ON THE STRUCTURE AND COHOMOLOGY RING OF CONNECTED HOPF ALGEBRAS

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ABSTRACT. Let p be a prime, and k be a field of characteristic p. We investigate the algebra structure and the structure of the cohomology ring for the connected Hopf algebras of dimension p^3, which appear in the classification obtained in [22]. The list consists of 23 algebras together with two infinite families. We identify the Morita type of the algebra, and in almost all cases this is sufficient to clarify the structure of the cohomology ring.

1. INTRODUCTION

The classification of finite-dimensional Hopf algebras in characteristic zero is well investigated by many people, see survey paper [1]. While this work is stimulating in its own right, many Hopf algebras of interest are, however, defined over a field k of positive characteristic, where the classification is much less known. Some work has been done in this direction. In [28], Scherotzke classified finite-dimensional pointed rank one Hopf algebras in positive characteristic which are generated by group-like and skew-primitive elements. Many Hopf algebras in positive characteristic also come from Nichols algebras which are of much interest, see [7, 15].

We recall that a Hopf algebra is called connected when it admits only one isomorphism class of simple comodules, or equivalently, its coradical is k. Note that finite-dimensional connected Hopf algebras only appear in characteristic p, for instance, group algebras of finite p-groups, restricted universal enveloping algebras of restricted Lie algebras, finite connected group schemes and others. In [16], all graded cocommutative connected Hopf algebras of dimension less than or equal to p^3 are classified. In the recent work of Nguyen, Wang and the third author [22], connected Hopf algebras of dimension p^3 are classified over an algebraically closed field k of characteristic p under some assumption on the primitive space of these Hopf algebras. The result is given below in Theorem 2.1.

We are interested in understanding the structure of the cohomology rings for these algebras. This only depends on the algebra structure, not on the Hopf structure. This motivates our work, namely we analyse the algebras, for those which are not local we obtain a presentation by quiver and relations. This allows us to obtain the cohomology rings in all cases except one.

Section 2 contains the description of the Hopf algebras in question, and a discussion of antipode, and Nakayama automorphism. Furthermore, we state our results on the algebra structure. It turns out that the algebras fall into six different classes, and in sections 3 to 8 we deal with these. Section 9 contains a reduction result which

relates finite generation of Ext-algebras related via adjoint functors. This is formulated for abelian categories with enough injective and projective objects. Section 10 shows that some of the algebras in our list can be viewed as twisted tensor products (in the sense of [8]), and using Section 9 it follows that their cohomology is Noetherian (with one exception). Section 11 gives the cohomology rings of the various algebras occuring in the classification.

For general background on quivers, path algebras and admissible quotients of path algebras we refer the reader to [2, 3]. For basic homological algebra we refer the reader to [4].

2. Connected Hopf algebras of dimension p^3

In this section we give the classification of connected Hopf algebras of dimension p^3, and explain the origin of the classification from the primitive space of the Hopf algebras. We recall some facts about the antipode and the Nakayama automorphism of finite dimensional Hopf algebras in general, and we apply this knowledge to the algebras in the classification to describe the antipode and the order of the Nakayama automorphism. Finally we explain how we classify them as algebras, which is the base of our further investigations. The classification theorem below assumes k is algebraically closed. In this paper, the field is usually arbitrary of characteristic p.

2.1. **Classification of connected Hopf algebras of dimension p^3.** The Hopf algebras in the classification of finite dimensional connected Hopf algebras of dimension p^3 are always presented in the form of $k\langle x, y, z \rangle / I$, where I is an ideal of $k\langle x, y, z \rangle$ defined by giving generators. The comultiplication is given by

$$\Delta(x) = x \otimes 1 + 1 \otimes x,$$
$$\Delta(y) = y \otimes 1 + 1 \otimes y + Y,$$
$$\Delta(z) = z \otimes 1 + 1 \otimes z + Z,$$

for some elements Y and Z in $(k\langle x, y, z \rangle / I) \otimes (k\langle x, y, z \rangle / I)$. In Theorem 2.1, proved in [22], the generators for the relations and the ideal I and the elements Y and Z are given explicitly. If Y and/or Z are not given, then they are zero. In order to express Y and Z, we use the notation

$$\omega(t) = \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} t^i \otimes t^{p-i}.$$

Theorem 2.1. *Let k be an algebraically closed field of characteristic p. The connected Hopf algebras over k of dimension p^3 are precisely*

A1: $k[x, y, z]/\langle x^p - x, y^p - y, z^p - z \rangle$ *with* $Y = \omega(x)$ *and* $Z = \omega(x)[y \otimes 1 + 1 \otimes y + \omega(x)]^{p-1} + \omega(y)$.
A2: $k[x, y, z]/\langle x^p, y^p - x, z^p - y \rangle$.
A3: $k[x, y, z]/\langle x^p, y^p, z^p \rangle$.
A4: $k[x, y, z]/\langle x^p, y^p, z^p - x \rangle$.
A5, $p = 2$: $k\langle x, y, z \rangle / \langle x^2, y^2, [x, y], [x, z], [y, z] - x, z^2 + xy \rangle$.
A5, $p > 2$: $A(\beta) = k\langle x, y, z \rangle / \langle x^p, y^p, [x, y], [x, z], [y, z] - x, z^p + x^{p-1}y - \beta x \rangle$ *for some $\beta \in k$ with $Y = \omega(x)$ and $Z = \omega(x)(y \otimes 1 + 1 \otimes y)^{p-1} + \omega(y)$. Any two $A(\beta)$ and $A(\beta')$ are isomorphic as Hopf algebras if and only if $\beta' = \gamma\beta$ for some $(p^2 + p - 1)$-th root of unity γ.*
B1: $k\langle x, y, z \rangle / \langle [x, y] - y, [x, z], [y, z], x^p - x, y^p, z^p \rangle$ *with* $Z = \omega(y)$.

B2: $k\langle x,y,z\rangle/\langle[x,y]-y,[x,z],[y,z]-yf(x),x^p-x,y^p,z^p-z\rangle$, where $f(x)=\sum_{i=1}^{p-1}(-1)^{i-1}(p-i)^{-1}x^i$ and $Z=\omega(x)$.

B3: $k\langle x,y,z\rangle/\langle[x,y]-y,[x,z]-z,[y,z]-y^2,x^p-x,y^p,z^p\rangle$ with $Z=-2x\otimes y$ for $p>2$.

C1: $k[x,y,z]/(x^p-x,y^p-y,z^p-z)$.

C2: $k[x,y,z]/(x^p-y,y^p-z,z^p)$.

C3: $k[x,y,z]/\langle x^p,y^p-z,z^p\rangle$.

C4: $k[x,y,z]/\langle x^p,y^p,z^p\rangle$.

C5: $k\langle x,y,z\rangle/\langle[x,y]-z,[x,z],[y,z],x^p,y^p,z^p\rangle$.

C6: $k\langle x,y,z\rangle/\langle[x,y]-z,[x,z],[y,z],x^p-z,y^p,z^p\rangle$ for $p>2$.

C7: $k[x,y,z]/\langle x^p,y^p,z^p-z\rangle$.

C8: $k[x,y,z]/\langle x^p-y,y^p,z^p-z\rangle$.

C9: $k[x,y,z]/\langle x^p,y^p-y,z^p-z\rangle$.

C10: $k\langle x,y,z\rangle/\langle[x,y]-z,[x,z],[y,z],x^p,y^p,z^p-z\rangle$.

C11: $k\langle x,y,z\rangle/\langle[x,y]-y,[x,z],[y,z],x^p-x,y^p,z^p\rangle$.

C12: $k\langle x,y,z\rangle/\langle[x,y]-y,[x,z],[y,z],x^p-x,y^p-z,z^p\rangle$.

C13: $k\langle x,y,z\rangle/\langle[x,y]-y,[x,z],[y,z],x^p-x,y^p,z^p-z\rangle$.

C14: $k\langle x,y,z\rangle/\langle[x,y]-y,[x,z],[y,z],x^p-x,y^p-z,z^p-z\rangle$.

C15: $k\langle x,y,z\rangle/\langle[x,y]-z,[x,z]-x,[y,z]+y,x^p,y^p,z^p-z\rangle$ for $p>2$.

C16: $C(\lambda,\delta)=k\langle x,y,z\rangle/\langle[x,y],[x,z]-\lambda x,[y,z]-\lambda^{-1}y,x^p,y^p,z^p-\delta z\rangle$ for some $\lambda\in k^\times$ such that $\delta=\lambda^{p-1}=\pm1$. Any two $C(\lambda_1,\delta_1)$ and $C(\lambda_2,\delta_2)$ are isomorphic as Hopf algebras if and only if $\delta_1=\delta_2$ and $\lambda_1=\lambda_2$ or $\lambda_1\lambda_2=1$.

Remark. 1) The antipode of the above Hopf algebras always exists, because the coalgebra structure is connected.

2) Let H be a connected Hopf algebra of dimension p^3 and denote by

$$P(H)=\{h\in H\mid \Delta(h)=h\otimes 1+h\otimes 1\},$$

the *primitive space of* H. The isomorphism classes of H given above satisfy the following conditions:

  (**A**) $\dim_k P(H)=1$.
  (**B**) $\dim_k P(H)=2$ with non-commuting elements.
  (**C**) $\dim_k P(H)=3$.

3) Hopf algebras of type **C** are exactly the restricted universal enveloping algebras of restricted Lie algebras of dimension 3. It includes all the p-nilpotent restricted Lie algebras of dimension 3 classified in [29, Theorem 2.1 (3/1),(3/2)], i.e., (up to isomorphisms) (3/1) (a) is **C4**, (3/1) (b) is **C3**, (3/1) (c) is **C2**; (3/2) (a) is **C5** ($p\geq 3$) and **C10** ($p=2$), and (3/2) (b) is **C6**.

2.2. **Antipode and Nakayama automorphism.** In this subsection we review some known results on the antipode and the Nakayama automorphism of finite dimensional Hopf algebras in general, and then apply them to the algebras in Theorem 2.1.

Let $H=(H,m,u,\Delta,\epsilon,S)$ be any finite-dimensional Hopf algebra over an arbitrary base field k. A *left integral* Λ *in* H is an element of H such that $h\Lambda=\epsilon(h)\Lambda$ for all $h\in H$; and a *right integral in* H is an element $\Lambda'\in H$ such that $\Lambda'h=\epsilon(h)\Lambda'$ for all $h\in H$. The space of left integrals and the space of right integrals are denoted by $\int_H^l$ and $\int_H^r$, respectively. A well-known result of Larson and Sweedler shows that $\dim\int_H^l=\dim\int_H^r=1$. We say H is *unimodular* if $\int_H^l=\int_H^r$.

For any algebra map $\alpha\colon H \to k$, we can define the left winding automorphism of H associated to α as $\Xi^l[\alpha](h) = \sum \alpha(h_1)h_2$ for any $h \in H$. Similarly, the right winding automorphism of H associated to α can be defined as $\Xi^r[\alpha](h) = \sum h_1\alpha(h_2)$ for any $h \in H$. Note that $\int_H^l \subset H$ is a right H-submodule of H. Since it is one-dimensional, the right H-module structure gives an algebra map $\alpha\colon H \to k$ such that $\Lambda h = \alpha(h)\Lambda$ for $0 \neq \Lambda \in \int_H^l$ and $h \in H$. We will use the term left winding automorphism of $\int_H^l$ instead of α.

The following results are due to Larson-Sweedler [20], Pareigis [25], and Brown-Zhang [6].

Theorem 2.2. *Let H be any finite-dimensional Hopf algebra. Then*

(a) *H is Frobenius with a nondegenerate associated bilinear form $\langle -, - \rangle\colon H \otimes H \to k$ given by $\langle a, b \rangle = \lambda(ab)$, where $0 \neq \lambda \in \int_{H^*}^l$ and $a, b \in H$.*

(b) *The Nakayama automorphism is given by $S^2\xi$, where ξ is the left winding automorphism of $\int_H^l$.*

(c) *H is symmetric if and only if H is unimodular and S^2 is inner.*

This has the following consequence for finite dimensional involutory Hopf algebras. Recall that a Hopf algebra H with antipode S is *involutory* if $S^2 = 1$.

Corollary 2.3. *The following finite-dimensional involutory Hopf algebras are symmetric:*

(a) *commutative algebras,*

(b) *group algebras,*

(c) *local algebras,*

(d) *semisimple algebras.*

Proof. Parts (a) and (b) are clear. Parts (c) and (d) hold because any local or semisimple Hopf algebra is unimodular. $\square$

The next result for restricted Lie algebras is from [30, 18].

Theorem 2.4. *Let $\mathfrak{g}$ be a finite dimensional restricted Lie algebra over k of characteristic $p > 0$. Then the restricted enveloping algebra $u(\mathfrak{g})$ is Frobenius with Nakayama automrophism σ given by $\sigma(x) = x + \mathrm{Tr}(\mathrm{ad}\, x)$ for all $x \in \mathfrak{g}$. As a consequence, $\sigma^p = 1$ and $u(\mathfrak{g})$ is symmetric if and only if $\mathrm{Tr}(\mathrm{ad}\, x) = 0$ for all $x \in \mathfrak{g}$.*

A direct calculation by using the antipode axiom

$$m(S \otimes 1)\Delta = m(1 \otimes S)\Delta = u\epsilon$$

yields that the antipode of Hopf algebras for all **A**, **B**, **C** types is given by

$$S(x) = -x, \qquad S(y) = -y, \qquad S(z) = -z$$

except for **B3** whose antipode is given by

$$S(x) = -x, \qquad S(y) = -y, \qquad S(z) = -z - 2xy.$$

Using this we have the following.

