Direct Predictive Boundary Control of a First-order Quasilinear Hyperbolic PDE

Timm Strecker[1], Ole Morten Aamo[2] and Michael Cantoni[1]

***Abstract*— We present a method for the boundary control of a system governed by one hyperbolic PDE with a non-local coupling term by state feedback. The method is an extension of recently developed controllers for semilinear systems. The design consists of three steps: predicting the states up to the time when they are affected by the delayed input; virtually moving the input to the uncontrolled boundary (which makes characterizing stability trivial); and constructing the inputs by, starting with the desired boundary values at the uncontrolled boundary, solving an ODE governing the dynamics on the system's characteristic lines *backwards* in time. The controller steers the system to the origin in finite time. A discussion of potential extensions of the presented method is given.**

I. INTRODUCTION

Boundary control of 1-d hyperbolic partial differential equations has received a significant amount of interest because they model many relevant systems, such as flow through one-dimensional conduits and traffic flows. One interesting property of hyperbolic systems is that they can be controlled to the origin in finite time that needs to be larger than some minimum [1]. Quasilinear systems are perhaps the most relevant case of hyperbolic systems in practice.

A constructive method for designing feedback controllers that achieve a finite-time convergence is available in form of the backstepping method. First developed for nonlinear ODEs and then parabolic PDEs, it was applied to construct a controller for a simple first-order hyperbolic PDE in [2] and has since been extended to several more general hyperbolic systems [3], [4], [5]. However, backstepping for hyperbolic PDEs is (still) limited to linear systems.

Recently, we presented an output feedback control scheme for semilinear hyperbolic systems [6], [7], [8], [9]. In this paper, we extend this method to quasilinear systems. For clarity of presentation, we restrict ourselves to the simplest case with only one state and input, with a non-local coupling term such that the uncontrolled system can be unstable.

There are some differences to the semilinear design method from [6], which can cause discontinuities in the state. Therefore, the system equations are satisfied in a weak instead of the classical sense. Such discontinuities might be undesirable in some applications as they are associated with shock waves, although for linear and semilinear systems they are fine from a mathematical point of view. For quasilinear systems where the transport speeds depend on the state, however, fast changes in the state can cause problems with existence of a solution of the PDEs when characteristic lines collide. Therefore, solutions of quasilinear systems must be at least Lipschitz continuous [10]. This has implications on the control inputs via the compatibility conditions of the chosen state space. In particular, the input at each time is uniquely defined by the state at that time, and the control law is not allowed to prescribe any other value. Therefore, a continuous-time control algorithm as in [6] is ill-posed in all appropriate state spaces for quasilinear systems. Instead, we propose that, somewhat similar to model predictive control (MPC) schemes, at time t the control law maps the current state into the input over the interval $[t, t+\theta]$ for some small sampling time $\theta > 0$. Such a scheme constitutes feedback as the current state is taken into account after every θ-interval. However, unlike MPC, the inputs are constructed directly by solving two PDEs instead of solving an optimization problem with PDE constraints. This sampled-time approach is also discussed in Remark 6 of [6].

In contrast to semilinear systems, the result in this paper is local because existence of a solution as well as controllability can only be guaranteed locally for quasilinear systems. This should not be confused with the local result from [11] where the control law is based on a linearization of the nonlinear system, which approximates the nonlinear dynamics only locally. In some cases where the system coefficients satisfy stronger conditions, the method presented in this paper works globally. See e.g. [12] for a global controllability result.

The remainder of this paper is organized as follows. The precise problem description and some existence and controllability results are given in Section II. Preliminary results on the dynamics on the characteristic lines are given in Section III. The control law is designed in Section IV, with the feedback control algorithm in Section IV-D and some discussion of the relation to other methods in Section IV-E. A numerical example is presented in Section V before concluding remarks are given in Section VI.

II. PROBLEM DESCRIPTION

A. System description

This paper is concerned with systems of the form

$$u_t(x,t) = \lambda\left(u(x,t)\right) u_x(x,t) + f\left(u(x,t), u(0,t)\right) \tag{1}$$

$$u(1,t) = U(t) \tag{2}$$

$$u(x,0) = u_0(x). \tag{3}$$

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[1]Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville 3010, Australia timm.strecker@unimelb.edu.au, cantoni@unimelb.edu.au

[2]Department of Engineering Cybernetics, Norwegian University of Science and Technology (NTNU), Trondheim N-7491, Norway ole.morten.aamo@ntnu.no

In (1)-(3), $t \geq 0$ and $x \in [0,1]$, u is the scalar-valued state, $_t$ and $_x$ denote partial derivatives with respect to t and x, and U is the control input.

