

# Conformality loss and quantum criticality in topological Higgs electrodynamics in 2+1 dimensions

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The electromagnetic response of topological insulators and superconductors is governed by a modified set of Maxwell equations that derive from a topological Chern-Simons (CS) term in the effective Lagrangian with coupling constant  $\kappa$ . Here we consider a topological superconductor or, equivalently, an Abelian Higgs model in 2 + 1 dimensions with a global  $O(2N)$  symmetry in the presence of a CS term, but without a Maxwell term. At large  $\kappa$ , the gauge field decouples from the complex scalar field, leading to a quantum critical behavior in the  $O(2N)$  universality class. When the Higgs field is massive, the universality class is still governed by the  $O(2N)$  fixed point. However, we show that the massless theory belongs to a completely different universality class, exhibiting an exotic critical behavior beyond the Landau-Ginzburg-Wilson paradigm. For finite  $\kappa$  above a certain critical value  $\kappa_c$ , a quantum critical behavior with continuously varying critical exponents arises. However, as a function  $\kappa$  a transition takes place for  $|\kappa| < \kappa_c$  where conformality is lost. Strongly modified scaling relations ensue. For instance, in the case where  $\kappa^2 > \kappa_c^2$ , leading to the existence of a conformal fixed point, critical exponents are a function of  $\kappa$ .

## I. INTRODUCTION

### A. Conformal phase transition

A conformal phase transition (CPT) [1] is defined as featuring a critical point with a non-power law diverging correlation length, and which exhibits a universal jump in some generalized stiffness of the system. The cardinal example of such a transition is the Berezinskii-Kosterlitz-Thouless (BKT) phase transition [2, 3] taking place in two-dimensional superfluids and superconductors when they transition from the low-temperature phase to the normal state, and in the melting transition of two-dimensional crystals [4, 5]. A key point of such phase transitions is the absence of a traditional Landau-type (local) order parameter with which to monitor the transition. In the above examples, the lack of a local order parameter is due to a fundamental theorem by Mermin and Wagner [6], which states that spontaneous breaking of continuous symmetries in two dimensions at any non-zero temperature cannot take place.

The presence of a strongly fluctuating gauge-field puts an even stronger limitation on the existence of a local order parameter than the Mermin-Wagner theorem does. Namely, in *any* gauge theory in *any dimension* such as for instance the Ginzburg-Landau theory of superconductors or, equivalently, the Abelian Higgs model (AHM) in 2+1 dimensions, an order parameter cannot be defined, unless it is gauge-invariant. This result, known as Elitzur's theorem [7] implies that *no local order parameter exists for a superconductor*.

On the other hand, response functions are gauge invariant. They are computed in terms of correlation functions of conserved currents. Response functions should

exhibit universal features at a phase transition, provided the phase transition occurs at a critical point, that is, there exists a diverging length in the problem rendering the system scale-free at the transition. In an ordinary second-order phase transition, with power-law divergence of some correlation length and susceptibilities, the universal aspects are associated with the exponents of the power laws. A much studied quantity both experimentally and theoretically is the current-current correlation function, which features an overall multiplicative constant, the superfluid density of the system. The universal aspect of this quantity at a standard second-order phase transition of a bulk superconductor is the exponent determining how the superfluid density vanishes as  $T \rightarrow T_c$  from below. For thin film superconductors, the universal aspect of the same response function is a universal jump in the superfluid density at the transition, and a concomitant diverging correlation length with an essential singularity [8].

### B. Conformality loss argument

A general framework to derive BKT-like scaling in other theories was provided by Kaplan *et al.* [9] who showed that a CPT can be understood in terms of a conformality loss argument. A more mathematically precise discussion can be found in Ref. [10]. Simply stated, it amounts to considering a renormalization group (RG) flow for a coupling  $g$  depending on some parameter  $\alpha$  such that  $\beta(g; \alpha) \equiv \mu dg/d\mu = \alpha - \alpha_* - (g - g_*)^2$  [9]. For  $\alpha > \alpha_*$  the fixed points  $g_{\pm} = g_* \pm \sqrt{\alpha - \alpha_*}$  are obtained, with  $g_-$  infrared stable (IR) and  $g_+$  ultraviolet (UV) stable. Such a hypothetical RG flow describes a CPT as  $\alpha$  is varied. Indeed, the IR and UV fixed points merge

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when  $\alpha = \alpha_*$ . But for  $\alpha < \alpha_*$  conformality is lost, since  $g_{\pm}$  become complex. The BKT-like scaling follows easily by integrating the RG equation, yielding  $\Lambda_{IR}/\Lambda_{UV} \approx \exp(-\pi/\sqrt{\alpha_* - \alpha})$  when  $|g_{IR,UV} - g_*| \gg \sqrt{|\alpha - \alpha_*|}$ . Recall that in the BKT scaling the inverse correlation length has the form,  $\xi^{-1} \sim \exp(-\text{const}/\sqrt{T - T_c})$  [8], so by comparison the parameter  $\alpha$  plays a role analogous to the inverse temperature in the BKT transition [11]. This situation is also reminiscent of the one occurring in spinor QED in 2+1 dimensions (QED<sub>3</sub>), where a gap generation occurs due to spontaneous chiral symmetry breaking, having the form,  $m_{\text{QED}} \sim \exp(-2\pi/\sqrt{N_c - N})$ , where  $N$  is the number of Dirac fermion species and  $N_c = 32/\pi^2$  [12]. This behavior of QED in 2+1 dimensions has been identified in Ref. [13] as a CPT. In this case the CPT is also a consequence of the non-locality of the Maxwell term at strong coupling, with the conformality lost point of view of merging of fixed points analyzed in Ref. [14]. A similar behavior leading to a CPT is also found in graphene in the presence of Coulomb interactions, where an excitonic gap is generated by spontaneous chiral symmetry breaking [15]. Interestingly, in the case of graphene the CPT occurs as the coupling constant is varied, rather than the number of components [16]. This is due to the fact that in graphene the bare Coulomb interaction is three-dimensional (i.e.,  $\sim 1/r$ ) rather than logarithmic. Also in the context of so called deconfined quantum critical points [17] a CPT may occur in an  $N$ -component AHM in 2+1 dimensions at low  $N$  and in the strongly coupled regime [18]. In this case when the number of components  $N$  is varied below a certain critical value  $N_c$  the fixed points become complex, resulting in conformality loss [18, 19]. Recently this behavior of the AHM has been addressed within the framework of the  $\epsilon$ -expansion up to four loops [20]. In the past this behavior of the AHM was interpreted as a weak first-order phase transition [21]. More recently the weak first-order phase transition in the Potts model with  $Q > 4$  has been also understood in terms of an approximate conformality loss due to its proximity to complex fixed points [22, 23].

