehat{U} := U \sqcup U^{-1} \sqcup \left(\mathcal{G}^{(0)} \setminus (r(U) \cup s(U))\right).$$

Then $\widehat{U}$ is a full bisection and its associated homeomorphism $\pi_{\widehat{U}}$ satisfies

$$\begin{aligned} \pi_{\widehat{U}}(s(U)) &= r(U), & \pi_{\widehat{U}}(r(U)) &= s(U), \\ \operatorname{supp}(\pi_{\widehat{U}}) &= r(U) \cup s(U), & \left(\pi_{\widehat{U}}\right)^2 &= \mathrm{id}_{\mathcal{G}^{(0)}}. \end{aligned}$$

It is clear that $\pi_{\widehat{U}} \in [\![\mathcal{G}]\!]$. If $\tau \in [\![\mathcal{G}]\!]$ is an element satisfying $\tau^2 = 1$ and the set $\{x \in \mathcal{G}^{(0)} \mid \tau(x) = x\}$ is clopen, then one can show that $\tau = \pi_{\widehat{U}}$ for some compact bisection U as above. Following [Mat15b], [Mat16] we call these elements *transpositions*. We let $\mathcal{S}(\mathcal{G})$ denote the (normal) subgroup of $[\![\mathcal{G}]\!]$ generated by all transpositions, as in [Nek19].

Remark B.2.1. Some authors define the topological full group to consist of the full bisections themselves, rather than their associated homeomorphisms, but for effective groupoids this is merely a matter of taste. Topological full groups are quite interesting objects in their own right and we refer to [Mat17] and [NO19] and the references therein for more details on the subject.

B.2.3 Homology for ample groupoids

For an ample Hausdorff groupoid $\mathcal{G}$, let us describe its homology with values in $\mathbb{Z}$, as popularized by Matui in [Mat12] building on the general theory of [CM00]. See also [FKPS18, Section 4] for an excellent account.

For a locally compact Hausdorff space X, let $C_c(X,\mathbb{Z})$ denote the compactly supported continuous $\mathbb{Z}$-valued functions on X. Suppose $\psi\colon X \to Y$ is a local homeomorphism between such spaces. Then ψ induces a homomorphism $\psi_*\colon C_c(X,\mathbb{Z}) \to C_c(Y,\mathbb{Z})$ which is given by $\psi_*(f)(y) = \sum_{x\in\psi^{-1}(y)} f(x)$ for $f \in C_c(X,\mathbb{Z})$. Only finitely many terms are nonzero in this sum.

For $n \geq 1$, let $\mathcal{G}^{(n)}$ denote the space of composable strings of n elements from $\mathcal{G}$, equipped with the relative topology induced by the product topology on n copies of $\mathcal{G}$. In particular, $\mathcal{G}^{(2)}$ is the composable pairs, $\mathcal{G}^{(1)} = \mathcal{G}$ and for $n = 0$, we have the unit space $\mathcal{G}^{(0)}$. Define local homeomorphisms $d_i\colon \mathcal{G}^{(n)} \to \mathcal{G}^{(n-1)}$ for $n \geq 2$ and $i = 0,\dots,n$ by

$$d_i(g_1,g_2,\dots,g_n) = \begin{cases} (g_2,g_3,\dots,g_n) & \text{if } i = 0,\\ (g_1,\dots,g_{i-1},g_ig_{i+1},g_{i+2},\dots,g_n) & \text{if } 1 \leq i \leq n-1,\\ (g_1,g_2,\dots,g_{n-1}) & \text{if } i = n. \end{cases}$$

From these we in turn define homomorphisms $\delta_n\colon C_c(\mathcal{G}^{(n)},\mathbb{Z}) \to C_c(\mathcal{G}^{(n-1)},\mathbb{Z})$ by setting $\delta_n = \sum_{i=0}^{n}(-1)^i(d_i)_*$, and for $n = 1$ set $\delta_1 = s_* - r_*$. Then

$$0 \longleftarrow C_c(\mathcal{G}^{(0)},\mathbb{Z}) \xleftarrow{\delta_1} C_c(\mathcal{G}^{(1)},\mathbb{Z}) \xleftarrow{\delta_2} C_c(\mathcal{G}^{(2)},\mathbb{Z}) \xleftarrow{\delta_3} \cdots \tag{B.2.1}$$

becomes a chain complex and the homology groups $H_n(\mathcal{G})$ is defined as the homology of this complex, i.e. $H_n(\mathcal{G}) = \ker\delta_n/\operatorname{im}\delta_{n+1}$. We will use $C_\bullet(\mathcal{G},\mathbb{Z})$ to denote the chain complex (B.2.1).

Since the zeroth and first homology groups will appear frequently in this text, by virtue of being ingredients in the AH conjecture, we describe the two homomorphisms δ_1 and δ_2 that define them in more detail. The former is the difference of the maps from $C_c(\mathcal{G},\mathbb{Z})$ to $C_c(\mathcal{G}^{(0)},\mathbb{Z})$ induced by the source and range maps, and these are in turn given by

$$s_*(f)(x) = \sum_{g\in\mathcal{G}_x} f(g) \quad\text{and}\quad r_*(f)(x) = \sum_{g\in\mathcal{G}^x} f(g)$$

for $f \in C_c(\mathcal{G},\mathbb{Z})$ and $x \in \mathcal{G}^{(0)}$. The latter is $\delta_2 = (d_0)_* - (d_1)_* + (d_2)_*$, where each of these summands are maps from $C_c(\mathcal{G}^{(2)},\mathbb{Z})$ to $C_c(\mathcal{G},\mathbb{Z})$ given by

$$\begin{aligned}
(d_0)_*(\psi)(g) &= \sum_{h\in\mathcal{G},\ s(h)=r(g)} \psi(h,g)\\
(d_1)_*(\psi)(g) &= \sum_{(h_1,h_2)\in\mathcal{G}^{(2)},\ h_1h_2=g} \psi(h_1,h_2)\\
(d_2)_*(\psi)(g) &= \sum_{h\in\mathcal{G},\ r(h)=s(g)} \psi(g,h)
\end{aligned}$$

for $\psi \in C_c(\mathcal{G}^{(2)}, \mathbb{Z})$ and $g \in \mathcal{G}$.

Observe that H_0 is spanned (over $\mathbb{Z}$) by equivalence classes of indicator functions of compact open subsets of the unit space. For any compact bisection $U \subseteq \mathcal{G}$ we have $[1_{s(U)}] = [1_{r(U)}]$ in $H_0(\mathcal{G})$, since $\delta_1(1_U) = 1_{s(U)} - 1_{r(U)}$. If we view a compact open set $A \subseteq \mathcal{G}^{(0)}$ as a subset of $\mathcal{G}$, then $1_A \in \ker \delta_1$ and $[1_A] = 0$ in $H_1(\mathcal{G})$ since $\delta_2(1_{\Delta A}) = 1_A$, where $\Delta A \subseteq \mathcal{G}^{(2)}$ denotes the diagonal in $A \times A$.

Any étale homomorphism (that is, a local homeomorphism which respects the groupoid operations) $\rho\colon \mathcal{G} \to \mathcal{H}$ between ample Hausdorff groupoids induces local homeomorphisms $\rho^{(n)}\colon \mathcal{G}^{(n)} \to \mathcal{H}^{(n)}$ for $n \geq 0$ by applying ρ in each coordinate. The induced maps $(\rho^{(n)})_*$ from $C_c(\mathcal{G}^{(n)}, \mathbb{Z})$ to $C_c(\mathcal{H}^{(n)}, \mathbb{Z})$ form a chain map $\rho_\bullet\colon C_\bullet(\mathcal{G}, \mathbb{Z}) \to C_\bullet(\mathcal{H}, \mathbb{Z})$ which in turn induce homomorphisms $H_n(\rho_\bullet)\colon H_n(\mathcal{G}) \to H_n(\mathcal{H})$. This assignment is functorial. In particular, if $\mathcal{G} \subseteq \mathcal{H}$ is an open subgroupoid, then the inclusion map $\iota\colon \mathcal{G} \hookrightarrow \mathcal{H}$ induces homomorphisms $H_n(\iota_\bullet)\colon H_n(\mathcal{G}) \to H_n(\mathcal{H})$. For $n = 0$ the map $H_0(\iota_\bullet)\colon H_0(\mathcal{G}) \to H_0(\mathcal{H})$ is given by $[1_A] \mapsto [1_A]$ for any compact open set $A \subseteq \mathcal{G}^{(0)} \subseteq \mathcal{H}^{(0)}$.

If $Y \subseteq \mathcal{G}^{(0)}$ is a $\mathcal{G}$-full clopen set, then the inclusion map $\iota\colon \mathcal{G}|_Y \hookrightarrow \mathcal{G}$ induces *isomorphisms* $H_n(\iota_\bullet)\colon H_n(\mathcal{G}|_Y) \xrightarrow{\cong} H_n(\mathcal{G})$ for all $n \geq 0$ [FKPS18, Lemma 4.3]. From this it is clear that Kakutani equivalent groupoids have the same homology. For $n = 0$ the inverse map $H_0(\iota_\bullet)^{-1}\colon H_0(\mathcal{G}) \to H_0(\mathcal{G}|_Y)$ can be described as follows. Let $A \subseteq \mathcal{G}^{(0)}$ be a compact open set. By fullness of Y, we can for each $x \in A$ find a compact bisection $U_x \subseteq \mathcal{G}$ with $x \in s(U_x) \subseteq A$ and $r(U_x) \subseteq Y$. By compactness and zero-dimensionality we can find finitely many compact bisections $U_1, \ldots, U_m$ so that the $s(U_i)$'s form a clopen partition of A and so that $r(U_i) \subseteq Y$. Now $[1_A] = \sum_{i=1}^m [1_{s(U_i)}] = \sum_{i=1}^m [1_{r(U_i)}]$ in $H_0(\mathcal{G})$, and we thus have

$$H_0(\iota_\bullet)^{-1}([1_A]) = \sum_{i=1}^{m} [1_{r(U_i)}] \in H_0(\mathcal{G}|_Y). \tag{B.2.2}$$

B.2.4 AF-groupoids and their homology

Let $\mathcal{R}_n$ denote the full equivalence relation on the finite set $\{1, 2, \ldots, n\}$, viewed as a discrete groupoid. When X is a locally compact Hausdorff space, Renault [Ren80] calls the product groupoid $X \times \mathcal{R}_n$ an *elementary groupoid of type* n, where we view X as a trivial groupoid $X = X^{(0)}$. We will call an étale groupoid $\mathcal{G}$ *elementary* if it is Hausdorff, principal and $\mathcal{G} \setminus \mathcal{G}^{(0)}$ is compact. Lemma 3.4 in [GPS04] shows that an ample elementary groupoid is isomorphic to a finite disjoint union of elementary groupoids of type n_i. An *AF-groupoid* is an ample groupoid which can be written as an increasing union of open elementary subgroupoids.

It is a well-known fact that when $\mathcal{G}$ is an AF-groupoid, its homology is given by

$$H_n(\mathcal{G}) \cong \begin{cases} K_0(C_r^*(\mathcal{G})) & n = 0, \\ 0 & n \geq 1, \end{cases}$$

where $C_r^*(\mathcal{G})$ denotes the reduced groupoid C^*-algebra of $\mathcal{G}$, which in this case is an AF-algebra. The H_0-group (and the K_0-group) coincides with the dimension group of any defining Bratteli diagram (as an ordered abelian group with distuingished order unit, see [Mat12, Theorem 4.10]). Stated like this it first appeared in [Mat12] (for compact unit spaces), but it can be traced back to the earlier works [Ren80] and [Kri80]. The case of a non-compact unit space is treated in [FKPS18].

Theorem B.2.2 ([FKPS18, Corollary 5.2])**.** *Let $\mathcal{G}$ be an AF-groupoid. Then the map*

$$[1_A]_{H_0} \mapsto [1_A]_{K_0}$$

for $A \subseteq \mathcal{G}^{(0)}$ compact open induces an isomorphism $H_0(\mathcal{G}) \cong K_0(C_r^*(\mathcal{G}))$.

B.2.5 Cocycles and skew products

When $\mathcal{G}$ is an étale groupoid and Γ is a discrete group, we call $c \colon \mathcal{G} \to \Gamma$ a *cocycle* if it is a continuous groupoid homomorphism. We shall be dealing exclusively with $\mathbb{Z}$-valued cocycles, as these are the ones that appear naturally for graph groupoids.

Definition B.2.3. Let $\mathcal{G}$ be an étale groupoid with a cocycle $c \colon \mathcal{G} \to \mathbb{Z}$. The *skew product groupoid* of $\mathcal{G}$ by c is the groupoid $\mathcal{G} \times_c \mathbb{Z} := \mathcal{G} \times \mathbb{Z}$ with operations

$$(g, m)(h, m + c(g)) := (gh, m) \text{ if } (g, h) \in \mathcal{G}^{(2)} \quad \text{and} \quad (g, m)^{-1} := (g^{-1}, m + c(g)),$$

so that $s(g, m) = (s(g), c(g) + m)$ and $r(g, m) = (r(g), m)$.

The skew product groupoid becomes an étale groupoid in the product topology. The unit space of $\mathcal{G} \times_c \mathbb{Z}$ can be identified with $\mathcal{G}^{(0)} \times \mathbb{Z}$. For each bisection $U \subseteq \mathcal{G}$ and $m \in \mathbb{Z}$, the set $U \times \{m\}$ is a bisection in $\mathcal{G} \times_c \mathbb{Z}$. We record the following elementary lemma about the kernel of the cocycle sitting inside the skew product.

Lemma B.2.4. *Let $\mathcal{G}$ be an étale groupoid with a cocycle $c \colon \mathcal{G} \to \mathbb{Z}$. Then* $\ker(c)$ *is a clopen subgroupoid of $\mathcal{G}$, and we have $(\mathcal{G} \times_c \mathbb{Z})|_{\mathcal{G}^{(0)} \times \{0\}} \cong \ker(c)$ via the map $(g, 0) \mapsto g$.*

Remark B.2.5. We emphasize that even though $\ker(c)$ is a clopen subgroupoid of $\mathcal{G}$, and embeds as a clopen subgroupoid of the skew product $\mathcal{G} \times_c \mathbb{Z}$, we can generally not embed $\mathcal{G}$ itself into $\mathcal{G} \times_c \mathbb{Z}$ in any way (e.g. $\mathcal{G} \times_c \mathbb{Z}$ can be principal while $\mathcal{G}$ is not.)

There is a canonical action $\widehat{c}$ by $\mathbb{Z}$ on $\mathcal{G} \times_c \mathbb{Z}$ defined by $\widehat{c}_k \cdot (g, m) = (g, m + k)$, i.e. shifting the integer coordinate. If one then forms the semi-direct product groupoid $(\mathcal{G} \times_c \mathbb{Z}) \rtimes_{\widehat{c}} \mathbb{Z}$, one gets that this semi-direct product is Kakutani equivalent to the groupoid $\mathcal{G}$ that we started with, and hence they have the same homology groups [Mat12]. This is what Matui uses when he computes the homology groups of $\mathcal{G}_E$ for a finite graph E by means of a spectral sequence [Mat15b]. We shall instead use a long exact sequence in homology from [Ort18], to be described in Section B.6.

B.3 Graphs and their groupoids

As this paper primarily concerns graph groupoids, we spend some time in this section recalling their definition and properties, as well as establishing notation. We refer to [BCW17] and [NO19] for additional details.

B.3.1 Graphs

A *(directed) graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0 and E^1, whose elements are called vertices and edges, respectively, in addition to range and source maps $r, s \colon E^1 \to E^0$. We say that E is *finite* if both E^0 and E^1 are finite sets.

A *path* is a sequence of edges $\mu = e_1 e_2 \ldots e_n$ such that $r(e_i) = s(e_{i+1})$ for $1 \leq i \leq n - 1$. The *length* of μ is $|\mu| := n$. The set of paths of length n is denoted E^n and the set of all finite paths is $E^* := \bigcup_{n=0}^{\infty} E^n$. The range and source maps extend to E^* by setting $r(\mu) = r(e_n)$ and $s(\mu) = s(e_1)$. For $v \in E^0$, we set $s(v) = r(v) = v$. If $\mu, \nu \in E^*$ satisfy $r(\mu) = s(\nu)$, then $\mu\nu \in E^*$ denotes their concatenation. We say that μ is a *subpath* of ν if $\nu = \mu\lambda$ for some path λ with $s(\lambda) = r(\mu)$. Two paths are called *disjoint* if neither is a subpath of the other. A graph E is called *strongly connected* if for each pair of vertices $v, w \in E^0$ there is a path from v to w. By a *strongly connected component* we mean a maximal subset of vertices such that there is a path between any two vertices in this subset. The strongly connected components form a partition of E^0.

An edge $e \in E^1$ with $r(e) = s(e)$ is called a *loop*. More generally, a *cycle* is a nontrivial path μ (i.e. $|\mu| \geq 1$) with $r(\mu) = s(\mu)$, and we say that μ is *based* at $s(\mu)$ or that $s(\mu)$ *supports* the cycle μ. By μ^k we mean μ concatenated k times. A graph is called *acyclic* if it has no cycles. An *exit* for a path $\mu = e_1 \ldots e_n$ is an edge $e \in E^1$ such that $s(e) = s(e_i)$ and $e \neq e_i$ for some $1 \leq i \leq n$. The graph E is said to satisfy *Condition (L)* if every cycle in E has an exit.

For a vertex $v \in E^0$ and $n \geq 1$ we define the sets $vE^n := \{\mu \in E^n \mid s(\mu) = v\}$ and $E^n v := \{\mu \in E^n \mid r(\mu) = v\}$. We call v a *sink* if $vE^1 = \emptyset$ and a *source* if $E^1 v = \emptyset$. Furthermore, v is called an *infinite emitter* if vE^1 is an infinite set. Sinks and infinite emitters are collectively referred to as *singular* vertices and the set of these is denoted E^0_{sing}. Non-singular vertices are called *regular*. A graph is *row-finite* if it has no infinite emitters, and *essential* if it has no sinks nor sources.

B.3.2 The boundary path space

An *infinite path* in a graph E is a sequence of edges $x = e_1e_2e_3\ldots$ such that $r(e_i) = s(e_{i+1})$ for all $i \in \mathbb{N}$. We define $s(x) \coloneqq s(e_1)$ and $|x| \coloneqq \infty$. The set of all infinite paths is denoted E^∞. We call E *cofinal* if for every vertex $v \in E^0$ and for every infinite path $e_1e_2\ldots \in E^\infty$, there is a path from v to $s(e_n)$ for some $n \in \mathbb{N}$. The *boundary path space* of E is

$$\partial E \coloneqq E^\infty \cup \{\mu \in E^* \mid r(\mu) \in E^0_{\text{sing}}\}.$$

The *cylinder set* of a finite path $\mu \in E^*$ is $Z(\mu) \coloneqq \{\mu x \mid x \in \partial E,\ s(x) = r(\mu)\}$. Given a finite subset $F \subseteq r(\mu)E^1$, we define the associated *punctured cylinder set* to be $Z(\mu \setminus F) \coloneqq Z(\mu) \setminus \left(\bigsqcup_{e\in F} Z(\mu e)\right)$. Note that two finite paths are disjoint if and only if their cylinder sets are disjoint sets.

The topology on the boundary path space ∂E is specified by the countable basis $\left\{Z(\mu \setminus F) \mid \mu \in E^*, F \subseteq_{\text{finite}} r(\mu)E^1\right\}$. This turns ∂E into a locally compact Hausdorff space in which each basic set $Z(\mu \setminus F)$ is compact open [Web14]. Note that the boundary path space ∂E itself is compact if and only if E^0 is finite. Existence of isolated points in ∂E is characterized in [CW18, Section 3].

Define $\partial E^{\geq n} \coloneqq \{x \in \partial E \mid |x| \geq n\}$ for $n \in \mathbb{N}$, which are open subsets of ∂E. The *shift map* on E is the map $\sigma_E \colon \partial E^{\geq 1} \to \partial E$ given by

$$\sigma_E(e_1e_2e_3\ldots) = e_2e_3e_4\ldots$$

for $e_1e_2e_3\ldots \in \partial E^{\geq 2}$ and $\sigma_E(e) = r(e)$ for $e \in \partial E \cap E^1$. The image $\sigma_E\left(\partial E^{\geq 1}\right)$ is also open in ∂E and the shift map is surjective precisely when E has no sources. We also set $\sigma_E^0 = \mathrm{id}_{\partial E}$. Then the iterates $\sigma_E^n \colon \partial E^{\geq n} \to \partial E$ are local homeomorphisms for each $n \geq 0$.

B.3.3 Graph groupoids

The *graph groupoid* of a graph E is

$$\mathcal{G}_E \coloneqq \{(x, m-n, y) \mid m, n \geq 0,\ x \in \partial E^{\geq m},\ y \in \partial E^{\geq n},\ \sigma_E^m(x) = \sigma_E^n(y)\},$$

equipped with the product $(x, k, y) \cdot (y, l, z) \coloneqq (x, k + l, z)$ (and undefined otherwise), and inverse $(x, k, y)^{-1} \coloneqq (y, -k, x)$. In other words, a triplet (x, k, y) in $\partial E \times \mathbb{Z} \times \partial E$ belongs to the graph groupoid $\mathcal{G}_E$ if and only if $x = \mu z$ and $y = \nu z$ for some finite paths $\mu, \nu \in E^*$ and a boundary path $z \in \partial E$ satisfying $|\mu| = |\nu| + k$.

Given two finite paths $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$ and a finite subset $F \subseteq r(\mu)E^1$ we define the associated *punctured double cylinder set* to be the following subset of $\mathcal{G}_E$:

$$Z(\mu, F, \nu) := \{(x, |\mu| - |\nu|, y) \mid x \in Z(\mu \setminus F),\ y \in Z(\nu \setminus F),\ \sigma_E^{|\mu|}(x) = \sigma_E^{|\nu|}(y)\}.$$

Equipping the graph groupoid $\mathcal{G}_E$ with the topology generated by the countable basis

$$\left\{Z(\mu, F, \nu) \mid \mu, \nu \in E^*, r(\mu) = r(\nu), F \subseteq_{\text{finite}} r(\mu)E^1\right\}$$

turns it into an ample Hausdorff groupoid, as each $Z(\mu, F, \nu)$ becomes a compact open bisection. That this indeed is the standard topology on $\mathcal{G}_E$, as in e.g. [BCW17], was shown in Lemma A.9.2.

The unit space of $\mathcal{G}_E$ is $\mathcal{G}_E^{(0)} = \{(x, 0, x) \mid x \in \partial E\}$, which we will freely identify with the boundary path space ∂E via the homeomorphism $(x, 0, x) \leftrightarrow x$. In terms of the bases we identify $Z(\mu, F, \mu)$ with $Z(\mu \setminus F)$. The range and source maps of $\mathcal{G}_E$ then become $r(x, k, y) = x$ and $s(x, k, y) = y$. For a basic compact open bisection as above we have $r(Z(\mu, F, \nu)) = Z(\mu \setminus F)$ and $s(Z(\mu, F, \nu)) = Z(\nu \setminus F)$.

A graph groupoid $\mathcal{G}_E$ is effective precisely when E satisfies Condition (L) ([BCW17, Proposition 2.3]), and $\mathcal{G}_E$ is minimal if and only if E is both cofinal and there exists a path from every vertex to every singular vertex (Proposition A.8.3). On any graph groupoid there is a canonical cocycle $c_E : \mathcal{G}_E \to \mathbb{Z}$ given by $(x, k, y) \mapsto k$. We define

$$\mathcal{H}_E := \ker(c_E) = \{(x, 0, y) \in \mathcal{G}_E\},$$

which is a clopen subgroupoid of $\mathcal{G}_E$. The subgroupoid $\mathcal{H}_E$ and the skew product groupoid $\mathcal{G}_E \times_{c_E} \mathbb{Z}$ will play important roles in the proof of the AH conjecture for $\mathcal{G}_E$.

The full and the reduced groupoid C^*-algebra of a graph groupoid coincide. There is a canonical isomorphism $C_r^*(\mathcal{G}_E) \cong C^*(E)$ which is given by mapping the indicator function $1_{Z(v,v)} \in C_c(\mathcal{G}_E, \mathbb{C})$ to the projection $p_v \in C^*(E)$ for each $v \in E^0$ and mapping $1_{Z(e,r(e))} \in C_c(\mathcal{G}_E, \mathbb{C})$ to the partial isometry $s_e \in C^*(E)$ for each $e \in E^1$ [BCW17, Proposition 2.2]. For an introduction to graph C^*-algebras, see [Rae05].

B.3.4 The skew graph

Let E be a graph. The *skew graph* of E, denoted $E \times \mathbb{Z}$, is the graph with vertices $(E \times \mathbb{Z})^0 = E^0 \times \mathbb{Z}$ and edges $(E \times \mathbb{Z})^1 = E^1 \times \mathbb{Z}$, such that $s(e, i) = (s(e), i)$ and $r(e, i) = (r(e), i - 1)$. See Figure B.1 for an example.

The skew graph $E \times \mathbb{Z}$ played a part in the computation of K-theory for graph C^*-algebras [RS04]. A useful fact is that the skew graph is always acyclic, and

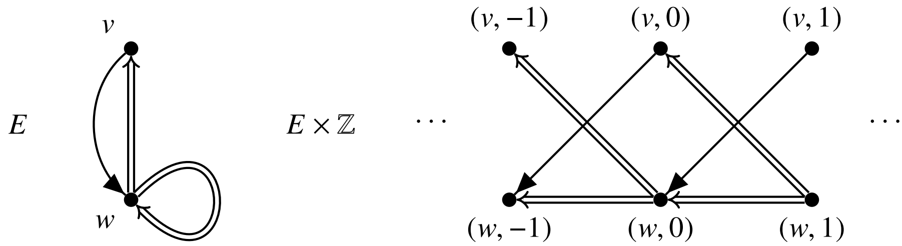


Figure B.1: An example of a graph and its skew graph. A double arrow indicates that there are infinitely many edges.

therefore its graph C^*-algebra, $C^*(E\times\mathbb{Z})$, is an AF-algebra [DT05, Corollary 2.13]. Thus its K_1 group vanishes, which in turn allows the K-theory of $C^*(E)$ to be computed from a suitable six-term exact sequence which relates the K-theory of the skew graph C^*-algebra with that of the original graph C^*-algebra. As Matui and others have noticed, one can do something similar for graph groupoids to compute their homology, see [Mat12], [Ort18], [FKPS18]. We will turn to this in Section B.7. For now, let us note that the skew graph corresponds to taking the skew product of the graph groupoid by the canonical graph cocycle.

Lemma B.3.1. *For any graph E we have that $\mathcal{G}_E \times_{c_E} \mathbb{Z} \cong \mathcal{G}_{E\times\mathbb{Z}}$ as étale groupoids via the map $((x,k,y),m) \mapsto (x^{(m)}, k, y^{(m+k)})$, where $x^{(m)} \in \partial(E\times\mathbb{Z})$ denotes the boundary path whose edges correspond to those in x, but which is anchored at level m in $E\times\mathbb{Z}$.*

Throughout this paper it will be crucial that the skew product of any graph groupoid is an AF-groupoid. This was observed for finite graphs in [Mat12] and for row-finite graphs it follows from [FKPS18, Lemma 6.1]. Since we are allowing infinite emitters in our graphs, we include an argument covering the general case.

Proposition B.3.2. *Let E be an acyclic graph. Then $\mathcal{G}_E$ is an AF-groupoid.*

Proof. Recall that all graphs are assumed to be countable. Therefore we can find an increasing sequence of finite subgraphs $F_1 \subseteq F_2 \subseteq F_3 \subseteq \ldots$ of E such that $\cup_{n=1}^{\infty} F_n = E$. From these we define the following finite sets of pairs of paths

$$\mathcal{E}_n := \{(\mu,\nu) \in (F_n)^* \times (F_n)^* \mid r(\mu) = r(\nu)\}.$$

We claim that the following subsets of $\mathcal{G}_E$ form an exhaustive sequence of open elementary subgroupoids:

$$\mathcal{K}_{E,n} \coloneqq \mathcal{G}_E^{(0)} \bigcup \bigcup_{(\mu,\nu)\in\mathcal{E}_n} Z(\mu,\nu).$$

A priori, it is not entirely clear that the $\mathcal{K}_{E,n}$'s are closed under multiplication (in $\mathcal{G}_E$). This relies on the acyclicity of E, and we provide an argument below.

Suppose $g, h \in \mathcal{K}_{E,n}$ and that the product $g \cdot h$ is defined (i.e. the source of h is the range of g). This means that $g = (\mu x, k, \nu x) \in Z(\mu,\nu)$ and that $h = (\rho y, l, \tau y) \in Z(\rho,\tau)$, where μ,ν,ρ,τ are finite paths in F_n and $\nu x = \rho y$. The latter equality implies that either $\nu \leq \rho$ or $\nu \geq \rho$. Assuming that $\nu \leq \rho$ (the other case proceeds similarly), there is a finite path γ, necessarily also in F_n, such that $\rho = \nu\gamma$. Then we have $x = \gamma y$, which means that $g \cdot h = (\mu\gamma y, k + l, \tau y)$. Since E is acyclic, $\mathcal{G}_E$ is principal and therefore we must have $k + l = |\mu\gamma| - |\tau|$. This shows that $g \cdot h \in Z(\mu\gamma, \tau) \subseteq \mathcal{K}_{E,n}$, as desired.

On the other hand, it is clear that $\mathcal{K}_{E,n}$ is closed under taking inverses, and hence $\mathcal{K}_{E,n}$ is a clopen subgroupoid of $\mathcal{G}_E$. It follows from the finiteness of $\mathcal{E}_n$ that $\mathcal{K}_{E,n} \setminus \mathcal{G}_E^{(0)}$ is compact. Finally, $\mathcal{K}_{E,n}$ is principal because $\mathcal{G}_E$ is. This shows that $\mathcal{G}_E$ is an AF-groupoid. □

Combining Lemma B.3.1 and Proposition B.3.2 together with the fact that $\mathcal{H}_E$ embeds as a clopen subgroupoid of $G_E \times_{c_E} \mathbb{Z}$ (Lemma B.2.4) we obtain the following corollary.