The transport speed λ is assumed to be continuously differentiable and satisfy

$$\lambda(0) > 0. \tag{4}$$

Similarly, the source term f is assumed to be continuously differentiable and satisfy

$$f(0,0) = 0. \tag{5}$$

As state-space we use the space of piece-wise continuously differentiable functions[1], denoted by $C^1_{pw}(\Omega)$ where Ω is the respective domain under consideration, equipped with norm (where $\|\cdot\|_\infty$ is the standard L^∞-norm, i.e. $\operatorname{ess\,sup}(\cdot)$)

$$\|h\|_{C^1_{pw}} = \max\{\|h\|_\infty, \|h_t\|_\infty, \|h_x\|_\infty\} \tag{6}$$

if $\Omega \subset [0,1] \times [0,\infty)$ and

$$\|h\|_{C^1_{pw}} = \max\{\|h\|_\infty, \|h'\|_\infty\} \tag{7}$$

if $\Omega \subset \mathbb{R}$. We assume that $u_0 \in C^1_{pw}([0,1])$.

The choice of state space C^1_{pw} immediately implies that, to ensure continuity, all solutions must satisfy the compatibility condition

$$U(t) = \lim_{x \to 1} u(x,t). \tag{8}$$

Remark 1: We decide to use state space C^1_{pw} instead of C^1, which is often used to obtain classical solution, because C^1 requires an additional compatibility condition on the first derivatives at the boundary that can limit the choice of control inputs. However, as opposed to semilinear systems, it is not possible to avoid compatibility condition (8) because solutions of quasilinear hyperbolic systems must be at least locally Lipschitz-continuous (which, by virtue of Rademacher's Theorem, already implies differentiability almost everywhere) [10].

B. Existence of solution and controllability

In general, solutions of quasilinear systems exist only locally (local in both time and state). Solutions cease to exist not only due to blow-up of the state but also due to collision of characteristic lines, which is associated with blow-up of the derivatives, see e.g. [10]. This can be problematic in a control context because the life-time of solutions might be less than the minimum time that is required to control the system to the origin. Therefore, the concept of semi-global solutions has been introduced [13], [14]. Here, for a pre-assigned time horizon T the existence of a solution on the interval $[0,T]$ is guaranteed under some smallness assumptions on initial and boundary conditions. Following [14] one can prove the following lemma.

Lemma 2: Assume u_0 and U are C^1_{pw}-functions satisfying (8). For all $T > 0$ there exist $\delta_1, \delta_2 > 0$ such that if $\|u_0\|_{C^1_{pw}} < \delta_1$ and $\|U\|_{C^1_{pw}} < \delta_2$ then (1)-(3) has a unique solution on the domain $[0,1] \times [0,T]$.

Remark 3: The estimates of δ_1 and δ_2 in the proof of Lemma 2 are usually very conservative. However, in some cases less conservative estimates can be obtained. In [12], for instance, results from [13] are applied to prove a global existence result for a very particular case of λ and f.

Based on the semi-global existence result, the methodology from [1] can be used to show null-controllability for system (1)-(3) for small initial data.

Lemma 4: For $T > T_0 = \frac{1}{\lambda(0)}$ there exists a $\delta_3 > 0$ such that if $\|u_0\|_{C^1_{pw}} < \delta_3$ then there exists an control signal U such that $u(\cdot, T) = 0$.

Remark 5: Choosing T too close to T_0 can come at the expense of a smaller bound δ_3 on the initial data. The proof of Lemma 4 is constructive in that it provides an open-loop control signal that drives the system to the origin, but it does not provide a way to update this signal in a feedback-fashion as time proceeds.

C. Control objective

The objective is to design a feedback law such that, for small-enough initial conditions such that the system is null-controllable, the closed-loop system converges to the origin in finite time.