### C. Topological Abelian Higgs model

Here we introduce a more subtle type of CPT driven by a *topological* term in the effective action. The main motivation comes from the modification of Maxwell electrodynamics in topological materials [24–26]. For instance, the surface of a topological superconductor corresponds to an AHM in presence of a CS term [26]. In the absence of a Maxwell term, this model has a soliton solution in the form of a self-dual CS vortex [27, 28]. The Lagrangian is simply given by,

$$\mathcal{L} = \frac{\kappa}{2} \epsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda + |(\partial_\mu - i a_\mu)\phi|^2 - m_0^2 |\phi|^2 - \frac{u_0}{2} |\phi|^4, \quad (1)$$

where  $\kappa$  is the Chern-Simons coupling and  $m_0^2$  and  $u_0$  are written with a "0" subscript to emphasize that they represent bare quantities at this stage. Note that  $a^\mu$  is a fluctuating field and not a background (external) gauge potential.

In the limit  $\kappa \rightarrow \infty$  the gauge field is frozen to zero and the theory becomes simply a globally  $U(1)$ -invariant scalar theory, which within an imaginary time formalism governs the universality class of a three-dimensional XY classical ferromagnet, which is the same as the universality class of superfluid Helium in three dimensions [29]. This theory is known to be exactly dual to an AHM without a CS term (note that in this case there is a Maxwell term) [30–33]. For  $\kappa = 1/(2\pi)$ , corresponding to level 1 CS AHM, it has been recently pointed out in several papers that the Lagrangian (1) maps via a bosonization duality to free Dirac fermions in 2+1 dimensions [34–43]. The duality is assumed to be valid also when the fields are massless. Since a genuine duality is supposed to map a strongly coupled theory on one side to a weakly coupled theory on the other side, the statement we just made might at first sight sound confusing, since the theory on one side does not interact. Actually, it is the IR fixed point of the model (1) that it is being mapped to the free fermion model. In this paper we show that the IR behavior underlying the duality transformation is subtle in the massless regime of Eq. (1), as the IR fixed point is dependent on the CS coupling  $\kappa$  and that conformality of the IR fixed point is lost as  $|\kappa|$  is varied below a certain critical value. However, this scaling regime typically corresponds to values of  $\kappa$  larger than the one associated to the level 1 theory. On the other hand, we will show that the massive theory implies a scaling behavior featuring a Wilson-Fisher fixed point for all values of  $\kappa$  and not only  $\kappa = 1/(2\pi)$ . An immediate consequence of this result is that there must be two different paths to constructing the continuum limit of the theory (1) from a lattice model. One example is provided by the bosonization duality derived using Wilson lattice fermions as discussed in Ref. [41].

## II. RENORMALIZATION GROUP FOR THE MASSLESS THEORY

Fixing the Landau gauge, the one-loop Feynman diagram contributing to the effective Higgs  $|\phi|^4$  coupling crucial for our analysis is given in Fig. 1, which is proportional to  $e^4 I(p)$ , where  $I(p)$  is a momentum space integral in Euclidean spacetime (see Appendix A). The wiggles represent photon propagators, while the external legs are Higgs scalars. Here we assume the presence of an evanescent ( $e^2 \rightarrow \infty$ ) Maxwell term as a regulator term in order to allow for a better analysis of the interplay between IR and UV energy scales. In absence of a CS term  $\lim_{|p| \rightarrow 0} I(p)$  is both IR and UV divergent, so usually we compute this diagram in this case assuming a nonzero momentum scale  $|p| = \mu$  [44]. On the other hand, dimen-

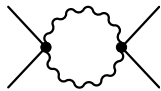


FIG. 1. Photon bubble contributing to the effective Higgs interaction.



FIG. 2. Other diagrams containing photon lines which vanish in the Landau gauge.

sional regularization [45] would in principle imply that  $\lim_{|p| \rightarrow 0} I(p) = 0$ , since such a regularization procedure usually compensates powers of IR and UV cutoffs [44]. Indeed, using cutoffs, we have  $I(0) = 2(\Lambda_{IR}^{-1} - \Lambda_{UV}^{-1})$  when  $\kappa = 0$ , so  $I(0)$  would vanish provided  $\Lambda_{IR} = \Lambda_{UV}$ . In the presence of the CS term, on the other hand, we have that  $I(0)$  vanishes identically at fixed dimension  $d = 2+1$ , being completely insensitive to IR and UV scales. Therefore, we expect that  $g(0)$  and  $g(\mu)$  lead to very different fixed points.

For the case of a CS AHM (1), the integral  $I(p)$  is calculated explicitly in the Appendix A assuming the Landau gauge, and is indeed found to vanish for  $|p| \rightarrow 0$  for all  $\kappa \neq 0$ . More importantly, we have,

$$\lim_{e^2 \rightarrow \infty} e^4 I(p) = -\frac{|p|}{8\kappa^2}. \quad (2)$$

Thus, what in the usual (non-topological) AHM yields a term of order  $e^4$  in a perturbation theory in terms of both  $u$  and  $e^2$ , becomes here a term of order  $1/\kappa^2$ , reflecting a perturbation in powers of  $1/|\kappa|$  instead. Note that the  $e^2 \rightarrow \infty$  limit yields a negative sign. This is to be contrasted to the  $\kappa \rightarrow 0$  limit,

$$\lim_{\kappa \rightarrow 0} e^4 I(p) = \frac{3}{16|p|}, \quad (3)$$

producing a positive sign. This is an important point, as precisely the positive sign of this term prevents the existence of charged fixed points for a small number of Higgs fields in 2+1 dimensions [21].

In order to put in perspective the role of the Maxwell term as a regulator in the AHM, let us assume the Lagrangian as it is given in Eq. (1), i.e., without a Maxwell term, but with  $e \neq 0$ . In this case the propagator in the Landau gauge is given simply by,

$$D_{\mu\nu}(p) = -\frac{1}{e^2\kappa} \frac{\epsilon_{\mu\nu\lambda\rho} p_\lambda}{p^2}, \quad (4)$$

leading to,

$$e^4 I(p) = \frac{2}{\kappa^2} \left[ \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2} - p^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{(p-q)^2 q^2} \right]. \quad (5)$$

The first integral on the right-hand side of the above equation is divergent and needs to be regularized, while the second integral is finite, yielding  $1/(8|p|)$ . We see that when a Maxwell term is present,  $I(p)$  produces a finite result for all  $e^2 \neq 0$ , including the limit case  $e^2 \rightarrow \infty$ . On the other hand, with the propagator (4) we can evaluate the divergent integral in Eq. (5) with a cutoff and absorb the result into the bare coupling  $u_0$ . In this way we obtain the same result as before when the limit  $e^2 \rightarrow \infty$  is taken.

