J. Math. Pures Appl. 142 (2020) 229–242

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

The Liouville theorem and linear operators satisfying the maximum principle

Nathaël Alibaud[a,b], Félix del Teso[c,1,2], Jørgen Endal[d,*,1], Espen R. Jakobsen[d,1]

[a] ENSMM, 26 Chemin de l'Epitaphe, 25030 Besançon cedex, France
[b] LMB, UMR CNRS 6623, Université de Bourgogne Franche-Comté (UBFC), France
[c] Departamento de Análisis Matemático y Matemática Aplicada, Universidad Complutense de Madrid (UCM), 28040 Madrid, Spain
[d] Department of Mathematical Sciences, Norwegian University of Science and Technology (NTNU), N-7491 Trondheim, Norway

A R T I C L E I N F O

Article history:
Received 14 October 2019
Available online 24 August 2020

MSC:
35B10
35B53
35J70
35R09
60G51
65R20

Keywords:
Nonlocal degenerate elliptic operators
Courrège theorem
Lévy-Khintchine formula
Liouville theorem
Periodic solutions
Propagation of maximum
Subgroups of $\mathbb{R}^d$
Kronecker theorem

A B S T R A C T

A result by Courrège says that linear translation invariant operators satisfy the maximum principle if and only if they are of the form $\mathcal{L} = \mathcal{L}^{\sigma,b} + \mathcal{L}^{\mu}$ where

$$\mathcal{L}^{\sigma,b}[u](x) = \operatorname{tr}(\sigma\sigma^{\mathsf{T}} D^2 u(x)) + b \cdot Du(x)$$

and

$$\mathcal{L}^{\mu}[u](x) = \int_{\mathbb{R}^d\setminus\{0\}} \big(u(x+z) - u(x) - z \cdot Du(x)\mathbf{1}_{|z|\leq 1}\big)\, \mathrm{d}\mu(z).$$

This class of operators coincides with the infinitesimal generators of Lévy processes in probability theory. In this paper we give a complete characterization of the operators of this form that satisfy the Liouville theorem: Bounded solutions u of $\mathcal{L}[u] = 0$ in $\mathbb{R}^d$ are constant. The Liouville property is obtained as a consequence of a periodicity result that completely characterizes bounded distributional solutions of $\mathcal{L}[u] = 0$ in $\mathbb{R}^d$. The proofs combine arguments from PDEs and group theory. They are simple and short.

* Corresponding author.
E-mail addresses: nathael.alibaud@ens2m.fr (N. Alibaud), fdelteso@ucm.es (F. del Teso), jorgen.endal@ntnu.no (J. Endal), espen.jakobsen@ntnu.no (E.R. Jakobsen).
URLs: https://lmb.univ-fcomte.fr/Alibaud-Nathael (N. Alibaud), https://sites.google.com/view/felixdelteso (F. del Teso), http://folk.ntnu.no/jorgeen (J. Endal), http://folk.ntnu.no/erj (E.R. Jakobsen).

[1] The authors were supported by the Toppforsk (research excellence) project Waves and Nonlinear Phenomena (WaNP), grant no. 250070 from the Research Council of Norway.
[2] The author was supported by the Basque Government through the BERC 2018-2021 program, and the Spanish Ministry of Science, Innovation and Universities through the BCAM Severo Ochoa accreditation SEV-2017-0718 and the Spanish research project PGC2018-094522-B-I00.

https://doi.org/10.1016/j.matpur.2020.08.008

R É S U M É

Selon un résultat de Courrège, les opérateurs linéaires invariants par translation satisfont le principe du maximum si et seulement si ils sont de la forme $\mathcal{L} = \mathcal{L}^{\sigma,b}+\mathcal{L}^{\mu}$, où

$$\mathcal{L}^{\sigma,b}[u](x) = \mathrm{tr}(\sigma\sigma^{\mathrm{T}} D^2u(x)) + b \cdot Du(x)$$

et

$$\mathcal{L}^{\mu}[u](x) = \int_{\mathbb{R}^d\setminus\{0\}} \big(u(x+z) - u(x) - z \cdot Du(x)\mathbf{1}_{|z|\leq 1}\big)\,\mathrm{d}\mu(z).$$

Cette classe d'opérateurs coïncide avec les générateurs infinitésimaux des processus de Lévy dans la théorie des probabilités. Dans cet article, nous donnons une caractérisation complète des opérateurs de cette forme qui satisfont le théorème de Liouville : les solutions bornées u de $\mathcal{L}[u] = 0$ dans $\mathbb{R}^d$ sont constantes. La propriété de Liouville est obtenue grâce à un résultat de périodicité qui caractérise complètement les solutions distributionnelles bornées de $\mathcal{L}[u] = 0$ dans $\mathbb{R}^d$. Les preuves combinent des arguments d'EDP et de la théorie des groupes. Ils sont simples et brefs.

1. Introduction and main results

The classical Liouville theorem states that bounded solutions of $\Delta u = 0$ in $\mathbb{R}^d$ are constant. The Laplace operator Δ is the most classical example of an operator $\mathcal{L} : C_c^\infty(\mathbb{R}^d) \to C(\mathbb{R}^d)$ satisfying the maximum principle in the sense that

$$\mathcal{L}[u](x) \leq 0 \text{ at any global maximum point } x \text{ of } u. \tag{1}$$

In the class of linear translation invariant[3] operators (which includes Δ), a result by Courrège [13][4] says that the maximum principle holds if and only if

$$\mathcal{L} = \mathcal{L}^{\sigma,b} + \mathcal{L}^{\mu}, \tag{2}$$

where

$$\mathcal{L}^{\sigma,b}[u](x) = \mathrm{tr}(\sigma\sigma^{\mathrm{T}} D^2u(x)) + b \cdot Du(x), \tag{3}$$

$$\mathcal{L}^{\mu}[u](x) = \int_{\mathbb{R}^d\setminus\{0\}} \big(u(x+z) - u(x) - z \cdot Du(x)\mathbf{1}_{|z|\leq 1}\big)\,\mathrm{d}\mu(z), \tag{4}$$

and

$$b \in \mathbb{R}^d, \quad \text{and} \quad \sigma = (\sigma_1, \dots, \sigma_P) \in \mathbb{R}^{d\times P} \text{ for } P \in \mathbb{N},\, \sigma_j \in \mathbb{R}^d, \tag{$\mathrm{A}_{\sigma,b}$}$$

$$\mu \geq 0 \text{ is a Radon measure on } \mathbb{R}^d \setminus \{0\}, \quad \int_{\mathbb{R}^d\setminus\{0\}} \min\{|z|^2, 1\}\,\mathrm{d}\mu(z) < \infty. \tag{A_μ}$$

[3] Translation invariance means that $\mathcal{L}[u(\cdot + y)](x) = \mathcal{L}[u](x + y)$ for all x, y.

[4] If (1) holds at any *nonnegative* maximum point, then by definition the *positive* maximum principle holds and by [13] there is an extra term $cu(x)$ with $c \leq 0$ in (3). For the purpose of this paper (Liouville and periodicity), the case $c < 0$ is trivial since then $u = 0$ is the unique bounded solution of $\mathcal{L}[u] = 0$.

These elliptic operators have a local part $\mathcal{L}^{\sigma,b}$ and a nonlocal part $\mathcal{L}^{\mu}$, either of which could be zero.[5]

Another point of view of these operators comes from probability and stochastic processes: Every operator mentioned above is the generator of a Lévy process, and conversely, every generator of a Lévy process is of the form given above. Lévy processes are Markov processes with stationary independent increments and are the prototypical models of noise in science, engineering, and finance. Well-known examples are Brownian motions, Poisson processes, stable processes, and various other types of jump processes.