III. Transformation to Dynamics on Characteristic Lines

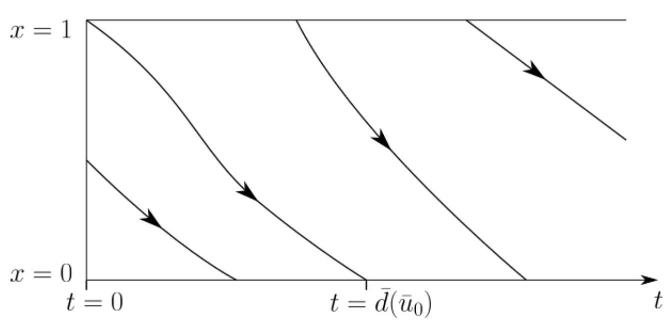

Fig. 1. Characteristic lines of system (1)-(3).

The characteristic lines of system (1)-(2) are sketched in Figure 1. Due to the hyperbolic nature of (1), the control input $U(t)$ which enters at $x = 1$ at time t propagates through the domain $[0,1]$ with finite speed λ. Therefore, the state in the interior of the domain $[0,1]$ is affected after a delay only. More precisely, the control input only affects the state on the characteristic line of (1) along which it propagates, but not earlier. Moreover, it is well-known that the dynamics of a hyperbolic system along its characteristic line reduces to an ODE. Therefore, rather than trying to control the current state, we design an algorithm to control the state on the characteristic lines of system (1) along which the inputs evolve.

[1] i.e. the space of continuous functions which have a continuous derivative almost everywhere

A. Definition of characteristic lines

The input $U(t)$ at time t evolves along the characteristic line $(x, t + \phi(u,t;x))$ defined by

$$\phi(u,t;x) = \int_x^1 \frac{1}{\lambda\left(u(\xi, t + \phi(u,t;\xi))\right)} d\xi \tag{9}$$

Note that ϕ depends on the trajectory of u and, in particular, on the input $U(t)$. This is a major difference to semilinear systems, where the characteristic lines are known a priori. Define the total delay by

$$d(u,t) = \phi(u,t;0). \tag{10}$$

Define the state on the characteristic line as

$$\bar{u}(x,t) = u(x, t + \phi(u,t;x)) \tag{11}$$

Also define

$$\bar{\phi}(\varphi;x) = \int_x^1 \frac{1}{\lambda\left(\varphi(\xi)\right)} d\xi, \qquad \bar{d}(\varphi) = \bar{\phi}(\varphi;0) \tag{12}$$

for $\varphi \in C^1_{pw}([0,1])$, which implies

$$\phi(u,t;x) = \bar{\phi}(\bar{u}(\cdot,t);x) \qquad d(u,t) = \bar{d}(\bar{u}(\cdot,t)). \tag{13}$$

Using this, we can write $\bar{u}$ as

$$\bar{u}(x,t) = u\left(x, t + \bar{\phi}(\bar{u}(\cdot,t);x)\right) \tag{14}$$

Note that in (12), $\bar{\phi}(\bar{u},t;x)$ depends on values $\bar{u}(\xi,t)$ for $\xi \in [x,1]$, i.e. $\bar{u}$ evaluated at time t only, while in (9), $\phi(u,t;x)$ depends on values $u(\xi, t+\phi(u,t;\xi))$ for $\xi \in [x,1]$. Consequently, the definition of $\bar{u}(x,t)$ depends on values of $\bar{u}(\xi,t)$ for $\xi \in [x,1]$.

B. Predictability of state on characteristic lines

As indicated above, for $x \in [0,1]$ the state up to time $t + \phi(u,t;x)$ is not affected by input $U(t)$. Furthermore, because of compatibility condition (8), the input $U(t)$ is actually uniquely defined by state $u(\cdot,t)$. Therefore, for $x \in [0,1]$ the state up to and including time $t + \phi(u,t;x)$, as well as the characteristic line $(x, t+\phi(u,t;x))$ itself, are predictable based on the state $u(\cdot,t)$ alone. However, for the control law we only need the predictions provided by the following lemma.

Lemma 6: If $u(\cdot,t) \in C^1_{pw}([0,1])$ with $\|u(\cdot,t)\|_{C^1_{pw}}$ sufficiently small, there exists an operator

$$\begin{aligned} \Phi :\, & C^1_{pw}([0,1]) \to \mathbb{R} \times C^1_{pw}([t, t+\bar{d}(\bar{u}(\cdot,t))]) \\ & u(\cdot,t) \mapsto \bar{d}(\bar{u}(\cdot,t)),\, u(0,\cdot). \end{aligned} \tag{15}$$

