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# Distributed Tracking via Simultaneous Perturbation Stochastic Approximation-based Consensus Algorithm

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*Abstract*— **Networked systems comprised of multiple nodes with sensing, processing, and communication capabilities are able to provide more accurate estimates of some state of a dynamic process through communication between the network nodes. This paper considers the distributed estimation or tracking problem and focuses on a class of consensus normalized algorithms. A distributed algorithm consisting of two well-studied parts, namely, Simultaneous Perturbation Stochastic Approximation (SPSA) and the consensus approach is proposed for networked systems with uncertainties. Such combination allows us to relax the assumption regarding the strong convexity of the minimized mean-risk functional, which may not be fulfilled in the distributed optimization problems. For the proposed algorithm we get a mean squared upper bound of residual between estimates and unknown states. The theoretically established properties of proposed algorithm are illustrated by simulation results.**

## I. INTRODUCTION

Current research directions related to parameter estimation problems are motivated by the ubiquity of networked systems. The need to control the behavior of such systems for real-world applications leads to the active development of various distributed algorithms [1]. The distributed problem domain, networked system constraints and uncertainties pose new challenges stimulating researchers to come up with the new theory or improve the existing one.

Stochastic optimization is commonly used to solve the problems involving different kinds of uncertainties, e.g., noisy measurements, external disturbances. Methods of this class include stochastic approximation [2], finite-difference stochastic approximation [3], *simultaneous perturbation stochastic approximation* (SPSA) [4] or randomized stochastic approximation [5]. In gradient-free conventional stochastic approximation algorithms, two measurements are used to approximate each component of the vector-gradient of the cost function (implying $2d$ measurements for $d$-dimension state space). SPSA can be used to solve optimization problems in the case when it is difficult or impossible to obtain a gradient of the loss function with respect to the parameters being optimized. In multidimensional case ($d >> 1$), SPSA requires only two measurements of a loss function on each iteration. In this algorithm, a current estimate is changed along randomly chosen direction $\Delta$ consisting of Bernoulli distributed components.

Traditionally, the decreasing to zero step-size is used in the stochastic approximation algorithms. A sufficiently small constant step-size is often used in the case of the non-stationary loss function minimization [6], [7], e.g. for the *tracking* of unknown system states or changes in the system parameters because they may vary over time. Stochastic approximation algorithms are used for tracking with a constant step-size in [8]–[11]. In [8], [11] the SPSA-like algorithms are considered for the case of the non-constrained optimization in the context of the minimum tracking problem. In other case the stochastic approximation method with a constant step-size is used in [12] to achieve the approximate mean-squared consensus in multi-agent systems operating under noisy conditions.

One of the main restrictions of SPSA-like algorithms is the assumption regarding the strong convexity of the minimized mean-risk functional. More recently, the research has moved to the combination of optimization or estimation methods and a consensus approach which is broadly used in networked systems [13]–[15]. This approach aims to find an agreement between all agents of a group to a common value across a networked system using only local information and communicating among neighboring agents. In [16], the authors presented the gossip optimization algorithm that minimizes a sum of functions when each component function is only known to a specific node in a networked system and utilizes the information exchange between nodes. The cyclic approach or a block scheme are natural extensions. A block stochastic gradient method that benefits of both stochastic gradient approximation and block-coordinate updates was proposed in [17]. Similarly, such approach was used in [18]. Cyclic SPSA algorithm was studied in several works under different problem settings [11], [19], [20]. In this paper, we have relaxed the mentioned above strong convexity assumption by combining SPSA with the consensus Local Voting Protocol from [12], [21]. The idea of the new algorithm is similar to joining Least Mean Squares with consensus algorithm in [22].

The rest of this paper is organized as follows. Section II provides notations used in the paper. The formal problem setting of a non-constrained time-varying mean-risk optimization is given in Section III. The SPSA-based consensus

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algorithm for tracking is introduced in Section IV. The assumptions and main result concerning stability properties of the proposed algorithm are given in Section V. In Section VI, we consider a numerical example and show the simulation results. Section VII concludes the paper.

## II. MATHEMATICAL PRELIMINARIES

In subsequent sections, we use the following notations.

Consider a dynamic network system of $n$ agents, which collaborate among themselves. Without loss of generality, agents in the network system are numbered. Let $\mathcal{N} = \{1, \ldots, n\}$ be the set of agents, and $i \in \mathcal{N}$ be the number of an agent. $\forall i \in \mathcal{N}$ let $\mathcal{N}_t^i$ be a subset of all agents: $\mathcal{N}_t^i \subset \mathcal{N}$, which are able to send information to agent $i$. Here and after, an upper index of agent $i$ is used as the corresponding number of an agent (while not as an exponent).

Let the network topology be modeled by a digraph $(\mathcal{N}, E)$, where $E$ denotes the set of edges of topology graph $(\mathcal{N}, E)$. The corresponding adjacency matrix is denoted as $A = [a^{i,j}]$, where $a^{i,j} > 0$ if agent $j$ is connected to agent $i$ (i.e. if there is an arc from $j$ to $i$) and $a^{i,j} = 0$ otherwise. Denote $\mathcal{G}_A$ the graph corresponding to adjacency matrix $A$.

To introduce some properties of the network topology, the following definitions from the graph theory will be used. Define *the weighted in-degree* of node $i$ as the sum of $i$-th row of matrix $A$: $\deg_i^+(A) = \sum_{j=1}^n a^{i,j}$; $\deg_{\max}^+(A)$ is the maximum node in-degree in graph $\mathcal{G}_A$; $\mathcal{D}(A) = \mathrm{diag}_n(\mathrm{col}\{\deg_1^+(A), \ldots, \deg_n^+(A)\})$ is the corresponding diagonal matrix. Here and further, $\mathrm{col}\{\mathbf{x}^1, \ldots, \mathbf{x}^n\}$ denotes a vector obtained by stacking the specified vectors; $\mathrm{diag}_n(\mathbf{b})$ is a square diagonal matrix with vector $\mathbf{b}$ as the main diagonal. Let $\mathcal{L}(A) = \mathcal{D}(A) - A$ denote the *Laplacian* of graph $\mathcal{G}_A$. $\cdot^{\mathrm{T}}$ is a vector or matrix transpose operation; $\|A\|$ is the Frobenius norm: $\|A\| = \sqrt{\sum_i \sum_j (a^{i,j})^2}$; $\mathrm{Re}(\lambda_2(A))$ is the real part of the second eigenvalue of matrix $A$ ordered by the absolute magnitude; $\lambda_{\max}(A)$ is the eigenvalue of matrix $A$ with maximum absolute magnitude; $\mathbf{1}_n = (1, \ldots, 1)^{\mathrm{T}}$ is the vector of $n$-times replication of ones; $\mathrm{I}_d$ is the identity matrix $d \times d$. $A \otimes B$ is the Kronecker product defined for any $m \times n$ and $p \times q$ matrices $A$ and $B$.

