Using discrete Darboux polynomials to detect and determine preserved measures and integrals of rational maps

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Abstract

In this Letter we propose a systematic approach for detecting and calculating integrals and preserved measures of rational maps. The approach is based on the use of cofactors and Discrete Darboux Polynomials and relies on the use of symbolic algebra tools. We show, in three examples, how to use this method to detect and determine integrals and preserved measures of the considered rational maps.

1 Introduction

The search for integrals and preserved measures of ordinary differential equations (ODEs) has been at the forefront of mathematical physics since the time of Galileo and Newton.

In this Letter our aim will be to develop an analogous theory for the (arguably more general) discrete-time case. This will lead to essentially linear algorithms for detecting and determining preserved measures and first and second integrals of (discrete) rational maps (both integrable and non-integrable). But before we consider the discrete case, let us look at the continuous case, i.e. ODEs.

Consider two polynomials P_1 and P_2. Then $I := P_1/P_2$ is a rational integral of the ODE $\frac{d\mathbf{x}}{dt} = f(\mathbf{x})$ if

$$\dot{P}_1 P_2 - P_1 \dot{P}_2 = 0$$

along solutions of the ODE. Here $\dot{c}$ denotes $\frac{dc}{dt}$. For a polynomial ODE, the problem of finding P_1 and P_2, as posed, is bilinear in the coefficients of the polynomials P_1, P_2.

1.1 Darboux polynomials (ODE case)

A very nice introduction to Darboux polynomials for ODEs was given by Goriely [6]. Darboux polynomials were already studied by Darboux, Poincaré, Painlevé and others, cf. [6], and are also known by several other names, including “second integrals” and “weak integrals”.

Let $P(\mathbf{x})$ and $C(\mathbf{x})$ be polynomials. Then $P(\mathbf{x})$ is called a Darboux polynomial of the ODE $\frac{d\mathbf{x}}{dt} = f(\mathbf{x})$, where f is polynomial, if

$$\dot{P}(\mathbf{x}) = C(\mathbf{x})P(\mathbf{x}),$$

along solutions of the ODE. Here $C(\mathbf{x})$ is called the cofactor of P. Note that $P(\mathbf{x}(0)) = 0$ implies $P(\mathbf{x}(t)) = 0$ for all t. Hence the set $P(\mathbf{x}) = 0$ is an invariant set in phase space.

Consider two Darboux polynomials with the same cofactor C:

$$\begin{array}{l} \dot{P}_1 = CP_1 \\ \dot{P}_2 = CP_2 \end{array} \Rightarrow \frac{d}{dt}\left(\frac{P_1}{P_2}\right) = \frac{\dot{P}_1 P_2 - P_1 \dot{P}_2}{P_2^2} = \frac{CP_1P_2 - P_1CP_2}{P_2^2} = 0, \tag{1}$$

i.e. the ratio of two Darboux polynomials with the same cofactor is a rational integral. The converse is also true. However, finding C, P_1 and P_2 involves one bilinear problem, plus one linear problem.

More generally,

$$\begin{array}{l} \dot{P}_1 = C_1P_1 \\ \dot{P}_2 = C_2P_2 \end{array} \Rightarrow \frac{d}{dt}(P_1P_2) = \dot{P}_1P_2 + P_1\dot{P}_2 = (C_1 + C_2)P_1P_2. \tag{2}$$

1.2 Discrete Darboux Polynomials (mapping case)

Instead of polynomial ODEs $\frac{d\mathbf{x}}{dt} = f(\mathbf{x})$, we now consider rational maps $\mathbf{x}_{n+1} = \phi(\mathbf{x}_n)$ (cf [4, 5]). Then we define $P(\mathbf{x})$ to be a Discrete Darboux Polynomial of the rational map $\mathbf{x}_{n+1} = \phi(\mathbf{x}_n)$ if

$$P(\mathbf{x}_{n+1}) = C(\mathbf{x}_n)P(\mathbf{x}_n), \tag{3}$$

where the cofactor C is now a rational function whose form will be presented in §1.3. We use the shorthand notation

$$P' = CP.$$

Note that, similarly to the continuous case, $P(\mathbf{x}) = 0$ is an invariant set in phase space.

