Adaptive Observer Design for an $n+1$ Hyperbolic PDE with Uncertainty and Sensing on Opposite Ends

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***Abstract*— An adaptive observer design for a system of $n+1$ coupled 1-D linear hyperbolic partial differential equations with an uncertain boundary condition is presented, extending previous results by removing the need for sensing collocated with the uncertainty. This modification is important and motivated by applications in oil & gas drilling where information about the down-hole situation is crucial in order to prevent or deal with unwanted incidents. Uncertainties are usually present down-hole while measurements are available top-side at the rig, only. Boundedness of the state and parameter estimates is proved in the general case, while convergence to true values requires bounded system states and, for parameter convergence, persistent excitation. The central tool for analysis is the infinite-dimensional backstepping method applied in two steps, the first of which is time-invariant, while the second is time-varying induced by the time-varying parameter estimates.**

I. Introduction

A. Problem formulation

We consider the system of linear first-order hyperbolic Partial Differential Equations (PDEs) with n positive convecting invariants and 1 negative convecting invariant given by

$$u_t + \Lambda u_x = \Sigma(x)u + \omega(x)v \tag{1a}$$
$$v_t - \mu v_x = \varpi(x)u \tag{1b}$$

where $u \in \mathbb{R}^n$ is the *upward* propagating Riemann invariants, $v \in \mathbb{R}$ is the single *downward* propagating Riemann invariant, $\Sigma : [0,1] \to \mathbb{R}^{n\times n}$ with diagonal terms being zero, $\omega : [0,1] \to \mathbb{R}^{n\times 1}$, $\varpi : [0,1] \to \mathbb{R}^{1\times n}$, and $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ and μ satisfying $-\mu < 0 < \lambda_1 < \cdots < \lambda_n$. We consider boundary conditions on the form

$$u(0,t) = qv(0,t) + d \tag{2a}$$
$$v(1,t) = \rho u(1,t) + U(t) \tag{2b}$$

where $q = \{q_i\}_{1\le i\le n} \in \mathbb{R}^n$ and $d = \{d_i\}_{1\le i\le n} \in \mathbb{R}^n$ are unknown, $\rho \in \mathbb{R}^{1\times n}$ is known, and $U : [0,\infty) \to \mathbb{R}$ can be any known time-varying function. In addition, we assume that

$$y(t) := u(1,t) \tag{3}$$

is measured. The initial conditions

$$u(x,0) =: u_{ic}(x) \tag{4a}$$
$$v(x,0) =: v_{ic}(x) \tag{4b}$$

satisfy certain compatibility conditions, making the Cauchy problem (1)–(4) well-posed (see e.g. [1, Theorem 3.1]).

B. A motivating application

The system (1) can be used to model, among others, various phenomena in multiphase fluid flows. An overview of other possible applications ranging from open-channel networks to transmission lines can be found in [2]. In this paper, we are concerned with the problem of state estimation in fluid flow systems where one of the boundaries is specified in terms of uncertain parameters and sensing is limited to the opposite boundary. The motivation is an application in oil & gas drilling where only top-side flow measurements are available and the bottom-hole flow is influenced by an oil/gas reservoir with unknown properties.

In the drilling application, mud is circulated down the drill-string, through the drill-bit at the bottom, and up in the open annulus surrounding the drill-string back up to the rig where flow is measured. See Figure 1. If the pressure down-hole is lower than the reservoir pressure, oil, water or even gas might start flowing into the well and up the annulus. This is called a kick and can, if not handled, lead to catastrophic consequences when the reservoir fluids reach the surface. If properties such as the reservoir pressure is unknown, handling and even detecting such influxes of reservoir fluids is a very challenging problem. Previous methods of detecting and estimating influxes have mainly focused on lumped-order models [3]–[11], were the distributed dynamics are neglected. Some results using the so called *early-lumping* approach where the PDE model is spatially discretized and approximated by a set of ODEs have also been explored [12]–[14]. In this paper, we propose to use the contrasting *late-lumping* approach where the observer is derived for the distributed model, and discretization is only necessary for computer implementation.

In the drift-flux model, which is the most commonly used model for drilling applications involving gas, all liquids (mud, oil, water) are lumped into a single phase, and gas is considered separately. Following [15], the drift flux model proposed in [16] can be written on conservative form in terms of pressure, gas fraction and gas velocity by the quasi-linear 3×3 system

$$w_t + A(w)w_x = S(w) \tag{5}$$

over the domain $(x,t) \in [0,1] \times [0,\infty)$, where A and S are complicated and given in [15]. System (5) can be linearized around a given operating profile, diagonalized and written in terms of Riemann invariants to obtain the form (1) with $n = 2$ [17].