Corollary B.3.3. *For any graph E, both $\mathcal{G}_E \times_{c_E} \mathbb{Z}$ and $\mathcal{H}_E$ are AF-groupoids.*

We end this section by describing a consequence of Theorem B.2.2 that we shall need in the proof of Lemma B.7.7. For an arbitrary graph E the K_0-group of its graph C^*-algebra is isomorphic to the abelian group generated by elements g_v for $v \in E^0$, subject to the relations

$$g_v = \sum_{e\in vE^1} g_{r(e)}$$

whenever v is a regular vertex [DT02]. This isomorphism is implemented by mapping $[p_v]_0$ to g_v, where p_v denotes the projection in $C^*(E)$ associated to v. Using the identification between K_0 and H_0 for AF-groupoids from Theorem B.2.2, together with the fact that the skew product $\mathcal{G}_E \times_{c_E} \mathbb{Z}$ is an AF-groupoid, we deduce the following.

Lemma B.3.4. *Let E be a graph. For each $w \in E^0_{sing}$ and $i \in \mathbb{Z}$, the element $[1_{Z(w)\times\{i\}}]$ generates a free summand of $H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$.*

B.4 The AH conjecture

It is time to define the AH conjecture properly, as well as discuss its current status and some aspects of how one can prove it. We will also define and discuss the HK property.

Matui's AH Conjecture ([Mat16])**.** *Let $\mathcal{G}$ be an effective minimal second countable Hausdorff étale groupoid whose unit space $\mathcal{G}^{(0)}$ is a Cantor space. Then the following sequence is exact*

$$H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \xrightarrow{j} [\![\mathcal{G}]\!]_{ab} \xrightarrow{I_{ab}} H_1(\mathcal{G}) \longrightarrow 0. \tag{B.4.1}$$

B.4.1 The maps in the AH conjecture

Let us recall the two maps that appear in (B.4.1). The *index map* $I\colon [\![\mathcal{G}]\!] \to H_1(\mathcal{G})$ is the homomorphism given by $\pi_U \mapsto [1_U]$, where U is a full bisection in $\mathcal{G}$. We denote the induced map on the abelianization $[\![\mathcal{G}]\!]_{\mathrm{ab}}$ by I_{ab}. The index map was introduced in the setting of Cantor minimal systems in [GPS99] and later generalized to étale groupoids over Cantor spaces in [Mat12].

Many of the results leading up to the main result do not require the unit space of the groupoid to be compact. In some of these results the index map appears, but the definition of the index map above does not make sense in the non-compact case. Indeed, if $\mathcal{G}$ is an ample Hausdorff groupoid with $\mathcal{G}^{(0)}$ non-compact, then any full bisection $U \subseteq \mathcal{G}$ is non-compact as well, and so 1_U is not compactly supported. However, there is a straightforward way to remedy this. As shown in [NO19], where we extended the definition of the topological full group to the non-compact setting, each full bisection $U \subseteq \mathcal{G}$ can be written as

$$U = U^{\perp} \bigsqcup \left(\mathcal{G}^{(0)} \setminus \operatorname{supp}(\pi_U)\right),$$

where $U^{\perp}$ is a compact bisection with $s(U^{\perp}) = r(U^{\perp}) = \operatorname{supp}(\pi_U)$. We extend the definition of the index map by setting

$$I(\pi_U) \coloneqq [1_{U^{\perp}}].$$

This agrees with the definition in the compact case because $[1_U] = [1_{U'}]$ if U is a compact bisection which decomposes as $U' \sqcup A$, where $A \subseteq \mathcal{G}^{(0)}$ [Mat12, Lemma 7.3]. The first homology group only "sees" the part of the groupoid that lies outside the unit space.

While the index map now is defined for all ample effective Hausdorff groupoids, the map $j\colon H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \to [\![\mathcal{G}]\!]_{\mathrm{ab}}$ is a priori only defined when every $\mathcal{G}$-orbit has at least 3 elements and $\mathcal{G}^{(0)}$ is a Cantor space. In this case, the group $H_0(\mathcal{G}) \otimes \mathbb{Z}_2$ is generated by elements of the form $[1_{s(U)}] \otimes 1$, where $U \subseteq \mathcal{G}$ is a compact bisection with $s(U) \cap r(U) = \emptyset$. The map j is given by $j([1_{s(U)}] \otimes 1) = [\pi_{\widehat{U}}] \in [\![\mathcal{G}]\!]_{\mathrm{ab}}$, where $\pi_{\widehat{U}} \in [\![\mathcal{G}]\!]$ is the transposition defined in Subsection B.2.2. That j is well defined is proven in [Nek19, Section 7] (see also the proof of [Mat16, Theorem 3.6]).

B.4.2 The AH conjecture for graph groupoids

Let us determine what the assumptions in the AH conjecture mean for graph groupoids. It follows from the results in Section A.8 that the following conditions exactly capture these assumptions.

Definition B.4.1. We say that a graph E satisfies the *AH criteria* if E^0 is finite, E has no sinks, is cofinal, satisfies Condition (L) and each vertex can reach all infinite emitters.

Proposition B.4.2. *Let E be a graph. Then $\mathcal{G}_E$ satisfies the assumptions in the AH conjecture if and only if E satisfies the AH criteria.*

Concretely, the AH criteria mean that E has exactly one nontrivial strongly connected component, in the sense that this is the only component which contains a cycle. In fact, there are at least two disjoint cycles based at each vertex in this component. This component also contains all infinite emitters (if there are any). Any vertex outside this component does not support a cycle, and any path from such a vertex eventually ends up in the nontrivial connected component. So if E is not strongly connected, then some of the vertices outside the nontrivial connected component must be sources. Also note that E is either finite or has an infinite emitter. In particular, a strongly connected graph with finitely many vertices satisfies the AH criteria as long as it is not one of the cycle graphs C_n (i.e. a single cycle with n vertices).

As mentioned in the introduction, the AH conjecture was proved for (restrictions of) graph groupoids arising from strongly connected finite graphs (which are not cycle graphs) in [Mat15b]. The main difficulty of extending this to all graphs satisfying the AH criteria lies in dealing with the presence of infinite emitters. Dealing with any sources in the graph, on the other hand, turns out to be quite easy. Many of the results leading up to the main result applies to more general graphs than those satisfying the AH criteria. Therefore we will not restrict to this until the very end.

Remark B.4.3. We mention in passing that, coincidentally, a graph E satisfies the AH criteria if and only if its graph C^*-algebra, $C^*(E)$, is a unital Kirchberg algebra (in the UCT class).

B.4.3 Status of the AH conjecture

The AH conjecture has so far been verified in a number of cases. In [Mat16] it was shown (generalizing prior results) that the AH conjecture holds for groupoids which are almost finite and principal, and for products of SFT-groupoids. The former class includes AF-groupoids, transformation groupoids of (free) d-dimensional Cantor minimal systems and groupoids associated to aperiodic quasicrystals (as described in [Nek19, Subsection 6.3]). The AH conjecture also holds for transformation groupoids associated odometers [Sca18].

In some cases the map j can even be shown to be injective, making (B.4.1) a short exact sequence. When this is the case the groupoid is said to have the *strong AH property* [Mat16]. If, moreover, j is split-injective, so that the sequence splits, then we say that $\mathcal{G}$ has the *split AH property*. AF-groupoids, groupoids of Cantor minimal systems ($d = 1$) and SFT-groupoids all have the split AH property [Mat17, Example 4.8]. The odometers in [Sca18] have the strong AH property, but it is unknown whether they all split. To the best of the authors' knowledge, there are currently no known examples of groupoids which have the strong AH property, but not the split AH property. There are, however, examples of groupoids for which the AH conjecture holds, yet they do not have the strong AH property. Two classes of such examples are groupoids arising from self-similar groups [Nek19, Example 7.6] and products of SFT-groupoids [Mat16, Subsection 5.5].

Remark B.4.4. Note that if the AH conjecture holds for a groupoid $\mathcal{G}$ and the homology groups $H_0(\mathcal{G})$ and $H_1(\mathcal{G})$ are finitely generated, then so is the abelianization $[\![\mathcal{G}]\!]_{\mathrm{ab}}$. In this case, the split AH property is equivalent to the strong AH property together with having any isomorphism $[\![\mathcal{G}]\!]_{\mathrm{ab}} \cong H_1(\mathcal{G}) \oplus (H_0(\mathcal{G}) \otimes \mathbb{Z}_2)$.

We also remark that if $H_1(\mathcal{G})$ is free abelian (i.e. projective in the category of abelian groups), then the split AH property is equivalent to the strong AH property.

B.4.4 The HK property

As mentioned in the introduction, the other conjecture from [Mat16], namely the HK conjecture, has recently been refuted. In order to reflect this, we make the following definition for groupoids satisfying its conclusion.

Definition B.4.5. We say that an ample Hausdorff groupoid $\mathcal{G}$ has the *HK property* if there are isomorphisms

$$K_0\left(C_r^*(\mathcal{G})\right) \cong \bigoplus_{n=0}^{\infty} H_{2n}(\mathcal{G}) \quad \text{and} \quad K_1\left(C_r^*(\mathcal{G})\right) \cong \bigoplus_{n=0}^{\infty} H_{2n+1}(\mathcal{G}).$$

We remark that the assumptions in the HK conjecture were exactly the same as in the AH conjecture. As mentioned in the introduction, the HK property has been established for several key classes of groupoids. Furthermore, the HK property is preserved under Kakutani equivalence. It is also preserved under products, as long as the factors are amenable, due to the Künneth formula from [Mat16]. Most pertinent to the present paper, however, is the fact that all graph groupoids have the HK property (even if they are not minimal or effective). More precisely, we have the following.

Theorem B.4.6. *Let E be any graph. Then*

$$H_0(\mathcal{G}_E) \cong K_0(C^*(E)), \quad H_1(\mathcal{G}_E) \cong K_1(C^*(E)) \quad \textit{and} \quad H_n(\mathcal{G}_E) = 0 \textit{ for } n \geq 2.$$

In particular, $\mathcal{G}_E$ has the HK property.

Theorem B.4.6 was established for finite essential graphs in [Mat12]. For row-finite graphs with no sinks it follows both from the results in [Ort18] and [FKPS18]. In [HL18] the description of $H_0(\mathcal{G}_E)$ was extended to arbitrary graphs. We add the finishing touch by noting that any graph groupoid is Kakutani equivalent to the groupoid of a row-finite graph with no sinks (namely its *desingularization* [DT05]). Since Kakutani equivalent groupoids have the same homology and their reduced groupoid C^*-algebras are Morita equivalent, the theorem follows from the aforementioned results.

The K-groups of graph C^*-algebras are relatively easy to compute. They are, roughly speaking, determined by the Smith normal form of the part of the adjacency matrix of E which only includes edges emitted by regular vertices. The group $K_0(C^*(E))$ is a quotient of $\mathbb{Z}^{|E^0|}$ and we have $\mathrm{rank}(K_0(C^*(E))) \geq |E^0_{\mathrm{sing}}|$. On the other hand, $K_1(C^*(E))$ is free abelian and

$$\mathrm{rank}(K_1(C^*(E))) = \mathrm{rank}(K_0(C^*(E))) - |E^0_{\mathrm{sing}}|.$$

Consult e.g. [Tom07, Chapter 2.3.1] for more details and examples.

Once we have established the AH conjecture for graph groupoids, the fact that we can compute the homology groups allows us to say something useful about the abelianization $[\![\mathcal{G}_E]\!]_{\mathrm{ab}}$, also when E has infinite emitters. See Section B.10 for a discussion of examples and consequences of the AH conjecture. For now we note the following.

Corollary B.4.7. *Let E be a graph. Then $\mathcal{G}_E$ has the strong AH property if and only if $\mathcal{G}_E$ has the split AH property.*

Proof. As $K_1(C^*(E))$ is always free [DT02], the assertion follows from Theorem B.4.6 and Remark B.4.4. □

B.4.5 Aspects of proving the AH conjecture

When it comes to verifying the AH conjecture for a groupoid $\mathcal{G}$, the hardest part is arguably to establish that $\ker(I_{\mathrm{ab}}) \subseteq \operatorname{im}(j)$. Indeed, the reverse inclusion $I_{\mathrm{ab}} \circ j = 0$ is always true, since all transpositions belong to $\ker(I)$. That is, $\mathcal{S}(\mathcal{G}) \leq \ker(I)$. To see this, suppose $U \subseteq \mathcal{G}$ is a compact bisection with disjoint source and range. Then

$$I\left(\pi_{\widehat{U}}\right) = [1_{\widehat{U}}] = \left[1_{U \sqcup U^{-1} \sqcup (\mathcal{G}^{(0)} \setminus \operatorname{supp}(\pi_{\widehat{U}}))}\right] = [1_U + 1_{U^{-1}}] = 0 \in H_1(\mathcal{G}),$$

using [Mat12, Lemma 7.3]. Surjectivity of the index map has already been established for two general classes of groupoids, namely for *almost finite* groupoids [Mat12, Theorem 7.5] and for *purely infinite* groupoids [Mat15b, Theorem 5.2]. Just as with SFT-groupoids, we will see that the more general graph groupoids studied here also belong to the latter class.

Definition B.4.8 ([Mat15b, Definition 4.9])**.** An effective ample groupoid $\mathcal{G}$ with compact unit space is said to be *purely infinite* if there for every clopen subset $A \subseteq \mathcal{G}^{(0)}$ exists compact bisections $U, V \subseteq \mathcal{G}$ satisfying $s(U) = s(V) = A$ and $r(U) \sqcup r(V) \subseteq A$.

Proposition B.4.9. *Let E be a graph satisfying the AH criteria. Then the groupoid $\mathcal{G}_E|_Y$ is purely infinite for each clopen $Y \subseteq \partial E$.*

Proof. Although the proof of [Mat15b, Lemma 6.1] remains valid with minor modifications in the presence of infinite emitters, we give a brief argument in our notation for the convenience of the reader. Since pure infiniteness passes to restrictions it suffices to consider $Y = \partial E$.

Let $A \subseteq \partial E$ be given. By compactness we can express $A = \sqcup_{i=1}^m Z(\mu_i \setminus F_i)$ as a finite union of punctured cylinder sets. By the description following Definition B.4.1, any vertex lying outside the nontrivial strongly connected component of E is regular. Any path from such a vertex eventually ends up in the nontrivial connected component. This means that by partitioning the cylinder set $Z(\mu_i \setminus F_i)$ into smaller cylinder sets, defined by paths extending μ_i, we may without loss of generality assume that $r(\mu_i)$ lie in the nontrivial connected component for each i. Thus we can, for each i, find two disjoint cycles ν_i, ν_i' based at $r(\mu_i)$. Using these we define bisections $U = \sqcup_{i=1}^m Z(\mu_i \nu_i, F_i, \mu_i)$ and $V = \sqcup_{i=1}^m Z(\mu_i \nu_i', F_i, \mu_i)$ which we see satisfy the conditions in Definition B.4.8. □

Remark B.4.10. Recently, more general notions of pure infiniteness for étale groupoids have appeared in the works of Suzuki [Suz17] and Ma [Ma20]. However, for ample minimal groupoids with compact unit space, as in the setting of this paper, both notions agree with Matui's. Furthermore, they imply Anantharaman-Delaroche's notion of *locally contracting* [AD97]. On a somewhat related note, there is also the recent preprint [ADS19] in which the (not necessarily simple) pure infiniteness of graph C^*-algebras (of row-finite graphs without sinks) is characterized solely in terms of the graph groupoid, by means of the paradoxicality notion from [BL20].

The inclusion $\ker(I_{\mathrm{ab}}) \subseteq \mathrm{im}(j)$ is intimately related to the kernel of the index map being generated by transpositions, as encapsulated by the following definition.

Definition B.4.11 ([Mat16, Definition 2.11])**.** Let $\mathcal{G}$ be an effective ample Hausdorff groupoid. We say that $\mathcal{G}$ has *Property TR* if $\mathcal{S}(\mathcal{G}) = \ker(I)$.

By Proposition B.4.9 and [Mat16, Theorem 4.4] it suffices to establish Property TR in order to verify the AH conjecture for graph groupoids. Therefore, the rest of the paper is mostly devoted to demonstrating that graph groupoids do have Property TR.

Remark B.4.12. In general, Property TR implies the inclusion $\ker(I_{\mathrm{ab}}) \subseteq \mathrm{im}(j)$, i.e. exactness at $[\![\mathcal{G}]\!]_{\mathrm{ab}}$ in (B.4.1). The converse holds if the commutator subgroup $\mathsf{D}([\![\mathcal{G}]\!])$ is simple, for in that case $\mathsf{D}([\![\mathcal{G}]\!]) = \mathcal{A}(\mathcal{G})$, where $\mathcal{A}(\mathcal{G})$ denotes the "alternating" subgroup of $\mathcal{S}(\mathcal{G})$ defined in [Nek19]. The group $\mathsf{D}([\![\mathcal{G}]\!])$ is known to be simple for minimal groupoids which are either almost finite or purely infinite [Mat15b]. So for these two classes of groupoids we see that Property TR is in fact equivalent to the AH conjecture.

We close this section by observing, as was done in [Mat15b], that to establish Property TR it suffices to only consider elements in the topological full group whose support is a proper subset of the unit space. Although an easy observation, this is needed for the proof of the main result to work.

Lemma B.4.13. *Let $\mathcal{G}$ be an ample effective Hausdorff groupoid. If all elements $\alpha \in [\![\mathcal{G}]\!]$ which satisfy $I(\alpha) = 0 \in H_1(\mathcal{G})$ and* $\mathrm{supp}(\alpha) \neq \mathcal{G}^{(0)}$ *are products of transpositions, then $\mathcal{G}$ has Property TR.*

Proof. Let $\alpha \in [\![\mathcal{G}]\!] \setminus \{\mathrm{id}\}$ be given and suppose $I(\alpha) = 0 \in H_1(\mathcal{G})$. As α is not the identity, $\mathrm{supp}(\alpha)$ is non-empty. Then there is some compact open set $Z \subseteq \mathcal{G}^{(0)}$ such that $\alpha(Z) \cap Z = \emptyset$. We define a transposition $\tau \in \mathcal{S}(\mathcal{G})$ by setting $\tau = \alpha$ on Z, $\tau = \alpha^{-1}$ on $\alpha(Z)$ and $\tau = \mathrm{id}$ elsewhere. Then $\mathrm{supp}(\tau) = \alpha(Z) \sqcup Z$ and $\mathrm{supp}(\tau\alpha) \subseteq \mathcal{G}^{(0)} \setminus (\alpha(Z) \sqcup Z) \subsetneq \mathcal{G}^{(0)}$. Since both α and τ (being a transposition) are in the kernel of the index map, so is their product, and by assumption $\tau\alpha$ is then a product of transpositions. Now α is clearly also a product of transpositions. $\square$

B.5 Cancellation for AF-groupoids

Cancellation for ample Hausdorff groupoids was introduced by Matui in [Mat16], and it bears resemblance to the cancellation property (in K-theory) for C^*-algebras (see [RLL00, Definition 7.3.1]).

Definition B.5.1. An ample Hausdorff groupoid $\mathcal{G}$ is said to have *cancellation* if whenever one has $[1_A] = [1_B]$ in $H_0(\mathcal{G})$ for $\emptyset \neq A, B \subseteq \mathcal{G}^{(0)}$ compact open, there exists a bisection $U \subseteq \mathcal{G}$ with $s(U) = A$ and $r(U) = B$.

In order to prove our main result we are going to need the fact that AF-groupoids have cancellation. This might be known to experts, but we were unable to locate a reference. Theorem 6.12 in [Mat12] covers minimal AF-groupoids with compact unit space, but we need cancellation for the skew product $\mathcal{G}_E \times_{c_E} \mathbb{Z}$, which is generally neither minimal nor does it have compact unit space. So we provide a proof here, which we divide into three lemmas in terms of permanence properties of cancellation.

Lemma B.5.2. *Let $\mathcal{G}$ be an ample Hausdorff groupoid. If $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{G}_3 \subseteq \ldots$ are open subgroupoids of $\mathcal{G}$ with $\cup_{n=1}^{\infty} \mathcal{G}_n = \mathcal{G}$, and each $\mathcal{G}_n$ has cancellation, then $\mathcal{G}$ has cancellation.*

Proof. Let $A, B \subseteq \mathcal{G}^{(0)}$ be compact open and suppose $[1_A] = [1_B]$ in $H_0(\mathcal{G})$. This means that $1_A - 1_B = \delta_1(f)$ for some $f \in C_c(\mathcal{G}, \mathbb{Z})$. As the support of f is compact we must have $\operatorname{supp}(f) \subseteq \mathcal{G}_n$ for some $n \in \mathbb{N}$. By possibly increasing n we may suppose that $A, B \subseteq \mathcal{G}_n^{(0)}$ as well. We have $f|_{\mathcal{G}_n} \in C_c(\mathcal{G}_n, \mathbb{Z})$ and $\delta_1(f|_{\mathcal{G}_n}) = \delta_1(f) = 1_A - 1_B$. Cancellation in $\mathcal{G}_n$ now provides a bisection $U \subseteq \mathcal{G}_n \subseteq \mathcal{G}$ with $s(U) = A$ and $r(U) = B$. □

Lemma B.5.3. *If $\mathcal{G}_1$ and $\mathcal{G}_2$ are ample Hausdorff groupoids with cancellation, then the disjoint union groupoid $\mathcal{G}_1 \sqcup \mathcal{G}_2$ has cancellation.*

Proof. Let $A, B \subseteq (\mathcal{G}_1 \sqcup \mathcal{G}_2)^{(0)}$ be compact open and suppose that $[1_A] = [1_B]$ in $H_0(\mathcal{G}) \cong H_0(\mathcal{G}_1) \oplus H_0(\mathcal{G}_2)$. Let $f \in C_c(\mathcal{G}_1 \sqcup \mathcal{G}_2, \mathbb{Z})$ be such that $\delta_1(f) = 1_A - 1_B$. We can write $(\mathcal{G}_1 \sqcup \mathcal{G}_2)^{(0)} = \mathcal{G}_1^{(0)} \sqcup \mathcal{G}_2^{(0)}$, $A = A_1 \sqcup A_2$, $B = B_1 \sqcup B_2$ and $f = f_1 + f_2$ respecting this decomposition. It is now clear that $\delta_1(f_1) = 1_{A_1} - 1_{B_1}$ and $\delta_1(f_2) = 1_{A_2} - 1_{B_2}$, so by cancellation in $\mathcal{G}_1$ and $\mathcal{G}_2$ we obtain bisections $U_1 \subseteq \mathcal{G}_1$ and $U_2 \subseteq \mathcal{G}_2$ with $s(U_1) = A_1$, $r(U_1) = B_1$, $s(U_2) = A_2$ and $r(U_2) = B_2$. Setting $U = U_1 \sqcup U_2$ does the trick. □

Lemma B.5.4. *Let X be a zero-dimensional compact Hausdorff space. Then the elementary groupoid of type n, $X \times \mathcal{R}_n$, has cancellation for each $n \in \mathbb{N}$.*

Proof. Denote $\mathcal{K} := X \times \mathcal{R}_n$, and write $\mathcal{K}^{(0)} = \sqcup_{i=1}^n X_i$, where $X_i = X \times \{i\}$. Then X_1 is a $\mathcal{K}$-full clopen subset of $\mathcal{K}^{(0)}$ and $\mathcal{K}|_{X_1} \cong X$, so we have

$$H_0(\mathcal{K}) \cong H_0(\mathcal{K}|_{X_1}) \cong H_0(X) = C(X, \mathbb{Z}).$$

Suppose that $A, B \subseteq \mathcal{K}^{(0)}$ are clopen subsets with $[1_A] = [1_B]$ in $H_0(\mathcal{K})$. We partition A by writing $A = \sqcup_{i=1}^n A_i \times \{i\}$ for $A_i \subseteq X$ clopen. Let B_i be similar for B. The bisections $A_i \times \{(1, i)\} \subseteq \mathcal{K}$ have source $A_i \times \{i\}$ and range $A_i \times \{1\}$. By (B.2.2) this means that under the isomorphism $H_0(\mathcal{K}) \cong C(X, \mathbb{Z})$ above, the element $[1_A] \in H_0(\mathcal{K})$ maps to the function $f_A := \sum_{i=1}^n 1_{A_i} \in C(X, \mathbb{Z})$, and similarly $[1_B] \mapsto f_B$.

Since $f_A = f_B$ and they are both sums of indicator functions we can find $m_j \in \mathbb{N}$ and $C_j \subseteq X$ clopen such that $f_A = f_B = \sum_{j=1}^J m_j 1_{C_j}$. We can think of f_A (and f_B) being produced by taking each of the parts A_i and "projecting" them down and then stacking them on top of each other. The height at a point becomes the function value of f_A. For each C_j we have that $C_j \times \{i\} \subseteq A_i$ for precisely m_j indices i, and we have the same for the B_i's. For fixed j denote these indices for A by $i_1, \ldots, i_{m_j}$, and denote them by $i'_1, \ldots, i'_{m_j}$ for B. Then define a bisection $U_j \subseteq \mathcal{K}$ by $U_j = \sqcup_{k=1}^{m_j} U_k$, where $U_k := C_j \times (i'_k, i_k)$. Finally setting $U = \sqcup_{j=1}^J U_j$ gives a bisection with $s(U) = A$ and $r(U) = B$. □

Theorem B.5.5. *Any AF-groupoid has cancellation.*

Proof. Let $\mathcal{G}$ be an AF-groupoid. Then we can write $\mathcal{G} = \cup_{n=1}^\infty \mathcal{G}_n$ as an increasing union of open elementary ample subgroupoids. By [GPS04, Lemma 3.4] each subgroupoid decomposes as

$$\mathcal{G}_n \cong \left(\sqcup_{i=1}^{I_n} X_{i,n} \times \mathcal{R}_{m_{i,n}}\right) \bigsqcup Y_n,$$

where each $X_{i,n}$ is a zero-dimensional compact Hausdorff space, and where Y_n is empty if $\mathcal{G}^{(0)}$ is compact and zero-dimensional, locally compact non-compact and Hausdorff if $\mathcal{G}^{(0)}$ is non-compact. Since the trivial groupoid Y_n clearly has cancellation, the result follows by combining the three lemmas above. □

We end this section by observing that in an AF-groupoid, a non-empty subset of the unit space always gives rise to a nonzero element in homology. This is not so for all groupoids with cancellation (e.g. the SFT-groupoid of the full 2-shift, $\mathcal{G}_{[2]}$).

Corollary B.5.6. *Let $\mathcal{G}$ be an AF-groupoid. If $A \subseteq \mathcal{G}^{(0)}$ is compact open, then $[1_A] = 0$ in $H_0(\mathcal{G})$ if and only if $A = \emptyset$.*

Proof. Follows from the proofs above by considering $B = \emptyset$, i.e. $1_B = 0$. □

B.6 Two long exact sequences in homology

Let us first describe a long exact sequence in homology coming from a cocycle. Let $\mathcal{G}$ be an ample Hausdorff groupoid with a cocycle $c\colon \mathcal{G} \to \mathbb{Z}$. Let π denote the canonical projection from $\mathcal{G} \times_c \mathbb{Z}$ onto $\mathcal{G}$, i.e. $\pi(g,m) = g$. Also, let $\rho := \widehat{c}_1 \colon \mathcal{G} \times_c \mathbb{Z} \to \mathcal{G} \times_c \mathbb{Z}$, i.e. $\rho(g,m) = (g, m+1)$. Since these are étale homomorphisms, they induce chain maps

$$\pi_\bullet \colon C_\bullet(\mathcal{G} \times_c \mathbb{Z}, \mathbb{Z}) \to C_\bullet(\mathcal{G}, \mathbb{Z}) \quad \text{and} \quad \rho_\bullet \colon C_\bullet(\mathcal{G} \times_c \mathbb{Z}, \mathbb{Z}) \to C_\bullet(\mathcal{G} \times_c \mathbb{Z}, \mathbb{Z})$$

on the chain complexes that define the homology groups. In fact, $\mathrm{id} - \rho_\bullet$ and $\pi_\bullet$ form a short exact sequence of complexes, which in turn induces a long exact sequence in homology.

Proposition B.6.1 ([Ort18, Lemma 1.4])**.** *Let $\mathcal{G}$ be an ample Hausdorff groupoid and let $c\colon \mathcal{G} \to \mathbb{Z}$ be a cocycle. Then there is a long exact sequence*

$$\cdots \xrightarrow{H_1(\pi_\bullet)} H_1(\mathcal{G}) \xrightarrow{\partial_1} H_0(\mathcal{G} \times_c \mathbb{Z}) \xrightarrow{\mathrm{id} - H_0(\rho_\bullet)} H_0(\mathcal{G} \times_c \mathbb{Z}) \xrightarrow{H_0(\pi_\bullet)} H_0(\mathcal{G}) \to 0,$$

where ∂_n denotes the connecting homomorphism.

The maps on the zeroth level are given by

$$H_0(\rho_\bullet)\left([1_{A\times\{i\}}]\right) = [1_{A\times\{i+1\}}] \quad \text{and} \quad H_0(\pi_\bullet)\left([1_{A\times\{i\}}]\right) = [1_A]$$

for $A \subseteq \mathcal{G}^{(0)}$ compact open and $i \in \mathbb{Z}$. In the case of graph groupoids, we will see later that the first connecting homomorphism $\partial_1 \colon H_1(\mathcal{G}) \to H_0(\mathcal{G} \times_c \mathbb{Z})$ can be described explicitly, and that this will allow us to describe the image of the index map. In order to do that, we are going to need a particular part of the proof of [Ort18, Lemma 1.4] pertaining to lifts by $\mathrm{id} - \rho_0$. We record this lifting in Lemma B.6.2 below, whose proof itself is an easy calculation.