The calculation using the propagator (4) has the advantage of making calculations easier, even though sometimes an explicit UV cutoff is needed. For instance, it is more easily shown that the diagrams of Fig. 2 vanish in the Landau gauge even when a CS term is present. For the usual AHM it is a well known fact that the diagrams of Fig. 2 vanish as  $p \rightarrow 0$  in the Landau gauge [46]. In 2+1 dimensions the same is true using massless scalars at nonzero external momenta. Indeed, dimensional analysis implies that diagrams have a value  $c/|p|$ , where  $c$  is some real constant. Since the diagrams must vanish in the  $|p| \rightarrow 0$  limit, we must necessarily have  $c = 0$ . The same behavior holds true for the model (1).

We may wonder about what happens if a different gauge, say, the Feynman gauge is chosen instead. In this case the calculations are longer, but gauge invariant results should not change. For example, while the wave function renormalization of the scalar field and the renormalization of the Higgs self-coupling are both gauge dependent, the RG  $\beta$  functions are gauge independent (for an example, see Refs. [47] and [48]).

Usually in order to obtain the RG  $\beta$  function we need also the wave function renormalization  $Z_\phi$  for the scalar field. Since  $\kappa$  is scale invariant,  $Z_\phi$  will at the end give no contribution to the one-loop  $\beta$  function. The one-loop self-energy  $\Sigma(p)$  excluding tadpole diagrams satisfies,

$$\lim_{e^2 \rightarrow \infty} \Sigma(p) = -\frac{2p^2}{3\pi|\kappa|}, \quad (6)$$

and has this form even if  $m \neq 0$  (see Appendix A). This implies,

$$Z_\phi = \frac{1}{1 - 2/(3\pi|\kappa|)}. \quad (7)$$

Note that due to the infrared bound [49],  $0 < Z_\phi/p^2 \leq 1/p^2$ , we have necessarily the bound  $|\kappa| > 2/(3\pi)$ . Thus, the limit  $e^2 \rightarrow \infty$  and quantum fluctuations at one-loop prevent us to allow  $\kappa$  to vanish, although this is certainly possible at the classical level.

The dimensionless effective coupling is given by  $g(\mu) = Z_\phi^2 u(\mu)/\mu$ , where  $\mu = |p|$ , and,

$$u(\mu) = u_0 - \left( \frac{N+4}{8} \right) \frac{u_0^2}{\mu} + \frac{\mu}{\kappa^2}, \quad (8)$$

corresponding to the sum of the one-loop diagrams with scalar and and photon bubbles. Thus,

$$g(\mu) = \left( 1 + \frac{4}{3\pi|\kappa|} \right) \frac{u_0}{\mu} - \left( \frac{N+4}{8} \right) \frac{u_0^2}{\mu^2} + \frac{1}{\kappa^2}, \quad (9)$$

where we have assumed that there are  $N$  complex scalar fields and that  $1/\kappa^2$  and  $u_0/\mu$  are of the same order. Therefore, the RG  $\beta$  function,  $\beta(g) = \mu dg/d\mu$  for the dimensionless coupling is given by (recall that  $\kappa$  does not flow [50]),

$$\beta(g) = - \underbrace{\left[ \left(1 + \frac{4}{3\pi|\kappa|}\right) \frac{u_0}{\mu} - \left(\frac{N+4}{8}\right) \frac{u_0^2}{\mu^2} + \frac{1}{\kappa^2} \right]}_{=g} + \left(\frac{N+4}{8}\right) \frac{u_0^2}{\mu^2} + \frac{1}{\kappa^2}. \quad (10)$$

The above equation can be rewritten within the accuracy of the one-loop approximation as,

$$\beta(g) = \frac{g_*}{2} \left[ \frac{\kappa_c^2}{\kappa^2} - 1 + \left(\frac{g}{g_*} - 1\right)^2 \right], \quad (11)$$

where  $g_* = 4/(N+4)$  and  $\kappa_c^2 = 2/g_*$ . The RG  $\beta$  function (11) has precisely the paradigmatic form discussed by Kaplan *et al.* [9] for theories featuring conformality lost. The only difference is that in our case the  $\beta$  function has the opposite sign. Depending on the range of  $\kappa$ , the theory may have a conformal fixed point or not. The  $\beta$  function profile is shown schematically in Fig. 3.

Nontrivial fixed points corresponding to quantum criticality exist whenever  $\kappa^2 \geq \kappa_c^2$ . We note that in contrast to the usual AHM in 2+1 dimensions [21], the CS AHM features a quantum critical point for all  $N$  if the CS coupling satisfies the inequality  $\kappa^2 \geq \kappa_c^2$ . Indeed, this regime features the IR ( $g_+$ ) and UV ( $g_-$ ) stable fixed points,

$$g_{\pm} = g_*(1 \pm \sqrt{1 - \kappa_c^2/\kappa^2}), \quad (12)$$

with the IR stable fixed point corresponding to the quantum critical point of the theory.

In order for perturbation theory to be well controlled we need a parameter to guarantee the smallness of both  $g$  and  $1/\kappa^2$ . As usual, such a smallness is dictated by the fixed point structure of the theory. Since  $g_{\pm} \propto g_*$ , we have that the fixed points are small for a large enough value of  $N$ . Indeed, for  $N$  large  $g_* \sim \mathcal{O}(1/N)$ . Thus, for  $g$  not too far from  $g_{\pm}$ , perturbation theory is well behaved. Similarly, we have that  $1/\kappa^2 \sim \mathcal{O}(1/N)$  for  $\kappa$  near  $\kappa_c$ . We can in principle also speculate that even for  $N = 1$  perturbation theory is well controlled, since we still have  $g_* < 1$ . However, without a careful large order behavior analysis, such a claim remains inconclusive.

