Constructing Transverse Coordinates for Orbital Stabilization of Periodic Trajectories

Maksim Surov[1], Sergei Gusev[2], Leonid Freidovich[3]

***Abstract*— An approach for introduction of transverse coordinates in a vicinity of a periodic trajectory is presented. The approach allows finding by numerical integration periodic normalized mutually-orthogonal vector-functions that form a continuously differentiable basis on moving Poincaré sections for a given periodic solution of a nonlinear dynamical system.**

The found moving frame is used to define new local (transverse) coordinates for an associated affine nonlinear control system in a neighborhood of the trajectory, and to proceed with orbital stability analysis and/or synthesis of a stabilizing feedback control law.

As a demonstrating example of the approach, the problem of orbital stabilization of a trajectory of a multibody car system is considered. The results of computer simulations of the system are presented.

I. Introduction

The transverse linearization approach is widely used in the problem of orbital stabilization [1], [2], [3], [4], [5] of periodic (and not only) trajectories of dynamical systems. In particular, the approach has been well-studied in application to Euler-Lagrange systems [6], [7], [8], while a constructive algorithm for defining transverse coordinates, and synthesis of feedback controllers was suggested in [6]. The algorithm was successfully used, for example, in solving the problem of orbital stabilization of periodic trajectories of a Pendubot [3], a "Butterfly" robot [4], etc. However, it was shown in [9] that the approach in some cases leads to singularities in transverse dynamics, and, hence, cannot be used. There, a distinct method for defining transverse coordinates for periodic trajectories of nonlinear dynamical systems of dimensions 2, 4, and 8 was suggested.

This paper proposes a new method for defining transverse coordinates that works for affine nonlinear systems of arbitrary dimension and is based on solving a well-known differential equation for a rotation matrix. In addition, a linearization of dynamics of the transverse coordinates is presented generalizing a result by Urabe [10] and giving a constructive alternative not only to analysis of orbital stability, that can be performed using the elegant Leonov's formula, see e.g. [11], [12], but also to design orbitally stabilizing feedback controllers.

[1]Maksim Surov is with the Department of Engineering Cybernetics, Norwegian University of Science and Technology (NTNU), Trondheim, Norway. E-mails: maksim.surov@ntnu.no, anton.shiriaev@ntnu.no

[2]Sergei Gusev is with the Faculty of Mathematics and Mechanics, St. Petersburg State University, Russia. E-mail: gusev@ieee.org

[3]Leonid Freidovich, Department of Applied Physics and Electronics, Umeå University, Umeå, Sweden. E-mail: E-mail:leonid.freidovich@umu.se

This paper uses the following notation. The value of a function f at argument x is usually written with the help of subscript index: $f_x \equiv f(x)$. The time derivative is usually denoted by $\dot{}$ symbol: $\dot{x} \equiv \frac{dx}{dt}$. The norm of a vector $v \in \mathbb{R}^n$ is written as $\|v\|$ and means the standard Euclidean norm: $\|v\| \equiv \sqrt{\sum_{i=1}^n |v_i|^2}$. The distance operator $\mathbf{dist}(a,b) : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ is induced by the norm operator: $\mathbf{dist}(a,b) \equiv \|a-b\|$. The operator $\mathbf{col}_i A$ returns i-th column of the matrix A.

A. Problem formulation

The paper studies the problem of *feedback stabilization* of non-trivial periodic solutions of dynamical systems of the form

$$\dot{x} = f(x) + g(x)\, u \tag{1}$$

with state space variable $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, functions $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, and $g \in C^1(\mathbb{R}^n, \mathbb{R}^{n\times m})$. As mentioned above, particular attention is paid to the *orbital stability* property. For completeness, let us recall its definition, see e.g. [13, Definition 8.2 on p.333].

Definition 1. *Let $\gamma_\tau \in C^2(\mathbb{R}, \mathbb{R}^n)$ be a T-periodic function (with $T > 0$) satisfying*

$$\frac{d}{d\tau}\gamma_\tau = f(\gamma_\tau) \quad \forall \tau \in \mathbb{R}. \tag{2}$$

The set

$$\gamma := \{x \in \mathbb{R}^n \mid \exists t \in \mathbb{R} : x = \gamma_t\}$$

is called the orbit *of the solution $x = \gamma_\tau$.*

The solution $x = \gamma_t$ of the open-loop system (2) or/and of a closed-loop system with a time-invariant feedback control law, vanishing along it, is asymptotically orbitally stable *within ϵ-neighborhood*

$$U_\epsilon := \{x \in \mathbb{R}^n \mid \mathbf{dist}(x,\gamma) < \epsilon\}$$

of γ, if for every $0 < \bar\epsilon \le \epsilon$ there exists $\bar\delta > 0$ such that every solution of the system, initiated inside $U_{\bar\delta}$, stays inside $U_{\bar\epsilon}$; while, every solution originated inside U_ϵ asymptotically converges to γ, i.e.

$$\mathbf{dist}\big(x(t), \gamma\big) \to 0 \quad as \quad t \to \infty.$$

Above, the distance *between the set γ and a point x is defined as*

$$\mathbf{dist}(x,\gamma) := \min_{\tau\in\mathbb{R}} \|x - \gamma_\tau\|.$$

The orbital stabilization problem is to define a (time-independent) feedback control law, achieving asymptotic orbital stability.