Lemma B.6.2. *Let $c\colon \mathcal{G} \to \mathbb{Z}$ be a cocycle on an ample Hausdorff groupoid $\mathcal{G}$. Then for any $A \subseteq \mathcal{G}^{(0)}$ compact open and $k \in \mathbb{Z}$ we have*

$$1_{A\times\{k\}} - 1_{A\times\{0\}} = \begin{cases} (\mathrm{id} - \rho_0)\left(-\sum_{i=0}^{k-1} 1_{A\times\{i\}}\right) & k > 0, \\ 0 & k = 0, \\ (\mathrm{id} - \rho_0)\left(\sum_{i=k}^{-1} 1_{A\times\{i\}}\right) & k < 0. \end{cases}$$

The next long exact sequence in homology arises from open invariant subsets of the unit space. This is akin to the six-term exact sequences arising from nested ideals in filtered K-theory of C^*-algebras [Res06]. Let $\mathcal{G}$ be an ample Hausdorff groupoid and let $Z \subseteq Y \subseteq \mathcal{G}^{(0)}$ be open sets. The inclusion $\iota\colon \mathcal{G}|_Z \hookrightarrow \mathcal{G}|_Y$ induces the chain map

$$\iota_n\colon C_c((\mathcal{G}|_Z)^{(n)}, \mathbb{Z}) \to C_c((\mathcal{G}|_Y)^{(n)}, \mathbb{Z})$$

which is given by extending functions to be 0 outside $\mathcal{G}|_Z$. Let

$$\kappa_n\colon C_c((\mathcal{G}|_Y)^{(n)}, \mathbb{Z}) \to C_c((\mathcal{G}|_{(Y\setminus Z)})^{(n)}, \mathbb{Z})$$

denote the canonical restriction maps. Taking such restrictions commute with the differentials δ_n, so $\kappa_\bullet$ is also a chain map.

We claim that when the sets Z and Y are $\mathcal{G}$-invariant, then $\iota_\bullet$ and $\kappa_\bullet$ form a short exact sequence of complexes as follows:

$$0 \longrightarrow C_\bullet\left(\mathcal{G}|_Z, \mathbb{Z}\right) \xrightarrow{\iota_\bullet} C_\bullet\left(\mathcal{G}|_Y, \mathbb{Z}\right) \xrightarrow{\kappa_\bullet} C_\bullet\left(\mathcal{G}|_{(Y\setminus Z)}, \mathbb{Z}\right) \longrightarrow 0.$$

It is clear that $\kappa_n \circ \iota_n = 0$. It is also clear that ι_n is injective and that κ_n is surjective. Suppose that we have $\kappa_n(f) = 0$ for some $f \in C_c\left((\mathcal{G}|_Y)^{(n)}, \mathbb{Z}\right)$. This means that f is identically zero on $\left(\mathcal{G}|_{(Y\setminus Z)}\right)^{(n)}$. The invariance of Z implies that there are no groupoid elements $g \in \mathcal{G}$ for which $s(g) \in Z$ while $r(g) \in Y \setminus Z$, or vice versa. This forces f to be supported solely on $(\mathcal{G}|_Z)^{(n)}$, which means that $f \in \operatorname{im}(\iota_n)$. The claim follows. We therefore obtain the following long exact sequence in homology.

Proposition B.6.3. *Let $\mathcal{G}$ be an ample Hausdorff groupoid and let $Z \subseteq Y \subseteq \mathcal{G}^{(0)}$ be open and $\mathcal{G}$-invariant subsets. Then there is a long exact sequence*

$$\cdots \xrightarrow{H_1(\kappa_\bullet)} H_1\left(\mathcal{G}|_{(Y\setminus Z)}\right) \to H_0\left(\mathcal{G}|_Z\right) \xrightarrow{H_0(\iota_\bullet)} H_0\left(\mathcal{G}|_Y\right) \xrightarrow{H_0(\kappa_\bullet)} H_0\left(\mathcal{G}|_{(Y\setminus Z)}\right) \to 0.$$

B.7 The homology groups of a graph groupoid

We have already seen that the homology groups of a graph groupoid coincide with the K-groups of its groupoid C^*-algebra. We will make use of this in the final section. However, in order to prove Property TR for the graph groupoid $\mathcal{G}_E$ we are going to relate the first homology group $H_1(\mathcal{G}_E)$ to the homology groups $H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$ and $H_0(\mathcal{H}_E)$. In this section we will use the long exact sequences from the previous section to deduce the following embeddings:

$$H_1(\mathcal{G}_E) \hookrightarrow H_0(\mathcal{H}_E) \hookrightarrow H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z}).$$

This will be done in three steps: first we show that $H_1(\mathcal{G}_E) \hookrightarrow H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$, then that $H_0(\mathcal{H}_E) \hookrightarrow H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$ and finally that $H_1(\mathcal{G}_E) \hookrightarrow H_0(\mathcal{H}_E)$. The reason we need three steps (and not two) is that the third embedding relies on the first two.

B.7.1 The first embedding

Let us begin by describing the zeroth homology group of the skew product $\mathcal{G}_E \times_{c_E} \mathbb{Z}$. Recall that $(\mathcal{G}_E \times_c \mathbb{Z})^{(0)}$ is identified with $\partial E \times \mathbb{Z}$. Observe that we have

$$\begin{aligned} H_0(\mathcal{G}_E \times_c \mathbb{Z}) &= \operatorname{span}\{[1_A] \mid A \subseteq \partial E \times \mathbb{Z} \text{ compact open}\} \\ &= \operatorname{span}\{\left[1_{Z(\mu \setminus F) \times \{i\}}\right] \mid \mu \in E^*,\ F \subseteq_{\text{finite}} r(\mu)E^1,\ i \in \mathbb{Z}\} \\ &= \operatorname{span}\{\left[1_{Z(\mu) \times \{i\}}\right] \mid \mu \in E^*,\ i \in \mathbb{Z}\}, \end{aligned}$$

since $1_{Z(\mu\setminus F)\times\{i\}} = 1_{Z(\mu)\times\{i\}} - \sum_{e\in F} 1_{Z(\mu e)\times\{i\}}$ (by *span* we mean linear combinations over $\mathbb{Z}$). These elements satisfy the following relations in $H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$:

$$\left[1_{Z(\mu)\times\{i\}}\right] = \left[1_{Z(\sigma_E(\mu))\times\{i+1\}}\right] \text{ if } |\mu| \geq 1, \tag{B.7.1}$$

$$\left[1_{Z(\mu)\times\{i\}}\right] = \left[1_{Z(e\mu)\times\{i-1\}}\right] \text{ for any } e \in E^1 s(\mu), \tag{B.7.2}$$

$$\left[1_{Z(\mu)\times\{i\}}\right] = \sum_{e\in r(\mu)E^1} \left[1_{Z(\mu e)\times\{i\}}\right] \text{ if } r(\mu) \text{ is a regular vertex,} \tag{B.7.3}$$

$$\left[1_{Z(\mu)\times\{i\}}\right] = [1_{Z(\nu)\times\{i\}}] \text{ if } |\mu| = |\nu| \text{ and } r(\mu) = r(\nu). \tag{B.7.4}$$

For all of the sets appearing in the indicator functions above it is easy to find a bisection in $\mathcal{G}_E \times_c \mathbb{Z}$ whose source is the left hand side and whose range is the right hand side. From repeated use of the relation (B.7.1) we see that we can even write

$$H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z}) = \operatorname{span}\{[1_{Z(v)\times\{i\}}] \mid v \in E^0,\ i \in \mathbb{Z}\},$$

since $\left[1_{Z(\mu)\times\{i\}}\right] = \left[1_{Z(r(\mu))\times\{i+|\mu|\}}\right]$.

Let us now consider the long exact sequence in homology that we get from the canonical cocycle c_E on a graph groupoid $\mathcal{G}_E$. Since $\mathcal{G}_E \times_{c_E} \mathbb{Z}$ is an AF-groupoid (Corollary B.3.3), its H_1 group vanishes, and therefore the first part of the long exact sequence from Proposition B.6.1 becomes

$$0 \to H_1(\mathcal{G}_E) \xrightarrow{\partial_1} H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z}) \xrightarrow{\operatorname{id} - H_0(\rho_\bullet)} H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z}) \xrightarrow{H_0(\pi_\bullet)} H_0(\mathcal{G}_E) \to 0. \tag{B.7.5}$$

The map $H_0(\rho_\bullet)\colon H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z}) \to H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$ is given by

$$H_0(\rho_\bullet)\left([1_{Z(v)\times\{i\}}]\right) = [1_{Z(v)\times\{i+1\}}]$$

for $v \in E^0$ and $i \in \mathbb{Z}$. The connecting homomorphism ∂_1 will be described explicitly in the proof of Lemma B.8.6. From the exactness of (B.7.5) we deduce the following.

Proposition B.7.1. *Let E be a graph and let $H_0(\rho_\bullet)$ be as above. Then*

$$H_0(\mathcal{G}_E) \cong \operatorname{coker}(\mathrm{id} - H_0(\rho_\bullet)) \qquad \text{and} \qquad H_1(\mathcal{G}_E) \cong \ker(\mathrm{id} - H_0(\rho_\bullet)).$$

Remark B.7.2. In the proof of [Mat12, Theorem 4.14], Matui obtained formulas similar to those in Proposition B.7.1 using a spectral sequence. This relied on the fact that $H_0(\mathcal{H}_E)$ and $H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$ can be identified when E is finite (or more generally row-finite) with no sinks. In this setting $\partial E \times \{0\}$ is $(\mathcal{G}_E \times_{c_E} \mathbb{Z})$-full, so $\mathcal{H}_E$ is Kakutani equivalent to $\mathcal{G}_E \times_{c_E} \mathbb{Z}$. This allowed Matui to immediately realize $H_1(\mathcal{G}_E)$ as a subgroup of $H_0(\mathcal{H}_E)$.

At this point we encounter a significant difference from the finite graph case. When E has singular vertices one can show that $\partial E \times \{0\}$ *never* is $(\mathcal{G}_E \times_{c_E} \mathbb{Z})$-full. So in our setting we cannot necessarily identify $H_0(\mathcal{H}_E)$ with $H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$. We will, however, be able to identify the former with a subgroup of the latter.

B.7.2 The second embedding

Recall that $\mathcal{H}_E = \ker(c_E) \subseteq \mathcal{G}_E$ and from Lemma B.2.4 we have that

$$\mathcal{H}_E \cong (\mathcal{G}_E \times_{c_E} \mathbb{Z})\,|_{\partial E \times \{0\}}$$

via the identification $(x, 0, y) \leftrightarrow ((x, 0, y), 0)$. In $H_0(\mathcal{H}_E)$ we have the relation

$$\left[1_{Z(\mu)}\right] = \left[1_{Z(\nu)}\right]$$

whenever $\mu, \nu \in E^*$ satisfy $|\mu| = |\nu|$ and $r(\mu) = r(\nu)$. The element $\left[1_{Z(\mu)}\right]$ in $H_0(\mathcal{H}_E)$ corresponds to $\left[1_{Z(\mu)\times\{0\}}\right] \in H_0((\mathcal{G}_E \times_{c_E} \mathbb{Z})\,|_{\partial E\times\{0\}})$ under the identification above. On the other hand, the indicator function $1_{Z(\mu)\times\{0\}}$ gives rise to an element $\left[1_{Z(\mu)\times\{0\}}\right]$ in $H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$ as well. A priori, these are different, but we will see that mapping $\left[1_{Z(\mu)}\right] \in H_0(\mathcal{H}_E)$ to $\left[1_{Z(\mu)\times\{0\}}\right] \in H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$ actually gives an embedding of groups. So that in the end, there is no ambiguity. The map from $H_0(\mathcal{H}_E)$ to $H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$ proposed above extends to arbitrary elements by

$$H_0(\mathcal{H}_E) \ni [f] \longmapsto [f \times 0] \in H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$$

for $f \in C_c(\partial E, \mathbb{Z})$, where $f \times 0 \in C_c(\partial E \times \mathbb{Z}, \mathbb{Z})$ is given by

$$(f \times 0)(x, m) = \begin{cases} f(x) & \text{if } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

By noting that $(\mathcal{G}_E \times_{c_E} \mathbb{Z})\,|_{\partial E\times\{0\}} = \mathcal{H}_E \times \{0\} \subseteq \mathcal{G}_E \times \mathbb{Z} = \mathcal{G}_E \times_{c_E} \mathbb{Z}$ as sets, it is not hard to see that this is a well-defined homomorphism. Its injectivity will be deduced using the second long exact sequence from Section B.6.

Lemma B.7.3. *Let E be a graph. The map $\phi\colon H_0(\mathcal{H}_E) \to H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$ given by $\phi([f]) = [f \times 0]$ for $f \in C_c(\partial E, \mathbb{Z})$ is an injective group homomorphism.*

Proof. In the setting of Proposition B.6.3, set $\mathcal{G} = \mathcal{G}_E \times_{c_E} \mathbb{Z}$, $Y = \mathcal{G}^{(0)} = \partial E \times \mathbb{Z}$ and $X = \partial E \times \{0\}$. The clopen set X is neither $\mathcal{G}$-full nor invariant, so we instead consider its saturation, namely $Z := r(s^{-1}(X))$. In words Z is the smallest $\mathcal{G}$-invariant subset containing X. Since the range map r is open, Z is open in $\partial E \times \mathbb{Z}$. By its very definition, X is clopen in Z and $\mathcal{G}|_Z$-full, hence $\mathcal{H}_E \cong \mathcal{G}|_X = (\mathcal{G}|_Z)|_X$ is Kakutani equivalent to $\mathcal{G}|_Z$. The induced isomorphism $H_0(\mathcal{H}_E) \cong H_0(\mathcal{G}|_Z)$ maps $[1_{Z(\mu)}]$ to $[1_{Z(\mu)\times\{0\}}]$, where we now consider $1_{Z(\mu)\times\{0\}} \in C_c(Z, \mathbb{Z})$. Since $\mathcal{G}$ is an AF-groupoid and the set $Y \setminus Z$ is closed in $\mathcal{G}^{(0)}$, the restriction $\mathcal{G}|_{(Y\setminus Z)}$ becomes an AF-groupoid (in the relative topology) as well. Its H_1 group then vanishes and the first part of the long exact sequence in Proposition B.6.3 becomes

$$
\begin{tikzcd}
0 \arrow[r] & H_0\left((\mathcal{G}_E \times_{c_E} \mathbb{Z})|_Z\right) \arrow[r, "H_0(\iota_\bullet)"] & H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z}) \arrow[dll, "H_0(\kappa_\bullet)"'] \\
H_0\left((\mathcal{G}_E \times_{c_E} \mathbb{Z})|_{(\partial E\times\mathbb{Z})\setminus Z}\right) \arrow[r] & 0
\end{tikzcd}
\tag{B.7.6}
$$

The map $H_0(\iota_\bullet)$ is given by inclusion (i.e. by extending to 0). So if we compose $H_0(\iota_\bullet)$ with the isomorphism $H_0(\mathcal{H}_E) \cong H_0(\mathcal{G}|_Z) = H_0\left((\mathcal{G}_E \times_{c_E} \mathbb{Z})|_Z\right)$ from above we get ϕ back. Its injectivity then follows from the injectivity of $H_0(\iota_\bullet)$. □

Remark B.7.4. We can actually describe the set Z from the proof of Lemma B.7.3 explicitly, assuming that E is strongly connected, as follows:

$$Z = \{(x,k) \mid x \in E^\infty, k \in \mathbb{Z}\} \bigsqcup \{(\mu, l) \mid \mu \in \partial E \cap E^*,\ l \geq -|\mu|\} \subseteq \partial E \times \mathbb{Z} = Y.$$

The complement is therefore

$$Y \setminus Z = (\partial E \times \mathbb{Z}) \setminus Z = \{(\mu, l) \mid \mu \in \partial E \cap E^*,\ l < -|\mu|\}.$$

If E has a singular vertex, then Z is an open and dense proper subset of $\partial E \times \mathbb{Z}$, as well as $\mathcal{G}_E \times_{c_E} \mathbb{Z}$-invariant. The complement is non-empty, closed, has empty interior and is also invariant.

B.7.3 The third embedding

From now on we will freely identify $H_0(\mathcal{H}_E)$ with the subgroup generated by the elements $\left[1_{Z(\mu)\times\{0\}}\right]$ for $\mu \in E^*$ inside $H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$. The first thing we shall note is that this copy of $H_0(\mathcal{H}_E)$ inside $H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$ is invariant under $H_0(\rho_\bullet)$, provided that E has no sources. Indeed, for $\mu \in E^*$

$$H_0(\rho_\bullet)\left(\left[1_{Z(\mu)\times\{0\}}\right]\right) = \left[1_{Z(\mu)\times\{1\}}\right] = \left[1_{Z(e\mu)\times\{0\}}\right],$$

where e is any edge whose range is $s(\mu)$ (and the equivalence class does not depend on which edge e is chosen). The restriction of $H_0(\rho_\bullet)$ to $H_0(\mathcal{H}_E)$ will be important in the sequel, so we give it a name of its own.

Definition B.7.5. Let E be an essential graph. By viewing $H_0(\mathcal{H}_E)$ as a subgroup of $H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$ we define an endomorphism $\varphi \colon H_0(\mathcal{H}_E) \to H_0(\mathcal{H}_E)$ by

$$\varphi\left(\left[1_{Z(\mu)\times\{0\}}\right]\right) = H_0(\rho_\bullet)\left(\left[1_{Z(\mu)\times\{0\}}\right]\right) = \left[1_{Z(e\mu)\times\{0\}}\right],$$

where $e \in E^1 s(\mu)$ is arbitrary.

In the next section we will see that the image of an element of the topological full group under the index map can be described in terms of the map φ.

Remark B.7.6. On page 56 of [Mat15b] Matui implicitly defines, for any finite strongly connected graph E, an automorphism of $H_0(\mathcal{H}_E)$ denoted δ. Explicitly, δ is given by

$$\begin{aligned}&\delta\left(\left[1_{Z(\mu)\times\{0\}}\right]\right) = \left[1_{Z(\sigma_E(\mu))\times\{0\}}\right] = \left[1_{Z(\mu)\times\{-1\}}\right]\\ &\text{for } \left[1_{Z(\mu)\times\{0\}}\right] \in H_0(\mathcal{H}_E) = \operatorname{span}\left\{\left[1_{Z(\mu)\times\{0\}}\right] \mid \mu \in E^{\geq 1}\right\}.\end{aligned}$$

Hence the homomorphism φ from Definition B.7.5 equals δ^{-1}. However, if the graph E has singular vertices, then δ is no longer globally defined on $H_0(\mathcal{H}_E)$. To see this, note that φ is generally not surjective. For example, the elements $[1_{Z(w)\times\{0\}}]$, where w is an infinite emitter, will generally not be in the image of φ.

We are now ready to prove the the third and final embedding of the homology groups.

Lemma B.7.7. *Let E be an essential graph. Then* $\ker(\mathrm{id} - H_0(\rho_\bullet)) = \ker(\mathrm{id} - \varphi)$ *as subsets of* $H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$.

Proof. With ϕ as in Lemma B.7.3 we have the commutative diagram

$$\begin{tikzcd} H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z}) \arrow[r, "\mathrm{id} - H_0(\rho_\bullet)"] & H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z}) \\ H_0(\mathcal{H}_E) \arrow[u, hook, "\phi"] \arrow[r, "\mathrm{id} - \varphi"] & H_0(\mathcal{H}_E) \arrow[u, hook, "\phi"'] \end{tikzcd}$$

under which we identify $H_0(\mathcal{H}_E)$ with $\phi(H_0(\mathcal{H}_E)) \subseteq H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$. From this it is clear that $\ker(\mathrm{id} - \varphi) \subseteq \ker(\mathrm{id} - H_0(\rho_\bullet))$.

To prove the reverse inclusion we first show that any element of $H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$ can be put in a certain "standard form". Each element $\omega \in H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$ can be written as

$$\omega = \sum_{i=-n}^{n} \sum_{j=1}^{k_i} \lambda_{i,j} \left[1_{Z(v_{i,j}) \times \{i\}}\right],$$

where $\lambda_{i,j}$ are integers and $v_{i,j} \in E^0$. When $i \geq 0$ we have

$$[1_{Z(v) \times \{i\}}] = \left[1_{Z(\mu) \times \{0\}}\right], \tag{B.7.7}$$

where μ is any path of length i in E which ends in v. When v is a regular vertex we have

$$[1_{Z(v) \times \{i\}}] = \sum_{e \in vE^1} [1_{Z(r(e)) \times \{i+1\}}]. \tag{B.7.8}$$

So when $i < 0$ we can, by repeated use of (B.7.8), write

$$[1_{Z(v) \times \{i\}}] = \sum_{j=i}^{-1} \sum_{k=1}^{K_j} \left[1_{Z(w_{j,k}) \times \{j\}}\right] + \sum_{k=1}^{K_0} [1_{Z(v_k) \times \{0\}}], \tag{B.7.9}$$

where each $w_{j,k}$ is an infinite emitter. Combining (B.7.7) and (B.7.9) we see that we can write the arbitrary element ω as

$$\omega = \sum_{i=-n}^{-1} \sum_{j=1}^{J_i} \lambda_{i,j} \left[1_{Z(w_{i,j}) \times \{i\}}\right] + \sum_{j=1}^{J_0} \lambda_{0,j} \left[1_{Z(\mu_j) \times \{0\}}\right],$$

where $n \in \mathbb{N}$, $\lambda_{i,j} \in \mathbb{Z}$, each $w_{i,j}$ is an infinite emitter and $\mu_j \in E^*$. We may assume that all the $w_{i,j}$'s are different for each fixed i.

Suppose now that $\omega \in \ker(\mathrm{id} - H_0(\rho_\bullet))$. We need to show that $\omega \in H_0(\mathcal{H}_E)$ (viewed as a subgroup of $H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$). We compute

$$\begin{aligned} H_0(\rho_\bullet)(\omega) &= \sum_{i=-n}^{-1} \sum_{j=1}^{J_i} \lambda_{i,j} \left[1_{Z(w_{i,j}) \times \{i+1\}}\right] + \sum_{j=1}^{J_0} \lambda_{0,j} \left[1_{Z(\mu_j) \times \{1\}}\right] \\ &= \sum_{i=-n+1}^{0} \sum_{j=1}^{J_{i-1}} \lambda_{i-1,j} \left[1_{Z(w_{i-1,j}) \times \{i\}}\right] + \sum_{j=1}^{J_0} \lambda_{0,j} \left[1_{Z(e_j \mu_j) \times \{0\}}\right], \end{aligned}$$

where e_j is any edge ending in $s(\mu_j)$. From this we get

$$\begin{aligned}
0 = \omega - H_0(\rho_\bullet)(\omega) &= \sum_{j=1}^{J_{-n}} \lambda_{-n,j} \left[1_{Z(w_{-n,j})\times\{-n\}}\right] \\
&+ \sum_{i=-n+1}^{-1} \left(\sum_{j=1}^{J_i} \lambda_{i,j} \left[1_{Z(w_{i,j})\times\{i\}}\right] - \sum_{j=1}^{J_{i-1}} \lambda_{i-1,j} \left[1_{Z(w_{i-1,j})\times\{i\}}\right] \right) \\
&+ \sum_{j=1}^{J_0} \left(\lambda_{0,j} \left[1_{Z(\mu_j)\times\{0\}}\right] - \lambda_{0,j} \left[1_{Z(e_j\mu_j)\times\{0\}}\right] \right) - \sum_{j=1}^{J_{-1}} \lambda_{-1,j} \left[1_{Z(w_{-1,j})\times\{0\}}\right].
\end{aligned} \tag{B.7.10}$$

By Lemma B.3.4, each element $\left[1_{Z(w_{-n,j})\times\{-n\}}\right]$ generates a free summand of $H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$, as $w_{-n,j}$ is singular. Since all the other terms have a strictly smaller second coordinate, in order for the right hand side of (B.7.10) to be 0 we must have $\lambda_{-n,j} = 0$ for all $1 \le j \le J_{-n}$. Thus we may replace $-n$ with $-n+1$ in the expression for ω. Arguing inductively we get that $\lambda_{i,j} = 0$ for all $-1 \le i \le -n$ and $1 \le j \le J_i$. Hence the expression for ω reduces to

$$\omega = \sum_{j=1}^{J_0} \lambda_{0,j} \left[1_{Z(\mu_j)\times\{0\}}\right],$$

from which we see that $\omega \in H_0(\mathcal{H}_E)$. □

B.8 The image of the index map

Recall the index map $I\colon [\![\mathcal{G}_E]\!] \to H_1(\mathcal{G}_E)$ described in Section B.4. Our main goal is to establish that the kernel of the index map is generated by transpositions (i.e. property TR) for minimal graph groupoids. To that end, the goal of this section is to describe the image $I(\alpha) \in H_1(\mathcal{G}_E)$ of an element $\alpha \in [\![\mathcal{G}_E]\!]$ under the identification $H_1(\mathcal{G}_E) \cong \ker(\mathrm{id} - \varphi)$ from Proposition B.7.1 and Lemma B.7.7.

B.8.1 Graded partitions

The identification desribed above will be done in terms of the following "graded partitions" as defined in [Mat15b, page 60].

Definition B.8.1. Let E be a graph. For $\alpha = \pi_U \in [\![\mathcal{G}_E]\!]$ and $k \in \mathbb{Z}$ we define the set

$$S_\alpha(k) := s\left(U \cap c_E^{-1}(k)\right) = \{x \in \partial E \mid (\alpha(x), k, x) \in U\}.$$

Note that each $S_\alpha(k)$ is clopen and that $\partial E \setminus \operatorname{supp}(\alpha) \subseteq S_\alpha(0)$, i.e. $S_\alpha(0)$ contains the largest (cl)open set fixed by α. As $\operatorname{supp}(\alpha)$ is compact, $S_\alpha(k)$ is also compact when $k \neq 0$. This implies that only finitely many $S_\alpha(k)$'s will be non-empty. Hence these form a finite partition of the boundary path space ∂E. We make a few more observations about these graded partitions that we are going to need in the proof of the main result.

Lemma B.8.2. *Let E be a graph and let $\alpha \in [\![\mathcal{G}_E]\!]$. We have $\alpha(S_\alpha(k)) = S_{\alpha^{-1}}(-k)$ for each $k \in \mathbb{Z}$.*

Proof. Recall that U_α denotes the unique bisection that satisfies $\alpha = \pi_{U_\alpha}$. Let $x \in S_\alpha(k)$, i.e. $(\alpha(x), k, x) \in U_\alpha$. Then $(x, -k, \alpha(x)) \in (U_\alpha)^{-1} = U_{\alpha^{-1}}$. This shows that $\alpha(x) \in S_{\alpha^{-1}}(-k)$, which means that $\alpha(S_\alpha(k)) \subseteq S_{\alpha^{-1}}(-k)$ for all integers k. Since these sets form partitions of the unit space we must necessarily have equality. □

The next observation is that when two elements of the topological full group have the same graded partitions, then their difference belongs to the AF-kernel of the cocycle.

Lemma B.8.3. *Let E be a graph and let $Y \subseteq \mathcal{G}_E^{(0)} = \partial E$ be clopen. Suppose that $\alpha, \beta \in [\![\mathcal{G}_E|_Y]\!]$ satisfy $S_\alpha(k) = S_\beta(k)$ for all $k \in \mathbb{Z}$. Then $\beta\alpha^{-1} \in [\![\mathcal{H}_E|_Y]\!]$, that is, $U_{\beta\alpha^{-1}} \subseteq c_E^{-1}(0)$.*

Proof. We claim that because the graded partitions of α and β are the same, we must have

$$S_{\beta\alpha^{-1}}(k) = \begin{cases} Y & k = 0, \\ \emptyset & k \neq 0. \end{cases}$$

Once we have this we immediately see that each element $g = (x, k, y) \in U_{\beta\alpha^{-1}}$ must have $k = 0$, i.e. that $U_{\beta\alpha^{-1}} \subseteq c_E^{-1}(0)$.

To prove the claim, take an arbitrary point $y \in Y$. Then $y \in S_{\alpha^{-1}}(k)$ for some k. By Lemma B.8.2 we have $\alpha^{-1}(y) \in S_\alpha(-k) = S_\beta(-k)$. Then $g = (\alpha^{-1}(y), k, y) \in U_{\alpha^{-1}}$ and $h = (\beta\alpha^{-1}(y), -k, \alpha^{-1}(y)) \in U_\beta$. From this we get $h \cdot g = (\beta\alpha^{-1}(y), 0, y) \in U_{\beta\alpha^{-1}}$, hence $y \in S_{\beta\alpha^{-1}}(0)$, which proves the claim. □

The third lemma describes what happens to the graded partition of an element of the topological full group when we perturb it with a particular transposition.

Lemma B.8.4. *Let E be a graph and let $Y \subseteq \mathcal{G}_E^{(0)} = \partial E$ be clopen. Let $V \subseteq \mathcal{G}_E|_Y$ be a compact bisection with disjoint source and range, and such that $V \subseteq c_E^{-1}(K)$ for some integer K. Let $\tau = \pi_{\widehat{V}} \in [\![\mathcal{G}_E|_Y]\!]$ be the associated transposition. If a homeomorphism $\alpha \in [\![\mathcal{G}_E|_Y]\!]$ satisfies* $\operatorname{supp}(\alpha) = s(V)$*, then* $\operatorname{supp}(\tau\alpha\tau) = r(V)$ *and $S_{\tau\alpha\tau}(k) = \tau(S_\alpha(k))$ for each $k \in \mathbb{Z}$.*

Proof. We first take care of the support of $\tau\alpha\tau$. If $x \notin r(V)$, then $\tau(x) \notin s(V) = \operatorname{supp}(\alpha)$. From this we see that $\tau\alpha\tau$ fixes x because

$$\tau\alpha\tau(x) = \tau\alpha(\tau(x)) = \tau\tau(x) = x.$$

This shows that $\operatorname{supp}(\tau\alpha\tau) \subseteq r(V)$. By definition, the set $\{x \in \partial E \mid \alpha(x) \neq x\}$ is dense in $\operatorname{supp}(\alpha) = s(V)$. Then the set $Z := \{\tau(x) \mid x \in \partial E \text{ and } \alpha(x) \neq x\}$ is dense in $r(V)$. Let $y \in Z$ and set $x = \tau(y)$, so that $y = \tau(x)$ and $\alpha(x) \neq x$. Then we have

$$\tau(\alpha(\tau(y))) = \tau(\alpha(\tau^2(x))) = \tau(\alpha(x)) \neq \tau(x) = y.$$

Hence $Z \subseteq \operatorname{supp}(\tau\alpha\tau) \subseteq r(V)$, and by the density of Z we get $\operatorname{supp}(\tau\alpha\tau) = r(V)$ as desired.