Conformality is lost when  $|\kappa| < \kappa_c$ , corresponding to the situation where the above fixed points become complex. The solution of the differential equation (11) for this case is,

$$\ln\left(\frac{\mu}{\Lambda}\right) = \frac{-2}{\sqrt{\kappa_c^2/\kappa^2 - 1}} \left[ \arctan\left(\frac{1 - g/g_*}{\sqrt{\kappa_c^2/\kappa^2 - 1}}\right) + \arctan\left(\frac{g_{\Lambda}/g_* - 1}{\sqrt{\kappa_c^2/\kappa^2 - 1}}\right) \right], \quad (13)$$

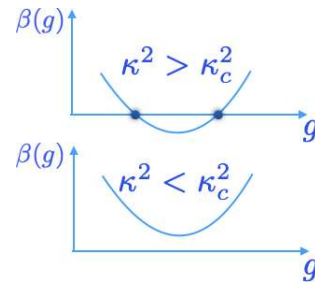


FIG. 3. Schematic behavior of the RG  $\beta$  function (11). Real fixed points exist only for  $\kappa^2 > \kappa_c^2$ , leading to a quantum critical behavior. Conformality is lost for  $|\kappa| < \kappa_c$  and a BKT-like quantum phase transition occurs in the topological superconductor.

where  $g_* < g_{\Lambda} = g(\Lambda)$ . When  $\kappa \rightarrow \kappa_c$  for  $|\kappa| < \kappa_c$  the complex fixed points merge and we obtain that for  $|g_{\Lambda}/g_* - 1| \gg \sqrt{\kappa_c^2/\kappa^2 - 1}$  the momentum scale satisfies,

$$\frac{\mu}{\Lambda} = \exp\left(-\frac{\pi + \theta(\kappa, g)}{\sqrt{\kappa_c^2/\kappa^2 - 1}}\right), \quad (14)$$

where  $\theta(\kappa, g) = 2 \arctan[(1 - g/g_*)/\sqrt{\kappa_c^2/\kappa^2 - 1}]$ , implying a BKT-like scaling when  $g < g_*$ . On the other hand, as  $\kappa \rightarrow \kappa_c$ , we obtain for all  $g > g_*$ ,

$$\frac{\mu}{\Lambda} \underset{\kappa \rightarrow \kappa_c}{=} \exp\left(-\frac{2g_*}{g - g_*}\right), \quad (15)$$

which features an essential singularity at  $g = g_*$ , representing a behavior similar to the one obtained in the case of deconfined quantum critical points [18]. Therefore, when  $\kappa \in [-\kappa_c, \kappa_c]$  Eqs. (14) and (15) imply that only for  $g < g_*$  the system becomes critical as  $\kappa$  approaches  $\kappa_c$ , implying a BKT-like critical point. For  $g > g_*$  the system does not become critical as  $\kappa \rightarrow \kappa_c$ , needing in addition that  $g \rightarrow g_*$ , corresponding to the fixed point in this case.

The solution (13) can also be used for  $\kappa^2 > \kappa_c^2$  after performing a simple manipulation with complex numbers. In this case we obtain a quantum critical scaling behavior,

$$\frac{\mu}{\Lambda} \approx \left(\frac{g_+ - g}{g - g_-}\right)^{1/\sqrt{1 - \kappa_c^2/\kappa^2}}, \quad (16)$$

where we have assumed once more that  $|g_{\Lambda}/g_* - 1| \gg \sqrt{1 - \kappa_c^2/\kappa^2}$ . The above solution makes it apparent that  $g = g_+$  is an IR stable fixed point corresponding to  $\mu/\Lambda \rightarrow 0$ , while  $g_-$  is a UV stable fixed point corresponding to  $\mu/\Lambda \rightarrow \infty$ . As  $\kappa \rightarrow \kappa_c$  leading to a merging of the IR and UV stable fixed points, Eq. (16) becomes Eq. (15) and the system undergoes a CPT.

### III. RENORMALIZATION GROUP FOR THE MASSIVE THEORY

For a nonzero renormalized mass  $m$  the one-loop scalar field bubble diagram yielding the  $u_0^2$  contribution for the vertex function does not diverge for  $\mu = |p| \rightarrow 0$ . Thus, we can use  $m$  as RG scale instead [29, 51]. However, as we have already seen, in this case the photon bubble in Fig. 1 vanishes identically. Furthermore, due to the  $e^2 \rightarrow \infty$  limit the wavefunction renormalization of the Higgs field does not contribute to  $\beta(g) = mdg/dm$ , just as before, since  $\kappa$  itself does not flow. As a result, the critical behavior as  $m \rightarrow 0$  is governed by the Wilson-Fisher fixed point. We conclude therefore that in the critical behavior of the topological AHM the limits  $\mu \rightarrow 0$  and  $m \rightarrow 0$  do not commute. Thus, the critical behavior of the massive theory does not feature conformality lost as  $\kappa$  is varied. This lack of commutativity in the scaling behavior is a unique feature of interacting CS field theories. Such a behavior is more explicit in the CS term itself, when the latter is generated by quantum fluctuations after integrating out Dirac fields in 2+1 dimensions [52–55]. In that case the odd parity contribution to the vacuum polarization yielding the CS term is only nonzero if the Dirac fermion is massive. However, after performing the loop integral the end result depends only on the sign of the mass  $M$  of the Dirac field, corresponding to a CS coupling  $\kappa = (4\pi)^{-1}M/|M|$  [53–55]. Thus, the CS term survives the  $M \rightarrow 0$  limit *after* the quantum fluctuations are calculated. Unlike the fermionic case, the mass  $m$  of the Higgs field has nothing to do with the presence or absence of a CS term in the Lagrangian. Nevertheless, a nonzero  $m$  has an indirect relation to the scaling behavior in an AHM with a CS term, due to the vanishing of the diagram Fig. 1 as  $|p| \rightarrow 0$ .

### IV. SUPERFLUID STIFFNESS

On the basis of the above we now provide a concrete and in principle testable prediction on the behavior of the superfluid stiffness in the topological Higgs superconductor. The superfluid stiffness  $\rho_s$  is a response function given quite generally by the current correlation function at zero momentum [56]. Its scaling behavior is given Josephson scaling relation,  $\rho_s \sim (T_c - T)^{\nu(D-2)}$  [57], where  $D$  is the dimension of space. In the two-dimensional case, the result of Ref. 57 immediately implies a jump as  $T_c$  is approached from the left, since the superfluid stiffness must vanish for  $T > T_c$ . A hallmark of the transition is that this jump is universal [8].