Now consider again two Discrete Darboux Polynomials P_1 and P_2 with the same cofactor C:

$$\begin{array}{l} P_1' = CP_1 \\ P_2' = CP_2 \end{array} \Rightarrow \frac{P_1'}{P_2'} = \frac{P_1}{P_2},$$

i.e. the ratio of the two Discrete Darboux Polynomials with the same cofactors is again an integral (and the converse is also true). More generally

$$\begin{array}{l} P_1' = C_1P_1 \\ P_2' = C_2P_2 \end{array} \Rightarrow (P_1P_2)' = C_1C_2(P_1P_2)$$

How is all this going to help us find integrals of a given map? The answer comes in two parts:

1. In the discrete case we use a non-trivial ansatz for the cofactors $C(\mathbf{x})$.
2. In the discrete case the cofactor of the product is the *product* of the cofactors.
   In the continuous case the cofactor of the product is the *sum* of the cofactors.

The latter point is crucial: It means that in the discrete case we can use the fact that the factorization of the cofactor C is unique. By contrast, in the ODE case we have addition, where splitting into summands is not unique.

1.3 Ansatz

Given a rational map ϕ with common denominator

$$D(\mathbf{x}) = \prod_{j=1}^{k} D_j^{a_j}(\mathbf{x})$$

and Jacobian determinant

$$J(\mathbf{x}) = \frac{\prod_{i=1}^{l} K_i^{b_i}(\mathbf{x})}{\prod_{j=1}^{k} D_j^{e_j}(\mathbf{x})},$$

where the $K_i(\mathbf{x})$ are l distinct factors[1], we try all cofactors (up to a certain polynomial degree d) of the form

$$C(\mathbf{x}) = \pm \frac{\prod_{i=1}^{l} K_i^{f_i}(\mathbf{x})}{\prod_{j=1}^{k} D_j^{g_j}(\mathbf{x})}, \tag{4}$$

where $f_j, g_j \in \mathbb{N}_0$.

Comments:

1. For the finite number of cofactors of degree $< d$, we only need to solve the linear problem (3) to determine the corresponding Darboux polynomial P (up to a chosen degree).
2. If $C(\mathbf{x}) = J(\mathbf{x})$, the corresponding Darboux polynomials are (inverse) densities of preserved measures.
3. Note that we include $\pm 1, \pm J$ in the list of potential cofactors to be examined.

The above approach is particularly useful for maps for which no other systematic method is known to derive integrals, prime examples being Kahan maps [2, 15] and dual maps [16]. In the remainder of this Letter we therefore study two maps that arise as Kahan-Hirota-Kimura (KHK) discretizations of quadratic ODEs (in section 2.1, resp. section 3) [9, 11], plus a third map that arises as the reduction of the dual AKP equation (in section 2.2) [19].

For any given cofactor, of the form (4), the equation (3) gives rise to a system of equations that are linear in the coefficients, c_i, of the discrete Darboux polynomial P, and may also depend on the parameters α_j in the mapping through both x_{n+1} and $C(x_n)$ in (3). One can either treat the resulting system as a linear system to determine the c_i (as we do e.g. in section 2.2) or solve the (nonlinear) system to simultaneously determine the c_i and detect the α_j (as we do in section 3). In this paper we use standard Maple routines to perform both kinds of computations.

2 Determining preserved measures and first and second integrals of rational maps

In this section, we study two examples of discrete maps for which we use the ansatz of section 1.3 to find their preserved integrals and measures.

[1] Factorisation is carried out over the field implied by the coefficients of the map ϕ.