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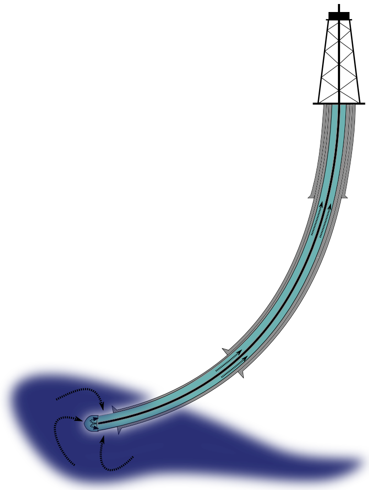

Fig. 1. Schematic of the drilling system.

Interaction with the reservoir is modeled by the boundary conditions which are assumed to equal the bottom-hole net liquid and gas inflow $q_L(0,t)$ and $q_G(0,t)$, respectively. It is common to model them as proportional to the pressure difference between the bottom-hole pressure $p(0,t)$ and reservoir pressure $p_{res}(t)$, that is,

$$q_L(0,t) = J_L(p_{res}(t) - p(0,t)) \tag{6a}$$
$$q_G(0,t) = J_G(p_{res}(t) - p(0,t)) \tag{6b}$$

where J_L and J_G are constants called *production indices* (PI). Both PI's J_G and J_L and the reservoir pressure p_{res} are assumed to be unknown and (6) can be rephrased in the form (2a). For the top-side boundary condition at $x = 1$, we assume that the pressure $p(1,t)$ and flow $q(1,t)$ are known (one is measured, the other is a control input).

C. Relevant previous results

We use the much celebrated backstepping method for infinite dimensional systems, first derived for hyperbolic systems in [18] and later extended to 2×2 systems [19] and $m+n$ systems [20]. In the adaptive setting with uncertain boundary parameters, an observer for a 2×2 system only relying on measurements on the boundary opposite of the boundary with the uncertain parameters is derived in [21], [22]. For $n+1$ systems, the adaptive observer problem has been solved in [23]–[25] utilizing sensing at the same boundary as the uncertain parameters. Particularly relevant to this paper is [25] which considers an application in under-balanced drilling. Worth mentioning is also [17] where the backstepping approach is used to control multiphase flows in drilling using state-feedback.

In this paper, we extend the observer in [21] and derive an observer for $n+1$ systems with parametric uncertainties on one boundary and measurements taken at the opposite boundary. A disadvantage of the method in [21] is that the observer injection gains have to be updated on-line with every new time-varying parameter estimate. We propose a design avoiding the on-line recalculation of injection gains by instead solving a set of computationally simpler transport equations on-line. The estimation scheme with state observer and adaptive laws including stability proofs are presented in Section II. Some concluding remarks are offered in Section III.

D. Notation

For a signal $z : [0,1]\times[0,\infty) \to \mathbb{R}^n$, partial derivatives with respect to i.e. space are denoted z_x or $\partial_x z_i$ for each element $i = 1,...,n$. The L_2-norm is denoted

$$||z|| := \sqrt{\int_0^1 z^T(x,t)z(x,t)dx}. \tag{7}$$

For $f : [0,\infty) \to \mathbb{R}$, we use the vector spaces

$$f \in \mathscr{L}_p \leftrightarrow \left(\int_0^\infty |f(t)|^p dt\right)^{\frac{1}{p}} < \infty \tag{8}$$

for $p \geq 1$ with the particular case

$$f \in \mathscr{L}_\infty \leftrightarrow \sup_{t\geq 0} |f(t)| < \infty. \tag{9}$$

Derivatives with respect to time are denoted $\dot{f}$. If not otherwise stated, a statement for a variable with subscript i refers to all variables with subscript $i = 1,...,n$.

II. State and parameter estimation

Let $(\hat{u},\hat{v}) \in \mathbb{R}^{n+1}$ denote the state estimates. We consider the observer system

$$\hat{u}_t + \Lambda\hat{u}_x = \Sigma(x)\hat{u} + \omega(x)\hat{v} + P_1(x,t)(y(t) - \hat{u}(1,t)) \tag{10a}$$
$$\hat{v}_t - \mu\hat{v}_x = \varpi(x)\hat{u} + P_2(x,t)(y(t) - \hat{u}(1,t)) \tag{10b}$$
$$\hat{u}(0,t) = \hat{q}(t)\hat{v}(0,t) + \hat{d}(t) \tag{10c}$$
$$\hat{v}(1,t) = \rho u(1,t) + U(t) \tag{10d}$$

with compatible initial conditions $(\hat{u}(x,0),\hat{v}(x,0)) = (\hat{u}_{ic}(x),\hat{v}_{ic}(x))$ and where $\hat{q}_i(t)$ and $\hat{d}_i(t)$ are the parameter estimates of q and d. The injection gains P_1 and P_2 have the structure