In the present case it is not the dimension of space that is relevant in the scaling of the stiffness, but rather the dimension of spacetime,  $d = 2+1$ , which in the context of quantum critical phenomena can be regarded as a theory with dynamical exponent  $z = 1$  [58], corresponding to a Lorentz-invariant system. The role of the temperature is played by the bare mass squared,  $m_0^2$ , which in the mass-

less case is tuned to a critical value  $m_{0c}^2$ . In the massless case we obtain that the critical exponent  $\nu$  is defined as usual via insertions of the operator  $|\phi|^2$ , whose anomalous dimension is  $3 - 1/\nu$  [29]. Clearly, this exponent is only defined for  $\kappa^2 \geq \kappa_c^2$ , with the result,

$$\nu(\kappa, N) = \frac{2}{4 - \left(\frac{N+1}{N+4}\right) (1 + \sqrt{1 - \kappa_c^2/\kappa^2})}. \quad (17)$$

We see that in this case the critical exponent is not a number, but a function of the CS coupling  $\kappa$ . Note that for  $\kappa \rightarrow \infty$  it agrees with the one-loop result for a  $O(2N)$  classical Heisenberg model, as expected, since for  $\kappa \rightarrow \infty$  the gauge field and the scalar field decouple. For the  $U(1)$  symmetric case ( $N = 1$ ) we obtain  $\nu = 5/9$  at the critical value  $\kappa = \kappa_c$ .

Continuously varying critical exponents is a well known feature of some CS theories. A closely related model where this occurs is the  $CP^{N-1}$  model with a CS term, which has been studied in detail for large  $N$  [59]. Note that in the model we have considered the massless regime does not smoothly connect to larger values of  $N$ , since in the large  $N$  limit  $\kappa_c^2$  becomes large and conformality is lost. The large  $N$  results of Ref. [59] were obtained in the massive regime implied by the  $CP^{N-1}$  constraint. We have seen that in the massive regime the model flows to a conformal fixed point.

When a CPT occurs the critical behavior of the stiffness is highly unusual due to the BKT scaling (14). Because the theory is 2 + 1-dimensional, the argument of the BKT universal jump in the superfluid stiffness is not exactly the same as in the case of a BKT transition [8]. In fact, the stiffness must scale as in Eq. (14), meaning that it vanishes continuously as  $\kappa \rightarrow \kappa_c$  if  $g < g_*$ . However, a jump would occur for  $g > g_*$ , since for  $\kappa \rightarrow \kappa_c$  Eq. (15) holds.

### V. DISCUSSION

Before concluding, it is worth putting into proper context the results we have found so far, especially due to the interest of the model within the framework of the so called "duality web" [34]. While the RG result for massive scalars seems to be largely consistent with the bosonization duality scenario, the RG result for the massless case does not give conclusive answers, as the case  $N = 1$  is not fully controlled perturbatively. In fact,  $\kappa = 1/(2\pi)$  is smaller than  $\kappa_c = \sqrt{(N+4)}/2$  for  $N = 1$ . Thus, even if we assume the validity of perturbation theory down to the case of a single scalar field, the level 1 CS theory would be in the regime of loss of conformality. On the other hand, the RG analysis raises interesting questions for massless scalars in the large  $N$  limit. First, note that the critical behavior clearly does not correspond to a Wilson-Fisher fixed point, although it is close to it for large  $N$ . This is in stark contrast to the massive case, where the Wilson-Fisher fixed point governs the critical behavior with  $\kappa$  having an arbitrary value.

There are several details about the duality scenario that further complicates the analysis. Although this is beyond the scope of the present paper, let us mention some of the issues. First, it must be noted that  $N > 1$  scalars necessarily implies a number of fermionic fields  $N_f \neq 1$  and eventually a non-Abelian structure for the CS terms on each side of the duality [37–39]. In this case, the phase structure may exhibit a confining phase depending on the value of  $N_f$ . The second point, related to the first, is that we are attempting to draw a comparison to the simpler  $N_f = N = 2\pi\kappa = 1$  case [34], based on an RG analysis of a theory having a global  $O(2N)$  symmetry and a local  $U(1)$  one. Here, we must note that a standard boson-boson duality of the type employed in lattice gauge theories is not known for the  $O(2N)$  group. However, we are seeking a duality mapping fixed points theories, especially a situation where one of the sides of the duality is a free fermion theory. The best scenario would be to find a tractable interacting fermion theory whose fixed point and current correlations match the corresponding ones of the boson theory.

## VI. CONCLUSIONS

We have shown that AHM with a CS term exhibits a much more peculiar quantum critical scaling behavior than has been realized previously. We have seen that the massless theory exhibits quantum critical behavior with power law scaling of physical quantities only for a CS coupling  $\kappa$  above a certain critical value  $\kappa_c$ . Although the critical behavior is governed by an IR stable fixed point leading to power law behavior, quantum criticality is highly unconventional, since critical exponents are a function of  $\kappa$ . For a CS coupling below the critical value  $\kappa_c$  there is a phase transition to a state featuring complex fixed points and conformality lost. The RG scale exhibits a BKT-like scaling in this case. On the other hand, if the model is massive and the critical point is approached by sending the mass to zero, conventional critical behavior with a Wilson-Fisher fixed point is obtained. Thus, the two limits of vanishing mass and momenta do not commute, leading to radically distinct forms of quantum criticality.

The results we have obtained are relevant in light of recently well studied bosonization duality in 2 + 1 dimensions [34–43]. While this bosonization duality seems to be well established in the massive case, it remains a conjecture in the massless case. The unconventional criticality of the massless case shows that in the conformality lost regime, a duality to free massless Dirac fermions is unlikely, as the bosonic theory features complex fixed points. However, as mentioned in Sect. V, our RG analysis of the massless case is essentially valid at large  $N$ , in a regime where the duality is anyway more complex [38].

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## Appendix A: Calculation of integrals

### 1. Calculation of $I(p)$

For the sake of convenience, here we will set  $M = e^2|\kappa|$ . The Feynman diagram from Fig. 1 is proportional to the integral,

$$I(p) = \int \frac{d^3k}{(2\pi)^3} D_{\mu\nu}(p-k) D_{\mu\nu}(k), \quad (\text{A1})$$

where,

$$D_{\mu\nu}(p) = \frac{1}{p^2 + M^2} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - \frac{M}{p^2} \epsilon_{\mu\nu\lambda p\lambda} \right), \quad (\text{A2})$$

is the propagator in the Landau gauge.

