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Approaching mean-variance efficiency for large portfolios

A replication on Norwegian data

Master's thesis in Financial economics
Supervisor: Snorre Lindset
June 2020

Master's thesis

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Kunnskap for en bedre verden

Preface

This Thesis concludes our MSc in Financial Economics at the Norwegian University of Technology and Science. We would like to thank Snorre Lindset for guiding us through the process of writing this thesis, and providing valuable feedback. We would also like to thank TITLON for supplying us with the necessary data. This thesis is written in co-operation by Carl Henrik Svendsen and Espen Kristiansen Paulsen.

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Abstract

This master thesis replicates the approach by Ao et al. (2018) to construct optimal mean-variance portfolios on the Norwegian stock market. Their model, which they call the maximum-Sharpe-ratio estimated and sparse regression (MAXSER) method, relies on a novel unconstrained regression representation of the mean-variance problem. Their findings show the model offering an advantage over previous models and is able to effectively control for risk. However, based on the Norwegian stock market our estimated portfolio does not attain the maximum expected returns, and is not able to effectively control for risk. However, when the number of assets and observations increase, the ability of the model to control for risk and attain the maximum expected returns increases. These findings are demonstrated through simulation and empirical analysis. We also find that when investing in individual stocks in addition to the Fama-French three-factor portfolios, the performance is improved.

Sammendrag

Denne masteroppgaven replikerer metoden til Ao et al. (2018) for å sette sammen optimal mean-variance porteføljer på det Norske aksjemarkedet. Deres modell, kalt maximum-Sharpe-ratio estimated and sparse regression (MAXSER), avhenger av en ny og ubegrenset regresjonspresentasjon av mean-variance problemet. Deres funn viser at modellen har økonomiske fordeler over tidligere modeller, og er i stand til å effektivt kontrollere for risiko. Vår estimerte portefølje, som er basert på det Norske aksjemarkedet, klarer ikke å oppnå maksimum forventet avkastning, og er heller ikke i stand til å effektivt kontrollerer for risiko. På en annen side, når antall aksjer og observasjoner øker evner modellen bedre å kontrollere for risiko, samt oppnå den maksimale forventede avkastningen. Våre funn blir demonstrert gjennom både simulering og empirisk analyse. Videre finner vi at å investere i individuelle aksjer i tillegg til Fama-French sin tre-faktor-portefølje øker prestasjonen betraktelig.

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1 Introduction

Ever since its introduction by Markowitz (1952), portfolio theory (or mean-variance analysis) has experienced immense growth and has had an impact on the fields of finance and economics among university scholars, portfolio managers, and individual investors as shown by Francis and Kim (2013). Markowitz's portfolio theory revolves around how an investor, with a certain level of risk, can maximize their expected return. This theory involves only two population characteristics: the mean and the covariance of asset returns. Even though it was groundbreaking during its time, many researchers now believe there are several shortcomings to Markowitz's theory, especially when portfolios become large. Thus, researchers have sought to improve his work either with better estimates for the mean and the covariance or with entirely new methodologies for portfolio selection. Among these contributions are the new estimations of the covariance proposed by Ledoit et al. (2002) and Ledoit and Wolf (2017). Furthermore, a relatively new approach to the mean-variance analysis has been developed by Ao et al. (2018). This new approach is known as the maximum Sharpe ratio estimated and sparse regression (MAXSER). This approach is equivalent to Markowitz's optimization, but it revolves around a novel unconstrained regression.

The model developed by Ao et al. (2018) has, as best as we know, only been tested on the US stock market. This is because most of the research involving portfolio theory is widely influenced by findings concerning the US stock market or other large stock markets. Ao et al. show that their methodology has an advantage in terms of risk control and return over the regular mean-variance analysis when creating large portfolios.

This thesis uses Ao et al.'s framework and seeks to evaluate the performance of their methodology when portfolios are created based on assets traded on the Norwegian stock market. In doing so, we address the following research question:

Does the use of the MAXSER methodology offer an advantage compared to previous portfolio theories when performed on the Norwegian stock market?

To answer this question, the model was analyzed and we found that the methodology gains no advantage when performed with smaller portfolios. However, we found that when portfolios become larger, the model performs better, which is consistent with the findings by Ao et al. (2018). Our findings raise three sub-questions:

1. Is MAXSER able to effectively control for risk when the portfolios are small?
2. At which point is the portfolio too small for MAXSER to be effective?
3. Does the use of MAXSER offer an advantage when including additional stocks to factor investing on the Norwegian stock market as well?

The format of this paper is as follows. In Section 2, we cover previous studies on portfolio theory and the challenges with large portfolios. In Section 3, we address all decisions made when cleaning and adjusting the data. In Section 4, we establish key definitions and cover the methodology behind Ao et al.'s model. In Section 5, we cover a simulation analysis and an empirical analysis as well as a discussion of the results. Last, in Section 6, we conclude the thesis.

2 Literature

In this section we cover the academic literature and existing alternative methods based on the mean-variance portfolio problem proposed by Markowitz (1952).

2.1 Literature review

Markowitz's mean-variance portfolio theory remains quite relevant to this day for both research and practice. This theory depends only on the expected mean and covariance matrix of asset returns. Since these parameters cannot be observed in the real world, the mean and covariance matrix have to be estimated, creating a sample mean and a sample covariance. The "plug-in" portfolio is a result of these sample estimates. Thus, portfolios calculated using the sample mean and sample covariance will hereby be referred to as "plug-in portfolios." There are several challenges when estimating these parameters. The use of a plug-in portfolio is a maximum-likelihood estimator of the optimal portfolio and is well justified by classical statistics theory. However, this plug-in portfolio tends to perform worse out of sample. This poor performance is worsened when the number of assets is increased. Due to this situation, the mean-variance portfolio proposed by Markowitz has been widely adopted by others by using better estimates for the sample mean and sample covariance. The poor performance of the plug-in portfolio is illustrated in Figure 1

The plug-in portfolio was constructed based on 10 years of monthly log returns generated from an i.i.d. multivariate normal distribution. The horizontal lines in the two panels illustrate the prespecified risk level

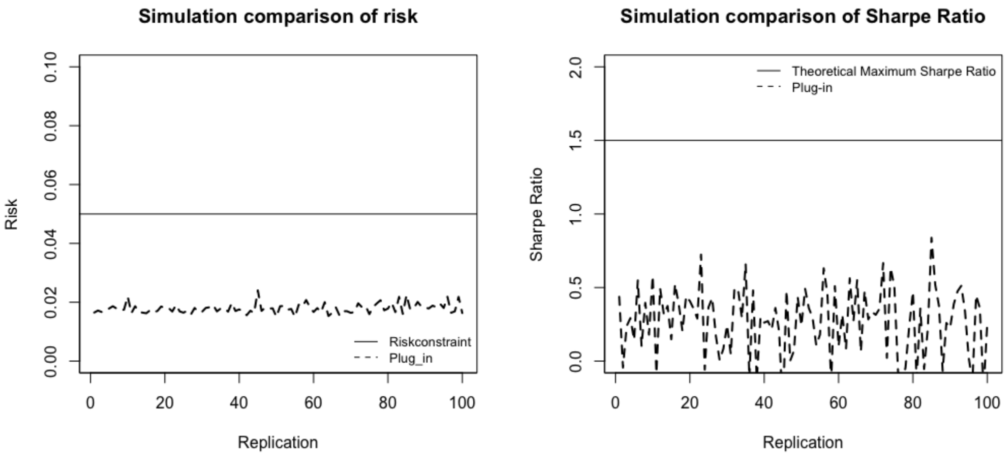

Figure 1: Performance of the plug-in portfolio

Figure 1 compares the risk and Sharpe ratio of the plug-in portfolio against a risk constraint and a theoretical maximum Sharpe ratio, respectively. The risk and Sharpe ratios are plotted on the y-axis of the two panels, while the number of replications are plotted on the x-axis. Our portfolios are based on data generated from an identical independent distributed multivariate normal distribution with parameters calibrated from real data (see Section 5.1.2 for details). Twenty-five stocks and three factors form the asset pool, and the number of observations is set to 240.

and the theoretical maximum Sharpe ratio[1]. The lines are used as a benchmark to compare the simulated portfolios against each other.

On the one hand, the left panel in Figure 1 shows that the plug-in portfolio carries approximately half the risk as the risk constraint [2] in nearly every replication. On the other hand, the right panel shows that the Sharpe ratio of the plug-in portfolio is approximately 20% of the theoretical Sharpe ratio. If we compare the results from Figure 1 to

[1]We were unable to identify the theoretical maximum Sharpe ratio, so we put 1.5 in order to illustrate the results.

[2]The risk constraint is set to be $\sigma = 0.05$ which is a bit higher than that to the original paper, and is the same for all figures.

the results achieved by Ao et al., we notice a few differences in terms of both the plug-in portfolio risk versus the constraint and the plug-in Sharpe ratio versus the theoretical Sharpe ratio.

2.2 Challenges with large portfolios or a large number of assets

An important topic in finance is estimating and assessing the risk of a portfolio. This risk can be measured as a volatility matrix, often known as the covariance matrix. Because modern portfolios often consist of a large number of assets, the mean-variance problem becomes high-dimensional and poses serious challenges. Fan et al. (2015) argue that when estimating the risk of large portfolios, the estimation of the covariance matrix becomes difficult. They show this by creating a portfolio with 2,000 assets; the covariance matrix involved would then contain over two million unknown parameters. The assessment of the estimation accuracy when the estimation errors from more than two million parameters are aggregated is hard. Thus, large portfolios with high-dimensional data pose crucial challenges when it comes to calculating mean-variance efficiency.

Furthermore, if we take the plug-in portfolio as an example, the risk can be very high compared to the perceived level of risk even when the portfolio weights are computed based on simulated independent and identically distributed returns. For this to be optimal, such high risk should be compensated by a high return. However, this is not the case, which results in a low Sharpe ratio (e.g. see Ao et al. (2018)). In their paper, Ao et al. (2018) argue that even in an ideal situation, when all assumptions regarding the mean-variance optimization are satisfied, there are still intrinsic challenges for the estimation of the mean-variance efficient portfolio.