We now turn to the second statement. Let $x \in S_\alpha(k)$. Then $(\alpha(x), k, x) \in U_\alpha$. Consider first the case $x \in \operatorname{supp}(\alpha) = s(V)$. It is clear from the assumptions on V that we have $S_\tau(K) = s(V)$, $S_\tau(-K) = r(V)$ and $S_\tau(0) = Y \setminus \operatorname{supp}(\tau)$. Thus both x and $\alpha(x)$ lie in $S_\tau(K)$. This means that $(\tau(x), K, x) \in U_\tau$ and that $(\tau\alpha(x), K, \alpha(x)) \in U_\tau$. We also have $(\tau(x), K, x)^{-1} = (x, -K, \tau(x)) \in U_\tau$, since $\tau = \tau^{-1}$. Multiplying these together we obtain

$$(\tau\alpha(x), K, \alpha(x)) \cdot (\alpha(x), k, x) \cdot (x, -K, \tau(x)) = (\tau\alpha(x), k, \tau(x)) \in U_{\tau\alpha\tau},$$

which shows precisely that $\tau(x) \in S_{\tau\alpha\tau}(k)$.

Lastly consider the case when $x \notin \operatorname{supp}(\alpha)$. Then we must have $k = 0$, and since $\alpha(x) = x$, we have $(x, 0, x) \in U_\alpha$. If x is not in the support of τ either (i.e. $x \notin r(V)$), then $\tau(x) = x \in S_{\tau\alpha\tau}(0)$ as desired. The final possibility is that $x \in r(V) = S_\tau(-K)$, and then $(\tau(x), -K, x) \in U_\tau$ and $(x, K, \tau(x)) \in U_\tau$. Multiplying these gives

$$(\tau(x), -K, x) \cdot (x, 0, x) \cdot (x, K, \tau(x)) = (\tau(x), 0, \tau(x)) \in U_{\tau\alpha\tau},$$

hence $\tau(x) \in S_{\tau\alpha\tau}(0)$.

We have shown that $\tau(S_\alpha(k)) \subseteq S_{\tau\alpha\tau}(k)$ for all k, but since both the $S_\alpha(k)$'s and the $S_{\tau\alpha\tau}(k)$'s are partitions, we must actually have equality. This finishes the proof. □

B.8.2 Identifying $I(\alpha)$

Let us now turn to describing the image of the index map. Recall the homomorphism $\varphi\colon H_0(\mathcal{H}_E) \to H_0(\mathcal{H}_E)$ from Definition B.7.5, where we view $H_0(\mathcal{H}_E)$ as a subgroup of $H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$. For $n \in \mathbb{N}$ its iterates are given by

$$\varphi^n\left(\left[1_{Z(\mu)\times\{0\}}\right]\right) = \left[1_{Z(\mu)\times\{n\}}\right] = \left[1_{Z(\nu\mu)\times\{0\}}\right],$$

where ν is any path of length n in E terminating in $s(\mu)$. For any path μ in E of length at least n the iterated inverses are also defined, and they are given by

$$\varphi^{-n}\left(\left[1_{Z(\mu)\times\{0\}}\right]\right)=\left[1_{Z(\mu)\times\{-n\}}\right]=\left[1_{Z(\sigma_E^n(\mu))\times\{0\}}\right].$$

In the setting of Definition B.8.1 we can write $U_\alpha \cap c_E^{-1}(k)=\bigsqcup_{j=1}^{J_k} Z(\mu_j,F_j,\nu_j)$, where for each j, $|\mu_j|-|\nu_j|=k$. When $k<0$ this entails that $|\nu_j|\geq |k|$. Since we have that $S_\alpha(k)=s\left(U_\alpha\cap c_E^{-1}(k)\right)=\bigsqcup_{j=1}^{J_k} Z(\nu_j\setminus F_j)$, the negative powers φ^i are then defined on the associated characteristic functions for $-|k|\leq i\leq -1$ and we have

$$\varphi^i\left(\left[1_{S_\alpha(k)\times\{0\}}\right]\right)=\left[1_{S_\alpha(k)\times\{i\}}\right]. \tag{B.8.1}$$

For $k\geq 0$ and $i\geq 0$ Equation (B.8.1) clearly holds as well. For $i=k$ we furthermore have

$$\varphi^k\left(\left[1_{S_\alpha(k)\times\{0\}}\right]\right)=\left[1_{\alpha(S_\alpha(k))\times\{0\}}\right]. \tag{B.8.2}$$

Definition B.8.5. For $k\in\mathbb{Z}$ we define the following expression

$$\varphi^{(k)}:=\begin{cases}-(\mathrm{id}+\varphi+\cdots+\varphi^{k-1}) & k>0,\\ 0 & k=0,\\ \varphi^{-1}+\varphi^{-2}+\cdots+\varphi^{k} & k<0.\end{cases}$$

The definition above is somewhat formal in the sense that for $k<0$ it is only defined on certain elements. However, we will only apply the negative powers as in Equation (B.8.1) where they are indeed defined. Observe that formally we have

$$(\mathrm{id}-\varphi)\circ\varphi^{(k)}=\varphi^k-\mathrm{id}. \tag{B.8.3}$$

Let us now show how an element $\alpha\in[\![\mathcal{G}_E]\!]$ gives rise to an element of $\ker(\mathrm{id}-\varphi)$ as on page 61 of [Mat15b]. Assume for simplicity that E^0 is finite, so that $S_\alpha(0)$ is compact. Since both the $S_\alpha(k)$'s and $\alpha(S_\alpha(k))$'s form partitions of ∂E we obtain the following using (B.8.2)

$$[1_{\partial E}]=\sum_{k\in\mathbb{Z}}\left[1_{S_\alpha(k)\times\{0\}}\right]=\sum_{k\in\mathbb{Z}}\left[1_{\alpha(S_\alpha(k))\times\{0\}}\right]=\sum_{k\in\mathbb{Z}}\varphi^k\left(\left[1_{S_\alpha(k)\times\{0\}}\right]\right).$$

Subtracting these using (B.8.3) we get

$$\sum_{k\in\mathbb{Z}}(\varphi^k-\mathrm{id})\left(\left[1_{S_\alpha(k)\times\{0\}}\right]\right)=(\mathrm{id}-\varphi)\left(\sum_{k\in\mathbb{Z}}\varphi^{(k)}\left(\left[1_{S_\alpha(k)\times\{0\}}\right]\right)\right)=0,$$

which shows that $\sum_{k\in\mathbb{Z}}\varphi^{(k)}\left(\left[1_{S_\alpha(k)\times\{0\}}\right]\right)\in\ker(\mathrm{id}-\varphi)$. Analogously to [Mat15b, Lemma 6.8] we will see that this is precisely the element to which $I(\alpha)$ corresponds.

$$
\begin{tikzcd}
 & 0 \arrow[r, dashed] \arrow[d] & 0 \arrow[r, dashed] \arrow[d] & H_1(\mathcal{G}_E) \arrow[d] \arrow[dddll, dashed, "\partial_1"] & \\
 & \frac{C_c(\mathcal{G}_E \times \mathbb{Z})}{\operatorname{im}(\delta_2)} \arrow[r, "\operatorname{id}-\rho_1/\operatorname{im}(\delta_2)"] \arrow[d, "\widetilde{\delta_1}"] & \frac{C_c(\mathcal{G}_E \times \mathbb{Z})}{\operatorname{im}(\delta_2)} \arrow[r, "\pi_1/\operatorname{im}(\delta_2)"] \arrow[d, "\widetilde{\delta_1}"] & \frac{C_c(\mathcal{G}_E)}{\operatorname{im}(\delta_2)} \arrow[r] \arrow[d, "\widetilde{\delta_1}"] & 0 \\
0 \arrow[r] & C_c(\partial E \times \mathbb{Z}) \arrow[r, "\operatorname{id}-\rho_0"] \arrow[d] & C_c(\partial E \times \mathbb{Z}) \arrow[r, "\pi_0"] \arrow[d] & C_c(\partial E) \arrow[d] & \\
 & H_0(\mathcal{G}_E \times \mathbb{Z}) \arrow[r, dashed] & H_0(\mathcal{G}_E \times \mathbb{Z}) \arrow[r, dashed] & H_0(\mathcal{G}_E) &
\end{tikzcd}
$$

Figure B.2: The connecting homomorphism ∂_1 from the exact sequence (B.7.5).

Lemma B.8.6. *Let E be an essential graph and let $\alpha = \pi_U \in [\![\mathcal{G}_E]\!]$. Under the identification $H_1(\mathcal{G}_E) \cong \ker(\operatorname{id}-\varphi)$, the element $I(\alpha) \in H_1(\mathcal{G}_E)$ corresponds to*

$$\sum_{k \in \mathbb{Z}} \varphi^{(k)}\left([1_{S_\alpha(k)\times\{0\}}]\right) \in \ker(\operatorname{id}-\varphi) \leq H_0(\mathcal{H}_E).$$

Proof. The identification $H_1(\mathcal{G}_E) \cong \ker(\operatorname{id}-H_0(\rho_\bullet))$ from Proposition B.7.1 is implemented by the (injective) connecting homomorphism

$$\partial_1 \colon H_1(\mathcal{G}_E) \to H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$$

from the exact sequence (B.7.5). Since $\ker(\operatorname{id}-\varphi) = \ker(\operatorname{id}-H_0(\rho_\bullet)) = \operatorname{im}(\partial_1)$ as subsets of $H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$, it suffices to compute $\partial_1(I(\alpha)) \in H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$. We will do this by stepwise going through the definition of ∂_1 in terms of the Snake Lemma applied to the diagram in Figure B.2. To save space we have shortened $C_c(\mathcal{G}, \mathbb{Z})$ to $C_c(\mathcal{G})$ and $\mathcal{G}_E \times_{c_E} \mathbb{Z}$ to $\mathcal{G}_E \times \mathbb{Z}$. The maps $\widetilde{\delta_1}$ in Figure B.2 are given by $\widetilde{\delta_1}(f + \operatorname{im}(\delta_2)) = \delta_1(f)$. The top and bottom rows are the kernels and cokernels of the $\widetilde{\delta_1}$'s, respectively.

We first treat the case when E^0 is finite, for then U and $S_\alpha(0)$ are both compact. We start with $\alpha = \pi_U \in [\![\mathcal{G}_E]\!]$ and look at $I(\alpha) = [1_U] \in H_1(\mathcal{G}_E)$. Now view $1_U + \operatorname{im}(\delta_2)$ as an element of $C_c(\mathcal{G}_E)/\operatorname{im}(\delta_2)$ (recall that $\delta_1(1_U) = 0$). A lift of this element by $\pi_1/\operatorname{im}(\delta_2)$ is given by the element $h := 1_{U\times\{0\}} \in C_c(\mathcal{G}_E \times_{c_E} \mathbb{Z})$, since $\pi_1(h) = 1_U$. At this point we have $h + \operatorname{im}(\delta_2) \in C_c(\mathcal{G}_E \times_{c_E} \mathbb{Z}) / \operatorname{im}(\delta_2)$. Before applying $\widetilde{\delta_1}$, we partition the full bisection U defining α in terms of its values under the cocycle c_E:

$$U = \bigsqcup_{k=-N}^{N} U_k, \text{ where } U_k = U \cap c_E^{-1}(k),$$

so that $s(U_k) = S_\alpha(k)$. Note that

$$1_{\partial E \times \{0\}} = \sum_{k=-N}^{N} 1_{s(U_k) \times \{0\}} = \sum_{k=-N}^{N} 1_{r(U_k) \times \{0\}}. \tag{B.8.4}$$

Using this we compute

$$\begin{aligned}
\widetilde{\delta_1}(h + \operatorname{im}(\delta_2)) = \delta_1(h) &= \delta_1(1_{U \times \{0\}}) \\
&= \sum_{k=-N}^{N} \delta_1(1_{U_k \times \{0\}}) \\
&= \sum_{k=-N}^{N} \left(s_*(1_{U_k \times \{0\}}) - r_*(1_{U_k \times \{0\}})\right) \\
&= \sum_{k=-N}^{N} \left(1_{s(U_k \times \{0\})} - 1_{r(U_k \times \{0\})}\right) \\
&= \sum_{k=-N}^{N} \left(1_{s(U_k) \times \{k\}} - 1_{r(U_k) \times \{0\}}\right) \\
&= \sum_{k=-N}^{N} \left(1_{s(U_k) \times \{k\}} - 1_{s(U_k) \times \{0\}}\right) \\
&= \sum_{k=-N}^{-1} \left(1_{S_\alpha(k) \times \{k\}} - 1_{S_\alpha(k) \times \{0\}}\right) + \sum_{k=1}^{N} \left(1_{S_\alpha(k) \times \{k\}} - 1_{S_\alpha(k) \times \{0\}}\right).
\end{aligned}$$

The next step is to find the unique lift of $\delta_1(h)$ by $\mathrm{id} - \rho_0$. Applying Lemma B.6.2 to each term in the sum above we see that this lift is

$$g := \sum_{k=-N}^{-1} \sum_{i=k}^{-1} 1_{S_\alpha(k) \times \{i\}} - \sum_{k=1}^{N} \sum_{i=0}^{k-1} 1_{S_\alpha(k) \times \{i\}} \in C_c(\partial E \times \mathbb{Z}, \mathbb{Z}).$$

The final step is to map the element g “downwards” into the cokernel of $\widetilde{\delta_1}$, which is precisely $H_0(\mathcal{G}_E \times_{c_E} \mathbb{Z})$. Using Equation (B.8.1) we find

$$\begin{aligned}
\partial_1(I(\alpha)) &= \partial_1([1_U]) = g + \operatorname{im}(\widetilde{\delta_1}) = [g] \\
&= \sum_{k=-N}^{-1} \sum_{i=k}^{-1} [1_{S_\alpha(k)\times\{i\}}] - \sum_{k=1}^{N} \sum_{i=0}^{k-1} [1_{S_\alpha(k)\times\{i\}}] \\
&= \sum_{k=-N}^{-1} \sum_{i=k}^{-1} \varphi^i\left([1_{S_\alpha(k)\times\{0\}}]\right) - \sum_{k=1}^{N} \sum_{i=0}^{k-1} \varphi^i\left([1_{S_\alpha(k)\times\{0\}}]\right) \\
&= \sum_{k\in\mathbb{Z}} \varphi^{(k)}\left([1_{S_\alpha(k)\times\{0\}}]\right)
\end{aligned}$$

In the case that E^0 is infinite, the proof above remains valid if we simply replace U with $U^\perp$ from Subsection 4.1, (as this makes all indicator functions above remain compactly supported) and replace ∂E with $\operatorname{supp}(\pi_U)$ in Equation (B.8.4). □

We emphasize that the sum in the lemma above really is a finite sum. Since we are aiming to establish Property TR for restrictions of graph groupoids, we need to verify that the description of the index map as above also works in this case.

Corollary B.8.7. *Let E be an essential graph and let $Y \subseteq \partial E$ be clopen and $\mathcal{G}_E$-full. Then the element $I(\alpha) \in H_1(\mathcal{G}_E|_Y)$ for $\alpha \in [\![\mathcal{G}_E|_Y]\!]$ corresponds to*

$$\sum_{k\in\mathbb{Z}} \varphi^{(k)}\left([1_{S_\alpha(k)\times\{0\}}]\right) \in \ker(\mathrm{id}-\varphi) \leq H_0(\mathcal{H}_E)$$

under the identification $H_1(\mathcal{G}_E|_Y) \cong H_1(\mathcal{G}_E) \cong \ker(\mathrm{id}-\varphi)$, and the $S_\alpha(k)$'s form a finite clopen partition of Y.

Proof. The inclusion $\mathcal{G}_E|_Y \hookrightarrow \mathcal{G}_E$ induces an isomorphism in homology due to the fullness of Y. We also have a canonical inclusion $[\![\mathcal{G}_E|_Y]\!] \hookrightarrow [\![\mathcal{G}_E]\!]$ given by $\pi_U \mapsto \pi_{\widetilde{U}}$, where $\widetilde{U} = U \sqcup \partial E \setminus Y$ for $U \subseteq \mathcal{G}_E|_Y$ a full bisection. In words, $\pi_{\widetilde{U}}$ simply extends π_U trivially to the identity on $\partial E \setminus Y$. Together with the respective index maps, we claim that from this we get a commutative diagram as follows:

$$\begin{tikzcd}
{[\![\mathcal{G}_E|_Y]\!]} \arrow[r, "I"] \arrow[d, hook] & H_1(\mathcal{G}_E|_Y) \arrow[d, "\cong"] \\
{[\![\mathcal{G}_E]\!]} \arrow[r, "I"] & H_1(\mathcal{G}_E)
\end{tikzcd}$$

To see that the diagram commutes, let $\alpha = \pi_U \in [\![\mathcal{G}_E|_Y]\!]$ be given, where $U \subseteq \mathcal{G}_E|_Y$ is a full bisection. The two paths in the diagram result in $\alpha \mapsto [1_{(\widetilde{U})^\perp}] \in H_1(\mathcal{G}_E)$

and $\alpha \mapsto [1_{U^\perp}] \in H_1(\mathcal{G}_E)$, respectively. These elements are indeed the same since the sets $(\widetilde{U})^\perp$ and $U^\perp$ are actually equal.

Let $\widetilde{\alpha} = \pi_{\widetilde{U}}$ denote the trivial extension of α. Then $S_\alpha(k) = S_{\widetilde{\alpha}}(k)$ for all $k \neq 0$. Recall that $\varphi^{(0)} = 0$, so $k = 0$ does not contribute. Appealing to Lemma B.8.6 we obtain

$$I(\alpha) \longleftrightarrow I(\widetilde{\alpha}) \longleftrightarrow \sum_{k\in\mathbb{Z}} \varphi^{(k)}\left(\left[1_{S_{\widetilde{\alpha}}(k)\times\{0\}}\right]\right) = \sum_{k\in\mathbb{Z}} \varphi^{(k)}\left([1_{S_\alpha(k)\times\{0\}}]\right),$$

under the correspondence $H_1(\mathcal{G}_E|_Y) \cong H_1(\mathcal{G}_E) \cong \ker(\mathrm{id} - \varphi) \leq H_0(\mathcal{H}_E)$. □

Remark B.8.8. For finite graphs one might expect all formulas in the present paper to recover those in [Mat15b, Section 6] after substituting $\varphi = \delta^{-1}$, since one has that

$$\ker(\mathrm{id} - \delta) = \ker(\mathrm{id} - \delta^{-1})$$

as sets. However, a small difference already appears in Corollary B.8.7 when compared to [Mat15b, Lemma 6.8], which will propagate in the sequel. After substituting $\varphi = \delta^{-1}$, the k'th term (for $k \neq 0$) in Corollary B.8.7 becomes

$$\varphi^{(k)} = \begin{cases} -(\mathrm{id} + \delta^{-1} + \cdots + \delta^{1-k}) & k > 0, \\ \delta + \delta^2 + \cdots + \delta^{|k|} & k < 0, \end{cases}$$

whereas the k'th term in [Mat15b, Lemma 6.8] is

$$\delta^{(-k)} = \begin{cases} -(\delta^{-1} + \delta^{-2} \cdots + \delta^{-k}) & k > 0, \\ \mathrm{id} + \delta + \cdots + \delta^{|k|-1} & k < 0. \end{cases}$$

The reason these are different is because identifying $H_1(\mathcal{G}_E)$ with $\ker(\mathrm{id} - \delta)$ instead of $\ker(\mathrm{id} - \delta^{-1})$ give different lifts of the element $\delta_1(h)$ in the proof of Lemma B.8.6.

B.9 Establishing Property TR

We are by now almost ready to prove that restrictions of graph groupoids have Property TR. Given what we have established so far, our proof will in broad strokes follow the proof of [Mat15b, Lemma 6.10] using the endomorphism φ instead of the automorphism δ mentioned in Remark B.7.6. However, there is another major difference, which we discuss below.

What is actually proved in [Mat15b, Lemma 6.10] is that if the adjacency matrix A_E of a finite graph E is primitive, then any restriction of $\mathcal{G}_E$ has Property TR. Recall that a non-negative integral matrix A is *primitive* if for some $n \in \mathbb{N}$ all entries in A^n are strictly positive. At the beginning of the proof of [Mat15b, Theorem 6.11] it is noted that the graph groupoid of a strongly connected finite graph is always Kakutani equivalent to graph groupoid whose adjacency matrix is primitive, from which it follows that restrictions of the former also have Property TR.

One reason why primitivity of the adjacency matrix is so useful is that this matrix then has a (strictly dominant) Perron eigenvalue $\lambda > 1$. Another reason is that the AF-groupoid $\mathcal{H}_E$ becomes minimal. This is if and only if, in fact, and also equivalent to the shift of finite type determined by A_E being topologically mixing. In this case the infinite path space E^∞ admits exactly one $\mathcal{H}_E$-invariant probability measure. This measure, lets denote it by ω, satisfies $\omega(s(U)) = \lambda\, \omega(rU))$ for any compact bisection $U \subseteq \mathcal{G}_E$ with $U \subseteq c_E^{-1}(1)$. This then allows one to compare clopen subsets and the image of the class of their characteristic functions under the automorphism δ and from this obtain bisections connecting them using [Mat12, Lemma 6.7]. The approach in [Mat15b] was subsequently generalized to an abstract setting in [Mat16, Proposition 4.5 (2)].

In the setting of the present paper, however, where we allow infinite emitters in the graphs, we are no longer dealing with a shift of finite type (or any shift space for that matter), nor do we have a Perron eigenvalue. Neither is the AF-groupoid $\mathcal{H}_E$ ever minimal (see Remark B.7.2). So the aforementioned [Mat16, Proposition 4.5 (2)] does not apply. We replace the notion of primitivity (or mixing) by the technical Lemma B.9.1 below. It prescribes necessary conditions on a graph E to guarantee the existence of certain disjoint paths in E from which we can explicitly define sets with similar properties as the sets $C_{n,i}$ and $D_{n,j}$ which are constructed using the invariant measure ω in [Mat15b, Lemma 6.10]. A key point is that these necessary conditions can always be arranged, without changing the isomorphism class of the groupoid, as demonstrated in Lemma B.9.2.

B.9.1 Technical lemmas

The following "combinatorial bookkeeping" lemma will allow us to explicitly describe the terms in the sum in Corollary B.8.7 and relate them to each other. As mentioned above, it will play a similar role as primitivity (or mixing) does in [Mat15b, Lemma 6.10].

Lemma B.9.1. *Let E be a strongly connected graph. Assume there is an infinite emitter in E which supports infinitely many loops and from which there is at least one edge to any other vertex in E. Let $\emptyset \neq Y \subseteq \partial E$ be clopen. Suppose we are given a clopen proper subset $\emptyset \neq A \subsetneq Y$, finite subsets $P \subset \mathbb{N}$ and $Q \subset -\mathbb{N}$, natural numbers $m_k \in \mathbb{N}$ and vertices $v_{k,i} \in E^0$ indexed over $k \in Q \cup \{0\} \cup P$ and $1 \leq i \leq m_k$. Then there exists a natural number $N \geq \max_{q \in Q} |q|$ and*

1. *mutually disjoint paths* $\gamma_{k,i}^{(0)} \in E^N v_{k,i}$ *such that* $Z\left(\gamma_{k,i}^{(0)}\right) \subseteq Y \setminus A$ *for all* k *in* $Q \cup \{0\} \cup P$ *and* $1 \leq i \leq m_k$,
2. *mutually disjoint paths* $\gamma_{p,i}^{(j)} \in E^{N+j} v_{p,i}$ *such that* $Z\left(\gamma_{p,i}^{(j)}\right) \subseteq A$ *for all* $p \in P$, $1 \leq i \leq m_p$ *and* $j = 1, 2, \ldots, p$,
3. *mutually disjoint paths* $\gamma_{q,i}^{(l)} \in E^{N-l} v_{q,i}$ *such that* $Z\left(\gamma_{q,i}^{(l)}\right) \subseteq A$ *for all* $q \in Q$, $1 \leq i \leq m_q$ *and* $l = 1, 2, \ldots, |q|$.

Proof. Pick an infinite emitter $w \in E^0_{\text{sing}}$ which satisfies the assumptions in the lemma. We enumerate the infinitely many loops based at w as $e_{k,i}$ (these are all distinct) where k and i both range over $\mathbb{N}$. Choose paths $\mu, \mu' \in E^*$ such that $Z(\mu) \subseteq Y \setminus A$ and $Z(\mu') \subseteq A$. By extending these paths we may assume that they both end in w, and by concatenating sufficiently many loops at w to the shortest one of these, we may furthermore assume that $|\mu| = |\mu'|$. For each $k \in Q \cup \{0\} \cup P$ and $1 \leq i \leq m_k$ we pick an edge $f_{k,i}$ which goes from w to $v_{k,i}$.

The paths we will define will all start with either μ or μ', which will ensure that their cylinder sets are contained in either A or $Y \setminus A$ as needed. Then they will have a certain number of the loops at w and it is these that will ensure the paths are mutually disjoint. Also, they will all end with an edge $f_{k,i}$ taking care of the range of the paths. We set $K := \max_{q \in Q} |q|$ and $M := |\mu| = |\mu'|$, and then define $N := M + K + 2$. Here M is present because all the paths start with μ or μ', K is a "buffer" we can subtract from for the $\gamma_{q,i}^{(l)}$'s (as these should have length $N - l$) and the term 2 comes from having at least one loop $e_{k,i}$ and then ending with $f_{k,i}$. We now define the desired paths as follows:

(1) $\gamma_{k,i}^{(0)} := \mu\, e_{k,i}^{K+1}\, f_{k,i}$ for $k \in Q \cup \{0\} \cup P$ and $1 \leq i \leq m_k$,
(2) $\gamma_{p,i}^{(j)} := \mu'\, e_{p,i}^{K+1+j}\, f_{p,i}$ for $p \in P$, $1 \leq i \leq m_p$ and $j = 1, 2, \ldots, p$,
(3) $\gamma_{q,i}^{(l)} := \mu'\, e_{q,i}^{K+1-l}\, f_{q,i}$ for $q \in Q$, $1 \leq i \leq m_q$ and $l = 1, 2, \ldots, |q|$.

It is clear that these satisfy the conditions in the lemma. □

The next lemma shows that for a graph E with finitely many vertices, the conditions in Lemma B.9.1 can always be arranged, by changing the graph, but without changing the (isomorphism class of the) groupoid. This is actually the only place where we need to assume that the graph has finitely many vertices (see also Remark B.9.6).

In order to prove it, we will appeal to one of Sørensen's geometric moves on graphs from [Sør13]. On page 1207 therein, a move on graphs called *move (T)* is described. In order to apply this move one needs a graph E with an infinite emitter

$w \in E^0_{\text{sing}}$. If there is a path $e_1 e_2 \cdots e_n$ in E from w to another vertex v such that w emits infinitely many edges to $r(e_1)$, then move (T) is the operation of adding a countably infinite number of new edges from w to v.

It is proved in [Sør13, Theorem 5.4] that move (T) can be expressed using the first four "standard moves" in Section 3 of [Sør13]. This means that move (T) produces *move equivalent* graphs, which in turn implies that the associated graph groupoids are Kakutani equivalent [CRS17]. By virtue of [BCW17, Lemma 6.5] we can deduce something even stronger, namely that move (T) actually produce *orbit equivalent* graphs. In our setting this in fact implies isomorphism of the graph groupoids.

Lemma B.9.2. *Let E be a strongly connected graph with finitely many vertices and suppose that E has an infinite emitter $w \in E^0_{sing}$. Let F denote the graph which is obtained from E by, for each $v \in E^0$, adding a countably infinite number of new edges from w to v. Then $\mathcal{G}_E \cong \mathcal{G}_F$ as étale groupoids.*

Proof. The strong connectedness of E guarantees, that for each vertex $v \in E^0$, there exists a path from w to v that starts with an edge to a vertex to which w emits infinitely already. Thus we see that the graph F is obtained from E by applying move (T) finitely many times. As mentioned in the paragraph above, this implies that the graphs E and F are orbit equivalent. The assumptions on E also ensure that E satisfies Condition (L), and therefore so does F. It now follows from the main result of [BCW17] that $\mathcal{G}_E \cong \mathcal{G}_F$. □

The final lemma describes in some sense a "graded cancellation" for the map φ on $H_0(\mathcal{H}_E)$. It is a straightforward extension of [Mat15b, Lemma 6.9], after having established cancellation for general AF-groupoids in Section B.5, but we have nevertheless included the short argument for completeness.

Lemma B.9.3. *Let E be an essential graph and let $A, B \subseteq \partial E$ be compact open subsets. If $\varphi^n([1_A]) = [1_B]$ in $H_0(\mathcal{H}_E)$ for some $n \in \mathbb{N}$, then there exists a bisection $U \subseteq \mathcal{G}_E$ satisfying $U \subseteq c_E^{-1}(n)$, $s(U) = A$ and $r(U) = B$.*

Proof. We first write A as a disjoint union of punctured cylinder sets as follows $A = \sqcup_{j=1}^{J} Z(\mu_j \setminus F_j)$. Now pick paths $\gamma_j \in E^n$ with $r(\gamma_j) = s(\mu_j)$ and set $C := \sqcup_{j=1}^{J} Z(\gamma_j \mu_j \setminus F_j)$. Then we have

$$[1_B] = \varphi^n([1_A]) = [1_C] \text{ in } H_0(\mathcal{H}_E)$$

by definition of φ. Invoking cancellation in the AF-groupoid $\mathcal{H}_E$ (Theorem B.5.5) produces a bisection $W \subseteq \mathcal{H}_E \subseteq \mathcal{G}_E$ with $s(W) = C$ and $r(W) = B$. Next define the bisection $V := \sqcup_{j=1}^{J} Z(\gamma_j \mu_j, F_j, \mu_j)$, which satisfies $s(V) = A$ and $r(V) = C$. Finally, setting $U := WV$ gives us the desired bisection since $s(U) = s(V) = A$, $r(U) = r(W) = B$ and $U \subseteq c_E^{-1}(n)$, because $W \subseteq c_E^{-1}(0)$ and $V \subseteq c_E^{-1}(n)$. □

B.9.2 The main result

We are now ready to give the proof of our main result.