The main contributions of this paper are the following:

1. *We give necessary and sufficient conditions for $\mathcal{L}$ to have the Liouville property: Bounded solutions u of $\mathcal{L}[u] = 0$ in $\mathbb{R}^d$ are constant.*
2. *For general $\mathcal{L}$, we show that all bounded solutions of $\mathcal{L}[u] = 0$ in $\mathbb{R}^d$ are periodic and we identify the set of admissible periods.*

Let us now state our results. For a set $S \subseteq \mathbb{R}^d$, we let $G(S)$ denote the smallest additive subgroup of $\mathbb{R}^d$ containing S and define the subspace $V_S \subseteq \overline{G(S)}$ by

$$V_S := \left\{ g \in \overline{G(S)} \ : \ tg \in \overline{G(S)} \ \forall t \in \mathbb{R} \right\}.$$

Then we take $\operatorname{supp}(\mu)$ to be the support of the measure μ and define

$$G_\mu := \overline{G(\operatorname{supp}(\mu))}, \quad V_\mu := V_{\operatorname{supp}(\mu)}, \quad \text{and} \quad c_\mu := -\int\limits_{\{|z|\le 1\}\setminus V_\mu} z \, \mathrm{d}\mu(z).$$

Here c_μ is well-defined and uniquely determined by μ, cf. Proposition 2.13. We also need the subspace $W_{\sigma,b+c_\mu} := \operatorname{span}_{\mathbb{R}}\{\sigma_1, \dots, \sigma_P, b + c_\mu\}$.

Theorem 1.1 *(General Liouville). Assume ($\mathrm{A}_{\sigma,b}$) and (A_μ). Let $\mathcal{L}$ be given by (2)–(3)–(4). Then the following statements are equivalent:*

(a) *If $u \in L^\infty(\mathbb{R}^d)$ satisfies $\mathcal{L}[u] = 0$ in $\mathcal{D}'(\mathbb{R}^d)$, then u is a.e. a constant.*
(b) $\overline{G_\mu + W_{\sigma,b+c_\mu}} = \mathbb{R}^d$.

The above Liouville result is a consequence of a periodicity result for bounded solutions of $\mathcal{L}[u] = 0$ in $\mathbb{R}^d$. For a set $S \subseteq \mathbb{R}^d$, a function $u \in L^\infty(\mathbb{R}^d)$ is a.e. *S-periodic* if $u(\cdot + s) = u(\cdot)$ in $\mathcal{D}'(\mathbb{R}^d)$ $\forall s \in S$. Our result is the following:

Theorem 1.2 *(General periodicity). Assume ($\mathrm{A}_{\sigma,b}$), (A_μ), and $u \in L^\infty(\mathbb{R}^d)$. Let $\mathcal{L}$ be given by (2)–(3)–(4). Then the following statements are equivalent:*

(a) $\mathcal{L}[u] = 0$ *in* $\mathcal{D}'(\mathbb{R}^d)$.
(b) *u is a.e. $\overline{G_\mu + W_{\sigma,b+c_\mu}}$-periodic.*

This result characterizes the bounded solutions for all operators $\mathcal{L}$ in our class, also those not satisfying the Liouville property. Note that if $\overline{G_\mu + W_{\sigma,b+c_\mu}} = \mathbb{R}^d$, then u is constant and the Liouville result follows. Both theorems are proved in Section 2.

[5] The representation (2)–(3)–(4) is unique up to the choice of a cut-off function in (4) and a square root σ of $a = \sigma\sigma^{\mathrm{T}}$. In this paper we always use $\mathbf{1}_{|z|\le 1}$ as a cut-off function.

We give examples in Section 3. Examples 3.2 and 3.5 provide an overview of different possibilities, and Examples 3.7 and 3.8 are concerned with the case where $\text{card}\,(\text{supp}(\mu)) < \infty$. The Liouville property holds in the latter case if and only if $\text{card}\,(\text{supp}(\mu)) \geq d - \dim\left(W_{\sigma,b+c_\mu}\right) + 1$ with additional algebraic conditions in relation with Diophantine approximation. The Kronecker theorem (Theorem 3.6) is a key ingredient in this discussion and a slight change in the data may destroy the Liouville property.

The class of operators $\mathcal{L}$ given by (2)–(3)–(4) is large and diverse. In addition to the processes mentioned above, it includes also discrete random walks, constant coefficient Itô- and Lévy-Itô processes, and most processes used as driving noise in finance. Examples of nonlocal operators are fractional Laplacians [23], convolution operators [14,1,5], relativistic Schrödinger operators [19], and the CGMY model in finance [12]. We mention that discrete finite difference operators can be written in the form (2)–(3)–(4), cf. [17]. For more examples, see Section 3.

There is a huge literature on the Liouville theorem. In the local case, we simply refer to the survey [20]. In the nonlocal case, the Liouville theorem is more or less understood for fractional Laplacians or variants [23,4,8,9,18], certain Lévy operators [2,27,30,28,16], relativistic Schrödinger operators [19], or convolution operators [10,5–7]. The techniques vary from Fourier analysis, potential theory, probabilistic methods, to classical PDE arguments.

To prove that solutions of $\mathcal{L}[u] = 0$ are G_μ-periodic, we rely on propagation of maximum points [10,14,11,15,16,22,6,7] and a localization technique à la [10,3,29,7]. As far as we know, Choquet and Deny [10] were the first to obtain such results. They were concerned with the equation $u * \mu - u = 0$ for some bounded measure μ. This is a particular case of our equation since $u * \mu - u = \mathcal{L}^\mu[u] + \int_{\mathbb{R}^d\setminus\{0\}} z\mathbf{1}_{|z|\leq 1}\,\mathrm{d}\mu(z)\cdot Du$. For general μ, the drift $\int_{\mathbb{R}^d\setminus\{0\}} z\mathbf{1}_{|z|\leq 1}\,\mathrm{d}\mu(z)\cdot Du$ may not make sense and the identification of the full drift $b + c_\mu$ relies on a standard decomposition of closed subgroups of $\mathbb{R}^d$, see e.g. [24]. The idea is to establish G_μ-periodicity of solutions of $\mathcal{L}[u] = 0$ as in [10], and then use that $G_\mu = V_\mu \oplus \Lambda$ for the vector space V_μ previously defined and some discrete group Λ. This will roughly speak remove the singularity $z = 0 \in V_\mu$ in the computation of c_μ because $\int_{\mathbb{R}^d\setminus\{0\}} \mathbf{1}_{z\in V_\mu} z\mathbf{1}_{|z|\leq 1}\,\mathrm{d}\mu(z)\cdot Du = 0$ for any G_μ-periodic function. See Section 2 for details.

Our approach then combines PDEs and group arguments, extends the results of [10] to Courrège/Lévy operators, yields necessary and sufficient conditions for the Liouville property, and provides short and simple proofs.

Outline of the paper

Our main results (Theorems 1.1 and 1.2.) were stated in Section 1. They are proved in Section 2 and examples are given in Section 3.

