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Three classes of quadratic vector fields for which the Kahan discretisation is the root of a generalised Manin transformation

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Abstract

We apply Kahan's discretisation method to three classes of 2-dimensional quadratic vector fields with quadratic, resp. cubic, resp. quartic Hamiltonians. We show that the maps obtained in this way can be geometrically understood as the composition of two involutions, one of which is a (linear) symmetry switch, and the other is a generalised Manin involution. Applications to 2-dimensional Suslov and reduced Nahm equations are included.

1 Introduction

Kahan's method for discretizing quadratic differential equations was introduced in [9]. It was rediscovered in the context of integrable systems by Hirota and Kimura [11]. Suris and collaborators extended the applications to integrable systems significantly in a series of papers [15], [16], [17], [18], [8]. Applications to both integrable as well as non-integrable Hamiltonian systems and the use of polarisation to discretise arbitrary degree Hamiltonian systems were studied in [2], [3] and [4]. For homogeneous quadratic vector fields,

$$\frac{dx_i}{dt} = \sum_{j,k} a_{ijk} x_j x_k$$

Kahan's method gives the following discretisation

$$\frac{x_i' - x_i}{h} = \sum_{j,k} a_{ijk}(x_j' x_k + x_j x_k')/2.$$

Two classes of 2-dimensional ODE systems of quadratic vector fields where the Kahan discretisation is integrable were presented in [1]. The latter systems are of the form

$$\frac{d\mathbf{x}}{dt} = \varphi(\mathbf{x})\mathcal{J}\nabla H(\mathbf{x}), \tag{1}$$

where

$$\mathbf{x} := \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \mathcal{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and $\varphi(\mathbf{x})$ and $H(\mathbf{x})$ are scalar functions of the components of $\mathbf{x}$. In the present paper we show that for one of these classes, and for two other classes, the Kahan discretisation can be geometrically

understood as the composition of two involutions, one of which is a symmetry switch and the other is a generalised Manin involution, both introduced in [10]. This implies that in each case the Kahan map is the root of a generalised Manin transformation, and hence that there is a (fractional affine) transformation which brings the map into symmetric Quispel-Roberts-Thompson (QRT) form [13, 14]. We briefly review all these concepts in the next section.

The three classes we consider are quite general. We include some illustrative applications to systems from the physics literature: a two-dimensional sub-system of the three-dimensional non-holonomic Suslov problem which describes the motion of a rigid body under the constraint that a certain component of the angular velocity vector vanishes [20],

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} x \mathcal{J} \nabla (x^2 + \alpha y^2) = \begin{pmatrix} \alpha xy \\ -x^2 \end{pmatrix};$$ (2)

the reduced Nahm equations [7] corresponding to tetrahedrally symmetric monopoles of charge 3,

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{J} \nabla y (x^2 - \frac{1}{3} y^2) = \begin{pmatrix} x^2 - y^2 \\ -2xy \end{pmatrix};$$ (3)

and the reduced Nahm equations for octahedrally symmetric monopoles of charge 4,

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{x - y} \mathcal{J} \nabla y (2x + 3y)(x - y)^2 = \begin{pmatrix} 2x^2 - 12y^2 \\ -6xy - 4y^2 \end{pmatrix}.$$ (4)

Their Kahan (or Hirota-Kimura) discretisations, together with an invariant two-form and an integral of motion, were given in [18]. In this paper we show that the Kahan discretisations of (2), (3), (4) are each equivalent to a symmetric QRT map, $(x, y) \to (y, y')$ with

$$y' = \frac{y^2 + \alpha(2h)^2}{x}, \quad y' = \frac{(x + y)y - (6h)^2}{3x - y}, \quad y' = \frac{xy - 2(2h)^2}{2x - y}$$

respectively.

This paper provides a geometric understanding of the Kahan discretisation of three distinct classes of ODEs, in particular it shows they possess the same geometric structure.