Theorem B.9.4. *Let E be a strongly connected graph with finitely many vertices and at least one infinite emitter. Let further $\emptyset \neq Y \subseteq \mathcal{G}_E^{(0)} = \partial E$ be clopen. Then the restricted graph groupoid $\mathcal{G}_E|_Y$ has Property TR.*

Proof. Let $\alpha = \pi_U \in [\![\mathcal{G}_E|_Y]\!] \setminus \{\mathrm{id}\}$ be given and suppose that $I(\alpha) = 0$ in $H_1(\mathcal{G}_E|_Y)$. We are going to show that α is a product of transpositions. In the previous section we saw that $I(\alpha)$ corresponds to an element in $\ker(\mathrm{id} - \varphi)$ which is described in terms of the finite clopen partition $\{S_\alpha(k)\}_{k\in\mathbb{Z}}$ of Y. Define

$$P := \{k > 0 \mid S_\alpha(k) \neq \emptyset\} \qquad \text{and} \qquad Q := \{k < 0 \mid S_\alpha(k) \neq \emptyset\}.$$

These are finite subsets of $\mathbb{N}$. Set $A := \mathrm{supp}(\alpha)$. By Lemma B.4.13 we may assume that $A \neq Y$. The set A is non-empty since $\alpha \neq \mathrm{id}$. We can write

$$A = \mathrm{supp}(\alpha) = (S_\alpha(0) \cap A) \bigsqcup \bigsqcup_{k\in Q\cup P} S_\alpha(k).$$

Now decompose these in terms of punctured cylinder sets as

$$S_\alpha(0) \cap A = \bigsqcup_{i=1}^{m_0} Z(\mu_{0,i} \setminus F_{0,i}) \qquad \text{and} \qquad S_\alpha(k) = \bigsqcup_{i=1}^{m_k} Z(\mu_{k,i} \setminus F_{k,i}),$$

where $\mu_{k,i} \in E^*$ and $F_{k,i} \subseteq_{\text{finite}} r(\mu_{k,i})$. It is possible for one of P, Q or $S_\alpha(0) \cap A$ to be empty (but not all of them). For now we assume that all three are non-empty, and we shall comment on what happens otherwise near the end of the proof.

At this point we want to invoke Lemma B.9.1. By Lemma B.9.2 we may assume that E satisfies the hypotheses of Lemma B.9.1. Setting $v_{k,i} = s(\mu_{k,i})$ in Lemma B.9.1 gives us a natural number N (larger in absolute value than all numbers in Q) and

1. mutually disjoint paths $\gamma_{k,i}^{(0)} \in E^N s(\mu_{k,i})$ such that $Z\left(\gamma_{k,i}^{(0)}\right) \subseteq Y \setminus A$ for all $k \in Q \cup \{0\} \cup P$ and $1 \leq i \leq m_k$,
2. mutually disjoint paths $\gamma_{p,i}^{(j)} \in E^{N+j} s(\mu_{p,i})$ such that $Z\left(\gamma_{p,i}^{(j)}\right) \subseteq A$ for all $p \in P$, $1 \leq i \leq m_p$ and $j = 1, 2, \ldots, p$,
3. mutually disjoint paths $\gamma_{q,i}^{(l)} \in E^{N-l} s(\mu_{q,i})$ such that $Z\left(\gamma_{q,i}^{(l)}\right) \subseteq A$ for all $q \in Q$, $1 \leq i \leq m_q$ and $l = 1, 2, \ldots, |q|$.

From these we define the compact open set

$$B := \bigsqcup_{k\in Q\cup\{0\}\cup P} \bigsqcup_{i=1}^{m_k} Z(\gamma_{k,i}^{(0)}\mu_{k,i} \setminus F_{k,i}) \subseteq Y \setminus A.$$

Next we define the bisection

$$V := \bigsqcup_{k\in Q\cup\{0\}\cup P} \bigsqcup_{i=1}^{m_k} Z(\gamma_{k,i}^{(0)}\mu_{k,i}, F_{k,i}, \mu_{k,i}).$$

As $s(V) = A$ is disjoint from $r(V) = B$ we get a transposition $\tau_V := \pi_{\widehat{V}} \in [\![\mathcal{G}_E|_Y]\!]$. This transposition satisfies $\tau_V(A) = B$, $\tau_V(B) = A$, $\mathrm{supp}(\tau_V) = A \sqcup B$ and

$$S_{\tau_V}(N) = A, \quad S_{\tau_V}(-N) = B, \quad S_{\tau_V}(0) = Y \setminus A \sqcup B.$$

We now define another element in $[\![\mathcal{G}_E|_Y]\!]$, namely $\beta := \tau_V\alpha\tau_V$. If we can prove that β is a product of transpositions, then the theorem follows. To do just that, we are going construct another element $\tau \in [\![\mathcal{G}_E|_Y]\!]$, which is itself a product of transpositions, but which also satisfies $S_\tau(k) = S_\beta(k)$ for all k. The construction of τ is a bit involved, so before we get to that, let us explain why having τ suffices. Given an element τ as above, we deduce from Lemma B.8.3 that $\beta\tau^{-1} \in [\![\mathcal{H}_E|_Y]\!]$. Making use of the fact that $I(\alpha) = 0$ we find that $I(\beta\tau^{-1}) = 0$ as well. Indeed,

$$I(\beta\tau^{-1}) = I(\tau_V\alpha\tau_V\tau^{-1}) = I(\tau_V) + I(\alpha) + I(\tau_V) - I(\tau),$$

which are all 0 as transpositions are in the kernel of the index map. As the groupoid $\mathcal{H}_E|_Y$ is an AF-groupoid and all AF-groupoids have Property TR [Mat16, Theorem 3.3.(4)], we deduce that $\beta\tau^{-1}$ is a product of transpositions (in $[\![\mathcal{H}_E|_Y]\!]$). It follows that β is a product of transpositions as well.

All that remains now is the construction of τ as above. The element τ will be of the form $\tau = \tau_- \circ \tau_+$, where τ_+ will be constructed from the $S_\beta(p)$'s for $p \in P$ and similarly τ_- comes from the $S_\beta(q)$'s for $q \in Q$. We begin by noting that $\mathrm{supp}(\beta) = B$ and that

$$S_\beta(k) = \tau_V(S_\alpha(k)) = \begin{cases} \bigsqcup_{i=1}^{m_k} Z(\gamma_{k,i}^{(0)}\mu_{k,i} \setminus F_{k,i}) & \text{for } k \neq 0 \\ \bigsqcup_{i=1}^{m_0} Z(\gamma_{0,i}^{(0)}\mu_{0,i} \setminus F_{0,i}) \bigsqcup Y \setminus B & \text{for } k = 0 \end{cases} \tag{B.9.1}$$

by Lemma B.8.4. Let us define the compact open sets

$$D_{p,j} := \bigsqcup_{i=1}^{m_p} Z(\gamma_{p,i}^{(j)}\mu_{p,i} \setminus F_{p,i})$$

for $p \in P$ and $1 \le j \le p$ and set

$$D := \bigsqcup_{p \in P} \left(\bigsqcup_{j=1}^{p-1} D_{p,j} \bigsqcup S_\beta(p) \right).$$

Henceforth we suppress the "$\times\{0\}$" from $S_\beta(p)\times\{0\}$, $D_{p,j}\times\{0\}$, etc., to increase readability. Observe that

$$\varphi^j\left(\left[1_{S_\beta(p)}\right]\right) = \left[1_{D_{p,j}}\right] \in H_0(\mathcal{H}_E) \tag{B.9.2}$$

for p, j as above. Furthermore, for $p \in P$ define the bisections

$$W_{p,j} := \bigsqcup_{i=1}^{m_p} Z(\gamma_{p,i}^{(j)}\mu_{p,i},\ F_{p,i},\ \gamma_{p,i}^{(j-1)}\mu_{p,i}) \subseteq \mathcal{G}_E \quad \text{for } 1 \le j \le p.$$

Using Equation (B.9.1) and the definition of the $D_{p,j}$'s we observe that

$$\begin{aligned} W_{p,j} \subseteq c_E^{-1}(-1), &\quad r(W_{p,j}) = D_{p,j} \text{ for } j \ge 1, \\ s(W_{p,1}) = S_\beta(p), &\quad s(W_{p,j}) = D_{p,j-1} \text{ for } j \ge 2. \end{aligned}$$

As the sources and ranges of these bisections are disjoint (mutually disjoint even) we obtain transpositions $\tau_{p,j} = \pi_{\widehat{W_{p,j}}}$ which swap them. Now we are ready to define the "first half" of τ, namely τ_+, as follows

$$\tau_+ := \prod_{p \in P} \tau_{p,p} \circ \tau_{p,p-1} \circ \cdots \circ \tau_{p,1}.$$

As a homeomorphism, τ_+ is the "disjoint union of the cycles"

$$S_\beta(p) \mapsto D_{p,p} \mapsto D_{p,p-1} \mapsto \cdots \mapsto D_{p,1} \mapsto S_\beta(p)$$

for $p \in P$. Observe that we have

$$\begin{aligned} \tau_+(S_\beta(p)) = D_{p,p}, \quad S_{\tau_+}(p) = S_\beta(p), \quad S_{\tau_+}(-1) = \bigsqcup_{p \in P} \bigsqcup_{j=1}^{p} D_{p,j}, \\ \operatorname{supp}(\tau_+) = \bigsqcup_{p \in P} S_{\tau_+}(p) \bigsqcup S_{\tau_+}(-1) = D \bigsqcup_{p \in P} D_{p,p}. \end{aligned}$$

Our next objective is to construct the other half of τ, namely τ_-. Combining

Corollary B.8.7 (Y is full because $\mathcal{G}_E$ is minimal) with Equation (B.9.2) we obtain

$$\begin{aligned}
0 = I(\alpha) = I(\beta) = \sum_{k\in\mathbb{Z}} \varphi^{(k)}\left(\left[1_{S_\beta(k)}\right]\right) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \varphi^{(k)}\left(\left[1_{S_\beta(k)}\right]\right) \\
\implies \sum_{q\in Q} \varphi^{(q)}\left(\left[1_{S_\beta(q)}\right]\right) = -\sum_{p\in P} \varphi^{(p)}\left(\left[1_{S_\beta(p)}\right]\right) \\
\implies \sum_{q\in Q} \left(\varphi^{-1}\left(\left[1_{S_\beta(q)}\right]\right) + \varphi^{-2}\left(\left[1_{S_\beta(q)}\right]\right) + \cdots + \varphi^{q}\left(\left[1_{S_\beta(q)}\right]\right)\right) \\
= \sum_{p\in P} \left(\left[1_{S_\beta(p)}\right] + \varphi\left(\left[1_{S_\beta(p)}\right]\right) + \cdots + \varphi^{p-1}\left(\left[1_{S_\beta(p)}\right]\right)\right) \\
= \sum_{p\in P} \left(\left[1_{S_\beta(p)}\right] + \left[1_{D_{p,1}}\right] + \cdots + \left[1_{D_{p,p-1}}\right]\right) = [1_D]\,. \qquad \text{(B.9.3)}
\end{aligned}$$

Similarly to the $D_{p,j}$'s, we define the compact open sets

$$X_{q,l} := \bigsqcup_{i=1}^{m_q} Z(\gamma_{q,i}^{(l)}\mu_{q,i} \setminus F_{q,i})$$

for $q \in Q$ and $1 \leq j \leq |q|$, and set

$$X := \bigsqcup_{q\in Q} \bigsqcup_{l=1}^{|q|} X_{q,l}.$$

These sets then satisfy

$$\varphi^{-l}\left(\left[1_{S_\beta(q)}\right]\right) = \left[1_{X_{q,l}}\right] \in H_0(\mathcal{H}_E) \qquad \text{(B.9.4)}$$

for q, l as above. Equation (B.9.3) now says that $[1_X] = [1_D]$ in $H_0(\mathcal{H}_E)$. Invoking cancellation in $\mathcal{H}_E$ (Theorem B.5.5) we can find a bisection $R \subseteq \mathcal{H}_E$ with $s(R) = X$ and $r(R) = D$. Now define $R_{q,l} := s^{-1}\left(X_{q,l}\right)$ and $C_{q,l} := r\left(R_{q,l}\right)$. Then the $R_{q,l}$'s are mutually disjoint bisections which witness that $\left[1_{C_{q,l}}\right] = \left[1_{X_{q,l}}\right]$. We also define

$$C := \bigsqcup_{q\in Q} \bigsqcup_{l=1}^{|q|} C_{q,l}.$$

Observe that we actually have $C = D$, as they both equal $r(R)$. Equation (B.9.4) implies that

$$\varphi\left(\left[1_{C_{q,1}}\right]\right) = \left[1_{S_\beta(q)}\right] \quad \text{and} \quad \varphi\left(\left[1_{C_{q,l}}\right]\right) = \left[1_{C_{q,l-1}}\right] \quad \text{for } l \geq 2$$

in $H_0(\mathcal{H}_E)$. Hence Lemma B.9.3 yields bisections $T_{q,l} \subseteq \mathcal{G}_E$ satisfying

$$T_{q,l} \subseteq c_E^{-1}(1), \quad s(T_{q,l}) = C_{q,l} \text{ for } l \geq 1,$$
$$r(T_{q,1}) = S_\beta(q), \quad r(T_{q,l}) = C_{q,l-1} \text{ for } l \geq 2.$$

Let $\tau_{q,l} := \pi_{\widehat{T_{q,l}}}$ denote the associated transpositions. From these we in turn define τ_- in a similar fashion as τ_+ by setting

$$\tau_- := \prod_{q \in Q} \tau_{q,|q|} \circ \tau_{q,|q|-1} \circ \cdots \circ \tau_{q,1}.$$

Just like τ_+, the homeomorphism τ_- is a "disjoint union of cycles"

$$S_\beta(q) \mapsto C_{q,|q|} \mapsto C_{q,|q|-1} \mapsto \cdots \mapsto C_{q,1} \mapsto S_\beta(q)$$

for $q \in Q$. We have that

$$\tau_-(S_\beta(q)) = C_{q,|q|}, \quad S_{\tau_-}(q) = S_\beta(q), \quad S_{\tau_-}(1) = \bigsqcup_{q \in Q} \bigsqcup_{l=1}^{|q|} C_{q,l} = C,$$
$$\operatorname{supp}(\tau_-) = \bigsqcup_{q \in Q} S_{\tau_-}(q) \bigsqcup S_{\tau_-}(1) = C \bigsqcup_{q \in Q} S_\beta(q).$$

Finally, we define $\tau := \tau_- \circ \tau_+$. In order to finish the proof, we need to show that $S_\tau(k) = S_\beta(k)$ for all $k \in \mathbb{Z}$. We start by noting that

$$\begin{aligned}\operatorname{supp}(\tau) &\subseteq \operatorname{supp}(\tau_+) \bigcup \operatorname{supp}(\tau_-) \\ &= \left(\bigsqcup_{q \in Q} S_\beta(q)\right) \bigsqcup \left(\bigsqcup_{p \in P} S_\beta(p)\right) \bigsqcup \left(\bigsqcup_{p \in P} \bigsqcup_{j=1}^{p} D_{p,j}\right).\end{aligned}$$

We are going to analyze this support piece by piece. We begin by fixing some $q \in Q$ and consider $S_\beta(q)$. Since $S_\beta(q) \subseteq Y \setminus \operatorname{supp}(\tau_+)$ we have

$$S_\beta(q) \xmapsto[\text{lag } 0]{\tau_+} S_\beta(q) \xmapsto[\text{lag } q]{\tau_-} C_{q,|q|}.$$

This means that $S_\beta(q) \subseteq S_\tau(q)$. We similarly have $S_\beta(p) \subseteq S_\tau(p)$ for each $p \in P$ since $D_{p,p} \subseteq Y \setminus \operatorname{supp}(\tau_-)$. For the last part, we consider the sets $D_{p,j}$ for $p \in P$ and $1 \leq j \leq p$. For $j = 1$ we find that

$$D_{p,1} \xmapsto[\text{lag } -1]{\tau_+} S_\beta(p) \xmapsto[\text{lag } 1]{\tau_-} \tau_-(S_\beta(p))$$

because $S_\beta(p) \subseteq D = C$, which maps with lag 1 by τ_-. As the total lag is $1 - 1 = 0$, we get that $D_{p,1} \subseteq S_\tau(0)$. When $j \geq 2$ we similarly have

$$D_{p,j} \xmapsto[\text{lag } -1]{\tau_+} D_{p,j-1} \xmapsto[\text{lag } 1]{\tau_-} \tau_-(D_{p,j-1})$$

since $D_{p,j-1} \subseteq C$. Hence $D_{p,j} \subseteq S_\tau(0)$ as well. If we now set

$$Z := \left(\bigsqcup_{q \in Q} S_\beta(q) \right) \bigsqcup \left(\bigsqcup_{p \in P} S_\beta(p) \right) \bigsqcup \left(\bigsqcup_{p \in P} \bigsqcup_{j=1}^{p} D_{p,j} \right) \subseteq Y \setminus \operatorname{supp}(\tau)$$

and decompose Y as

$$Y = \left(\bigsqcup_{q \in Q} S_\beta(q) \right) \bigsqcup \left(\bigsqcup_{p \in P} S_\beta(p) \right) \bigsqcup \left(\bigsqcup_{p \in P} \bigsqcup_{j=1}^{p} D_{p,j} \right) \bigsqcup (Y \setminus Z)$$

then we have seen that

$$S_\beta(q) \subseteq S_\tau(q), \quad S_\beta(p) \subseteq S_\tau(p), \quad D_{p,j} \subseteq S_\tau(0), \quad Y \setminus Z \subseteq S_\tau(0).$$

Since both of these form partitions of Y we must actually have equality here. This means that $S_\beta(k) = S_\tau(k)$ for all $k \neq 0$, and then $S_\beta(0) = S_\tau(0)$ as well.

Let us now comment on what happens if one of P, Q or $S_\alpha(0) \cap A$ are empty. All three cannot be empty since $\operatorname{supp}(\alpha) \neq \emptyset$. We claim that $P = \emptyset$ if and only if $Q = \emptyset$. Arguing by contradicition, if $P \neq \emptyset$ and $Q = \emptyset$, then Equation (B.9.3) says that $[1_D] = 0$ in $H_0(\mathcal{H}_E)$, so by Corollary B.5.6 $D = \emptyset$. This forces $P = \emptyset$. Having $P = \emptyset$ and $Q \neq \emptyset$ is ruled out similarly. In the case of $P = \emptyset = Q$ we have that $A = \operatorname{supp}(\alpha) \subseteq S_\alpha(0)$, which means that $\alpha \in [\![\mathcal{H}_E|_Y]\!]$ (since $U_\alpha \subseteq c_E^{-1}(0)$). Now we are done since this groupoid is AF and hence has Property TR. The last possibility is that $S_\alpha(0) \cap A = \emptyset$ and P, Q are both non-empty. In this case the proof above goes through by removing everything indexed by $k = 0$. This finishes the proof at large. □

Having established Property TR for strongly connected graphs with infinite emitters, we deduce the AH conjecture for these from [Mat16, Theorem 4.4]. As we saw in Proposition B.4.2 the assumptions in the AH conjecture for graph groupoids are slightly weaker than strong connectedness. For completeness we want to show that all graph groupoids covered by the assumptions satisfy the conjecture. Using another one of Sørensen's moves on graphs, namely *source removal*, we can actually reduce this to the strongly connected situation.

Corollary B.9.5. *Let E be a graph satisfying the AH criteria and let $Y \subseteq \mathcal{G}_E^{(0)} = \partial E$ be clopen. Then the AH conjecture is true for $\mathcal{G}_E|_Y$.*

Proof. As discussed in Subsection 4.2, the graph E has a single nontrivial strongly connected component which contain all infinite emitters. The vertices which lie outside this component and the edges they emit form an acyclic subgraph with sources which connect to the nontrivial connected component. By repeatedly applying Sørensen's move (S) from [Sør13, Section 3] we can remove all the vertices lying outside the strongly connected component of E. This results in a graph F which is strongly connected and which is move equivalent to E. By the results in [CRS17] $\mathcal{G}_F$ is Kakutani equivalent to $\mathcal{G}_E$. Hence there are full clopen subsets $X \subseteq \mathcal{G}_E^{(0)}$ and $Z \subseteq \mathcal{G}_F^{(0)}$ such that $\mathcal{G}_E|_X \cong \mathcal{G}_F|_Z$. Appealing to [Mat15b, Proposition 4.11] we can find a compact bisection $U \subseteq \mathcal{G}_E$ satisfying $s(U) = Y$ and $r(U) \subseteq X$. Then

$$\mathcal{G}_E|_Y \cong \mathcal{G}_E|_{r(U)} = (\mathcal{G}_E|_X)|_{r(U)} \cong (\mathcal{G}_F|_Z)|_{Z'} = \mathcal{G}_F|_{Z'}$$

for some clopen set $Z' \subseteq Z \subseteq \mathcal{G}_F^{(0)}$.

If E has infinite emitters, then the result follows from applying Theorem B.9.4 to $\mathcal{G}_F|_{Z'}$. Similarly, if E is finite it follows from applying the results in [Mat15b, Subsection 6.4]. □

Remark B.9.6. The finiteness assumption on the set of vertices is actually only needed to guarantee that we can apply Lemma B.9.1, by first applying Lemma B.9.2. Hence Theorem B.9.4 also applies to strongly connected graphs with infinitely many vertices, provided that the graph satisfies the hypotheses of Lemma B.9.1. Namely that there exists an infinite emitter which supports infinitely many loops and from which there is at least one edge to any other vertex.

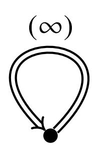
B.10 Examples and applications

B.10.1 Groupoid models for Cuntz algebras

Let E_n denote the graph with one vertex and n loops for $2 \leq n \leq \infty$. The graph C^*-algebras of these graphs are the Cuntz algebras, that is $C^*(E_n) \cong \mathcal{O}_n$, whose K-theory is given by $\mathbb{Z}_n$ and 0 respectively (where $\mathbb{Z}_\infty$ means $\mathbb{Z}$).

Let us now consider our main motivating example, namely the graph

E_∞



and its graph groupoid $\mathcal{G}_{E_\infty}$. By Theorem B.4.6 $H_0(\mathcal{G}_{E_\infty}) \cong \mathbb{Z}$ and $H_1(\mathcal{G}_{E_\infty}) \cong 0$. So the exact sequence in the AH conjecture for $\mathcal{G}_{E_\infty}$ collapses to

$$\mathbb{Z}_2 \xrightarrow{j} [\![\mathcal{G}_{E_\infty}]\!]_{\mathrm{ab}} \longrightarrow 0.$$

This leaves two possibilities for the abelianization $[\![\mathcal{G}_{E_\infty}]\!]_{\mathrm{ab}}$:

1. Either $[\![\mathcal{G}_{E_\infty}]\!]_{\mathrm{ab}}$ is trivial (in which case $[\![\mathcal{G}_{E_\infty}]\!]$ is simple),
2. or $[\![\mathcal{G}_{E_\infty}]\!]_{\mathrm{ab}}$ is isomorphic to $\mathbb{Z}_2$ (in which case $\mathcal{G}_{E_\infty}$ has the strong AH property).

For $2 \leq n < \infty$ the topological full group $[\![\mathcal{G}_{E_n}]\!]$ is isomorphic to the Higman–Thompson group $V_{n,1}$ [Mat15b], and we have

$$[\![\mathcal{G}_{E_n}]\!]_{\mathrm{ab}} \cong (V_{n,1})_{\mathrm{ab}} \cong \begin{cases} \mathbb{Z}_2 & n \text{ odd}, \\ 0 & n \text{ even}. \end{cases}$$

Although we have not been able to decide which is the case for $[\![\mathcal{G}_{E_\infty}]\!]_{\mathrm{ab}}$, we can still deduce some structural properties of the topological full group $[\![\mathcal{G}_{E_\infty}]\!]$.

Theorem 4.16 in [Mat15b] shows not only that the commutator subgroup $\mathsf{D}([\![\mathcal{G}_{E_\infty}]\!])$ is simple, it is also contained in any nontrivial normal subgroup of $[\![\mathcal{G}_{E_\infty}]\!]$. This means that $[\![\mathcal{G}_{E_\infty}]\!]$ either is simple itself, or contains precisely one nontrivial normal subgroup, namely $\mathsf{D}([\![\mathcal{G}_{E_\infty}]\!])$ (of index 2). The group $[\![\mathcal{G}_{E_\infty}]\!]$ is nonamenable [Mat15b], but does have the Haagerup property (Corollary A.11.11). One can also deduce that $[\![\mathcal{G}_{E_\infty}]\!]$ is C^*-simple by the results in [BS19]. Finally, it is shown below that $[\![\mathcal{G}_{E_\infty}]\!]$ is not finitely generated.

B.10.2 Simplicity and non-finite generation of topological full groups

We would have liked to decide whether all graph groupoids of graphs satisfying the AH criteria have the strong AH property, as we know SFT-groupoids do. Matui's proof of this for SFT-groupoids in [Mat15b] relies on the construction of a finite presentation for their topological full groups. However, if a graph has infinite emitters, then the topological full group of its graph groupoid will not even be finitely generated.

Proposition B.10.1. *Let E be a graph with at least one infinite emitter and suppose E satisfies Condition (L). Then $[\![\mathcal{G}_E]\!]$ is not finitely generated.*

Proof. Let $w \in E^0_{\mathrm{sing}}$ be an infinite emitter and enumerate the edges emitted by w as $wE^1 = \{e_1, e_2, e_3, \ldots\}$. Suppose we are given finitely many elements

$\alpha_1, \alpha_2, \ldots, \alpha_N$ from $[\![\mathcal{G}_E]\!]$. According to Proposition A.9.4 we can decompose each full bisection defining these elements as

$$U_{\alpha_j} = \left(\bigsqcup_{i=1}^{k_j} Z(\mu_{i,j}, F_{i,j}, \nu_{i,j}) \right) \sqcup (\partial E \setminus \operatorname{supp}(\alpha_j)).$$

Among the paths $\mu_{i,j}$ and $\nu_{i,j}$ and in the sets of forbidden edges $F_{i,j}$, only finitely many of the edges in wE^1 can occur. Pick an $M \in \mathbb{N}$ such that $e_M, e_{M+1}, \ldots$ do not occur in any of these. Any product of the α_j's and their inverses will again result in an element of $[\![\mathcal{G}_E]\!]$ whose defining bisection decomposes similarly as above. The crucial point is that none of the edges $e_M, e_{M+1}, \ldots$ will occur in its decomposition either. This means that elements such as $\pi_{\widehat{V}}$ for $V = Z(e_M, e_{M+1})$ does not belong to the subgroup generated by the elements $\alpha_1, \alpha_2, \ldots, \alpha_N$, and consequently $[\![\mathcal{G}_E]\!]$ cannot be finitely generated. □

A consequence of SFT-groupoids having the strong AH property is that their topological full groups are simple if and only if the zeroth homology group is 2-divisible [Mat15b, Corollary 6.24.(3)]. This is the case for e.g. the graphs E_n above when n is even. For graphs with infinite emitters, however, the sitatuation is quite different. What we observed for $\mathcal{G}_{E_\infty}$ above, namely that the strong AH property rules out the simplicity of the topological full group and vice versa, is actually a general phenomenon. This is due to $H_0(\mathcal{G}_E)$ never being 2-divisible when E has singular vertices.

Proposition B.10.2. *Let E be a graph satisfying the AH criteria and having at least one infinite emitter. If $\mathcal{G}_E$ has the strong AH property, then $[\![\mathcal{G}_E]\!]$ is not simple.*

Proof. By Theorem B.4.6 $H_0(\mathcal{G}_E)$ is a finitely generated abelian group whose rank is greater than or equal to the number of singular vertices in E. So if E has an infinite emitter, then $H_0(\mathcal{G}_E) \otimes \mathbb{Z}_2$ is nonzero. If $\mathcal{G}_E$ has the strong AH property, then this forces $[\![\mathcal{G}_E]\!]_{\mathrm{ab}} \neq 0$ too. Thus $[\![\mathcal{G}_E]\!]$ cannot be simple (being non-abelian). □

Whether or not graph groupoids of graphs with infinite emitters all have the strong AH property can therefore be decided in the negative by finding such a groupoid whose topological full group is simple.

B.10.3 Describing the abelianization of the topological full group

We first note that by Remark B.4.4, the abelianization $[\![\mathcal{G}_{E_\infty}]\!]_{\mathrm{ab}}$ is a finitely generated abelian group for any graph E satisfying the AH criteria. Let us next consider an example where both $H_0(\mathcal{G}_E)$ and $H_1(\mathcal{G}_E)$ are nontrivial.

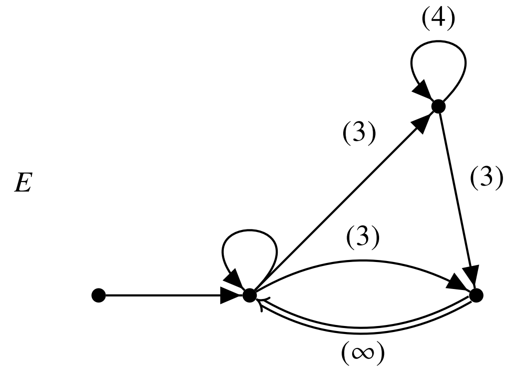


Figure B.3: An infinite graph for which $H_0(\mathcal{G}_E)$ and $H_1(\mathcal{G}_E)$ are both nontrivial. The numbers in paranthesis indicate the number of edges.

Example B.10.3. Consider the graph E in Figure B.3. From Theorem B.4.6 we find that $H_0(\mathcal{G}_E) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_3$ and $H_1(\mathcal{G}_E) \cong \mathbb{Z}$. Hence the AH exact sequence becomes

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{\ j\ } [\![\mathcal{G}_E]\!]_{\mathrm{ab}} \xrightarrow{I_{\mathrm{ab}}} \mathbb{Z} \longrightarrow 0.$$

This implies that $[\![\mathcal{G}_E]\!]_{\mathrm{ab}} \cong \mathbb{Z} \oplus \mathrm{im}(j)$. Thus $[\![\mathcal{G}_E]\!]_{\mathrm{ab}}$ is isomorphic to either $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}_2$ or $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

The previous example generalizes to the following partial description of the abelianization $[\![\mathcal{G}_E]\!]_{\mathrm{ab}}$.