Notation and preliminaries

The *support* of a measure μ is defined as

$$\text{supp}(\mu) := \left\{z \in \mathbb{R}^d \setminus \{0\}\ :\ \mu(B_r(z)) > 0,\ \forall r > 0\right\}, \tag{5}$$

where $B_r(z)$ is the ball of center z and radius r. To continue, we assume ($\mathrm{A}_{\sigma,b}$), (A_μ), and $\mathcal{L}$ is given by (2)–(3)–(4).

Definition 1.3. For any $u \in L^\infty(\mathbb{R}^d)$, $\mathcal{L}[u] \in \mathcal{D}'(\mathbb{R}^d)$ is defined by

$$\langle \mathcal{L}[u], \psi\rangle := \int_{\mathbb{R}^d} u(x)\mathcal{L}^*[\psi](x)\,\mathrm{d}x \quad \forall \psi \in C_c^\infty(\mathbb{R}^d)$$

with $\mathcal{L}^* := \mathcal{L}^{\sigma,-b} + \mathcal{L}^{\mu^*}$ and $\mathrm{d}\mu^*(z) := \mathrm{d}\mu(-z)$.

The above distribution is well-defined since $\mathcal{L}^* : W^{2,1}(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is bounded.

Definition 1.4. Let $S \subseteq \mathbb{R}^d$ and $u \in L^\infty(\mathbb{R}^d)$, then u is a.e. *S-periodic* if

$$\int_{\mathbb{R}^d} \big(u(x+s) - u(x)\big)\psi(x)\,\mathrm{d}x = 0 \quad \forall s \in S, \forall \psi \in C_c^\infty(\mathbb{R}^d).$$

The following technical result will be needed to regularize distributional solutions of $\mathcal{L}[u] = 0$ and a.e. periodic functions. Let the mollifier $\rho_\varepsilon(x) := \frac{1}{\varepsilon^d}\rho(\frac{x}{\varepsilon})$, $\varepsilon > 0$, for some $0 \le \rho \in C_c^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \rho = 1$.

Lemma 1.5. *Let $u \in L^\infty(\mathbb{R}^d)$ and $u_\varepsilon := \rho_\varepsilon * u$. Then:*

(a) *$\mathcal{L}[u] = 0$ in $\mathcal{D}'(\mathbb{R}^d)$ if and only if $\mathcal{L}[u_\varepsilon] = 0$ in $\mathbb{R}^d$ for all $\varepsilon > 0$.*
(b) *u is a.e. S-periodic if and only if u_ε is S-periodic for all $\varepsilon > 0$.*

Proof. The proof of (a) is standard since $\mathcal{L}[u_\varepsilon] = \mathcal{L}[u] * \rho_\varepsilon$ in $\mathcal{D}'(\mathbb{R}^d)$. Moreover (b) follows from (a) since for any $s \in S$ we can take $\mathcal{L}[\phi](x) = \phi(x+s) - \phi(x)$ by choosing $\sigma, b = 0$ and $\mu = \delta_s$ (the Dirac measure at s) in (2)–(3)–(4). □

2. Proofs

This section is devoted to the proofs of Theorems 1.1 and 1.2. We first reformulate the classical Liouville theorem for local operators in terms of periodicity, then study the influence of the nonlocal part.

2.1. $W_{\sigma,b}$-periodicity for local operators

Let us recall the Liouville theorem for operators of the form (3), see e.g. [26,25]. In the result we use the set

$$W_{\sigma,b} = \operatorname{span}_{\mathbb{R}}\{\sigma_1, \dots, \sigma_P, b\}.$$

Note that $\operatorname{span}_{\mathbb{R}}\{\sigma_1, \dots, \sigma_P\}$ equals the span of the eigenvectors of $\sigma\sigma^{\mathtt{T}}$ corresponding to nonzero eigenvalues.

Theorem 2.1 *(Liouville for $\mathcal{L}^{\sigma,b}$). Assume ($\mathrm{A}_{\sigma,b}$) and $\mathcal{L}^{\sigma,b}$ is given by (3). Then the following statements are equivalent:*

(a) *If $u \in L^\infty(\mathbb{R}^d)$ solves $\mathcal{L}^{\sigma,b}[u] = 0$ in $\mathcal{D}'(\mathbb{R}^d)$, then u is a.e. constant in $\mathbb{R}^d$.*
(b) *$W_{\sigma,b} = \mathbb{R}^d$.*

Let us now reformulate and prove this classical result as a consequence of a periodicity result, a type of argument that will be crucial in the nonlocal case. We will consider $C_b^\infty(\mathbb{R}^d)$ solutions, which will be enough later during the proofs of Theorem 1.1 and 1.2, thanks to Lemma 1.5.

Proposition 2.2 *(Periodicity for $\mathcal{L}^{\sigma,b}$). Assume ($\mathrm{A}_{\sigma,b}$), $\mathcal{L}^{\sigma,b}$ is given by (3), and $u \in C_b^\infty(\mathbb{R}^d)$. Then the following statements are equivalent:*

(a) *$\mathcal{L}^{\sigma,b}[u] = 0$ in $\mathbb{R}^d$.*
(b) *u is $W_{\sigma,b}$-periodic.*

Note that part (b) implies that u is constant in the directions defined by the vectors $\sigma_1, \dots, \sigma_P, b$. If their span then covers all of $\mathbb{R}^d$, Theorem 2.1 follows trivially. To prove Proposition 2.2, we adapt the ideas of [25] to our setting.

Proof of Proposition 2.2. (b) $\Rightarrow$ (a) We have $b \cdot Du(x) = \frac{\mathrm{d}}{\mathrm{d}t} u(x+tb)_{|t=0} = 0$ for any $x \in \mathbb{R}^d$ since the function $t \mapsto u(x+tb)$ is constant. Similarly $(\sigma_j \cdot D)^2 u(x) := \frac{\mathrm{d}^2}{\mathrm{d}t^2} u(x+t\sigma_j)_{|t=0} = 0$ for any $j = 1, \dots, P$. Using then that $\operatorname{tr}(\sigma\sigma^{\mathrm{T}} D^2 u) = \sum_{j=1}^{P} (\sigma_j \cdot D)^2 u$, we conclude that $\mathcal{L}^{\sigma,b}[u] = 0$ in $\mathbb{R}^d$.

(a) $\Rightarrow$ (b) Let $v(x,y,t) := u(x+\sigma y - bt)$ for $x \in \mathbb{R}^d$, $y \in \mathbb{R}^P$, and $t \in \mathbb{R}$. Direct computations show that

$$\Delta_y v(x,y,t) = \sum_{j=1}^{P} (\sigma_j \cdot D)^2 u(x+\sigma y - bt) = \operatorname{tr}\left[\sigma\sigma^{\mathrm{T}} D^2 u(x+\sigma y - bt)\right]$$

and $\partial_t v(x,y,t) = -b \cdot Du(x+\sigma y - bt)$. Hence for all $(x,y,t) \in \mathbb{R}^d \times \mathbb{R}^P \times \mathbb{R}$,

$$\Delta_y v(x,y,t) - \partial_t v(x,y,t) = \mathcal{L}^{\sigma,b}[u](x+\sigma y - bt) = 0.$$

Since $v(x,\cdot,\cdot)$ is bounded, we conclude by uniqueness of the heat equation that for any $s < t$,

$$v(x,y,t) = \int_{\mathbb{R}^P} v(x,z,s) K_P(y-z,t-s)\,\mathrm{d}z, \tag{6}$$

where K_P is the standard heat kernel in $\mathbb{R}^P$. But then

$$\|\Delta_y v(x,\cdot,t)\|_\infty \leq \|v(x,\cdot,s)\|_\infty \|\Delta_y K_P(\cdot,t-s)\|_{L^1(\mathbb{R}^P)},$$

and since $\|\Delta_y K_P(\cdot,t-s)\|_{L^1} \to 0$ as $s \to -\infty$, we deduce that $\Delta_y v = 0$ for all x,y,t.