2.3 Existing alternative methods

Due to the challenges with large portfolios, researchers have devised several other methods to calculate mean-variance efficient portfolios. Researchers seek to improve portfolio performance by using other estimations of the underlying mean and covariance matrix. Thus, the differences in these strategies are how one estimates the covariance matrices and the mean. In estimating the covariance matrix, an alternative estimator that is widely used is the linear shrinkage proposed by Ledoit and Wolf (2003). Ledoit and Wolf argue that the sample covariance matrix imposes too little structure and is, therefore, seldom used. Their answer to this problem is to impose some sort of structure on the estimator. They do this by shrinking the unbiased but very variable sample covariance matrix towards the biased but less variable single-index model covariance matrix. In doing so, they obtain a more efficient estimator. Furthermore, this estimator is invertible and well-conditioned, which is crucial. More recently, Ledoit and Wolf (2017) have proposed a new covariance estimator, the nonlinear shrinkage estimator. Here they push up the small eigenvalues of the sample covariance matrix and pull down the large ones by an amount that is determined individually for each eigenvalue. Here lies a challenge in choosing the correct loss intensity of each eigenvalue. However, they show that when using their estimator with back testing on historical data, their estimator outperforms previously suggested estimators and in fact dominates their linear estimator.

For estimating the mean, Black and Litterman (1990) present a new model where they compare their outlook for the asset market with expected asset returns based on the capital asset pricing model. This approach has been further developed by Lai et al. (2011), whose solution opens new possibilities for solving the portfolio optimization problem in cases where the means and covariances for the next investment period is unknown. Lai et al.'s solution only requires the posterior mean and second moment matrix of the return vector for the next period, which then can be combined with the Black-Litterman approach to develop a Bayesian model with good predictive properties and to maximize a certain utility function.

To improve portfolio performance, other methods have also been incorporated. Some of these methods involve modifying the original optimization by imposing certain restrictions on the portfolio weights. This research is mainly focused on the global-minimum-variance portfolio. Jagannathan and Ma (2003) argue that constructing a minimum-risk portfolio, which is subject to the constraint that portfolio weights should be positive, is equivalent to constructing a portfolio without any constraints on portfolio weights after the covariance matrix has been modified in a particular way. Furthermore, they show through simulation that imposing a non-negativity constraint to the portfolio weights could substantially benefit the performance even if the constraints are wrong.

All the aforementioned methods should lead to improved portfolio performance, but there are still challenges that remain. In Figure 2, we

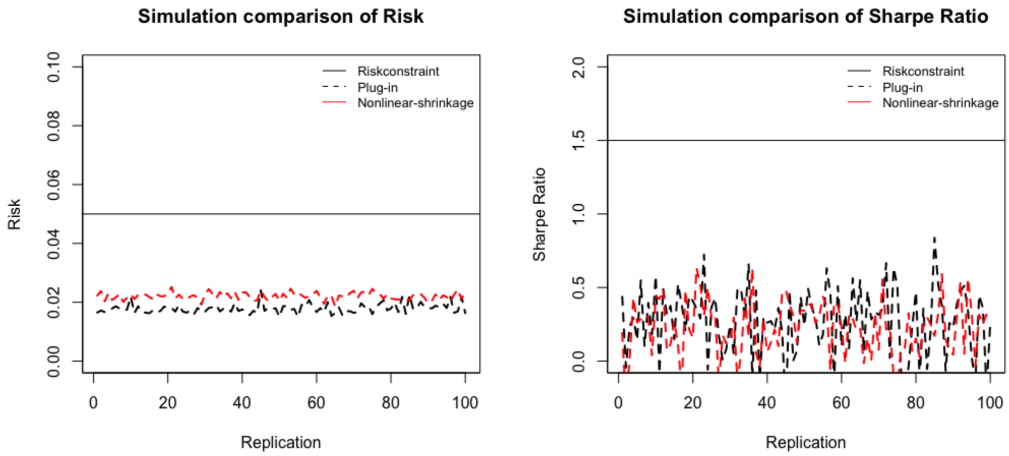

Figure 2: Performance of the nonlinear shrinkage portfolio

Figure 2 compares the risk and Sharpe ratio of the plug-in portfolio and nonlinear shrinkage portfolio against a risk constraint and a theoretical maximum Sharpe ratio, respectively. The portfolios are constructed the same way and use the same data, as explained in Figure 1.

illustrate the result of implementing the proposed nonlinear shrinkage method by Ledoit and Wolf (2017), and compare it to the plug-in portfolio. We observe that the nonlinear shrinkage method carries slightly more risk than the plug-in, but still only half the risk as the risk constraint. The difference in terms of the Sharpe ratio is nearly nonexistent.

2.4 The MAXSER

MAXSER was developed by Ao et al. (2018) and it is a new methodology to estimate the mean-variance efficient portfolio. This model can be applied as a general approach to various situations when the number of assets is large. In their paper, Ao et al. show that using this model, under mild assumptions, will achieve mean-variance efficiency and satisfy the risk constraint. According to Ao et al. (2018), MAXSER is the first methodology that can achieve these two objectives at the same time for large portfolios. Furthermore, they make several other contributions such as the equivalent unconstrained regression representation of the mean-variance portfolio problem as well as optimal portfolios when they consist of assets only and when factors are included.

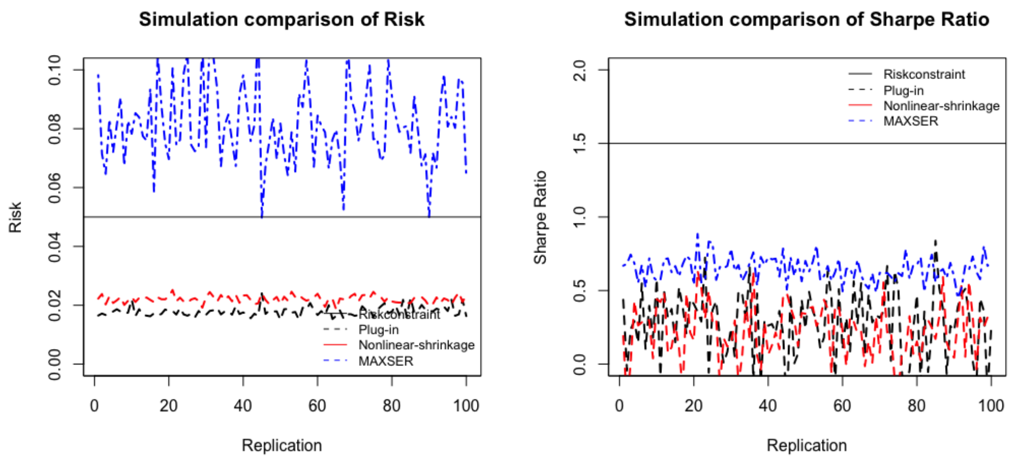

Figure 3: Performance of the MAXSER portfolio

Figure 3 compares the MAXSER portfolio against the plug-in portfolio and nonlinear shrinkage portfolio. The portfolios are constructed the same way and use the same data as explained in Figure 1 and 2. The replications were conducted 100 times.

In Figure 3, we observe that the constructed MAXSER portfolio outperforms the others in terms of the Sharpe ratio, but it fails to control for risk as efficiently as the others do.

In this thesis, we will use this model to check whether Ao et al.'s results in achieving mean-variance efficiency and the ability to control for risk also applies to the Norwegian stock market.

3 Data

Evaluating large portfolios is a data-centered exercise. This section explains our decisions when working with data available. We address all relevant steps regarding data cleaning and adjustments to samples, with the intention of enabling the interested reader to fully replicate our study.

3.1 Data cleaning

Forming and testing portfolios with the MAXSER methodology requires stock market and accounting data. Using the definitions of variables found in 4.1, we needed data containing monthly observations of stock prices and market capitalization, as well as data on a company's book value of equity and the risk-free rate. Since we have restricted access to such data, the financial database for Norwegian academic institutions, TITLON, is the best source of data for this analysis. The TITLON database offers data on daily stock prices from 1980 to the present. In addition, TITLON offers sufficient accounting data from 1997 to 2019. As our analysis requires the use of the three-factor model which consists of accounting-based factors, our analysis is limited to the time period from June 1997 to June 2019, a total of 263 months. In addition, TITLON offers daily data on the risk-free rate. The risk-free rate has been estimated from the Norwegian overnight weighted average since 2013 and, before that, from the Norwegian InterBank Offered Rate.

The raw TITLON data provides a sample of about 2,100,000 stock market data points and about 1,400,000 accounting data points. However, many companies had multiple entries under both the same and different International Securities Identification Numbers. Thus, we first removed all duplicated entries from the original sample. Second, to make the data easier to work with, we merged stock market data and accounting data into one data frame for the entire period. This data cleaning left us with about 1,800,000 stock market observations and 662 unique firms over the period from 1980 to 2019. This data cleaning was conducted using the programming language R; in particular, the data manipulation package "dplyr" by Wickham et al. (2020) has been used for the tasks concerning data manipulation and writing algorithms. Cleaning the data was both long and challenging at times.

Ao et al. (2018) picked 100 stocks at random in their portfolio for their analysis. This makes their portfolio quite large. However, the Oslo Stock Exchange Benchmark Index(OSEBX) consists of only 66 stocks. This implies that our portfolio can consist of 66 stocks at a maximum, but we could not include all, as that would not be a random selection. Thus, we estimated that we could create portfolios that consist of at least 50 random stocks. However, after the data cleaning process, we found that there were stocks that were missing data. Because stocks were missing data, we had to exclude them from the portfolio. After exclusion of the stocks that were missing data, our portfolio could only have half of the stocks included in OSEBX, implying that our choice of stocks would be even smaller than before. Our using of Norwegian data and our selection of possible stocks, the number of which is much lower than that of Ao et al. (2018), can have a severe impact on our analysis, causing our results to differ substantially.

3.2 Adjusted prices

The MAXSER methodology considers two scenarios, one of which is where the asset pool contains only individual stocks. Thus, we needed the monthly data of each stock traded on the Oslo Stock Exchange up to the present day. The database TITLON provides daily closing prices for these companies. Hence, we had to convert these observations into monthly returns, as seen in 4.1 and 4.2. However, some companies have either had stock splits, reverse stock splits or paid out dividends over the time period provided by TITLON. If we neglected this, we would have errors in our calculations, and our analysis would become biased. To correct for this, we instead used the adjusted closing price which takes both dividends and splits into consideration.