## III. PROBLEM STATEMENT

Let $(\Omega, \mathcal{F}, P)$ be the underlying probability space corresponding to the sample space $\Omega$ with $\sigma$-algebra of all events $\mathcal{F}$ and the probability measure $P$, and $\mathbb{E}$ be a mathematical expectation symbol.

Let $\Xi$ be a set, $\forall i \in \mathcal{N}$ $\{f_\xi^i(\theta)\}_{\xi \in \Xi}$, be a family of differentiable functions: $f_\xi^i(\theta) : \mathbb{R}^d \to \mathbb{R}$, and let $\mathbf{x}_1^i, \mathbf{x}_2^i, \ldots$ be a sequence of measurement points chosen by the experimenter (observation plan), where the values $y_1^i, y_2^i, \ldots$ of functions $f_\xi^i(\cdot)$ are accessible to observations at every time instant $t = 1, 2, \ldots$, with additive external noise $v_t^i$

$$y_t^i = f_{\xi_t}^i(\mathbf{x}_t^i) + v_t^i, \tag{1}$$

where $\{\xi_t\}$ is a non-controllable sequence: $\xi_t \in \Xi$ (e.g., $\Xi = \mathbb{N}$ and $\xi_t = t$, or $\Xi \subset \mathbb{R}^p$ and $\{\xi_t\}$ is a sequence of some random elements).

Let $\mathcal{F}_{t-1}$ be the $\sigma$-algebra of all probabilistic events which happened up to time instant $t = 1, 2, \ldots$, $\mathbb{E}_{\mathcal{F}_{t-1}}$ is a symbol of the conditional mathematical expectation with respect to the $\sigma$-algebra $\mathcal{F}_{t-1}$.

*Non-stationary problem formulation*. The time-varying point of minimum $\theta_t$ of the distributively computed mean-risk function

$$\bar{F}_t(\theta) = \sum_{i \in \mathcal{N}} F_t^i(\theta) = \mathbb{E}_{\mathcal{F}_{t-1}} \sum_{i \in \mathcal{N}} f_{\xi_t}^i(\theta) \to \min_\theta, \tag{2}$$

*needs to be estimated.*

More precisely, based on the observations $y_1^i, y_2^i, \ldots, y_t^i$ and inputs $\mathbf{x}_1^i, \mathbf{x}_2^i, \ldots, \mathbf{x}_t^i$, $i \in \mathcal{N}$, we consider the problem of constructing an estimate $\widehat{\theta}_t$ of an unknown vector $\theta_t$ minimizing the time-varying *mean-risk functional* (2) which is a conditional expectation of the sum of distributed sub-functions $f_{\xi_t}^i(\theta)$.

Minimization of the functional $F_t(\theta)$ is usually studied with simpler observation models

$$y_t^i = F_t^i(\mathbf{x}_t) + v_t^i \quad \text{or} \quad y_t^i = f_{\xi_t}^i(\mathbf{x}_t),\ i \in \mathcal{N}.$$

The generalization used in model (1) allows separation of any uncertainties and observation disturbances with "good" (e.g., zero-mean and independent and identically distributed — i.i.d.) statistical properties $\{\xi_t\}$ and arbitrary additive external noise $\{v_t^i\}$. Of course, this separation is not needed when we can assume that $\{v_t^i\}$ is a random zero-mean and independent and identically distributed as well.

Centralized algorithms usually require the distributed agent network to transmit the whole system information $y_1^i, y_2^i, \ldots, y_t^i$, $\mathbf{x}_1^i, \mathbf{x}_2^i, \ldots, \mathbf{x}_t^i$, $i \in \mathcal{N}$, into a fusion center to estimate the unknown vector $\theta_t$, which may lack robustness at the fusion center and need strong communication capability over the agent networks. In many practical situations, agents may only have the capability to exchange information locally with their neighbors with noise and delays, and the network topology may switch over time. Moreover, a lot of practical reasons lead to the problem setting with cost constraints for the using network topology. In sensor networks, the set of agents $\mathcal{N}$ is a set of $n$ nodes distributed over the geographic region.

We assume that to form its current estimates $\widehat{\theta}_t^i$ at time instant $t$ agent $i$ has its own noisy observation (1) and, if the set $\mathcal{N}_t^i$ is not empty, information about its neighbors' current estimates $\widehat{\theta}_t^j$, $j \in \mathcal{N}_t^i$.

## IV. SPSA-BASED CONSENSUS ALGORITHM

Let $\mathbf{\Delta}_k^i$, $k = 1, 2, \ldots$, $i \in \mathcal{N}$, be an observed sequence of independent Bernoulli random vectors from $\mathbb{R}^d$ with each component independently taking values $\pm \frac{1}{\sqrt{d}}$ with probabilities $\frac{1}{2}$. This sequence is usually called the *simultaneous test perturbation*. Let us take fixed nonrandom initial vectors $\widehat{\theta}_0^i \in \mathbb{R}^d$, positive step-size $\alpha$, gain coefficient $\gamma$, and choose the scale of perturbation $\beta > 0$. We consider the algorithm with two observations of distributed sub-functions $f_{\xi_t}^i(\theta)$ for each agent $i \in \mathcal{N}$ for constructing sequences of points of observations $\{\mathbf{x}_t^i\}$ and estimates $\{\widehat{\theta}_t^i\}$:

$$\begin{cases} \mathbf{x}_{2k}^i = \widehat{\theta}_{2k-2}^i + \beta \mathbf{\Delta}_k^i,\ \mathbf{x}_{2k-1}^i = \widehat{\theta}_{2k-2}^i - \beta \mathbf{\Delta}_k^i, \\ \widehat{\theta}_{2k-1}^i = \widehat{\theta}_{2k-2}^i, \\ \widehat{\theta}_{2k}^i = \widehat{\theta}_{2k-1}^i - \alpha \Big[ \mathbf{\Delta}_k^i \frac{y_{2k}^i - y_{2k-1}^i}{2\beta} + \\ \qquad\qquad \gamma \sum_{j \in \mathcal{N}_t^i} a^{i,j} (\widehat{\theta}_{2k-1}^i - \widehat{\theta}_{2k-1}^j) \Big]. \end{cases} \tag{3}$$