By the classical Liouville theorem (see e.g. [26]), v is constant in y. It is also constant in t by (6) since $\int_{\mathbb{R}^P} K_P(z,t-s)\,\mathrm{d}z = 1$. We conclude that u is $W_{\sigma,b}$-periodic since

$$u(x) = v(x,0,0) = v(x,y,t) = u(x+\sigma y - bt)$$

and $W_{\sigma,b} = \{\sigma y - bt : y \in \mathbb{R}^P, t \in \mathbb{R}\}$. $\square$

2.2. G_μ-periodicity for general operators

Proposition 2.2 might seem artificial in the local case, but not so in the nonlocal case. In fact we will prove our general Liouville result as a consequence of a periodicity result. A key step in this direction is the lemma below.

Lemma 2.3. *Assume* ($\mathrm{A}_{\sigma,b}$), (A_μ), *$\mathcal{L}$ is given by* (2)–(3)–(4), *and $u \in C_\mathrm{b}^\infty(\mathbb{R}^d)$. If $\mathcal{L}[u] = 0$ in $\mathbb{R}^d$, then u is* $\operatorname{supp}(\mu)$*-periodic.*

To prove this result, we use propagation of maximum (see e.g. [10,14,11]).

Lemma 2.4. *If $u \in C_\mathrm{b}^\infty(\mathbb{R}^d)$ achieves its global maximum at some $\bar{x}$ such that $\mathcal{L}[u](\bar{x}) \geq 0$, then $u(\bar{x}+z) = u(\bar{x})$ for any $z \in \operatorname{supp}(\mu)$.*

Proof. At $\bar{x}$, $u = \sup u$, $Du = 0$ and $D^2 u \le 0$, and hence $\mathcal{L}^{\sigma,b}[u](\bar{x}) \le 0$ and

$$0 \le \mathcal{L}[u](\bar{x}) \le \mathcal{L}^{\mu}[u](\bar{x}) = \int\limits_{\mathbb{R}^d\setminus\{0\}} \big(u(\bar{x}+z) - \sup_{\mathbb{R}^d} u\big)\, d\mu(z).$$

Using that $\int_{\mathbb{R}^d\setminus\{0\}} f \,\mathrm{d}\mu \ge 0$ and $f \le 0$ implies $f = 0$ μ-a.e., we deduce that $u(\bar{x}+z) - \sup_{\mathbb{R}^d} u = 0$ for μ-a.e. z. Since u is continuous, this equality holds for all $z \in \operatorname{supp}(\mu)$.[6] □

To exploit Lemma 2.4, we need to have a maximum point. For this sake, we use a localization technique à la [10,3,29,7].

Proof of Lemma 2.3. Fix an arbitrary $\bar{z} \in \operatorname{supp}(\mu)$, define

$$v(x) := u(x+\bar{z}) - u(x),$$

and let us show that $v(x) = 0$ for all $x \in \mathbb{R}^d$. We first show that $v \le 0$. Take M and a sequence $\{x_n\}_n$ such that

$$v(x_n) \overset{n\to\infty}{\longrightarrow} M := \sup v,$$

and define

$$u_n(x) := u(x+x_n) \quad \text{and} \quad v_n(x) := v(x+x_n).$$

Note that $\mathcal{L}[v_n] = 0$ in $\mathbb{R}^d$. Now since $v \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^d)$, the Arzelà-Ascoli theorem implies that there exists v_∞ such that $v_n \to v_\infty$ locally uniformly (up to a subsequence). Taking another subsequence if necessary, we can assume that the derivatives up to second order converge and pass to the limit in the equation $\mathcal{L}[v_n] = 0$ to deduce that $\mathcal{L}[v_\infty] = 0$ in $\mathbb{R}^d$. Moreover, v_∞ attains its maximum at $x = 0$ since $v_\infty \le M$ and

$$v_\infty(0) = \lim_{n\to\infty} v_n(0) = \lim_{n\to\infty} v(x_n) = M.$$

A similar argument shows that there is a u_∞ such that $u_n \to u_\infty$ as $n \to \infty$ locally uniformly. Taking further subsequences if necessary, we can assume that u_n and v_n converge along the same sequence. Then by construction

$$v_\infty(x) = u_\infty(x+\bar{z}) - u_\infty(x).$$

By Lemma 2.4 and an iteration, we find that $M = v_\infty(m\bar{z}) = u_\infty((m+1)\bar{z}) - u_\infty(m\bar{z})$ for any $m \in \mathbb{Z}$. Then by another iteration,

$$u_\infty((m+1)\bar{z}) = u_\infty(m\bar{z}) + M = \ldots = u_\infty(0) + (m+1)M.$$

But since u_∞ is bounded, the only choice is $M = 0$ and thus $v \le M = 0$. A similar argument shows that $v \ge 0$, and hence, $0 = v(x) = u(x+\bar{z}) - u(x)$ for any $\bar{z} \in \operatorname{supp}(\mu)$ and all $x \in \mathbb{R}^d$. □

We can give a more general result than Lemma 2.3 if we consider groups.

[6] If not, we would find some z_0 and $r_0 > 0$ such that $f(z) := u(\bar{x}+z) - \sup u < 0$ in $B_{r_0}(z_0)$ where as $\mu(B_{r_0}(z_0)) > 0$ by (5).

Definition 2.5.

(a) A set $G \subseteq \mathbb{R}^d$ is an *additive subgroup* if $G \neq \emptyset$ and

$$\forall g_1, g_2 \in G, \quad g_1 + g_2 \in G \quad \text{and} \quad -g_1 \in G.$$

(b) The *subgroup generated* by a set $S \subseteq \mathbb{R}^d$, denoted $G(S)$, is the smallest additive group containing S.

Now we return to a key set for our analysis:

$$G_\mu = \overline{G(\mathrm{supp}(\mu))}. \tag{7}$$

This set appears naturally because of the elementary result below.

Lemma 2.6. *Let $S \subseteq \mathbb{R}^d$. Then $w \in C(\mathbb{R}^d)$ is S-periodic if and only if w is $\overline{G(S)}$-periodic.*

Proof. It suffices to show that $G := \{g \in \mathbb{R}^d : w(\cdot + g) = w(\cdot)\}$ is a closed subgroup of $\mathbb{R}^d$. It is obvious that it is closed by continuity of w. Moreover, for any $g_1, g_2 \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$,

$$w(x + g_1 - g_2) = w(x - g_2) = w(x - g_2 + g_2) = w(x),$$

which ends the proof. □

By Lemmas 2.3 and 2.6, we have proved that:

Proposition 2.7 *(G_μ-periodicity). Assume ($\mathrm{A}_{\sigma,b}$), (A_μ), $\mathcal{L}$ is given by (2)–(3)–(4), and G_μ by (7). Then any solution $u \in C_b^\infty(\mathbb{R}^d)$ of $\mathcal{L}[u] = 0$ in $\mathbb{R}^d$ is G_μ-periodic.*

2.3. The role of c_μ

Propositions 2.2 and 2.7 combined may seem to imply that $\mathcal{L}[u] = 0$ gives $(G_\mu + W_{\sigma,b})$-periodicity of u, but this is not true in general. The correct periodicity result depends on a new drift $b + c_\mu$, where c_μ is defined in (9) below. To give this definition, we need to decompose G_μ into a direct sum of a vector subspace and a relative lattice.