Remark 1. *To simplify various expressions, it is assumed here that $x = \gamma_t$ is a (forward complete) bounded open-loop solution of (1), i.e. with $u \equiv 0$, and, moreover, $\forall t \geq 0$: $\gamma_t = \gamma_{t+T}$. It is possible to handle similarly the case when $x = \gamma_t$ is a solution of (1) with some $T-$periodic $u(t) \not\equiv 0$. In such a case, one may either introduce a time-invariant feedback control transformation as in [7] or follow the idea presented in the illustrative multibody car example below. However, we proceed here without such a rather straightforward generalization.*

The solution presented below is a variation of the so-called *transverse linearization* approach for control systems, see e.g. [14], [15]. The core idea of the approach is to define, in a neighborhood of the orbit γ, new local coordinates $(\tau, \xi) \simeq x$, $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^{n-1}$, called the *transverse coordinates* such that

- The scalar coordinate $\tau = \pi(x)$ defines the "closest" to the given x point on the orbit and $\pi(\gamma_t) = t \mod T$.
- The $n-1$ coordinates $\xi = \Xi(x)$ locate points in "perpendicular" (or transverse) to the trajectory direction. They vanish at any point on the orbit: $\Xi(\gamma_t) = 0, \ \forall t$.

There is an equivalent to (1) system

$$\frac{d(\tau, \xi)}{dt} = \alpha(\tau, \xi) + \beta(\tau, \xi)\, u, \tag{3}$$

written in new coordinates. The linearization of system (3) allows defining a linear time-varying system

$$\frac{d\bar{\xi}}{d\tau} = A_\tau \bar{\xi} + B_\tau \bar{u} \tag{4}$$

with coefficients $A_\tau \in C^1\left(\mathbb{R}, \mathbb{R}^{(n-1)\times(n-1)}\right)$ and $B_\tau \in C^1\left(\mathbb{R}, \mathbb{R}^{(n-1)\times m}\right)$ that are periodic functions of time. The problem of asymptotic orbital stabilization of γ_τ can be solved using a stabilizing feedback control law for the trivial solution $\bar{\xi} = 0$ of (4) and exploiting the fact that under certain conditions, see e.g. [11], $\xi(t) = \bar{\xi}(\tau)\big|_{\tau=\pi(x(t))} + o\big(d(x(t), \gamma)\big)$. The last problem is well-studied and has general solutions (see for example [16], [17]).

Our main contribution here is an algorithm for constructing functions $\tau = \pi(x)$ and $\xi = \Xi(x)$ for a given periodic trajectory γ_τ, and deriving formulas for linearized transverse dynamics written in these coordinates.

B. Transverse coordinates

To define transverse coordinates, let us make preliminary geometrical constructions. For each $\tau \in [0, T)$ let us define a hyperplane $\mathcal{S}_\tau$ that transversally intersects the orbit γ at point γ_τ. Let the vectors $e_\tau^i \in \mathbb{R}^n, i = 1..n-1$ be an orthonormal basis of the hyperplane $\mathcal{S}_\tau$. Then, any x belong to $\mathcal{S}_\tau \cap U_\epsilon$ can be decomposed as

$$x = \gamma_\tau + \sum_{i=1}^{n-1} e_\tau^i \xi_i, \tag{5}$$

where the decomposition coefficients $\xi \in \mathbb{R}^{n-1}$, and the curve parameter τ play the role of transverse coordinates (the sufficient conditions, the mapping $(\tau, \xi) \mapsto x$ is a diffeomorphism will be given below).

It is convenient in practice to define the hyperplane $\mathcal{S}_\tau$ in implicit form

$$\mathcal{S}_\tau = \left\{ x \in U_\epsilon | (x - \gamma_\tau)^T P \frac{d\gamma_\tau}{d\tau} = 0 \right\}$$

with the help of a square matrix $P \in \mathbb{R}^{n \times n}$, such that $\frac{d\gamma_\tau}{d\tau}$ does not belong to nullspace of $P\frac{d\gamma_\tau}{d\tau}$. For example, if the hyperplane is taken to be orthogonal to the orbit at each point, then P will be the identity matrix. Such a type of section was used in the problem of orbital stabilization of the cart-pole system [9].

In some cases, it makes sense to construct the sections $\mathcal{S}_\tau$ in such a way that the map $\tau = \pi(x)$ (solution of the eqution $(x - \gamma_\tau)^T P \frac{d\gamma_\tau}{d\tau} = 0$) does not depend on some components of x. This can be helpful, if these componets are not measured well. In this case, the matrix P will be diagonal with zeros at the corresponding positions. In the problem of orbital stabilization of the "Butterfly" robot [4], the matrix P was taken as $P = \mathbf{diag}(0, 1, 0, 0)$.

Generalizing these, let us assume that P is a constant matrix, symmetric, and satisfying $\frac{d\gamma_\tau^T}{d\tau} P \frac{d\gamma_\tau}{d\tau} \neq 0 \, \forall \tau \in [0, T)$.

To be able to represent dynamics (1) in an equivalent form in coordinates τ, ξ, it is necessary for the transformation (5) to be a diffeomorphism within a neighborhood, so that

- the inverse transformation $\tau = \pi(x)$, $\xi = \Xi(x)$ exists;
- the Jacobian matrix of the transformation must be non-singular.

These conditions mean that vector functions e_τ^i must be periodic with period T, and the vectors $e_\tau^i, \frac{d\gamma_\tau}{d\tau}$ must be linearly independent for all τ.

In the most simple case when the phase space of a dynamical system is $\mathbb{R}^2$, and the space $\mathcal{S}_\tau$ is the orthogonal complement of $\frac{d\gamma_\tau}{d\tau}$, the basis of $\mathcal{S}_\tau$ is given by the vector

$$e_\tau^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\frac{d\gamma_\tau}{d\tau}}{\left\| \frac{d\gamma_\tau}{d\tau} \right\|}.$$

defining a normalized rotation of $f(\gamma_\tau)$ by $\pi/2$.