3.3 Fama-French three factor model

In addition to considering individual assets, the MAXSER methodology also considers factor investing in an investment pool. Regarding factors, we could use any factor identified in the large literature on asset pricing. However, in this analysis, we have decided to use the Fama-French three-factor model. This model is specified as

$$R_{i_t} = \alpha + \sum_{j=1}^{K} \beta_{i_j f_j} + e_i, \tag{3.1}$$

where f_j's are factor excess return, β_{i_j}'s are the individual stock sensitivities to the factors and e_i are the models remaining errors. The factors included here are small minus big (SMB), high minus low (HML) and the excess return on the market portfolio, in our case the excess return on the OSEBX. Monthly data on the Fama-French three factors can be found on Bernt Arne Ødegaard's webpage. However, as the use of his data was cumbersome for us, we decided to use daily data of these factors provided by TITLON, which we transformed into monthly data.

3.3.1 Small minus big

The additional return received when investing in stocks of companies with relatively small market capitalization is often referred to as the "size premium," which is captured by the SMB factor. This additional return is computed as the average return of the stocks with the 30% smallest market capitalization minus the stocks with the 30% largest market capitalization as shown by Womack and Zhang (2003). A large capitalization stock outperforms a small capitalization stock if the SMB factor is negative. A large capitalization stock has outperformed a stock with small capitalization if the SMB factor is negative.

3.3.2 High minus low

The value premium is measured by the HML factor, which captures the additional return investors receive by investing in companies with high book-to-market values, expressed as B/M. Here, the book-to-market ratio is the value placed on the company by accountants relative to the value placed by the public. As with the SMB factor, the HML factor is computed as the average return of stocks with the 50% highest B/M ratio (typically value stocks) minus the average return of stocks with the lowest 50% B/M ratio (typically growth stocks) as shown by Womack and Zhang (2003). In the same manner as the SMB factor, a negative HML implies that growth stocks have outperformed high value stocks.

3.3.3 Market portfolio

The excess return on the market portfolio is measured by the market factor, normally expressed as Mkt. This factor is computed as the return on the market portfolio (in our case, the Oslo Stock Exchange) minus the risk-free rate. Both these variables are provided by TITLON but in daily returns. Thus, we had to convert both of them to monthly returns. How we have done this can be seen in Equation (4.1)

4 Methodology

To examine how the MAXSER methodology performs on portfolios based on Norwegian stock returns compared to previous models, we need a solid methodological foundation. This section is aimed at providing the reader with central definitions and the framework for the MAXSER model. The last part establishes important tools needed to empirically evaluate the performance of the model on Norwegian Stocks.

4.1 Central definitions

4.1.1 Monthly returns

The MAXSER model considers two scenarios, one were the portfolio consists of only individual assets and the second when factor investing is allowed. Both scenarios revolve around monthly returns. Thus, we first have to compute all assets' compounded daily returns, which is the percentage change in price from one day to the next:

$$Daily\ Return_{i,d} = \ln\left(\frac{Adjusted\ price_{i,d}}{Adjusted\ price_{i,d-1}}\right), \tag{4.1}$$

where i represents assets and d days, and we have to convert this into monthly return, which is computed as follows:

$$Monthly\ Return_{it} = \ln\left(\frac{Daily\ Return_{i,M}}{Daily\ Return_{i,1}}\right), \tag{4.2}$$

where $Daily\ Return_{i,M}$ is the daily return, M indicates the last day of the month, while 1 indicates the first day of the month. Equation (4.2) is used to calculate both the monthly risk-free rate and monthly market return as well. The calculations were conducted in R using the *Return.calculate* function by Peterson and Carl (2020).

4.1.2 Excess return

The model uses stocks' random excess returns, which are computed as the assets' monthly return minus the monthly return of the risk free rate:

$$\textit{Excess Return}_{i,t} = \textit{Monthly Return}_{i,t} - \textit{Risk-Free Rate}_t, \tag{4.3}$$

where i indicates assets and t indicates months .

4.1.3 Factor returns

In Scenario II the MAXSER model includes factor return as well as return on individual assets. The definitions of each factor return is defined in Section 3.3. But, the computations of each factor return is defined below:

$$\textit{SMB Return}_t = \textit{Return on small firms}_t - \textit{Return on big firms}_t, \tag{4.4}$$

$$\textit{HML Return}_t = \textit{Return on high B/M}_t - \textit{Return on low B/M}_t, \tag{4.5}$$

$$\textit{Market Return}_t = \textit{Return on OSEBX}_t - \textit{Risk-Free Rate}_t, \tag{4.6}$$

where t in (4.4), (4.5), (4.6) indicates months.

4.1.4 Sharpe ratio

The Sharpe ratio is a measure to calculate risk-adjusted returns for a portfolio, and in our case a measure to compare different portfolios. The Sharpe ratio is calculated as follows:

$$\textit{Sharpe ratio} = \frac{E[R_p]}{\sigma_p}, \tag{4.7}$$

where $E[R_p]$ is the expected portfolio return and σ_p is the standard deviation of the portfolio excess return.

4.1.5 Spread

When trading in stocks there are certain costs, known as transactions costs. One of the most known measure for transaction costs is the spread, where spread is the difference in the best bid price and the best asking price. A stocks spread is often calculated relative to its price in order to find the relative spread. The relative spread is calculated as follows:

$$Relative\ spread = \frac{Ask\ price - Bid\ price}{\frac{1}{2}(Ask\ price + Bid\ price)}, \tag{4.8}$$

4.2 The MAXSER methodology

4.2.1 An unconstrained regression representation

For any given risk constraint σ, the mean-variance portfolio problem for a pool of N risky assets is

$$\arg\max_{w} \mathrm{E}(\boldsymbol{w}'\boldsymbol{r}) = \boldsymbol{w}'\boldsymbol{\mu} \quad \text{subject to} \quad \mathrm{Var}(\boldsymbol{w}'\boldsymbol{r}) = \boldsymbol{w}'\textstyle\sum\boldsymbol{w} \leq \sigma^2 \ , \tag{4.9}$$

where $\boldsymbol{r} = (r_1, r_2, ..., r_N)'$ is an asset's random excess return, and for any vector $\boldsymbol{v}$, $\boldsymbol{v}'$ is the transpose. Furthermore, let $\boldsymbol{\mu}$ and $\sum$ represent the mean vector and covariance matrix of $\boldsymbol{r}$, respectively, and the weights on the portfolio be the vector $\boldsymbol{w}$.

Then let $\theta = \mu' \sum^{-1} \mu$ represent the square of the maximum Sharpe ratio of the optimal portfolio. Following Ao et al. (2018), the optimization problem (4.9) can be represented in its dual form, given a return constraint as:

$$\arg\min_{w} \; \boldsymbol{w}'\sum\boldsymbol{w} \quad \text{subject to} \quad \boldsymbol{w}'\boldsymbol{\mu} \geq r^* = \sigma\sqrt{\theta},^{3} \tag{4.10}$$

where r^* is the required rate of return.

Thus, the optimal portfolio, $\boldsymbol{w^*}$, can be expressed as:

$$\boldsymbol{w^*} = \frac{\sigma}{\sqrt{\theta}}\sum^{-1}\boldsymbol{\mu}, \tag{4.11}$$

Because we are working with large amount of data, this becomes high dimensional and this causes difficulties in estimating the mean and covariance matrix. Thus, Ao et al. (2018) propose a *novel* and *unconstrained* regression representation of the mean-variance representation instead of using plug-in estimates for $\boldsymbol{\mu}$ and $\sum$ in formula (4.11).

Proposition 1. *The unconstrained regression* [4]

$$\arg\min_{w} \; \mathrm{E}(r_c \text{ - } \boldsymbol{w'r})^2, \quad \textit{where} \;\; r_c = \frac{1+\theta}{\theta}r^* \equiv \sigma\frac{1+\theta}{\sqrt{\theta}}, \tag{4.12}$$

which equals the mean-variance optimizations in (4.9) *and* (4.10).

There are several regression representations for the mean-variance portfolio estimation, but Ao et al. (2018) emphasize that their representation is very different from existing representations. Their representation is identical to the mean-variance portfolio problem and unconstrained. Eliminating the constraint leads to the use of sparse regression, which becomes helpful for large portfolios due to high dimensionality.

[3] The equations below are defined by Ao et al. (2018)

[4] Please note that we assume the propositions listed in this paper are correct. See the appendix of Ao et al. (2018) for complete proof.

4.2.2 The sparse regression

The original mean-variance problem can be transformed into an equivalent and unconstrained regression problem by proposition 1. To utilize this regression we need the sample version of $\arg\min_{w} \mathrm{E}(r_c$ - $\boldsymbol{w'r})^2$ which is:

$$\arg\min_{w} \frac{1}{T} \sum_{t=1}^{T} (r_c - \boldsymbol{w'R_t})^2, \tag{4.13}$$

where $\boldsymbol{R_t} = (R_{t1}, ..., R_{tN})'$, $t = 1, ..., T$. Implying that $\boldsymbol{R_t}$ is the return matrix including i.i.d copies of the return vector $\boldsymbol{r}$ for all N assets. When dealing with a high-dimensional regression, estimating the coefficients becomes nearly impossible. In order to solve the problem one would require to set an upper bound on the ℓ_1-norm of the regression coefficients. Which for us, corresponds to assuming $||w^*||_1$ is bounded, where $||v||_1 = \sum_{i=1}^{N} |v_i|$ for any $\boldsymbol{v} = (v_1, ..., v_N)'$. We adopt the sparse regression technique called LASSO by Tibshirani (1996), in order to estimate the optimal portfolio w^*:

$$\boldsymbol{w(r_c)} = \arg\min_{w} \frac{1}{T} \sum_{\boldsymbol{t=1}}^{\boldsymbol{T}} (r_c - \boldsymbol{w'R_t})^2 \quad \text{subject to} \quad ||\boldsymbol{w}||_1 \leq \lambda, \tag{4.14}$$

where λ represents the ℓ_1-norm constraint tuning parameter. However, the solution obtained in equation (4.14) is infeasible because the response variable r_c is unknown. Thus, it needs to be estimated. Details surrounding the estimation of the response variable can be found in section 4.3.1.

In theory, investors can invest in assets and factors. This implies that they can invest only in assets, only in factors or both. Due to this we will look at two scenarios. In Scenario I we will look at an asset pool which only consists of individual assets, while in Scenario II we will look at the case where factor investing is allowed leading to the asset pool comprises both individual stocks and factors. We will mainly focus on the second scenario in our analysis later on.