2.1 Determining measures and integrals of a 2D Lotka-Volterra system

In this sub-section we study the following two-dimensional ODE as an example:

$$\begin{aligned} \frac{dx}{dt} &= x(x+6y-3) \\ \frac{dy}{dt} &= y(-3y-2x+3) \end{aligned} \tag{5}$$

The KHK discretization of (5) reads (cf. [2, 3, 7, 9, 11, 10, 15])

$$\begin{aligned} x' &= \frac{x(1+h(x+6y-3)+\frac{h^2}{4}(9-6x))}{D(x)} \\ y' &= \frac{y(1+h(3-2x-3y)+\frac{9h^2}{4}(1-2y))}{D(x)} \end{aligned} \tag{6}$$

where the common denominator $D(\mathbf{x})$ of the map is given by

$$D(\mathbf{x}) := 1 - \frac{h^2}{4}(9-12x-36y+4x^2+12xy+36y^2) \tag{7}$$

The Jacobian determinant $J(\mathbf{x})$ of the mapping (6) is

$$J(\mathbf{x}) = \frac{K_1(\mathbf{x})K_2(\mathbf{x})K_3(\mathbf{x})}{D^3(\mathbf{x})} \tag{8}$$

where

$$\begin{aligned} K_1 &= 1+h(x-3y)-\frac{3}{4}h^2(3-2x-6y) \\ K_2 &= 1+h(x+6y-3)+\frac{3}{4}h^2(3-2x) \\ K_3 &= 1+h(3-2x-3y)+\frac{9}{4}h^2(1-2y) \end{aligned} \tag{9}$$

We have used cofactors $C_1 = \frac{K_1}{D}$, $C_2 = \frac{K_2}{D}$, $C_3 = \frac{K_3}{D}$, $C_4 = J$ to find the corresponding Discrete Darboux Polynomials for the map (6):

$$\begin{aligned} p_{1,1} &= x+3y-3 \\ p_{2,1} &= x \\ p_{3,1} &= y \\ p_{4,1} &= xy(x+3y-3) \\ p_{4,2} &= 1-\frac{h^2}{4}(9-12x-36y+4x^2+12xy+36y^2) \end{aligned} \tag{10}$$

Here, and in section 2.2, $p_{i,j}$ denotes the j^{th} Darboux polynomial corresponding to the cofactor C_i.

A phase plot for the map (6), clearly exhibiting the linear Darboux polynomials $p_{1,1}$, $p_{2,1}$, and $p_{3,1}$, is given in Figure 1.

It follows that the map (6) preserves the integral

$$\tilde{I}(\mathbf{x}) = \frac{xy(x+3y-3)}{1-\frac{h^2}{4}(9-12x-36y+4x^2+12xy+36y^2)} \tag{11}$$

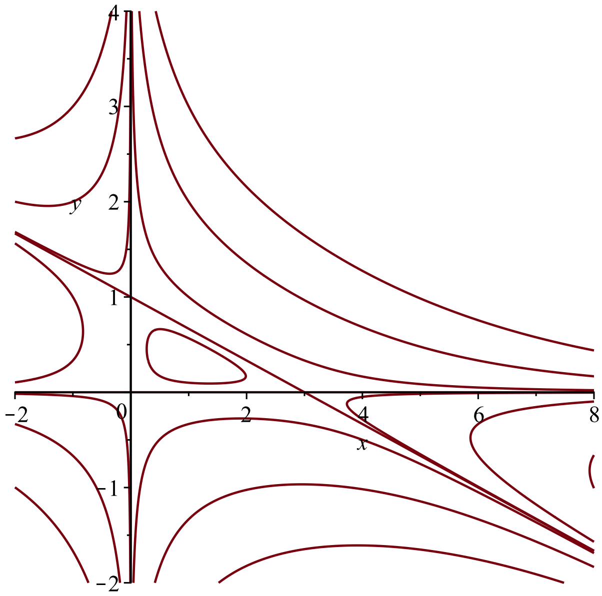

Figure 1: Phase plot for map (6), for $h = \frac{1}{17}$

and the measure

$$\frac{dxdy}{1 - \frac{h^2}{4}(9 - 12x - 36y + 4x^2 + 12xy + 36y^2)} \tag{12}$$

Taking the continuum limit $h \to 0$, we obtain the cofactors $\tilde{C}_1 = x - 3y$, $\tilde{C}_2 = x + 6y - 3$, $\tilde{C}_3 = 3 - 2x - 3y$, $\tilde{C}_4 = 0$, and the corresponding Darboux polynomials

$$p_{1,1} = x + 3y - 3, \quad p_{2,1} = x, \quad p_{3,1} = y, \quad p_{4,1} = xy(x + 3y - 3), \quad p_{4,2} = 1$$

It follows that the ODE (5) preserves the integral

$$I(\mathbf{x}) = xy(x + 3y - 3) \tag{13}$$

and the measure

$$dxdy. \tag{14}$$

It thus turns out that our original ODE (5) is Hamiltonian, with $H(\mathbf{x}) = xy(x + 3y - 3)$.

