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## THE GENERIC PROPERTIES OF EQUILIBRIUM CORRECTION MECHANISMS

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# The generic properties of equilibrium correction mechanisms\*

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## **Abstract**

Linear dynamic equilibrium correction mechanisms are shown to follow from the discretisation of continuous economic processes with steady-state solutions. In addition, the proposed procedure provides testable restrictions on the coefficients of the dynamic econometric model.

## **Keywords**

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# Introduction

Since the publication of Davidson, Hendry, Srba and Yeo (1978), equilibrium correction mechanisms (ECMs) such as

$$\Delta y_t = a(y - bx - c)_{t-1} + p\Delta y_{t-1} + q\Delta x_{t-1} + u_t \quad (1)$$

have been used successfully to model dynamic economic relationships. The rationale for such econometric specifications has been derived either from economic theory, as in Nickell (1985), or from the properties of time series models, as in the seminal work on cointegration by Engle and Granger (1987). Hendry (1995, pp. 286-294) provides an introduction to and overview of these motivations, while Alogoskoufis and Smith (1991) give an extensive list.

In contrast, we point out that a dynamic ECM follows naturally from the discretisation of a differential equation with a steady-state solution. In addition, the linear case provides testable restrictions on the coefficients of the ECM formulation.

## Backward-difference schemes

Let  $t_1, t_2, \dots, t_k, \dots$  be a sequence of times spaced  $h$  apart and let  $y_1, y_2, \dots, y_k, \dots$  be the values of a continuous real variable  $y(t)$  at these times. The backward-difference operator  $\Delta$  is defined by the rule

$$\Delta y_k = y_k - y_{k-1}, \quad k \geq 1. \quad (2)$$

By observing that  $y_k = (1 - \Delta)^0 y_k$  and  $y_{k-1} = (1 - \Delta)^1 y_k$ , the value of  $y$  at the intermediate point  $t = t_k - sh$  ( $0 < s < 1$ ) may be estimated by the interpolation formula

$$y(t_k - sh) = y_{k-s} = (1 - \Delta)^s y_k, \quad s \in [0, 1]. \quad (3)$$

When  $s$  is not an integer,  $(1 - \Delta)^s$  should be interpreted as the power series in the backward-difference operator obtained from the binomial expansion of  $(1 - x)^s$ . This is an infinite series of differences. Specifically

$$(1 - \Delta)^s = 1 - s\Delta - \frac{s(1-s)}{2!}\Delta^2 - \frac{s(1-s)(2-s)}{3!}\Delta^3 - \dots \quad (4)$$

With this preliminary background, the differential equation

$$\frac{dy}{dt} = f(y, x), \quad x = x(t), \quad (5)$$

may be integrated over the time interval  $[t_k, t_{k+1}]$  to obtain

$$y(t_{k+1}) - y(t_k) = \Delta y_{k+1} = \int_{t_k}^{t_{k+1}} f(y(t), x(t)) dt \quad (6)$$

in which the integral on the right hand side of this equation is to be estimated by using the backward-difference interpolation formula given in equation (4). The substitution  $t = t_k + sh$

is now used to change the variable of this integral from  $t \in [t_k, t_{k+1}]$  to  $s \in [0, 1]$ . The details of this change of variable are

$$\int_{t_k}^{t_{k+1}} f(y(t), x(t)) dt = \int_0^1 f(y(t_k + sh), x(t_k + sh)) (h ds) = h \int_0^1 f_{k+s} ds$$

where  $f_{k+s} = f(y(t_k + sh), x(t_k + sh))$ . The value of this latter integral is now computed using the interpolation formula based on (4). Thus

$$\begin{aligned} \int_0^1 f_{k+s} ds &= \int_0^1 (1 - \Delta)^{-s} f_k ds \\ &= \int_0^1 \left( f_k + s\Delta f_k + \frac{s(1+s)}{2!} \Delta^2 f_k + \frac{s(1+s)(2+s)}{3!} \Delta^3 f_k + \dots \right) ds \\ &= f_k + \frac{1}{2} \Delta f_k + \frac{5}{12} \Delta^2 f_k + \frac{3}{8} \Delta^3 f_k + \dots \end{aligned}$$