2 Preliminaries on QRT maps and generalised Manin transformations

QRT map. Let $P(\mathbf{x}) = \alpha F_a(\mathbf{x}) + \beta F_b(\mathbf{x}) = 0$ be a pencil of biquadratic curves. The horizontal switch ι_1 switches the two points on the curve $P(\mathbf{x}) = 0$ with the same y-coordinate and the vertical switch ι_2 switches the two points on the curve $P(\mathbf{x}) = 0$ with the same x-coordinate, cf. Figure 1 in the preface of [5]. The QRT-map is the composition of the two involutions given by $\iota_2 \circ \iota_1$. In [5, 21] it is shown that every smooth member of the pencil P is an elliptic curve, on which the QRT map acts as a translation.

Symmetric QRT map. When P is symmetric, i.e. invariant under the (standard) symmetry switch $\bar{\sigma} : (x, y) \to (y, x)$, the map $\rho = \bar{\sigma} \circ \iota_1 = \iota_2 \circ \bar{\sigma}$ is the square root of τ. It is called the QRT root of the symmetric QRT map in [5], but commonly known as the symmetric QRT map. A rational formula for the (12-parameter) symmetric QRT-map is $(x, y) \to (y, y')$ where

$$y' = \frac{f_1(y) - x f_2(y)}{f_2(y) - x f_3(y)}$$

with $\mathbf{f} = A\mathbf{v} \times B\mathbf{v}$, where $\mathbf{v}^T = (x^2, x, 1)$, and A and B are symmetric 3×3 matrices.

Manin transformation. Let $\mathbf{p}$ be a base point of a pencil of cubic curves $\alpha F_a(\mathbf{x}) + \beta F_b(\mathbf{x}) = 0$, i.e. we have $F_a(\mathbf{p}) = F_b(\mathbf{p}) = 0$. A Manin involution, $\iota_{\mathbf{p}}$, maps a point $\mathbf{r}$ to the point $\mathbf{s} = \iota_{\mathbf{p}}(\mathbf{r})$

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uniquely given by the third intersection of the line $\mathbf{pr}$ and the curve of the pencil that contains $\mathbf{r}$ [5, 12]. We call $\iota_{\mathbf{p}}$ a $\mathbf{p}$-switch, and the point $\mathbf{p}$ its involution point. A Manin transformation is the composition of two Manin involutions, cf. e.g. Example 6 in [10].

Generalised Manin transformation. A generalised Manin involution [10] preserves a pencil of degree N, where N is not necessarily 3. When $N = 2$ the involution point p can be chosen arbitrarily, for $N > 3$ the degree N pencil should have a base point which is a singular point of multiplicity $N - 2$. It was shown in [10] that it suffices to consider pencils of degree $N < 5$ and that a generalised Manin transformation can be written in QRT-form by a projective collineation.

Root of a generalised Manin transformation. A transformation σ is called a symmetry switch of the pencil $P = 0$ if σ is a symmetry of P and it is an involution. The following result was proven in [10]. Let σ be a symmetry switch of a pencil $P = 0$ which maps lines to lines (so it is a projective collineation). Then

$$\tau_{\mathbf{p}} = \iota_{\sigma(\mathbf{p})} \circ \iota_{\mathbf{p}} = \rho_{\mathbf{p}}^2, \quad \text{with } \rho_{\mathbf{p}} = \sigma \circ \iota_{\mathbf{p}} = \iota_{\sigma(\mathbf{p})} \circ \sigma.$$

The map $\rho_{\mathbf{p}}$ is called the root of $\tau_{\mathbf{p}}$.