Interpreted conversely, one can say that the KHK discretization (6) preserves the three affine Darboux polynomials of the ODE (5), as well as the modified integral (11) and the modified density (12). These results are no coincidences.

Indeed, the preservation of the three affine Darboux polynomials is the consequence of the following theorem (whose proof is presented in Appendix A).

Theorem 1. *The KHK discretization preserves all affine Darboux polynomials of a given quadratic ODE.*

But there is more to be said. Looking at eqs (8), (9) and (10), one notices that to each affine Darboux polynomial ($x+3y-3$, resp. x, resp. y) there is a corresponding factor in the numerator of $J(\mathbf{x})$ (viz. K_1, resp. K_2, resp. K_3). This again is no coincidence. Indeed, the latter fact is a consequence of the following Theorem (whose proof is presented in Appendix B).

Theorem 2. *Let* $\frac{d\mathbf{x}}{dt} = f(\mathbf{x})$ *be a quadratic differential equation in 1, 2, 3 or 4 dimensions, that possesses an affine Darboux polynomial* $P(\mathbf{x})$. *Then the numerator of the discrete cofactor of* P *in the corresponding Kahan map* $\mathbf{x}' = \phi_h(\mathbf{x})$ *is a factor of the numerator of the Jacobian determinant of* ϕ_h.

Theorems 1 and 2 are a very significant step towards the full resolution of the open problem posed in 2002 in [13]: 'How does one preserve more than $n-1$ integrals and weak integrals (of an n-dimensional vector field)?'

The preservation of the modified integral and measure is an example of a general result in [2] giving a modified integral for all systems with a cubic Hamiltonian in any dimension.

2.2 Determining measures and integrals of a 9D dual AKP-reduction

In [19], Van der Kamp, Quispel and Zhang derived a dual AKP equation. Here we consider its $(1,2,4)$-reduction, given by the 9D map:

$$\begin{cases} x_i' = x_{i+1} & 1 \le i \le 8 \\ x_9' = \frac{\alpha_1}{\alpha_3}\left(\frac{{x_7}^2}{x_4} - \frac{x_4x_6x_8}{x_3x_5}\right) + \frac{\alpha_2}{\alpha_3}\left(\frac{x_5x_8}{x_3} - \frac{{x_6}^2x_7}{x_4x_5}\right) + \frac{x_1x_6x_7x_8}{x_4x_3x_5} + \frac{\alpha_4}{\alpha_3}\left(\frac{x_2x_7x_8}{x_4x_3} - \frac{x_6x_9}{x_5}\right), \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are parameters. Its Jacobian determinant is $J(\mathbf{x}) = \frac{x_6x_7x_8}{x_4x_3x_5}$. We have used cofactors

$$C_1(\mathbf{x}) = \frac{x_7}{x_4}, \quad C_2(\mathbf{x}) = \frac{x_8}{x_3}, \quad C_3(\mathbf{x}) = \frac{x_6x_8}{x_3x_5}$$

to find Darboux polynomials. Seven of these are listed below.