4.3 Scenario I: When the asset pool comprises individual assets only

4.3.1 Estimating the maximum Sharpe ratio and the regression response

The response variable needs to be estimated, as it is unknown in the regression representation and the estimation of the maximum Sharpe ratio is closely related to this variable. To estimate this we randomly pick 25 assets, from OSEBX, with 15 years of data by using the *sample* function in R. Then, by creating a function in R that extracts 120 months from the data-pool, hereby *extractRand*, we generate a random 10 year period, which will be the same for all the assets. We then create the $\boldsymbol{R} = (\boldsymbol{R_{t1}}, ..., \boldsymbol{R_{TN}})'$ matrix, where $t = 1, ..., T$ and $\boldsymbol{R}$ are i.i.d copies of the monthly excess return vectors $\boldsymbol{r}$, for all N assets. From here we calculate the sample mean and sample covariance, denoted as $\hat{\mu}$ and $\hat{\sum}$ respectively. Now the unbiased estimator, proposed by Kan and Zhou (2007), is the following:

$$\hat{\theta} = \frac{(T - N - 2)\hat{\theta}_s - N}{T}, \tag{4.15}$$

where $\hat{\theta} = \boldsymbol{\hat{\mu}'}\hat{\sum}^{\boldsymbol{-1}}\boldsymbol{\hat{\mu}}$ is the sample estimate of $\theta = \boldsymbol{\mu'} \sum^{\boldsymbol{-1}} \boldsymbol{\mu}$.

Proposition 2. *Ao et al. (2018) shows that the response variable we will be using in the unconstrained regression becomes:*

$$\widehat{r_c} = \sigma \frac{1+\widehat{\theta}}{\sqrt{\widehat{\theta}}}. \tag{4.16}$$

They emphasize that the estimation of θ is *not* obtained through consistently estimating $\boldsymbol{\mu}$ and $\sum$, which would be *impossible* without imposing strong structural assumptions on them. However, we have estimated θ directly which makes the estimator $\hat{\theta}$ consistent.

4.3.2 A LASSO-type estimator

Now that we have consistently estimated the response variable, r_c in proposition 2, following (4.14) we can construct our feasible LASSO-type estimator $\hat{\boldsymbol{w}}^* = (\hat{w}_1^*, ..., \hat{w}_N^*)'$ as follows:

$$\hat{\boldsymbol{w}}^* = \arg\min_{w} \frac{1}{T} \sum_{\boldsymbol{t=1}}^{\boldsymbol{T}} (r_c - \boldsymbol{w'R_t})^2 \quad \text{subject to} \quad ||\boldsymbol{w}||_1 \leq \lambda, \tag{4.17}$$

where the expression for $\hat{\boldsymbol{w}}^*$ is the basis of the MAXSER methodology.

4.4 Scenario II: When factor investing is allowed

An investor might be able to invest in more than just assets, as they might be able to invest in factors as well. This implies that factors must be included in the asset pool. We will now illustrate the implementation in the case where one can also invest in factors. To do this consider the following model:

$$r_i = \alpha_i + \sum_{\boldsymbol{j=1}}^{\boldsymbol{K}} \beta_{ij} f_j + e_i = \sum_{\boldsymbol{j=1}}^{\boldsymbol{K}} \beta_{ij} f_j + u_i, \tag{4.18}$$

where the excess return on factors are denoted as f_j and each stocks individual sensitivity to the factors are denoted as β_{ij}'s, the remaining errors in the model which are independent from the factor returns are denoted as e_i. We will allow the idiosyncratic returns (u_i) to admit factor structure, unlike the approximate factor model where the "idiosyncratic returns" $(u_i = \alpha_i + e_i)$ are assumed to have no factor structure. The factors in the model can be any factors identified in the literature of asset pricing. Statistical factors based on historical returns of larger asset pools can also be used. However, we have chosen the Fama-French three factor model (FF3) which is the same as Ao et al. (2018) are using in their analysis. The expression above (4.18) can be rewritten into a compact form as:

$$\boldsymbol{r} = \beta \boldsymbol{f} + \boldsymbol{u}, \tag{4.19}$$

where $\boldsymbol{\beta} = (\beta_{ij})_{NxK}$, $\boldsymbol{f} = (f_1, ..., f_K)'$, and $\boldsymbol{u} = (u_1, ..., u_N)'$. We denote the mean of factor returns as μ_f, the mean of idiosyncratic returns $\boldsymbol{u}$ as $\boldsymbol{\alpha} = (\alpha_1, ..., _N)'$. We denote the covariance matrix of factor returns and idiosyncratic returns as $\sum_{\boldsymbol{f}}$ and $\sum_{\boldsymbol{u}}$ respectively. Following this, the return vector $\boldsymbol{r}$ will have the following mean $\boldsymbol{\mu}$ and covariance matrix $\sum$:

$$\boldsymbol{\mu} = \boldsymbol{\beta}\boldsymbol{\mu}_f + \boldsymbol{\alpha} \qquad \Sigma = \boldsymbol{\beta}\Sigma_f\boldsymbol{\beta}' + \Sigma_u. \tag{4.20}$$

Now we can include the full pool of factors and assets and denote the mean and covariance matrix of this return as:

$$\boldsymbol{\mu}_{all} = \begin{pmatrix} \boldsymbol{\mu}_f \\ \boldsymbol{\mu} \end{pmatrix}, \qquad \Sigma_{all} = \begin{pmatrix} \Sigma_f & \boldsymbol{\beta}'\Sigma_f \\ \Sigma_f\boldsymbol{\beta} & \Sigma \end{pmatrix}. \tag{4.21}$$

The aim now is to find the optimal allocation of weights for both factors and individual assets $(\boldsymbol{w}_f, \boldsymbol{w})$, where weights on the factors and weights on the individual assets are denoted $\boldsymbol{w}_f$ and $\boldsymbol{w}$ respectively. Ao et al. (2018) show that this optimal allocation leads to a third proposition.

Proposition 3. For any given risk constraint level σ, *the optimal portfolio* $\boldsymbol{w}_{all} = (\boldsymbol{w}_f, \boldsymbol{w})$ *is given by:*

$$\sigma \left(\sqrt{\frac{\theta_f}{\theta_{all}}} \boldsymbol{w}_f^* - \sqrt{\frac{\theta_u}{\theta_{all}}} \boldsymbol{\beta}' \boldsymbol{w}_u^*, \quad \sqrt{\frac{\theta_u}{\theta_{all}}} \boldsymbol{w}_u^* \right), \tag{4.22}$$

where $\theta_f = \boldsymbol{\mu}_f' \Sigma_f^{-1} \boldsymbol{\mu}_f$, $\theta_u = \boldsymbol{\alpha}' \Sigma_u^{-1} \boldsymbol{\alpha}$, *and* $\theta_{all} = \boldsymbol{\mu}_{all}' \Sigma_{all}^{-1} \boldsymbol{\mu}_{all}$ *are the squared maximum Sharpe ratios of portfolios on the factors, the idiosyncratic components, and the full set of factors and individual assets, respectively. Moreover,* $\boldsymbol{w}_f^*$ and $\boldsymbol{w}_u^*$ *admits the following expressions:*

$$\boldsymbol{w}_f^* = \frac{1}{\sqrt{\theta_f}} \Sigma_f^{-1} \boldsymbol{\mu}_f, \qquad \boldsymbol{w}_u^* = \frac{1}{\sqrt{\theta_u}} \Sigma_u^{-1} \boldsymbol{\alpha}. \tag{4.23}$$

Proposition 3 implies that we need estimates for $\theta_f, \theta_u, \theta_{all}, \boldsymbol{w}_f^*$ and $\boldsymbol{w}_u^*$ to be able to estimate the optimal portfolio. How these are estimated will be covered in the next section.

4.4.1 Estimating the maximum Sharpe ratio and the regression response

Now we have to estimate the response variable to use in the unconstrained regression when factors are included. To estimate this we randomly pick 25 assets, from OSEBX, with 15 years of data by using the same *sample* function mentioned earlier. Then, we use *extractRand* once more, in order to obtain a 10 year time period, which will be the same for all assets. This will be identical to what we did in Scenario I. However, we now have to include factors as well. Since we are using the Fama-French three factor model there are three factors to be included, high minus low (HML), small minus big (SMB) and the market factor. The asset return and factor return can be denoted as $\boldsymbol{R_t} = (\boldsymbol{R_{t1}}, ..., \boldsymbol{R_{tN}})'$ and $\boldsymbol{F_t} = (\boldsymbol{F}_{t1}, ..., \boldsymbol{F}_{tK})'$ respectively. Here both the asset return and factor return have the same time horizon. The $\boldsymbol{\beta}$ coefficient in (4.19) is estimated by regressing the return matrix of the individual assets, $\boldsymbol{R}$, on the factor return matrix, $\boldsymbol{F}$, and the estimated coefficient matrix is stored and denoted as $\hat{\boldsymbol{\beta}}$. The estimator of $\boldsymbol{U} = \boldsymbol{R} - \boldsymbol{F\beta}$ is thus $\hat{\boldsymbol{U}} = (\hat{\boldsymbol{U}}_1, ..., \hat{\boldsymbol{U}}_T)' = \boldsymbol{R} - \boldsymbol{F}\hat{\boldsymbol{\beta}}$. Now we calculate the sample mean and sample covariance of the factor return and denote them as $\hat{\boldsymbol{\mu}}_f$ and $\hat{\sum}_f$ respectively.

We can now calculate the three Sharpe ratios needed to calculate the response variable, these are $\sqrt{\theta_f}$, $\sqrt{\theta_u}$ and $\sqrt{\theta_{all}}$, which will be the maximum Sharpe ratio on factors, idiosyncratic returns and all assets respectively. Due to the fact that there is only three factors the maximum Sharpe ratio on the factors can be estimated consistently by its plug-in estimator:

$$\hat{\theta}_f = \hat{\boldsymbol{\mu}}_f' \hat{\Sigma}_f^{-1} \hat{\boldsymbol{\mu}}_f. \tag{4.24}$$

The next two Sharpe ratios, $\sqrt{\theta_u}$ and $\sqrt{\theta_{all}}$, involves a large number of assets and are thus high dimensional. The issue of high dimensionality can be solved by adjusting for bias in the plug-in estimator. Ao et al. (2018) defines the following proposition:

Proposition 4. *Define the following estimator of* $\sqrt{\theta_{all}}$:

$$\hat{\theta}_{all} = \frac{(T - N - K - 2)\hat{\theta}_{s,all} - N - K}{T}, \tag{4.25}$$

where $\hat{\theta}_{s,all} = \hat{\boldsymbol{\mu}}_{all}' \hat{\Sigma}_{all}^{-1} \hat{\boldsymbol{\mu}}_{all}$ *is the sample estimate of* θ_{all}.