3   Concomitants of linear and quadratic forms

We define linear and quadratic forms

$$L = L(\mathbf{x}) := ax + by, \qquad Q = Q(\mathbf{x}) := cx^2 + 2dxy + ey^2.$$

The three classes of quadratic vector fields we consider are of the form (1) with $\varphi(\mathbf{x}) = L^{2-i}$ and $H(\mathbf{x}) = L^{i-1}Q$ and $i = 1, 2, 3$. All relevant quantities, e.g. modified Hamiltonian for the Kahan map and involution point for the Manin involutions will be given in terms of the concomitants (i.e. invariants, covariants, symmetry) defined here, cf. [6, Page 252].

Let η be an element of $SL(2)$ acting on $\mathbf{x}$. This induces an action of $SL(2)$ on the coefficients a, b, c, d, e which we denote by η'. The discriminant of Q,

$$D := ce - d^2$$

and the eliminant (resultant of L and Q),

$$E := 2abd - a^2e - b^2c,$$

are invariants, and (half of) the Jacobian determinant $\partial(L, Q)/\partial(x, y)$,

$$G = G(\mathbf{x}) := (ad - bc)x + (ae - bd)y,$$

(which is the harmonic conjugate of L with respect to Q) is covariant, i.e.

$$\eta'(D) = D, \quad \eta'(E) = E, \quad \eta'(G) = \eta(G).$$

In terms of

$$\mathbf{v} := (b, -a), \quad \mathbf{w} := (ad - bc, ae - bd)$$

we have $G = \mathbf{x} \cdot \mathbf{w}$ and $E = G(\mathbf{v})$.

A particular linear symmetry switch, introduced in [10], is relevant here. We define

$$\sigma_{a,b,c,d,e} : \mathbf{x} \to \mathbf{x} - \frac{2G(\mathbf{x})}{E}\mathbf{v}. \tag{5}$$

A special case of σ is $\sigma_{a,a,c,d,c}(\mathbf{x}) = (y, x)$ and the matrices of $\sigma_{a,a,c,d,c}$ and $\sigma_{a,b,c,d,e}$ are conjugate. In the sequel we will omit the index $_{a,b,c,d,e}$. Geometrically, the linear transformation σ given by (5) is a reflection in the line through $(0, 0)$ perpendicular to $\mathbf{w}$ along a line with direction $\mathbf{v}$, i.e. we have

$$\sigma(\mathbf{v}) = -\mathbf{v}, \qquad \sigma(\mathcal{J}\mathbf{w}) = \mathcal{J}\mathbf{w}.$$

Importantly, σ (5) leaves the forms L and Q invariant (and it also negates the linear form G), that is

$$L(\sigma(\mathbf{x})) = L(\mathbf{x}), \qquad Q(\sigma(\mathbf{x})) = Q(\mathbf{x}), \qquad G(\sigma(\mathbf{x})) = -G(\mathbf{x}).$$

4 A quadratic Hamiltonian

Consider the 2-dimensional ODE (1) where $\varphi(\mathbf{x}) = L$, and the homogeneous Hamiltonian has the form $H = H(\mathbf{x}) = Q$. The Kahan map for this system is explicitly given by

$$\kappa_1(\mathbf{x}) = \frac{\mathbf{x} - h(G\mathbf{x} - LJ\nabla H)}{1 - hG + 2h^2 DL^2}. \tag{6}$$

It preserves the modified Hamiltonian $\widetilde{H}(\mathbf{x}) = Q/T$ with $T = T(\mathbf{x}) = 1 + h^2 DL^2$, cf. [3, Eq. (18)], and it is measure preserving with density

$$\frac{1}{LQ}, \tag{7}$$

cf. [3, Eq. (17)].