The first part of the algorithm (3) is similar to SPSA-like algorithm from [8] and the second one coincides with a local voting protocol (LVP) from [12], where it was studied for stochastic networks in the context of load balancing problem. The SPSA part represents a stochastic gradient descent of sub-functions $f_{\xi_t}^i(\theta)$, and LVP part is determined for each agent $i$ by the weighted sum of differences between the information about the current estimate $\widehat{\theta}_{2k-1}^i$ of agent $i$ and information about the estimates of its neighbors.

Further, we use notation $\bar{\theta}_t = \mathrm{col}\{\widehat{\theta}_t^1, \dots, \widehat{\theta}_t^n\}$ for the common vector of estimates of all agents at time instant $t$. Also we introduce the following notations:

$$\bar{\mathbf{y}}_t = \mathrm{diag}_n(\mathrm{col}\{y_t^1, \dots, y_t^n\}), \bar{\Delta}_{t \div 2} = \mathrm{col}\{\Delta_{t \div 2}^1, \dots, \Delta_{t \div 2}^n\}.$$

Using the notations introduced above, the algorithm (3) can be rewritten in the following form

$$\bar{\theta}_{2k} = \bar{\theta}_{2k-1} - \alpha \Big[ \left( \frac{\bar{\mathbf{y}}_{2k} - \bar{\mathbf{y}}_{2k-1}}{2\beta} \otimes \mathrm{I}_d \right) \bar{\Delta}_k + \gamma (\mathcal{L}(A) \otimes \mathrm{I}_d) \bar{\theta}_{2k-1} \Big]. \tag{4}$$

## V. ASSUMPTIONS AND MAIN RESULT

This section presents assumptions and Theorem 1 for the algorithm (3).

First, let us formulate assumptions about the functions $F_t^i(\mathbf{x})$, $f_{\xi_t}^i(\mathbf{x})$, $\forall i \in \mathcal{N}$, noise, disturbances, and network topology.

*1:* Functions $F_t^i(\cdot)$ are convex and there is a common minimum point $\theta_t$ and

$$\forall \mathbf{x} \in \mathbb{R}^d \ \ \langle \mathbf{x} - \theta_t, \mathbb{E}_{\mathcal{F}_{t-1}} \nabla f_{\xi_t}^i(\mathbf{x}) \rangle \geq 0.$$

Here and further $\langle \cdot, \cdot \rangle$ is a scalar product of two vectors.

*2:* $\forall \xi \in \Xi$ the gradient $\nabla f_\xi^i(\mathbf{x})$ satisfies the Lipschitz condition: $\forall \mathbf{x}', \mathbf{x}'' \in \mathbb{R}^d$

$$\|\nabla f_{\xi_t}^i(\mathbf{x}') - \nabla f_{\xi_t}^i(\mathbf{x}'')\| \leq M \|\mathbf{x}' - \mathbf{x}''\|$$

with the same constant $M > 0$.

*3:* The gradient $\nabla f_{\xi_t}^i$ is uniformly bounded in the mean-squared sense at the minimum points $\theta_t$: $\mathbb{E}\|\nabla f_{\xi_t}^i(\theta_t)\|^2 \leq g_2^2$, $\mathbb{E}\langle \nabla f_{\xi_t}^i(\theta_t), \nabla f_{\xi_{t-1}}^i(\theta_{t-1}) \rangle \leq g_2^2$ ($g_2 = 0$ if $\xi_t$ is not a random parameter, i.e. $f_{\xi_t}^i(\mathbf{x}) = F_t(\mathbf{x})$).

*4:* The drift is bounded: a) $\|\theta_t - \theta_{t-1}\| \leq \delta_\theta < \infty$, or $\mathbb{E}\|\theta_t - \theta_{t-1}\|^2 \leq \delta_\theta^2$ and $\mathbb{E}\|\theta_t - \theta_{t-1}\| \|\theta_{t-1} - \theta_{t-2}\| \leq \delta_\theta^2$ if a sequence $\{\xi_t\}$ is random;

b) $\mathbb{E}_{\mathcal{F}_{2k-2}} |f_{\xi_{2k}}^i(\mathbf{x}) - f_{\xi_{2k-1}}^i(\mathbf{x})|^q \leq \delta_\theta^q (g_0^q + g_1^q \|\mathbf{x} - \theta_{2k-2}\|^q)$ for $q = 1, 2$ and for any $i \in \mathcal{N}$.

*5:* For $n = 1, 2, \dots$, the successive differences $\tilde{v}_k^i = v_{2k}^i - v_{2k-1}^i$ of observation noise are bounded: $|\tilde{v}_k^i| \leq c_v < \infty$, or $\mathbb{E}(\tilde{v}_k^i)^2 \leq c_v^2$ if a sequence $\{\tilde{v}_t^i\}$ is random.

*6:* For any $i, j \in \mathcal{N}$ a) vectors $\mathbf{\Delta}_k^i$ are mutually independent; b) $\mathbf{\Delta}_k^i$ and $\xi_{2k-1}, \xi_{2k}$ (if they are random) do not depend on the $\sigma$-algebra $\mathcal{F}_{2k-2}$; c) if $\xi_{2k-1}, \xi_{2k}, \bar{v}_n^i$ are random, then random vectors $\mathbf{\Delta}_k^i$ and elements $\xi_{2k-1}, \xi_{2k}, \bar{v}_n^i$ are independent.

*7:* Graph $\mathcal{G}_A$ is strongly connected.