A similar approach for defining basis vectors $e_\tau^i = Q_i \frac{\frac{d\gamma_\tau}{d\tau}}{\left\| \frac{d\gamma_\tau}{d\tau} \right\|}$ with some constant matrices $Q_i \in \mathbb{R}^{n \times n}, i = 1..n-1$ works also in $\mathbb{R}^4$ and $\mathbb{R}^8$ (see [9] for details).

In the differential geometry of curves the basis of moving frame of a curve is usually given by the Frenet–Serret equations [18]. The basis vectors depend on derivatives of γ_τ up to $(n-1)$-order. Practically, the numerical estimation of derivatives of a function is an ill-posed problem [19] and may lead to computational difficulties. At the same time, a disadvantage of this approach is that the first $n-1$ derivatives of γ_τ must be linearly independent. This means, for example, that the approach cannot be used for trajectories having zero curvature at some points.

In [10, Theorem 5.1] an approach for constructing periodic basis functions for systems without control inputs is presented. This approach uses the fact that in phase space of dimension $n > 2$ there exists a constant vector e_1 such that tangent vectors of the trajectory never coincide with it (see Lemma 5.1 there).

One more approach for constructing basis functions e_τ^i for periodic trajectories is presented below. The method is based on the fact, that a smooth matrix function $R_\tau : \mathbb{R} \to O(n)$, columns of which are vectors $\frac{d\gamma_\tau}{d\tau} / \left\| \frac{d\gamma_\tau}{d\tau} \right\|, e_\tau^1, ..., e_\tau^{n-1}$, can be found solving (numerically) a linear matrix differential equation. The found solution R_τ is further modified to satisfy periodicity criteria.

In the next section the approach for constructing functions e_τ^i is formulated in detail. Moreover, transverse dynamics written in the associated new coordinates, and its linearization, are presented.

II. Transverse dynamics

Let us consider the coordinate transformation (5) in ϵ-neighborhood U_ϵ of orbit γ. The trajectory γ_τ is considered as a twice continuously differentiable closed curve: $\gamma_\tau \in C^2(\mathbb{R}, \mathbb{R}^n)$, $\gamma_{\tau+T} = \gamma_\tau \ \forall \tau \in \mathbb{R}$. Since the curve γ_τ is a periodic solution of an ordinary differential equation, it does not have self-intersections, and equilibrium points. Therefore, $\frac{d\gamma_\tau}{d\tau} = f(\gamma_\tau) \neq 0$.

The inverse transformation $x \mapsto (\tau, \xi)$ within U_ϵ is defined with the help of the projection operator

$$\tau = \pi(x) \in [0, T), \text{ satisfying} \qquad (6)$$
$$\left(x - \gamma_{\tau=\pi(x)}\right)^T P \left(\frac{d\gamma_\tau}{d\tau}\right)_{\tau=\pi(x)} = 0,$$

and $\xi = \Xi(x)$ as the solution of the linear algebraic equation

$$\sum_i e_{\pi(x)}^i \xi_i = x - \gamma_{\pi(x)}.$$

As it was stated above, the matrix $P \in \mathbb{R}^{n\times n}$ is assumed to be constant, symmetric, and satisfying $\frac{d\gamma_\tau^T}{d\tau} P \frac{d\gamma_\tau}{d\tau} \neq 0 \ \forall \tau \in \mathbb{R}$. It is also assumed that ϵ is small enough so that the equation

$$(x - \gamma_\tau)^T P \frac{d\gamma_\tau}{d\tau} = 0 \quad \forall x \in U_\epsilon$$

has a unique solution $\tau \in [0, T)$. Then, the projection operator $\pi(x)$ within U_ϵ is a continuously differentiable function. Moreover, the following claim holds.

Claim 1. *Let $\left\{e_\tau^i\right\}_{i=1}^{n-1}$ be mutually-orthogonal unit vectors, and form a basis of the hyperplane $\mathcal{S}_\tau$. Let also the vectors e_τ^i be continuously differentiable T-periodic functions. Then, there exists an $\epsilon > 0$, so that for any $x \in U_\epsilon$ the coordinate transformation (5) is a diffeomorphism.*

The question of constructing vectors e_τ^i satisfying criteria of Claim 1 is considered next.

A. Constructing a Moving Orthonormal Basis

The problem of constructing vectors e_τ^i can be formulated as following.

Problem 1. *It is necessary to find the functions $e_\tau^i \in C^1(\mathbb{R}, \mathbb{R}^n)$, $i = 1, \dots, n-1$ satisfying for all $\tau \in \mathbb{R}$:*

- *orthonormality criteria: $\left(e_\tau^i\right)^T e_\tau^j = \delta_{ij}$;*
- *periodicity criteria: $e_{\tau+T}^i = e_\tau^i$;*
- *orthogonality criteria: $\left(P \frac{d\gamma_\tau}{d\tau}\right)^T e_\tau^i = 0$,*

where δ_{ij} is the Kronecker delta.

According to the requirements of Problem 1, the vectors $e_\tau^1, e_\tau^2, ..., e_\tau^{n-1}$ together with

$$v_\tau := \frac{P \frac{d\gamma_\tau}{d\tau}}{\left\| P \frac{d\gamma_\tau}{d\tau} \right\|} = \frac{P f(\gamma_\tau)}{\|P f(\gamma_\tau)\|} \qquad (7)$$

form an orthonormal basis (note that the vector $P \frac{d\gamma_\tau}{d\tau} \neq 0$). Then, the matrix function

$$R_\tau = \left(v_\tau, e_\tau^1, e_\tau^2, ..., e_\tau^{n-1}\right)$$

is a map $\mathbb{R} \to O(n)$. Moreover, without loss of generality[1] let us assume that $R_\tau \in SO(n)$. Then, a solution of Problem 1 can be found using (7) with a solution of the next problem.