The last Sharpe ratio is a bit trickier to estimate. Due to the fact that the idiosyncratic returns $\boldsymbol{U}$ is not observable in our model. One solution would be to use the sample estimate, U. However, applying (4.25) to the sample estimate of $\boldsymbol{U}$ will become biased. However, we can use model (4.19), and through this it can be shown that

$$\theta_{all} = \theta_f + \theta_u. \tag{4.26}$$

Now, through (4.24) and Proposition 4, θ_u and θ_f can both be estimated consistently. This leads to the following proposition.

Proposition 5. *Define* $\hat{\theta}_u = \hat{\theta}_{all} - \hat{\theta}_f$. Using proposition 4, r_c, becomes $r_c = (1+\theta_u)/\sqrt{\theta_u}$, which leads to the estimate of the response variable becoming:

$$\hat{r_c} = \frac{1+\hat{\theta_u}}{\sqrt{\hat{\theta_u}}}. \tag{4.27}$$

4.4.2 Estimating the optimal portfolio on idiosyncratic components

Since the idiosyncratic returns are not observable, we have to use the estimated idiosyncratic returns $\hat{\boldsymbol{U}}$. However, since we have estimated the response variable we can consistently estimate the weights on the idiosyncratic components using the LASSO-type estimator in (4.17) as the following:

$$\hat{\boldsymbol{w}}_{\boldsymbol{u}}^{*} = \arg\min_{w} \frac{1}{T} \sum_{\boldsymbol{t=1}}^{\boldsymbol{T}} (\hat{r_c} - \boldsymbol{w}'\hat{\boldsymbol{U}}_t)^2 \quad \text{subject to} \quad ||\boldsymbol{w}||_1 \leq \lambda, \tag{4.28}$$

where λ represents the ℓ_1-norm constraint tuning parameter.

We have now consistently estimated θ_f, θ_u, θ_{all} and $\boldsymbol{w}_u^*$. However, there is one final item which needs to be estimated, namely the weights on factors, $\boldsymbol{w}_f^*$. This is relatively easy since there are few factors, and thus we can compute a consistent estimator on the factor weights using the plug-in estimator as

$$\hat{\boldsymbol{w}_f^*} = \frac{1}{\sqrt{\hat{\theta_f}}} \Sigma_f^{-1} \hat{\boldsymbol{u}}_f. \tag{4.29}$$

Now, combining all results together with Proposition 3 we achieve our estimator for the optimal full portfolio $\hat{\boldsymbol{w}}_{all}$:

$$\hat{\boldsymbol{w}}_{all} = (\hat{\boldsymbol{w}}_f, \hat{\boldsymbol{w}}) = \sigma \left(\sqrt{\frac{\hat{\theta_f}}{\hat{\theta_{all}}}} \hat{\boldsymbol{w}_f^*} - \sqrt{\frac{\hat{\theta_u}}{\hat{\theta_{all}}}} \hat{\boldsymbol{\beta}}' \hat{\boldsymbol{w}}_u^*, \quad \sqrt{\frac{\hat{\theta_u}}{\hat{\theta_{all}}}} \hat{\boldsymbol{w}}_u^* \right). \tag{4.30}$$

4.4.3 Choosing λ in (4.17) or (4.28)

When implementing MAXSER it is important to select an appropriate λ in (4.17) or (4.28). Since the goal of the MAXSER is to meet the risk constraint, the selection of λ must be such that the risk of the estimated portfolio is close to the given risk constraint. The underlying covariance matrix Σ or Σ_{all} is unknown in practice. Thus, in order to circumvent this difficulty, a cross-validation procedure is used. A 10-fold cross-validation procedure randomly split the sample into 10 groups in order to form 10 testing sets. For each testing set, the rest of the observations form the training set. Each training set i, has a ℓ_1-norm ratio ζ, which is defined as:

$$\zeta = \frac{||\boldsymbol{w}||_1}{||\boldsymbol{w}_{ols}||_1}. \tag{4.31}$$

Let ζ vary from 0 to 1 to obtain the whole solution path of $(\hat{\boldsymbol{w}}_\zeta^*)_{0 \leqslant \zeta \leqslant 1}$ ($(\hat{\boldsymbol{w}}_{all,\zeta}^*)_{0 \leqslant \zeta \leqslant 1}$ in scenario II), and find the value $\zeta(i)$ such that the difference in the risk computed using the testing set and the given risk constraint is minimized in the estimated portfolio. The $\hat{\lambda}$ is then the average of $(\lambda(i), i = 1, ..., 10)$.

4.4.4 Implementation steps of MAXSER

When implementing the MAXSER methodology there are several steps to follow. However, as the number of assets with enough data is relatively low compared to that of the S&P500 we do not need to conduct a subpool selection procedure. Thus, this step can be dropped from both scenarios.

Scenario I: When the asset pool consist of stocks only

1. First step is to estimate the response variable, $\hat{r_c}$, in (4.16). To do this we need to compute the square of the maximum Sharpe ratio, $\hat{\theta}$
2. Second step is to use cross validation and choose the appropriate tuning parameter, λ and denote this as $\hat{\lambda}$
3. Lastly, use the selected $\hat{\lambda}$ in the LASSO-type estimator and solve for the MAXSER portfolio $\hat{\boldsymbol{w}}^*$ in (4.17)

Scenario II: Asset pool consists of both assets and factors

1. First step is to obtain $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{U}}$ by performing an OLS regression of asset returns on factor returns.
2. Second step is to estimate the squares of the maximum Sharpe ratios of $\hat{\boldsymbol{\theta}}_f$, $\hat{\boldsymbol{\theta}_{all}}$ and $\hat{\boldsymbol{\theta}}_u$ to estimate the response variable, $\hat{r_c}$, in (4.27)
3. Third step is again to use cross validation and choose the appropriate tuning parameter, λ and denote this as $\hat{\lambda}$
4. Fourth step is to use the selected $\hat{\lambda}$ in (4.28) and solve for $\hat{\boldsymbol{w}}_u^*$
5. Lastly, obtain the MAXSER portfolio $\hat{\boldsymbol{w}_{all}}$ by computing $\hat{\boldsymbol{w}_f}$ and plug in this estimate and the previous ones into (4.30)

5 Analysis

In this section, we cover a simulation analysis and an empirical analysis of our data. Ao et al. (2018) show that the MAXSER methodology performs better than previous models on US stock returns. Thus, our analysis was carried out with the aim of checking whether this is the case when their methodology is implemented on Norwegian stock returns.

5.1 Simulation analysis

5.1.1 Benchmark portfolios

In our analysis, we have chosen the same strategies that Ao et al. (2018) use for comparison in their analysis. These strategies include the plug-in, linear shrinkage and nonlinear shrinkage portfolios discussed above. Table 1 gives a complete list of the strategies used for comparison.

One special portfolio among the portfolios under comparison is the "Factor portfolio," which here is the mean-variance portfolio on factors. This portfolio is special because it only contains a small number of assets, three in our case. Since the number of assets is low, the plug-in formula will yield nearly optimal portfolio weights. This is because we are no longer in a high-dimensional world. By including this portfolio, we can compare our results and check whether additional investments in assets are beneficial. The Factor portfolio is defined as:

$$\hat{\boldsymbol{w}}_{Fac} = \frac{\sigma}{\sqrt{\hat{\boldsymbol{\mu}}'_{\boldsymbol{f}} \hat{\sum}_{\boldsymbol{f}}^{\boldsymbol{-1}} \hat{\boldsymbol{\mu}}_{\boldsymbol{f}}}} \hat{\sum}_{\boldsymbol{f}}^{\boldsymbol{-1}} \hat{\boldsymbol{\mu}}_{\boldsymbol{f}} (= \sigma \hat{\boldsymbol{w}}_{\boldsymbol{f}}^{*}), \tag{5.1}$$

Table 1: *List of benchmark portfolios and their abbreviations. The mean-variance portfolio, represented by "MV", and the global minimum variance portfolio, represented by "GMV".*

Portfolio	Abbreviation
Plug-in MV on factors	Factor
MV/GMV with different covariance matrix estimates	
MV with sample cov	MV-P
MV with linear shrinkage cov	MV-LS
MV with nonlinear shrinkage cov	MV-NLS
MV with nonlinear shrinkage cov adjusted for factor models	MV-NLSF
GMV with linear shrinkage cov	GMV-LS
GMV with nonlinear shrinkage cov	GMV-NLS
MV with short-sale constraint	
MV with sample cov and short-sale	MV-P-SS
MV with linear shrinkage cov and short-sale	MV-LS-SS
MV with nonlinear shrinkage cov and short-sale	MV-NLS-SS
MV with ℓ_1-norm constraint and cross-validation	
MV with sample cov and ℓ_1-CV	MV-P-L1CV
MV with linear shrinkage cov and ℓ_1-CV	MV-LS-L1CV
MV with nonlinear shrinkage cov and ℓ_1-CV	MV-NLS-L1CV

here, $\hat{\mu_f}$ and $\hat{\sum}_f^{-1}$ are the sample mean and sample covariance matrix of the factor returns, respectively.

The other portfolios are constructed using mean-variance (MV) and global minimum-variance (GMV), but are replaced by the covariance matrix with the sample linear shrinkage estimator by Ledoit and Wolf

(2003) and the nonlinear shrinkage estimator by Ledoit and Wolf (2017) in the respective MV and GMV portfolio weights formulas. In addition to the regular mean-variance portfolios with different covariance estimators, we include portfolios with either a short-sale constraint or a ℓ_1-norm constraint. These portfolios use the sample covariance, linear shrinkage covariance and the nonlinear shrinkage covariance. The short-sale constraint implies that an investor can only hold long positions in their portfolio. These portfolios are included so that they hold the same benefit (in terms of risk control) as the MAXSER portfolio. In this way, one can compare the result not solely based on the constraint imposed on the portfolios, but also based on the fundamental methodology of the MAXSER model.

5.1.2 Simulation comparison

In this section, we present our results when the data are generated under a multivariate normal distribution. Here, the data was generated using the *mvnorm* command from the *MASS* package in R by Venables and Ripley (2002). This command allows us to generate new data for all assets included in our asset pool. This implies that we can create new return matrices for our asset pool. We then constructed our portfolios based on the new return matrices. For our simulation comparison, we created 1,000 entirely different return matrices, with both $T = 120$ and $T = 240$, which are calibrated using real data, namely μ_{all} and $\sum_{all}$ from section 4.4. When implementing the MAXSER methodology, we need both a return matrix with stock returns and a return matrix with factor returns. This implies that we have to generate 1,000 new return matrices on both stock returns and factor returns. This is done as mentioned above, but instead of using μ_{all} and $\sum_{all}$, we generated stock return μ and $\sum$, and the factor returns are generated using μ_f and $\sum_f$. The formulas for these parameters can be found in section 4.3.1 and 4.4. The level of risk constraint is set as $\sigma = 0.05$, because the number of stocks is relatively low compared to the study we are replicating. Thus, when implementing the MAXSER methodology, we start from step one described in Section 4.4.4.