Theorem 1. *The map (6) can be written as a composition $\kappa_1 = \sigma \circ \iota_{\mathbf{b}}$, where σ is given by (5) and*

$$\iota_{\mathbf{b}}(\mathbf{x}) = \mathbf{x} + \left(1 + \frac{1 + hG}{1 + hG - 2T}\right)(\mathbf{b} - \mathbf{x})$$

where

$$\mathbf{b} = \frac{\mathbf{v}}{hE}. \tag{8}$$

The projective collineation

$$\pi : (x, y) \to (u, v) = \left(\frac{1 + hG}{L}, \frac{1 - hG}{L}\right) \tag{9}$$

brings the map κ_1 into QRT form

$$\overline{\kappa_1} = \overline{\sigma} \circ \iota_1 : (u, v) \to \left(v, \frac{v^2 + k}{u}\right), \tag{10}$$

where $\overline{\sigma}(u, v) = (v, u)$ and $k = 4h^2 D$. The modified Hamiltonian transforms into the integral of (10),

$$\overline{Q/T} = \frac{(u - v)^2 + k}{(u + v)^2 + k},$$

Proof. This is verified by direct calculation. The map $\iota_{\mathbf{b}}$ is the generalised Manin involution with involution point $\mathbf{b}$, cf. [10, equation (2)] with $N = 2$ and $F_a = Q$ and $F_b = T$. The point $\mathbf{b}$ is the intersection point of the lines $L = 0$ and $1 - hG = 0$, and we have $\sigma(\mathbf{b}) = -\mathbf{b}$. The projective collineation π brings the point $\mathbf{b}$ to the point at infinity $(\infty, 0)$. Hence the $\mathbf{b}$-switch is transformed into the horizontal switch

$$\overline{\iota_{\mathbf{b}}} = \pi \circ \iota_{\mathbf{b}} \circ \pi^{-1} = \iota_1 : (u, v) \to \left(\frac{v^2 + k}{u}, v\right).$$

Moreover, the symmetry switch σ is transformed into the standard symmetry switch $\overline{\sigma} : (u, v) \to (v, u)$, and thus π brings the map κ_1 into symmetric QRT form. $\qquad\square$

Example, 2-dimensional Suslov system. We take $a = \frac{1}{2}$, $b = d = 0$, $c = 1$, and $e = \alpha$. Then $L = \frac{1}{2}x$ and $Q = x^2 + \alpha y^2$,

$$\widetilde{H} = \frac{Q}{T} = \frac{x^2 + \alpha y^2}{1 + \alpha\left(\frac{hx}{2}\right)^2},$$

and

$$\kappa_1(\mathbf{x}) = \left(\frac{x(2 + \alpha hy)}{2 + \alpha h(hx^2 - y)}, \frac{2y - h(2x^2 + \alpha y^2)}{2 + \alpha h(hx^2 - y)}\right).$$

The symmetry switch is $\sigma(\mathbf{x}) = (x, -y)$. Taking $\alpha = -1$ the curves $Q = 0$ and $T = 0$ intersect in four points, namely $(\pm\frac{2}{h}, \pm\frac{2}{h})$. The involution point is $\mathbf{b} = (0, -\frac{2}{h})$, which is not one of the base points. We choose $h = 2$ and have drawn three level sets of the modified Hamiltonian in Figure 1. We have also plotted the points $(\sqrt{2}, 2)$, $(\sqrt{2}, 0)$ and $(2, 2)$, together with their images under the Manin involution

$$\iota_{\mathbf{b}}(\mathbf{x}) = \sigma \circ \kappa_1(\mathbf{x}) = \left(\frac{x(y-1)}{2x^2 - y - 1}, -\frac{2x^2 - y(y+1)}{2x^2 - y - 1} \right).$$