Definition 2.8.

(a) If two subgroups $G, \tilde{G} \subseteq \mathbb{R}^d$ satisfy $G \cap \tilde{G} = \{0\}$, their sum is said to be *direct* and we write $G + \tilde{G} = G \oplus \tilde{G}$.
(b) A *full lattice* is a subgroup $\Lambda \subseteq \mathbb{R}^d$ of the form $\Lambda = \oplus_{n=1}^d a_n \mathbb{Z}$ for some basis $\{a_1, \ldots, a_d\}$ of $\mathbb{R}^d$. A *relative lattice* is a lattice of a vector subspace of $\mathbb{R}^d$.

Theorem 2.9 *(Theorem 1.1.2 in [24]). If G is a closed subgroup of $\mathbb{R}^d$, then $G = V \oplus \Lambda$ for some vector space $V \subseteq \mathbb{R}^d$ and some relative lattice $\Lambda \subseteq \mathbb{R}^d$ such that $V \cap \mathrm{span}_{\mathbb{R}} \Lambda = \{0\}$.*

In this decomposition the space V is unique and can be represented by (8) below.

Lemma 2.10. *Let V be a vector subspace and Λ a relative lattice of $\mathbb{R}^d$ such that $V \cap \mathrm{span}_{\mathbb{R}} \Lambda = \{0\}$. Then for any $\lambda \in \Lambda$, there is an open ball B of $\mathbb{R}^d$ containing λ such that $B \cap (V \oplus \Lambda) = B \cap (V + \lambda)$.*

Proof. If the lemma does not hold, there exists $v_n + \lambda_n \to \lambda$ as $n \to \infty$ where $v_n \in V$, $\lambda_n \in \Lambda$, $\lambda_n \neq \lambda$. Note that $v_n, \lambda_n, \lambda \in V \oplus \operatorname{span}_{\mathbb{R}} \Lambda$, and that

$$\lambda = \underset{\in V}{0} + \underset{\in \Lambda}{\lambda}.$$

By continuity of the projection from $V \oplus \operatorname{span}_{\mathbb{R}} \Lambda$ onto $\operatorname{span}_{\mathbb{R}} \Lambda$, $\lambda_n \to \lambda$ and this contradicts the fact that each point of Λ is isolated. □

Lemma 2.11. *Let G, V and Λ be as in Theorem 2.9. Then*

$$V = V_G := \{g \in G \ : \ tg \in G \ \forall t \in \mathbb{R}\}. \tag{8}$$

Proof. It is clear that $V \subseteq V_G$. Now given $g \in V_G$, there is $(v, \lambda) \in V \times \Lambda$ such that $g = v + \lambda$. For any $t \in \mathbb{R}$, $tg = tv + t\lambda \in G$ and thus $t\lambda \in G$ since $tv \in V \subseteq G$. Let B be an open ball containing λ such that $B \cap G = B \cap (V + \lambda)$. Choosing t such that $t \neq 1$ and $t\lambda \in B$, we infer that $t\lambda = \tilde{v} + \lambda$ for some $\tilde{v} \in V$. Hence $\lambda = (t-1)^{-1}\tilde{v} \in V$ and this implies that $\lambda = 0$. In other words $V_G \subseteq V$, and the proof is complete. □

Remark 2.12. Any G-periodic function $w \in C^1(\mathbb{R}^d)$ is such that $z \cdot Dw(x) = \lim_{t \to 0} \frac{w(x+tz)-w(x)}{t} = 0$ for any $x \in \mathbb{R}^d$ and $z \in V_G$.

By Theorem 2.9 and Lemma 2.11, we decompose the set G_μ in (7) into a lattice and the subspace $V_\mu := V_{G_\mu}$. The new drift can then be defined as

$$c_\mu = - \int\limits_{\{|z| \leq 1\} \setminus V_\mu} z \, \mathrm{d}\mu(z). \tag{9}$$

Proposition 2.13. *Assume* (A_μ) *and c_μ is given by* (9)*. Then $c_\mu \in \mathbb{R}^d$ is well-defined and uniquely determined by μ.*

Proof. Using that $\operatorname{supp}(\mu) \subset G_\mu = V_\mu \oplus \Lambda$,

$$\begin{aligned} \int\limits_{\{|z| \leq 1\} \setminus V_\mu} |z| \, \mathrm{d}\mu(z) &= \int\limits_{G_\mu \setminus (V_\mu + 0)} |z| \mathbf{1}_{|z| \leq 1} \, \mathrm{d}\mu(z) \\ &\leq \int\limits_{G_\mu \setminus B} |z| \mathbf{1}_{|z| \leq 1} \, \mathrm{d}\mu(z) \end{aligned}$$

for some open ball B containing 0 given by Lemma 2.10. This integral is finite by (A_μ) which completes the proof. □

Proposition 2.14. *Assume* (A_μ) *and $\mathcal{L}^\mu$, G_μ, c_μ are given by* (4), (7), (9)*. If $w \in C_{\mathrm{b}}^\infty(\mathbb{R}^d)$ is G_μ-periodic, then*

$$\mathcal{L}^\mu[w] = c_\mu \cdot Dw \quad \text{in} \quad \mathbb{R}^d.$$

Proof. Using that $\int_{\mathbb{R}^d \setminus \{0\}} f \, \mathrm{d}\mu = \int_{\operatorname{supp}(\mu)} f \, \mathrm{d}\mu$, we have

$$\mathcal{L}^\mu[w](x) = - \int\limits_{\mathbb{R}^d \setminus \{0\}} z \cdot Dw(x) \mathbf{1}_{|z| \leq 1} \, \mathrm{d}\mu(z)$$

because $w(x+z) - w(x) = 0$ for all $x \in \mathbb{R}^d$ and $z \in \operatorname{supp}(\mu) \subset G_\mu$. The result is thus immediate from Remark 2.12 and Proposition 2.13. □

2.4. Proofs of Theorems 1.1 and 1.2

We are now in a position to prove our main results. We start with Theorem 1.2 which characterizes all bounded solutions of $\mathcal{L}[u] = 0$ in $\mathbb{R}^d$ as periodic functions and specifies the set of admissible periods.

Proof of Theorem 1.2. By Lemma 1.5 we can assume that $u \in C_b^\infty(\mathbb{R}^d)$.

(a) ⇒ (b) Since $\mathcal{L}[u] = 0$ in $\mathbb{R}^d$, u is G_μ-periodic by Proposition 2.7. Proposition 2.14 then implies that

$$0 = \mathcal{L}[u] = \mathcal{L}^{\sigma,b}[u] + c_\mu \cdot Du = \mathcal{L}^{\sigma,b+c_\mu}[u] \quad \text{in} \quad \mathbb{R}^d,$$

which by Proposition 2.2 shows that u is also $W_{\sigma,b+c_\mu}$-periodic. It is now easy to see that u is $\overline{G_\mu + W_{\sigma,b+c_\mu}}$-periodic.

(b) ⇒ (a) Since u is both G_μ and $W_{\sigma,b+c_\mu}$-periodic, by first applying Proposition 2.14 and then Proposition 2.2, $\mathcal{L}[u] = \mathcal{L}^{\sigma,b+c_\mu}[u] = 0$ in $\mathbb{R}^d$. □

We now prove Theorem 1.1 on necessary and sufficient conditions for $\mathcal{L}$ to satisfy the Liouville property. We will use the following consequence of Theorem 2.9.