Proposition 2.5. *For the Hopf algebras listed in* Theorem 2.1, *the following are true.*

(a) *All Hopf algebras are involutory except for **B3** where $S^{2p} = 1$.*

(b) *The Hopf algebras of type* **A** *are unimodular and symmetric.*
(c) *The Hopf algebras of type* **B1** *and* **B3** *are not symmetric.*
(d) *The Hopf algebras* **C1**–**C10** *and* **C15** *are symmetric,* **C11**–**C14** *are not symmetric, and* **C16** *is symmetric if and only if* $\lambda^2 = -1$.

Proof. (a) is clear since for **B3**, we have $S^{2n}(z) = z + 2ny$ for all $n \geq 1$.

(b) The algebras **A1**–**A4** are commutative. One checks easily that **A5** is local. Then it follows from Corollary 2.3(c).

(c) **B1** and **C11** are isomorphic as augmented algebras so it follows from (d) later. For **B3**, one sees that $S^2(z) = z + xy$ is not inner.

(d) This follows from Theorem 2.4. □

Remark. From a Hopf algebra view point it is not known whether **B2** is symmetric or not because it is hard to compute its left or right integral and to check whether it is unimodular. However we shall show that through finding another presentation of algebras of type **B2** none of them are symmetric (See Proposition 5.1 (a)).

2.3. **Algebra structure.** The cohomology ring of a finite dimensional Hopf algebra only depends on the algebra structure, not the Hopf structure. So in order to determine the structure of the cohomology ring it is enough to study the algebra structure. We will do this in the following six sections, where we show that the algebras in this classification fall into 6 different classes, (0) semisimple algebras, (1) group algebras (tensored or direct sum with a semisimple algebra), (2) (direct sums of) selfinjective Nakayama algebras, (3) enveloping algebra of restricted Lie algebras, (4) coverings of local algebras, (5) other local algebras. We will show that

Semisimple algebras	**A1 = C1**
Group algebras	**A2 = C2**, **A3 = C4**, **A4 = C3**, **C7**, **C8**, **C9**, **C10**
Selfinjective Nakayama algebras	**B2**, **C12**, **C13**, **C14**
Enveloping algebra of restricted Lie algebras	**C5**, **C6**, **C15**
Coverings of local algebras	**B1 = C11**, **B3**, **C16**
Other local algebras	**A5**

Note that here we focus on the algebra structure (and we do not consider the comultiplication).

3. Semisimple algebras

In this section we classify the semisimple algebras occurring among the algebras in the classification of the finite dimensional connected Hopf algebras of dimension p^3.

We show the following.

Proposition 3.1. *Let k be a field of characteristic p. The algebras of type* **A1** *and* **C1** *are equal, and they are isomorphic to* k^{p^3}.

Proof. The algebra of type **A1** (or **C1**) are given as

$$\Lambda = k[x, y, z]/\langle x^p - x, y^p - y, z^p - z\rangle$$

for a field k of characteristic p. The polynomial $u^p - u$ in u has p different roots in $\mathbb{Z}_p$, so that $k[u]/\langle u^p - u\rangle \simeq k^p$. Furthermore we have that

$$\Lambda \simeq k[x]/(x^p - x) \otimes_k k[y]/(y^p - y) \otimes_k k[z]/(z^p - z) \simeq k^{p^3}.$$

$\square$

Remark 3.2. As we already said, the p-dimensional algebra $B = k[u]/\langle u^p - u \rangle$ is isomorphic to k^p for k of characteristic p. There is an elementary explicit formula for the orthogonal primitive idempotents of this algebra, and we will use this later. Namely for $r \in \mathbb{Z}_p$ let

$$e_r := \prod_{\substack{s=0 \\ s \neq r}}^{p-1} \frac{(u-s)}{(r-s)}.$$

Then $ue_r = re_r$, and the e_r for $r \in \mathbb{Z}_p$ are pairwise orthogonal idempotents and their sum is the identity of the algebra. We mention two consequences.

(1) Suppose $0 \neq \zeta \in B$ and $u\zeta = r\zeta$, then ζ is a scalar multiple of e_r. Namely, we have $\zeta = \sum_i \lambda_i e_i$ for $\lambda_i \in k$, and then $0 = (u-r)\zeta = \sum_i \lambda_i (u-r) e_i$. From the formula, $(u-r)e_i = 0$ for $i = r$ and otherwise it is a non-zero scalar multiple of e_i, and the claim follows.

(2) Suppose B is contained in some algebra Λ, and $y \in \Lambda$ satisfies $[u, y] = y$. Then $ye_i = e_{i+1}ye_i$. Namely we have $uy = y(u+1)$ and therefore $u(ye_i) = y(u+1)e_i = (i+1)ye_i$. Variations of this argument will be used later.

4. Group algebras

In this section we find group algebras occurring in the classification of the finite dimensional connected Hopf algebras of dimension p^3. Here we denote by C_n the cyclic group of order n, and for a finite group G we denote by kG the group algebra of G over a field k.

Proposition 4.1. *Let k be a field of characteristic p.*

(a) *The algebras $\mathbf{A2}$ and $\mathbf{C2}$ are equal, and they are isomorphic to kC_{p^3}.*

(b) *The algebras $\mathbf{A3}$ and $\mathbf{C4}$ are equal, and they are isomorphic to*
$$k(C_p \times C_p \times C_p).$$

(c) *The algebras $\mathbf{A4}$ and $\mathbf{C3}$ are isomorphic, and they are isomorphic to*
$$k(C_p \times C_{p^2}).$$

(d) *The algebra $\mathbf{C7}$ is isomorphic to the direct sum of p copies of $k(C_p \times C_p)$.*

(e) *The algebra $\mathbf{C8}$ is isomorphic to the direct sum of p copies of kC_{p^2}.*

(f) *The algebra $\mathbf{C9}$ is isomorphic to the direct sum of p^2 copies of kC_p.*

(g) *The algebra $\mathbf{C10}$ is isomorphic to the direct sum of $k(C_p \times C_p)$ and $p-1$ copies of $M_p(k)$.*

Proof. (a) The algebras of type $\mathbf{A2}$ and $\mathbf{C2}$ are given as (with a possible change of variables)
$$\Lambda = k[x, y, z]/\langle x^p - y, y^p - z, z^p \rangle.$$
The algebra homomorphism $k[u] \to \Lambda$ sending $u \mapsto x + \langle x^p - y, y^p - z, z^p \rangle$ is surjective with kernel $\langle u^{p^3} \rangle$. Hence $\Lambda \simeq k[u]/\langle u^{p^3} \rangle$, which is isomorphic to kC_{p^3}.

(b) The algebras of type $\mathbf{A3}$ and $\mathbf{C4}$ are given as $\Lambda = k[x, y, z]/\langle x^p, y^p, z^p \rangle$. Then
$$\Lambda \simeq k[x]/\langle x^p \rangle \otimes_k k[y]/\langle y^p \rangle \otimes_k k[z]/\langle z^p \rangle.$$
This is in turn isomorphic to $k(C_p \times C_p \times C_p)$.

(c) The algebra of type **A4** and **C3** are given as (with a possible change of variables) $\Lambda = k[x, y, z]/\langle x^p, y^p - z, z^p \rangle$. Then

$$\Lambda \simeq k[x]/\langle x^p \rangle \otimes_k k[y, z]/\langle y^p - z, z^p \rangle$$
$$\simeq k[x]/\langle x^p \rangle \otimes_k k[u]/\langle u^{p^2} \rangle,$$

since the algebra homomorphism $k[u] \to k[y, z]/\langle y^p - z, z^p \rangle$ sending $u \mapsto y + \langle y^p - z, z^p \rangle$ is surjective with kernel $\langle u^{p^2} \rangle$. This algebra is isomorphic to $k(C_p \times C_{p^2})$.

(d) The algebra of type **C7** is given as $\Lambda = k[x, y, z]/\langle x^p, y^p, z^p - z \rangle$. Then

$$\Lambda \simeq k[x]/\langle x^p \rangle \otimes_k k[y]/\langle y^p \rangle \otimes_k k[z]/\langle z^p - z \rangle$$
$$\simeq k[x]/\langle x^p \rangle \otimes_k k[y]/\langle y^p \rangle \otimes_k k^p$$

The last algebra is isomorphic to $k(C_p \times C_p) \otimes_k k^p$, which is isomorphic to p copies of $k(C_p \times C_p)$.

(e) The algebra of type **C8** is given as $\Lambda = k[x, y, z]/\langle x^p - y, y^p, z^p - z \rangle$. Then

$$\Lambda \simeq k[x, y]/\langle x^p - y, y^p \rangle \otimes_k k[z]/\langle z^p - z \rangle$$
$$\simeq k[x, y]/\langle x^p - y, y^p \rangle \otimes_k k^p$$
$$\simeq k[u]/\langle u^{p^2} \rangle \otimes_k k^p,$$

since the algebra homomorphism $k[u] \to k[x, y]/\langle x^p - y, y^p \rangle$ sending $u \mapsto x + \langle x^p - y, y^p \rangle$ is surjective with kernel $\langle u^{p^2} \rangle$. This last algebra occurring above is in turn isomorphic to $kC_{p^2} \otimes_k k^p$, which is isomorphic to p copies of kC_{p^2}.

(f) The algebra of type **C9** is given as $\Lambda = k[x, y, z]/\langle x^p, y^p - y, z^p - z \rangle$. Then

$$\Lambda \simeq k[x]/\langle x^p \rangle \otimes_k k[y]/\langle y^p - y \rangle \otimes_k k[z]/\langle z^p - z \rangle$$
$$\simeq k[x]/\langle x^p \rangle \otimes_k k^{p^2}$$

This last algebra is isomorphic to $kC_p \otimes_k k^{p^2}$, which in turn is isomorphic to p^2 copies of kC_p.

(g) The algebra of type **C10** is given as

$$\Lambda = k\langle x, y, z \rangle/\langle [x, y] - z, [x, z], [y, z], x^p, y^p, z^p - z \rangle.$$

It is clear from the relations that the subalgebra generated by z is in the center of Λ and is giving rise to p orthogonal central idempotents $\{e_i\}_{i=0}^{p-1}$ with $ze_i = ie_i$ as in Remark 3.2. Then $\Lambda = \oplus_{i=0}^{p-1} \Lambda e_i$, and we have that $ie_i = ze_i = [x, y]e_i$. Furthermore Λe_i is generated by $\{xe_i, ye_i\}$, hence we have a surjective homomorphism of rings $\varphi_i \colon k\langle x, y \rangle \to \Lambda e_i$ sending $x \mapsto xe_i$ and $y \mapsto ye_i$. The kernel of φ_i is generated by the elements $x^p, y^p, [x, y] - i$ giving that $\Lambda e_i \simeq k\langle x, y \rangle/\langle x^p, y^p, [x, y] - i \rangle$.

For $i = 0$ we get that $\Lambda e_0 \simeq k[x, y]/\langle x^p, y^p \rangle \simeq k[x]/\langle x^p \rangle \otimes_k k[y]/\langle y^p \rangle$, which is isomorphic to $k(C_p \times C_p)$.

Now consider the case when i is in $[1, \ldots, p-1]$. We have that

$$\Lambda \simeq k \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \langle x, y, z \rangle/\langle [x, y] - z, [x, z], [y, z], x^p, y^p, z^p - z \rangle,$$

and all the arguments we used above are valid over $\mathbb{Z}_p$. So we first analyze the algebra over $\mathbb{Z}_p$ and then induce up to k. So to reduce the amount of notation we also call the algebra Λ when considering it given over the field $\mathbb{Z}_p$.

Any element in Λe_i can be written as a linear combination of $\{x^r y^s e_i\}_{r,s=0}^{p-1,p-1}$. Given any non-zero ideal I in Λe_i, we can assume that $x^{p-1} y^{p-1} e_i$ is in I by multiplying with a power of x from the left and a power of y from the right.

Using the identity $xye_i - ie_i = yxe_i$, we obtain that $y^s e_i x = xy^s e_i - siy^{s-1}e_i$ for $s \geqslant 1$. Starting with multiplying the element $x^{p-1}y^{p-1}e_i$ with x from the right, we infer that $x^{p-1}y^{p-2}e_i$ is in I. Inductively it follows that $x^{p-1}e_i$ (and by similar arguments that $y^{p-1}e_i$) is in I. Consequently, $x^{p-1}y^s e_i$ is in I for all $1 \leqslant s \leqslant p-1$. By multiplying $x^{p-1}e_i$ from the left with y, we infer that $x^{p-2}e_i$ is in I. Continuing this way we obtain that e_i is in I and $I = \Lambda e_i$. It follows that Λe_i is a simple ring for i in $[1, \ldots, p-1]$. By Wedderburn-Artin Λe_i is isomorphic to $M_n(D)$ for $n \geq 1$ and a division ring D. Using that $\dim_{\mathbb{Z}_p} \Lambda e_i = p^2$, one can show that the only possibility for Λe_i is $M_p(\mathbb{Z}_p)$. Inducing up to k, we infer that the algebra of type **C10** is isomorphic to $k(C_p \times C_p) \oplus M_p(k)^{p-1}$. In fact, a simple Λe_i-module S with basis $\mathcal{B} = \{b_0, b_1, \ldots, b_{p-1}\}$ can be constructed in the following way. Let
$$xb_j = \begin{cases} b_{j-1}, & j \neq 0 \\ 0, & j = 0 \end{cases} \text{ and } yb_j = \begin{cases} \lambda_{j+1}b_{j+1}, & j \neq p-1 \\ 0, & j = p-1 \end{cases} \text{ for } \lambda_{j+1} = -(j+1)i. \text{ Then}$$
one can show that the vector space spanned by $\mathcal{B}$ is a simple Λe_i-module. $\square$

5. Selfinjective Nakayama algebras

In this section we find selfinjective Nakayama algebras occurring in the classification of the finite dimensional connected Hopf algebras of dimension p^3. Here we denote by $\widetilde{\mathbb{A}}_n$ the quiver consisting of $n+1$ vertices with one arrow starting in each vertex forming an oriented cycle. For a field k and an integer $n \geq 1$ we denote by J the ideal in $k\widetilde{\mathbb{A}}_n$ generated by the arrows.