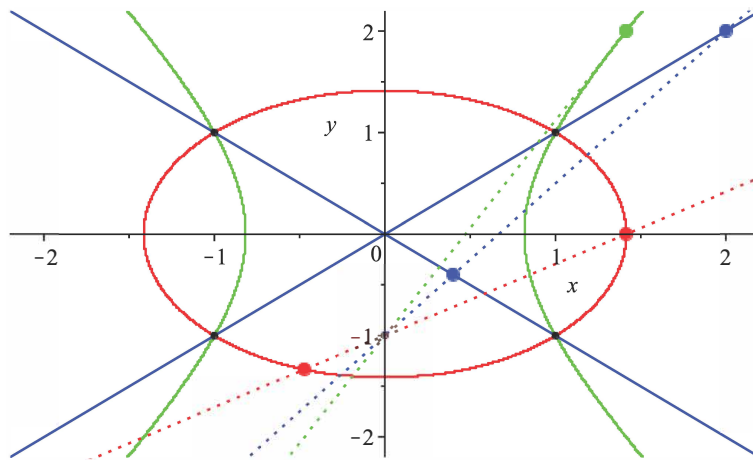Note that the point $(\sqrt{2}, 2)$ is a fixed point of $\iota_{\mathbf{b}}$.

Figure 1: The curves $\widetilde{H} = 2$, $\widetilde{H} = -2$, $\widetilde{H} = 0$, in resp. green, red and blue. Here $h = 2$ and $\alpha = -1$.

5 A cubic Hamiltonian

Next we consider the 2-dimensional ODE (1) where $\varphi(\mathbf{x}) = 1$, and the homogeneous Hamiltonian has the form $H = H(\mathbf{x}) = LQ$. The Kahan map for this system,

$$\kappa_2(\mathbf{x}) = \frac{\mathbf{x} + h\mathcal{J}\nabla H}{R}, \qquad R = R(\mathbf{x}) = 1 + h^2(3DL^2 - G^2), \qquad (11)$$

is measure preserving with density (7), and it preserves the modified Hamiltonian $\widetilde{H} = H/R$ [3, Eq. (4)].

Theorem 2. *The map (11) can be written as a composition $\kappa_2 = \sigma \circ \iota_{\mathbf{b}}$, where σ is given by (5) and $\iota_{\mathbf{b}}$ is the Manin transformation*

$$\iota_{\mathbf{b}} = \mathbf{x} + \left(1 - \frac{1 + 2hG}{R} \right)(\mathbf{b} - \mathbf{x})$$

where $\mathbf{b}$ is given by (8). The projective collineation π given by (9) brings the map (11) in QRT form,

$$\overline{\kappa_2} : (u, v) \rightarrow \left(v, \frac{(u + v)v + 3k}{3u - v} \right),$$

where $k = 4h^2 D$. The QRT-invariant is

$$\overline{H/R} = \frac{(u - v)^2 + k}{(u + v)(2uv + \frac{3}{2}k)}.$$

Proof. The expression for $\iota_\mathbf{b}$ can be obtained from [10, equation (2)], taking $N = 3$ and $F_a = LQ$ and $F_b = R$. Note that here the involution point $\mathbf{b}$ is a base point, of the pencil $\alpha H + \beta R = 0$, as it is the intersection of the lines $L = 0$ and $hG = 1$. $\qquad\square$

Example, tetrahedrally symmetric Nahm equations. We take $a = 0$, $b = \frac{1}{3}$, $c = 3$, $d = 0$, and $e = -1$. Then $H = y(x^2 - \frac{1}{3}y^2)$,

$$\widetilde{H} = \frac{H}{R} = \frac{y(x^2 - \frac{1}{3}y^2)}{1 - h^2(x^2 + y^2)},$$

and

$$\kappa_2(\mathbf{x}) = \left(\frac{x + h^2(x^2 - y^2)}{1 - h^2(x^2 + y^2)}, \frac{y(1 - 2hx)}{1 + h^2(x^2 + y^2))} \right).$$