Proof: Analogously to Corolary 2.1 in [15] or Lemma 3.1 in [16] and also similar to Theorem 3.8 in [10], one can show that the Cauchy problem specified by (1) with "initial" condition $u(\cdot,t)$ and without a boundary condition of form (2) has a unique solution on the domain

$$\mathcal{A}(u;t) = \{(x,s) :\, x \in [0,1],\, s \in [t, t+\phi(u,t;x)]\} \tag{16}$$

This set includes the boundary values $\bar{u}(0,s)$ for $s \in [t, t+\bar{d}(\bar{u}(\cdot,t))]$ as well as all states that are required to determine the delay time $\bar{d}(\bar{u}(\cdot,t))$. ∎

Remark 7: The set $\mathcal{A}$ has been called the maximum determinate set by some [15], [16], or also a domain of determinacy [10]. The output arguments of Φ could be modified to include all values of $u(x,t)$ for $(x,t) \in \mathcal{A}(u;t)$. This would be relevant if λ and f when evaluated at x was dependent on $u(\xi,t)$ for $\xi \in [0,x]$, but is not needed here.

C. Dynamics on characteristic lines

As indicated above, the dynamics of u on the characteristic line $(x, t+\phi(u,t;x))$, i.e $\bar{u}$, simplify to an ODE as formalized by the following lemma.

Lemma 8: The state $\bar{u}$ is governed by the ODE

$$\bar{u}_x(x,t) = -\frac{1}{\lambda\left(\bar{u}(x,t)\right)} f\left(\bar{u}(x,t), u\left(0, t+\bar{\phi}(\bar{u}(\cdot,t);x)\right)\right) \tag{17}$$

$$\bar{u}(1,t) = U(t). \tag{18}$$

Proof: From definition (9) we directly get

$$\begin{aligned} \frac{d}{dx}\phi(u,t;x) &= -\frac{1}{\lambda\left(u(x, t+\phi(u,t,x))\right)} \\ &= -\frac{1}{\lambda\left(\bar{u}(x,t)\right)} \end{aligned} \tag{19}$$

Differentiating (11) with respect to x, using (19) and inserting the dynamics (1) gives

$$\begin{aligned} \frac{d}{dx}\bar{u}(x,t) &= \frac{d}{dx}u(x, t+\phi(u,t;x)) \\ &= u_x(x, t+\phi(u,t;x)) + u_t(x, t+\phi(u,t;x)) \\ &\quad \times \left(-\frac{1}{\lambda\left(u(x,t+\phi(u,t,x))\right)}\right) \\ &= -\frac{1}{\lambda\left(u(x,t+\phi(u,t,x))\right)} \\ &\quad \times f\left(u(x,t+\phi(u,t,x)), u(0,t+\phi(u,t,x))\right) \\ &= -\frac{1}{\lambda\left(\bar{u}(x,t)\right)} f\left(\bar{u}(x,t), u\left(0, t+\phi(u,t;x)\right)\right). \end{aligned} \tag{20}$$

The boundary condition (18) follows directly from (2) and the fact that, due to $\phi(u,t;1) = 0$, $\bar{u}(1,t) = u(1,t)$. ∎

IV. Control Design

The control design can be split into two steps: First, a target system is developed for the dynamics of $\bar{u}$ for which it is straightforward to characterize stability. The target system is the dynamics that $\bar{u}$ as defined in (11) with u governed by (1)-(2) in closed loop with the control law for $U(t)$ shall satisfy. Second, the control inputs $U(t)$ that ensure that in closed loop $\bar{u}$ is equal to the target system are constructed.

A. Target system for $\bar{u}$

An essential part of the design method from [6] is to virtually move the input $U(t)$ from $x = 1$ to $x = 0$. For this purpose, the virtual control inputs $U^*(t)$, which are the desired boundary values for $u(0,t)$ (of course only after they are affected by control, i.e. for times later than the delay), are introduced. Due to the assumption that λ is positive, the propagation direction in (1) is given and the boundary condition is specified at the inflow boundary at

$x = 1$ as in (2). However, in the ODE (17) governing $\bar{u}$ there is no direction of propagation and the solution is uniquely prescribed by a "boundary" value at arbitrary $x \in [0, 1]$ (including in the interior of the domain).