$$\begin{aligned}
p_{1,1}(\mathbf{x}) =& \alpha_1x_3x_7x_5 + \left(-x_3{x_6}^2 - {x_4}^2x_7\right)\alpha_2 + \left(x_1x_6x_8 + x_2x_4x_9\right)\alpha_3 + \alpha_4x_2x_8x_5 \\
p_{1,2}(\mathbf{x}) =& x_4x_6x_5 \\
p_{2,1}(\mathbf{x}) =& -{x_4}^2{x_6}^2\alpha_1x_5 + {x_5}^3a_2x_4x_6 + \left(x_1x_5{x_6}^2x_7 + x_2x_3x_6{x_7}^2 + {x_3}^2x_4x_7x_8 + x_3{x_4}^2x_5x_9\right)\alpha_3 \\
&+ \left(x_2{x_5}^2x_6x_7 + {x_3}^2{x_6}^2x_7 + x_3{x_4}^2{x_7}^2 + x_3x_4{x_5}^2x_8\right)\alpha_4 \\
p_{2,2}(\mathbf{x}) =& \left(-x_3x_4{x_6}^3 - x_3{x_5}^3x_7 - {x_4}^3x_6x_7\right)\alpha_1 + \left(x_3{x_5}^2{x_6}^2 + {x_4}^2{x_5}^2x_7\right)\alpha_2 \\
&+ \left(x_1x_4x_6{x_7}^2 + x_2x_3x_5x_7x_8 + {x_3}^2x_4x_6x_9\right)\alpha_3 + \left(x_2x_4x_5{x_7}^2 + {x_3}^2x_5x_6x_8\right)\alpha_4 \\
p_{2,3}(\mathbf{x}) =& x_3x_7x_4x_5x_6 \\
p_{3,1}(\mathbf{x}) =& \left(-x_2x_3x_5x_6{x_7}^2 + x_2{x_4}^2{x_6}^2x_8 - {x_3}^2x_4x_5x_7x_8\right){\alpha_1}^2 + \left(x_2x_3{x_6}^3x_7 - x_2x_4{x_5}^2x_6x_8 + x_3{x_4}^3x_7x_8\right)\alpha_1\alpha_2 \\
&+ \left(-x_1x_2{x_6}^2x_7x_8 - x_1x_3{x_5}^2x_7x_9 - x_2x_3{x_4}^2x_8x_9\right)\alpha_1\alpha_3 + \left(-x_1x_3x_5{x_7}^3 + x_1{x_4}^2x_6x_7x_8 - {x_2}^2x_5x_6x_7x_8\right. \\
&\left.-x_2x_3x_4x_5{x_8}^2 + x_2x_3x_4{x_6}^2x_9 - {x_3}^3x_5x_7x_9\right)\alpha_1\alpha_4 + {x_3}^2{x_7}^2{\alpha_2}^2x_4x_6 - x_3x_7\left(x_1x_4x_7x_8 + x_2x_3x_6x_9\right)\alpha_2\alpha_3 \\
&+ \left(x_1x_3{x_6}^2{x_7}^2 - x_1x_4{x_5}^2x_7x_8 - x_2x_3{x_5}^2x_6x_9 + {x_3}^2{x_4}^2x_7x_9\right)\alpha_2\alpha_4 + {\alpha_3}^2x_1x_2x_8x_9x_7x_3 + \left(-{x_1}^2x_6{x_7}^2x_8\right. \\
&\left.-x_2{x_3}^2x_4{x_9}^2\right)\alpha_4\alpha_3 + \left(-x_1x_2x_5{x_7}^2x_8 + x_1x_3x_4x_6x_7x_9 - x_2{x_3}^2x_5x_8x_9\right){\alpha_4}^2 \\
p_{3,2}(\mathbf{x}) =& x_3x_4{x_5}^2x_6x_7.
\end{aligned}$$

The four integrals

$$k_1(\mathbf{x}) = \frac{p_{1,1}}{p_{1,2}}, \quad k_2(\mathbf{x}) = \frac{p_{2,1}}{p_{2,3}}, \quad k_3(\mathbf{x}) = \frac{p_{2,2}}{p_{2,3}}, \quad k_4(\mathbf{x}) = \frac{p_{3,1}}{p_{3,2}}$$

are functionally independent.

Since $C_1(\mathbf{x})C_3(\mathbf{x}) = J(\mathbf{x})$, it follows that the map also preserves the measure with density $p_{1,2}(\mathbf{x})^{-1}p_{3,2}(\mathbf{x})^{-1} = (x_3x_4^2x_5^3x_6^2x_7)^{-1}$.

3 Detecting preserved measures and first and second integrals of rational maps

In this section we consider the following three-dimensional ODE as an example:

$$\frac{dx}{dt} = x(y-\alpha_1 z), \quad \frac{dy}{dt} = y(\alpha_2 z - x), \quad \frac{dz}{dt} = z(\alpha_1 x - \alpha_2 y), \tag{15}$$

where α_2 and α_1 are arbitrary parameters.