After performing the straightforward indices contraction in the integral (A1), we obtain,

$$I(p) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{[(p-q)^2 + M^2](q^2 + M^2)} \times \left\{ 1 + \frac{[q \cdot (p-q)]^2}{(p-q)^2 q^2} + \frac{2M^2 q \cdot (p-q)}{q^2 (p-q)^2} \right\}. \quad (\text{A3})$$

In what follows we use a series of simple algebraic manipulations to reduce  $I$  to a combination of integrals,

$$I_0 = \int \frac{d^3q}{(2\pi)^3} \frac{1}{(p-q)^2 q^2}, \quad (\text{A4})$$

$$\begin{aligned} I_1 &= \int \frac{d^3q}{(2\pi)^3} \frac{1}{(p-q)^2 (q^2 + M^2)} \\ &= \int \frac{d^3q}{(2\pi)^3} \frac{1}{[(p-q)^2 + M^2] q^2}, \end{aligned} \quad (\text{A5})$$

$$I_2 = \int \frac{d^3q}{(2\pi)^3} \frac{1}{[(p-q)^2 + M^2](q^2 + M^2)}, \quad (\text{A6})$$

which can be solved using the method of Feynman parameters [45] in a standard way to give,

$$I_0 = \frac{1}{8|p|}, \quad (\text{A7})$$

$$I_1 = \frac{1}{4\pi|p|} \arctan\left(\frac{|p|}{|M|}\right), \quad (\text{A8})$$

$$I_2 = \frac{1}{4\pi|p|} \arctan\left(\frac{|p|}{2|M|}\right). \quad (\text{A9})$$

The reduction of  $I(p)$  to a combination of the above integrals is achieved by means of simple algebraic ticks, for instance, by using repeatedly relations like,

$$\begin{aligned} q \cdot (p - q) &= \frac{p^2 - q^2 - (p - q)^2}{2} \\ &= M^2 + \frac{p^2 - (q^2 + M^2) - [(p - q)^2 + M^2]}{2}, \end{aligned} \quad (\text{A10})$$

and,

$$\frac{1}{q^2(q^2 + M^2)} = \frac{1}{M^2} \left( \frac{1}{q^2} - \frac{1}{q^2 + M^2} \right), \quad (\text{A11})$$

in which case we obtain,

$$\begin{aligned} e^4 I(p) &= e^4 \left[ \frac{1}{e^2|\kappa|} + \frac{|p|}{8e^4\kappa^2} \left( \frac{p^2}{4e^4\kappa^2} - 1 \right) \right. \\ &\quad + \frac{1}{4\pi|p|} \left( -\frac{5}{2} + \frac{p^2}{e^4\kappa^2} - \frac{p^4}{2e^8\kappa^4} \right) \arctan\left(\frac{|p|}{e^2|\kappa|}\right) \\ &\quad \left. + \frac{p^4 + 16e^8\kappa^4}{16\pi e^8\kappa^4|p|} \arctan\left(\frac{|p|}{2e^2|\kappa|}\right) \right], \end{aligned} \quad (\text{A12})$$

It is easily obtained that,

$$\lim_{|p| \rightarrow 0} I(p) = 0, \quad (\text{A13})$$

for all  $\kappa \neq 0$ . On the other hand, we have,

$$\lim_{\kappa \rightarrow 0} I(p) = \frac{3}{8|p|}. \quad (\text{A14})$$

Furthermore, we have,

$$\lim_{e^2 \rightarrow \infty} e^4 I(p) = -\frac{|p|}{8\kappa^2}, \quad (\text{A15})$$

corresponding to Eq. (2).

## 2. Calculation of $\Sigma(p)$

The self-energy  $\Sigma(p)$ , excluding tadpole diagrams, is given by,

$$\begin{aligned} \Sigma(p) &= -e^2 \int \frac{d^3q}{(2\pi)^3} \frac{(2p_\mu - q_\mu)(2p_\nu - q_\nu)}{[(p - q)^2 + m^2](q^2 + e^4\kappa^2)} \\ &\quad \times \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} - \frac{e^2 \kappa \epsilon_{\mu\nu\lambda} q_\lambda}{q^2} \right) \\ &= 4e^2 \int \frac{d^3q}{(2\pi)^3} \frac{(q \cdot p)^2 / q^2 - p^2}{[(p - q)^2 + m^2](q^2 + e^4\kappa^2)} \end{aligned} \quad (\text{A16})$$

and involves two integrals, namely,

$$J_1 = \int \frac{d^3q}{(2\pi)^3} \frac{1}{[(p - q)^2 + m^2](q^2 + M^2)}, \quad (\text{A17})$$

and,

$$J_2 = \int \frac{d^3q}{(2\pi)^3} \frac{(2p \cdot q)^2}{[(p - q)^2 + m^2](q^2 + M^2)q^2}. \quad (\text{A18})$$

The first integral is easily calculated with the method of Feynman parameters [45], yielding,

$$\begin{aligned} J_1 &= \frac{1}{8\pi|p|} \left[ \arctan\left(\frac{p^2 + m^2 - M^2}{2|M||p|}\right) \right. \\ &\quad \left. + \arctan\left(\frac{p^2 + M^2 - m^2}{2|m||p|}\right) \right]. \end{aligned} \quad (\text{A19})$$

The calculation of  $J_2$  takes more time, but it is also straightforward. First we rewrite it as,

$$\begin{aligned} J_2 &= \int \frac{d^3q}{(2\pi)^3} \frac{(2p \cdot q)[p^2 + q^2 - (p - q)^2]}{[(p - q)^2 + m^2](q^2 + M^2)q^2} \\ &= \frac{2(p^2 + m^2)}{M^2} \int \frac{d^3q}{(2\pi)^3} \frac{q \cdot p}{[(p - q)^2 + m^2]q^2} \\ &\quad - \frac{2(p^2 + m^2 - M^2)}{M^2} \int \frac{d^3q}{(2\pi)^3} \frac{q \cdot p}{[(p - q)^2 + m^2](q^2 + M^2)}. \end{aligned} \quad (\text{A20})$$

Now the fastest way to proceed is to use the method of Feynman parameters once more to obtain,

$$\begin{aligned} &\int \frac{d^3q}{(2\pi)^3} \frac{q_\mu}{[(p - q)^2 + m^2](q^2 + M^2)} \\ &= \frac{p_\mu}{8\pi} \int_0^1 \frac{d\alpha \alpha}{\alpha(1 - \alpha)p^2 + \alpha m^2 + (1 - \alpha)M^2} \\ &= \frac{p_\mu}{16\pi|p|^3} \left\{ 2|p|(|M| - |m|) + (p^2 + m^2 - M^2) \right. \\ &\quad \times \left[ \arctan\left(\frac{p^2 + m^2 - M^2}{2|M||p|}\right) \right. \\ &\quad \left. \left. + \arctan\left(\frac{p^2 + M^2 - m^2}{2|m||p|}\right) \right] \right\}. \end{aligned} \quad (\text{A21})$$