$$P_1(x,t) = M(x,1)\Lambda + G(x,1,t)\Lambda + \int_x^1 M(x,\xi)G(\xi,1,t)\Lambda d\xi \tag{11a}$$
$$P_2(x,t) = N(x,1)\Lambda + \int_x^1 N(x,\xi)G(\xi,1,t)\Lambda d\xi \tag{11b}$$

where (M,N,G) are Volterra integral kernels and will be specified further in the next sections. The state estimation error $\tilde{u} = u - \hat{u}$, $\tilde{v} = v - \hat{v}$ then satisfies

$$\tilde{u}_t + \Lambda\tilde{u}_x = \Sigma(x)\tilde{u} + \omega(x)\tilde{v} - P_1(x,t)\tilde{u}(1,t) \tag{12a}$$
$$\tilde{v}_t - \mu\tilde{v}_x = \varpi(x)\tilde{u} - P_2(x,t)\tilde{u}(1,t) \tag{12b}$$
$$\tilde{u}(0,t) = \hat{q}(t)\tilde{v}(0,t) + \tilde{q}(t)v(0,t) + \tilde{d}(t) \tag{12c}$$
$$\tilde{v}(1,t) = 0 \tag{12d}$$

where $\tilde{q}(t) = q - \hat{q}(t)$ and $\tilde{d} = d - \hat{d}(t)$. The design strategy is as follows: In Section II-A, we specify the (M,N)-kernels and show that the estimation error system (12) is equivalent to a simpler target system. This target system is used to derive a parametric model relating the unknown parameters to some known signals in Section II-B. Equivalence to yet another target system are shown in Section II-C by specifying the remaining G-kernel. Properties of this final target system together with appropriate adaptive laws based on the parametric model are used in Theorem 1 in Section II-D to state the main contribution of this paper on state end parameter estimation.

A. Backstepping transformation

Lemma 1: Let $\bar{q} = \{\bar{q}_i\}_{1\leq i\leq n} \in \mathbb{R}^n$. On the triangular domain $\mathscr{T}_1 = \{(x,\xi)|0 \leq x \leq \xi \leq 1\}$, the backstepping transformation

$$\tilde{u}(x,t) = \alpha(x,t) + \int_x^1 M(x,\xi)\alpha(\xi,t)d\xi \tag{13a}$$

$$\tilde{v}(x,t) = \beta(x,t) + \int_x^1 N(x,\xi)\alpha(\xi,t)d\xi, \tag{13b}$$

with kernels $M = \{M_{ij}(x,t)\}_{1\leq i,j\leq n} : \mathscr{T}_1 \to \mathbb{R}^{n\times n}$, $N = \{N_i(x,t)\}_{1\leq i\leq n} : \mathscr{T}_1 \to \mathbb{R}^{1\times n}$ satisfying

$$M_\xi\Lambda + \Lambda M_x = \Sigma(x)M + \omega(x)N \tag{14a}$$

$$N_\xi\Lambda - \mu N_x = \varpi(x)M, \tag{14b}$$

$$\Sigma(x) = M(x,x)\Lambda - \Lambda M(x,x) \tag{15a}$$

$$\varpi(x) = N(x,x)\Lambda + \mu N(x,x), \tag{15b}$$

$$M_{ij}(0,\xi) = \bar{q}_i N_j(0,\xi),\ 1 \leq i \leq j \leq n \tag{16}$$

and

$$M_{ij}(x,1) = \frac{\Sigma_{ij}(x)}{\lambda_j - \lambda_i}, \quad 1 \leq j < i \leq n \tag{17}$$

is invertible and maps the target system

$$\alpha_t(x,t) + \Lambda\alpha_x(x,t) = \omega(x)\beta(x,t) - G(x,1,t)\Lambda\alpha(1,t) - \int_x^1 A(x,\xi)\beta(\xi,t)d\xi \tag{18a}$$

$$\beta_t(x,t) - \mu\beta_x(x,t) = -\int_x^1 B(x,\xi)\beta(\xi,t)d\xi \tag{18b}$$

$$\alpha(0,t) = \hat{q}(t)\beta(0,t) + \int_0^1 H(\xi,t)\alpha(\xi,t)d\xi + \tilde{q}(t)v(0,t) + \tilde{d}(t) \tag{18c}$$

$$\beta(1,t) = 0 \tag{18d}$$

where $G = \{g_{ij}(x,t)\}_{1\leq i,j\leq n}$ is an upper triangular matrix to be decided, A and B satisfy