Problem 2. *For a given T-periodic vector function $v_\tau : \mathbb{R} \to \{v \in \mathbb{R}^n \mid \|v\| = 1\}$ find a matrix function $R_\tau \in C^1(\mathbb{R}, SO(n))$, satisfying for all $\tau \in \mathbb{R}$:*

- $R_{\tau+T} = R_\tau$,
- $\mathbf{col}_1 R_\tau = v_\tau$.

By the first step let us find matrix $\bar{R}_\tau$ satisfying requirements of Problem 2 except the periodicity.

Claim 2. *A solution $\bar{R}_\tau$ of the initial value problem*

$$\begin{aligned} \frac{d\bar{R}_\tau}{d\tau} &= \left(\frac{dv_\tau}{d\tau} \wedge v_\tau\right) \bar{R}_\tau \\ \bar{R}_0 &\in \{R \in SO(n) \,|\, \mathbf{col}_1 R = v_0\} \end{aligned} \qquad (8)$$

satisfies: $\bar{R}_\tau \in C^1(\mathbb{R}, SO(n))$, $\mathbf{col}_1 \bar{R}_\tau = v_\tau$ for all $\tau \in \mathbb{R}$. Here $a \wedge b$ is the exterior product, in matrix form it can be written as $a \wedge b \equiv ab^T - ba^T$.

Above, construction of $\bar{R}_0$ can be done following the standard Gram-Schmidt orthonormalization for an arbitrary basis containing v_0. This also can be done for every fixed value of τ but the issue here is to have a reliable algorithm to generate a continuum for $\tau \in [0, T)$ as well as to ensure periodicity, continuity, and differentiability.

However, solution $\bar{R}_\tau$ of the initial value problem (8) for an arbitrary function v_τ is not necessarily a periodic function. The following claim shows how to modify the function $\bar{R}_\tau$ so that it satisfies the periodicity criteria and provides a solution for Problem 2.

Claim 3. *Let us assume that the matrix $\bar{R}_T^T \bar{R}_0$ has no eigenvalues equal to minus one[2]. Then the matrix function*

$$R_\tau = \bar{R}_\tau \exp\left\{\frac{\tau}{T} \log \bar{R}_T^T \bar{R}_0\right\}, \qquad (9)$$

[1] It is always possible to ensure that the determinant of matrix R_τ is equal to one by changing the sign of one of the columns of matrix R_τ.

[2] Otherwise the function R_τ can be defined, for example, as $R_\tau = \bar{R}_\tau \exp\left\{\frac{2\tau}{T} \log X\right\}$, where X is a suitable solution of $X^2 = \bar{R}_T^T \bar{R}_0$.

where $\exp\{\cdot\}$ *is the matrix exponential, and* $\log\{\cdot\}$ *is the matrix logarithm (see definition of these functions, e.g., in [21]), satisfies:* $R_\tau \in C^1(\mathbb{R}, SO(n))$, $\mathbf{col}_1 R_\tau = v_\tau$, $R_{\tau+T} = R_\tau$, $\frac{dR_\tau}{d\tau}|_{\tau+T} = \frac{dR_\tau}{d\tau}|_\tau$ *for all* $\tau \in \mathbb{R}$. *Hence, the vectors*

$$e_\tau^i = \mathbf{col}_{i+1} R_\tau$$

form a solution of Problem 1.

B. Dynamics in Transverse Coordinates

Claim 1 allows to represent dynamics (1) in coordinates ξ, τ. For brevity let us denote

$$E_\tau := \left(e_\tau^1, e_\tau^2, ..., e_\tau^{n-1}\right) \in \mathbb{R}^{n \times n-1}.$$

Since the vectors e_τ^i are mutually orthogonal, we have $E_\tau^T E_\tau = I_{n-1\times n-1}$. Then, the coordinate transformation (5) can be written in the compact form as

$$x = \gamma_\tau + E_\tau \xi, \tag{10}$$

and its inverse is

$$\begin{aligned} \tau &= \pi(x), \qquad \text{defined in (6)} \\ \xi &= \Xi(x) = \left[E_\tau^T(x-\gamma_\tau)\right]_{\tau=\pi(x)}. \end{aligned}$$

Differentiating the expression for ξ, computed along a solution $x(t)$ with respect to t, and substituting (1), one obtains dynamics for ξ:

$$\begin{aligned} \dot{\xi} &= E_\tau^T\left(f(\gamma_\tau + E_\tau\xi) - f(\gamma_\tau)\dot{\tau}\right) \\ &\quad + \frac{dE_\tau^T}{d\tau}E_\tau\xi\dot{\tau} + E_\tau^T g(\gamma_\tau + E_\tau\xi)u. \end{aligned} \tag{11}$$

The time derivative $\dot{\tau}$ can be found as

$$\dot{\tau} = \left(\frac{\partial\pi(x)}{\partial x}(f(x) + g(x)u)\right)_{x=\gamma_\tau + E_\tau\xi} \tag{12}$$

The equations (11),(12) form the equivalent dynamics of (1) within U_ϵ in coordinates ξ, τ. The linear approximation of this dynamics is given next.