*Examples*. Assumption *4* about the drift holds for the drift with model $\theta_t = \theta_{t-1} + \zeta_{t-1}$, $\theta_t \in \mathbb{R}^d$, where $\{\zeta_t\}$ is a sequence of random i.i.d. vectors which have symmetrical distribution on the ball: $\|\zeta_t\| \leq \delta_\theta$, $E\zeta_t = 0$, $E\|\zeta_t\|^2 = \sigma_\zeta^2$, $E\|\zeta_t\|^4 = M_\zeta^4$. If at time instant $t$ for $i \in \mathcal{N}$ we can measure the squared distance of projections $\|Proj^i(\mathbf{x} - \theta_t)\|^2$ between a chosen point $\mathbf{x}$ and $\theta_t$ with additive bounded non-random noise $v_t^i$: $|v_t^i| < 1$, then we have $\Xi = \mathbb{N}$ and $F_t^i(\mathbf{x}) = \|Proj^i(\mathbf{x} - \theta_t)\|^2$. Here $\{Proj^i(\cdot)\}$ are a set of projection operators into the set of subspaces of $\mathbb{R}^d$. Assumptions 2 and 3 hold with constants $M = 2$ and $g_0 = 3$, $g_1 = 2$, $g_2 = 0$.

To analyze the quality of estimates we apply the following definition for the problem of minimum tracking for mean-risk functional (2).

*Definition.* A sequence of estimates $\{\bar{\theta}_{2k}\}$ has *an asymptotically efficient upper bound* $\bar{L} > 0$ *of residuals of estimation* if $\forall \varepsilon > 0\ \exists \bar{k}$ such that $\forall k > \bar{k}$

$$\sqrt{E\|\bar{\theta}_{2k} - \mathbf{1}_n \otimes \theta_{2k}\|^2} \leq \bar{L} + \varepsilon.$$

Denote $\bar{\lambda}_2 = \mathrm{Re}(\lambda_2(\mathcal{L}(A)))$, $\bar{\lambda}_m = \lambda_{\max}^{\frac{1}{2}}(\mathcal{L}(A)^{\mathrm{T}} \mathcal{L}(A))$, $\delta_\beta = \frac{\delta_\theta}{2\beta}$, $c_1 = \delta_\beta g_1 + 1$, $c_2 = \delta_\beta^2 g_1^2 / M^2 + 1$, $c_\mu = (\bar{\lambda}_2 - \alpha \bar{\lambda}_m M c_1) / \bar{\lambda}_m^2$, $c_d = \sqrt{1 - 2\alpha^2 M^2 c_2 \bar{\lambda}_m^2 / (\bar{\lambda}_2 - \alpha \bar{\lambda}_m M c_1)^2}$.

The following theorem shows the asymptotically efficient upper bound of estimation residuals provided by algorithm (3).

*Theorem 1: If* Assumptions *1–8* hold, positive constant $\alpha$ is sufficiently small:

$$\alpha < \frac{\bar{\lambda}_2}{\bar{\lambda}_m M (c_1 + \sqrt{2 c_2})} \tag{5}$$

and

$$c_\mu (1 - c_d) < \alpha \gamma < c_\mu (1 + c_d) \tag{6}$$

*then* the sequence of estimates provided by algorithm (3) has an asymptotically efficient upper bound which equals to

$$\bar{L} = \frac{1}{\mu} \left( h + \sqrt{h^2 + l\mu} \right), \tag{7}$$

where $\mu = 2\gamma\bar{\lambda}_2 - \alpha(\bar{\lambda}_m^2 \gamma^2 + 2\alpha M(\gamma \bar{\lambda}_m c_1 + M c_2))$, $h = \gamma(2\sqrt{n}\bar{\lambda}_m \delta_\theta + \alpha \bar{\lambda}_m M(\delta_\beta g_0 + 2\beta)) + M(3.5 + g_1/2)\delta_\theta + 2M^2\beta$, $l = \frac{4n\delta_\theta^2}{\alpha} + Mn(\delta_\theta^2 + 4\beta) + n\left(2.25\delta_\theta^2 + g_2^2 + 4M^2\beta^2\right) + \alpha n \left( 2\frac{c_v^2 + \delta_\theta^2 g_0^2}{\beta^2} + M(c_v + \delta_\theta g_0) + 4M\beta(\delta_\theta(M + 0.5) + g_2) \right)$.

See the proof of Theorem 1 in Appendix.

*Remarks*. 1. The observation noise $v_t$ in Theorem 1 can be said to be almost arbitrary since it may either be nonrandom but bounded or it may also be a realization of some stochastic process with arbitrary internal dependencies. In particular, to prove the results of Theorem 1, there is no need to assume that $v_t$ and $\mathcal{F}_{t-1}$ are not dependent.

2. The result of the Theorem 1 shows that for the case without drift ($\delta_\theta = 0$) we have $c_1 = c_2 = 1$ and the asymptotic upper bound is $\bar{L} = \frac{2M^2\beta+\sqrt{4M\beta(M^3\beta+(1+M\beta)n\mu)+\alpha(2\frac{c_v^2}{\beta^2}+Mc_v)n\mu}}{\mu}$. Under any noise level $c_v$ this bound can be made infinitely small by choosing sufficiently small $\alpha$ and $\beta$. At the same time, in the case of drift, the bigger drift norm $\delta_\theta$ can be compensated by choosing a bigger step-size $\alpha$. This leads to a tradeoff between making $\alpha$ smaller because of noisy observations and making $\alpha$ bigger due to the drift of optimal points.

## VI. SIMULATION

In this section, we show the numerical experiment, which illustrates the performance of the suggested algorithm (3).

We consider a networked system consisting of $n = 10$ nodes. Each node tries to estimate the multidimensional moving point coordinates: $\theta_t = \theta_{t-1} + \zeta_{t-1},\ \theta_t \in \mathbb{R}^d$. Let $\zeta_{t-1}$ be a random vector uniformly distributed on the ball of radius equal to 1, and the dimension of vector $\theta_t$ is $d = 10$.

We assume that the nodes cannot estimate all components of the vector $\theta_t$. In practical applications, this situation may arise due to several reasons. For example, in multi-target tracking problem, the targets may be out of range for some sensor nodes estimating their positions and velocities. In our simulation, the nodes estimate only one component of the vector $\theta_t$. The indices of these components are equal to the indices of the corresponding nodes, e.g., the node $i = 1$ estimates the first component of $\theta_t$.