In our simulation analysis, we performed a portfolio selection on 1,000 replications to evaluate the portfolio performance in terms of both risk and the Sharpe ratio. To calculate the risk and Sharpe ratios for the mean-variance portfolios (except L1CV), we created an empty list and we used the *portfolio.optim* function, from the *tseries* package by Trapletti and Hornik (2019), on each simulated set of returns with the corresponding covariance matrices for each strategy. Each iteration was saved in the empty list. Then, we extracted the portfolio returns, portfolio weights, and the standard deviations from the newly created list. By finding the mean for the standard deviations, we produced the "Risk" column in Table 2. We found the Sharpe ratio by dividing the returns by the corresponding standard deviation, and we calculated the mean of all the Sharpe ratios in order to produce the "Sharpe Ratio" column in Table 2.

We used the SRISK function by Koenker (2020) on the sample returns, by setting our $\lambda = 0.528$ and $\hat{r}^* = 0.0412$ in order to find the mean-variance portfolios with ℓ_1-norm constraint. Furthermore, we extracted the returns and standard deviations by using *sapply*. Once we extracted the standard deviations and returns, we determined the values for the Risk column by taking the mean of the standard deviations. We determined the values for the Sharpe Ratio column by dividing the returns by the standard deviations and taking the mean of that quotient.

The global minimum-variance portfolios were found by first generating data using the *mvrnorm* function, then by finding the covariance matrix of each iteration. We find the covariance matrices in order to use the *minvar* function by Schumann (2020), which finds the portfolio weights that minimize the variance. The expected return was calculated by using the *mapply* function on the generated returns and the the optimal portfolio weights. To determine the risk, we first had to obtain the variance of each portfolio. This was done by extracting the attributes from the *minvar* function. Once we had the variance, we could find the risk by taking the square root of the variance on each corresponding portfolio before finding the mean. The Sharpe ratio was found by using the same method as with the mean-variance portfolios. The comparison results for sample sizes $T = 120$ and $T = 240$ are summarized in Table 2.

When $T = 120$, the risk of the MAXSER portfolio is relatively high compared to the standard mean-variance portfolio with plug-in estimates. Similarly, the factor portfolio is fairly close to the given risk constraint because it is a low-dimensional portfolio where the plug-in estimator would be sufficient. Next, for the mean-variance portfolios with different covariance matrix estimates, we observe that none of them violate the given risk constraint. In fact, they are relatively low compared to the risk constraint. The risks of the plug-in, linear shrinkage and non-linear shrinkage are approximately 66%, 46% and 58% lower than the risk constraint, respectively. By imposing a short-sale constraint, we see that the risk constraint is still not violated for any of them, but the

Table 2:

Summary of risks and Sharpe ratios of the portfolios under comparison based on 1,000 replications. The sample size of the returns are $T = 120$ and $T = 240$, and are generated from a multivariate normal distribution. The risk constraint is set to be $\sigma = 0.05$. The average values of the Sharpe ratio and risk of each portfolio are recorded, and its standard deviation(in brackets)

$\sigma = 0.05$	T = 120		T = 240	
Portfolio	Risk	Sharpe Ratio	Risk	Sharpe Ratio
Factor	0.031 (0.002)	0.129 (0.094)	0.032 (0.002)	0.129 (0.068)
MAXSER	0.077 (0.118)	0.653 (0.129)	0.076 (0.013)	0.662 (0.081)
MV/GMV with different covariance matrix estimates				
MV-P	0.017 (0.002)	0.308 (0.313)	0.018 (0.001)	0.286 (0.219)
MV-LS	0.027 (0.002)	0.189 (0.202)	0.028 (0.001)	0.170 (0.132)
MV-NLS	0.021 (0.002)	0.233 (0.272)	0.022 (0.001)	0.230 (0.180)
GMV-LS	0.026 (0.002)	0.175 (0.157)	0.029 (0.001)	0.166 (0.093)
GMV-NLS	0.020 (0.002)	0.229 (0.163)	0.022 (0.001)	0.212 (0.101)
MV with short-sale constraint				
MV-P-SS	0.025 (0.003)	0.209 (0.212)	0.026 (0.002)	0.206 (0.157)
MV-LS-SS	0.031 (0.002)	0.165 (0.175)	0.031 (0.002)	0.154 (0.132)
MV-NLS-SS	0.027 (0.002)	0.184 (0.209)	0.027 (0.002)	0.189 (0.146)
MV with ℓ_1-norm constraint and cross-validation				
MV-P-L1CV	0.016 (0.001)	0.327 (0.146)	0.017 (0.000)	0.299 (0.090)
MV-LS-L1CV	0.026 (0.002)	0.175 (0.163)	0.028 (0.001)	0.160 (0.096)
MV-NLS-L1CV	0.020 (0.001)	0.232 (0.166)	0.022 (0.001)	0.215 (0.101)

risk is marginally larger than without the short-sale constraint. This indicates that, concerning risk control, the cross-validation procedure might not have worked as effectively as for the MAXSER portfolio.

Furthermore, we can see from the simulations that the Sharpe ratios of the mean-variance portfolios are very low compared to MAXSER; in fact, the mean-variance with plug-in estimates performs the best among them, even when considering different constraints. This result is a contradiction to what Ao et al. (2018) found in their analysis. This might be because our asset pool consists of fewer assets or because of the overall performance of the Oslo Stock Exchange.

When $T = 240$, most of the portfolios perform worse than when $T = 120$, both in terms of risk and the Sharpe ratio. However, the MAXSER portfolio is one of the few portfolios that achieves better results both in terms of lower risk and higher Sharpe ratios. However, the improvements are marginal. Furthermore, we can see that when $T = 240$, we still get the same performance results for both risk and the Sharpe ratio. The mean-variance portfolio with plug-in estimates performs best in terms of risk, while the MAXSER portfolio performs best in terms of the Sharpe ratio. Moreover, the MAXSER portfolio achieves the highest Sharpe ratio out of the other portfolios tested both when $T = 120$ and $T = 240$.

In short, the MAXSER portfolio achieves significantly higher Sharpe ratios compared to the other benchmark portfolios, both when $T = 120$ and $T = 240$, , but it fails to efficiently control risk as the risk is always higher than the given risk constraint.

5.2 Empirical analysis

In this section, we investigate the performance of the MAXSER methodology based on the out-of-sample return of the stocks included in the two Oslo Stock Exchange indices OBX and OSEBX, and we compare our results to that of previous models. The portfolios used for comparison are the same as those used in the simulation analysis in Section 5.1.1, in addition to the index and an equally weighted portfolio. The results are outlined in Section 5.2.1 and 5.2.5

5.2.1 OBX

First, we evaluated the MAXSER strategy based on the stocks included in the OBX Index, where OBX is an index consisting of the 25 most liquid shares (ranked after six months of trading) and the Fama-French three factors. We began by retrieving all the stock data from the companies currently listed on the OBX Index, and then we formed an asset pool consisting of all the stocks. Using the asset monthly excess return of each stock during the prior T-months training period, we performed a one-step-ahead forecast using a rolling-window scheme. Here, T represents the sample size. In our case, $T = 30$. If a stock has missing data in the prior T-months training period, it gets excluded from the asset pool. This led to several stocks being excluded from our asset pool, and our asset pool consisted of 19 stocks and the Fama-French three factors. Using the one-step-ahead forecast, we could form a return matrix consisting of our out-of-sample data, which would be our testing data. Next, we calculated the standard deviation of the monthly excess return on the OBX Index in the first training period in order to obtain the risk constraint. Thus, the risk constraint σ was set as 0.1. Last, we created the portfolios for comparison. These were created using the prior T-months training data and were rebalanced with new optimal weights monthly. Our findings based on these portfolios are recorded in Table 3.

5.2.2 Performance evaluation

The performance of the MAXSER portfolio and the other benchmark portfolios is evaluated based on their Sharpe ratio and risk, which are computed from their respective (out-of-sample) monthly returns. We conducted two testing periods: the first was a 10 year period from 2009 to 2018, and the second one was a 5 year period from 2014 to 2018. Thus, we had 120 and 60 out-of-sample monthly returns, respectively, for each strategy.

To verify the statistical significance of the advantage of the MAXSER portfolio, we conducted hypothesis testing based on the difference between the MAXSER portfolio's Sharpe ratio and the other benchmark portfolios' Sharpe ratio. This test was conducted by using the $sharpeTesting$ command in R, included in the package $PeerPerformance$ by David and Boudt (2020). The results is outlined in the tables under the $p$value. The test is formed as:

$$H_0 : SR_{MAXSER} \leqslant SR_0 \quad vs \quad H_1 : SR_{MAXSER} > SR_0, \qquad (5.2)$$

where SR_{MAXSER} denotes the Sharpe ratio of the MAXSER portfolio, and SR_0 denotes the comparable portfolio's Sharpe ratio. The $sharpeTesting$ function is a correction by Ledoit et al. (2002) to the test of Jobson and Korkie (1981); it tests the Sharpe ratio difference between two portfolios.

5.2.3 Comparison summary

Table 3:

Outlines the risk, Sharpe ratio and $p-values$ of the Sharpe ratio test (5.2) for the selection of comparison portfolios on the OBX Index and the FF3 factors. The two testing periods are 2009-2018 and 2014-2018. The risk constraint is calculated as the standard deviation of the monthly excess returns from July 2006 - December 2008 on the OBX Index, the first training period.