Proposition B.10.4. *Let E be a graph satisfying the AH criteria and let $\emptyset \neq Y \subseteq \partial E$ be clopen. Then*

$$[\![\mathcal{G}_E|_Y]\!]_{ab} \cong H_1(\mathcal{G}_E) \oplus \mathrm{im}(j),$$

where $H_1(\mathcal{G}_E) \cong \mathbb{Z}^M$ and $\mathrm{im}(j) \cong (\mathbb{Z}_2)^N$ for nonnegative integers M, N.

Remark B.10.5. The integer N in the preceding proposition is necessarily bounded above by the number of "even summands" in $H_0(\mathcal{G}_E)$, which in turn is at least $M + |E^0_{\mathrm{sing}}|$ and at most $|E^0|$. In general, we may only say that $0 \leq N \leq |E^0|$.

B.10.4 The cycle graphs

The statement in Theorem B.1.1 would look cleaner if we did not have to specify that E cannot be a cycle graph. However, this is necessary, as we will see shortly. Let C_n denote the graph consisting of a single cycle with n vertices. Observe that $\mathcal{G}_{C_n} \cong \mathcal{R}_n \times \mathbb{Z}$ (where $\mathbb{Z}$ is viewed as a group), which is a discrete transitive groupoid with unit space consisting of n points (a groupoid with only one orbit is called *transitive*). This is consistent with the C^*-algebraic side of things, as we have that

$$C_r^*(\mathcal{G}_{C_n}) \cong C^*(C_n) \cong M_n(C(\mathbb{T})),$$
$$C_r^*(\mathcal{R}_n \times \mathbb{Z}) \cong M_n(\mathbb{C}) \otimes C(\mathbb{T}) \cong M_n(C(\mathbb{T})).$$

Since $\mathcal{G}_{C_n}$ is Kakutani equivalent to $\mathbb{Z}$ and $K_*(M_n(C(\mathbb{T}))) \cong K_*(C(\mathbb{T})) \cong (\mathbb{Z}, \mathbb{Z})$, Theorem B.4.6 gives

$$H_0(\mathbb{Z}) \cong H_0(\mathcal{G}_{C_n}) \cong \mathbb{Z} \qquad \text{and} \qquad H_1(\mathbb{Z}) \cong H_1(\mathcal{G}_{C_n}) \cong \mathbb{Z}.$$

(We could also have deduced the homology of $\mathcal{G}_{C_n}$ from the group homology $\mathbb{Z}$, as these coincide due to their Kakutani equivalence.) However, the unit space of $\mathcal{G}_{C_n}$ is finite, hence so is $[\![\mathcal{G}_{C_n}]\!]$ (it is isomorphic to the symmetric group S_n), and then clearly the index map $I \colon [\![\mathcal{G}_{C_n}]\!] \to H_1(\mathcal{G}_{C_n})$ cannot be surjective.

Paper C

Katsura–Exel–Pardo Groupoids and the AH Conjecture

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Submitted for publication.

Paper C

Katsura–Exel–Pardo Groupoids and the AH Conjecture

Abstract

It is proven that Matui's AH conjecture is true for Katsura–Exel–Pardo groupoids $\mathcal{G}_{A,B}$ associated to integral matrices A and B. This conjecture relates the topological full group of an ample groupoid with the homology groups of the groupoid. We also give a criterion under which the topological full group $[\![\mathcal{G}_{A,B}]\!]$ is finitely generated.

C.1 Introduction

The *AH conjecture* is one of two conjectures formulated by Matui in [Mat16] concerning certain ample groupoids over Cantor spaces. This conjecture predicts that the abelianization of the topological full group of such a groupoid together with its first two homology groups fit together in an exact sequence as follows:

$$H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \xrightarrow{\ j\ } [\![\mathcal{G}]\!]_{\text{ab}} \xrightarrow{\ I_{\text{ab}}\ } H_1(\mathcal{G}) \longrightarrow 0.$$

So far, the AH conjecture has been confirmed in a number of cases. For instance, it holds for groupoids which are both almost finite and principal [Mat12]. This includes AF-groupoids, transformation groupoids of higher-dimensional Cantor minimal systems and groupoids associated to aperiodic quasicrystals (as described in [Nek19, Subsection 6.3]). At the opposite end of the spectrum, the AH conjecture is also true for (products of) SFT-groupoids [Mat16]. The same goes for transformation groupoids associated to odometers [Sca18], which incidentally, provided counterexamples to the other conjecture from [Mat16], namely the *HK conjecture*. In the recent paper [NO20], we showed that the AH conjecture holds for graph groupoids of infinite graphs, complementing Matui's result in the finite case [Mat15b].

The present paper may be viewed as a follow-up to [NO20]. Here we investigate the validity of the AH conjecture for a class of groupoids known as *Katsura–Exel–Pardo groupoids*. These groupoids are built from two equal-sized row-finite integer matrices A and B, where A has no negative entries, and are denoted $\mathcal{G}_{A,B}$. Their origins stem from Katsura's paper [Kat08b], in which he constructed C^*-algebras $\mathcal{O}_{A,B}$—which we call *Katsura algebras*—from such matrices. Katsura showed that every Kirchberg algebra (in the UCT class) is stably isomorphic to some $\mathcal{O}_{A,B}$ and used this concrete realization to prove results pertaining to lifts of actions on the K-groups of Kirchberg algebras. The Katsura algebras $\mathcal{O}_{A,B}$ first appear as examples of topological graph C^*-algebras in [Kat08a].

Some years later later, Exel and Pardo introduced the notion of a *self-similar graph*, and showed how to construct a C^*-algebra from this data, in [EP17]. This generalized Nekrashevych's construction from self-similar groups in [Nek09], as a self-similar group may be viewed as a self-similar graph where the graph has only one vertex [EP17, Example 3.3]. On the other hand, the construction of Exel and Pardo also encompassed the Katsura algebras. They realized that the matrices A and B could be used to describe a self-similar action by the integer group $\mathbb{Z}$ on the graph whose adjacency matrix is A in such a way that the associated C^*-algebra becomes $\mathcal{O}_{A,B}$. Exel and Pardo also gave a groupoid model for their C^*-algebras, and it is the groupoid associated with the aforementioned $\mathbb{Z}$-action that we call the *Katsura–Exel–Pardo groupoid*. See Section C.3 for details.

The second author computed the homology groups of the Katsura–Exel–Pardo groupoids in [Ort18] (under the assumption of pseudo-freeness, see Subsection C.3.3), and found that the homology groups of $\mathcal{G}_{A,B}$ sum up to the K-theory of $C_r^*(\mathcal{G}_{A,B}) \cong \mathcal{O}_{A,B}$ in accordance with Matui's HK conjecture [Mat16, Conjecture 2.6].

In the present paper we make use of the description of the homology groups of $\mathcal{G}_{A,B}$ from [Ort18] to show that the AH conjecture holds whenever $\mathcal{G}_{A,B}$ is Hausdorff and effective and the matrix A is finite and irreducible (Corollary C.5.8).

There are two subgroupoids of $\mathcal{G}_{A,B}$ that play important roles in the proof. One is the SFT-groupoid $\mathcal{G}_A \cong \mathcal{G}_{A,0}$ associated to the matrix A. The other is the kernel of the canonical cocycle on $\mathcal{G}_{A,B}$, denoted $\mathcal{H}_{A,B}$. Unlike the case of SFT-groupoids (or graph groupoids), the kernel of the cocycle is no longer an AF-groupoid. This means that we also need to take $H_1(\mathcal{H}_{A,B})$ into account when describing $H_1(\mathcal{G}_{A,B})$. A key observation that drives our proof is that the topological full group $[\![\mathcal{G}_{A,B}]\!]$ can be decomposed as $[\![\mathcal{G}_{A,B}]\!] = [\![\mathcal{H}_{A,B}]\!][\![\mathcal{G}_A]\!]$, when viewing $[\![\mathcal{H}_{A,B}]\!]$ and $[\![\mathcal{G}_A]\!]$ as subgroups of $[\![\mathcal{G}_{A,B}]\!]$.

We also investigate whether the topological full group $[\![\mathcal{G}_{A,B}]\!]$ is finitely generated. Matui has shown that topological full groups of (irreducible) SFT-groupoids are finitely presented [Mat15b]. In the same vein, topological full groups associated to self-similar groups were shown to be finitely presented by Nekrashevych whenever the self-similar group is *contracting* [Nek18a]. We extend Nekrashevych's notion of a *contracting* self-similar group to self-similar graphs and show that the self-similar graph associated to the pair of matrices A and B is contracting, assuming that B is entrywise smaller than A. Combining this with the finite generation of $[\![\mathcal{G}_A]\!]$, we show in Theorem C.6.6 that $[\![\mathcal{G}_{A,B}]\!]$ is then indeed finitely generated. In contrast, if E is a graph with an infinite emitter, then the topological full group $[\![\mathcal{G}_E]\!]$ is not finitely generated (Proposition B.10.1).

We emphasize that the Katsura–Exel–Pardo groupoids are merely prominent special cases of the tight groupoids constructed from self-similar graphs in [EP17]. Moreover, this construction was further generalized to non-row-finite graphs in [EPS18]. It is therefore a natural question whether the results of this paper can be generalized to other groupoids arising from self-similar graphs. A few things that make the Katsura–Exel–Pardo groupoids particularly nice to work with is that the self-similar action is explicitly given in terms of the matrices A and B, the action does not move vertices, and the acting group is abelian (the "most elementary" abelian group even). We believe that the methods employed in this paper could work well for other self-similar graphs where the acting group is abelian and the action fixes the vertices.

This paper is organized as follows. In Section C.2, we briefly recall Matui's AH conjecture and give references to the necessary preliminaries. The construction of the Katsura–Exel–Pardo groupoid is recalled in detail in Section C.3. Then Hausdorffness, effectiveness and minimality of $\mathcal{G}_{A,B}$ is characterized in terms of the matrices A and B. We also observe that if $\mathcal{G}_{A,B}$ satisfies the assumptions in the AH conjecture, then $\mathcal{G}_{A,B}$ must be purely infinite. In Section C.4, we describe the first two homology groups of $\mathcal{G}_{A,B}$. This is done using a long exact sequence that relates the homology groups of $\mathcal{G}_{A,B}$ to those of the kernel groupoid $\mathcal{H}_{A,B}$. Our main result, namely that the AH conjecture is true for Katsura–Exel–Pardo groupoids, is proved in Section C.5. Finally, in Section C.6, we prove that $[\![\mathcal{G}_{A,B}]\!]$ is finitely generated, provided that B is entrywise smaller than A.

C.2 The AH conjecture

As mentioned in the introduction, this paper is a follow-up to Paper B. We treat the same problem—namely the AH conjecture—for a related, but different, class of groupoids. Since the setting is so similar we have chosen to not give an extensive section covering preliminaries, but rather refer the reader to Section B.2 and adapt all notation and conventions from there. Topics covered there include ample groupoids, topological full groups, homology of ample groupoids, cocycles and skew products. The reader is hereby warned that notation from Section B.2 henceforth will be used directly without reference.

Let us move on to describing the AH conjecture, which predicts a precise relationship between the topological full group and the first two homology groups. For further details, consult Section B.4.

Matui's AH Conjecture ([Mat16, Conjecture 2.9])**.** *Let $\mathcal{G}$ be an effective minimal second countable Hausdorff ample groupoid whose unit space $\mathcal{G}^{(0)}$ is a Cantor space. Then the following sequence is exact:*

$$H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \xrightarrow{\ j\ } [\![\mathcal{G}]\!]_{ab} \xrightarrow{\ I_{ab}\ } H_1(\mathcal{G}) \longrightarrow 0. \tag{C.2.1}$$

The *index map* $I\colon [\![\mathcal{G}]\!] \to H_1(\mathcal{G})$ is the homomorphism given by $\pi_U \mapsto [1_U]$, where U is a full bisection in $\mathcal{G}$, and the induced map on the abelianization $[\![\mathcal{G}]\!]_{\mathrm{ab}}$ is denoted I_{ab}. The map j will not be used directly (see e.g. Subsection B.4.1 for its definition).

Recall the notion of transpositions in the topological full group from Subsection B.2.2. We will let $\mathcal{T}(\mathcal{G})$ denote the subgroup of $[\![\mathcal{G}]\!]$ generated by all transpositions. Beware that in Paper B, the subgroup generated by all transpositions is denoted $\mathcal{S}(\mathcal{G})$, but for $\mathcal{G} = \mathcal{G}_{A,B}$ we find this to be too similar to the set $\mathcal{S}_{A,B}$ that is defined in Subsection C.3.2 below. One always has $\mathcal{T}(\mathcal{G}) \subseteq \ker(I)$, and having equality is closely related to the AH conjecture.

Definition C.2.1 ([Mat16, Definition 2.11])**.** Let $\mathcal{G}$ be an effective ample Hausdorff groupoid. We say that $\mathcal{G}$ has *Property TR* if $\mathcal{T}(\mathcal{G}) = \ker(I)$.

In the next section, we will see that the Katsura–Exel–Pardo groupoids are purely infinite (in the sense of [Mat15b, Definition 4.9]). It then follows that the AH conjecture is equivalent to having Property TR for these (see Remark B.4.12). The main goal therefore becomes to establish Property TR for $\mathcal{G}_{A,B}$.

C.3 The Katsura–Exel–Pardo groupoid

In this section we recall the construction of the The Katsura–Exel–Pardo groupoid $\mathcal{G}_{A,B}$ from [EP17], and we recall some of its properties.

C.3.1 The self similar action by $\mathbb{Z}$ on the graph E_A

Let us begin explaining the construction. Let $N \in \mathbb{N} \cup \{\infty\}$ and let A and B be two row-finite $N \times N$ integral matrices. We require that all entries in A are non-negative and that A has no zero-rows. For the construction we may also assume without loss of generality that $B_{i,j} = 0$ whenever $A_{i,j} = 0$. Let E_A denote the (directed) graph whose adjacency matrix is A. For graphs we freely adopt notation and conventions from Section B.3. In addition to that, given a finite path $\mu = e_1 e_2 \cdots e_k \in E_A^*$ and an index $1 \leq j \leq k$, the subpath $e_1 e_2 \ldots e_j$ is denoted $\mu|_j$. We will call a matrix *essential* if it has no zero-rows and no zero-columns.

We will now describe how the matrices A and B give rise to a self-similar action by the integer group $\mathbb{Z}$ on the graph E_A as in the framework of [EP17]. In the next subsection, we will describe the associated (tight) groupoid.

Remark C.3.1. We remark that Exel and Pardo use the opposite convention for paths in [EP17], which means that their paths go “backwards” in the graph.

To describe the action $\kappa\colon \mathbb{Z} \curvearrowright E_A$ we need to fix an (arbitrary) enumeration of the edges in E_A as follows

$$E_A^1 = \{e_{i,j,n} \mid 1 \leq i,j \leq N,\ 0 \leq n < A_{i,j}\}.$$

Then we have $s(e_{i,j,n}) = i$ and $r(e_{i,j,n}) = j$, when enumerating the vertices as $E_A^0 = \{1, 2, \ldots, N\}$. Let $m \in \mathbb{Z}$ and $e_{i,j,n} \in E_A^1$ be given. By the division algorithm there are unique integers q and r satisfying

$$mB_{i,j} + n = qA_{i,j} + r \quad \text{and} \quad 0 \leq r < A_{i,j}.$$

The action κ is defined to be trivial on the vertices (i.e. $\kappa_m(i) = i$), and on edges it is given by

$$\kappa_m(e_{i,j,n}) \coloneqq e_{i,j,r}.$$

In words κ_m maps the n’th edge between the vertices i and j to the r’th edge, where r is the remainder of $mB_{i,j} + n$ modulo $A_{i,j}$. The associated *one-cocycle* $\varphi\colon \mathbb{Z} \times E_A^1 \to \mathbb{Z}$ is given by

$$\varphi(m, e_{i,j,n}) \coloneqq q.$$

The cocycle condition

$$\varphi(m_1 + m_2, e) = \varphi(m_1, \kappa_{m_2}(e)) + \varphi(m_2, e)$$

is easily seen to be satisfied. That same computation shows that $\kappa_{m_1+m_2} = \kappa_{m_1} \circ \kappa_{m_2}$. Furthermore, the standing assumption (2.3.1) on page 1051 of [EP17] is trivially satisfied since κ fixes the vertices. Note that $\varphi(0, e) = 0$ and $\kappa_0(e) = e$ for all $e \in E_A^1$.

As in [EP17, Proposition 2.4] κ and φ extends inductively to finite paths by setting

$$\kappa_m(\mu e) \coloneqq \kappa_m(\mu)\kappa_{\varphi(m,\mu)}(e) \quad \text{and} \quad \varphi(m, \mu e) \coloneqq \varphi(\varphi(m, \mu), e)$$

for $\mu \in E_A^*$ and $e \in r(\mu)E_A^1$. Explicitly, for a finite path $\mu = e_1 e_2 \cdots e_k \in E_A^*$ we have

$$\kappa_m(\mu) = \kappa_m(e_1)\kappa_{\varphi(m,e_1)}(e_2)\kappa_{\varphi(m,e_1e_2)}(e_3)\cdots\kappa_{\varphi(m,\mu|_{k-1})}(e_k) \tag{C.3.1}$$

and

$$\varphi(m,\mu) = \varphi\left(\varphi(\ldots(\varphi(\varphi(m,e_1),e_2),\ldots),e_{k-1}),e_k\right). \tag{C.3.2}$$

By allowing Equation (C.3.1) to go on ad infinitum, κ extends to an action on the infinite path space E_A^∞. Note that we still have

$$\varphi(m_1+m_2,\mu) = \varphi(m_1,\kappa_{m_2}(\mu)) + \varphi(m_2,\mu)$$

and

$$\kappa_m(\mu\nu) = \kappa_m(\mu)\kappa_{\varphi(m,\mu)}(\nu)$$

for $\mu,\nu \in E_A^*$ with $r(\mu) = s(\nu)$. The latter formula also holds if ν is replaced by an infinite path.

C.3.2 Describing the tight groupoid

Define the set

$$\mathcal{S}_{A,B} := \{(\mu,m,\nu) \in E_A^* \times \mathbb{Z} \times E_A^* \mid r(\mu) = r(\nu)\}.$$

In [EP17], the set $\mathcal{S}_{A,B}$ is given the structure of an inverse $*$-semigroup which acts (partially) on the infinite path space E_A^∞. In brief terms this partial action is given by

$$(\mu,m,\nu)\cdot\nu y = \mu\kappa_m(y) \text{ for } y \in r(\nu)E_A^\infty.$$

Following [Ort18] we skip directly to the concrete description of the tight groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{A,B})$ given in [EP17, Section 8].

Consider the set of all quadruples $(\mu,m,\nu;x)$ where $(\mu,m,\nu) \in \mathcal{S}_{A,B}$ and $x \in Z(\nu)$. Then we can write $x = \nu e z$ for some $e \in E_A^1$ and $z \in E_A^\infty$. Let $\sim$ be the equivalence relation on this set of quadruples generated by the basic relation

$$(\mu,m,\nu;x) \sim (\mu\kappa_m(e),\varphi(m,e),\nu e;x). \tag{C.3.3}$$

Denote the equivalence class of $(\mu,m,\nu;x)$ under $\sim$ by $[\mu,m,\nu;x]$. In particular, we have

$$[\mu,m,\nu;x] = \left[\mu\kappa_m(y|_j),\varphi(m,y|_j),\nu y|_j;x\right]$$

for each $j \in \mathbb{N}$, where y is the infinite path satisfying $x = \nu y$. It is somewhat cumbersome to explicitly write this equivalence relation out, but it can be done as follows. Let $(\mu,m,\nu),(\lambda,n,\tau) \in \mathcal{S}_{A,B}$, $x \in Z(\nu)$ and $z \in Z(\tau)$. Then

$$[\mu,m,\nu;x] = [\lambda,n,\tau;z]$$

if and only if

- $x = z$, so then $x = \nu y = \tau w$ for some infinite paths y and w. In particular, ν is a subpath of τ or vice versa.

- $|\mu| - |\nu| = |\lambda| - |\tau|$.

- $\mu\kappa_m(y) = \lambda\kappa_n(w)$.

- $\varphi(m, y|_j) = \varphi(n, w|_l)$ for some $j, l \in \mathbb{N}$ with $l - j = |\mu| - |\nu|$.

We define the *Katsura–Exel–Pardo groupoid* to be

$$\mathcal{G}_{A,B} := \{[\mu, m, \nu; x] \mid (\mu, m, \nu) \in \mathcal{S}_{A,B},\ x \in Z(\nu)\}.$$

Writing $x = \nu y$, the inverse operation is given by

$$[\mu, m, \nu; x]^{-1} := [\nu, -m, \mu; \mu\kappa_m(y)].$$

The composable pairs are

$$\mathcal{G}_{A,B}^{(2)} := \{([\lambda, n, \tau; z], [\mu, m, \nu; \nu y]) \in \mathcal{G}_{A,B} \times \mathcal{G}_{A,B} \mid \mu\kappa_m(y) = z\}$$

and the product is given by

$$[\lambda, n, \tau; z] \cdot [\mu, m, \nu; x] := [\lambda\kappa_m(\tau'), \varphi(n, \tau') + m, \nu; x],$$

in the case that $\mu = \tau\tau'$. In the case that $\tau = \mu\mu'$ the formula is slightly more complicated, so let us instead use the equivalence relation $\sim$ to state a simpler "standard form" for the product. Using the basic relation (C.3.3) we can choose representatives with $|\tau| = |\mu|$, which forces $\tau = \mu$. Hence every composable pair and their product can be represented as

$$[\lambda, n, \mu; \mu\kappa_m(y)] \cdot [\mu, m, \nu; \nu y] = [\lambda, n + m, \nu; \nu y].$$

The source and range maps are given by

$$\begin{aligned} s([\mu, m, \nu; \nu y]) &= [\nu, 0, \nu; \nu y] = [s(\nu), 0, s(\nu); \nu y], \\ r([\mu, m, \nu; \nu y]) &= [\mu, 0, \mu; \mu\kappa_m(y)] = [s(\mu), 0, s(\mu); \mu\kappa_m(y)]. \end{aligned}$$

Thus we may identify the unit space $\mathcal{G}_{A,B}^{(0)}$ with the infinite path space E_A^∞ under the correspondence $[s(x), 0, s(x); x] \leftrightarrow x$. This correspondence is also compatible with the topology on $\mathcal{G}_{A,B}$ that will be specified shortly. The source and range maps become

$$s([\mu, m, \nu; x]) = x \quad \text{and} \quad r([\mu, m, \nu; \nu y]) = \mu\kappa_m(y).$$

For a triple $(\mu, m, \nu) \in \mathcal{S}_{A,B}$ we define

$$Z(\mu, m, \nu) := \{[\mu, m, \nu; x] \mid x \in Z(\nu)\}.$$

These sets form a basis for the topology on $\mathcal{G}_{A,B}$, in which each basic set $Z(\mu, m, \nu)$ is a compact open bisection [EP17, Proposition 9.4]. Note that

$$s(Z(\mu, m, \nu)) = Z(\nu) \quad \text{and} \quad r(Z(\mu, m, \nu)) = Z(\mu).$$

The Katsura–Exel–Pardo groupoid $\mathcal{G}_{A,B} \cong \mathcal{G}_{\text{tight}}(\mathcal{S}_{A,B})$ is ample, second countable and amenable [EP17]. However, it is not always Hausdorff. This, and other properties, will be characterized in the next subsection.

An important observation that will be exploited in several of the coming proofs is that the graph groupoid $\mathcal{G}_{E_A}$ is isomorphic to $\mathcal{G}_{A,0}$, and moreover embeds canonically into $\mathcal{G}_{A,B}$ for any matrix B. Observe that in $\mathcal{G}_{A,0}$ we have $[\mu, m, \nu; \nu y] = [\mu, 0, \nu; \nu y]$ for each $m \in \mathbb{Z}$. Hence mapping $[\mu, 0, \nu; \nu y]$ to $(\mu y, |\mu| - |\nu|, \nu y)$ yields an isomorphism between $\mathcal{G}_{A,0}$ and $\mathcal{G}_{E_A}$. Furthermore, it is clear that $[\mu, 0, \nu; x] \mapsto [\mu, 0, \nu; x]$ gives an étale embedding $\mathcal{G}_{A,0} \hookrightarrow \mathcal{G}_{A,B}$ which preserves the unit space.

Another special case is when $A = B$. Then we have $\mathcal{G}_{A,A} \cong \mathcal{G}_A \times \mathbb{Z}$ (where $\mathbb{Z}$ is viewed as a group(oid)). These groupoids fall outside of the scope of the AH conjecture, however, for they are far from being effective.

C.3.3 When is $\mathcal{G}_{A,B}$ Hausdorff, effective and minimal?

We begin by noting that $\mathcal{G}_{A,B}$ has compact unit space if and only if $N < \infty$ (i.e. A and B are finite matrices). In this case it is a Cantor space precisely when E_A satisfies Condition (L).

Before characterizing Hausdorfness precisely, we discuss a sufficient condition known as *pseudo-freeness*. This is an underlying assumption in [Ort18]. The action $\kappa\colon \mathbb{Z} \curvearrowright E_A$ is called *pseudo-free* if $\kappa_m(e) = e$ and $\varphi(m, e) = 0$ implies $m = 0$, for $m \in \mathbb{Z}$ and $e \in E_A^1$ (see [EP17, Definition 5.4] for the general definition). Combining Lemma 18.5 and Proposition 12.1 from [EP17] yields the following.

Proposition C.3.2 ([EP17]). *The action $\kappa\colon \mathbb{Z} \curvearrowright E_A$ is pseudo-free if and only if $A_{i,j} = 0$ whenever $B_{i,j} = 0$. When this is the case, $\mathcal{G}_{A,B}$ is Hausdorff.*

A precise characterization of when $\mathcal{G}_{A,B}$ is Hausdorff is the following.

Proposition C.3.3 ([EP17, Theorem 18.6]). *The following are equivalent:*

1. *The Katsura–Exel–Pardo groupoid $\mathcal{G}_{A,B}$ is Hausdorff.*

2. *Whenever* $B_{i,j} = 0$ *while* $A_{i,j} \geq 1$, *then for any* $m \in \mathbb{Z} \setminus \{0\}$ *the set*

$$\left\{ \mu \in E_A^* \mid r(\mu) = i \text{ and } m\frac{B_{\mu|_t}}{A_{\mu|_t}} \in \mathbb{Z} \setminus \{0\} \text{ for } 1 \leq t \leq |\mu| \right\}$$

is finite.

Remark C.3.4. There is a small misprint in the statement of [EP17, Theorem 18.6], which is why the statement above differs slightly (even after reversing the direction of the edges).

The minimality of $\mathcal{G}_{A,B}$ turns out to be independent of the matrix B, and is only governed by the minimality of the graph groupoid $\mathcal{G}_{E_A}$.

Proposition C.3.5 ([EP17, Theorem 18.7])**.** *The Katsura–Exel–Pardo groupoid* $\mathcal{G}_{A,B}$ *is minimal if and only if the graph* E_A *is cofinal.*

In particular, if the matrix A is irreducible (which is equivalent to E_A being strongly connected), then $\mathcal{G}_{A,B}$ is minimal. The converse holds if E_A has no sources (nor sinks).

Remark C.3.6. Proposition C.3.5 actually holds for any self-similar graph in which the vertices are fixed. A general characterization is given in [EP17, Theorem 13.6].

Let us move on to characterizing when $\mathcal{G}_{A,B}$ is effective (the term “essentially principal” is used in [EP17]).

Proposition C.3.7 ([EP17, Theorem 18.8])**.** *The following are equivalent:*

1. *The Katsura–Exel–Pardo groupoid* $\mathcal{G}_{A,B}$ *is effective.*
2. (a) *The graph* E_A *satisfies Condition (L).*

   (b) *If* $1 \leq i \leq N$, $m \in \mathbb{Z} \setminus \{0\}$, *and for all* $x \in Z(i)$ *we have* $m\frac{B_{x|_t}}{A_{x|_t}} \in \mathbb{Z}$ *for all* $t \in \mathbb{N}$, *then there exists* $T \in \mathbb{N}$ *such that* $B_{x|_T} = 0$ *for all* $x \in Z(i)$.

The premise in (2b) above is fairly strong, as it stipulates that $\kappa_m(x) = x$ for all $x \in Z(i)$. In many cases this will not happen for any vertex i, which means (2b) is trivially satisfied. One such case is the following.

Corollary C.3.8 ([EP17, Corollary 18.9])**.** *If* E_A *satisfies Condition (L) and for each* $1 \leq i \leq N$ *there exists* $x \in Z(i)$ *such that* $B_{x|_t} \neq 0$ *for all* $t \in \mathbb{N}$ *and* $\lim_{t\to\infty} \frac{B_{x|_t}}{A_{x|_t}} = 0$, *then* $\mathcal{G}_{A,B}$ *is effective.*

The following is a class of examples to which Corollary C.3.8 applies.

Example C.3.9. If the matrices A, B satisfy $A_{i,i} \geq 2$ and $0 < |B_{i,i}| < A_{i,i}$ for all $1 \leq i \leq N$, then $\mathcal{G}_{A,B}$ is effective. If A is irreducible it suffices that this condition holds for a single vertex i.

The following remark illustrates that the class of examples above is already fairly rich.

Remark C.3.10. It suffices to consider matrices A, B satisfying $A_{i,i} \geq 2$ and $B_{i,i} = 1$ for each $1 \leq i \leq N$ with A irreducible for $\mathcal{O}_{A,B}$ to exhaust all Kirchberg algebras up to stable isomorphism [Kat08a, Proposition 4.5].

Next we observe that the Katsura–Exel–Pardo groupoids that satisfy the assumptions of the AH conjecture are purely infinite (in the sense of [Mat15b, Definition 4.9]). This means that the index map is surjective [Mat15b, Theorem 5.2], so we only need to establish Property TR in order to prove that the AH conjecture hold for these groupoids.

Proposition C.3.11. *Let $N < \infty$ and assume that $\mathcal{G}_{A,B}$ is Hausdorff, effective and minimal. Then $\mathcal{G}_{A,B}$ is purely infinite.*

Proof. Since the SFT-groupoid $\mathcal{G}_A \cong \mathcal{G}_{A,0}$ is an open ample subgroupoid of $\mathcal{G}_{A,B}$, the pure infiniteness of $\mathcal{G}_{A,B}$ follow from that of $\mathcal{G}_A$, which is established in [Mat15b, Lemma 6.1]. □

As in Paper B we make the following ad hoc definition for brevity.