Let at time instant $t$ agent $i$ be able to measure the squared distance $\|Proj^i(\mathbf{x} - \theta_t)\|^2$ between projections of chosen point $\mathbf{x}$ and $\theta_t$ into the basis line corresponding to $i$-th coordinate ($Proj^i$ is $1 \times n$ row with $d-1$ zeros components and 1 at $i$-th position): $F_t^i(\mathbf{x}) = \|Proj^i(\mathbf{x} - \theta_t)\|^2$. The measurements are corrupted by additive noise $v_t^i$.

*Simulation:* We consider the tracking of the process including drift, i.e., $\delta_\theta = 1$. For the described application, Assumptions 2, 3, and 4 hold if the corresponding constants are as follows: $M = 2$, $g_0 = 3$, $g_1 = 2$, $g_2 = 0$, $c_v = 1$. Let the communication graph $\mathcal{G}_A$ be full, i.e. all nodes are connected to each other and there are no self-loops. In this case, $\bar{\lambda}_2 = 10$ and $\bar{\lambda}_m = 10$.

The algorithm (3) working on each node has the following parameters: $\alpha = 0.18$, $\beta = 4$, $\gamma = 0.285$. We consider three types of noise: uniformly distributed random variable falling within the interval $[-1, 1]$, periodic oscillation (e.g, sine or cosine), and an unknown constant. In the simulation presented in the paper, we use $v_t^i = sin(\psi^i t)$, $|v_t^i| < 1$. Let $\psi^i$ be equal to the index of the node, i.e., $\psi^i = i$.

Fig. 1 illustrates how the 10-th component of the vector $\theta_t$ (blue line) and the estimates of this component calculated by each node $i$ (red and green lines) evolve over time. The duration of the experiment is 5000 discrete time steps. The initial value of $\widehat{\theta}_0^i$ on each node $i$ was chosen randomly from the interval $[350; 550]$. The point $\theta_t$ starts its movement at the position consisting of randomly chosen components from the interval $[100; 200]$. Figure shows that there exists the time instant $t$ starting with which the estimations converge to the real value and move next to it. Fig. 1 also contains a zoomed representation of the estimates for a small time window. It can be seen that most of the components depicted in red color have more smooth changes. These components are not directly measurable by the $n-1$ nodes, i.e., $i = 1, \ldots, 9$, they are estimated through consensus part of the algorithm (3). The component depicted in green color is directly estimated by the 10-th node. The upper bound is $\bar{L} = 57$.

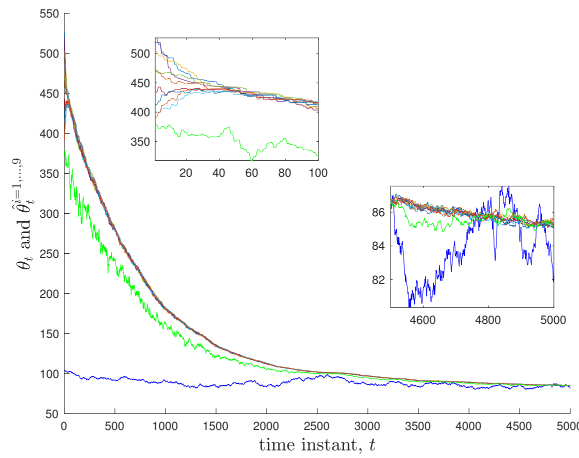


Fig. 1. Ninth component of the vector $\theta_t$ and the estimates of this component obtained by nodes $i = 1, \ldots, 10$. Blue line: 10-th component of the vector $\theta_t$; Green line: the estimate of this component calculated by node $i = 10$; Other colors: the estimates of this component calculated by each node $i = 1, \ldots, 9$

## VII. CONCLUSIONS

In this paper, we propose the new state estimation method for networked systems combining Simultaneous Perturbation Stochastic Approximation and the consensus algorithm. The SPSA algorithm itself is well-studied and may be used in various applications. However, the new approach makes it possible to relax the assumption regarding the strong convexity of the minimized mean-risk functional. This assumption may not be fulfilled in the distributed optimization problems. We have obtained a finite bound of residual between estimates and time-varying unknown parameters. We have also validated the new algorithm through simulation. The new algorithm is suitable for traffic optimization in road networks. For modern road networks corresponding OD-matrices representing traffic demand between origin-destination pairs have large dimension and may require utilization of distributed methods for the weights values estimation and tracking.

## APPENDIX

The following Lemma 1 in [23] is instrumental to the proof of Theorem 1.

*Lemma 1* [23]: *If* $e_k > 0$, $\mu, \alpha > 0$, $0 < \mu\alpha < 1$, $h, l \geq 0$,

$$e_k^2 \leq (1-\mu\alpha)e_{k-1}^2 + 2\alpha h e_{k-1} + \alpha l,\ k = 1, 2, \dots$$

*then* $\forall \varepsilon > 0\ \exists K$ such that $\forall k > K$ the following inequality holds: $e_k \leq \frac{1}{\mu}(h + \sqrt{h^2 + l\mu}) + \varepsilon$.

*Proof:* [Theorem 1] Denote $\bar{\mathbf{s}}_k = \frac{\alpha}{\beta_k}((\bar{\mathbf{y}}_{2k} - \bar{\mathbf{y}}_{2k-1}) \otimes \mathrm{I}_d)\bar{\Delta}_k$, $\mathbf{d}_t^i = \widehat{\theta}^i_{2\lceil \frac{t-1}{2} \rceil} - \theta_t$, $\bar{\mathbf{d}}_t = \mathrm{col}\{\mathbf{d}_t^1, \dots, \mathbf{d}_t^n\}$, where $\lceil \cdot \rceil$ is a ceiling function, $\bar{\mathbf{v}}_t = \mathrm{col}\{\tilde{v}_t^1, \dots, \tilde{v}_t^n\}$.