The final form for the backward-difference approximation to the solution of this differential equation is therefore

$$\Delta y_{k+1} = hf_k + \frac{h}{2} \Delta f_k + \frac{5h}{12} \Delta^2 f_k + \frac{3h}{8} \Delta^3 f_k + \dots \quad (7)$$

## Model specification

The backward-difference scheme (7) is valid for all suitably differentiable functions  $f(y, x)$ . For ease of exposition, consider the simplest case in which a constant input  $x = X$  induces  $y(t)$  to approach asymptotically a constant state  $Y$  as  $t \rightarrow \infty$ . Clearly  $X$  and  $Y$  satisfy  $f(Y, X) = 0$ . The usual procedure is to expand the differential equation about this steady-state solution (see Gandolfo, 1997, p260). Employing this procedure yields

$$f(y, x) = f(Y, X) + \frac{\partial f(Y, X)}{\partial y} (y - Y) + \frac{\partial f(Y, X)}{\partial x} (x - X) + R \quad (8)$$

where

$$R = \frac{1}{2!} \left( \frac{\partial^2 f(\xi, \eta)}{\partial x^2} (x - X)^2 + 2 \frac{\partial^2 f(\xi, \eta)}{\partial x \partial y} (x - X)(y - Y) + \frac{\partial^2 f(\xi, \eta)}{\partial y^2} (y - Y)^2 \right)$$

and  $(\xi, \eta)$  is a point such that  $\xi$  lies between  $y$  and  $Y$  while  $\eta$  lies between  $x$  and  $X$ . Since  $Y$  and  $X$  are the steady-state values for  $y$  and  $x$  respectively, then the expression for  $f(y, x)$  takes the simplified form

$$f(y, x) = a(y - Y) + \beta(x - X) + R \quad (9)$$

where  $a = \partial f(Y, X)/\partial y$  and  $\beta = \partial f(Y, X)/\partial x$  are constants.

If  $f$  is a linear function of  $y$  and  $x$  then  $R = 0$  and so

$$f(x, y) = a \left( y - Y + \frac{\beta}{a} (x - X) \right) = a(y - bx - c), \quad (10)$$

in which  $b = -\beta/a$  and  $c = Y + (\beta/a)X$ . This expression for  $f$ , when applied to the backward-difference scheme (7), gives the ECM representation (with  $k = t - 1$  and  $h = 1$ )

$$\Delta y_t = a(y - bx - c)_{t-1} + \frac{1}{2} a(\Delta y_{t-1} - b \Delta x_{t-1}) + u_t, \quad (11)$$

$$u_t = \frac{5}{12} a(\Delta^2 y_{t-1} - b \Delta^2 x_{t-1}) + \dots. \quad (12)$$

Equation (11), expressed in reduced form, is identical to the ECM equation (1). In particular, if  $f$  is a linear function then theory predicts that  $p = a/2$  and  $q = ab/2$ . These restrictions should, in principle, be testable.

On the other hand, if  $f$  is a nonlinear function of  $y$  and  $x$ , the backward-difference analysis is still valid but now the appropriate scheme is

$$\Delta y_t = a(y - bx - c)_{t-1} + R_{t-1} + \frac{1}{2} a(\Delta y_{t-1} - b \Delta x_{t-1}) + \frac{1}{2} \Delta R_{t-1} + u_t,$$

$$u_t = \frac{5}{12} a(\Delta^2 y_{t-1} - b \Delta^2 x_{t-1}) + \frac{5}{12} \Delta^2 R_{t-1} + \dots.$$

It can now be expected that the presence of non-negligible contributions from  $R$  contaminates  $p$  and  $q$  so that the linear restrictions derived previously are no longer applicable.

## Conclusion

This note shows that the ECM representation popular in the dynamic econometric literature follows from the discretisation of a continuous process with a steady-state solution. This recognition should prompt further research—perhaps the most immediate relates to the practical feasibility of testing the linear restrictions identified previously.

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