The symmetry switch is $\sigma(\mathbf{x}) = (-x, y)$. The involution point is $\mathbf{b} = (-1/h, 0)$. Choosing $h = 1$ the curves $H = 0$ and $R = 0$ intersect in six points on the unit circle $(\pm 1, 0)$ and $\frac{1}{2}(\pm 1, \pm\sqrt{3})$. We have drawn three level sets of the modified Hamiltonian in Figure 2, where we have also indicated the images of $(\frac{\sqrt{3}}{6}, -\frac{1}{2} + \sqrt{3})$, $(\frac{1}{2}, -\frac{3}{10})$ and $(1, \frac{3}{2} - \frac{\sqrt{21}}{2})$ under the Manin involution $\iota_\mathbf{b} = \sigma \circ \kappa_2$. Note that the image of the point $(\frac{1}{2}, -\frac{3}{10})$ is $\mathbf{b}$.
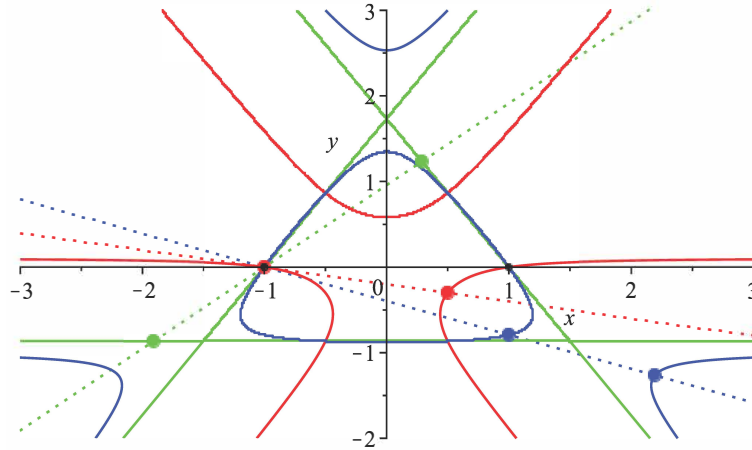

Figure 2: The curves $\widetilde{H} = \frac{\sqrt{3}}{2}$, $\widetilde{H} = -\frac{1}{10}$, $\widetilde{H} = 1$, in resp. green, red and blue. Here $h = 1$.

6   A quartic Hamiltonian

Consider the 2-dimensional ODE (1) where $\varphi(\mathbf{x}) = \frac{1}{L}$, and the homogeneous Hamiltonian has the form $H(\mathbf{x}) = L^2Q$. Then the Kahan map for this system,

$$\kappa_3(\mathbf{x}) = \frac{\mathbf{x} + h(G\mathbf{x} + L^{-1}\mathcal{J}\nabla H)}{(1 - hG)(1 + 2hG) + 4h^2 DL^2} \tag{12}$$

preserves the modified Hamiltonian $\widetilde{H}(\mathbf{x}) = \frac{H}{S}$ with $S = S(\mathbf{x}) = (1 - h^2G^2)(1 + h^2(8DL^2 - G^2))$ and it is measure preserving with density (7), cf. [1, Section 2].

Theorem 3. *The map (12) can be written as a composition $\kappa_3 = \sigma \circ \iota_\mathbf{b}$, where σ is given by (5) and $\iota_\mathbf{b}$ is the Manin involution, with involution point $\mathbf{b}$ given by (8),*

$$\iota_\mathbf{b} = \mathbf{x} + \left(1 - \frac{1 + 3hG}{(1 - hG)(1 + 2hG) + 4h^2DL^2} \right)(\mathbf{b} - \mathbf{x}).$$

The projective collineation π given by (9) brings the map (12) in QRT form,

$$\overline{\kappa_3} : (u, v) \rightarrow \left(v, \frac{uv + 2k}{2u - v} \right),$$

where $k = 4h^2 D$. The QRT-invariant is

$$\overline{H/S} = \frac{(u - v)^2 + k}{4uv(uv + 2k)}.$$

Proof. By direct calculation. The expression for $\iota_\mathbf{b}$ agrees with [10, equation (2)] taking $N = 4$ and $F_a = L^2 Q$ and $F_b = S$. The involution point $\mathbf{b}$ is a double base point, as $\mathbf{b}$ is also on the curve $h^2 E Q = 1$. $\qquad\square$