Proposition 5.1. *Let k be a field of characteristic p.*
  (a) *The algebra of type* **B2** *is isomorphic to $k\widetilde{\mathbb{A}}_{p^2-1}/J^p$, which is not a symmetric algebra.*
  (b) *The algebra of type* **C12** *is isomorphic to $k\widetilde{\mathbb{A}}_{p-1}/J^{p^2}$.*
  (c) *The algebra of type* **C13** *is isomorphic to a direct sum of p copies of the selfinjective Nakayama algebra $k\widetilde{\mathbb{A}}_{p-1}/J^p$ of dimension p^2.*
  (d) *The algebra of type* **C14** *is isomorphic to the direct sum of $k\widetilde{\mathbb{A}}_{p-1}/J^p$ and $p-1$ copies of $M_p(k)$.*

Proof. (a) The algebra of type **B2** is given as
$$\Lambda = k\langle x, y, z\rangle / \langle [x,y] - y, [x,z], [y,z] - yf(x), x^p - x, y^p, z^p - z\rangle,$$
where $f(x) = \sum_{i=1}^{p-1}(-1)^{i-1}(p-i)^{-1}x^i$. We have that $yx = xy - y$, $zx = xz$ and $zy = yz - yf(x)$. This implies that any monomial in the variables $\{x, y, z\}$ can be written as a linear combination of monomials of the form $x^r y^s z^t$ for r, s and t in $\{0, 1, \ldots, p-1\}$. Hence the set $\mathcal{B} = \{x^r y^s z^t\}_{r,s,t=1}^{p-1}$ spans Λ, which consists of p^3 elements. Since the dimension of Λ is p^3, the set $\mathcal{B}$ is a basis for Λ.

All the relations for the algebra are homogeneous in y. It follows from this that $\langle y\rangle^i = \langle y^i\rangle$. Since $y^p = 0$, it follows that $\langle y\rangle^p = (0)$. Furthermore,
$$\Lambda/\langle y\rangle \simeq k[x]/\langle x^p - x\rangle \otimes_k k[z]/\langle z^p - z\rangle \simeq k^{p^2}.$$
It follows that $\mathrm{rad}\,\Lambda = \langle y\rangle$.

The subalgebra of Λ generated by x and z is
$$k\langle x, z\rangle/\langle x^p - x, [x,z], z^p - z\rangle \simeq k[x]/\langle x^p - x\rangle \otimes_k k[z]/\langle z^p - z\rangle,$$
which is commutative. Let $e_0, \ldots, e_{p-1}$ be the orthogonal primitive idempotents of the algebra generated by x so that $xe_i = ie_i$. Let also $g_0, \ldots, g_{p-1}$ be the orthogonal

primitive idempotents of the algebra generated by z so that $zg_j = jg_j$. Then these span a commutative semisimple subalgebra of dimension p^2 of Λ, with orthogonal primitive idempotents $e_i g_j$. These correspond to the vertices in the quiver we shall construct (as they are liftings of the primitive idempotents in $\Lambda / \operatorname{rad} \Lambda$).

These orthogonal idempotents decompose $\Lambda = \oplus_{i,j} \Lambda e_i g_j$ into a direct sum of p^2 left modules. The summand $\Lambda e_i g_j$ has basis $\{y^r e_i g_j\}_{r=0}^{p-1}$ and the radical is generated by $ye_i g_j$ (and $\Lambda e_i g_j$ is indecomposable as a left module). Hence $\operatorname{rad} \Lambda / \operatorname{rad}^2 \Lambda$ is generated by $\{ye_i g_j\}_{i,j=0}^{p-1,p-1}$, and they correspond to the arrows in the quiver we shall construct.

We claim that $ye_i g_j = (e_{i+1} g_{j-f(i)}) ye_i g_j (e_i g_j)$. Clearly $ye_i g_j = ye_i g_j (e_i g_j)$.

We have the relation $xy = y(x+1)$ which implies that $x \cdot ye_i = (i+1)ye_i$ and hence $x \cdot (ye_i g_j) = (i+1)ye_i g_j$. Using Remark 3.2 we obtain that

$$e_{i+1} ye_i g_j = ye_i g_j.$$

We have $zy = yz - y \cdot f(x)$ and z, x commute, so

$$z(yg_j) = yzg_j - yf(x)g_j = j \cdot yg_j - yg_j \cdot f(x)$$

and therefore

$$\begin{aligned} z(ye_i g_j) = z(yg_j e_i) &= (jyg_j)e_i - (yg_j f(x))e_i \\ &= jyg_j e_i - yg_j f(i)e_i \\ &= (j - f(i)) \cdot (ye_i g_j) \end{aligned}$$

and again by Remark 3.2, we have that $yg_j e_i = g_{j-f(i)} \cdot yg_j e_i$. It follows that $ye_i g_j = (e_{i+1} g_{j-f(i)}) ye_i g_j (e_i g_j)$.

It follows from the above that if Q is a quiver with p^2 vertices $v_{i,j} = ye_i g_j$ and p^2 arrows $a_{i,j} \colon v_{i,j} \to v_{i+1,j-f(i)}$, then Λ is a quotient of kQ. To show that Q is $\widetilde{\mathbb{A}}_{p^2-1}$, we need to show that the orbit of $(0,0)$ under the map $(i,j) \mapsto (i+1, j-f(i))$ is all of $\mathbb{Z}_p \times \mathbb{Z}_p$. After kp steps one gets

$$(kp, -f(1) - f(2) - \cdots - f(kp-1)) \tag{1}$$

To compute this, note that for $1 \leq m \leq p-1$ one has $\sum_{j=1}^{p-1} j^m \equiv -1 \bmod p$ if $m = p-1$ and is zero otherwise (To see this, one can for example use [12, Lemma 4.3]). Then one gets that (1) is equal to $(0, -k)$ and it follows that the orbit of $(0,0)$ must have size p^2 as required. Hence the quiver Q is $\widetilde{\mathbb{A}}_{p^2-1}$ with the orientation as an oriented cycle. Since the Loewy length of the indecomposable projectives $\Lambda e_i g_j$ are all equal, the relation ideal is J^p.

A Nakayama algebra $k\widetilde{\mathbb{A}}_n / J^t$ is symmetric if and only if $n+1$ divides $t-1$ by [31, Corollary IV.6.16]. Since $p^2 \nmid p-1$ for every prime p, all the algebras of type **B2** are not symmetric.

Remark. Note that the field k can be arbitrary of characteristic p.

(b) The algebra of type **C12** is given as

$$\Lambda = k\langle x, y, z \rangle / \langle [x,y] - y, [x,z], [y,z], x^p - x, y^p - z, z^p \rangle.$$

Any element in Λ can be written as a linear combination of elements in the set $\{x^r y^s\}_{r,s=0}^{p-1,p^2-1}$. Let $\mathfrak{a} = \langle y \rangle$ in Λ. Since the relations are homogeneous in y, it follows from the above that $\mathfrak{a}$ is a nilpotent ideal in Λ. Furthermore it is easy to see that $\Lambda / \mathfrak{a} \simeq k[x]/\langle x^p - x \rangle \simeq k^p$. We infer that $\operatorname{rad} \Lambda = \mathfrak{a}$. By the above comments

we have that $\operatorname{rad}\Lambda$ has basis $\{x^r y^s\}_{r=0,s\geqslant 1}^{p-1,p^2-1}$ (dimension $p^3 - p$) and $\operatorname{rad}^2 \Lambda$ has basis $\{x^r y^s\}_{r=0,s\geqslant 2}^{p-1,p^2-1}$ (dimension $p^3 - 2p$). The $\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda$ has a basis given by the residue classes of the elements $\{x^r y\}_{r=0}^{p-1}$. Hence Λ is isomorphic to a quotient of a path algebra kQ over k, where Q has p vertices

$$\left\{ v_\alpha = \frac{\prod_{\beta\in\mathbb{Z}_p\setminus\{\alpha\}}(x-\beta)}{\prod_{\beta\in\mathbb{Z}_p\setminus\{\alpha\}}(\alpha-\beta)} \right\}_{\alpha\in\mathbb{Z}_p}$$

and p arrows given by $\{v_\alpha y\}_{\alpha\in\mathbb{Z}_p}$. By part (2) of Remark 3.2 we have that

$$v_\alpha y = y v_{\alpha-1},$$

so that $v_{\alpha+1} y$ is an arrow from vertex v_α to vertex $v_{\alpha+1}$ for all α in $\mathbb{Z}_p$. It follows that $\Lambda \simeq k\widetilde{\mathbb{A}}_{p-1}/J^{p^2}$.

(c) The algebra of type **C13** is given as

$$\Lambda = k\langle x,y,z\rangle/\langle [x,y]-y, [x,z],[y,z], x^p-x, y^p, z^p-z\rangle.$$

This is isomorphic to the tensor product

$$\Lambda \cong k\langle x,y\rangle/\langle [x,y]-y, x^p-x, y^p\rangle \otimes_k k[z]/\langle z^p-z\rangle.$$

The second tensor factor is semisimple and isomorphic to k^p, hence Λ is the direct sum of p copies of $B = k\langle x,y\rangle/\langle [x,y]-y, x^p-x, y^p\rangle$. Let $\mathfrak{a} = \langle y\rangle$ in B. Since the relations are homogeneous in y, it follows from the above that $\mathfrak{a}$ is a nilpotent ideal in B. Furthermore it is easy to see that $B/\mathfrak{a} \simeq k[x]/\langle x^p-x\rangle \simeq k^p$. We infer that $\operatorname{rad}B = \mathfrak{a}$. The subalgebra generated by x gives rise to a complete set of orthogonal idempotents e_i with $xe_i = ie_i$ (see Remark 3.2), so that $B = \oplus_{i=0}^{p-1} Be_i$. Then Be_i has basis $\{y^s e_i\}_{s=0}^{p-1}$ and its radical is generated by ye_i. We have $x(ye_i) = y(x+1)e_i = (i+1)ye_i$, hence by Remark 3.2 $ye_i = e_{i+1}ye_i$ and ye_i gives an arrow $e_i \to e_{i+1}$. It follows from this that B is isomorphic to the algebra $k\widetilde{\mathbb{A}}_{p-1}/J^p$ of dimension p^2. By the above, this completes the proof of (c).

(d) The algebra of type **C14** is given as

$$\Lambda = k\langle x,y,z\rangle/\langle [x,y]-y, [x,z],[y,z], x^p-x, y^p-z, z^p-z\rangle.$$

It is clear from the relations that the subalgebra generated by z is in the center of Λ and is giving rise to p orthogonal central idempotents $\{e_i\}_{i=0}^{p-1}$ with $ze_i = ie_i$. Then $\Lambda = \oplus_{i=0}^{p-1}\Lambda e_i$. We claim that Λe_0 is isomorphic to $k\widetilde{\mathbb{A}}_{p-1}/J^p$, while Λe_i is isomorphic to $M_p(k)$ for $i \geq 1$. To do this, we first consider the algebra Λ as an algebra over the prime field $\mathbb{Z}_p$. The idempotents constructed above are given over this field. By abuse of notation we still denote the algebra by Λ when considered over the field $k' = \mathbb{Z}_p$.

The subalgebra generated by x gives rise to a complete set of orthogonal idempotents $\{f_i\}_{i=1}^{p-1}$ with $xf_i = if_i$. This implies that $\{y^r e_i f_s\}_{r,s=0}^{p-1,p-1}$ is basis for Λe_i. Let $i \geq 1$, then we want to show that Λe_i is a simple ring. Let I be a non-zero ideal in Λe_i with a non-zero element $m = \sum_{r,s=0}^{p-1,p-1} \alpha_{r,s} y^r e_i f_s$. Assume that $\alpha_{r_0,s_0} \neq 0$. Then $mf_{s_0} = \sum_{r=0}^{p-1} \alpha_{r,s_0} y^r e_i f_{s_0}$ is in I. Since $f_j y = y f_{j-1}$, it follows that $f_{r_0+s_0} m f_{s_0} = \alpha_{r_0,s_0} y^{r_0} e_i f_{s_0}$ is in I and consequently $y^{r_0} e_i f_{s_0}$ is in I. Multiplying this last element from the right with powers of y, we obtain that $y^{r_s} e_i f_s$ is in I for some r_s for $s = 0,1,\ldots,p-1$. Multiplying from the left with y^{p-r_s} we obtain that $e_i f_s$ is in I for $s = 0,1,\ldots,p-1$, hence e_i is in I and $I = \Lambda e_i$ and Λe_i is a simple ring. Furthermore, $\Lambda e_i \simeq M_n(D)$ for some $n \geq 1$ and a division

ring D. A simple module over Λe_i then has dimension $n \dim_{k'} D$ over k'. Since Λe_i is non-commutative and D is commutative, $n > 1$. Using similar arguments as above one can show that $S_t = k'\{y^r e_i f_t\}_{r=0}^{p-1}$ is a simple Λe_i-module. Hence $n \dim_{k'} D = p$. It follows that $n = p$ and $D = k'$, and therefore when inducing up to the field k we get that $\Lambda e_i \simeq M_p(k)$ for $i \geq 1$.