With reference to (15), let

$$\left(\bar{d}(\bar{u}_0), v_0\right) = \Phi(u_0). \tag{21}$$

The target system that $\bar{u}$ as defined in (11) with u governed by (1)-(2) in closed loop with the control law for $U(t)$ shall satisfy is

$$\bar{u}_x^*(x,t) = -\frac{1}{\lambda\left(\bar{u}^*(x,t)\right)} f\left(\bar{u}^*(x,t), v\left(t + \bar{\phi}(\bar{u}^*(\cdot,t); x)\right)\right) \tag{22}$$

$$\bar{u}^*(0,t) = U^*\left(t + \bar{d}(\bar{u}^*(\cdot,t))\right) \tag{23}$$

with

$$v(t) = \begin{cases} v_0(t) & t < \bar{d}(\bar{u}_0) \\ U^*(t) & t \geq \bar{d}(\bar{u}_0) \end{cases} \tag{24}$$

where U^* must satisfy the C^1_{pw}-compatibility condition

$$U^*(\bar{d}(\bar{u}_0)) = v_0(\bar{d}(\bar{u}_0)). \tag{25}$$

Note that the boundary value $u(0,t)$ for $t < \bar{d}(\bar{u}_0)$ (which enter in f in (22) and (24)) is uniquely determined by the initial condition $u(\cdot, 0) = u_0$.

Virtually moving the input from $x = 1$ to $x = 0$ significantly simplifies the task of characterizing stability of (22)-(23) compared to (1)-(2) and (17)-(18). In (1)-(2) and also in (17)-(18), the boundary input $U(t)$ interacts with, and needs to compensate the potentially destabilizing effect of, the opposite boundary value $u(0, \cdot)$ through the f-term. By contrast, in (22)-(23) the second argument of f is just the time-shifted input U^*. In fact, target system (22)-(23) is stabilized if U^* becomes zero.

Lemma 9: If $U^*(t) = 0$ for all $t \geq t_0$ for some $t_0 \geq \bar{d}(\bar{u}_0)$, then $\bar{u}^*(x,t) = 0$ for all (x,t) satisfying $t + \bar{\phi}(\bar{u}^*(\cdot,t); x) \geq t_0$.

Proof: If $t+\bar{\phi}(\bar{u}^*(\cdot,t); x) \geq t_0$ then $t+\bar{\phi}(\bar{u}^*(\cdot,t); \xi) \geq t_0$ for all $\xi \in [0, x]$. Therefore, for these points (22)-(23) is of the form

$$\bar{u}_x^*(x,t) = g(\bar{u}^*(x,t)) \qquad \bar{u}^*(0,t) = 0 \tag{26}$$

with $g(0) = 0$. Clearly, the (unique) solution of (26) is the zero-function. ■

Remark 10: One can show that $\bar{u}^*(\cdot,t)$ is exponentially stable, i.e. $\|\bar{u}^*(\cdot,t)\|_{C^1_{pw}} \leq c_1 e^{-c_2 t}$ with $c_1, c_2 > 0$ if and only if $|U^*(\cdot)|$ decreases exponentially with rate c_2.

B. Construction of control inputs

The subject of this section is the design of the control inputs $U(t)$ such that $\bar{u}$ as governed by (17)-(18) in closed loop with the control law for $U(t)$ is equivalent to the target system $\bar{u}^*$ governed by (22)-(23). Clearly, the only difference between (17)-(18) and (22)-(23) are in the boundary conditions. The input $U(t)$ that ensures (23) can be constructed by solving the ODE (22) with boundary condition (23) in the direction from $x = 0$ to $x = 1$, i.e. in opposite direction to the propagation of the original input U, and setting $U(t) = \bar{u}^*(1,t)$.

Lemma 11: For $\theta > 0$ and $t_1 \geq 0$ and small w, consider the operator

$$\begin{aligned} \Psi_\theta^{t_1} : C^1_{pw}\left([t_1, t_1 + \theta + \bar{d}(\bar{u}(\cdot, t_1 + \theta))]\right) &\to C^1_{pw}([t_1, t_1 + \theta]) \\ w &\mapsto \varphi_1 \end{aligned} \tag{27}$$

with $\varphi_1(t) = \varphi(1,t)$ where φ is the solution of

$$\varphi_x(x,s) = -\frac{1}{\lambda\left(\varphi(x,s)\right)} f\left(\varphi(x,t), w\left(t + \bar{\phi}(\varphi(\cdot,t); x)\right)\right) \tag{28}$$

$$\varphi(0,t) = w\left(t + \bar{d}(\varphi(\cdot,t))\right) \tag{29}$$

on $[0,1] \times [t_1, t_1 + \theta]$. Let

$$v^{t_1}(t) = \begin{cases} u(0,t) & t < t_1 + \bar{d}(\bar{u}(\cdot,t_1)) \\ U^*(t) & t \geq t_1 + \bar{d}(\bar{u}(\cdot,t_1)) \end{cases} \tag{30}$$