Theorem 1. *Linearization of dynamics (11), (12) around trajectory* γ_τ *is*

$$\begin{aligned} \dot{\bar{\xi}} &= A_{\bar{\tau}}\bar{\xi} + B_{\bar{\tau}}\bar{u} &(13)\\ \dot{\bar{\tau}} &= 1 + C_{\bar{\tau}}\bar{\xi} + D_{\bar{\tau}}\bar{u} &(14) \end{aligned}$$

with

$$A_\tau = E_\tau^T \frac{f_{\gamma_\tau}^T P f_{\gamma_\tau} J_\tau - f_{\gamma_\tau} f_{\gamma_\tau}^T P J_\tau - f_{\gamma_\tau} f_{\gamma_\tau}^T J_\tau^T P}{f_{\gamma_\tau}^T P f_{\gamma_\tau}} E_\tau +$$

$$\frac{dE_\tau^T}{d\tau}E_\tau, \quad B_\tau = E_\tau^T\left(I - \frac{f_{\gamma_\tau} f_{\gamma_\tau}^T P}{f_{\gamma_\tau}^T P f_{\gamma_\tau}}\right) g_{\gamma_\tau},$$

$$C_\tau = \frac{f_{\gamma_\tau}^T\left[PJ_\tau + J_\tau^T P\right]E_\tau}{f_{\gamma_\tau}^T P f_{\gamma_\tau}}, \quad D_\tau = \frac{f_{\gamma_\tau}^T P g_{\gamma_\tau}}{f_{\gamma_\tau}^T P f_{\gamma_\tau}},$$

where $J_\tau := \left(\frac{\partial f}{\partial x}\right)_{\gamma_\tau}$, $f_{\gamma_\tau} := (f \circ \gamma)(\tau)$, $g_{\gamma_\tau} := (g \circ \gamma)(\tau)$.

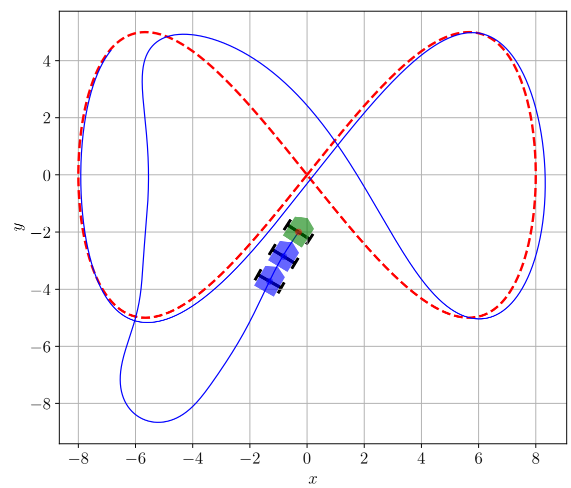

Fig. 1. Car with two trailers: reverse drive.

Remark 2. *In the case* $P = I$ *the expressions for matrices* A_τ, B_τ *can be simplified:*

$$\begin{aligned} A_\tau &= \frac{dE_\tau^T}{d\tau}E_\tau + E_\tau^T J_\tau E_\tau \\ B_\tau &= E_\tau^T g_{\gamma_\tau}, \end{aligned}$$

and the linearized system is equivalent to the published in [10].

C. Orbital stabilization

From (13) we can conclude that (see [11] for a relevant discussion)

$$\frac{d\xi}{d\tau} = A_\tau\xi + B_\tau u + o\left(\|\xi\|^2\right). \tag{15}$$

It can be shown that asymptotic stabilization of the trivial solution of (13) can be used to design an asymptotically orbitally stabilizing controller for (15), see e.g. [14], [15], [5], [7], [11].

Notice that matrices A_τ, B_τ are smooth periodic functions. Thus, the system (13) is linear with periodic coefficients. The problem of feedback stabilization of such a system can be solved as in [16], see also [22].

III. Example: Multibody Car System

Let us consider the car-trailers system (see [20]) to illustrate applicability of the method of orbital stabilization exploiting the proposed construction of transverse coordinates. The system consists of a two-wheeled car with two trailers (see Figure 1). The wheels of the car are actuated, while the trailers are freely moving. Thus, the system has five degrees of freedom and two control inputs. The equations of motion are

$$\dot{\chi} = h(\chi) + g(\chi)w \tag{16}$$

with state variable $\chi = (x, y, \theta, \phi_1, \phi_2)$, and control input $w = (\alpha, \beta)$; where x, y are the Cartesian coordinates of the car, θ is the orientation of the car, the angles ϕ_1, ϕ_2 define

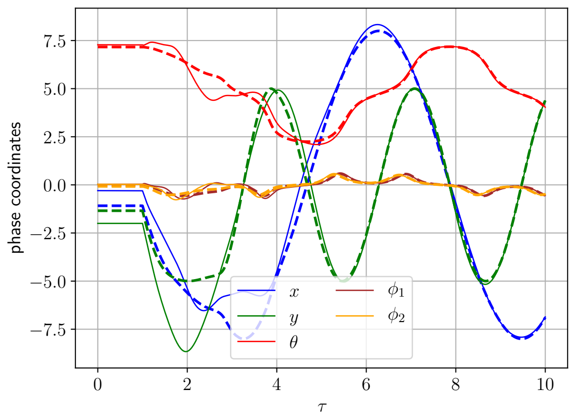

Fig. 2. Target trajectory (dashed), and transient process (solid).

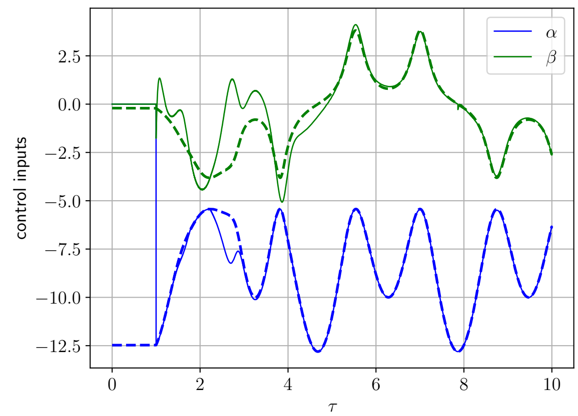

Fig. 3. Nominal control inputs (dashed), and actual (solid).