For Λe_0, let $\varphi \colon k\langle x, y \rangle \to \Lambda e_0$ be induced by the inclusion $k\langle x, y \rangle \hookrightarrow k\langle x, y, z \rangle$. Then $\{[x,y] - y, x^p - x, y^p\}$ are contained in $\mathrm{Ker}\, \varphi$. Since

$$\dim_k \Lambda e_0 = \dim_k k\langle x, y \rangle / \langle [x,y] - y, x^p - x, y^p \rangle = p^2,$$

it follows that $\Lambda e_0 \simeq k\langle x, y \rangle / \langle [x,y] - y, x^p - x, y^p \rangle$, which we consider as an identification. The ideal generated by y is the radical of Λe_0. View the primitive orthogonal idempotents $\{f_i\}_{i=0}^{p-1}$ as elements in Λe_0. Then $\{y^r f_s\}_{r,s=0}^{p-1,p-1}$ is a k-basis for Λe_0. Therefore $\{f_s\}_{s=0}^{p-1}$ is a k-basis for $\Lambda e_0 / \mathrm{rad}\, \Lambda e_0$ and $\{y f_s\}_{s=0}^{p-1}$ is a k-basis for $\mathrm{rad}\, \Lambda e_0 / \mathrm{rad}^2 \Lambda e_0$. We let these sets define the vertices and the arrows in a quiver Q, which is $\widetilde{\mathbb{A}}_{p-1}$. All the indecomposable projective modules have the same Loewy length and Λe_0 has dimension p^2, so that Λe_0 is isomorphic to $k\widetilde{\mathbb{A}}_{p-1}/J^p$.

It follows from the above that Λ is isomorphic to direct sum of $k\widetilde{\mathbb{A}}_{p-1}/J^p$ and $p - 1$ copies of $M_p(k)$. $\qquad\square$

6. Enveloping algebra of restricted Lie algebras

In this section we find enveloping algebras of restricted Lie algebras occurring in the classification of the finite dimensional connected Hopf algebras of dimension p^3.

First we give a quick review of the definition of the enveloping algebra of a restricted Lie algebra L. A restricted Lie algebra L is a (finite dimensional) Lie algebra over a field k of characteristic p with Lie bracket $[-,-] \colon L \times L \to L$ and a p-operation $(-)^{[p]} \colon L \to L$ (for details see [19]). Suppose L has a k-basis $\{x_1, x_2, \ldots, x_n\}$. By abuse of notation we let $k\langle x_1, x_2, \ldots, x_n \rangle$ be the free algebra of the indeterminants $\{x_1, x_2, \ldots, x_n\}$. Then the universal enveloping algebra $U^{[p]}(L)$ of L is given by

$$U^{[p]}(L) = k\langle x_1, x_2, \ldots, x_n \rangle / \langle \{x_i x_j - x_j x_i - [x_i, x_j]\}_{i,j=1}^{n,n}, \{x_i^p - x_i^{[p]}\}_{i=1}^{n} \rangle.$$

Next we give the description of the algebras **C5**, **C6** and **C15** as enveloping algebras of restricted Lie algebras all of which are 3-dimensional.

Proposition 6.1. (a) *Let L be the 3-dimensional restricted p-nilpotent Lie algebra with basis $\{x_1, x_2, x_3\}$, the only non-zero bracket being $[x_1, x_2] = x_3$ and the p-operation given by $x_i^{[p]} = 0$ for $i = 1, 2, 3$ for $p > 2$. Then the algebra of type **C5** is isomorphic to the enveloping algebra of the restricted Lie algebra of L.*

(b) *Let L be the 3-dimensional restricted Lie algebra with basis $\{x_1, x_2, x_3\}$ with the only non-zero bracket being $[x_1, x_2] = x_3$ and the p-operation given by $x_1^{[p]} = x_3$ and $x_i^{[p]} = 0$ for $i = 2, 3$, given in [29, Theorem 2.1 (3/2) (b)]. Then the algebra of type **C6** is isomorphic to the enveloping algebra of the restricted Lie algebra of L.*

(c) *Let L be the restricted Lie algebra of type $\mathfrak{sl}_2(k)$. The algebra of type **C15** is isomorphic to the enveloping algebra of the restricted Lie algebra of L.*

Proof. (a) The algebra of type **C5** is given by

$$\Lambda = k\langle x, y, z\rangle / \langle [x,y] - z, [x,z], [y,z], x^p, y^p, z^p\rangle.$$

Let L be the 3-dimensional Lie algebra with basis $\{x_1, x_2, x_3\}$ with the only non-zero bracket being $[x_1, x_2] = x_3$. We want to show that we can choose a zero p-operation to obtain a restricted Lie algebra.

We have that

$$\operatorname{ad} x_1 = [-, x_1] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad (\operatorname{ad} x_1)^2 = 0,$$

$$\operatorname{ad} x_2 = [-, x_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (\operatorname{ad} x_2)^2 = 0,$$

$$\operatorname{ad} x_3 = [-, x_3] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \operatorname{ad} x_3 = 0.$$

By [19, Chapter V, Theorem 11] there is a p-operation $(-)^{[p]} \colon L \to L$ such that $(x_i)^{[p]} = 0$ for $i = 1, 2, 3$. This shows that L is the 3-dimensional restricted Lie algebra, and therefore Λ is the enveloping algebra of the 3-dimensional restricted Lie algebra L.

(b) The algebra of type **C6** is given by

$$\Lambda = k\langle x, y, z\rangle / \langle [x,y] - z, [x,z], [y,z], x^p - z, y^p, z^p\rangle$$

for $p > 2$. Recall the 3-dimensional restricted Lie algebra L with basis $\{x_1, x_2, x_3\}$ with the only non-zero bracket being $[x_1, x_2] = x_3$ and the p-operation given by $x_1^{[p]} = x_3$ and $x_i^{[p]} = 0$ for $i = 2, 3$, given in [29, Theorem 2.1 (3/2) (b)]. Then the algebra Λ is isomorphic to the restricted enveloping algebra of the 3-dimensional restricted Lie algebra L.

(c) The algebra of type **C15** is given by

$$\Lambda = k\langle x, y, z\rangle / \langle [x,y] - z, [x,z] - x, [y,z] + y, x^p, y^p, z^p - z\rangle$$

for $p > 2$. We shall prove that this is the enveloping algebra of the 3-dimensional restricted Lie of type $\mathfrak{sl}_2(k)$.

Let $L = \mathfrak{sl}_2(k) = ke_+ \oplus kh \oplus ke_-$, where the standard presentation usually is give as

$$[h, e_\pm] = \pm 2e_\pm, \quad [e_+, e_-] = h.$$

By letting

$$h' = -\frac{1}{2}h, \quad e'_+ = \alpha e_+, \quad e'_- = \beta e_-,$$

we obtain the equations

$$[h', e'_+] = -e_+, \quad [h', e'_-] = e'_-, \quad [e'_+, e'_-] = h',$$

whenever $\alpha\beta = -\frac{1}{2}$ in k.

Now we investigate if there exists a p-operation on L. We have that

$$\operatorname{ad} e'_+ = [-, e'_+] = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad (\operatorname{ad} e'_+)^3 = 0,$$

$$\operatorname{ad} e'_- = [-, e'_-] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (\operatorname{ad} e'_-)^3 = 0,$$

$$\operatorname{ad} h' = [-, h'] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (\operatorname{ad} h')^p = \begin{pmatrix} 1^p & 0 & 0 \\ 0 & (-1)^p & 0 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{ad} h',$$

since p is odd. By [19, Chapter V, Theorem 11] there is a p-operation $(-)^{[p]} \colon L \to L$ such that

$$(e'_\pm)^{[p]} = 0 \quad \text{and} \quad (h')^{[p]} = h'.$$

This shows that L is the 3-dimensional restricted Lie algebra of type $\mathfrak{sl}_2(k)$, and therefore Λ is the enveloping algebra of the 3-dimensional restricted Lie algebra of type $\mathfrak{sl}_2(k)$. $\square$

7. Coverings of local algebras

In this section we find algebras in the classification of the finite dimensional connected Hopf algebras of dimension p^3 which are coverings of local algebras by a cyclic group. We start off by recalling the notion of a covering of a path algebra (for details see [13]).

Let $\Lambda = kQ/I$ be an admissible quotient of the path algebra kQ over a field k. Let G be a (finite) group, and let $w\colon Q_1 \to G$ be a *weight function*, that is, just a function $w\colon Q_1 \to G$. This is extended to a weight function on all non-trivial paths by defining the *weight of a path* $p = a_n a_{n-1} \cdots a_2 a_1$ in Q to be

$$\prod_{i=1}^{n} w(a_i) = w(a_1)w(a_2)\cdots w(a_{n-1})w(a_n),$$

and for a trivial path p the weight is $w(p) = e$, the identity in G. Assume that the relations in I are homogeneous with respect to the weight. With these data we define a covering $\widetilde{Q}(w)$ of Q with the vertex set

$$\widetilde{Q}(w)_0 = \{(v, g) \mid v \in Q_0, g \in G\}$$

and the arrow set

$$\widetilde{Q}(w)_1 = \{(a, g)\colon (\mathfrak{o}(a), g) \to (\mathfrak{t}(a), gw(a)) \mid a \in Q_1, g \in G\},$$

where $\mathfrak{o}(a)$ and $\mathfrak{t}(a)$ denote the origin and the terminus of the arrow a, respectively. Define a map $\pi\colon \widetilde{Q}(w) \to Q$ induced from letting

$$\pi(v, g) = v \quad \text{and} \quad \pi(a, g) = a$$

for all vertices v in Q_0, all arrows a in Q_1 and all g in G. It is straightforward to see that given an element g in G and a path p in Q, there is a unique path $\widetilde{p}_g = (p, g)\colon (\mathfrak{o}(p), g) \to (\mathfrak{t}(p), gw(p))$ in $\widetilde{Q}(w)$ such that $\pi(\widetilde{p}_g) = p$. This we extend to a lifting from kQ to $k\widetilde{Q}(w)$. Hence, given g in G and a weight homogeneous relation σ in I, there is a unique lifting $\widetilde{\sigma}_g$ of σ to $k\widetilde{Q}(w)$. This is a uniform element, that is, all paths occuring in $\widetilde{\sigma}_g$ start in the same vertex and end in the same vertex. Furthermore, if I' is the ideal generated by the liftings of a minimal generating set of I and $\widetilde{I}(w)$ is the set of liftings of elements in I, then $I' = \widetilde{I}(w)$ and $\widetilde{I}(w)$ is an ideal in $k\widetilde{Q}(w)$. In particular, there is a well-defined algebra map $\pi\colon k\widetilde{Q}(w)/\widetilde{I}(w) \to kQ/I$.
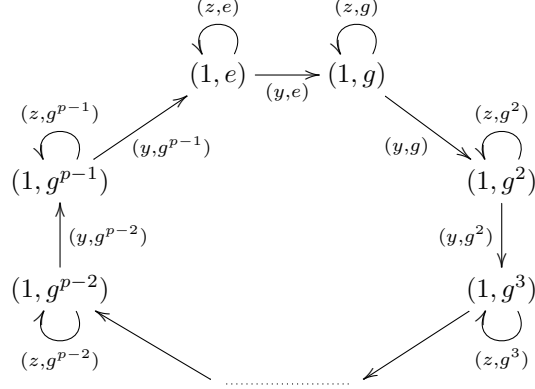
We give the following example which will occur in the main result of this section.

Example 7.1. Let

$$\Lambda = k(\ y \overset{\frown}{\underset{\smile}{\ }} 1 \overset{\frown}{\underset{\smile}{\ }} z \ )/\langle y^p, yz - zy, z^p \rangle,$$

and define $w\colon \{y, z\} \to C_p = \langle g \rangle$ by letting $w(y) = g$ and $w(z) = e$. Let $Q\colon y \overset{\frown}{\underset{\smile}{\ }} 1 \overset{\frown}{\underset{\smile}{\ }} z$ and $I = \langle y^p, yz - zy, z^p \rangle$. Then the covering $k\widetilde{Q}(w)/\widetilde{I}(w)$ is

given by the quiver

and relations

$$(y, g^{i+p-1}) \cdots (y, g^{i+2})(y, g^{i+1})(y, g^i),$$
$$(z, g^i)^p,$$
$$(y, g^i)(z, g^i) - (z, g^{i+1})(y, g^i),$$

for all i in $\{0, 1, \ldots, p-1\}$.

Now we are ready to give the algebras in the classification of finite dimensional connected Hopf algebras of dimension p^3 which are coverings of a local algebra by a cyclic group.