Corollary 2.15. *A subgroup G of $\mathbb{R}^d$ is dense if and only if there are no $c \in \mathbb{R}^d$ and codimension 1 subspace $H \subset \mathbb{R}^d$ such that $G \subseteq H + c\mathbb{Z}$.*

Proof. Let us argue by contraposition for both the "only if" and "if" parts.

(⇒) Assume $G \subseteq H + c\mathbb{Z}$ for some codimension 1 space H and $c \in \mathbb{R}^d$. If $c \in H$, then $\overline{G} \subseteq \overline{H} = H \neq \mathbb{R}^d$. If $c \notin H$, then $\mathbb{R}^d = H \oplus \operatorname{span}_\mathbb{R}\{c\}$, and each $x \in \mathbb{R}^d$ can be written as $x = x_H + \lambda_x c$ for a unique $(x, \lambda_x) \in H \times \mathbb{R}$. Hence $H + c\mathbb{Z} = \{x : \lambda_x \in \mathbb{Z}\}$ is closed by continuity of the projection $x \mapsto \lambda_x$, and $\overline{G} \subseteq H + c\mathbb{Z} \neq \mathbb{R}^d$.

(⇐) Assume $\overline{G} \neq \mathbb{R}^d$. By Theorem 2.9, $\overline{G} = V \oplus \Lambda$ for a subspace V and lattice Λ with $V \cap \operatorname{span}_\mathbb{R}\Lambda = \{0\}$. It follows that the dimensions n of V and m of the vector space $\operatorname{span}_\mathbb{R}\Lambda$ satisfy $n < d$ and $n + m \leq d$. If $m = 0$, $G \subseteq V \subseteq H$ for some codimension 1 space H. If $m \geq 1$, then $\Lambda = \oplus_{i=1}^m a_i\mathbb{Z}$ for some basis $\{a_1, \ldots, a_m\}$ of $\operatorname{span}_\mathbb{R}\Lambda$. Let $W := V \oplus \operatorname{span}_\mathbb{R}\{a_i : i \neq m\}$ for $m > 1$ and $W := V$ for $m = 1$. Then W is of dimension $n + m - 1 \leq d - 1$ and contained in some codimension 1 space H. Hence $G \subseteq H + c\mathbb{Z}$ with $c = a_m$. □

Proof of Theorem 1.1. (b) ⇒ (a) If $u \in L^\infty(\mathbb{R}^d)$ satisfy $\mathcal{L}[u] = 0$ in $\mathcal{D}'(\mathbb{R}^d)$, then u is $\overline{G_\mu + W_{\sigma,b+c_\mu}}$-periodic by Theorem 1.2. Hence u is constant by (b).

(a) ⇒ (b) Assume (b) does not hold and let us construct a nontrivial $\overline{G_\mu + W_{\sigma,b+c_\mu}}$-periodic L^∞-function. By Corollary 2.15,

$$\overline{G_\mu + W_{\sigma,b+c_\mu}} \subseteq H + c\mathbb{Z}, \tag{10}$$

for some $c \in \mathbb{R}^d$ and codimension 1 subspace $H \subset \mathbb{R}^d$. We can assume $c \notin H$ since otherwise (10) will hold if we redefine c to be any element in H^c. As before, each $x \in \mathbb{R}^d$ can be written as $x = x_H + \lambda_x c$ for a unique pair $(x_H, \lambda_x) \in H \times \mathbb{R}$. Now let $U(x) := \cos(2\pi\lambda_x)$ and note that for any $h \in H$ and $n \in \mathbb{Z}$,

$$x + h + nc = \underbrace{(x_H + h)}_{\in H} + \underbrace{(\lambda_x + n)}_{\in \mathbb{R}} c,$$

so that

$$U(x + h + nc) = \cos(2\pi(\lambda_x + n)) = \cos(2\pi\lambda_x) = U(x).$$

This proves that U is $(H + c\mathbb{Z})$-periodic and thus also $\overline{G_\mu + W_{\sigma,b+c_\mu}}$-periodic. By Theorem 1.2, $\mathcal{L}[U] = 0$, and we have a nonconstant counterexample of (a). Note indeed that $u \in L^\infty(\mathbb{R}^d)$ since it is everywhere bounded by construction and C^∞ (thus measurable) because the projection $x \mapsto \lambda_x$ is linear. We therefore conclude that (a) implies (b) by contraposition. □

3. Examples

Let us give examples for which the Liouville property holds or fails. We will use Theorem 1.1 or the following reformulation:

Corollary 3.1. *Under the assumptions of Theorem 1.1, $\mathcal{L}$ does* not *satisfy the Liouville property if and only if*

$$\operatorname{supp}(\mu) + W_{\sigma,b+c_\mu} \subseteq H + c\mathbb{Z}, \tag{11}$$

for some codimension 1 subspace H and vector c of $\mathbb{R}^d$.

Proof. Just note that $\overline{G(\operatorname{supp}(\mu) + W_{\sigma,b+c_\mu})} = \overline{G_\mu + W_{\sigma,b+c_\mu}}$ and apply Theorem 1.1 and Corollary 2.15. □

Example 3.2.

(a) For nonlocal operators $\mathcal{L} = \mathcal{L}^\mu$ with μ symmetric, (11) reduces to

$$\operatorname{supp}(\mu) \subseteq H + c\mathbb{Z}, \tag{12}$$

for some H of codimension 1 and c. This fails for fractional Laplacians, relativistic Schrödinger operators, convolution operators, or most nonlocal operators appearing in finance whose Lévy measures contain an open ball in their supports. In particular all these operators have the Liouville property.

(b) Even if $\operatorname{supp}(\mu)$ has an empty interior, (12) may fail and Liouville still holds. This is e.g. the case for the mean value operator

$$\mathcal{M}[u](x) = \int_{|z|=1} \big(u(x+z) - u(x)\big)\, \mathrm{d}S(z), \tag{13}$$

where S denotes the $d-1$-dimensional surface measure.

(c) We may have in fact the Liouville property with just a finite number of points in the support of μ, see Example 3.7.

(d) The way we have defined the nonlocal operator, if $\mathcal{L} = \mathcal{L}^\mu$ with general μ, (11) reduces to

$$\operatorname{supp}(\mu) \subseteq H + c\mathbb{Z} \quad \text{and} \quad c_\mu \in H, \tag{14}$$

for some H of codimension 1 and $c \in \mathbb{R}^d$. We can have (12) without (14) as e.g. for the 1–d measure $\mu = \delta_{-1} + 2\delta_1$. Indeed $\operatorname{supp}(\mu) \subset \mathbb{Z}$ but $c_\mu = 1 \neq 0$. The associated operator $\mathcal{L}^\mu$ then has the Liouville property even though it would not for any symmetric measure with the same support.

(e) A general operator $\mathcal{L} = \mathcal{L}^{\sigma,b} + \mathcal{L}^\mu$ may satisfy the Liouville property even though each part $\mathcal{L}^{\sigma,b}$ and $\mathcal{L}^\mu$ does not. A simple 3–d example is given by $\mathcal{L} = \partial^2_{x_1} + \partial_{x_2} + (\partial^2_{x_3})^\alpha$, $\alpha \in (0,1)$.
Indeed $\sigma = (1,0,0)^{\mathsf{T}}$, $b = (0,1,0)$, $\mathrm{d}\mu(z) = \frac{c(\alpha)\,\mathrm{d}z_3}{|z_3|^{1+2\alpha}}$ with $c(\alpha) > 0$, thus $c_\mu = 0$, $W_{\sigma,b} = \mathbb{R} \times \mathbb{R} \times \{0\}$, and $G_\mu = \{0\} \times \{0\} \times \mathbb{R}$, so the result follows from Theorem 1.1.