Portfolio performance based on OBX constituents and FF3 factors

OBX constituents and FF3 *(without transaction costs)*				$T = 30$		$\sigma = 0.1$
Period	2009-2018			2014-2018		
Portfolio	Risk	Sharpe Ratio	p-value	Risk	Sharpe Ratio	p-value
Index	0.044	0.215	0.189	0.033	0.170	0.001
Equally Weighted	0.045	0.261	0.370	0.048	0.354	0.003
Factor	0.019	0.247	0.259	0.018	0.278	0.001
MAXSER	0.176	**0.407**	-	0.106	**0.890**	-
MV/GMV with different covariance matrix estimates						
MV-P	0.019	0.266	0.417	0.011	1.282	0.055
MV-LS	0.022	0.247	0.309	0.023	0.546	0.036
MV-NLS	0.017	0.265	0.368	0.014	0.861	0.862
GMV-LS	0.019	0.446	0.823	0.019	0.573	0.050
GMV-NLS	0.014	0.508	0.557	0.012	0.664	0.161
MV with short-sale constraint						
MV-P-SS	0.019	0.246	0.335	0.012	1.077	0.188
MV-LS-SS	0.024	0.201	0.191	0.023	0.527	0.025
MV-NLS-SS	0.020	0.189	0.170	0.014	0.769	0.406
MV with ℓ_1-norm constraint and cross-validation						
MV-P-L1CV	0.012	0.529	0.467	0.012	0.596	0.204

Here, $p < 0.1$ represents the statistical significance at the 10% significance level, $p < 0.05$ at the 5% significance level and $p < 0.01$ 1% at the significance level.

Table 3 provides a summary of the risk, Sharpe ratio and p-value for the MAXSER portfolio and for each benchmark portfolio. From the table we observe the following.

During the first testing period, we can see that the risk of the MAXSER portfolio is much higher than the given risk constraint of 0.1 and is almost four times higher than that of the index during the same period. This might imply that the MAXSER portfolio is not able to effectively control for the risk constraint. Furthermore, we observe that nearly every mean-variance portfolio has almost 10 times lower risk than the MAXSER portfolio, making the mean-variance portfolios attractive to risk-averse investors. When comparing the Sharpe ratios, we observe that the MAXSER portfolio does not achieve the highest Sharpe ratio out of the comparable portfolios. In fact, the global minimum-variance portfolios and the mean-variance portfolio with ℓ_1-norm constraint and cross validation are the portfolios that achieve the highest Sharpe ratio, and for much lower risk. Thus, this makes these portfolios attractive to many investors. Furthermore, this might imply that there is no advantage of using the MAXSER methodology to create a portfolio comprised of stocks traded on the OBX Index. We observe that the portfolio achieving the lowest Sharpe ratio is the mean-variance portfolio with a nonlinear covariance estimate and a short-sale constraint.

Furthermore, when observing data in the second testing period, we can see that the risk of the MAXSER portfolio is still higher than that of the benchmark portfolios. However, when comparing it to the first training period, we can see that the risk is closer to the given risk constraint. This might imply that the MAXSER methodology is able to effectively control for the given risk constraint during the second period. However, this change might be because the first training period contains data from the financial crisis, which the second training period does not. However, there does not seem to be such an impact in risk regarding the other portfolios, justifying MAXSER's ability to control for risk. Furthermore, the risk of the benchmark portfolios is still very low compared to MAXSER. Next, looking at the Sharpe ratio, we observe an increase for almost all portfolios. Compared to the first testing period, MAXSER still does not achieve the highest Sharpe ratio. However, now the standard mean-variance with plug-in estimates both with and without constraints achieves a higher Sharpe ratio.

By looking at the p-values from the Sharpe ratio test, we can see that they are large for all the portfolios during the first testing period and large for all mean-variance and global minimum-variance portfolios during the second testing period. This implies that the eventual advantages of using the MAXSER methodology when creating portfolios is not economically large or statistically significant when looking at the OBX Index.

5.2.4 Accounting for transaction costs

Next, we evaluate the performance of the portfolios when transaction costs are taken into account by computing the net returns of transaction costs. Following the formula used by Ao et al. (2018),the portfolio net returns of transaction costs are computed as

$$r_{net}(t) = \left(1 - \sum_j c_{t,j} \mid w_j(t+1) - w_j(t+) \mid \right)(1 + r(t)) - 1, \quad (5.2.1)$$

where $w_j(t+1)$ is the weight of asset j at the beginning of period $t+1$, and $w_j(t+)$ is the weight of the same asset at the end of period t. $c_{t,j}$ is a cost level that measures the transaction cost per NOK traded for asset j, and $r(t)$ is the portfolio return without transaction cost in period t. The cost level $c_{t,j}$ is based on the work of Ødegaard (2009). We assume the average relative spread price for the period from 2000 to 2008 is applicable through 2018 when calculating the net return of transaction costs.This average relative spread is used on all portfolios, when looking at both OBX and OSEBX. However, there might be a different cost level when trading in factors. Furthermore, Ao et al. (2018) use a cost level that changes during the testing period, which affects their results. The fact that we use the same cost level over the entire testing period and for all assets might have an impact that makes our results differ from their analysis.

Compared to the results without transaction costs, we see an increase in risk for almost all portfolios for both testing periods and a decrease in the Sharpe ratio for all portfolios except the equally weighted portfolio. Furthermore, looking at Table 4, we can see that the global minimum-variance portfolio with linear shrinkage covariance outperforms the other portfolios in terms of Sharpe ratios for both periods. We also observe that the MAXSER portfolio is severely impacted by the transaction costs, both in terms of an increase in risk and also a severe reduction in the Sharpe ratio. The MAXSER portfolio dropped from one of the best-performing portfolios, when transaction costs were excluded, to one of the worst-performing portfolios when transaction costs were included. Furthermore, as in the case without transaction costs, we observe that the MAXSER portfolio during the second testing period is much closer to the given risk constraint, showing the MAXSER methodology's ability to control for risk.

There is an impact on the p-values when transaction costs are included during both periods. This implies that when transaction costs are included, the advantage of using MAXSER compared to some of the benchmark portfolios is statistically significant.

First, by analyzing the use of MAXSER on the OBX Index, we see that the risk of the MAXSER portfolio is very high compared to the other benchmark portfolios. The reason is that the MAXSER methodology tries to control for risk. Thus, when calculating the weights, the main goal is to minimize the difference between the portfolio's risk and the given risk constraint. Therefore, the risk of this portfolio will lie around the given risk constraint if it is able to effectively control for risk. However, the other portfolios have no given risk constraint, and this might be why they are so much lower than the MAXSER portfolio. Second, we can see that the advantage of using the MAXSER methodology when creating a portfolio is not statistically significant. A reason for this might be that the asset pool comprises of so few assets, as OBX includes only the 25 most liquid stocks and we had to exclude six of them. Additionally, during the first testing period, the MAXSER portfolio did not seem to be able to effectively control for the given risk constraint.

Table 4:

Outlines the risk, Sharpe ratio and $p-values$ of the Sharpe ratio test (5.2) with portfolio returns with transaction costs for the selection of comparison portfolios on the OBX Index and the FF3 factors. The two testing periods are 2009-2018 and 2014-2018. The risk constraint is calculated as the standard deviation of the monthly excess returns from July 2006 - December 2008 on the OBX Index, the first training period.

Portfolio performance based on OBX constituents and FF3 factors

OBX constituents and FF3 *(with transaction costs)*					$T = 30$	$\sigma = 0.1$
Period	2009-2018			2014-2018		
Portfolio	Risk	Sharpe Ratio	p-value	Risk	Sharpe Ratio	p-value
Index	0.044	0.215	0.001	0.033	0.187	0.014
Equally Weighted	0.045	0.247	0.003	0.048	0.350	0.014
Factor	0.018	0.077	0.020	0.017	0.126	0.041
MAXSER	0.206	**-0.228**	-	0.140	**-0.279**	-
MV/GMV with different covariance matrix estimates						
MV-P	0.043	-0.936	0.000	0.047	-0.846	0.004
MV-LS	0.024	-0.007	0.115	0.022	0.445	0.000
MV-NLS	0.024	-0.667	0.002	0.022	-0.366	0.586
GMV-LS	0.020	0.301	0.000	0.019	0.491	0.000
GMV-NLS	0.016	-0.044	0.142	0.014	-0.137	0.403
MV with short-sale constraint						
MV-P-SS	0.043	-0.925	0.000	0.046	-0.876	0.003
MV-LS-SS	0.025	0.012	0.083	0.023	0.434	0.000
MV-NLS-SS	0.018	-0.294	0.601	0.015	0.073	0.048
MV with ℓ_1-norm constraint and cross-validation						
MV-P-L1CV	0.030	-1.00	0.000	0.022	-1.191	0.000

Here, $p < 0.1$ represents the statistical significance at the 10% significance level, $p < 0.05$ at the 5% significance level and $p < 0.01$ 1% at the significance level.

5.2.5 OSEBX

Next, we look at the OSEBX Index. The OSEBX Index is a larger stock universe, which contains a representative selection of all stocks traded on the Oslo Stock Exchange. We look at OSEBX to perform the same strategy on larger portfolios.

Again, we used a rolling-window scheme to perform the analysis similar to the one mentioned in Section 5.2.1. We recorded the out-of-sample returns from the period from 2009 to 2018 and rebalanced the portfolios monthly based on the prior T-months training period.

We began by randomly forming a pool of 35 stocks currently traded on the OSEBX Index and of the Fama-French three factors. Using the assets' excess monthly returns during the prior T-months training period, we performed the one-step-ahead forecast, where T represents the sample size and, in our case, $T = 60$ As before, if a stock had missing data in the prior T-month training period, it was excluded from the asset pool. This led to several stocks being excluded, with our asset pool then consisting of 29 stocks and the Fama-French three factors. Using the same procedure as before, we formed a return matrix based on our rolling-window scheme, which again became our testing data. Next, we calculated the standard deviation on the OSEBX Index for the first training period in order to obtain the risk constraint. The risk constraint was set as 0.08. Then we calculated our portfolio return based on the monthly optimal weights calculated using the prior T-month training data and the testing data. When implementing the MAXSER portfolio, we started with step one in Section 4.4.4, as the number of stocks is still relatively low compared to that of Ao et al. (2018). Last, we recorded our findings in Table 5 for all our portfolios under comparison.

As with the analysis done on the OBX Index, the performance of the

MAXSER portfolio and the other benchmark portfolios were evaluated based on their Sharpe ratio and risk, which were computed from their respective (out-of-sample) monthly returns. The two testing periods were the same, one with 10 years of testing data and the other with five years of testing data. This led to the same out-of-sample monthly returns, 120 and 60, respectively. We also conducted the Sharpe ratio test mentioned earlier on every portfolio under comparison.