Example, octahedrally symmetric Nahm equations. Taking $a = -b = d = 1$, $c = 0$ and $e = 3$ yields $H = (x - y)^2(2xy + 3y^2)$,

$$\widetilde{H} = \frac{H}{S} = \frac{(x - y)^2(2xy + 3y^2)}{(1 - h^2(x + 4y)^2)(1 - h^2(8(x - y)^2 + (x + 4y)^2))},$$

and

$$\kappa_3(\mathbf{x}) = \left(\frac{x + h(3x^3 + 4xy - 12y^2)}{1 + h(x + 4y) - 2h^2(3x^2 + 4xy + 18y^2)}, \frac{y(1 - 5hx)}{1 + h(x + 4y) - 2h^2(3x^2 + 4xy + 18y^2)} \right).$$

The symmetry switch is $\sigma(\mathbf{x}) = \frac{1}{5}(3x - 8y, -2x - 3y)$, and $\mathbf{b} = \frac{1}{5h}(1, 1)$. Taking $h = \frac{1}{5}$, the curves $H = 0$ and $S = 0$ intersect in 10 real points,

$$\pm (5, 0), \qquad \pm \left(\frac{5}{3}, 0 \right), \qquad \pm (1, 1), \qquad \pm (3, -2), \qquad \pm \left(1, -\frac{2}{3} \right).$$

We have drawn three level sets of the modified Hamiltonian in Figure 3, as well as the points $(-1, \frac{192}{103})$, $(-3, -\frac{4}{3})$, $(1, -\frac{256}{179})$ and their images under the Manin involution $\iota_\mathbf{b} = \sigma \circ \kappa_3$.
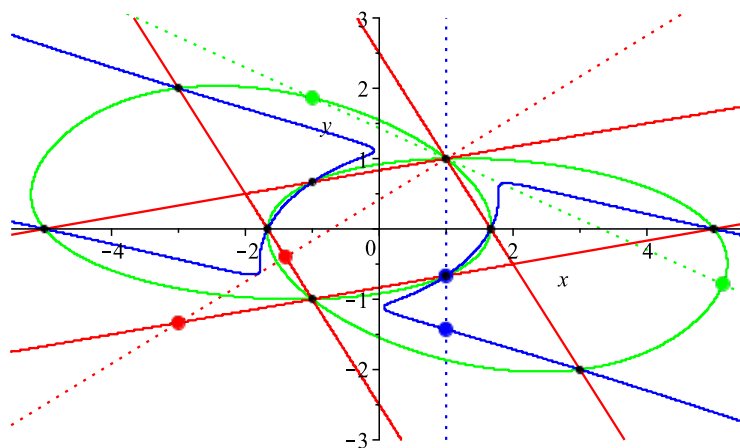

Figure 3: The curves $\widetilde{H} = 25$, $\widetilde{H} = \frac{125}{16}$, $\widetilde{H} = -100$, in resp. green, red and blue. Here $h = \frac{1}{5}$.

7 Summary

We have shown that the Kahan discretisation of the ODE (1) with $\varphi(\mathbf{x}) = L^{2-i}$ and $H(\mathbf{x}) = L^{i-1}Q$ for each $i = 1, 2, 3$ takes the form $\kappa = \sigma \circ \iota_{\mathbf{b}}$ where $\iota_{\mathbf{b}}$ is the $\mathbf{b}$-switch with involution point $\mathbf{b} = \mathbf{v}/(hE)$, and σ is both a linear map and a symmetry of the preserved pencil which has degree $i+1$. Therefore, in each case the Kahan map is the root of the generalised Manin transformation $\tau_{\mathbf{b}} = \iota_{\sigma(\mathbf{b})} \circ \iota_{\mathbf{b}}$. According to [10] a generalised Manin involution $\iota_{\mathbf{p}}$ which preserves a pencil $\alpha F_a(\mathbf{x}) + \beta F_b(\mathbf{x}) = 0$ of degree $2 \leq N \leq 4$ is measure preserving with density L^{N-3}/F_a, where L is any line through $\mathbf{p}$. This implies, as we have $F_a = H$, that the density of the measure preserved by the Kahan map is the same for each i, namely $1/(LQ)$. For each Kahan map we have provided its symmetric QRT form.