Definition C.3.12. We say that the matrices A, B satisfy the *AH criteria* if $N < \infty$ and $\mathcal{G}_{A,B}$ is Hausdorff, effective and minimal.

A large class of pairs of matrices satisfying the AH criteria are given in the following example.

Example C.3.13. Let $N \in \mathbb{N}$ and let $A \in M_N(\mathbb{Z}_+)$, $B \in M_N(\mathbb{Z})$. Assume that A is irreducible and that $B_{i,j} = 0$ if and only if $A_{i,j} = 0$. Assume further that there exists some i between 1 and N such that $|B_{i,i}| < A_{i,i} \geq 2$. Then the matrices A, B satisfy the AH criteria.

C.4 The homology of $\mathcal{G}_{A,B}$

In this section we will describe the homology groups of the Katsura–Exel–Pardo groupoids, following [Ort18]. Although the action is assumed to be pseudo-free throughout in [Ort18], most of what we need here also work without this assumption, with one notable exception which is adressed in Equation (C.4.6) below.

Assumption C.4.1. *We assume throughout that $N < \infty$ and that $\mathcal{G}_{A,B}$ is Hausdorff.*

C.4.1 The kernel subgroupoid $\mathcal{H}_{A,B}$

Similarly to the canonical cocycle on an SFT-groupoid (see page 37 of [Mat12]) we can define a cocycle (that is, a continuous groupoid homomorphism into a group) $c\colon \mathcal{G}_{A,B} \to \mathbb{Z}$ on a Katsura–Exel–Pardo groupoid by setting

$$c([\mu, m, \nu; x]) = |\mu| - |\nu|.$$

This is well defined since the difference $|\mu| - |\nu|$ is preserved under $\sim$. Now define

$$\mathcal{H}_{A,B} \coloneqq \ker(c) = \{[\mu, m, \nu; x] \in \mathcal{G}_{A,B} \mid |\mu| = |\nu|\},$$

which is a clopen ample subgroupoid of $\mathcal{G}_{A,B}$. In contrast to the case of graph groupoids, this kernel is generally not an AF-groupoid (it need not be principal), but it is still key to computing the homology of $\mathcal{G}_{A,B}$.

Next, for each $n \in \mathbb{N}$ we define the open subgroupoid

$$\mathcal{H}_{A,B,n} \coloneqq \{[\mu, m, \nu; x] \in \mathcal{G}_{A,B} \mid |\mu| = |\nu| = n\} \subseteq \mathcal{H}_{A,B}.$$

Observe that $\mathcal{H}_{A,B,n} \subseteq \mathcal{H}_{A,B,n+1}$ by (C.3.3) and that $\cup_{n=1}^{\infty} \mathcal{H}_{A,B,n} = \mathcal{H}_{A,B}$. Hence

$$H_i(\mathcal{H}_{A,B}) \cong \varinjlim \left(H_i(\mathcal{H}_{A,B,n}), H_i(\iota_n)\right) \tag{C.4.1}$$

by [FKPS18, Proposition 4.7], where ι_n is the inclusion map.

It follows from the proof of [Ort18, Proposition 2.3] that if $\mu, \nu \in E_A^n$ and $r(\mu) = r(\nu)$, then

$$\left[1_{Z(\mu)}\right] = \left[1_{Z(\nu)}\right] \in H_0(\mathcal{H}_{A,B,n})$$

and that we have

$$H_0(\mathcal{H}_{A,B,n}) = \operatorname{span}\left\{\left[1_{Z(\mu)}\right] \mid \mu \in E_A^n\right\} \cong \mathbb{Z}^N, \tag{C.4.2}$$

even without the assumption of pseudo-freeness. The isomorphism in (C.4.2) is given by mapping $\left[1_{Z(\mu)}\right]$ to $1_{r(\mu)}$, where by $1_w \in \mathbb{Z}^N \cong \oplus_{v \in E_A^0} \mathbb{Z}$ for $w \in E_A^0$, we mean the tuple with 1 in the w'th coordinate and 0 elsewhere.

As for $H_1(\mathcal{H}_{A,B,n})$, for paths μ and ν as above it similarly follows from the proof of [Ort18, Proposition 2.4] that

$$\begin{aligned}
\left[1_{Z(\mu,m,\mu)}\right] &= \left[1_{Z(\mu,m,\nu)}\right] = \left[1_{Z(\nu,m,\nu)}\right] \in H_1(\mathcal{H}_{A,B,n}), \\
\left[1_{Z(\mu,m,\mu)}\right] &= m\left[1_{Z(\mu,1,\mu)}\right] \in H_1(\mathcal{H}_{A,B,n}),
\end{aligned} \tag{C.4.3}$$

and hence

$$H_1(\mathcal{H}_{A,B,n}) = \operatorname{span}\left\{\left[1_{Z(\mu,1,\mu)}\right] \mid \mu \in E_A^n\right\}. \tag{C.4.4}$$

If the action is pseudo-free, then

$$H_1(\mathcal{H}_{A,B,n}) \cong \mathbb{Z}^N$$

by identifying $\left[1_{Z(\mu,1,\mu)}\right]$ with $1_{r(\mu)}$.

However, when the action is not pseudo-free, we need to take care. The group $H_1(\mathcal{H}_{A,B,n})$ will still be a free abelian group, but its rank may be smaller than N. To explain this phenomenon, let us call a vertex $i \in \{1, 2, \ldots, N\}$ a B-*sink* if $B_{i,j} = 0$ for all j with $A_{i,j} > 0$. Any path passing through a B-sink will be strongly fixed by the action, meaning that $\kappa_m(\mu) = \mu$ and $\varphi(m, \mu) = 0$ ([EP17, Definition 5.2]). To see the impact this has on $H_1(\mathcal{H}_{A,B,n})$, suppose that i is a B-sink and that $\mu \in E_A^n$ has $r(\mu) = i$. Then we have the counter-intuitive equality

$$Z(\mu, 1, \mu) = Z(\mu, 0, \mu) \subseteq \mathcal{G}_{A,B}^{(0)}, \tag{C.4.5}$$

since for any $x = \mu e z \in Z(\mu)$ with $e \in r(\mu)E_A^1$ we have

$$(\mu, 1, \mu; x) \sim (\mu\kappa_1(e), \varphi(1, e), \mu e; x) = (\mu e, 0, \mu e; x) \sim (\mu, 0, \mu; x).$$

This in turn means that $\left[1_{Z(\mu,1,\mu)}\right] = 0 \in H_1(\mathcal{H}_{A,B,n})$, so this part of $H_1(\mathcal{H}_{A,B,n})$ collapses. More generally, the same will happen to any path $\mu \in E_A^n$ for which every infinite path $x \in Z(r(\mu))$ passes through a B-sink. To have a name for vertices for which this does not happen, let us define a vertex $1 \leq i \leq N$ to be a B-*regular* if there exists a path μ, containing no B-sinks, starting at i which connects to a cycle that contain no B-sinks. This is the same as saying that there is some infinite path starting at i which does not pass through any B-sink. Bisections $Z(\mu, 1, \mu)$ with $r(\mu)$ B-regular behave just like in the pseudo-free case, while those with $r(\mu)$ not B-regular vanish in $H_1(\mathcal{H}_{A,B,n})$ as explained above. Let R_B denote the number of B-regular vertices. Then we have that

$$H_1(\mathcal{H}_{A,B,n}) = \operatorname{span}\left\{\left[1_{Z(\mu,1,\mu)}\right] \mid \mu \in E_A^n \text{ with } r(\mu)\ B\text{-regular}\right\} \cong \mathbb{Z}^{R_B}. \tag{C.4.6}$$

This particular description (as opposed to (C.4.4)) is only used in the proof of Lemma C.5.3.

Remark C.4.2. By viewing the matrices A and B as endomorphisms of $\mathbb{Z}^N$ (via left multiplication) we may consider the inductive limits

$$\mathbb{Z}_A := \varinjlim \left(\mathbb{Z}^N, A\right) \quad \text{and} \quad \mathbb{Z}_B := \varinjlim \left(\mathbb{Z}^N, B\right). \tag{C.4.7}$$

Let $\phi_{n,\infty}^A \colon Z^N \to Z_A$ and $\phi_{n,\infty}^B \colon Z^N \to Z_B$ denote the canonical maps into the inductive limits. Propositions 2.3 and 2.4 in [Ort18] remain valid without pseudo-freeness and they show that the inductive limits in (C.4.1) for $i = 0$ and $i = 1$ turn into the limits in (C.4.7), respectively. This means that

$$H_0(\mathcal{H}_{A,B}) \cong \mathbb{Z}_A \quad \text{and} \quad H_1(\mathcal{H}_{A,B}) \cong \mathbb{Z}_B,$$

where the isomorphisms are given by

$$\left[1_{Z(\mu)}\right] \mapsto \phi^A_{n,\infty}\left(1_{r(\mu)}\right) \quad \text{and} \quad \left[1_{Z(\mu,1,\mu)}\right] \mapsto \phi^B_{n,\infty}\left(1_{r(\mu)}\right),$$

respectively, for $\mu \in E^n_A$. This is still compatible with Equation (C.4.6), because if v is a non-B-regular vertex, then 1_v is eventually annihilated in the inductive limit $\mathbb{Z}_B$. What does not necessarily hold without pseudo-freeness is [Ort18, Proposition 2.2], which says that $H_n\left(\mathcal{H}_{A,B}\right) = 0$ when $n \geq 2$. This part, however, is not needed for the results in the present paper.

Let $\mathcal{G}_{A,B} \times_c \mathbb{Z}$ denote the skew product associated to the cocycle c (see Subsection B.2.5).

Lemma C.4.3. *The clopen set $E^\infty_A \times \{0\} \subseteq \left(\mathcal{G}_{A,B} \times_c \mathbb{Z}\right)^{(0)}$ is $\left(\mathcal{G}_{A,B} \times_c \mathbb{Z}\right)$-full.*

Proof. The same proof as for SFT-groupoids works here (see [FKPS18, Lemma 6.1] for a more general result). Let $(x,k) \in E^\infty_A \times \mathbb{Z} = \left(\mathcal{G}_{A,B} \times_c \mathbb{Z}\right)^{(0)}$ be given. If $k < 0$, then the groupoid element $\left[x|_{-k}, 0, r\left(x|_{-k}\right); x_{[-k+1,\infty]}\right] \in \mathcal{G}_{A,B} \times_c \mathbb{Z}$ has range (x,k) and source $(x_{[-k+1,\infty]}, 0)$, which shows that $E^\infty_A \times \{0\}$ meets the $\left(\mathcal{G}_{A,B} \times_c \mathbb{Z}\right)$-orbit of (x,k). In the case that $k > 0$ we can, since E_A is a finite graph without sinks, find an index $n \in \mathbb{N}$ for which $r(x_n)$ supports a cycle. By concatenating along this cycle we can find a path $\nu \in E^*_A$ with $r(\nu) = r(x_n)$ and $|\nu| = n + k$. Then the element $\left[x|_n, 0, \nu; \nu x_{[n+1,\infty]}\right]$ has range (x,k) and source $(\nu x_{[n+1,\infty]}, 0) \in E^\infty_A \times \{0\}$. □

Recall that

$$\mathcal{H}_{A,B} \cong \left(\mathcal{G}_{A,B} \times_c \mathbb{Z}\right)|_{E^\infty_A \times \{0\}}$$

via the map

$$[\mu, m, \nu; x] \mapsto ([\mu, m, \nu; x], 0).$$

Composing this with the inclusion of the restriction we obtain an embedding ι of $\mathcal{H}_{A,B}$ into the skew product $\mathcal{G}_{A,B} \times_c \mathbb{Z}$. Lemma C.4.3 says that $\mathcal{H}_{A,B}$ is Kakutani equivalent to $\mathcal{G}_{A,B} \times_c \mathbb{Z}$ from which we have the following consequence (by [FKPS18, Lemma 4.3]).

Proposition C.4.4. *The embedding $\iota\colon \mathcal{H}_{A,B} \to \mathcal{G}_{A,B} \times_c \mathbb{Z}$ induces isomorphisms*

$$H_i\left(\mathcal{H}_{A,B}\right) \cong H_i\left(\mathcal{G}_{A,B} \times_c \mathbb{Z}\right)$$

for each $i \geq 0$.

C.4.2 A long exact sequence in homology

From Proposition B.6.1 applied to the cocycle $c\colon \mathcal{G}_{A,B} \to \mathbb{Z}$ we obtain the following long exact sequence in homology:

$$\begin{tikzcd}
\cdots \arrow[r] & H_2(\mathcal{G}_{A,B}) \arrow[r, "\partial_2"] & H_1(\mathcal{G}_{A,B} \times_c \mathbb{Z}) \arrow[r, "\mathrm{id} - H_1(\rho_\bullet)"] & H_1(\mathcal{G}_{A,B} \times_c \mathbb{Z}) \arrow[dll, "H_1(\pi_\bullet)"'] \\
& H_1(\mathcal{G}_{A,B}) \arrow[r, "\partial_1"] & H_0(\mathcal{G}_{A,B} \times_c \mathbb{Z}) \arrow[r, "\mathrm{id} - H_0(\rho_\bullet)"] & H_0(\mathcal{G}_{A,B} \times_c \mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd} \tag{C.4.8}$$

Consult Section B.6 for a description of the maps. Appealing to Proposition C.4.4 we can replace $H_i\left(\mathcal{G}_{A,B} \times_c \mathbb{Z}\right)$ with $H_i\left(\mathcal{H}_{A,B}\right)$ and extract the following exact sequence from the one above:

$$H_1(\mathcal{H}_{A,B}) \xrightarrow{\rho^1} H_1(\mathcal{H}_{A,B}) \xrightarrow{\Phi} H_1(\mathcal{G}_{A,B}) \xrightarrow{\Psi} H_0(\mathcal{H}_{A,B}) \xrightarrow{\rho^0} H_0(\mathcal{H}_{A,B}). \tag{C.4.9}$$

The maps Φ and Ψ are the unique maps satisfying

$$H_1(\pi_\bullet) \circ H_1(\iota) = \Phi \quad \text{and} \quad \partial_1 = H_0(\iota) \circ \Psi,$$

respectively. Similarly, the maps ρ^i are defined by

$$H_i(\iota) \circ \rho^i = (\mathrm{id} - H_i(\rho_\bullet)) \circ H_i(\iota) \quad \text{for } i = 0, 1.$$

In the next section we are going to need explicit descriptions of the maps in (C.4.9). This is provided in the lemmas below. Some of them are given in terms of "prefixing an edge" to a path, and therefore we need to assume that the graph E_A has no sources.

Assumption C.4.5. *For the remainder of this section we assume that the matrix A is essential.*

Lemma C.4.6. *The map $\Phi\colon H_1(\mathcal{H}_{A,B}) \to H_1(\mathcal{G}_{A,B})$ is given by*

$$\Phi\left(\left[1_{Z(\mu,1,\mu)}\right]\right) = \left[1_{Z(\mu,1,\mu)}\right] = \left[1_{Z(r(\mu),1,r(\mu))}\right] \in H_1(\mathcal{G}_{A,B})$$

for $\left[1_{Z(\mu,1,\mu)}\right] \in H_1(\mathcal{H}_{A,B})$. In particular, $I(\alpha) = \Phi\left(I_{\mathcal{H}}(\alpha)\right) \in H_1(\mathcal{G}_{A,B})$ for each $\alpha \in [\![\mathcal{H}_{A,B}]\!]$.

Proof. Straightforward. $\square$

Lemma C.4.7. *The map $\rho^0\colon H_0(\mathcal{H}_{A,B}) \to H_0(\mathcal{H}_{A,B})$ is given by*

$$\rho^0\left(\left[1_{Z(\mu)}\right]\right) = \left[1_{Z(\mu)}\right] - \left[1_{Z(e\mu)}\right],$$

where $e \in E_A^1$ is any edge with $r(e) = s(\mu)$.

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
H_0(\mathcal{G}_{A,B} \times_c \mathbb{Z}) \arrow[r, "\mathrm{id} - H_0(\rho_\bullet)"] & H_0(\mathcal{G}_{A,B} \times_c \mathbb{Z}) \\
H_0(\mathcal{H}_{A,B}) \arrow[u, "H_0(\iota)", "\cong"'] \arrow[r, "\rho^0"] & H_0(\mathcal{H}_{A,B}) \arrow[u, "H_0(\iota)", "\cong"']
\end{tikzcd}$$

The maps are given by

$$H_0(\iota)\left(\left[1_{Z(\mu)}\right]\right) = \left[1_{Z(\mu)\times\{0\}}\right] \in H_0\left(\mathcal{G}_{A,B} \times_c \mathbb{Z}\right)$$

and

$$H_0\left(\rho_\bullet\right)\left(\left[1_{Z(\mu)\times\{0\}}\right]\right) = \left[1_{Z(\mu)\times\{1\}}\right] = \left[1_{Z(e\mu)\times\{0\}}\right] \in H_0\left(\mathcal{G}_{A,B} \times_c \mathbb{Z}\right),$$

where $e \in E_A^1$ is any edge with $r(e) = s(\mu)$. Combining these we obtain the desired description of ρ^0. ◻

When $B = 0$, the map $\rho^0\colon H_0(\mathcal{H}_{A,0}) \to H_0(\mathcal{H}_{A,0})$ coincides with the map

$$(\mathrm{id} - \varphi)\colon H_0\left(\mathcal{H}_{E_A}\right) \to H_0\left(\mathcal{H}_{E_A}\right),$$

where φ is from Definition B.7.5. We apologize for the conflicting notation of φ with the 1-cocycle from Section C.3, but since the 1-cocycle makes no appearence for the rest of this section we believed it better to stick with the notation from Paper B to make it easier to compare with results therein. Below, we (trivially) extend the definition of φ, as well as $\varphi^{(k)}$ from Definition B.8.5, to Katsura–Exel–Pardo groupoids. The automorphism φ is the one induced by $H_0\left(\rho_\bullet\right)$ when identifying $H_0\left(\mathcal{G}_{A,B} \times_c \mathbb{Z}\right)$ with $H_0\left(\mathcal{H}_{A,B}\right)$.

Definition C.4.8. Define $\varphi\colon H_0\left(\mathcal{H}_{A,B}\right) \to H_0\left(\mathcal{H}_{A,B}\right)$ by for each $\mu \in E_A^*$ setting

$$\varphi\left(\left[1_{Z(\mu)}\right]\right) = \left[1_{Z(e\mu)}\right],$$

where $e \in E_A^1$ is any edge with $r(e) = s(\mu)$. For $k \in \mathbb{Z}$ we further define

$$\varphi^{(k)} := \begin{cases} -(\mathrm{id} + \varphi + \cdots + \varphi^{k-1}) & k > 0, \\ 0 & k = 0, \\ \varphi^{-1} + \varphi^{-2} + \cdots + \varphi^{k} & k < 0. \end{cases}$$

Remark C.4.9. In the case that $B = 0$ the map $\varphi\colon H_0(\mathcal{H}_{A,0}) \to H_0(\mathcal{H}_{A,0})$ coincides the inverse δ^{-1} of Matui's map δ from [Mat15b, page 56]. See Remarks B.7.6 and B.8.8 for more on this.

The next lemma is essentially the same as Lemma B.8.6.

Lemma C.4.10. *Let $[f] \in H_1(\mathcal{G}_{A,B})$ and write $f = \sum_{i=1}^k n_i 1_{Z(\mu_i,1,\nu_i)}$. Then the map $\Psi\colon H_1(\mathcal{G}_{A,B}) \to H_0(\mathcal{H}_{A,B})$ is given by*

$$\Psi([f]) = \sum_{i=1}^k n_i \varphi^{(|\nu_i|-|\mu_i|)}\left([1_{Z(\nu_i)}]\right).$$

Proof. Recall that $\partial_1 = H_0(\iota)\circ\Psi$, where $\partial_1\colon H_1(\mathcal{G}_{A,B}) \to H_0(\mathcal{G}_{A,B}\times_c \mathbb{Z})$ is the connecting homomorphism in (C.4.8). We are going to describe ∂_1 in a similar way as in the proof of Lemma B.8.6. It may be helpful to consult Figure B.2, as we will adopt the notation from there.

Let $[f] \in H_1(\mathcal{G}_{A,B})$ be given, where $f \in C_c(\mathcal{G}_{A,B},\mathbb{Z})$ satisfies $\delta_1(f) = 0$. Then we can write $f = \sum_{i=1}^k n_i 1_{Z(\mu_i,1,\nu_i)}$, where $\sum_{i=1}^k n_i 1_{Z(\mu_i)} = \sum_{i=1}^k n_i 1_{Z(\nu_i)}$. Now view $f + \operatorname{im}(\delta_2)$ as an element in $C_c(\mathcal{G}_{A,B},\mathbb{Z})/\operatorname{im}(\delta_2)$.

The element $\pi_1(h) + \operatorname{im}(\delta_2)$, where

$$h \coloneqq f\times 0 = \sum_{i=1}^k n_i 1_{Z(\mu_i,1,\nu_i)\times\{0\}} \in C_c(\mathcal{G}_{A,B}\times_c\mathbb{Z},\mathbb{Z}),$$

provides a lift of $f + \operatorname{im}(\delta_2)$ by $\pi_1 + \operatorname{im}(\delta_2)$. Next, we need to compute

$$\widetilde{\delta_1}(h + \operatorname{im}(\delta_2)) = \delta_1(h) \in C_c\left((\mathcal{G}_{A,B}\times_c\mathbb{Z})^{(0)},\mathbb{Z}\right) \cong C_c\left(E_A^\infty\times\mathbb{Z},\mathbb{Z}\right).$$

Setting $l_i \coloneqq |\mu_i| - |\nu_i|$ to save space we have

$$\begin{aligned}
\delta_1(h) &= \sum_{i=1}^k n_i(s_* - r_*)\left(1_{Z(\mu_i,m_i,\nu_i)\times\{0\}}\right)\\
&= \sum_{i=1}^k n_i\left(1_{s(Z(\mu_i,m_i,\nu_i)\times\{0\})} - 1_{r(Z(\mu_i,m_i,\nu_i)\times\{0\})}\right)\\
&= \sum_{i=1}^k n_i\left(1_{Z(\nu_i)\times\{|\mu_i|-|\nu_i|\}} - 1_{Z(\mu_i)\times\{0\})}\right)\\
&= \sum_{i=1}^k n_i\left(1_{Z(\nu_i)\times\{l_i\}} - 1_{Z(\nu_i)\times\{0\})}\right),
\end{aligned}$$

where we have used that $\sum_{i=1}^{k} n_i 1_{Z(\mu_i)} = \sum_{i=1}^{k} n_i 1_{Z(\nu_i)}$. By Lemma B.6.2 the (unique) lift of $\delta_1(h)$ by $\mathrm{id} - \rho_0$ is the function

$$g := \sum_{i=1}^{k} n_i L_i,$$

where

$$L_i = \begin{cases} -\sum_{j=0}^{l_i-1} 1_{Z(\nu_i)\times\{j\}} & l_i > 0, \\ 0 & l_i = 0, \\ \sum_{j=l_i}^{-1} 1_{Z(\nu_i)\times\{j\}} & l_i < 0. \end{cases}$$

Observe that

$$[L_i] = \varphi^{(l_i)}\left([1_{Z(\nu_i)\times\{0\}}]\right) \in H_0(\mathcal{G}_{A,B} \times_c \mathbb{Z}).$$

This means that

$$\partial_1([f]) = [g] = \sum_{i=1}^{k} n_i \varphi^{(|\mu_i|-|\nu_i|)}\left([1_{Z(\nu_i)\times\{0\}}]\right) \in H_0(\mathcal{G}_{A,B} \times_c \mathbb{Z}),$$

and hence

$$\Psi([f]) = \sum_{i=1}^{k} n_i \varphi^{(|\nu_i|-|\mu_i|)}\left([1_{Z(\nu_i)}]\right) \in H_0\left(\mathcal{H}_{A,B}\right).$$

□

Lemma C.4.11. *Assume that $U \subseteq \mathcal{G}_{A,0} \subseteq \mathcal{G}_{A,B}$ is a full bisection. Let I and I_A denote the index maps of $\mathcal{G}_{A,B}$ and $\mathcal{G}_{A,0}$, respectively. If $\Psi\left(I\left(\pi_U\right)\right) = 0 \in H_0\left(\mathcal{H}_{A,B}\right)$, then $I_A\left(\pi_U\right) = 0 \in H_1\left(\mathcal{G}_{A,0}\right)$.*

Proof. We can write $U = \sqcup_{i=1}^{k} Z(\mu_i, 0, \nu_i)$, where $E_A^{\infty} = \sqcup_{i=1}^{k} Z(\mu_i) = \sqcup_{i=1}^{k} Z(\nu_i)$. By Lemma C.4.10 we have

$$0 = \Psi\left(I\left(\pi_U\right)\right) = \Psi\left([1_U]\right) = \sum_{i=1}^{k} \varphi^{(|\nu_i|-|\mu_i|)}\left([1_{Z(\nu_i)}]\right) \in \ker\left(\rho^0\right) \subseteq H_0\left(\mathcal{H}_{A,B}\right).$$

On the other hand we have that $H_1\left(\mathcal{G}_{A,0}\right) \cong \ker\left(\rho^0\right) \cong \ker\left(\mathrm{id} - H_0\left(\rho_\bullet\right)\right)$ since $H_1\left(\mathcal{G}_{A,0} \times_c \mathbb{Z}\right) = 0$ (see Section B.7). This isomorphism is implemented by the connecting homomorphism ∂_1 from (C.4.8) for $B = 0$. Lemma B.8.6 (or the proof of Lemma C.4.10 with $B = 0$) says that under this isomorphism the element $I_A\left(\pi_U\right) \in H_1\left(\mathcal{G}_{A,0}\right)$ corresponds to $\Psi\left(I\left(\pi_U\right)\right) \in \ker\left(\rho^0\right)$. Hence $I_A\left(\pi_U\right) = 0$. □

The following lemma is part of the proof of [Ort18, Proposition 2.5], but we nevertheless sketch the proof for completness.

Lemma C.4.12. *The map $\rho^1 \colon H_1(\mathcal{H}_{A,B}) \to H_1(\mathcal{H}_{A,B})$ is given by*

$$\rho^1\left(\left[1_{Z(\mu,m,\mu)}\right]\right) = \left[1_{Z(\mu,m,\mu)}\right] - \left[1_{Z(e\mu,m,e\mu)}\right],$$

where $e \in E_A^1$ is any edge with $r(e) = s(\mu)$.

Proof. Arguing similarly as in the proof of Lemma C.4.7 it suffices to show that

$$\left[1_{Z(\mu,1,\mu)\times\{1\}}\right] = \left[1_{Z(e\mu,1,e\mu)\times\{0\}}\right]$$

in $H_1\left(\mathcal{G}_{A,B} \times_c \mathbb{Z}\right)$.

Suppose U, V are compact bisections with $s(U) = r(V)$ in some ample groupoid $\mathcal{G}$. Denote

$$U \circ V := (U \times V) \cap \mathcal{G}^{(2)} = \left\{(g,h) \in \mathcal{G}^{(2)} \mid g \in U,\ h \in V\right\}.$$

By [Mat12, Lemma 7.3], we have

$$\delta_2\left(1_{U\circ V}\right) = 1_U - 1_{U\cdot V} + 1_V. \tag{C.4.10}$$

Let $e \in E_A^1$ be any edge with $r(e) = s(\mu)$ and define the following bisections in $\mathcal{G}_{A,B} \times_c \mathbb{Z}$:

$$\begin{aligned}
U_1 &:= Z(\mu,1,\mu) \times \{1\}, & V_1 &:= Z(\mu,0,e\mu) \times \{1\},\\
U_2 &:= Z(e\mu,0,\mu) \times \{0\}, & V_2 &:= Z(\mu,1,e\mu) \times \{1\},\\
U_3 &:= U_2, & V_3 &:= V_1,\\
U_4 &:= Z(e\mu,0,e\mu) \times \{0\}, & V_4 &:= U_4.
\end{aligned}$$

From these we define the indicator functions $f_i := 1_{U_i \circ V_i} \in C_c\left(\mathcal{G}_{A,B}^{(2)}, \mathbb{Z}\right)$ for $i = 1,2,3,4$. Using (C.4.10) it is easy to check that

$$\delta_2\left(f_1 + f_2 - f_3 - f_4\right) = 1_{Z(\mu,1,\mu)\times\{1\}} - 1_{Z(e\mu,1,e\mu)\times\{0\}},$$

which shows that $\left[1_{Z(\mu,1,\mu)\times\{1\}}\right] = \left[1_{Z(e\mu,1,e\mu)\times\{0\}}\right]$ in $H_1\left(\mathcal{G}_{A,B} \times_c \mathbb{Z}\right)$. □

Remark C.4.13. The main result of [Ort18] is the following description of the homology groups of $\mathcal{G}_{A,B}$, assuming that the self-similar graph is pseudo-free:

$$\begin{aligned}
H_0\left(\mathcal{G}_{A,B}\right) &\cong \operatorname{coker}\left(I_N - A\right),\\
H_1\left(\mathcal{G}_{A,B}\right) &\cong \ker\left(I_N - A\right) \oplus \operatorname{coker}\left(I_N - B\right),\\
H_2\left(\mathcal{G}_{A,B}\right) &\cong \ker\left(I_N - B\right),\\
H_i\left(\mathcal{G}_{A,B}\right) &= 0, \quad i \geq 3.
\end{aligned}$$

Here I_N is the $N\times N$ identity matrix and $I_N - A$, $I_N - B$ are viewed as endomorphisms of $\mathbb{Z}^N$. When the self-similar graph is pseudo-free, [Ort18, Lemma 2.2] shows that $H_i(\mathcal{H}_{A,B}) = 0$ for $i \geq 2$. This truncates the long exact sequence (C.4.8) into (identifying as in (C.4.9)):

$$\begin{tikzcd}
0 \arrow[r] & H_2(\mathcal{G}_{A,B}) \arrow[r] & H_1(\mathcal{H}_{A,B}) \arrow[r, "\rho^1"] & H_1(\mathcal{H}_{A,B}) \arrow[dll, "\Phi"'] & \\
& H_1(\mathcal{G}_{A,B}) \arrow[r, "\Psi"] & H_0(\mathcal{H}_{A,B}) \arrow[r, "\rho^0"] & H_0(\mathcal{H}_{A,B}) \arrow[r] & H_0(\mathcal{G}_{A,B}) \arrow[r] & 0.
\end{tikzcd}$$

It follows that $H_2(\mathcal{G}_{A,B}) \cong \ker\left(\rho^1\right)$, $H_0(\mathcal{G}_{A,B}) \cong \operatorname{coker}\left(\rho^0\right)$ and that

$$0 \longrightarrow \operatorname{coker}\left(\rho^1\right) \xrightarrow{\widetilde{\Phi}} H_1(\mathcal{G}_{A,B}) \xrightarrow{\Psi} \ker\left(\rho^0\right) \longrightarrow 0 \tag{C.4.11}$$

is exact. It is also shown in [Ort18] that

$$\begin{aligned}
\ker\left(\rho^0\right) &\cong \ker\left(I_N - A\right), & \operatorname{coker}\left(\rho^0\right) &\cong \operatorname{coker}\left(I_N - A\right), \\
\ker\left(\rho^1\right) &\cong \ker\left(I_N - B\right), & \operatorname{coker}\left(\rho^1\right) &\cong \operatorname{coker}\left(I_N - B\right).
\end{aligned}$$

Since $\ker\left(\rho^0\right)$ is free, the exact sequence (C.4.11) splits, and we therefore obtain $H_1(\mathcal{G}_{A,B}) \cong \ker\left(\rho^0\right) \oplus \operatorname{coker}\left(\rho^1\right)$.