relative orientations of trailers with respect to the car, α, β are the sum and difference of car wheels angular velocities (see [20, chapter 3] for more details). The functions h, g are defined as

$$h(\chi) = 0_{5\times 1}, \quad g(\chi) = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \\ -\sin\phi_1 & -1 \\ \sin\phi_1 - \cos\phi_1 \sin\phi_2 & 0 \end{pmatrix}.$$

Let us consider a 2π-periodic trajectory of system (16)

$$\gamma_\tau = (x_\tau, y_\tau, \theta_\tau, \phi_{1\tau}, \phi_{2\tau}),$$

such that the car moves along the curve

$$x_\tau = -8\sin\tau, \qquad y_\tau = -5\sin 2\tau,$$

and trailers are always in front of the car. The function θ_τ is expressed as $\theta_\tau = \arctan(\dot{y}_\tau, \dot{x}_\tau)$. The functions $\phi_{1\tau}, \phi_{2\tau}$ are also 2π-periodic, they were found numerically. All the components of the trajectory γ_τ are depicted with dashed curves in Figure 2. The feed-forward control input $w_\tau = (\alpha_\tau, \beta_\tau)$ with

$$\alpha_\tau = -\sqrt{\dot{x}_\tau^2 + \dot{y}_\tau^2}, \qquad \beta_\tau = -\frac{\ddot{y}_\tau \dot{x}_\tau - \dot{y}_\tau \ddot{x}_\tau}{\dot{x}_\tau^2 + \dot{y}_\tau^2}$$

is depicted in Figure 3. Then, the control goal is formulated as follows: Find a feedback $w(\chi)$, such that the trajectory γ_τ of the closed-loop system is asymptotically orbitally stable.

In Definition 1 it was assumed that the trajectory γ_τ is a solution of time-invariant unforced autonomous equation $\frac{d\gamma_\tau}{d\tau} = f(\gamma_\tau)$. To satisfy this assumption, let us introduce a new control law $u(\chi)$ with the help of projection operator $\pi(\chi)$, defined as in (6):

$$u(\chi) := w(\chi) - w_{\tau=\pi(\chi)},$$

then the system (16) can be written as

$$\dot{\chi} = f(\chi) + g(\chi)\, u, \tag{17}$$

where $f(\chi) := h(\chi) + g(\chi)\, w_{\tau=\pi(\chi)}$. It is easy to prove, that $\chi = \gamma_t$ is a solution of system (17) with $u = 0$, thus the results of Claim 3 and Theorem 1 can be used.

By the first step, let us construct the basis functions $\{e_\tau^i\}_{i=1}^4$ for the trajectory γ_τ. To this end, let us take the matrix $\bar{R}_0$ composed of $Pf(\gamma_0)/\left\|Pf(\gamma_0)\right\|$, and its orthocomplement as it was stated in Claim 2:

$$\bar{R}_0 = \begin{pmatrix} -0.625 & -0.781 & 0.0 & -0.00112 & 0.00179 \\ -0.781 & 0.625 & 0.0 & -0.000536 & 0.000862 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ -0.00112 & -0.000536 & 0.0 & 0.999 & 0.000001 \\ -0.00179 & -0.000862 & 0.0 & 0.0 & -0.999 \end{pmatrix}.$$

The matrix P here is taken diagonal: $P = \mathrm{diag}(1, 1, 1, 0.1, 0.1)$. After that we solve the initial value problem (8) numerically, and obtain $\bar{R}_\tau$. Finally, we construct the matrix R_τ as stated in Claim 3, its columns $2-5$ form the basis vectors $E_\tau = (e_\tau^1, e_\tau^2, e_\tau^3, e_\tau^4) = \mathrm{col}_{2..5} R_\tau$, and transverse coordinates $\xi = E_\tau^T(\chi - \gamma_\tau)$.

By the second step we find matrices A_τ, B_τ according to formula (13). To stabilize the trivial solution of the obtained linear time-varying system the linear quadratic regulator of the form $u(\tau, \xi) = k(\tau)\xi$ is constructed using the method described in [17]. The resulting feedback for stabilizing the trajectory of system (16) is then

$$w(\chi) = w_{\tau=\pi(\chi)} + k(\pi(\chi))\, \Xi(\chi).$$

As seen, the feedback needs to compute value of the function $\pi(x)$ in real-time. To this end in practice, the trajectory γ_τ is convenient to represent as a B-Spline of degree n with control points c_j and knots u_j, $j = 1..N$ as

$$\gamma_\tau = \sum_{j=1}^{N} c_j B_{j,n}(\tau).$$

In this case, the projection operator can be evaluated using the two-steps algorithm. In the first step, the algorithm finds the closest to the given x knot as

$$j_* = \arg\min_{j=0..N} \|(x - \gamma_{u_j}) P \dot{\gamma}_{u_j}\|.$$

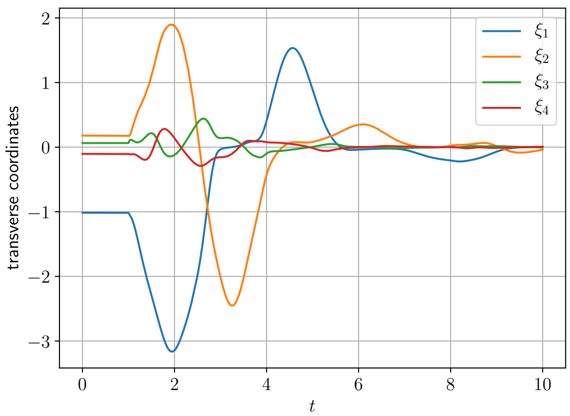

Fig. 4. Transverse coordinates.

In the second step, the algorithm iteratively finds a root τ_* of the equation

$$(x-\gamma_{\tau_*})P\dot{\gamma}_{\tau_*}=0$$

on the interval $\tau_* \in [u_{j_*-1}, u_{j_*+1}]$ using the Brent's method [23].