Acknowledgment

References

[1] E Celledoni, R I McLachlan, D I McLaren, B Owren, G R W Quispel, *Two classes of quadratic vector fields for which the Kahan discretization is integrable*, MI Lecture Notes **74**: Kyushu University (2017) 60–62.

[2] E Celledoni, R I McLachlan, B Owren and G R W Quispel, *Geometric properties of Kahan's method* J Phys A **46** (2013) 025001.

[3] E Celledoni, R I McLachlan, D I McLaren, B Owren and G R W Quispel, *Integrability properties of Kahan's method,* J Phys A **47** (2014) 365202.

[4] E Celledoni, R I McLachlan, D I McLaren, B Owren and G R W Quispel, *Discretization of polynomial vector fields by polarisation*, Proc Roy Soc A **471** (2015) 20150390, 10pp.

[5] J J Duistermaat, QRT Maps and Elliptic Surfaces, Springer, 2010.

[6] E Elliott, An introduction to the algebra of quantics, Oxford University Press, 1895.

[7] Hitchin, N. J., Manton, N. S., and Murray, M.K., Symmetric Monopoles, Nonlinearity **8** (1995) 661–692.

[8] A Hone and M Petrera, *Three-dimensional discrete systems of Hirota-Kimura type and deformed Lie-Poisson algebras*, Journal of Geometric Mechanics **1** (2009) 55–85.

[9] W Kahan, *Unconventional numerical methods for trajectory calculations*, Unpublished lecture notes, 1993.

[10] P H van der Kamp, D I McLaren and G R W Quispel, *Generalised Manin transformations and QRT maps*, arXiv:1806.05340 [nlin.SI].

[11] K Kimura and R Hirota, *Discretization of the Lagrange top*, J Phys Soc Jap **69** (2000) 3193–3199.

[12] Y I Manin, *The Tate height of points on an Abelian variety*, Izv Akad Nauk SSSR Ser Mat **28** (1964) 1363–1390. English translation in A.M.S. Translations Ser **2** 59 (1966) 82–110.

[13] G R W Quispel, J A G Roberts and C J Thompson, *Integrable mappings and soliton equations*, Phys Lett A **126** (1988) 419–421.

[14] G R W Quispel, J A G Roberts and C J Thompson, *Integrable mappings and soliton equations II*, Physica D: Nonl Phen **34** (1989) 183–192.

[15] M Petrera and Y B Suris, *On the Hamiltonian structure of Hirota-Kimura discretization of the Euler top*, Mathematische Nachrichten **283** (11) (2010) 1654–1663.

[16] M Petrera and Y B Suris, *SV Kovalevskaya system, its generalization and discretization*, Frontiers of Mathematics in China **8** (2012) 1047–1065.

[17] M Petrera and Y B Suris, *Spherical geometry and integrable systems*, Geometriae Dedicata **169** (2014) 83–98.

[18] M Petrera, A Pfadler and Y B Suris, *On integrability of Hirota-Kimura type discretizations*, Regular and Chaotic Dynamics **16** (2011) 245–289.

[19] M Petrera and R Zander, *New classes of quadratic vector fields admitting integral-preserving Kahan-Hirota-Kimura discretizations*, J Phys A: Math Theor **50** (2017) 205203.

[20] G Suslov, Theoretical Mechanics, Moscow-Leningrad: Gostekhizdat, 1946.

[21] T Tsuda, Integrable mappings via rational elliptic surfaces, J Phys A: Math Gen **37** (2004) 2721-2730.