$$A(x,\xi) = M(x,\xi)\omega - \int_x^\xi M(x,s)A(s,\xi)ds \tag{19a}$$

$$B(x,\xi) = N(x,\xi)\omega - \int_x^\xi N(x,s)A(s,\xi)ds, \tag{19b}$$

and $H(\xi,t) = \{h_{ij}(\xi,t)\}_{1\leq i,j\leq n}$ is defined by

$$h_{ij}(\xi,t) := \hat{q}_i(t)N_j(0,\xi) - M_{ij}(0,\xi), \tag{20}$$

into the error system (12) with injection gains (11). Moreover, the kernel equation (14)–(17) has a unique solution. The target system (18), but without the $G(x,1,t)\alpha(1,t)$ term, and injection gains $P_1(x) = M(x,1)\Lambda$ and $P_2(x) = N(x,1)\Lambda$, was first used in [24] for non-collocated observer design for $n+1$ systems, which itself was a straightforward application of the kernel equations derived in [20]. The effect of including the $G(x,1,t)\alpha(1,t)$ term in the target system can be seen by substituting $G(x,1,t)\alpha(1,t)$ for $\alpha(\xi,t)$ in (13) showing the origin of the injection gains (11). The proof of Lemma 1 is therefore omitted.

Remark 1: The constant $\bar{q}$ can be chosen arbitrarily, but better performance is expected if it is chosen as close to q as possible, i.e. $\bar{q} = \hat{q}(0)$, presuming $\hat{q}(0)$ is our best guess of q at $t = 0$. Due to (16), we have

$$h_{ij}(\xi,t) = (\hat{q}_i - \bar{q}_i)N_j \tag{21}$$

for all $j \geq i$, meaning that $H(\xi,t)$ will in general only be strictly lower triangular for all $\xi \in [0,1]$ if $\bar{q} = \hat{q}(t)$. The remaining injection gain $G(x,1,t)\Lambda$ left unspecified in (18a) (specified later in Section II-C) will be used to handle the time-varying discrepancy $(\bar{q} - \hat{q}(t))$.

B. Parametric model

The advantage of transforming the error system to the form (18) is that, for $t \geq \mu^{-1}$, $\beta \equiv 0$ and the α-dynamics can be solved independently for each element α_i. This solution is exploited in the next lemma to obtain a bilinear parametric model relating the unknown parameters to known signals.

Lemma 2: Let $\lambda = \min_i \lambda_i = \lambda_1$. For $t \geq t_F := \mu^{-1} + 2\lambda^{-1}$,

$$\psi_i(t) = q_i\phi(t) + d_i, \quad 1 \leq i \leq n \tag{22}$$

where

$$\begin{aligned}\psi_i(t) = &\tilde{y}_i(t+\lambda_i^{-1}-\lambda^{-1}) \\ &+ \hat{q}_i(t-\lambda^{-1})\phi(t) + \hat{d}_i(t-\lambda^{-1}) \\ &- \sum_{j=i}^n \int_{t-\lambda^{-1}}^{t+\lambda_i^{-1}-\lambda^{-1}} g_{ij}((\tau - t + \lambda^{-1})\lambda_i, \tau)\lambda_j\tilde{y}_j(\tau)d\tau \\ &- \sum_{j=1}^n \int_0^1 h_{ij}(\xi, t-\lambda^{-1}) \Bigg(\tilde{y}_j(t+\lambda^{-1}(1-\xi)-\lambda^{-1}) \\ &\quad - \sum_{l=j}^n \int_{t-\lambda^{-1}}^{t+\lambda_j^{-1}(1-\xi)-\lambda^{-1}} k_{jl}(\xi + \lambda_j(\tau+\lambda^{-1}-t),\tau) \\ &\quad \times \tilde{y}_l(\tau)d\tau\Bigg)d\xi,\end{aligned} \tag{23}$$

$$\begin{aligned}\phi(t) \equiv &-\hat{v}(0,t-\lambda^{-1}) + \sum_{i=1}^n \int_0^1 N_i(0,\xi,t-\lambda^{-1}) \\ &\times \Bigg(\sum_{j=1}^n \int_{t-\lambda^{-1}}^{t+\lambda_i^{-1}(1-\xi)-\lambda^{-1}} g_{ij}(\xi+\lambda_i(\tau+\lambda^{-1}-t),\tau) \\ &\times \lambda_j\tilde{y}_j(\tau)d\tau - \tilde{y}_i(t+\lambda_i^{-1}(1-\xi)-\lambda^{-1})\Bigg)d\xi,\end{aligned} \tag{24}$$

and

$$\tilde{y}_i(t) := \alpha_i(1,t) = y_i(t) - \hat{u}_i(1,t). \tag{25}$$

Proof: Consider the target system (18) in Lemma 1. Since $\beta \equiv 0$ for $t \geq \mu^{-1}$, we have on component form