An integral corresponding to the limit  $|M| \rightarrow 0$  of the above result is also needed in the expression for  $J_2$ . After carrying out some straightforward simplifications, we obtain,

$$\begin{aligned} \Sigma(p) &= \frac{e^2}{4\pi} \left\{ e^2|\kappa| - |m| - \frac{p^2 + m^2}{e^2|\kappa|} \right. \\ &\quad + \frac{(p^2 + m^2)^2}{2e^4\kappa^2|p|} \left[ \frac{\pi}{2} + \arctan\left(\frac{p^2 - m^2}{2|m||p|}\right) \right] \\ &\quad - \frac{(p^2 + m^2 - e^4\kappa^2)^2 + 4e^4\kappa^2 p^2}{2e^4\kappa^2|p|} \\ &\quad \times \left[ \arctan\left(\frac{p^2 + e^4\kappa^2 - m^2}{2|m||p|}\right) \right. \\ &\quad \left. \left. + \arctan\left(\frac{p^2 + m^2 - e^4\kappa^2}{2e^2|\kappa||p|}\right) \right] \right\}. \end{aligned} \quad (\text{A22})$$

The wavefunction renormalization is obtained by expanding  $\Sigma(p)$  up to  $p^2$ ,

$$\Sigma(p) = -\frac{2e^2}{3\pi} \frac{p^2}{m + e^2|\kappa|} + \mathcal{O}(p^4), \quad (\text{A23})$$

and we see that,

$$\lim_{m \rightarrow 0} \frac{\partial \Sigma}{\partial p^2} \Big|_{p=0} = \lim_{e^2 \rightarrow \infty} \frac{\partial \Sigma}{\partial p^2} \Big|_{p=0} = -\frac{2}{3\pi|\kappa|}, \quad (\text{A24})$$

as asserted in the main text. Furthermore, we note that,

$$\lim_{e^2 \rightarrow \infty} \Sigma(p) = -\frac{2p^2}{3\pi|\kappa|}. \quad (\text{A25})$$

## Appendix B: Renormalized mass

For completeness we give here the expression for the renormalized mass, which in the main text is assumed to vanish,

$$\begin{aligned} m^2 &= m_0^2 + (N+1)u_0 \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2 + m^2} \\ &+ 2e^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2 + e^4\kappa^2} \\ &= m_0^2 + \frac{(N+1)u_0 + 2e^2}{2\pi^2} \Lambda \\ &- \frac{(N+1)u_0|m| + 2e^4|\kappa|}{4\pi}, \end{aligned} \quad (\text{B1})$$

where  $\Lambda = \Lambda_{UV}$  and  $N$  is the number of complex scalars. Thus, we see that the bare mass  $m_0$  has to be chosen in such a way as to have a finite renormalized mass as  $e^2 \rightarrow \infty$ . We might be worry that this is a somewhat artificial fine-tuning. However, we should note that  $e^2$  actually behaves as a UV cutoff scale and can be considered as such.

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- [1] V. A. Miransky and K. Yamawaki, Conformal phase transition in gauge theories, *Phys. Rev. D* **55**, 5051 (1997).
- [2] V. Berezinskii, Destruction of long-range order in one-dimensional and two-dimensional systems having a continuous symmetry group i. classical systems, *Sov. Phys. JETP* **32**, 493 (1971).
- [3] J. M. Kosterlitz and D. J. Thouless, Ordering, metastability and phase transitions in two-dimensional systems, *Journal of Physics C: Solid State Physics* **6**, 1181 (1973).
- [4] H. Kleinert, *Gauge Fields in Condensed Matter: Vol. 1: Superflow and Vortex Lines (Disorder Fields, Phase Transitions) Vol. 2: Stresses and Defects (Differential Geometry, Crystal Melting)* (World Scientific, 1989).
- [5] D. R. Nelson, *Defects and geometry in condensed matter physics* (Cambridge University Press, 2002).
- [6] N. D. Mermin and H. Wagner, Absence of ferromagnetism or antiferromagnetism in one- or two-dimensional isotropic heisenberg models, *Phys. Rev. Lett.* **17**, 1133 (1966).
- [7] S. Elitzur, Impossibility of spontaneously breaking local symmetries, *Phys. Rev. D* **12**, 3978 (1975).
- [8] D. R. Nelson and J. M. Kosterlitz, Universal jump in the superfluid density of two-dimensional superfluids, *Phys. Rev. Lett.* **39**, 1201 (1977).
- [9] D. B. Kaplan, J.-W. Lee, D. T. Son, and M. A. Stephanov, Conformality lost, *Phys. Rev. D* **80**, 125005 (2009).
- [10] V. Gorbenko, S. Rychkov, and B. Zan, Walking, weak first-order transitions, and complex cfts, *Journal of High Energy Physics* **2018**, 108 (2018).
- [11] It is important to emphasize, however, that technically there are important distinctions between the BKT transition and conformality loss, as discussed by Gorbenko *et al.* [10].
- [12] T. Appelquist, D. Nash, and L. C. R. Wijewardhana, Critical behavior in (2+1)-dimensional qed, *Phys. Rev. Lett.* **60**, 2575 (1988).
- [13] V. P. Gusynin, V. A. Miransky, and A. V. Shpagin, Effective action and conformal phase transition in three-dimensional qed, *Phys. Rev. D* **58**, 085023 (1998).
- [14] I. F. Herbut, Chiral symmetry breaking in three-dimensional quantum electrodynamics as fixed point annihilation, *Phys. Rev. D* **94**, 025036 (2016).
- [15] D. V. Khveshchenko, Ghost excitonic insulator transition in layered graphite, *Phys. Rev. Lett.* **87**, 246802 (2001).
- [16] E. V. Gorbar, V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Magnetic field driven metal-insulator phase transition in planar systems, *Phys. Rev. B* **66**, 045108 (2002).
- [17] T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P. Fisher, Deconfined quantum critical points, *Science* **303**, 1490 (2004).
- [18] F. S. Nogueira and A. Sudbø, Deconfined quantum criticality and conformal phase transition in two-dimensional antiferromagnets, *EPL (Europhysics Letters)* **104**, 56004 (2013).
- [19] S. Benvenuti and H. Khachatryan, Qed's in 2+1 dimensions: complex fixed points and dualities, arXiv preprint arXiv:1812.01544 (2018).
- [20] B. Ihrig, N. Zerf, P. Marquard, I. F. Herbut, and M. M. Scherer, Abelian higgs model at four loops, fixed-point collision and deconfined criticality, arXiv preprint arXiv:1907.08140 (2019).
- [21] B. I. Halperin, T. C. Lubensky, and S.-k. Ma, First-order phase transitions in superconductors and smectic-*a* liquid crystals, *Phys. Rev. Lett.* **32**, 292 (1974).
- [22] H. Ma and Y.-C. He, Shadow of complex fixed point: Approximate conformality of  $q > 4$  potts model, *Phys. Rev. B* **99**, 195130 (2019).