(f) For other kinds of interactions between the local and nonlocal parts, see Example 3.8.

Remark 3.3. The Liouville property for the nonlocal operator (13) implies the classical Liouville result for the Laplacian, since $\mathcal{M}[u] = 0$ for harmonic functions u.

In the 1–d case, the general form of the operators which do not satisfy the Liouville property is very explicit.

Corollary 3.4. *Assume $d = 1$ and $\mathcal{L} : C_c^\infty(\mathbb{R}) \to C(\mathbb{R})$ is a linear translation invariant operator satisfying the maximum principle (1). Then the following statements are equivalent:*

(a) *There are nonconstant $u \in L^\infty(\mathbb{R})$ satisfying $\mathcal{L}[u] = 0$ in $\mathcal{D}'(\mathbb{R})$.*
(b) *There are $g > 0$ and a nonnegative $\{\omega_n\}_n \in l^1(\mathbb{Z})$ such that*

$$\mathcal{L}[u](x) = \sum_{n \in \mathbb{Z}} (u(x + ng) - u(x))\omega_n.$$

Proof. If (b) holds, any g-periodic function satisfies $\mathcal{L}[u] = 0$ in $\mathbb{R}$. Conversely, if (a) holds then $\mathcal{L}$ is of the form (2)–(3)–(4) by [13]. By Corollary 3.1, there is $g \geq 0$ such that $\operatorname{supp}(\mu) + W_{\sigma,b+c_\mu} \subseteq g\mathbb{Z}$. In particular $\sigma = b + c_\mu = 0$ and μ is a sum of Dirac measures: $\mu = \sum_{n \in \mathbb{Z}} \omega_n \delta_{ng}$.[7] By ($\mathrm{A}_\mu$), each $\omega_n \geq 0$ and $\sum_{n \in \mathbb{Z}} \omega_n < \infty$. Injecting these facts into (2)–(3)–(4), we can easily rewrite $\mathcal{L}$ as in (b). □

Example 3.5.

(a) In 1–d, the Liouville property holds for any nontrivial operator with nondiscrete Lévy measure.
(b) For discrete Lévy measures, we need $\sigma \neq 0$ or $b \neq -c_\mu$ or $G_\mu = \mathbb{R}$ for Liouville to hold. The condition $G_\mu = \mathbb{R}$ is typically satisfied if $\overline{\operatorname{supp}(\mu)}^{\mathbb{R}}$ has an accumulation point or if $\operatorname{supp}(\mu)$ contains two points z_1, z_2 with irrational ratio $\frac{z_1}{z_2}$ (see Theorem 3.6). Another example is when $\operatorname{supp}(\mu) = \{\frac{n^2+1}{n}\}_{n \geq 1}$, which has no accumulation point or contains any pair with irrational ratio.

Let us continue with interesting consequences of the Kronecker theorem on Diophantine approximation (p. 507 in [21]).

Theorem 3.6 *(Kronecker theorem). Let $c = (c_1, \ldots, c_d) \in \mathbb{R}^d$. Then $\overline{c\mathbb{Z} + \mathbb{Z}^d} = \mathbb{R}^d$ if and only if $\{1, c_1, \ldots, c_d\}$ is linearly independent over $\mathbb{Q}$.*

We can use this result to get the Liouville property with just a finite number of points in the support of the Lévy measure.

Example 3.7.

[7] If $g = 0$ then $\mu = 0$ and the rest of the proof is trivial.

(a) Consider the operator

$$\mathcal{L}[u](x) = u(x+c) + \sum_{i=1}^{d} u(x+e_i) - (d+1)u(x)$$

for some $c = (c_1, \dots, c_d) \neq 0$ where $\{e_1, \dots, e_d\}$ is the canonical basis. Liouville holds if and only if $\{1, c_1, \dots, c_d\}$ is linearly independent over $\mathbb{Q}$. Indeed $G_\mu = \overline{c\mathbb{Z} + \mathbb{Z}^d}$, so the result follows from Theorems 1.1 and 3.6.

(b) For more general operators $\mathcal{L}[u](x) = \sum_{z \in S}(u(x+z) - u(x))\omega(z)$, with S finite and $\omega(\cdot) > 0$, we may have similar results by applying Theorem 3.6 (or variants) and changing coordinates.

Let us end with an illustration of how the local part may interact with such nonlocal operators. We give 2–d examples of the form

$$\mathcal{L}[u](x) = \tilde{b}_1 u_{x_1} + \tilde{b}_2 u_{x_2} + u(x+z_1) + u(x+z_2) - 2u(x)$$

where $\tilde{b}$ represents the full drift $b + c_\mu$.

Example 3.8.

(a) If $\tilde{b}, z_1, z_2$ are collinear, Liouville does not hold by Theorem 1.1.

(b) If z_1 and z_2 are collinear and linearly independent of $\tilde{b}$ as in

$$\mathcal{L}[u](x) = u_{x_1}(x) + u(x_1, x_2+\alpha) + u(x_1, x_2+\beta) - 2u(x),$$

then the Liouville property holds if and only if $\frac{\alpha}{\beta} \notin \mathbb{Q}$.
Indeed, here we have $G_\mu = \{0\} \times \overline{\alpha\mathbb{Z} + \beta\mathbb{Z}}$ and $\operatorname{span}_{\mathbb{R}}\{b + c_\mu = (1,0)\} = \mathbb{R} \times \{0\}$, so we conclude by Theorems 1.1 and 3.6.

(c) If $\{z_1, z_2\}$ is a basis of $\mathbb{R}^2$ as in

$$\mathcal{L}[u](x) = \tilde{b}_1 u_{x_1}(x) + \tilde{b}_2 u_{x_2}(x) + u(x_1+1, x_2) + u(x_1, x_2+1) - 2u(x),$$

then Liouville holds if and only if $\tilde{b}_1 \neq 0$ and $\frac{\tilde{b}_2}{\tilde{b}_1} \notin \mathbb{Q}$.
Indeed, let us define $G := G_\mu + W_{\sigma, b+c_\mu}$ where we note that $G_\mu = \mathbb{Z}^2$ and $W_{\sigma,b+c_\mu} = \operatorname{span}_{\mathbb{R}}\{(\tilde{b}_1, \tilde{b}_2)\}$. If $\tilde{b}_1 = 0$ or $\tilde{b}_2 = 0$, then $\overline{G} \subseteq \mathbb{Z} \times \mathbb{R}$ or $\mathbb{R} \times \mathbb{Z}$ which is not $\mathbb{R}^2$. Assume now that $\tilde{b}_1, \tilde{b}_2 \neq 0$ and $\frac{\tilde{b}_2}{\tilde{b}_1} \in \mathbb{Q}$, i.e., $\frac{\tilde{b}_2}{\tilde{b}_1} = \frac{p}{q}$ with $p, q \neq 0$. Then

$$G \subseteq T := \Big(\frac{1}{p}, 0\Big)\mathbb{Z} + \operatorname{span}_{\mathbb{R}}\Big\{\Big(1, \frac{\tilde{b}_2}{\tilde{b}_1}\Big)\Big\} = \Big\{\Big(\frac{k}{p} + r, r\frac{p}{q}\Big) : k \in \mathbb{Z},\ r \in \mathbb{R}\Big\}$$