$$\partial_t \alpha_i + \lambda_i \partial_x \alpha_i = \sum_{j=i}^{n} g_{ij}(x,t)\lambda_j \tilde{y}_j(t) \tag{26a}$$

$$\alpha_i(0,t) = \sum_{j=1}^{n} \int_0^1 h_{ij}(\xi,t)\alpha_j(\xi,t)d\xi + \tilde{q}_i(t)v(0,t) + \tilde{d}_i(t) \tag{26b}$$

with the solution

$$\alpha_i(x,t) = \sum_{j=i}^{n} \int_{t+\lambda_i^{-1}(x_0-x)}^{t} g_{ij}(x+\lambda_i(\tau-t),\tau)\lambda_j \tilde{y}_j(\tau)d\tau + \alpha_i(x_0, t+\lambda_i^{-1}(x_0-x)) \tag{27}$$

valid for all $t \geq \mu^{-1} + \lambda_i^{-1}$ and some $x_0 \in [0,1]$. Selecting $x_0 = 0$ and inserting (18c) yield

$$\begin{aligned}\alpha_i(x,t) =& \sum_{j=i}^{n} \int_{t-\lambda_i^{-1}x}^{t} g_{ij}(x+\lambda_i(\tau-t),\tau)\lambda_j \tilde{y}_j(\tau)d\tau \\ &+ \sum_{j=1}^{n} \int_0^1 h_{ij}(\xi, t-\lambda_i^{-1}x)\alpha_j(\xi, t-\lambda_i^{-1}x)d\xi \\ &+ \tilde{q}_i(t-\lambda_i^{-1}x)v(0,t-\lambda_i^{-1}x) + \tilde{d}_i(t-\lambda_i^{-1}x). \end{aligned}\tag{28}$$

Selecting $x_0 = 1$ and inserting (25) yield

$$\begin{aligned}\alpha_i(x,t) =& -\sum_{j=i}^{n} \int_{t}^{t+\lambda_i^{-1}(1-x)} g_{ij}(x+\lambda_i(\tau-t),\tau)\lambda_j \tilde{y}_j(\tau)d\tau \\ &+ \tilde{y}_i(t+\lambda_i^{-1}(1-x)). \end{aligned}\tag{29}$$

We have from (13b) for $t \geq \mu^{-1}$, and (29) that

$$\begin{aligned}&v(0,t) = \hat{v}(0,t) + \int_0^1 N(0,\xi)\alpha(\xi,t)d\xi \\ &= \hat{v}(0,t) + \sum_{i=1}^{n} \int_0^1 N_i(0,\xi)\Bigg(\tilde{y}_i(t+\lambda_i^{-1}(1-\xi)) \\ &- \sum_{j=i}^{n} \int_t^{t+\lambda_i^{-1}(1-\xi)} g_{ij}(\xi+\lambda_i(\tau-t),\tau)\lambda_j \tilde{y}_j(\tau)d\tau\Bigg)d\xi. \end{aligned}\tag{30}$$

Thus,

$$v(0,t-\lambda^{-1}) = -\phi(t). \tag{31}$$

Next, inserting the right hand side of (29) into the left hand side of (28) evaluated at $x = 1$ and $t = t + \lambda_i^{-1} - \lambda^{-1} \leq t$ yields

$$\begin{aligned}&\tilde{y}_i(t+\lambda_i^{-1}-\lambda^{-1}) = \tilde{q}_i(t-\lambda^{-1})v(0,t-\lambda^{-1}) + \tilde{d}_i(t-\lambda^{-1}) \\ &+ \sum_{j=i}^{n} \int_{t-\lambda^{-1}}^{t+\lambda_i^{-1}-\lambda^{-1}} g_{ij}((\tau-t+\lambda^{-1})\lambda_i,\tau)\lambda_j \tilde{y}_j(\tau)d\tau \\ &+ \sum_{j=1}^{n} \int_0^1 h_{ij}(\xi,t-\lambda^{-1})\alpha_j(\xi,t-\lambda^{-1})d\xi \\ &= \sum_{j=i}^{n} \int_{t-\lambda^{-1}}^{t+\lambda_i^{-1}-\lambda^{-1}} g_{ij}((\tau-t+\lambda^{-1})\lambda_i,\tau)\lambda_j \tilde{y}_j(\tau)d\tau \\ &+ q_i v(0,t-\lambda^{-1}) + d_i \\ &- \hat{q}_i(t-\lambda^{-1})v(0,t-\lambda^{-1}) - \hat{d}_i(t-\lambda^{-1}) \\ &+ \sum_{j=1}^{n} \int_0^1 h_{ij}(\xi,t-\lambda^{-1})\Bigg(\tilde{y}_j(t+\lambda^{-1}(1-\xi)-\lambda^{-1}) \\ &- \sum_{l=j}^{n} \int_{t-\lambda^{-1}}^{t+\lambda_j^{-1}(1-\xi)-\lambda^{-1}} k_{jl}(\xi+\lambda_j(\tau+\lambda^{-1}-t),\tau) \\ &\times \tilde{y}_l(\tau)d\tau\Bigg)d\xi \end{aligned}\tag{32}$$

which is equivalent to (22) in view of (23)–(25) and (31). ■

C. Properties of estimation error target system

We now show equivalence to yet another target system by specifying the G-kernel.