Proposition 7.2. *Let k be a field of characteristic p. Let*

$$\Lambda_1 = k\langle y, z \rangle / \langle y^p, yz - zy, z^p \rangle \quad and \quad \Lambda_2 = k\langle y, z \rangle / \langle y^p, yz - zy - y^2, z^p \rangle.$$

Consider the group $G = \langle \sigma \rangle = C_p$.

    (a) *The algebras of type **B1** and **C11** are isomorphic, and they are isomorphic to a covering of the local algebra Λ_1 with the weight function $w \colon \{y, z\} \to G$ given by $w(y) = \sigma$ and $w(z) = e$.*

    (b) *The algebra of type **B3** is a covering of the local algebra Λ_2 with the weight function $w(y) = w(z) = \sigma$.*

    (c) *Assume that k contains the field of order p^2 if $p > 2$. The algebra of type **C16** is a covering of the local algebra Λ_1 with a weight function w of the form $w(y) = \sigma^{-1}$ and $w(z) = \sigma^{-a}$ for some a (precisely, a such that $\lambda^{-1} = a\lambda$ when λ is the defining parameter occuring in an algebra of type **C16**).*

Proof. (a) The algebra of type **B1** and **C11** are given as

$$\Lambda = k\langle x, y, z \rangle / \langle [x, y] - y, [x, z], [y, z], x^p - x, y^p, z^p \rangle.$$

Any element in Λ can be written as a linear combination of elements in the set $\{x^r y^s z^t\}_{r,s,t=0}^{p-1,p-1,p-1}$. Let $\mathfrak{a} = \langle y, z \rangle$ in Λ. Since the relations are homogeneous in y and z, it follows from the above that $\mathfrak{a}$ is a nilpotent ideal in Λ. Furthermore it is easy to see that $\Lambda/\mathfrak{a} \simeq k[x]/\langle x^p - x \rangle \simeq k^p$. We infer that $\operatorname{rad}\Lambda = \mathfrak{a}$. By the above comments we have that $\operatorname{rad}^2\Lambda$ has basis $\{x^r y^s z^t\}_{r=0, s+t \geq 2}^{p-1, p-1, p-1}$. The factor $\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda$ has a basis given by the residue classes of the elements $\{x^r y, x^r z\}_{r=0}^{p-1}$.

Hence Λ is isomorphic to a quotient of a path algebra kQ over k, where Q has p vertices

$$\left\{ v_\alpha = \frac{\prod_{\beta\in\mathbb{Z}_p\setminus\{\alpha\}}(x-\beta)}{\prod_{\beta\in\mathbb{Z}_p\setminus\{\alpha\}}(\alpha-\beta)} \right\}_{\alpha\in\mathbb{Z}_p}$$

and $2p$ arrows spanned by $\{x^r y, x^r z\}_{r=0}^{p-1}$. The linear span of $\{x^i\}_{i\in\mathbb{Z}_p}$ is equal to the linear span of $\{v_\alpha\}_{\alpha\in\mathbb{Z}_p}$. Hence, a basis for $\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda$ is $\{v_\alpha y, v_\alpha z\}_{\alpha\in\mathbb{Z}_p}$. Since z is in $Z(\Lambda)$, we have that $v_\alpha z = z v_\alpha$. By Remark 3.2 (2) the element $v_\alpha y$ gives rise to an arrow from vertex v_α to vertex $v_{\alpha+1}$, and the element $v_\alpha z$ to a loop at vertex v_α. Hence the quiver Q of Λ is
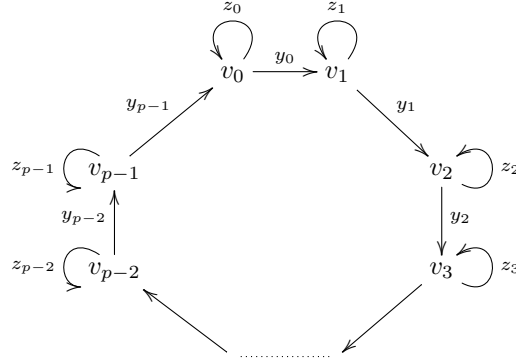
The relations are $\{y_{i+p-1}\cdots y_{i+2}y_{i+1}y_i, z_i^p, y_i z_i - z_{i+1}y_i\}_{i=0}^{p-1}$. Hence

$$\Lambda \simeq kQ/\langle\{y_{i+p-1}\cdots y_{i+2}y_{i+1}y_i, z_i^p, y_i z_i - z_{i+1}y_i\}_{i=0}^{p-1}\rangle.$$

Therefore Λ is isomorphic to the covering of the local algebra Λ_1 with the weight function $w\colon \{y,z\} \to C_p = \langle\sigma\rangle$ given by $w(y) = \sigma$ and $w(z) = e$ (see Example 7.1 taking $\sigma = g$).
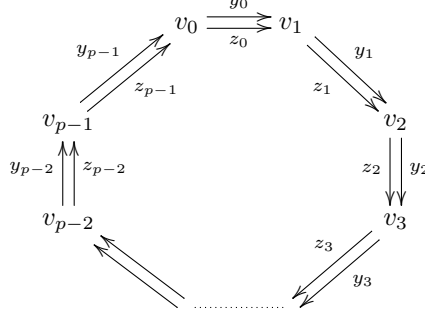
(b) The algebra of type **B3** is given by

$$\Lambda = k\langle x,y,z\rangle/\langle[x,y]-y, [x,z]-z, [y,z]-y^2, x^p-x, y^p, z^p\rangle.$$

The relations $[x,y]-y$, $[x,z]-z$ and $[y,z]-y^2$ imply that x's can be moved across y's, x's can be moved across z's and that z's can be moved across y's. From this we obtain that $\langle y,z\rangle$ is a nilpotent ideal in Λ. It is easy to see that $\Lambda/\langle y,z\rangle \simeq k[x]/\langle x^p-x\rangle \simeq k^p$. Hence $\langle y,z\rangle$ is the radical in Λ, and $\Lambda \simeq kQ/I$, where Q is a quiver with p vertices given by

$$\left\{ v_\alpha = \frac{\prod_{\beta\in\mathbb{Z}_p\setminus\{\alpha\}}(x-\beta)}{\prod_{\beta\in\mathbb{Z}_p\setminus\{\alpha\}}(\alpha-\beta)} \right\}_{\alpha\in\mathbb{Z}_p}.$$

The arrows are given by $v_\alpha y v_\beta$ and $v_\alpha z v_\beta$ which are non-zero when α and β runs through $\mathbb{Z}_p$. Using Remark 3.2 we get that $v_\alpha y = y v_{\alpha-1}$ and $v_\alpha z = z v_{\alpha-1}$ for all α

in $\mathbb{Z}_p$. Hence the quiver is

The relations are

$$\{y_{i+1}z_i - z_{i+1}y_i - y_{i+1}y_i, y_{i+p-1}y_{i+p-2}\cdots y_{i+1}y_i, z_{i+p-1}z_{i+p-2}\cdots z_{i+1}z_i\}_{i\in\mathbb{Z}_p}.$$

Therefore Λ is a covering of the local algebra Λ_2 with the weight function $w(y) = w(z) = \sigma$.

(c) The algebra of type **C16** is given as

$$\Lambda = C(\lambda, \delta) = k\langle x, y, z\rangle / \langle [x, y], [x, z] - \lambda x, [y, z] - \lambda^{-1}y, x^p, y^p, z^p - \delta z\rangle$$

for some $\lambda \in k^\times$ such that $\delta = \lambda^{p-1} = \pm 1$. Any element in Λ can be written as a linear combination of elements in the set $\{x^r y^s z^t\}_{r,s,t=0}^{p-1,p-1,p-1}$ of p^3 elements, hence it is a k-basis for Λ. Let $\mathfrak{a} = \langle x, y\rangle$ in Λ. Since the relations are homogeneous in x and y, it follows from the above that $\mathfrak{a}$ is a nilpotent ideal in Λ. Furthermore it is easy to see that $\Lambda/\mathfrak{a} \simeq k[z]/\langle z^p - \delta z\rangle \simeq k^p$ (note that λ lies in the field of order p^2 if $\delta = -1 \neq 1$). We infer that $\operatorname{rad}\Lambda = \mathfrak{a}$. By the above comments we have that $\operatorname{rad}\Lambda$ has basis $\{x^r y^s z^t\}_{r+s\geqslant 1,t=0}^{p-1,p-1,p-1}$ (dimension $p^3 - p$) and $\operatorname{rad}^2\Lambda$ has basis $\{x^r y^s z^t\}_{r+s\geqslant 2,t=0}^{p-1,p-1,p-1}$ (dimension $p^3 - 3p$). The vector space $\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda$ has a basis given by the residue classes of the elements $\{xz^t, yz^t\}_{t=0}^{p-1}$.

Consider the map $\varphi\colon k[u] \to \Lambda$ given by $\varphi(u) = z$. Then $\operatorname{Ker}\varphi = \langle u^p - \delta u\rangle$, so that $k[z]/\langle z^p - \delta z\rangle$ is a subalgebra of Λ. This is a commutative semisimple algebra, since the polynomial $z^p - \delta z$ is separable.

Note that the set of roots of $z^p - \delta z$ is closed under addition and that 0 is a root, so it is an additive subgroup of order p of the field. We take λ as in the definition of the algebra. This is a non-zero root of the polynomial $z^p - \delta z$, and with this, the set of roots is precisely

$$\{0, \lambda, 2\lambda, \ldots, (p-1)\lambda\}.$$

Since λ^{-1} also is a root, there is a unique integer a with $1 \leq a \leq p - 1$ such that $\lambda^{-1} = a\lambda$.

Next we find a set of complete orthogonal idempotents in Λ. We have fixed the roots of $z^p - \delta z$ above. With this, we have a set of complete orthogonal idempotents $\{e_0, e_1, \ldots, e_{p-1}\}$ of $k[z]/\langle z^p - \delta z\rangle$ such that

$$ze_i = i\lambda \cdot e_i.$$

(for a formula take a variation of Remark 3.2, but we don't need it). So $\Lambda = \oplus_{i=0}^{p-1}\Lambda e_i$ as a left module.

The set $\{x^u y^v e_i\}_{u,v=0}^{p-1,p-1}$ a k-basis for Λe_i (recall that x, y commute). In particular xe_i and ye_i are both non-zero and they generate the radical of Λe_i.

We claim that xe_i and ye_i are eigenvectors for left multiplication with z: To see this, we use the relations

$$zx = xz - \lambda x \quad \text{and} \quad zy = yz - \lambda^{-1}y.$$

Therefore

$$zxe_i = (xz - \lambda x)e_i = x(i\lambda e_i) - \lambda(xe_i) = (i-1)\lambda(xe_i)$$

and the eigenvalue is $(i-1)\lambda$. This implies that $xe_i = e_{i-1}xe_i$, by the Remark 3.2. Similarly

$$zye_i = (yz - \lambda^{-1}y)e_i = (i\lambda - a\lambda)ye_i$$

and the eigenvalue is $(i-a)\lambda$. So $ye_i = e_{i-a}ye_i$.

As we have found a basis for the algebra modulo its radical, and the radical modulo the radical square, we can find the quiver Q of Λ. The vertices of Q correspond to $\{e_i\}_{i=1}^{p-1}$, and we take as arrows $\alpha_i = xe_i$ and $\beta_i = ye_i$ starting at i. By the above, α_i ends at vertex $i-1$ and β_i ends at vertex $i-a$, where $\lambda^{-1} = a\lambda$. It follows from the above that $\alpha_{i-(p-1)} \cdots \alpha_{i-1}\alpha_i$, $\beta_{i-(p-1)a} \cdots \beta_{i-a}\beta_i$ and $\beta_{i-1}\alpha_i - \alpha_{i-a}\beta_i$ are relations. It is easy to see that $kQ/\langle\{\alpha_{i-(p-1)} \cdots \alpha_{i-1}\alpha_i, \beta_{i-(p-1)a} \cdots \beta_{i-a}\beta_i, \beta_{i-1}\alpha_i - \alpha_{i-a}\beta_i\}_{i=0}^{p-1}\rangle$ has dimension p^3, hence

$$\Lambda \simeq kQ/\langle\{\alpha_{i-(p-1)} \cdots \alpha_{i-1}\alpha_i, \beta_{i-(p-1)a} \cdots \beta_{i-a}\beta_i, \beta_{i-1}\alpha_i - \alpha_{i-a}\beta_i\}_{i=0}^{p-1}\rangle.$$

Therefore Λ is isomorphic to the covering of the local algebra Λ_1 with weight function w of the form $w(y) = \sigma^{-1}$ and $w(z) = \sigma^{-a}$ for $C_p = \langle\sigma\rangle$. $\qquad\square$

8. OTHER LOCAL ALGEBRAS

In this section we discuss the algebras which we cannot classify as we have done with all the other algebras. The only class of algebras left are the algebras of type **A5**. Here it is only in characteristic 2 that we can identify this algebra.

Proposition 8.1. *For $p = 2$ the algebra of type **A5** is isomorphic to the semidihedral algebra of dimension 8 labelled as* Alg III.1 (d) *in* [9, page 298].