In Figure 4 dynamics of transverse coordinates is shown. As can be seen, all the components of ξ converge to zero. Dynamics of phase coordinates is presented in Figure 2 (solid curves).

The computer simulation of the feedback controlled system can be found at `https://youtu.be/rTHUgWFLplE`.

IV. Conclusion

We have proposed a new approach for generating a basis for a moving Poincaré section along a periodic solution of a rather general nonlinear dynamical system. Our approach is based on solving a matrix differential equation for a rotation matrix, columns of which are used to define the basis vectors. It should be stressed that it is possible to proceed in the suggested way only if not only the "nominal" dynamics has no uncertainty but also if a periodic solution is known. As soon as the basis functions are computed, they can be used to define nominal transverse coordinates and a linearization of the corresponding transverse dynamics. The transverse coordinates and the linearized transverse dynamics are instrumental in synthesizing a state feedback controller making this coordinates vanishing and leading to orbital asymptotic stability of the corresponding nonlinear closed-loop system. We have illustrated applicability of our approach on a challenging simulation example stabilizing a periodic motion for a kinematic model of a multibody car system. While robustness of the approach can be verified via numerical simulations, and although exponential orbital stability can be verified, the formal theoretical investigation with quantification of this property is left for a future investigation.

V. Appendix

A. Proof of Claim 1

The Jacobian matrix of the coordinate transformation map

$$x(\tau,\xi)=\gamma_\tau+\sum_i e_\tau^i \xi_i$$

is

$$J:=\frac{\partial x}{\partial(\xi,\tau)}=v\left(e_\tau^1,e_\tau^2,...,e_\tau^{n-1},\frac{d\gamma_\tau}{d\tau}+\sum_i\frac{de_\tau^i}{d\tau}\xi_i\right).$$

Since, $\|\xi\|<\epsilon$, the term $\sum_i \frac{de_\tau^i}{d\tau}\xi_i$ can be made arbitrarily small. Then, rank of

$$J\approx\left(e_\tau^1,e_\tau^2,...,e_\tau^{n-1},\frac{d\gamma_\tau}{d\tau}\right)$$

is equal to n since e_τ^i are mutually orthogonal, and $\frac{d\gamma_\tau}{d\tau}$ does not belong to $\mathbf{span}\left\{e_\tau^i\right\}_{i=1}^{n-1}$.

B. Proof of Claim 2

Since, by definition, the matrix

$$S(\tau)=\frac{dv_\tau}{d\tau}\wedge v_\tau=\frac{dv_\tau^T}{d\tau}v_\tau-\frac{dv_\tau}{d\tau}v_\tau^T$$

is skew-symmetric, and the initial value $\bar{R}_0$ belongs to $SO(n)$, the solution $\bar{R}_\tau$ also belongs to $SO(n)$. This is obvious since

$$\frac{d\bar{R}_\tau}{d\tau}=S(\tau)\bar{R}_\tau \;\Leftrightarrow\; \frac{d\bar{R}_\tau^T}{d\tau}=-\bar{R}_\tau^T S(\tau) \;\Rightarrow\; \frac{d}{d\tau}(\bar{R}_\tau^T\bar{R}_\tau)=0$$

Moreover, since

$$\|v_\tau\|^2=v_\tau^T v_\tau=1 \quad\Rightarrow\quad v_\tau^T\frac{dv_\tau}{d\tau}=0$$

the equation has the first integral $I=v_\tau^T\bar{R}_\tau=\mathbf{const}$:

$$\begin{aligned}\frac{d}{d\tau}I &= \frac{dv_\tau^T}{d\tau}\bar{R}_\tau+v_\tau^T\frac{d\bar{R}_\tau}{d\tau}\\ &= \frac{dv_\tau^T}{d\tau}\bar{R}_\tau+\left(v_\tau^T\frac{dv_\tau}{d\tau}v_\tau^T-v_\tau^T v_\tau\frac{dv_\tau^T}{d\tau}\right)\bar{R}_\tau\\ &= \frac{dv_\tau^T}{d\tau}\bar{R}_\tau-\frac{dv_\tau^T}{d\tau}\bar{R}_\tau=0.\end{aligned}$$

Since $\mathbf{col}_1\bar{R}_0=v_0$ one concludes that $I=v_0^T\bar{R}_0=(1,0_{n-1})=v_\tau^T\bar{R}_\tau$. The last equality gives $\mathbf{col}_1\bar{R}_\tau=v_\tau$.

C. Proof of Claim 3

Substituting $\tau=0$ and $\tau=T$, one can see that $R_0=R_T$:

$$\begin{aligned}&R_0=\bar{R}_0\\ &R_T=\bar{R}_T\exp\left\{\log\bar{R}_T^T\bar{R}_0\right\}=\bar{R}_T\bar{R}_T^T\bar{R}_0=\bar{R}_0.\end{aligned}$$

Using $v_\tau^T R_\tau=(1,0_{n-1})$ we also see that $v_\tau^T R_\tau=(1,0_{n-1})\exp\left\{\frac{\tau}{T}\log\bar{R}_T^T\bar{R}_0\right\}$. Moreover, obviously, the matrix $\bar{R}_T^T\bar{R}_0$ has the structure

$$\bar{R}_T^T\bar{R}_0=\begin{pmatrix}1 & 0_{1\times n-1}\\ 0_{n-1\times 1} & X_{n-1\times n-1}\end{pmatrix}$$

because the first columns $v_\tau = \frac{P\frac{d\gamma_\tau}{d\tau}}{\left\|P\frac{d\gamma_\tau}{d\tau}\right\|}$ of $\bar{R}_\tau$ is periodic. The matrix $\exp\left\{\frac{\tau}{T}\log \bar{R}_T^T\bar{R}_0\right\}$ has the same structure. Then, $v_\tau^T R_\tau = (1, 0_{n-1})$. The continuity of the first derivative

$$\frac{dR_\tau}{d\tau} = \left(\frac{dv_\tau}{d\tau}\wedge v_\tau\right)\bar{R}_\tau \exp\left\{\frac{\tau}{T}\log \bar{R}_T^T\bar{R}_0\right\} + \frac{1}{T}\bar{R}_\tau \exp\left\{\frac{\tau}{T}\log \bar{R}_T^T\bar{R}_0\right\}\log\left\{\bar{R}_T^T\bar{R}_0\right\}$$

at $\tau = T$ is proved by the direct computations:

$$\left(\frac{dR_\tau}{d\tau}\right)_{\tau=0} - \left(\frac{dR_\tau}{d\tau}\right)_{\tau=T} = 0.$$