Lemma 3: For $t \geq \mu^{-1}$, the backstepping transformation

$$\alpha(x,t) = \eta(x,t) + \int_x^1 G(x,\xi,t)\eta(\xi,t)d\xi \tag{33}$$

with kernel $G = \{g_{ij}\}_{1\leq i,j\leq n} : \mathscr{T}_1 \times [0,\infty) \to \mathbb{R}^{n\times n}$ satisfying

$$\partial_t g_{ij} = -\lambda_j \partial_\xi g_{ij} - \lambda_i \partial_x g_{ij} \tag{34a}$$

$$g_{ij}(x,x,t) = 0 \tag{34b}$$

$$g_{ij}(0,\xi,t) = h_{ij}(\xi,t) + \sum_{k=1}^{j} \int_0^\xi h_{ik}(s,t)g_{kj}(s,\xi,t)ds \tag{34c}$$

for $1 \leq i \leq j \leq n$ and $g_{ij} \equiv 0$ for $1 \leq j < i \leq n$, which has a unique, bounded solution for every bounded h_{ij}, maps the sub-system (18a) and (18c) (recall that $\beta \equiv 0$ for $t \geq \mu^{-1}$) into the target system

$$\eta_t(x,t) + \Lambda\eta_x(x,t) = 0 \tag{35a}$$

$$\eta(0,t) = \int_0^1 \bar{H}(\xi,t)\eta(\xi,t)d\xi + \tilde{q}(t)v(0,t) + \tilde{d}(t) \tag{35b}$$

where $\bar{H}$ is the strictly lower triangular matrix

$$\bar{H}(\xi,t) := H(\xi,t) - G(0,\xi,t) + \int_0^\xi H(s,t)G(s,\xi,t)ds. \tag{36}$$

Proof: Differentiating (33) with respect to time and space, inserting the dynamics (35), and integrating by parts yield

$$\begin{aligned}&\alpha_t(x,t) + \Lambda\alpha_x(x,t) + G(\xi,1,t)\Lambda\alpha(1,t) \\ &= \eta_t(x,t) + \Lambda\eta_x(x,t) + G(\xi,1,t)\Lambda\alpha(1,t) \\ &\quad + \int_x^1 G_t(x,\xi,t)\eta(\xi,t)dsd\xi \\ &\quad + G(x,x,t)\Lambda\eta(1,t) - G(x,1,t)\Lambda\eta(1,t) \\ &\quad + \int_x^1 G_\xi(x,\xi,t)\Lambda\eta(\xi,t)d\xi \\ &\quad - \Lambda G(x,x,t)\eta(x,t) + \int_x^1 \Lambda G_x(x,\xi,t)\eta(\xi,t)d\xi \\ &= \int_x^1 \left(G_t(x,\xi,t) + G_\xi(x,\xi,t)\Lambda + \Lambda G_x(x,\xi,t)\right)\eta(\xi,t)d\xi \\ &\quad + (G(x,x,t)\Lambda - \Lambda G(x,x,t))\eta(x,t) \\ &\quad + G(x,1,t)\Lambda(\alpha(1,t) - \eta(1,t)) \end{aligned}\tag{37}$$

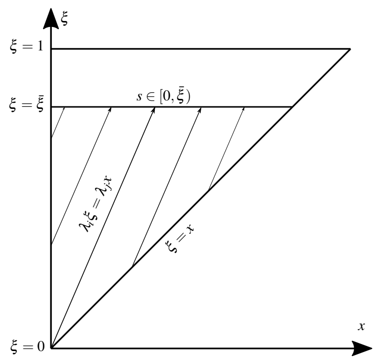

Fig. 2. Characteristic lines of g_{ij} for $j > i$.

which in view of (34) and $\alpha(1,t) = \eta(1,t)$ verifies (18) (with $\beta \equiv 0$). For the boundary condition, we have

$$\begin{aligned}\eta(0,t) =& \tilde{q}(t)v(0,t) + \tilde{d}(t) - \int_0^1 G(0,\xi,t)\eta(\xi,t)d\xi \\ &+ \int_0^1 H(\xi,t)\left(\eta(\xi,t) + \int_\xi^1 G(\xi,s,t)\eta(s,t)ds\right)d\xi \\ =& \int_0^1 \left(H(\xi,t) - G(0,\xi,t) + \int_0^\xi H(s,t)G(s,\xi,t)ds\right) \\ &\times \eta(\xi,t)d\xi + \tilde{q}(t)v(0,t) + \tilde{d}(t). \end{aligned} \tag{38}$$

Defining $\bar{H}$ as in (36), which due to (34c) is strictly lower triangular, yields (35b).