Proof. For $p = 2$ the algebra of type **A5** is given as

$$\Lambda = k\langle x, y, z\rangle/\langle x^2, y^2, [x, y], [x, z], [y, z] - x, z^2 + xy\rangle.$$

We observe that x is central and not a generator. We substitute $x = [y, z]$, this is central, and the other relations translate to

$$[y, z]^2, y^2, z^2 + [y, z]y.$$

Noting that $\text{char}(k) = 2$, the last two relations mean that in the algebra $y^2 = 0$ and $z^2 = yzy$. With these, the first relation is equivalent to $(yz)^2 = (zy)^2$ in the algebra. Moreover if $y^2 = 0$ and $z^2 = yzy$, then it follows that $[y, z]$ is central. We get that

$$\Lambda \simeq k\langle y, z\rangle/\langle(yz)^2 - (zy)^2, y^2, z^2 - yzy\rangle,$$

and these relations imply that $(zy)^2z = 0$.

The radical layers of this algebra have basis $\{1\}$, $\{y, z\}$, $\{yz, zy\}$, $\{yzy, zyz\}$, $\{yzyz\}$, hence of dimensions $\{1, 2, 2, 2, 1\}$ and structure of the projective is
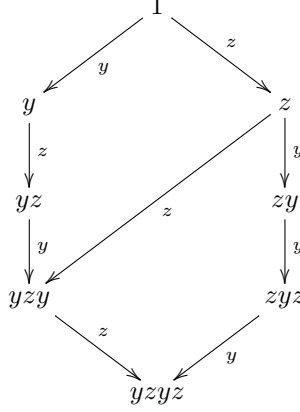
This algebra is isomorphic to a semidihedral algebra of dimension 8 labelled as Alg III.1 (d) in [9, page 298]. $\qquad\square$

Remark. The algebra Λ of type **A5** for $p = 2$ is a local algebra and not semisimple. It is not a selfinjective Nakayama algebra, since the radical of the indecomposable projective is generated by two elements. We don't know if it is an enveloping algebra of a restricted Lie algebra. It is not a non-trivial covering of any other algebra, since it is local.

We also claim that it is not a group algebra as we argue as follows. If Λ is isomorphic to a group algebra kG, then $|G| = 8$. The algebra is not commutative, and the only non-abelian groups of order 8 are the dihedral group D_8 and the quaternion group Q_8. If $G = D_8$, then kG is isomorphic to the algebra labelled III.1 (c) in [9, page 294], which is not isomorhpic to Λ. If $G = Q_8$, then kG is isomorphic to the algebra labelled III.1 (e) or (e'), which again is not isomorphic to Λ.

Remark. For $p > 2$ the algebra of type **A5** is given as

$$\Lambda(\beta) = k\langle x, y, z\rangle / \langle x^p, y^p, [x, y], [x, z], [y, z] - x, z^p + x^{p-1}y - \beta x\rangle.$$

We can see that the ideal $\langle x, y, z\rangle$ in Λ is the radical of Λ. Furthermore, Λ is generated as an algebra by $\{y, z\}$, and direct computations show that

$$\Lambda \simeq k\langle y, z\rangle / \langle [y, z]^p, y^p, 2yzy - zy^2 - y^2z, yz^2 + z^2y - 2zyz, z^p + (yz - zy)^{p-1}y - \beta(yz - zy)\rangle.$$

Let $\beta \neq 0$. Then one can show directly that the dimension of the radical quotients are

$$1, 2, 3, \ldots, p-1, \underbrace{p, p, \ldots, p}_{p^2 - p + 1}, p - 1, \ldots, 3, 2, 1.$$

If $\Lambda(\beta) \simeq kG$, then G is a non-abelian group of order p^3. It is well-known that up to isomorphism there are two non-abelian groups of order p^3, namely $C_{p^2} \rtimes C_p$ and $(C_p \times C_p) \rtimes C_p$. The theorem of Jennings determines explicitly in terms of the group data a basis of kG for G a group, compatible with the radical series.

The group algebra of $G = C_{p^2} \rtimes C_p$ has the same dimensions of the radical quotients as $\Lambda(\beta)$. If there exists an isomorphism $\psi \colon \Lambda(\beta) \to kG$, the element

$\psi([y, z])$ must be in the center. One can show using a presentation of kG that this is not possible.

Let $\beta = 0$. Then the algebra $\Lambda(0)$ has the same radical quotients as the group algebra $k((C_p \times C_p) \rtimes C_p)$ and similarly the algebras are not isomorphic.

9. Eckmann-Shapiro Lemma and Ext-algebras

This section is devoted to studying a situation for an adjoint pair of functors where Noetherianity of Ext-algebras can be transferred from one category to the other via the first or the second functor in the adjunction. This is the Eckmann-Shapiro Lemma in the setting of abelian categories.

Let $\mathcal{C}$ and $\mathcal{D}$ be two abelian categories with enough projective and enough injective objects. Let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$ be a pair (F, G) of adjoint functors, and denote the adjunction by

$$\psi = \psi_{C,D} \colon \operatorname{Hom}_{\mathcal{D}}(F(C), D) \to \operatorname{Hom}_{\mathcal{C}}(C, G(D))$$

for all C in $\mathcal{C}$ and D in $\mathcal{D}$. With this setup we are ready to formulate the transfer of Noetherianity of Ext-algebras across the adjunction using the functor F.

Proposition 9.1. *Let (F, G) be an adjoint pair of functors as above, and assume that F is exact and preserves projective objects.*

(a) *The adjunction induces a functorial isomorphism*

$$\operatorname{Ext}_{\mathcal{D}}^i(F(C), D) \simeq \operatorname{Ext}_{\mathcal{C}}^i(C, G(D))$$

for all $i \geq 1$ and for all C in $\mathcal{C}$ and D in $\mathcal{D}$. We also denote this isomorphism by ψ.

(b) *For every element η in $\operatorname{Ext}_{\mathcal{C}}^m(C', C)$ and θ in $\operatorname{Ext}_{\mathcal{C}}^n(C, G(D))$ the following equality holds in $\operatorname{Ext}_{\mathcal{D}}^{m+n}(F(C'), D)$:*

$$\psi^{-1}(\theta) \cdot F(\eta) = \psi^{-1}(\theta \cdot \eta).$$

(c) *Let n be a non-negative integer. If $\operatorname{Ext}_{\mathcal{C}}^{\geq n}(C, G(D))$ is a finitely generated right $\operatorname{Ext}_{\mathcal{C}}^*(C, C)$-module, then $\operatorname{Ext}_{\mathcal{D}}^{\geq n}(F(C), D)$ is a finitely generated right $\operatorname{Ext}_{\mathcal{D}}^*(F(C), F(C))$-module.*

(d) *If $\operatorname{Ext}_{\mathcal{C}}^*(C, C)$ is a right Noetherian ring and $\operatorname{Ext}_{\mathcal{C}}^*(C, G(F(C)))$ is a finitely generated right $\operatorname{Ext}_{\mathcal{C}}^*(C, C)$-module, then $\operatorname{Ext}_{\mathcal{D}}^*(F(C), F(C))$ is a right Noetherian ring.*

Proof. (a) Let $\cdots \to P_1 \to P_0 \to C \to 0$ be a projective resolution of C in $\mathcal{C}$. Then $\cdots \to F(P_1) \to F(P_0) \to F(C) \to 0$ is a projective resolution of $F(C)$ in $\mathcal{D}$. Applying the functor $\operatorname{Hom}_{\mathcal{C}}(-, G(D))$ to the first projective resolution and $\operatorname{Hom}_{\mathcal{C}}(-, D)$ to the second projective resolution, we obtain the following commutative diagram where all vertical maps are isomorphisms:

$$
\begin{array}{ccccccc}
0 \longrightarrow \operatorname{Hom}_{\mathcal{C}}(C, G(D)) & \longrightarrow & \operatorname{Hom}_{\mathcal{C}}(P_0, G(D)) & \longrightarrow & \operatorname{Hom}_{\mathcal{C}}(P_1, G(D)) & \longrightarrow & \cdots \\
\Big\downarrow{\psi^{-1}} & & \Big\downarrow{\psi^{-1}} & & \Big\downarrow{\psi^{-1}} & & \\
0 \longrightarrow \operatorname{Hom}_{\mathcal{D}}(F(C), D) & \longrightarrow & \operatorname{Hom}_{\mathcal{D}}(F(P_0), D) & \longrightarrow & \operatorname{Hom}_{\mathcal{D}}(F(P_1), D) & \longrightarrow & \cdots
\end{array}
$$

The claim follows immediately from this commutative diagram.

(b) Let η be an element in $\operatorname{Ext}_{\mathcal{C}}^m(C', C)$ and θ an element in $\operatorname{Ext}_{\mathcal{C}}^n(C, G(D))$. Let

$$\mathbb{P}' \colon \cdots \to P'_m \to P'_{m-1} \to \cdots \to P'_0 \to C' \to 0$$

and

$$\mathbb{P}\colon \cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to C \to 0$$

be projective resolutions of C' and C, respectively. Here P_0' and P_0 are in degree 0. Then η and θ can be represented by maps $P_m' \to C$ and $P_n \to G(D)$, which we also denote by η and θ, respectively. These give rise to chain maps $\widetilde{\eta}\colon \mathbb{P}_{\geq 0}' \to \mathbb{P}_{\geq 0}[-m]$ and $\theta\colon \mathbb{P}_{\geq 0} \to G(D)[-n]$, viewing $G(D)$ as a stalk complex concentrated in degree 0. Then the Yoneda products $\theta\eta$ and $\psi^{-1}(\theta)F(\eta)$ can be represented by the morphisms $\theta[-m]\widetilde{\eta} = \theta\eta_n$ and $\psi^{-1}(\theta)F(\eta_n)$, where η_n is a n-th lifting of η. We have the following commutative diagram of complexes

$$
\begin{array}{ccc}
\operatorname{Hom}_{\mathcal{D}}(F(\mathbb{P}_{\geq 0}), D[-n]) & \xrightarrow{\ \psi\ } & \operatorname{Hom}_{\mathcal{C}}(\mathbb{P}_{\geq 0}, G(D)[-n]) \\
\Big\downarrow{\scriptstyle F(\widetilde{\eta}[m])^*} & & \Big\downarrow{\scriptstyle \widetilde{\eta}[m]^*} \\
\operatorname{Hom}_{\mathcal{D}}(F(\mathbb{P}_{\geq 0}'[m]), D[-n]) & \xrightarrow{\ \psi\ } & \operatorname{Hom}_{\mathcal{C}}(\mathbb{P}_{\geq 0}'[m], G(D)[-n])
\end{array}
$$

Starting with $\psi^{-1}(\theta)$ in the upper left corner and taking homology, it follows that the Yoneda products $\psi^{-1}(\theta)F(\eta)$ and $\psi^{-1}(\theta\eta)$ are equal in $\operatorname{Ext}_{\mathcal{D}}^{m+n}(F(C'), D)$.

(c) Let n be a non-negative integer. Assume that $\operatorname{Ext}_{\mathcal{C}}^{\geq n}(C, G(D))$ is a finitely generated right $\operatorname{Ext}_{\mathcal{C}}^*(C, C)$-module. Suppose $\{\theta_1, \ldots, \theta_t\}$ is a set of generators of $\operatorname{Ext}_{\mathcal{C}}^{\geq n}(C, G(D))$ as a $\operatorname{Ext}_{\mathcal{C}}^*(C, C)$-module. Consider the set $\{\psi^{-1}(\theta_1), \ldots, \psi^{-1}(\theta_t)\}$ in $\operatorname{Ext}_{\mathcal{D}}^{\geq n}(F(C), D)$. Then it follows immediately from (b) that this is a generating set as a $\operatorname{Ext}_{\mathcal{D}}^*(F(C), F(C))$-module.

(c) Let $\Sigma = F(\operatorname{Ext}_{\mathcal{C}}^*(C, C))$. Then Σ is a subalgebra of $\operatorname{Ext}_{\mathcal{D}}^*(F(C), F(C))$. Let U be a right ideal in $\operatorname{Ext}_{\mathcal{D}}^*(F(C), F(C))$. In particular, U is a right Σ-submodule of $\operatorname{Ext}_{\mathcal{D}}^*(F(C), F(C))$. This corresponds to a right $\operatorname{Ext}_{\mathcal{C}}^*(C, C)$-submodule U' of $\operatorname{Ext}_{\mathcal{C}}^*(C, GF(C))$. Since $\operatorname{Ext}_{\mathcal{C}}^*(C, C)$ is Noetherian and $\operatorname{Ext}_{\mathcal{C}}^*(C, GF(C))$ is a finitely generated $\operatorname{Ext}_{\mathcal{C}}^*(C, C)$-module, the submodule U' is a finitely generated $\operatorname{Ext}_{\mathcal{C}}^*(C, C)$-module. Then we infer that U is a finitely generated Σ-module and therefore a finitely generated $\operatorname{Ext}_{\mathcal{D}}^*(F(C), F(C))$-module, hence a finitely generated right ideal. This shows that $\operatorname{Ext}_{\mathcal{D}}^*(F(C), F(C))$ is right Noetherian. $\qquad\square$

Next we state the dual result, where we leave the proof to the reader.

Proposition 9.2. *Let (F, G) be an adjoint pair of functors as above, and assume that G is exact and preserves injective objects.*

(a) *The adjunction induces a functorial isomorphism*

$$\operatorname{Ext}_{\mathcal{D}}^i(F(C), D) \simeq \operatorname{Ext}_{\mathcal{C}}^i(C, G(D))$$

for all $i \geq 1$ and for all C in $\mathcal{C}$ and D in $\mathcal{D}$. We also denote this isomorphism by ψ.