since $\operatorname{span}_{\mathbb{R}}\{(\tilde{b}_1, \tilde{b}_2)\} = \operatorname{span}_{\mathbb{R}}\{(1, \frac{\tilde{b}_2}{\tilde{b}_1})\} \subset T$ and $\mathbb{Z}^2 \subset T$. The last statement follows since for any $(m, n) \in \mathbb{Z}^2$, we can take $k = pm - qn \in \mathbb{Z}$ and $r = n\frac{q}{p} \in \mathbb{R}$. Since $\overline{T} \neq \mathbb{R}^2$, Liouville does not hold by Theorem 1.1 and Corollary 2.15.
Conversely, assume $\tilde{b}_1, \tilde{b}_2 \neq 0$ and $\frac{\tilde{b}_2}{\tilde{b}_1} \notin \mathbb{Q}$. Then $(0, \frac{\tilde{b}_2}{\tilde{b}_1}) = (-1, 0) + (1, \frac{\tilde{b}_2}{\tilde{b}_1}) \in G$ and since $(0,1) \in G$, we get that $\{0\} \times (\mathbb{Z} + \frac{\tilde{b}_2}{\tilde{b}_1}\mathbb{Z}) \subset G$. By Theorem 3.6, $\{0\} \times \mathbb{R} \subset \overline{G}$. Arguing similarly with $(\frac{\tilde{b}_1}{\tilde{b}_2}, 0)$, we find that $\mathbb{R} \times \{0\} \subset \overline{G}$. Hence $\overline{G} = \mathbb{R}^2$ and Liouville holds by Theorem 1.1.

Acknowledgements

F.d.T. and J.E. are grateful to Laboratoire de Mathématiques de Besançon (LMB, UBFC) and Ecole Nationale Supérieure de Mécanique et des Microtechniques (ENSMM) for hosting them during their visit in May 2018.

During the final preparation of this paper, we appreciated the feedback from the community which helped us to put the paper in context and also to improve the presentation.

References

[1] F. Andreu-Vaillo, J.M. Mazón, J.D. Rossi, J.J. Toledo-Melero, Nonlocal Diffusion Problems, Mathematical Surveys and Monographs, vol. 165, American Mathematical Society/Real Sociedad Matemática Española, Providence, RI/Madrid, 2010.
[2] M.T. Barlow, R.F. Bass, C. Gui, The Liouville property and a conjecture of De Giorgi, Commun. Pure Appl. Math. 53 (8) (2000) 1007–1038.
[3] H. Berestycki, F. Hamel, L. Rossi, Liouville-type results for semilinear elliptic equations in unbounded domains, Ann. Mat. Pura Appl. (4) 186 (3) (2007) 469–507.
[4] K. Bogdan, T. Kulczycki, A. Nowak, Gradient estimates for harmonic and q-harmonic functions of symmetric stable processes, Ill. J. Math. 46 (2) (2002) 541–556.
[5] C. Brändle, E. Chasseigne, F. Quirós, Phase transitions with midrange interactions: a nonlocal Stefan model, SIAM J. Math. Anal. 44 (4) (2012) 3071–3100.
[6] J. Brasseur, J. Coville, A counterexample to the Liouville property of some nonlocal problems, preprint, arXiv:1804.07485v1 [math.AP], 2018.
[7] J. Brasseur, J. Coville, F. Hamel, E. Valdinoci, Liouville type results for a nonlocal obstacle problem, Proc. Lond. Math. Soc. 119 (2) (2019) 291–328.
[8] X. Cabré, Y. Sire, Nonlinear equations for fractional Laplacians, I: regularity, maximum principles, and Hamiltonian estimates, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 31 (1) (2014) 23–53.
[9] W. Chen, L. D'Ambrosio, Y. Li, Some Liouville theorems for the fractional Laplacian, Nonlinear Anal. 121 (2015) 370–381.
[10] G. Choquet, J. Deny, Sur l'équation de convolution $\mu = \mu * \sigma$, C. R. Acad. Sci. Paris 250 (1960) 799–801.
[11] A. Ciomaga, On the strong maximum principle for second-order nonlinear parabolic integro-differential equations, Adv. Differ. Equ. 17 (7–8) (2012) 635–671.
[12] R. Cont, P. Tankov, Financial Modelling with Jump Processes, Chapman & Hall/CRC Financial Mathematics Series, Chapman & Hall/CRC, Boca Raton, FL, 2004.
[13] P. Courrège, Générateur infinitésimal d'un semi-groupe de convolution sur R^n, et formule de Lévy-Khinchine, Bull. Sci. Math. (2) 88 (1964) 3–30.
[14] J. Coville, Remarks on the strong maximum principle for nonlocal operators, Electron. J. Differ. Equ. 66 (2008) 10.
[15] F. del Teso, J. Endal, E.R. Jakobsen, On distributional solutions of local and nonlocal problems of porous medium type, C. R. Math. Acad. Sci. Paris 355 (11) (2017) 1154–1160.
[16] F. del Teso, J. Endal, E.R. Jakobsen, Uniqueness and properties of distributional solutions of nonlocal equations of porous medium type, Adv. Math. 305 (2017) 78–143.
[17] F. del Teso, J. Endal, E.R. Jakobsen, Robust numerical methods for nonlocal (and local) equations of porous medium type. Part II: schemes and experiments, SIAM J. Numer. Anal. 56 (6) (2018) 3611–3647.
[18] M.M. Fall, Entire s-harmonic functions are affine, Proc. Am. Math. Soc. 144 (6) (2016) 2587–2592.
[19] M.M. Fall, T. Weth, Liouville theorems for a general class of nonlocal operators, Potential Anal. 45 (1) (2016) 187–200.
[20] A. Farina, Liouville-type theorems for elliptic problems, in: Handbook of Differential Equations: Stationary Partial Differential Equations. Vol. IV, in: Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2007, pp. 61–116.
[21] S.M. Gonek, H.L. Montgomery, Kronecker's approximation theorem, Indag. Math. (N. S.) 27 (2) (2016) 506–523.
[22] Q. Huang, J. Duan, J.-L. Wu, Maximum principles for nonlocal parabolic Waldenfels operators, Bull. Math. Sci. (2018).
[23] N.S. Landkof, Foundations of Modern Potential Theory, Springer-Verlag, New York-Heidelberg, 1972.
[24] J. Martinet, Perfect Lattices in Euclidean Spaces, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 327, Springer-Verlag, Berlin, 2003.
[25] Y. Miyazaki, Liouville's theorem and heat kernels, Expo. Math. 33 (1) (2015) 101–104.
[26] E. Nelson, A proof of Liouville's theorem, Proc. Am. Math. Soc. 12 (1961) 995.
[27] E. Priola, J. Zabczyk, Liouville theorems for non-local operators, J. Funct. Anal. 216 (2) (2004) 455–490.
[28] X. Ros-Oton, J. Serra, Boundary regularity for fully nonlinear integro-differential equations, Duke Math. J. 165 (11) (2016) 2079–2154.
[29] L. Rossi, Liouville type results for periodic and almost periodic linear operators, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 26 (6) (2009) 2481–2502.
[30] R.L. Schilling, J. Wang, On the coupling property and the Liouville theorem for Ornstein-Uhlenbeck processes, J. Evol. Equ. 12 (1) (2012) 119–140.