The system (34) is a set of Riemann invariants parameterized by (ξ,t) with characteristic lines $(\lambda_i\lambda_j^{-1}s, \xi + s, t + \lambda_j^{-1}s)$ originating from the $(0,\xi,t)$-boundary for $\lambda_i\xi - \lambda_j x \geq 0$. See Figure 2. Since (34c) is causal for all $\xi \in [0,1]$ in the sense that $g_{ij}(0,\xi,t)$ is uniquely specified by $h_{ij}(\xi,t)$, $h_{ik} = (s,t)$ and $g_{kj}(0,s,t-\lambda_j^{-1})$ for $s \in [0,\xi), k = 1,...,j$, it follows that there exist a unique solution $g_{ij}(0,\xi,t)$ to (34c), which in turn implies the existence of a unique solution $g_{ij}(x,\xi,t)$ to (34) for all bounded h_{ij}. Since $g_{ij}(0,\xi,t)$ can be upper bounded in terms of h_{ij}, $g_{ij}(x,\xi,t)$ is bounded. ■

D. Adaptive law and stability of estimation error

Theorem 1: Consider the system (1) and observer (10). Let $\tilde{q}_i(t) = q_i - \hat{q}_i(t)$, $\tilde{d}_i(t) = d - \hat{d}_i(t)$ and $\tilde{\psi}_i(t) := \psi_i(t) - \hat{q}_i(t)\phi(t) + \hat{d}_i(t)$. If

$$\dot{\hat{q}}_i = \gamma_{q_i}\frac{\tilde{\psi}_i(t)\phi(t)}{2+\phi^2(t)}, \quad \dot{\hat{d}}_i = \gamma_{d_i}\frac{\tilde{\psi}_i(t)}{2+\phi^2(t)} \tag{39}$$

for $t \geq t_F$ and $\dot{\hat{q}}_i = \dot{\hat{d}} = 0$ otherwise, where $\gamma_{q_i}, \gamma_{d_i} > 0$ are the adaptation gains, then

$$\hat{q}_i, \hat{d}_i \in \mathscr{L}_\infty, \quad \dot{\hat{q}}_i, \dot{\hat{d}}_i \in \mathscr{L}_\infty \cap \mathscr{L}_2 \tag{40}$$

and

$$(\tilde{u}(x,\cdot), \tilde{v}(x,\cdot)) \in \mathscr{L}_\infty. \tag{41}$$

If in addition $v(0,t)$ is bounded for all $t \geq 0$, then

$$||\tilde{u}||, ||\tilde{v}|| \to 0. \tag{42}$$

Lastly, if the *persistence of excitation* condition

$$c_1 I_{2\times 2} \geq \frac{1}{T}\int_t^{t+T} [\phi(\tau),1]^T[\phi(\tau),1]d\tau \geq c_2 I_{2\times 2} \tag{43}$$

is satisfied for some constants $c_1, c_2, T > 0$, the parameter estimates $\hat{q}_i$ and $\hat{d}_i$ converge exponentially to their true values q_i and d_i.

Proof: Consider the parametric model (22) in Lemma 2. The properties (40) of $\hat{q}$ and $\hat{d}$ follow from [26, Theorem 4.3.2], along with the fact that $\tilde{\psi}_i(2+\phi^2)^{-1} \in \mathscr{L}_2 \cap \mathscr{L}_\infty$. Let

$$\tilde{\Theta}_i(t) =: \left[\tilde{q}_i(t), \tilde{d}_i(t)\right]^T \tag{44}$$

$$\Phi_i(t) =: \frac{1}{\sqrt{1+\phi^2(t)}}\left[\phi(t),1\right]^T. \tag{45}$$

so that $\tilde{\psi}_i(t)(2+\phi^2(t))^{-1} = \Phi_i^T(t)\tilde{\Theta}_i(t)$. We have

$$\Phi_i^T(t)\tilde{\Theta}_i(t) = \Phi_i^T(t)\left(\int_{t-\lambda^{-1}}^t \dot{\tilde{\Theta}}_i(\tau)d\tau + \tilde{\Theta}_i(t-\lambda^{-1})\right) \tag{46}$$