D. Proof of Theorem 1

First of all let us find the expression for $\dot{\tau}$ by differentiating with respect to time the identity

$$\left(x(t) - \gamma_{\tau(t)}\right)^T P\frac{d\gamma_\tau}{d\tau} \equiv \left(x(t) - \gamma_{\tau(t)}\right)^T Pf(\gamma_{\tau(t)}) = 0$$

that defines $\tau(t)$ as $\pi(x(t))$

Using the chain rule and solving the result for $\dot{\tau}$ we obtain

$$\dot{\tau} \;=\; \frac{f_\chi^T P\left(f_x + g_x u\right)}{f_\chi^T P f_\chi - f_\chi^T\left(\frac{\partial f}{\partial x}\right)_\chi^T P\left(x - \chi\right)}.$$

where $\chi := (\gamma\circ\pi)(x)$, $f_\chi := (f\circ\gamma\circ\pi)(x)$.

Substituting $\gamma_\tau + E_\tau\xi$ instead of x we obtain

$$\dot{\tau} \;\approx\; \frac{f_{\gamma_\tau}^T P\left(f_{\gamma_\tau} + \left(\frac{\partial f}{\partial x}\right)_{\gamma_\tau} E_\tau\xi + g_{\gamma_\tau + E_\tau\xi}u\right)}{f_{\gamma_\tau}^T P f_{\gamma_\tau} - f_{\gamma_\tau}^T\left(\frac{\partial f}{\partial x}\right)_{\gamma_\tau}^T P E_\tau\xi}.$$

Linearizing the fraction, we have

$$\begin{aligned}\dot{\tau} \;&=\; \frac{\partial\pi}{\partial x}f_x + \frac{\partial\pi}{\partial x}g_x u\\ &\approx\; 1 + \frac{f_{\gamma_\tau}^T\left[P\left(\frac{\partial f}{\partial x}\right)_{\gamma_\tau} + \left(\frac{\partial f}{\partial x}\right)_{\gamma_\tau}^T P\right]E_\tau}{f_{\gamma_\tau}^T P f_{\gamma_\tau}}\xi + \\ &\quad\frac{f_{\gamma_\tau}^T P g_{\gamma_\tau}}{f_{\gamma_\tau}^T P f_{\gamma_\tau}}u.\end{aligned}$$

To find the linearization of dynamics for ξ we substitute (12) into (11):

$$\begin{aligned}\dot{\xi} \;&=\; \frac{dE_\tau^T}{d\tau}E_\tau\xi\frac{\partial\pi}{\partial x}f_x + \frac{dE_\tau^T}{d\tau}E_\tau\xi\frac{\partial\pi}{\partial x}g_x u + \\ &\quad E_\tau^T\left(f_x - f_{\gamma_\tau}\frac{\partial\pi}{\partial x}f_x\right) - E_\tau^T f_{\gamma_\tau}\frac{\partial\pi}{\partial x}g_x u + E_\tau^T g_x u.\end{aligned}$$

Linearizing all the terms:

- $\frac{dE_\tau^T}{d\tau}E_\tau\xi\frac{\partial\pi}{\partial x}f_x \approx \frac{dE_\tau^T}{d\tau}E_\tau\xi$ due to $\frac{\partial\pi}{\partial x}f_x = 1 + O\left(\xi\right)$
- $\frac{dE_\tau^T}{d\tau}E_\tau\xi\frac{\partial\pi}{\partial x}g_x u \approx 0$ due to $u = O\left(\xi\right)$
- $f_x - f_{\gamma_\tau}\frac{\partial\pi}{\partial x}f_x \approx$ $\frac{f_{\gamma_\tau}^T P f_{\gamma_\tau}\left(\frac{\partial h}{\partial x}\right)_{x_\star} - f_{\gamma_\tau}f_{\gamma_\tau}^T P\left(\frac{\partial f}{\partial x}\right)_{\gamma_\tau} - f_{\gamma_\tau}f_{\gamma_\tau}^T\left(\frac{\partial h}{\partial x}\right)_{\gamma_\tau}^T P}{f_{\gamma_\tau}^T P f_{\gamma_\tau}}E_\tau\xi$
- $E_\tau^T g_x u \approx E_\tau^T g_{\gamma_\tau}u$
- $E_\tau^T f_{\gamma_\tau}\frac{\partial\pi}{\partial x}g_x u \approx E_\tau^T f_{\gamma_\tau}\left(\frac{f_{\gamma_\tau}^T P g_x}{f_{\gamma_\tau}^T P f_{\gamma_\tau} - f_{\gamma_\tau}^T\left(\frac{\partial f}{\partial x}\right)_{\gamma_\tau}^T P E\xi}\right)u \approx$ $E_\tau^T f_{\gamma_\tau}\frac{f_{\gamma_\tau}^T P g_x}{f_{\gamma_\tau}^T P f_{\gamma_\tau}}u$

Collecting all the terms together leads to (13).

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