Applying the Kahan-Hirota-Kimura discretization to (15), we obtain

$$\begin{aligned} \frac{x'-x}{h} &= \frac{x'(y-\alpha_1 z)+x(y'-\alpha_1 z')}{2} \\ \frac{y'-y}{h} &= \frac{y'(\alpha_2 z - x)+y(\alpha_2 z'-x')}{2} \\ \frac{z'-z}{h} &= \frac{z'(\alpha_1 x-\alpha_2 y)+z(\alpha_1 x'-\alpha_2 y')}{2}. \end{aligned} \tag{16}$$

Solving equation (16) for x', y', and z' we obtain the (rational) Kahan map discretizing (15). Using the Jacobian determinant $J(\mathbf{x})$ of the Kahan map as cofactor, our algorithm finds that for all (α_1, α_2), the map preserves the measure $\frac{dxdydz}{xyz}$ and the first integral $x+y+z$.

Moreover, the algorithm also detects the following special values of the parameters (α_1, α_2) where the map preserves an additional integral, and outputs the formula for the integral (see Table 1).

Table 1: Integrable parameter values and corresponding functionally independent additional first integrals detected by our algorithm.

(α_1, α_2)	additional first integral
$(-1,0)$	y/z
$(1,0)$	$yz/(1-\frac{h^2}{4}x^2)$
$(0,1)$	$xz/(1-\frac{h^2}{4}y^2)$
$(0,-1)$	z/x
$(1,1)$	$xyz/(1-\frac{h^2}{4}(x^2+y^2+z^2-2xy-2xz-2yz))$
$(1,-1)$	x/yz
$(-1,-1)$	z/xy
$(-1,1)$	y/xz

4 Concluding remarks

In this Letter we have presented a method for detecting and determining first and second integrals of rational maps. There are in the literature several other methods for *determining* first and second integrals of discrete systems, cf. [4, 5, 8, 14, 18] and references therein. There are also in the literature several other methods for *detecting* first and second integrals of discrete systems, cf. [1, 7, 17] and references therein.

However, to our knowledge none of the above combine all the following properties of the method presented in this Letter:

1. It is algorithmic, and requires no other input than the rational map in question. At heart the algorithm is linear and, to some extent apart from birationality, requires no knowledge about the map (such as symplecticity, measure preservation, time-reversal symmetry, integrability, Lax pairs, etc) on the part of the user.

2. Up to a certain prescribed degree, it determines and outputs all

   (a) rational first integrals

   (b) polynomial second integrals

   (c) preserved measures of the form $P(x)dx$ or $\frac{dx}{P(x)}$, where P is a polynomial.

   whose cofactors are of the form (4).

3. It can detect special parameter values where additional first and/or second integrals and/or preserved measures exist, and output those integrals, preserved measures and parameter values.

4. It works for both integrable and non-integrable cases, in arbitrary dimension.

5. It allows one to take the continuum limit, if appropriate.

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Appendix A. Proof of Theorem 1

For quadratic vector fields $f(\mathbf{x})$, Kahan's method is equivalent to the B-series method

$$\mathbf{x}' = \phi(\mathbf{x}) := \mathbf{x} + h\left(\mathbf{I} - \frac{1}{2}hf'(\mathbf{x})\right)^{-1} f(\mathbf{x}) \tag{A.1}$$

where f' is the Jacobian matrix of f [2].

Lemma 1 The B-series method (A.1) preserves all affine Darboux polynomials of any polynomial ODE.

Proof. Since B-series methods commute with affine transformations [12], we may assume that the Darboux polynomial to be preserved equals x_1, i.e.

$$\frac{dx_1}{dt} = x_1 \cdot C(\mathbf{x}), \tag{A.2}$$

where the cofactor C is polynomial. Using Cramer's rule it follows, using (A.1), that

$$\frac{x_1' - x_1}{h} = \left(M_1 f_1 + \frac{h}{2} \sum_{j=2}^{n} M_j f_j \frac{\partial f_1}{\partial x_j} \right) / \det \left(\mathbf{I} - \frac{h}{2} f' \right) \tag{A.3}$$

where the M_i are (determinantal) polynomials depending on h and x. Substituting (A.2) in (A.3), we obtain

$$x_1' = x_1 \left(1 + hCM_1 + \frac{h^2}{2} \sum_{j=2}^{n} M_j f_j \frac{\partial C}{\partial x_j} \right) / \det \left(\mathbf{I} - \frac{h}{2} f' \right)$$

This shows that the discrete cofactor is of the form (4), and concludes the proof of Lemma 1. □

Since Kahan's method is equivalent to the B-series method (A.1) for quadratic vector fields, Theorem 1 follows as a corollary.