which after rearranging and squaring both sides give the inequality

$$\begin{aligned}&(\Phi_i^T(t)\tilde{\Theta}_i(t-\lambda^{-1}))^2 \\ &\leq 2(\Phi_i^T(t)\tilde{\Theta}_i(t))^2 + 2\left(\Phi_i^T(t)\int_{t-\lambda^{-1}}^t \dot{\tilde{\Theta}}_i(\tau)d\tau\right)^2 \\ &\leq 2(\Phi_i^T(t)\tilde{\Theta}_i(t))^2 + c\int_{t-\lambda^{-1}}^t \dot{\tilde{q}}_i^2(\tau) + \dot{\tilde{d}}_i^2(\tau)d\tau \end{aligned} \tag{47}$$

for some constant $c > 0$. As already stated, the first term is integrable. For the second term, we have by changing the order of integration

$$\begin{aligned}&\lim_{T\to\infty}\int_{\lambda^{-1}}^T\int_{t-\lambda^{-1}}^t \dot{\tilde{q}}_i^2(\tau)d\tau dt \\ &= \lim_{T\to\infty}\int_0^{\lambda^{-1}}\int_{\lambda^{-1}}^{\tau+\lambda^{-1}} dt\lambda\dot{\tilde{q}}_i^2(\tau)d\tau \\ &+ \int_{\lambda^{-1}}^{T-\lambda^{-1}}\int_\tau^{\tau+\lambda^{-1}} dt\lambda\dot{\tilde{q}}_i^2(\tau)d\tau \\ &+ \int_{T-\lambda^{-1}}^T\int_\tau^T dt\lambda\dot{\tilde{q}}_i^2(\tau)d\tau \end{aligned} \tag{48}$$

Since all the inner integrals evaluate to λ^{-1} or less,

$$\lim_{T\to\infty}\int_{\lambda^{-1}}^T\int_{t-\lambda^{-1}}^t \dot{\tilde{q}}_i^2(\tau)d\tau dt \leq \lim_{T\to\infty}\lambda^{-1}\int_{\lambda^{-1}}^T \dot{\tilde{q}}_i^2(\tau)d\tau \tag{49}$$

which by (40) is bounded. The term involving $\dot{\tilde{d}}_i$ can similarly be shown to be bounded and integrable, showing that the left hand side of (47) is bounded and integrable. That is

$$\pi_i := \frac{\tilde{q}_i v(0,\cdot) + \tilde{d}_i}{\sqrt{2+v^2(0,\cdot)}} \in \mathscr{L}_2 \cap \mathscr{L}_\infty. \tag{50}$$

We construct the Lyapunov function candidate

$$V(t) = \int_0^1 e^{-x}\eta^T(x,t)\Pi\eta(x,t)dx \tag{51}$$

where Π is a positive definite diagonal matrix. Differentiating (51) with respect to time inserting the system dynamics (35) and integrating by parts give the upper bound

$$\dot{V}(t) \leq -\int_0^1 \eta^T(x,t)\left[\Pi\lambda_1 - \bar{H}^T(x,t)\Pi\bar{H}(x,t)\lambda_n\right]\eta(x,t)dx - c_1\eta^T(1,t)\eta(1,t) + c_2\pi^T(t)\pi(t). \tag{52}$$

where $\pi = \{\pi_i\}_{1\leq i\leq n}$. Since $\bar{H}(x,t)$ is strictly lower triangular and by (40) bounded for all $t \geq 0$, it is possible to (recursively, see e.g. [27, Appendix B.2.]) select Π such that $\Pi\lambda_1 - \bar{H}^T(x,t)\Pi\bar{H}(x,t)\lambda_n \succ 0$ yielding

$$\dot{V}(t) \leq -c_3V_1(t) - c_1\eta^T(1,t)\eta(1,t) + c_2\pi^T(t)\pi(t) \tag{53}$$

for some constants $c_1, c_2, c_3 > 0$. The bound (53) is of the form considered [28, Lemma 3] yielding $V \to 0$ which in turn implies $||\eta|| \to 0$. Invertibility of the transformations (13) and (33) and boundedness of all kernels finally give (42). ■

III. Concluding remarks

We have designed an adaptive observer estimating boundary parameters and distributed states in an $n+1$ linear hyperbolic system using measurements on the boundary opposite the uncertainty. Boundedness of state estimates are proved. The state and parameter estimates are shown to converge to their true value assuming bounded system states and persistence of excitation, respectively. The observer can be applied to a multiphase fluid flow system in drilling to estimate distributed flow and pressure profiles and reservoir properties. The drift-flux model mentioned in Section I-B is quasi-linear and linearization is only a valid simplification for limited variations around an equilibrium profile. This is violated for kick and loss scenarios, but may be fine for underbalanced drilling operations. An interesting option to pursue in applications, is to allow the non-linear terms in the observer even though output injections are designed for the linearization.

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