Let $\bar{\mathcal{F}}_{k-1} = \sigma\{\mathcal{F}_{k-1}, \bar{\mathbf{v}}_{2k-1}, \bar{\mathbf{v}}_{2k}, \xi_{2k-1}, \xi_{2k}, \bar{\Delta}_k\}$ be the $\sigma$-algebra of probabilistic events generated by $\mathcal{F}_{k-1}, \bar{\mathbf{v}}_{2k-1}, \bar{\mathbf{v}}_{2k}, \xi_{2k-1}, \xi_{2k}, \bar{\Delta}_k$, and $\tilde{\mathcal{F}}_{k-1} = \sigma\{\mathcal{F}_{k-1}, \bar{\mathbf{v}}_{2k-1}, \bar{\mathbf{v}}_{2k}, \xi_{2k-1}, \xi_{2k}\}$,

$$\mathcal{F}_{k-1} \subset \tilde{\mathcal{F}}_{k-1} \subset \bar{\mathcal{F}}_{k-1} \subset \mathcal{F}_k.$$

According to the algorithm (4), we have $\nu_k =$

$$\|\bar{\theta}_{2k-2} - \mathbf{1}_n \otimes \theta_{2k} - \bar{\mathbf{s}}_k - \alpha\gamma\bar{\mathcal{L}}\mathbf{1}_n \otimes \bar{\theta}_{2k-2}\| = \|\bar{\mathbf{g}}_k - \bar{\mathbf{s}}_k\|$$

where $\bar{\mathcal{L}} = \mathcal{L}(A)$, $\bar{\mathbf{g}}_k = (\mathrm{I}_{nd} - \alpha\gamma\bar{\mathcal{L}} \otimes \mathrm{I}_d)\bar{\mathbf{d}}_{2k-2} + \mathbf{1}_n \otimes (\theta_{2k-2} - \theta_{2k})$ since it is not so hard to prove that $\bar{\mathcal{L}} \otimes \mathrm{I}_d)\mathbf{1}_n \otimes \theta_{2k-2} = 0$ based on the properties of Laplasian matrix $\bar{\mathcal{L}}$. Taking the conditional expectation over $\sigma$-algebra $\bar{\mathcal{F}}_{k-1}$, we obtain we obtain

$$\mathbb{E}_{\bar{\mathcal{F}}_{k-1}}\nu_k^2 = \|\bar{\mathbf{g}}_k\|^2 + \|\bar{\mathbf{s}}_k\|^2 - 2\langle \bar{\mathbf{g}}_k, \bar{\mathbf{s}}_k\rangle. \tag{8}$$

By virtue of Assumption *6* we have $\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}\tilde{v}_k\Delta_k^i = \mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}\tilde{v}_k\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}\Delta_k^i = \mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}\tilde{v}_k \cdot 0 = 0$. Hence, taking the conditional expectation over $\sigma$-algebra $\tilde{\mathcal{F}}_{k-1}$ of both sides of the (8) and using observation model (1), we can assert the bound for $\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}\nu_k^2$ as follows:

$$\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}\nu_k^2 \leq \mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}\|\bar{\mathbf{g}}_k\|^2 - \frac{\alpha}{\beta}\sum_{i\in\mathcal{N}}\langle \mathbf{d}_{2k}^i, \mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}\tilde{f}_k^i\Delta_k^i\rangle + $$
$$+\frac{\alpha}{\beta}\sum_{i\in\mathcal{N}}\langle \alpha\gamma(\bar{\mathcal{L}}\mathbf{d}_{2k-2}^i, \mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}\tilde{f}_k^i\Delta_k^i\rangle +$$
$$+\frac{\alpha^2}{4\beta^2}\sum_{i\in\mathcal{N}}\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}\left(\tilde{v}_k^i + \tilde{f}_k^i\right)^2\|\Delta_k^i\|^2 \tag{9}$$

where $\tilde{f}_k^i = f_{\xi_{2k}}^i(\mathbf{x}_{2k}) - f_{\xi_{2k-1}}^i(\mathbf{x}_{2k-1})$.

Under fulfilment of Assumption *7*, we have $\bar{\lambda}_2 > 0$ (see [24]). Hence, for the first term in (8) we derive

$$\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}\|\bar{\mathbf{g}}_k\|^2 \leq \mathbf{d}_{2k-2}^{\mathrm{T}}(\mathbf{I}_{nd} - \alpha\gamma(\bar{\mathcal{L}} \otimes \mathbf{I}_d))^{\mathrm{T}} \times$$
$$(\mathbf{I}_{nd} - \alpha\gamma(\bar{\mathcal{L}} \otimes \mathbf{I}_d))\mathbf{d}_{2k-2} + \mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}2\alpha\gamma \times$$
$$\mathbf{d}_{2k-2}^{\mathrm{T}}(\mathbf{I}_{nd} - \alpha\gamma(\bar{\mathcal{L}} \otimes \mathbf{I}_d))^{\mathrm{T}}\mathbf{1}_n \otimes (\theta_{2k-2} - \theta_{2k}) +$$
$$\|\mathbf{1}_n \otimes (\theta_{2k-2} - \theta_{2k})\|^2 \leq (1 - 2\alpha\gamma\bar{\lambda}_2 + \alpha^2\gamma^2\bar{\lambda}_m^2)\nu_{k-1}^2 +$$
$$4\alpha\gamma\sqrt{n}\bar{\lambda}_m\delta_\theta\nu_{k-1} + 4n\delta_\theta^2. \tag{10}$$

For any $\mathbf{x}, \mathbf{z} \in \mathbb{R}^d$, by virtue of Taylor representation of $f_{\xi_t}^i(\mathbf{x})$ for $t^\pm = 2k - \frac{1}{2} \pm \frac{1}{2}$, we have

$$f_{\xi_{t^\pm}}^i(\mathbf{x}) = f_{\xi_{t^\pm}}^i(\mathbf{z}) + \langle \nabla f_{\xi_{t^\pm}}^i(\mathbf{z} + \rho_{\xi_{t^\pm}}^\pm(\mathbf{x} - \mathbf{z})), \mathbf{x} - \mathbf{z}\rangle, \tag{11}$$

where $\rho_{\xi_{t^\pm}}^\pm \in (0, 1)$.