5.2.6 Comparison summary

Table 5 shows the summary of the comparison of the MAXSER and the other benchmark portfolios in terms of risk, Sharpe ratio and p-values of the Sharpe ratio tests. For the first testing period, we observed that, as with the analysis on the OBX Index, the MAXSER portfolio carried the most risk, at 0.083. This is because the given risk constraint is set at 0.08, the standard deviation of the OSEBX Index during the first training period. The risk carried by the MAXSER portfolio implies that the MAXSER methodology effectively controlled for the given risk constraint. However, we can clearly see that no other portfolio reached this amount of risk; in fact, the next highest risk level was carried by the index during the testing period, which was nearly half of what the MAXSER portfolio carried. The portfolio that carried the least amount of risk was the mean-variance with linear shrinkage covariance matrix with a short-sale constraint. The risk of the two global minimum-variance portfolios and the mean-variance portfolio with the ℓ_1-norm constraint was, however, just marginally higher. Further, the risk of the mean-variance portfolios with different covariance estimates and no restrictions was somewhat the same. For the second testing pe-

Table 5:

Outlines the risk, Sharpe ratio and $p - values$ of the Sharpe ratio test (5.2) with portfolio returns net of transaction costs for the selection of comparison portfolios on the OSEBX Index and the FF3 factors. The two testing periods are 2009-2018 and 2014-2018. The risk constraint is calculated as the standard deviation of the monthly excess returns from 2004 - 2008 on the OSEBX Index, the first training period.

Portfolio performance based on OSEBX constituents and FF3 factors

OSEBX constituents and FF3 *(without transaction costs)*					$T = 60$	$\sigma = 0.08$
Period	2009-2018			2014-2018		
Portfolio	Risk	Sharpe Ratio	p-value	Risk	Sharpe Ratio	p-value
Index	0.043	0.245	0.000	0.032	0.198	0.004
Equally Weighted	0.031	0.319	0.000	0.018	0.467	0.018
Factor	0.018	0.184	0.000	0.019	0.182	0.001
MAXSER	0.083	**0.997**	-	0.064	**0.883**	-
MV/GMV with different covariance matrix estimates						
MV-P	0.019	0.189	0.000	0.015	0.695	0.219
MV-LS	0.018	0.183	0.000	0.012	0.733	0.354
MV-NLS	0.019	0.157	0.000	0.013	0.722	0.300
GMV-LS	0.014	0.543	0.004	0.011	0.804	0.660
GMV-NLS	0.013	0.572	0.004	0.011	0.813	0.698
MV with short-sale constraint						
MV-P-SS	0.017	0.270	0.000	0.012	0.873	0.958
MV-LS-SS	0.012	0.264	0.000	0.011	0.753	0.452
MV-NLS-SS	0.017	0.264	0.000	0.011	0.800	0.612
MV with ℓ_1-norm constraint and cross-validation						
MV-P-L1CV	0.014	0.625	0.002	0.015	0.624	0.118

Here, $p < 0.1$ represents the statistical significance at the 10% significance level, $p < 0.05$ at the 5% significance level and $p < 0.01$ 1% at the significance level.

riod, we can see the same results in terms of risk. However, there is a decrease in the amount of risk carried for every portfolio except for the mean-variance with ℓ_1-norm constraint and cross validation, which was marginally higher. A contradiction to what Ao et al. (2018) found in their analysis is that the MAXSER portfolio carried the most risk. One of the reasons behind this is that, as mentioned before, the portfolios under comparison were not constrained to a given level of risk. Had these portfolios also been given a risk constraint, our results might have been completely different.

When looking at the Sharpe ratios, we see that the portfolio achieving the highest Sharpe ratio during the first testing period is the MAXSER portfolio. This Sharpe ratio is nearly 75% more than that of the mean-variance portfolio with ℓ_1-norm constraint and cross validation, which achieves the next highest Sharpe ratio out of the comparable portfolios. Both the global minimum-variance portfolios achieve almost half the Sharpe ratio of the MAXSER portfolio. Furthermore, the mean-variance portfolios without constraints achieve relatively low Sharpe rations compared to the MAXSER portfolio; they are in fact lower than that of the equally weighted portfolio and the Index. However, the portfolio achieving the lowest Sharpe ratio is the portfolio consisting of only factors. In terms of the Sharpe ratios during the second testing period, the results are the same. However, all the mean-variance portfolios with different covariance estimates and constraints achieve much higher Sharpe ratios than during the first testing period, while the Sharpe ratio of the MAXSER portfolio decreases. In fact, the mean-variance portfolio with plug-in estimates and short-sale constraints achieves a Sharpe ratio only marginally lower than that of the MAXSER portfolio. We can see from the two indexes that when the MAXSER portfolio consists of fewer stocks, the portfolio's Sharpe ratio is lower. This might only be due to the stocks selected from each index. However, Ao et al. (2018) show that their methodology performs better when the portfolio comprises a larger asset pool.

Furthermore, we can see that the p-values of the Sharpe ratio test are small during the first testing period. This implies that the advantage of the MAXSER methodology is not only economically large, but also statistically significant. However, during the second testing period, the benchmark portfolios perform much better than before and the p-values are no longer small, which implies that the methodology is not statistically significant.

5.2.7 Accounting for transactions costs

Looking at the benchmark portfolios for both testing periods when transaction costs are included, we can see that the risk of the portfolios increases or stays the same and that the Sharpe ratio decreases. The MAXSER portfolio is still the portfolio carrying the most risk, nearly double that of the portfolio carrying the next highest amount of risk for both periods. However, we can see that the MAXSER portfolio is no longer achieving the highest Sharpe ratio. With transaction costs, the portfolios performing best in terms of the Sharpe ratio are the global minimum-variance portfolios and the equally weighted portfolio. However, in our analysis, the equally weighted portfolio has no transaction costs, as the weights on the assets are the same throughout the testing periods and are therefore excluded when looking at the whole picture regarding performance of the MAXSER methodology.

When it comes to the p-values of the Sharpe ratio test, they varied greatly during the two testing periods. During the first testing period, most of the p-values were small. However, there were some that were high enough to imply that the advantage of MAXSER is not statistically significant when transaction costs are included. We observe that almost all the p-values are high enough to show no statistical significance for the advantage of MAXSER. Due to the fact that the p-values show no statistical significance of an advantage of MAXSER, it becomes difficult to argue that MAXSER dominates the benchmark strategies when it comes to mean-variance efficiency on the Norwegian stock market. However, we observe that the methodology effectively controls for risk, as the difference between the portfolio risk and the given risk constraint was always relatively low, except during the first testing period on the OBX Index.

Table 6:

Outlines the risk, Sharpe ratio and $p-$values of the Sharpe ratio test (5.2) for the selection of comparison portfolios on the OSEBX Index and the FF3 factors. The two testing periods are 2009-2018 and 2014-2018. The risk constraint is calculated as the standard deviation of the monthly excess returns from 2004 - 2008 on the OSEBX Index, the first training period.

Portfolio performance based on OSEBX constituents and FF3 factors

OSEBX constituents and FF3 *(with transaction costs)*					$T = 60$	$\sigma = 0.08$
Period	2009-2018			2014-2018		
Portfolio	Risk	Sharpe Ratio	p-value	Risk	Sharpe Ratio	p-value
Index	0.043	0.246	0.000	0.032	0.214	0.696
Equally Weighted	0.031	0.330	0.000	0.018	0.463	0.082
Factor	0.018	0.084	0.000	0.018	0.152	0.938
MAXSER	0.085	**-0.022**	-	0.066	**0.138**	-
MV/GMV with different covariance matrix estimates						
MV-P	0.040	-1.043	0.000	0.046	-0.954	0.000
MV-LS	0.019	-0.109	0.479	0.012	0.342	0.163
MV-NLS	0.024	-0.611	0.000	0.019	-0.471	0.000
GMV-LS	0.014	0.373	0.002	0.011	0.532	0.012
GMV-NLS	0.013	0.190	0.075	0.011	0.272	0.391
MV with short-sale constraint						
MV-P-SS	0.039	-1.019	0.000	0.045	-0.971	0.000
MV-LS-SS	0.018	0.058	0.540	0.011	0.432	0.057
MV-NLS-SS	0.018	-0.060	0.771	0.011	0.246	0.479
MV with ℓ_1-norm constraint and cross-validation						
MV-P-L1CV	0.018	-0.202	0.118	0.019	-0.271	0.027

Here, $p < 0.1$ represents the statistical significance at the 10% significance level, $p < 0.05$ at the 5% significance level and $p < 0.01$ 1% at the significance level.

6 Conclusion

In this thesis, we replicated the approach of Ao et al. (2018) when estimating the mean-variance efficient portfolio using data from the Oslo Stock Exchange. Ao et al.'s approach builds upon a novel unconstrained regression representation of the mean-variance problem proposed by Markowitz (1952). When MAXSER is used to create small portfolios based on assets traded on the Oslo Stock Exchange, we find that the strategy has no advantage compared to previous models, both with and without factor investing. However, the strategy is somewhat able to control for risk.

Looking at the simulation analysis, we see that the MAXSER portfolio outperforms the benchmark portfolios in terms of the Sharpe ratio. However, the MAXSER portfolio carries much higher risk because it tries to effectively control for a given risk constraint. The empirical analysis showcases some of the weaknesses that the MAXSER strategy has when used on a smaller stock market. With the OBX Index, which in our case consisted of 19 stocks and three factors, it is clear that the strategy is not able to effectively control for risk. However, when the strategy is used on the OSEBX Index, consisting of a larger number of stocks than the OBX Index, it is more able to control for risk. Furthermore, when comparing the portfolios, we see that the MAXSER portfolio is outperformed by some of the benchmark portfolios in terms of the Sharpe ratio. This observation implies that the use of the MAXSER strategy does not gain an economic advantage compared to previous strategies. However, when assets are included with factor investing, the MAXSER portfolio gains an advantage.

The Norwegian stock market is smaller than the US stock market and contains fewer observations due to missing data. These facts enabled us to roughly determine how large the portfolios must be for MAXSER to be effective in terms of risk control. It would be interesting to discover if this thesis' findings would change if no data were missing.

As pointed out, a weakness of this study is that a risk constraint is not imposed on the benchmark portfolios. Had these portfolios been restricted by a risk constraint, our results in terms of economic gain might have been different. This is because an increase in risk would yield a lower Sharpe ratio, and thus, the use of MAXSER could have been statistically significant. To control for this, future research could impose such a constraint on the benchmark portfolios.

A Appendix

A.1 R-function

```
extractRand ← function(v, p){
firstIndex ← sample(seq(length(v) - p +1), 1)
v[firstIndex:(firstIndex + p - 1)]}
```

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