Appendix B. Proof of Theorem 2.

In order to prove Theorem 2, we first prove the following:

Lemma Let ϕ_h be given by[2]

$$x_1' = x_1 \frac{N_1(x,h)}{D(x,h)}, \text{ with } N_1 \text{ and D coprime,} \tag{B.1}$$

$$x_i' = \frac{N_i(x,h)}{D(x,h)}, \quad i = 2, \ldots, n \tag{B.2}$$

Also assume that the Jacobian Determinant J of ϕ_h satisfies

$$J = \frac{D(x', -h)}{D(x,h)}, \tag{B.3}$$

and let

$$\phi_h \circ \phi_{-h} = id, \tag{B.4}$$

i.e. the system is invariant under

$$x \leftrightarrow x', h \leftrightarrow -h. \tag{B.5}$$

Then N_1 divides the numerator of the Jacobian Determinant J.

Proof. Applying (B.5) to (B.1), we get

$$x_1 = x_1' \frac{N_1(x', -h)}{D(x', -h)} = x_1 \frac{N_1(x,h)}{D(x,h)} \frac{N_1(x', -h)}{D(x', -h)}.$$

Hence

$$N_1(x,h) N_1(x', -h) = D(x,h) D(x', -h). \tag{B.6}$$

From (B.1, B.2) it follows that $D(x', -h)$ can be written as

$$D(x', -h) = \frac{L(x,h)}{D^k(x,h)}. \tag{B.7}$$

[2]Capitalised functions are polynomial in their arguments.

for some $k \in \mathbb{N}_0$. It follows from (B.3) that

$$J = \frac{L(x,h)}{D^{k+1}(x,h)} =: \frac{\overline{L}(x,h)}{\overline{D}(x,h)}, \tag{B.8}$$

where $\overline{L}$ and $\overline{D}$ are coprime. Using (B.7) and (B.8) in (B.6) we get

$$\overline{D}(x,h)N_1(x,h)N_1(x',-h) = D^2(x,h)\overline{L}(x,h).$$

From (B.1, B.2) it also follows that $N_1(x',-h)$ can be written as

$$N_1(x',-h) = \frac{M(x,h)}{D^m(x,h)}, \quad \text{where, for Kahan's method } m = 1.$$

Hence

$$\overline{D}(x,h)N_1(x,h)M(x,h) = D^{2+m}(x,h)\overline{L}(x,h).$$

Hence $N_1|\overline{L}$, i.e. the numerator of the cofactor of x_1 divides the numerator of the Jacobian Determinant J.

□

What remains to prove is that in dimensions 1, 2, 3, and 4 the Kahan map satisfies Theorem 2. The first step towards proving this is to use the fact that if ϕ_h is a Kahan map, then we may assume w.l.o.g. that the affine Darboux polynomial is x_1, and hence the Kahan map has the form (B.1), (B.2) and its Jacobian determinant satisfies (B.3) and (B.4) [2].

The only thing we still need to show is that N_1 and D are coprime, where $D(x,h) = det\left(I - \frac{1}{2}hf'(x)\right)$. This we have done explicitly, using Maple for the most general quadratic ODEs with Darboux polynomial x_1.

The 5D case being outside the capabilities of the supercomputer at our disposal, we have performed 1000, resp. 50, resp. 6 computations similar to the ones above in 5D, resp. 6D, resp. 7D, but with randomly chosen integer values between -100 and 100 for the parameters instead of symbolic ones.

In all computations, N_1 and D are coprime.

Based on Theorem 2 and on the computations mentioned above, we make the following:

Conjecture. Theorem 2 (which we proved in dimensions 1, 2, 3, and 4) remains true in any dimension.

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