Then, $\bar{u}$ as governed by (17)-(18) satisfies $\bar{u}(x,t) = \bar{u}^*(x,t)$ for all $x \in [0,1]$, $t \in [t_1, t_1 + \theta]$ if and only if

$$U(\cdot) = \Psi_\theta^{t_1}\left(v^{t_1}\right). \tag{31}$$

Proof: As (28) is a copy of (17) and (22) after a change of notation, this follows directly from uniqueness of the solution of (22)-(23). ■

C. Design of U^*

In this section we discuss one option for the design of U^* that satisfies the C^1_{pw} compatibility condition (25), although for an arbitrary starting time, finite-time convergence to zero and has a sufficiently small C^1_{pw}-norm. Moreover, the trajectory of U^* should not be changed if it is re-initialized during a feedback scheme.

For t_1, $t_2 \geq 0$, $\delta > 0$ and $w_0 \in \mathbb{R}$, consider the following set of candidate functions

$$\mathbb{U}_\delta^{t_1,t_2}(w_0) = \left\{ U^* \in C^1_{pw} : \begin{array}{c} U^*(t_1) = w_0 \\ U^*(s) = 0 \quad \forall s \geq t_2 \\ \|U^*\|_{C^1_{pw}} \leq \delta \end{array} \right\} \tag{32}$$

For all $t_1 \geq 0$ and if $|w_0| < \delta$ there always exists an t_2 such that $\mathbb{U}_\delta^{t_1,t_2}(w_0)$ is non-empty. The minimum time to reach zero is given by

$$t^* = \min_{\mathbb{U}_\delta^{t_1,t_2}(w_0) \neq \emptyset} t_2. \tag{33}$$

In fact, the minimizing U^* is given by

$$U^*_{min}(w_0, t_1, \delta; t) = \begin{cases} w_0\left(1 - \frac{\delta}{|w_0|}(t - t_1)\right), & t < t_1 + \frac{|w_0|}{\delta} \\ 0, & t \geq t_1 + \frac{|w_0|}{\delta} \end{cases} \tag{34}$$

for $t \geq t_1$, with minimum convergence time $t^* = t_1 + \frac{|w_0|}{\delta}$.

Note that (34) satisfies the invariance condition under re-initialization because, defining

$$U_1(s) = U^*_{min}(w_0, t_1, \delta; s) \tag{35}$$

$$U_2(s) = U^*_{min}(U_1(t_1 + \theta), t_1 + \theta, \delta; s) \tag{36}$$

for $\theta > 0$, it is straightforward to show that $U_1(s) = U_2(s)$ for all $s \geq t_1 + \theta$.

D. Feedback control algorithm

We propose the following feedback law where the control inputs are updated based on the current state every time an interval of length θ has passed (see also Section VI for a discussion of the choice of θ). We also propose to re-initialize the virtual inputs U^* at every time-step in order to be able to react to potential disturbances and still ensure compatibility and smallness of the C^1_{pw}-norm. Without disturbances, U^* as constructed in (34) is invariant under re-initiailization as mentioned above, while with disturbances it must be expected that the convergence time is affected.

Evaluating the control law consists of the following steps.

1) At time t, evaluate Φ to obtain the prediction for $u(0,s)$, $s \in [t, t+\bar{d}(\bar{u}(\cdot,t))]$, as

$$\left(\bar{d}(\bar{u}(\cdot,t)), u(0,\cdot)\right) = \Phi(u(\cdot,t)). \quad (37)$$

Evaluating Φ amounts to solving the Cauchy problem (1) with initial data $u(\cdot,t)$ over the maximum determinate set (16).

2) Obtain $U^*(s)$ for $s \in [t+\bar{d}(\bar{u}(\cdot,t)), t+\theta+\bar{d}(\bar{u}(\cdot,t+\theta))]$ as in (34) using $t_1 = t+\bar{d}(\bar{u}(\cdot,t))$ and $w_0 = u(0, t+\bar{d}(\bar{u}(\cdot,t))$ and sufficiently small δ.