(b) *For every element η in $\operatorname{Ext}_{\mathcal{D}}^m(D, D')$ and θ in $\operatorname{Ext}_{\mathcal{D}}^n(F(C), D)$ the following equality holds in $\operatorname{Ext}_{\mathcal{C}}^{m+n}(C, G(D'))$:*

$$\psi(\eta \cdot \theta) = G(\eta) \cdot \psi(\theta).$$

(c) *Let n be a non-negative integer. If $\operatorname{Ext}_{\mathcal{D}}^{\geq n}(F(C), D)$ is a finitely generated left $\operatorname{Ext}_{\mathcal{D}}^*(D, D)$-module, then $\operatorname{Ext}_{\mathcal{C}}^{\geq n}(C, G(D))$ is a finitely generated left $\operatorname{Ext}_{\mathcal{C}}^*(G(D), G(D))$-module.*

(d) If $\mathrm{Ext}_{\mathcal{D}}^*(D, D)$ is a left Noetherian ring and $\mathrm{Ext}_{\mathcal{D}}^*(F(G(D)), D)$ is a finitely generated left $\mathrm{Ext}_{\mathcal{D}}^*(D, D)$-module, then $\mathrm{Ext}_{\mathcal{C}}^*(G(D), G(D))$ is a left Noetherian ring.

We end this section by an application of the above.

Let $\nu \colon \Gamma \to \Lambda$ be an algebra inclusion of finite dimensional k-algebras for a field k. Then we have the induction functor

$$F = \Lambda \otimes_\Gamma -\colon \, \mathrm{mod}\,\Gamma \to \mathrm{mod}\,\Lambda$$

and the restriction functor

$$H = \mathrm{Hom}_\Lambda({}_\Lambda\Lambda_\Gamma, -)\colon \, \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma,$$

which clearly is an adjoint pair (F, H) of functors.

Corollary 9.3. *Let $\nu \colon \Gamma \to \Lambda$ be an algebra inclusion of finite dimensional k-algebras for a field k. Let $\mathfrak{r}$ denote the Jacobson radical $\mathrm{rad}\,\Gamma$ of Γ, and assume that $\Lambda\mathfrak{r} = \mathrm{rad}\,\Lambda$. Moreover assume that Λ_Γ is a projective Γ-module.*

If $\mathrm{Ext}_\Gamma^*(\Gamma/\mathfrak{r}, \Gamma/\mathfrak{r})$ is right Noetherian, then $\mathrm{Ext}_\Lambda^*(\Lambda/\mathrm{rad}\,\Lambda, \Lambda/\mathrm{rad}\,\Lambda)$ is right Noetherian.

Proof. Note that F is exact and takes projectives to projectives since Λ_Γ is projective. By assumption $\mathrm{Ext}_\Gamma^*(\Gamma/\mathfrak{r}, \Gamma/\mathfrak{r})$ is a right Noetherian algebra. Using induction on the length of a module and that $\mathrm{Ext}_\Gamma^*(\Gamma/\mathfrak{r}, \Gamma/\mathfrak{r})$ is right Noetherian, it follows that $\mathrm{Ext}_\Gamma^*(\Gamma/\mathfrak{r}, M)$ is a finitely generated right $\mathrm{Ext}_\Gamma^*(\Gamma/\mathfrak{r}, \Gamma/\mathfrak{r})$-module for all finitely generated Γ-modules M.

We have $\Lambda\mathfrak{r} = \mathrm{rad}\,\Lambda$, the functor F is exact and $F(\Gamma) = \Lambda \otimes_\Gamma \Gamma \xrightarrow{\mathrm{mult}} \Lambda$ is an isomorphism. It follows that $F(\Gamma/\mathfrak{r}) \simeq \Lambda/\mathrm{rad}\,\Lambda$. Then Proposition 9.1 (d) implies that $\mathrm{Ext}_\Lambda^*(\Lambda/\mathrm{rad}\,\Lambda, \Lambda/\mathrm{rad}\,\Lambda)$ is right Noetherian. $\square$

10. Twisted tensor product algebras

We recall the notion of a twisted tensor product of two algebras discussed in [8], and the fact that a covering of an algebra can be viewed as a twisted tensor product.

Let Λ and Γ be two algebras over a commutative ring k. Let $\tau \colon \Gamma \otimes_k \Lambda \to \Lambda \otimes_k \Gamma$ be a linear map. Define the following operation on $\Lambda \otimes_k \Gamma$ by letting

$$(\lambda \otimes \gamma) \cdot_\tau (\lambda' \otimes \gamma') = \lambda\tau(\gamma \otimes \lambda')\gamma'$$

for λ, λ' in Λ and γ, γ' in Γ. Denote $\Lambda \otimes_k \Gamma$ with this structure by $\Lambda \otimes_\tau \Gamma$. If the linear map τ satisfies the following equalities

$$(1 \otimes \gamma') \cdot_\tau \tau(\gamma \otimes \lambda) = \tau(\gamma'\gamma \otimes \lambda)$$
$$\tau(\gamma \otimes \lambda) \cdot_\tau (\lambda' \otimes 1) = \tau(\gamma \otimes \lambda\lambda')$$
$$\tau(\gamma \otimes 1) = 1 \otimes \gamma$$
$$\tau(1 \otimes \lambda) = \lambda \otimes 1$$

for all λ, λ' in Λ and all γ, γ' in Γ, then $\Lambda \otimes_\tau \Gamma$ is an associative algebra. It is called a *twisted tensor product of Λ and Γ over the twisting τ.*

Assume from now on that $\Lambda \otimes_\tau \Gamma$ is a twisted tensor product over the twisting τ. Then the natural maps $\nu_\Lambda \colon \Lambda \to \Lambda \otimes_\tau \Gamma$ and $\nu_\Gamma \colon \Gamma \to \Lambda \otimes_\tau \Gamma$ given by $\lambda \mapsto \lambda \otimes 1$

and $\gamma \mapsto 1 \otimes \gamma$, respectively are homomorphisms of algebras. Then $\Lambda \otimes_\tau \Gamma$ is a $(\Lambda \otimes_\tau \Gamma)$-$\Gamma$-bimodule, so that we have the induction and the restriction functors

$$F = \Lambda \otimes_\tau \Gamma \otimes_\Gamma -\colon \operatorname{mod} \Gamma \to \operatorname{mod} \Lambda \otimes_\tau \Gamma$$

and

$$G = \operatorname{Hom}_{\Lambda \otimes_\tau \Gamma}(\Lambda \otimes_\tau \Gamma, -)\colon \operatorname{mod} \Lambda \otimes_\tau \Gamma \to \operatorname{mod} \Gamma$$

respectively. These functors are an adjoint pair, so we have functorial isomorphisms

$$\operatorname{Hom}_{\Lambda \otimes_\tau \Gamma}(\Lambda \otimes_\tau \Gamma \otimes_\Gamma M, N) \simeq \operatorname{Hom}_\Gamma(M, \operatorname{Hom}_{\Lambda \otimes_\tau \Gamma}(\Lambda \otimes_\tau \Gamma, N)) \simeq \operatorname{Hom}_\Gamma(M, {}_\Gamma N).$$

The functor F preserves projective modules. Since $\Lambda \otimes_\tau \Gamma$ is isomorphic to $\Lambda \otimes_k \Gamma$ as a right Γ-module, it is a projective right Γ-module if and only if Λ is a projective k-module. Assume that Λ is a projective k-module. Then F is an exact functor and if $\mathbb{P} \to M$ is a projective resolution of M as a Γ-module, then $F(\mathbb{P}) \to F(M)$ is a projective resolution of $F(M)$ as a $\Lambda \otimes_\tau \Gamma$-module. Using the above adjunction we infer that

$$\operatorname{Ext}^i_{\Lambda \otimes_\tau \Gamma}(\Lambda \otimes_\tau \Gamma \otimes_\Gamma M, N) \simeq \operatorname{Ext}^i_\Gamma(M, \operatorname{Hom}_{\Lambda \otimes_\tau \Gamma}(\Lambda \otimes_\tau \Gamma, N))$$
$$\simeq \operatorname{Ext}^i_\Gamma(M, {}_\Gamma N)$$

We shall apply twisted tensor products to coverings of algebras. To this end we need to recall the dual of a group algebra. Let G be a (finite) group, then

$$k[G]^* = \{\sum_{g \in G} \alpha_g \epsilon_g \mid \text{almost all } \alpha_g \text{ are zero}\},$$

where $\epsilon_g \epsilon_h = \delta_{g,h} \epsilon_g$. With these preliminaries we can state one of the main results from [8], namely [8, Theorem 5.7], taking into account the final comment in the paper concerning the extension of that result to coverings of quivers with relations.

Theorem 10.1 ([8, Theorem 5.7])**.** *Let kQ/I be an admissible quotient of the path algebra kQ. Let $w\colon Q_1 \to G$ be a weight function into a (finite) group, and assume that I is generated by weight homogeneous elements. Then the covering $k\widetilde{Q}(w)/\widetilde{I}(w)$ is isomorphic to $k[G]^* \otimes_\tau kQ/I$, where the twisting map $\tau\colon kQ/I \otimes_k k[G]^* \to k[G]^* \otimes_k kQ/I$ is given by $\tau(\overline{p} \otimes \epsilon_g) = \epsilon_{gw(\overline{p})} \otimes \overline{p}$.*

For an illustration see Example 7.1.

Using the general setup with an adjoint pair of functors for a twisted tensor product of algebras reviewed above, we apply Proposition 9.1 to coverings of quivers with relations by a finite group.

Proposition 10.2. *Let $\Gamma = kQ/I$ be an admissible quotient of the path algebra kQ. Let $w\colon Q_1 \to G$ be a weight function into a finite group, and assume that I is generated by weight homogeneous elements. Assume that $\operatorname{Ext}^*_\Gamma(\Gamma/\mathfrak{r}, \Gamma/\mathfrak{r})$ is a right Noetherian algebra. Let $\Lambda = k\widetilde{Q}(w)/\widetilde{I}(w)$.*

   (a) *Let T be the direct sum of all isomorphism classes of simple Λ-modules. Then $\operatorname{Ext}^*_\Lambda(T, T)$ is a right Noetherian algebra.*

   (b) *For each simple Λ-module S the algebra $\operatorname{Ext}^*_\Lambda(S, S)$ is right Noetherian.*

Proof. (a) We have that $\Lambda \simeq k[G]^* \otimes_\tau \Gamma$ as defined above. Then we have an inclusion of finite dimensional k-algebras $\nu\colon \Gamma \to \Lambda$. The right Γ-module Λ is isomorphic to $\Gamma_\Gamma^{|G|}$, hence is free and in particular projective. By the definition of the twisting τ, the subset $k[G]^* \otimes_\tau \mathfrak{r}$ is an ideal in Λ and in fact a nilpotent ideal in Λ. Furthermore,

since the weight of the trivial paths is the identity in G, it is straightforward to see that

$$(k[G]^* \otimes_\tau \Gamma)/(k[G]^* \otimes_\tau \mathfrak{r}) \simeq k[G]^* \otimes_\tau \Gamma/\mathfrak{r} \simeq k[G]^* \otimes_k \Gamma/\mathfrak{r},$$

where in the last isomorphism we use that the weight of any vertex is the identity in G. The last algebra in the above formula is clearly semisimple. It follows that

$$\operatorname{rad} \Lambda = k[G]^* \otimes_\tau \mathfrak{r} = (k[G]^* \otimes_k \Gamma)(1 \otimes \mathfrak{r}).$$

Now we can apply Corollary 9.3 to obtain (a).

(b) Let $T = \oplus_{i=1}^n S_i$, where S_i is a simple module. Let $f = f_1$ be the idempotent in $\operatorname{Hom}_{k[G]^* \otimes_\tau \Gamma}(T,T)$ corresponding to the direct summand $S = S_1$ of T. Then $P = f \operatorname{Ext}^*_{k[G]^* \otimes_\tau \Gamma}(T,T)$ is a finitely generated projective right $\operatorname{Ext}^*_{k[G]^* \otimes_\tau \Gamma}(T,T)$-module. Since $\operatorname{Ext}^*_{k[G]^* \otimes_\tau \Gamma}(T,T)$ is a right Noetherian algebra, it follows from [26, Proposition 2.3 (i)] (see also [14, 10]) that

$$\operatorname{End}_R(P) \simeq f \operatorname{Ext}^*_{k[G]^* \otimes_\tau \Gamma}(T,T)f \simeq \operatorname{Ext}^*_{k[G]^* \otimes_\tau \Gamma}(S,S)$$

is right Noetherian. This completes the proof of the proposition. $\qquad\square$

11. Cohomology rings

In this final section we show that the cohomology ring of the various algebras occurring in the classification of the finite dimensional connected Hopf algebras of dimension p^3 are Noetherian, except for the algebras of type **A5** for $p > 2$ and $\beta = 0$.

Before reviewing the cohomology rings of the algebras in the classification we recall the following general result on cohomology of tensor products of algebras.