For difference $\tilde{f}_k^i$, adding and subtracting $\langle \nabla f_{\xi_{t^\pm}}^i(\mathbf{z}), \mathbf{x}_{t^\pm}^i - \mathbf{z}\rangle$, we derive:

$$\tilde{f}_k^i = \sum_{t^\pm} \pm f_{t^\pm}^i(\mathbf{z}) \pm \langle \nabla f_{\xi_{t^\pm}}^i(\mathbf{z}), \mathbf{x}_{t^\pm}^i - \mathbf{z}\rangle \pm \bar{M}_{t^\pm}^i(\mathbf{z}) \tag{12}$$

where $\bar{M}_{t^\pm}^i(\mathbf{z}) = \langle \nabla f_{\xi_{t^\pm}}^i(\mathbf{z} + \rho_{\xi_{t^\pm}}^\pm(\mathbf{x}_{t^\pm}^i - \mathbf{z})) - \nabla f_{\xi_{t^\pm}}^i(\mathbf{z}), \mathbf{x}_{t^\pm}^i - \mathbf{z}\rangle$. Hence, for $\mathbf{z} = \widehat{\theta}_{2k-2}^i$, by virtue Assumption *6*, we have $\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}\tilde{f}_k^i\mathbf{\Delta}_k^i = \sum_{t^\pm} \pm\nabla f_{\xi_{t^\pm}}^i(\widehat{\theta}_{2k-2}^i)\beta \pm \mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}\bar{M}_{t^\pm}^i(\widehat{\theta}_{2k-2}^i)\mathbf{\Delta}_k^i$, since $\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}f_{t^\pm}^i(\mathbf{z})\mathbf{\Delta}_k^i = 0$.

According to the Assumption *2*, we have

$$\|\bar{M}_{t^\pm}^i(\widehat{\theta}_{2k-2}^i)\| \leq M\|\rho_{\xi_{t^\pm}}^\pm(\mathbf{x}_{t^\pm}^i - \widehat{\theta}_{2k-2}^i)\|\beta\|\mathbf{\Delta}_k^i\| \leq$$
$$M\beta^2\|\mathbf{\Delta}_k^i\|^2. \tag{13}$$

We can evaluate the second term in (9), using formula (13) and applying Assumptions *2*,

$$\dots \leq -\frac{\alpha}{\beta}\sum_{i\in\mathcal{N}}\sum_{t^\pm}\langle \widehat{\theta}_{2k-2}^i - \theta_{t^\pm}, \nabla f_{\xi_{t^\pm}}^i(\widehat{\theta}_{2k-2}^i)\beta\rangle -$$
$$\alpha\sum_{i\in\mathcal{N}}\langle \theta_{2k} - \theta_{2k-1}, \nabla f_{\xi_{2k-1}}^i(\widehat{\theta}_{2k-2}^i)\rangle + 2\alpha M\beta.$$

Here the conditional expectation over $\sigma$-algebra $\mathcal{F}_{k-1}$ for first terms with minus is not above zero by Assumption *1*. By virtue the definition we have $\mathbb{E}_{\mathcal{F}_{k-1}}\nabla f_{\xi_{2k-1}}^i(\theta_{2k-1}) = 0$. Hence, applying the first part of Assumption *4*, we get $\dots \leq \alpha M\mathbb{E}_{\mathcal{F}_{k-1}}\sum_{i\in\mathcal{N}}\delta_\theta\|\mathbf{d}_{2k-1}^i\| + 2\beta \leq \alpha M\left(\delta_\theta(\nu_{k-1} + n\delta_\theta) + 4n\beta\right)$.

To evaluate the conditional expectation over $\sigma$-algebra $\tilde{\mathcal{F}}_{k-1}$ of the third term in (9) we use the following representation for the difference $\tilde{f}_k^i$

$$\tilde{f}_k^i = f_{\xi_{2k}}^i(\mathbf{x}_{2k}) - f_{\xi_{2k-1}}^i(\mathbf{x}_{2k}) + f_{\xi_{2k-1}}^i(\mathbf{x}_{2k}) - f_{\xi_{2k-1}}^i(\mathbf{x}_{2k-1})$$
$$= \sum_{t^\pm} \pm f_{\xi_{t^\pm}}^i(\mathbf{x}_{2k}) + \langle \nabla f_{\xi_{2k-1}}^i(\widehat{\theta}_{2k-2}^i \pm \rho_{\xi_{t^\pm}}^\pm\beta\mathbf{\Delta}_k^i), \beta\mathbf{\Delta}_k^i\rangle$$

which is based on Taylor formula (11). By adding and subtraction $\sum_{t^\pm}\langle \nabla f_{\xi_{2k-1}}^i(\theta_{2k-1}), \beta\mathbf{\Delta}_k^i\rangle$, using the first part of Assumption *9*, we derive $\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}\tilde{f}_k^i\mathbf{\Delta}_k^i =$

$$\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}\sum_{t^\pm}(\pm f_{\xi_{t^\pm}}^i(\mathbf{x}_{2k}) + \langle \nabla f_{\xi_{2k-1}}^i(\widehat{\theta}_{2k-2}^i \pm \rho_{\xi_{t^\pm}}^\pm\beta\mathbf{\Delta}_k^i), \beta\mathbf{\Delta}_k^i\rangle)$$
$$\times\mathbf{\Delta}_k^i + \langle \nabla f_{\xi_{2k-1}}^i(\theta_{2k-1}), \mathbf{1}_d\rangle\mathbf{1}_d.$$

Taking the conditional expectation over $\sigma$-algebra $\mathcal{F}_{k-1}$, by virtue the properties $\mathbb{E}_{\mathcal{F}_{k-1}}\nabla f_{\xi_{2k-1}}^i(\theta_{2k-1}) = 0$ and the Assumptions *2,4,7*, we get

$$\mathbb{E}_{\mathcal{F}_{k-1}}\|\tilde{f}_k^i\mathbf{\Delta}_k^i\| \leq (\delta_\theta(g_0 + g_1\|\mathbf{d}_{2k-2}^i\|) +$$
$$\sum_{t^\pm} M(\mathbb{E}_{\mathcal{F}_{k-1}}\|\mathbf{d}_{2k-1}^i\| + \beta)\beta) \tag{14}$$

Hence, for the third therm in (9) we have $\dots \leq \frac{\alpha^2\gamma}{\beta}\times$

$$\bar{\lambda}_m Mc_\Delta\nu_{k-1}\left(\delta_\theta(g_0 + g_1\nu_{k-1}) + 2\beta(\nu_{k-1} + 2\beta)\right) \leq$$
$$\frac{\alpha^2\gamma}{\beta}\bar{\lambda}_m Mc_\Delta\left((\delta_\theta g_1 + 2\beta)\nu_{k-1}^2 + (\delta_\theta g_0 + 4\beta^2)\nu_{k-1}\right).$$