- [23] V. Gorbenko, S. Rychkov, and B. Zan, Walking, Weak first-order transitions, and Complex CFTs II. Two-dimensional Potts model at  $Q > 4$ , *SciPost Phys.* **5**, 50 (2018).
- [24] X. G. Wen and Q. Niu, Ground-state degeneracy of the fractional quantum hall states in the presence of a random potential and on high-genus riemann surfaces, *Phys. Rev. B* **41**, 9377 (1990).
- [25] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Topological field theory of time-reversal invariant insulators, *Phys. Rev. B* **78**, 195424 (2008).
- [26] X.-L. Qi, E. Witten, and S.-C. Zhang, Axion topological field theory of topological superconductors, *Phys. Rev. B* **87**, 134519 (2013).
- [27] J. Fröhlich and P. Marchetti, Quantum field theories of vortices and anyons, *Communications in Mathematical Physics* **121**, 177 (1989).
- [28] R. Jackiw and E. J. Weinberg, Self-dual chern-simons vortices, *Phys. Rev. Lett.* **64**, 2234 (1990).
- [29] J. Zinn-Justin, *Quantum field theory and critical phenomena* (Clarendon Press, 1996).
- [30] M. E. Peskin, Mandelstam-'t hooft duality in abelian lattice models, *Annals of Physics* **113**, 122 (1978).
- [31] P. R. Thomas and M. Stone, Nature of the phase transition in a non-linear  $o(2)3$  model, *Nuclear Physics B* **144**, 513 (1978).
- [32] C. Dasgupta and B. I. Halperin, Phase transition in a lattice model of superconductivity, *Phys. Rev. Lett.* **47**, 1556 (1981).
- [33] H. Kleinert, Disorder version of the abelian higgs model and the order of the superconductive phase transition, *Lettre al Nuovo Cimento* (1971-1985) **35**, 405 (1982).
- [34] N. Seiberg, T. Senthil, C. Wang, and E. Witten, A duality web in 2+1 dimensions and condensed matter physics, *Annals of Physics* **374**, 395 (2016).
- [35] A. Karch and D. Tong, Particle-vortex duality from 3d bosonization, *Phys. Rev. X* **6**, 031043 (2016).
- [36] D. F. Mross, J. Alicea, and O. I. Motrunich, Symmetry and duality in bosonization of two-dimensional dirac fermions, *Phys. Rev. X* **7**, 041016 (2017).
- [37] P.-S. Hsin and N. Seiberg, Level/rank duality and chern-simons-matter theories, *Journal of High Energy Physics* **2016**, 95 (2016).
- [38] Z. Komargodski and N. Seiberg, A symmetry breaking scenario for qcd 3, *Journal of High Energy Physics* **2018**, 109 (2018).
- [39] O. Aharony, F. Benini, P.-S. Hsin, and N. Seiberg, Chern-simons-matter dualities with so and usp gauge groups, *Journal of High Energy Physics* **2017**, 72 (2017).
- [40] F. Benini, Three-dimensional dualities with bosons and fermions, *Journal of High Energy Physics* **2018**, 68 (2018).
- [41] J.-Y. Chen, J. H. Son, C. Wang, and S. Raghu, Exact boson-fermion duality on a 3d euclidean lattice, *Phys. Rev. Lett.* **120**, 016602 (2018).
- [42] Y. Ferreiros and E. Fradkin, Bosonfermion duality in a gravitational background, *Annals of Physics* **399**, 1 (2018).
- [43] H. Nastase and C. Nez, Deriving three-dimensional bosonization and the duality web, *Physics Letters B* **776**, 145 (2018).
- [44] J. C. Collins, *Renormalization: an introduction to renormalization, the renormalization group and the operator-product expansion* (Cambridge university press, 1985).
- [45] C. Itzykson and J.-B. Zuber, *Quantum field theory*, International series in pure and applied physics (McGraw-Hill International Book Co, 1987).
- [46] S. Coleman and E. Weinberg, Radiative corrections as the origin of spontaneous symmetry breaking, *Phys. Rev. D* **7**, 1888 (1973).
- [47] J. S. Kang, Gauge invariance of the scalar-vector mass ratio in the coleman-weinberg model, *Phys. Rev. D* **10**, 3455 (1974).
- [48] A. M. Schakel, Boulevard of broken symmetries, arXiv preprint cond-mat/9805152 (1998).
- [49] J. Froelich, B. Simon, and T. Spencer, Infrared bounds, phase transitions and continuous symmetry breaking, *Commun. Math. Phys* **50**, 79 (1976).
- [50] S. Coleman and B. Hill, No more corrections to the topological mass term in qcd3, *Physics Letters B* **159**, 184 (1985).
- [51] G. Parisi, Field-theoretic approach to second-order phase transitions in two-and three-dimensional systems, *Journal of statistical physics* **23**, 49 (1980).
- [52] S. Deser, R. Jackiw, and S. Templeton, Topologically massive gauge theories, *Annals of Physics* **140**, 372 (1982).
- [53] A. N. Redlich, Parity violation and gauge noninvariance of the effective gauge field action in three dimensions, *Phys. Rev. D* **29**, 2366 (1984).
- [54] G. W. Semenoff and L. C. R. Wijewardhana, Dynamical mass generation in 3d four-fermion theory, *Phys. Rev. Lett.* **63**, 2633 (1989).
- [55] F. S. Nogueira and I. Eremin, Semimetal-insulator transition on the surface of a topological insulator with in-plane magnetization, *Phys. Rev. B* **88**, 085126 (2013).
- [56] P. B. Weichman, Crossover scaling in a dilute bose superfluid near zero temperature, *Phys. Rev. B* **38**, 8739 (1988).
- [57] B. Josephson, Relation between the superfluid density and order parameter for superfluid he near tc, *Physics Letters* **21**, 608 (1966).
- [58] S. S., *Quantum Phase Transitions*, 2nd ed. (Cambridge University Press, 2011).
- [59] G. Ferretti and S. G. Rajeev, cpn-1 model with a chern-simons term, *Modern Physics Letters A* **07**, 2087 (1992).