We remark that these results are valid for $N = \infty$ as well. Moreover, the descriptions of $H_0(\mathcal{G}_{A,B})$ and $H_1(\mathcal{G}_{A,B})$ are valid even when the self-similar graph is not pseudo-free.

C.5 Property TR for $\mathcal{G}_{A,B}$

The aim of this section is to show that the Katsura–Exel–Pardo groupoid $\mathcal{G}_{A,B}$ has Property TR. This means that given $\alpha \in [\![\mathcal{G}_{A,B}]\!]$ with $I(\alpha) = 0$, we need to show that $\alpha \in \mathcal{T}(\mathcal{G}_{A,B})$. In a nutshell, the strategy is to decompose the topological full group as $[\![\mathcal{G}_{A,B}]\!] = [\![\mathcal{H}_{A,B}]\!]\,[\![\mathcal{G}_{A,0}]\!]$ and show that Property TR is inherited from the kernel groupoid $\mathcal{H}_{A,B}$ and the SFT-groupoid $\mathcal{G}_{A,0}$. In what follows we will view $\mathcal{G}_{A,0} \cong \mathcal{G}_A$ as a subgroupoid of $\mathcal{G}_{A,B}$.

Assumption C.5.1. *In this whole section we fix $N \times N$ matrices A, B which satisfy the AH criteria and where A is essential. In particular $N < \infty$ and A is an irreducible non-permutation matrix.*

Proposition C.5.2. *The index map $I_{\mathcal{H}} \colon [\![\mathcal{H}_{A,B}]\!] \to H_1(\mathcal{H}_{A,B})$ is surjective.*

Proof. Let $\mu \in E_A^*$ and consider the bisection $V := Z(\mu, 1, \mu) \subseteq \mathcal{H}_{A,B}$. Since $s(V) = r(V) = Z(\mu)$ we can define a full bisection $U := V \sqcup \left(E_A^\infty \setminus Z(\mu)\right) \subseteq \mathcal{H}_{A,B}$. By [Mat12, Lemma 7.3] we have $I_{\mathcal{H}}(\pi_U) = [1_V]$. The result now follows since these elements span $H_1(\mathcal{H}_{A,B})$ (by (C.4.1) and (C.4.4)). ◻

Lemma C.5.3. *For each $n \in \mathbb{N}$ the groupoid $\mathcal{H}_{A,B,n}$ has Property TR.*

Proof. Let $U \subseteq \mathcal{H}_{A,B,n}$ be a full bisection. Then $U = \sqcup_{i=1}^k Z(\mu_i, m_i, \nu_i)$, where $\mu_i, \nu_i \in E_A^{\leq n}$ satisfy $|\mu_i| = |\nu_i|$, $r(\mu_i) = r(\nu_i)$ and $E_A^\infty = \sqcup_{i=1}^k Z(\mu_i) = \sqcup_{i=1}^k Z(\nu_i)$. Using the fact that each basic bisection decomposes as

$$Z(\mu, m, \nu) = \bigsqcup_{e \in s^{-1}(r(\nu))} Z(\mu\kappa_m(e), \varphi(m, e), \nu e) \tag{C.5.1}$$

we can assume without loss of generality that $|\mu_i| = |\nu_i| = n$ for all $1 \leq i \leq k$. We may also set $m_i = 0$ whenever $r(\mu_i)$ is not B-regular, for then $Z(\mu_i, m_i, \nu_i) = Z(\mu_i, 0, \nu_i)$, by the same reasoning as in Equation (C.4.5).

Let us now consider the index map $I_{\mathcal{H},n} \colon [\![\mathcal{H}_{A,B,n}]\!] \to H_1(\mathcal{H}_{A,B,n})$. Using (C.4.3) we compute

$$I_{\mathcal{H},n}(\pi_U) = [1_U] = \sum_{i=1}^k \left[1_{Z(\mu_i, m_i, \nu_i)}\right] = \sum_{i=1}^k m_i \left[1_{Z(\mu_i, 1, \mu_i)}\right] \in H_1(\mathcal{H}_{A,B,n}).$$

For each vertex $v \in E_A^0$ let $\mathcal{I}_v := \{1 \leq i \leq k \mid r(\mu_i) = v\}$. Using the identification in (C.4.6) (where only the B-regular vertices matter) we see that $I_{\mathcal{H},n}(\pi_U) = 0$ if and only if $\sum_{i \in \mathcal{I}_v} m_i = 0$ for each vertex $v \in E_A^0$.

We define two more full bisections in $\mathcal{H}_{A,B,n}$, namely

$$U_{\mathcal{H}} := \sqcup_{i=1}^k Z(\mu_i, m_i, \mu_i) \quad \text{and} \quad U_A := \sqcup_{i=1}^k Z(\mu_i, 0, \nu_i).$$

Observe that $U_{\mathcal{H}} \cdot U_A = U$. We claim that π_{U_A} is a product of transpositions, i.e. $\pi_{U_A} \in \mathcal{T}(\mathcal{H}_{A,B,n})$. Indeed, since $E_A^\infty = \sqcup_{i=1}^k Z(\mu_i) = \sqcup_{i=1}^k Z(\nu_i)$ and $|\mu_i| = |\nu_i| = n$, we must have that $E_A^n = \{\mu_1, \mu_2, \ldots, \mu_k\} = \{\nu_1, \nu_2, \ldots, \nu_k\}$. Hence the homeomorphism π_{U_A} on E_A^∞ can be identified with a permutation on a finite set of k symbols which maps ν_i to μ_i. The claim then follows.

Next we will show that $\pi_{U_{\mathcal{H}}}$ is a product of transpositions when $I_{\mathcal{H},n}(\pi_U) = 0$. Let $\mathcal{I}^0$ denote the set of vertices v for which $\mathcal{I}_v \neq \emptyset$ and pick a distinguished index $i_v \in \mathcal{I}_v$ for each vertex $v \in \mathcal{I}^0$. Suppose that $r(\mu_{i_1}) = v = r(\mu_{i_v})$ for some index $i_1 \neq i_v$. Set $V_1 := Z(\mu_{i_v}, m_{i_1}, \mu_{i_1})$ and $W_1 := Z(\mu_{i_v}, 0, \mu_{i_1})$. Then

$$U_{\mathcal{H}} \cdot \widehat{V_1} \cdot \widehat{W_1} = \left(\bigsqcup_{i \neq i_v, i_1} Z(\mu_i, m_i, \mu_i)\right) \bigsqcup Z(\mu_{i_v}, m_{i_v} + m_{i_1}, \mu_{i_v}) \bigsqcup Z(\mu_{i_1}, 0, \mu_{i_1}).$$

By iterating this process enough times for each vertex we can write

$$U_{\mathcal{H}} \cdot \widehat{V_1} \cdot \widehat{W_1} \cdots \widehat{V_K} \cdot \widehat{W_K} = \bigsqcup_{v \in \mathcal{I}^0} \left(Z\left(\mu_{i_v}, \textstyle\sum_{i \in \mathcal{I}_v} m_i, \mu_{i_v}\right) \sqcup \bigsqcup_{i \in \mathcal{I}_v \setminus \{i_v\}} Z(\mu_i, 0, \mu_i) \right), \tag{C.5.2}$$

where the V_i, W_i's are compact bisections with disjoint source and range, so that $\pi_{\widehat{V_i}}, \pi_{\widehat{W_i}}$ are transpositions. Now if $I_{\mathcal{H},n}(\pi_U) = 0$, then each $\sum_{i \in \mathcal{I}_v} m_i = 0$, in which case (C.5.2) says that

$$\pi_{U_{\mathcal{H}}} \left(\pi_{\widehat{V_1}} \pi_{\widehat{W_1}} \cdots \pi_{\widehat{V_K}} \pi_{\widehat{W_K}} \right) = \mathrm{id}_{E_A^\infty}.$$

This shows that $\pi_{U_{\mathcal{H}}} \in \mathcal{T}(\mathcal{H}_{A,B,n})$ and hence $\pi_U = \pi_{U_{\mathcal{H}}} \pi_{U_A} \in \mathcal{T}(\mathcal{H}_{A,B,n})$ too. ◻

Proposition C.5.4. *The groupoid $\mathcal{H}_{A,B}$ has Property TR.*

Proof. Since $\mathcal{H}_{A,B} = \cup_{n=1}^\infty \mathcal{H}_{A,B,n}$ and ${\mathcal{H}_{A,B,n}}^{(0)} = \mathcal{H}_{A,B}^{(0)}$ is compact we have $[\![\mathcal{H}_{A,B}]\!] = \cup_{n=1}^\infty [\![\mathcal{H}_{A,B,n}]\!]$ as well. Suppose that $I_{\mathcal{H}}(\pi_U) = 0 \in H_1(\mathcal{H}_{A,B})$ for some $\pi_U \in [\![\mathcal{H}_{A,B}]\!]$. We have $\pi_U \in [\![\mathcal{H}_{A,B,n}]\!]$ for some n. By (C.4.1) we must have $I_{\mathcal{H},n'}(\pi_U) = 0$ for some $n' \geq n$. The result now follows from Lemma C.5.3. ◻

Remark C.5.5. Even though $\mathcal{H}_{A,B}$ is minimal and has Property TR, Proposition 4.5 in [Mat16] does not apply, because $\mathcal{H}_{A,B}$ is not purely infinite and generally not principal.

Recall the exact sequence (C.4.9) from the previous section, as it is going to be used in the proofs of the next two results. The following lemma is inspired by [Mat16, Lemma 4.7].

Lemma C.5.6. *Let $U \subseteq \mathcal{H}_{A,B}$ be a full bisection and view π_U as an element of $[\![\mathcal{G}_{A,B}]\!]$. If $I(\pi_U) = 0 \in H_1(\mathcal{G}_{A,B})$, then $\pi_U \in \mathcal{T}(\mathcal{G}_{A,B})$.*

Proof. Set $\alpha := \pi_U$. By Lemma C.4.6 we have $\Phi(I_{\mathcal{H}}(\alpha)) = I(\alpha) = 0$, so $I_{\mathcal{H}}(\alpha) \in \ker(\Phi) = \mathrm{im}(\rho^1)$. Let $[f] \in H_1(\mathcal{H}_{A,B})$ be such that $I_{\mathcal{H}}(\alpha) = \rho^1([f])$. By Proposition C.5.2 there is some $\beta \in [\![\mathcal{H}_{A,B}]\!]$ such that $I_{\mathcal{H}}(\beta) = [f]$.

Now $\beta = \pi_V$ for some full bisection $V = \sqcup_{i=1}^k Z(\mu_i, m_i, \nu_i) \subseteq \mathcal{H}_{A,B}$, where $E_A^\infty = \sqcup_{i=1}^k Z(\mu_i) = \sqcup_{i=1}^k Z(\nu_i)$ and $|\mu_i| = |\nu_i| = n$ for all i, for some $n \in \mathbb{N}$. Employing the same argument and notation as in the proof of Lemma C.5.3 we can find a product of transpositions $\beta_0 \in \mathcal{T}(\mathcal{H}_{A,B})$ such that $\beta\beta_0 = \pi_W$, where $W = (\sqcup_{v \in \mathcal{I}^0} Z(\mu_{i_v}, l_{i_v}, \mu_{i_v})) \sqcup A$ with $A \subseteq \mathcal{H}_{A,B}^{(0)}$ and $l_{i_v} \in \mathbb{Z}$. In particular, the paths μ_{i_v} all have different ranges.

For each $v \in \mathcal{I}^0$ pick an edge $e_v \in E_A^1$ with $r(e_v) = s(\mu_{i_v}) \neq s(e_v)$, so that e_v is not a loop. Then for each v, the path $e_v\mu_{i_v}$ is disjoint from μ_{i_v}. Since all the μ_{i_v}'s are mutually disjoint, so are all the $e_v\mu_{i_v}$'s too. A priori, it is not guaranteed that

μ_{i_v} is disjoint from $e_w\mu_{i_w}$ when $v \neq w \in \mathcal{I}^0$. However, this (i.e. that $\mu_{i_v} \nleq e_w\mu_{i_w}$) can be arranged if we at the start ensure that n is chosen large enough (which in turn can be done by (C.5.1)) so that $\left|E_A^{n-1}v\right| \geq 2N$ for each $v \in E_A^0$. This gives enough options when choosing the distinguished indices i_v to ensure that the total collection of paths $\cup_{v\in\mathcal{I}^0}\{\mu_{i_v}, e_v\mu_{i_v}\}$ are mutually disjoint (independent of the choice of the e_v's).

By the above paragraph we may define the compact bisection

$$T := \sqcup_{v\in\mathcal{I}^0} Z\left(e_v\mu_{i_v}, 0, \mu_{i_v}\right) \subseteq \mathcal{G}_{A,B},$$

which has disjoint source and range. Define $\tau_T := \pi_{\widehat{T}} \in \mathcal{T}(\mathcal{G}_{A,B})$. Observe that $\widehat{T}\cdot W\cdot\widehat{T} = \left(\sqcup_{v\in\mathcal{I}^0} Z\left(e_v\mu_{i_v}, l_{i_v}, e_v\mu_{i_v}\right)\right) \sqcup A'$ with $A' \subseteq \mathcal{H}_{A,B}^{(0)}$. Combining this with the description of ρ^1 from Lemma C.4.12 we see that

$$\begin{aligned}\rho^1\left(I_{\mathcal{H}}\left(\pi_W\right)\right) &= \rho^1\left([1_W]\right) = [1_W] - \left[1_{\widehat{T}\cdot W\cdot\widehat{T}}\right] \\ &= I_{\mathcal{H}}\left(\pi_W\right) - I_{\mathcal{H}}\left(\tau_T\pi_W\tau_T\right) = I_{\mathcal{H}}\left(\pi_W\tau_T\pi_W^{-1}\tau_T\right).\end{aligned} \tag{C.5.3}$$

At the same time we have

$$I_{\mathcal{H}}\left(\pi_W\right) = I_{\mathcal{H}}\left(\beta\right) = [f], \tag{C.5.4}$$

since $\pi_W = \beta\beta_0$ and $\beta_0 \in \mathcal{T}\left(\mathcal{H}_{A,B}\right)$. Next we observe that

$$W\cdot\widehat{T}\cdot W^{-1} = \bigsqcup_{v\in\mathcal{I}^0}\left(Z\left(e_v\mu_{i_v}, -l_{i_v}, \mu_{i_v}\right) \sqcup Z\left(\mu_{i_v}, l_{i_v}, e_v\mu_{i_v}\right)\right) \sqcup A'',$$

where $A'' \subseteq \mathcal{G}_{A,B}^{(0)}$. This actually shows that $\pi_W\tau_T\pi_W^{-1} \in \mathcal{T}\left(\mathcal{G}_{A,B}\right)$ since we have $W\cdot\widehat{T}\cdot W^{-1} = \widehat{R}$, where $R = \sqcup_{v\in\mathcal{I}^0} Z\left(\mu_{i_v}, l_{i_v}, e_v\mu_{i_v}\right)$. Define the element $\gamma := \pi_W\tau_T\pi_W^{-1}\tau_T \in \mathcal{T}\left(\mathcal{G}_{A,B}\right)$. Equations (C.5.3) and (C.5.4) now says that

$$I_{\mathcal{H}}\left(\gamma\right) = \rho^1\left(I_{\mathcal{H}}\left(\pi_W\right)\right) = \rho^1\left([f]\right) = I_{\mathcal{H}}\left(\alpha\right).$$

This means that $I_{\mathcal{H}}\left(\alpha\gamma^{-1}\right) = 0 \in H_1\left(\mathcal{H}_{A,B}\right)$, and hence $\alpha\gamma^{-1} \in \mathcal{T}\left(\mathcal{H}_{A,B}\right)$ by Proposition C.5.4. Then $\alpha \in \mathcal{T}\left(\mathcal{G}_{A,B}\right)$ and we are done. □

Theorem C.5.7. *The Katsura–Exel–Pardo groupoid $\mathcal{G}_{A,B}$ has Property TR.*

Proof. Let $U \subseteq \mathcal{G}_{A,B}$ be a full bisection. Then $U = \sqcup_{i=1}^k Z(\mu_i, m_i, \nu_i)$, where $E_A^\infty = \sqcup_{i=1}^k Z(\mu_i) = \sqcup_{i=1}^k Z(\nu_i)$ (but the paths μ_i and ν_i may now have different lengths). As in the proof of Lemma C.5.3 we define the full bisections

$$U_{\mathcal{H}} := \sqcup_{i=1}^k Z(\mu_i, m_i, \mu_i) \subseteq \mathcal{H}_{A,B} \quad \text{and} \quad U_A := \sqcup_{i=1}^k Z(\mu_i, 0, \nu_i) \subseteq \mathcal{G}_{A,0},$$

where we view both $\mathcal{H}_{A,B}$ and $\mathcal{G}_{A,0}$ as subgroupoids of $\mathcal{G}_{A,B}$. Recall that we have $U_{\mathcal{H}} \cdot U_A = U$ and $\pi_U = \pi_{U_{\mathcal{H}}} \pi_{U_A} \in [\![\mathcal{G}_{A,B}]\!]$. We will be considering all three index maps:

$$\begin{aligned} I \colon [\![\mathcal{G}_{A,B}]\!] &\to H_1(\mathcal{G}_{A,B}), \\ I_{\mathcal{H}} \colon [\![\mathcal{H}_{A,B}]\!] &\to H_1(\mathcal{H}_{A,B}), \\ I_A \colon [\![\mathcal{G}_{A,0}]\!] &\to H_1(\mathcal{G}_{A,0}). \end{aligned}$$

We have $I(\pi_U) = I(\pi_{U_{\mathcal{H}}}) + I(\pi_{U_A}) \in H_1(\mathcal{G}_{A,B})$, but by viewing $\pi_{U_{\mathcal{H}}} \in [\![\mathcal{H}_{A,B}]\!]$ and $\pi_{U_A} \in [\![\mathcal{G}_{A,0}]\!]$ we may also consider $I_{\mathcal{H}}(\pi_{U_{\mathcal{H}}})$ and $I_A(\pi_{U_A})$ as elements of $H_1(\mathcal{H}_{A,B})$ and $H_1(\mathcal{G}_{A,0})$, respectively. The idea is to show that if $I(\pi_U) = 0$, then both $I(\pi_{U_{\mathcal{H}}})$ and $I_A(\pi_{U_A})$ vanish as well. At this point we may appeal to Lemma C.5.6 and Property TR for $\mathcal{G}_{A,0} \cong \mathcal{G}_A$, respectively, to conclude that π_U itself must be a product of transpositions.

Assume now that $I(\pi_U) = 0 \in H_1(\mathcal{G}_{A,B})$. By Lemma C.4.6 and the exactness of (C.4.9) have

$$\Psi(I(\pi_{U_{\mathcal{H}}})) = \Psi(\Phi(I_{\mathcal{H}}(\pi_{U_{\mathcal{H}}}))) = 0.$$

This means that $\Psi(I(\pi_{U_A})) = \Psi(I(\pi_U)) = 0$. From Lemma C.4.11 we conclude that $I_A(\pi_{U_A}) = 0 \in H_1(\mathcal{G}_{A,0})$. Hence $\pi_{U_A} \in \mathcal{T}(\mathcal{G}_{A,0}) \subseteq \mathcal{T}(\mathcal{G}_{A,B})$ by appealing to Property TR for SFT-groupoids [Mat15b]. It follows that $I(\pi_{U_A}) = 0 \in H_1(\mathcal{G}_{A,B})$ too, and then $I(\pi_{U_{\mathcal{H}}}) = I(\pi_U) = 0 \in H_1(\mathcal{G}_{A,B})$. By Lemma C.5.6 we then get $\pi_{U_{\mathcal{H}}} \in \mathcal{T}(\mathcal{G}_{A,B})$ as well. This finishes the proof, since $\pi_U = \pi_{U_{\mathcal{H}}} \pi_{U_A}$. □

Corollary C.5.8. *The AH conjecture holds for the Katsura–Exel–Pardo groupoid $\mathcal{G}_{A,B}$ whenever the matrices A, B satisfy the AH criteria and A is irreducible.*

Proof. Since $\mathcal{G}_{A,B}$ has Property TR (Theorem C.5.7) and is purely infinite (Proposition C.3.11) and minimal, the result follows from [Mat16, Theorem 4.4]. □

Remark C.5.9. To get rid of the assumption of A being essential, i.e. allowing for sources in E_A, we need to prove Property TR for restrictions, as is done for graph groupoids in Paper B. This should be doable.

C.6 Finite generation of $[\![\mathcal{G}_{A,B}]\!]$

In this section we will show that the topological full group $[\![\mathcal{G}_{A,B}]\!]$ is finitely generated, under the following hypotheses on A and B.

Assumption C.6.1. *In this whole section we fix $N \times N$ matrices A, B which satisfy the AH criteria and where A is essential. In particular $N < \infty$ and A is an irreducible non-permutation matrix. Furthermore, we assume that $|B_{i,j}| < A_{i,j}$ whenever $A_{i,j} \neq 0$.*

In [Nek18a, Definition 5.1], Nekrashevych defined the notion of a self-similar group being *contracting*. He showed that for a contracting self-similar group, the topological full group of the associated groupoid of germs is finitely presented. We will extend Nekrashevych's definition to cover the self-similar graphs of Exel and Pardo, and show that the self similar graph associated to matrices A and B as above is contracting. However, we will settle for showing that $[\![\mathcal{G}_{A,B}]\!]$ is finitely generated. A crucial ingredient in our argument is the fact that the topological full group $[\![\mathcal{G}_{A,0}]\!]$ is finitely generated [Mat15b].

Definition C.6.2. Let (G, E, φ) be a self-similar graph as in [EP17, Section 2]. We say that (G, E, φ) is *contracting* if there exists a finite subset $\mathcal{N} \subset G$ with the property that for every $g \in G$ there is some $n \in \mathbb{N}$ such that $\varphi(g, \mu) \in \mathcal{N}$ for all $\mu \in E^{\geq n}$.

The following rudimentary lemma will be used when showing that the self-similar graph from Section C.3 is contracting.

Lemma C.6.3. *Assume that $a, b, m, t \in \mathbb{Z}$ are integers satisfying $a \geq 1$ and*

$$(b-a)m - a < at < (b-a)m + a,$$
$$1 - 2a \leq b - a \leq -1.$$

Then

$$|m+t| < |m| \quad \textit{when } |m| \geq 2a,$$
$$|m+t| \leq |m| \quad \textit{when } |m| < 2a.$$

Proof. Assume first that $m \geq 0$. Then

$$(b-a)m - a \geq (1-2a)m - a = m - a - 2ma$$

and

$$(b-a)m + a \leq (-1)m + a = a - m,$$

so

$$m - a - 2ma < at < a - m.$$

Now if $m \geq 2a$, then

$$a - m \leq -a \quad \text{and} \quad m - a - 2ma \geq a - 2ma = (1-2m)a.$$

We infer from this that

$$1 - 2m < t < -1 \implies -2m < t < 0 \implies |m+t| < |m|.$$

Next, if $0 \leq m < a$, then

$$a - m \leq a \quad \text{and} \quad m - a - 2ma \geq -a - 2ma = (-1 - 2m)a.$$

From this we get

$$-1 - 2m < t < 1 \implies -2m \leq t \leq 0 \implies |m + t| \leq |m|.$$

The case $m < 0$ proceeds in essentially the same way. ◻

Lemma C.6.4. *Let $e = e_{i,j,n} \in E_A^1$ and $m \in \mathbb{Z}$ be given. Then we have*

$$|\varphi(m, e)| < |m| \quad \textit{when} \ |m| \geq 2A_{i,j},$$
$$|\varphi(m, e)| \leq |m| \quad \textit{when} \ |m| < 2A_{i,j}.$$

Proof. We have that $A_{i,j} \geq 1$ and that $0 \leq n < A_{i,j}$. Let t and r be the unique integers satisfying

$$mB_{i,j} + n = (m + t)A_{i,j} + r \quad \text{and} \quad 0 \leq r < A_{i,j}.$$

Recall that then $\varphi(m, e) = m + t$. We now have

$$\left(B_{i,j} - A_{i,j}\right) m + n - r = A_{i,j} t$$

where $-A_{i,j} < n - r < A_{i,j}$. From this we see that

$$\left(B_{i,j} - A_{i,j}\right) m - A_{i,j} < A_{i,j} t < \left(B_{i,j} - A_{i,j}\right) m + A_{i,j}.$$

We also have

$$1 - 2A_{i,j} \leq B_{i,j} - A_{i,j} \leq -1,$$

since $\left|B_{i,j}\right| < A_{i,j}$. We are now in the setting of Lemma C.6.3 and so the result follows. ◻

Proposition C.6.5. *The self-similar graph $(\mathbb{Z}, E_A, \varphi)$ associated to the matrices A and B is contracting.*

Proof. Define $R := 2 \cdot \max\{A_{i,j} \mid 1 \leq i, j \leq N\}$. Let $m \in \mathbb{Z}$ be given. Combining Lemma C.6.4 with Equation (C.3.2) we see that $|\varphi(m, \mu)| \leq R$ whenever $\mu \in E_A^{\geq |m|}$. So by setting $\mathcal{N} = [-R, R] \cap \mathbb{Z}$ and $n = |m|$ in Definition C.6.2 we find that $(\mathbb{Z}, E_A, \varphi)$ is contracting. ◻

Before establishing the finite generation of $[\![\mathcal{G}_{A,B}]\!]$ we introduce some notation. Given $\gamma \in E_A^*$ and $m \in \mathbb{Z}$ we denote the full bisection $Z(\gamma, m, \gamma) \sqcup \left(\mathcal{G}_{A,B}^{(0)} \setminus Z(\gamma)\right)$ by $U_{\gamma,m}$. Given two disjoint paths $\mu, \gamma \in E_A^*$ with $r(\mu) = r(\gamma)$ we define the transposition $\tau_{\mu,\gamma} := \pi_{\widehat{V}} \in [\![\mathcal{G}_{A,0}]\!]$, where $V = Z(\mu, 0, \gamma)$. Observe that we have

$$\tau_{\mu,\gamma} \circ \pi_{U_{\gamma,m}} \circ \tau_{\mu,\gamma} = \pi_{U_{\mu,m}}.$$

Theorem C.6.6. *Let A, B be matrices satisfying the AH criteria with A irreducible. Assume that $\left|B_{i,j}\right| < A_{i,j}$ whenever $A_{i,j} \neq 0$. Then the topological full group $[\![\mathcal{G}_{A,B}]\!]$ is finitely generated.*

Proof. First pick $n \in \mathbb{N}$ large enough so that $E_A^n v \geq 2$ for each $v \in E_A^0$. As above, set $R := 2 \cdot \max\{A_{i,j} \mid 1 \leq i, j \leq N\}$. Let S be a finite generating set for $[\![\mathcal{G}_{A,0}]\!]$ ([Mat15b, Theorem 6.21]) and define the finite set

$$T := \left\{\pi_{U_{\gamma,m}} \mid \gamma \in E_A^n \ \& \ -R \leq m \leq R\right\}.$$

We claim that $S \cup T$ generates $[\![\mathcal{G}_{A,B}]\!]$.

To prove the claim let $\pi_U \in \mathcal{G}_{A,B}$ be given. Write $U = \sqcup_{i=1}^k Z(\mu_i, m_i, \nu_i)$. By applying the splitting in Equation (C.5.1) enough times to each basic bisection in U we may assume without loss of generality that $|\mu_i| \geq n$ for each i. Similarly, by Proposition C.6.5 we may assume that $|m_i| \leq R$. As we have done a few times already we split the full bisection U into the two full bisections

$$U_{\mathcal{H}} := \sqcup_{i=1}^k Z(\mu_i, m_i, \mu_i) \quad \text{and} \quad U_A := \sqcup_{i=1}^k Z(\mu_i, 0, \nu_i),$$

making $\pi_U = \pi_{U_{\mathcal{H}}} \pi_{U_A}$. Since $\pi_{U_A} \in [\![\mathcal{G}_{A,0}]\!]$ and $\pi_{U_{\mathcal{H}}} = \Pi_{i=1}^k \pi_{U_{\mu_i,m_i}}$ it suffices to consider each $\pi_{U_{\mu_i,m_i}}$. By the assumption on n we can for each i find a path $\gamma_i \in E_A^n r(\mu_i)$ which is disjoint from μ_i. The equation

$$\pi_{U_{\mu_i,m_i}} = \tau_{\mu_i,\gamma_i} \circ \pi_{U_{\gamma_i,m_i}} \circ \tau_{\mu_i,\gamma_i}$$

then proves the claim, since $\tau_{\mu_i,\gamma_i} \in [\![\mathcal{G}_{A,0}]\!]$ and $\pi_{U_{\gamma_i,m_i}} \in T$. □

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