Summing up the conditional expectations over $\sigma$-algebra $\mathcal{F}_{k-1}$ of the second and third terms in (9) we derive

$$\ldots \leq 2\alpha^2\gamma\bar{\lambda}_m M\left(\delta_\beta g_1+1\right)\nu_{k-1}^2+\alpha M\left(\delta_\theta+\right.$$
$$\left.2\alpha\gamma\bar{\lambda}_m\left(\delta_\beta g_0+2\beta\right)\right)\nu_{k-1}+\alpha Mn(\delta_\theta^2+4\beta). \qquad (15)$$

Consider the squared difference $(\tilde{v}_k^i+\tilde{f}_k^i)^2$. Using formula (12) with $\mathbf{z}=\widehat{\theta}_{2k-2}$, the sum $(\tilde{v}_k^i+\tilde{f}_k^i)$ can be represented as sum of five terms

$$\tilde{v}_k^i+\tilde{f}_k^i=a_1+a_2+a_3+a_4$$

where $a_1=\tilde{v}_k^i$, $a_2=\sum_{t^\pm}\pm f_{t^\pm}^i(\widehat{\theta}_{2k-2})$, $a_3=\sum_{t^\pm}\langle\nabla f_{\xi_{t^\pm}}^i(\widehat{\theta}_{2k-2}),\mathbf{\Delta}_k^i\beta\rangle$, $a_4=\sum_{t^\pm}\pm\bar{M}_{t^\pm}^i(\widehat{\theta}_{2k-2})$.

The first two terms do not depend on $\mathbf{\Delta}_k^i$ and $\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}a_q\mathbf{\Delta}_k^i\|\Delta_k^i\|^2=0,\ q=1,2$, by virtue the Assumption *7*. Hence, we derive $\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}(\tilde{v}_k^i+\tilde{f}_k^i)^2\|\Delta_k^i\|^2\leq$

$$\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}(a_1+a_2)^2+2(a_1+a_2+a_3)a_4+a_3^2+a_4^2\leq$$
$$\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}2(a_1^2+a_2^2+(|a_1|+|a_2|+|a_3|)|a_4|)+a_3^2+a_4^2.$$

We need to estimate $\mathbb{E}_{\mathcal{F}_{k-1}}a_q^2,\ q=1,\ldots,4$ and we can use the formula $\mathbb{E}_{\mathcal{F}_{k-1}}|a_q|\leq\sqrt{\mathbb{E}_{\mathcal{F}_{k-1}}a_q^2},\ q=1,\ldots,4$ for the rest terms. Taking the conditional expectation over $\sigma$-algebra $\mathcal{F}_{k-1}$, by virtue Assumptions *2–4* and (13), we evaluate

$$\mathbb{E}_{\mathcal{F}_{k-1}}a_1^2\leq c_v^2,\ \mathbb{E}_{\mathcal{F}_{k-1}}|a_2|^q\leq\delta_\theta^q(g_0^q+g_1^q\|\mathbf{d}_{2k-2}^i\|^q),\ q=1,2,$$
$$\mathbb{E}_{\mathcal{F}_{k-1}}a_3^q\leq q\mathbb{E}_{\mathcal{F}_{k-1}}(\sum_{t^\pm}\langle\nabla f_{\xi_{t^\pm}}^i(\widehat{\theta}_{2k-2})-\nabla f_{\xi_{t^\pm}}^i(\theta_{t^\pm}),\mathbf{\Delta}_k^i\beta\rangle)^q$$
$$+q\left(\sum_{t^\pm}\langle\nabla f_{\xi_{t^\pm}}^i(\theta_{t^\pm}),\mathbf{\Delta}_k^i\beta\rangle\right)^q\leq 2c_\Delta^q\left(\left(2M\beta(\|\mathbf{d}_{2k-2}^i\|+\delta_\theta)\right.\right.$$
$$\left.\left.+\delta_\theta\beta)\right)^q+2^q\beta^q g_2^q\right),\ q=1,2,\ \mathbb{E}_{\mathcal{F}_{k-1}}a_4^2\leq 4M^2\beta^4.$$

Taking the conditional expectation over $\sigma$-algebra $\mathcal{F}_{k-1}$ for the fourth term in (9) we get using Assumptions *2–5*

$$\frac{\alpha^2}{2\beta^2}\mathbb{E}_{\mathcal{F}_{k-1}}\sum_{i\in\mathcal{N}}(\tilde{v}_k^i+\tilde{f}_k^i)^2\|\Delta_k^i\|^2\leq\frac{\alpha^2}{4\beta^2}\left(2\left(nc_v^2+\right.\right.$$
$$n\delta_\theta^2g_0^2+\delta_\theta^2g_1^2\nu_{k-1}^2+(nc_v+n\delta_\theta g_0+\delta_\theta g_1\nu_{k-1}+$$
$$2M\beta\nu_{k-1}+n\beta\left(2M\delta_\theta+\delta_\theta+2g_2\right))\times$$
$$2M\beta^2+\left(4M^2\nu_{k-1}^2\beta+10M\delta_\theta\beta^2\nu_{k-1}+\right.$$
$$\left.\left.\left.n(6.25\delta_\theta^2\beta^2+4\beta^2g_2^2)\right)\right)+4nM^2\beta^4\right). \qquad (16)$$

Summing up the findings bounds (10), (15), (16) and taking the conditional expectation over $\sigma$-algebra $\mathcal{F}_{k-1}$, we derive the following from (9)

$$\mathbb{E}_{\mathcal{F}_{k-1}}\nu_k^2\leq(1-\mu\alpha)\nu_{k-1}^2+2\alpha h\nu_{k-1}+\alpha l. \qquad (17)$$

Consider the condition $0<\mu\alpha<1$ of Lemma 2. The right part holds since $\bar{\lambda}_2\leq\bar{\lambda}_m^2$. The left part is satisfied by virtue condition (5)–(6). Hence, taking the unconditional expectation of both sides of (17), we see that all conditions of Lemma 2 hold for $e_k=\sqrt{\mathbb{E}\nu_k^2}$.

By virtue condition (5), we have $\mu<1$. Taking the unconditional expectation of both sides of (17), we see that all conditions of Lemma 1 hold for $e_k=\sqrt{\mathbb{E}\nu_k^2}$.

This completes the proof of Theorem 1. ■

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