Let Λ and Γ be two (finite dimensional) k-algebras for a field k. Then we can form their tensor product $\Lambda \otimes_k \Gamma$ with componentwise multiplication. Given a Λ-module M and a Γ-module N, their tensor product $M \otimes_k N$ over k is in a natural way a module over $\Lambda \otimes_k \Gamma$. Furthermore it is well-known that

$$\operatorname{Ext}^*_{\Lambda \otimes_k \Gamma}(M \otimes_k N, M \otimes_k N) \simeq \operatorname{Ext}^*_\Lambda(M,M) \overline{\otimes}_k \operatorname{Ext}^*_\Gamma(N,N),$$

where $\overline{\otimes}_k$ denotes the usual tensor product with the product of homogeneous elements of degrees d_1 and d_2 is twisted by $(-1)^{d_1 d_2}$ (that is, odd degree elements anti-commute). In particular, if Λ and Γ are augmented k-algebras, then k is in a natural way a Λ-module and a Γ-module, so that we can consider the cohomology rings $\operatorname{Ext}^*_\Lambda(k,k)$ and $\operatorname{Ext}^*_\Gamma(k,k)$. Then it follows from the above that

$$(2) \qquad \operatorname{Ext}^*_{\Lambda \otimes_k \Gamma}(k,k) \simeq \operatorname{Ext}^*_\Lambda(k,k) \overline{\otimes}_k \operatorname{Ext}^*_\Gamma(k,k).$$

When k is a field of characteristic p, then $kC_p \simeq k[x]/\langle x^p \rangle$. Hence it is of interest to discuss the cohomology ring of algebras of the form $k[x]/\langle x^n \rangle$ for $n \geq 2$. When $\Lambda_n = k[x]/\langle x^n \rangle$ for $n \geqslant 2$, then we have (see [5, Lemma 5.2])

$$(3) \qquad \operatorname{Ext}^*_{\Lambda_n}(k,k) \simeq \begin{cases} k[y], & \text{when } n = 2, \\ k[y,z]/\langle y^2 \rangle, & \text{when } n > 2 \end{cases}$$

where the degree of y is 1 and the degree of z is 2.

11.1. **Semisimple algebras.** Let H be a semisimple Hopf algebra. Since the global dimension of a semisimple algebra is zero, it is well-known that cohomology algebra of H is isomorphic to k. This gives the cohomology ring of the only semisimple algebra in the classification, namely the case **C1**. The cohomology ring is clearly Noetherian.

11.2. **Group algebras.** The algebras of type **A3** = **C4**, **C7**, **C9** and **C10** are elementary abelian groups, $(C_p)^r$ for some $r \geq 1$. For a field k of characteristic p and $k(C_p)^r$ it is known that

$$(4) \qquad H^*((C_p)^r, k) = \operatorname{Ext}^*_{k(C_p)^r}(k, k) = \wedge(x_1, x_2, \ldots, x_r) \otimes_k k[y_1, y_2, \ldots, y_r]$$

where the degree of x_i is 1 and the degree of y_i is 2. Hence this describes the cohomology rings of the above mentioned algebras.

The algebras of type **A2** = **C2** and **C8** are group algebras of cyclic groups of order p^3 and p^2, respectively. Since the field k has characteristic p they are isomorphic to $k[x]/\langle x^{p^3} \rangle$ and $k[x]/\langle x^{p^2} \rangle$. The structure of the cohomology ring of these algebras are then described by (3).

For the algebra of type **A4** = **C3** we have to use both (2) and (3) to describe the cohomology ring.

In all cases the cohomology ring of the algebra is Noetherian.

11.3. **Selfinjective Nakayama algebras.** Let $\Lambda_{n,t} = k\widetilde{\mathbb{A}}_{n-1}/J^t$ for a field k, which is a selfinjective algebra with n indecomposable projective modules of length t. The Ext-algebra of a simple module over $\Lambda_{1,t}$ is described in (3). So we assume that $n > 1$.

Lemma 11.1. *Let S be a simple module over $\Lambda_{n,t}$, and $t = qn + r$ for $q \geq 0$ and $0 \leq r < n$ with $n > 1$. Define l_1 to be the order of the subgroup $\langle r + \mathbb{Z}n \rangle$ of $\mathbb{Z}_n$, and define $l_2 \geq 0$ smallest possible such that $l_2 r + 1 \equiv 0 \bmod n$, if such an l_2 exists. Then the Ext-algebra of S is given as*

$$\operatorname{Ext}^*_{\Lambda_{n,t}}(S, S) \simeq \begin{cases} k[x], & \text{if } l_2 \text{ doesn't exist,} \\ k[x, y]/\langle y^2 \rangle, & \text{if } l_2 \text{ exists,} \end{cases}$$

where x has degree $2l_1$ and y has degree $2l_2 + 1$.

Proof. Each indecomposable module over $\Lambda_{n,t}$ can be identitified with the radical series, for instance, the projective associated to vertex 1 is

$$(\underbrace{1, 2, \ldots, n, 1, 2, \ldots, n, \ldots, 1, 2, \ldots, n}_{q}, 1, 2, \ldots, r),$$

where 1 symbolize the simple module associated to vertex 1, and so on. It is easy to see that the top of the $2l$-th syzygy is $lr + 1$ and the top of the $(2l+1)$-th syzygy is $lr + 2$. These numbers are always taken module n, where we choose $\{1, 2, \ldots, n\}$ as representatives for the different equivalence classes. The smallest $l \geq 1$ such that $lr + 1 \equiv 1 \bmod n$ is the order of the subgroup $\langle r + \mathbb{Z}n \rangle$ of $\mathbb{Z}_n$, that is, the top of the $2l$-th syzygy is 1. This means that $\operatorname{Ext}^{2l_1 r}_{\Lambda_{n,t}}(S, S) \simeq k$ for all $r \geq 1$. If there exists a smallest l_2 such that $l_2 r + 2 \equiv 1 \bmod n$, that is, the top of the $(2l+1)$-th syzygy is 1. This means that $\operatorname{Ext}^{2(l_2 + l_1 r)+1}_{\Lambda_{n,t}}(S, S) \simeq k$ for all $r \geq 1$. Direct computations show that the structure of the Ext-algebra of the simple module is as claimed. $\square$

Using this we find that the cohomology ring of the algebra of type **B2** is $k[x]$ with the degree of x equal to $2p$, of the algebra of type **C12** is $k[x]$ with the degree of x equal to 2, of the algebra of type **C13** is p copies of $k[x]$ with the degree of x equal to 2 and of the algebra of type **C14** is $k[x] \oplus (M_p(k))^{p-1}$ with the degree of x equal to 2.

In all cases the cohomology ring of the algebra is Noetherian.

11.4. Enveloping algebras of restricted Lie algebras.
The enveloping algebras of restricted Lie algebras are finite dimensional cocommutative Hopf algebras, and for such algebras the cohomology ring is finitely generated by [11, Theorem 1.1]. This shows that the cohomology rings of the algebras of type **C5**, **C6** and **C15** are Noetherian.

11.5. Coverings of local algebras.
Let k be a field of characteristic p. Let

$$\Lambda_1 = k\langle y, z\rangle / \langle y^p, yz - zy, z^p\rangle \quad \text{and} \quad \Lambda_2 = k\langle y, z\rangle / \langle y^p, yz - zy - y^2, z^p\rangle.$$

The algebra Λ_1 is isomorphic to $k[y]/\langle y^p\rangle \otimes_k k[z]/\langle z^p\rangle \simeq k(C_p \times C_p)$, so that using (3) and (2) or (4) one can describe the structure of the cohomology ring of this algebra and it is a Noetherian algebra.

For $p = 2$ the algebra Λ_2 is given by $\Lambda_2 = k\langle y, z\rangle / \langle y^2, yz - zy, z^2\rangle$, which is a Koszul algebra. Then $\operatorname{Ext}^*_{\Lambda_2}(k, k) \simeq k\langle y, z\rangle / \langle yz + zy\rangle$, which is Noetherian. For $p > 2$ the cohomology ring of Λ_2 is shown to be Noetherian in [23, Section 4] (see also [24]).

The algebras H of type **B1 = C11** and type **C16** are coverings of the algebra Λ_1 and the algebras of type **B3** are coverings of Λ_2. Then by Proposition 10.2 (b) it follows that cohomology ring $S = \operatorname{Ext}^*_H(k, k)$ of H is right Noetherian. Since S is graded commutative, it follows that S is Noetherian.

11.6. Other local algebras.
The cohomology ring of the algebra of type **A5** is to our knowledge unknown in general. When $p = 2$, it is a semidihedral algebra of dimension 8. This algebra is the smallest algebra in a family which contains group algebras of semidihedral 2-groups. The cohomology of semidihedral 2-groups is known and a presentation can be found in [27]. The cohomology of the algebra of type **A5** for $p = 2$ is closely related. With lengthy calculations we can show that **A5** for $p > 2$ and $\beta \neq 0$ also has finitely generated cohomology, we will not include the details.

11.7. Conclusion.
Our investigations show that there is only one case where we do not know that the cohomology ring is Noetherian, namely the algebras of type **A5** for $p > 2$ and $\beta = 0$.

References

[1] N. Andruskiewitsch, On finite-dimensional Hopf algebras, *Proc. Int. Cong. Math*, Seoul, 2 (2014), 117–141.

[2] I. Assem, D. Simson, A. Skowronski, *Elements of the representation theory of associative algebras*, London Mathematics Society, Student Texts, vol. 65.

[3] M. Auslander, I. Reiten, S. O. Smalø, *Introduction to representation theory of artin algebras*, Cambridge Studies in Advanced Mathematics, vol. 36.

[4] D. J. Benson, *Representations and cohomology* I, Cambridge Studies in Advanced Mathematics, vol. 30.

[5] P. A. Bergh, S. Oppermann, *Cohomology of twisted tensor products*, J. Algebra 320 (2008), no. 8, 3327–3338.

[6] K. A. Brown and J.-J. Zhang, Dualising complexes and twisted Hochschild (co)homology for noetherian Hopf algebras, *J. Algebra* 320 (5) (2008) 1814–1850.

[7] C. Cibils, A. Lauve and S. Witherspoon, Hopf quivers and Nichols algebras in positive characteristic, *Proc. Amer. Math. Soc.*, 137 (2009), 4029–4041.

[8] J. M. Corson, T. J. Ratkovich, *Quiver algebras of coverings and twisted tensor products*, Comm. Alg., 42, 4435–4450, 2014.

[9] K. Erdmann, *Blocks of tame representation type and related algebras*, Springer Lecture Notes in Mathematics, 1428 (1990).

[10] J. W. Fisher, *Finiteness conditions for projective and injective modules*, Proc. Amer. Math. Soc., 40 (1973), 389–394.

[11] E. M. Friedlander, A. Suslin, *Cohomology of finite group schemes over a field*, Invent. math. 127, 209–270 (1997).

[12] E. L. Green, *Constructing quantum groups and Hopf algebras from coverings*, J. Algebra **176**(1995), no. 1, 12–33.

[13] E. L. Green, *Graphs with relations, coverings and group-graded algebras*, Trans. Amer. Math. Soc., 279 (1983), 297–310.

[14] Harada, *On semi-primary abelian categories*, Osaka J. Math., 5 (1968), 189-198.

[15] I. Heckenberger and J. Wang, Rank 2 Nichols algebras of diagonal type over fields of positive characteristic, *SIGMA*, 11 (2015), 011, 24 pages.

[16] G. Henderson, Low-dimensional cocommutative connected Hopf algebras, *J. Pure Appl. Alg.*, 102 (1995), 173–193.

[17] T. Holm, *Hochschild cohomology rings of algebras $k[X]/(f)$*, Beiträge Algebra Geom. 41 (2000), no. 1, 291–301.

[18] J. Humphreys, Symmetry for finite dimensional Hopf algebras, *Proc. Amer. Math. Sot.* 6 (1978), 143–146.

[19] N. Jacobson, *Lie algebras*, Dover Publications Inc., 1979, ISBN: 0-486-63832-5.

[20] R. G. Larson and M. E. Sweedler, An associative orthogonal bilinear form for Hopf algebras, *Amer. J. Math.* 91 (1969), 75–94.

[21] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, CBMS Regional Conference Series in Mathematics 82, Amer. Math. Soc., Providence, RI, 1993.

[22] V. C. Nguyen, L.-H. Wang and X.-T. Wang, Classification of connected Hopf algebras of dimension p^3 I, *J. Algebra*, 424 (2015), 473–505.

[23] V. C. Nguyen, S. Witherspoon, *Finite generation of the cohomology of some skew group rings*, Algebra Number Theory 8, 1647–1657 (2014). Erratum 12(2018) No. 2, 491-492.

[24] _____, *Correction to the article Finite generation of the cohomology of some skew group algebras*, Algebra Number Theory 12 (2018), no. 2, 491–492.

[25] B. Pareigis, When Hopf algebras are Frobenius algebras, *J. Algebra*18(1971), 588–596.

[26] F. L., Sandomierski, *Modules over the endomorphism ring of a finitely generated projective module*, Proc. Amer. Math. Soc. 31, 27–31 (1972).

[27] H. Sasaki, *The mod 2 cohomology algebras of finite groups with semidihedral Sylow 2-subgroups*, Comm. Alg., 22 (1994), 4123–4156.

[28] S. Scherotzke, Classification of pointed rank one Hopf algebras, *J. Algebra*, 319 (2008), 2889–2912.

[29] C. Schneider, H. Usefi, *The classification of p-nilpotent restricted Lie algebras of dimension at most 4*, Forum Math. 28 (2016), no. 4, 713–727.

[30] R. Schue, Symmetry for the enveloping algebra of a restricted Lie algebra, *Proc. Amer. Math. Soc.*16 (1965), 1123–1124.

[31] A. Skowronski, K. Yamagata, *Frobenius algebras* I*, Basic representation theory*, EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2011. xii+650 pp. ISBN: 978-3-03719-102-6

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