3) Evaluate Ψ to obtain the inputs $U(s)$, $s \in [t, t+\theta]$, as

$$U(\cdot) = \Psi^t_\theta\left(v^t\right) \quad (38)$$

with v^t as in (30) using the predictions of $u(0,\cdot)$ from step 1 and $U^*(\cdot)$ from step 2. Evaluating Ψ amounts to solving (22)-(23) over the domain $[0,1]\times[t,t+\theta]$.

4) Then, steps 1-3 are repeated for time $t+\theta$.

Theorem 12: There exist $\delta_0 > 0$ and $\delta_U > 0$ such that if $\|u_0\|_{C^1_{pw}} \leq \delta_0$, the closed-loop system consisting of (1)-(3) in feedback with $U(t)$ as constructed in steps 1-4 above using $\delta \leq \delta_U$ reaches the origin for $t \geq \bar{d}(\bar{u}_0) + \frac{|u(0,\bar{d}(\bar{u}_0))|}{c}$. Moreover, the solution exists for all times.

Proof: By Lemmas 6 and 11, the construction in steps 1-4 ensures that $\bar{u}(\cdot,t) = \bar{u}^*(\cdot,t)$ for all $t \geq 0$. Moreover, for all (x,t) satisfying $t \geq \bar{\phi}(\bar{u}_0; x)$ there exists some $s \geq 0$, implicitly defined as the solution of

$$t = s + \bar{\phi}(\bar{u}(\cdot,s); x), \quad (39)$$

such that

$$u(x,t) = u(x, s+\bar{\phi}(\bar{u}(\cdot,s);x)) = \bar{u}(x,s) = \bar{u}^*(x,s). \quad (40)$$

Combining (39)-(40) with the fact that, by virtue of Lemma 9, $s + \bar{\phi}(\bar{u}(\cdot,s);x) \geq t_0$ implies $\bar{u}^*(x,s) = 0$, proves convergence to zero. Small u_0 and δ ensure that the outputs of Φ and Ψ, and thus U, have sufficiently small C^1_{pw}-norm up to the time the origin is reached. Thus, Lemma 2 ensures existence of the solution up to that time. After that, the solution is the zero-solution. ■

E. Relation to backstepping and open-loop control

In this section we discuss the relation of the control design method presented in this paper with previous results, namely the backstepping method for linear systems [2] and the open-loop control method from [1]. It turns out that all methods lead to equivalent control inputs in an appropriate sense.

1) Backstepping: For linear systems, which can always be written in the form

$$u_t(x,t) = u_x(x,t) + g(x)u(0,t) \quad (41)$$

$$u(1,t) = U(t) \quad (42)$$

$$u(x,0) = u_0(x) \quad (43)$$

(there is an additional integral term in [2] that is not included in (1); this term can be included in (1) but is not done here for simplicity) the backstepping method was applied in [2] to obtain a feedback law of the form

$$U(t) = \int_0^1 k(1,\xi)u(\xi,t) \quad (44)$$

that achieves $u(x,t) = 0$ for $x+t \geq 1$. The conditions that the kernel k must satisfy were derived as

$$k_x(x,\xi) + k_\xi(x,\xi) = 0, \qquad x \in [0,1],\ \xi \in [0,x] \quad (45)$$

$$k(x,0) = \int_0^x k(x,\xi)g(\xi)d\xi - g(x), \qquad x \in [0,1]. \quad (46)$$

For linear system (41), the maximum determinate domain as defined in (16) is know a priori. In fact, the characteristic lines simplify to $\phi(u,t;x) = \phi^{lin}(x) = 1-x$. Moreover, the state space can be changed from C^1_{pw} to L^∞. In L^∞, compatibility condition (8) is not needed. Therefore, one can choose $U^* \equiv 0$ and set the sampling time θ to zero. Then, it is possible to find an explicit expression for the control input constructed in Section IV-D by use of the ansatz

$$\bar{u}(x,t) = \int_0^x k(x,\xi)u(\xi, t+\phi^{lin}(x))d\xi \quad (47)$$

with $U(t) = \bar{u}(1,t)$. Differentiating (47) with respect to x, inserting the dynamics (1), integrating by parts and equating the result with $\bar{u}_x(x,t)$ as given in (17) gives the conditions that k must satisfy. It turns out that these conditions are equivalent to (45)-(46). Therefore, after modification of the state space and U^*, the control method presented in this paper delivers the same control inputs as the backstepping controller if the system is linear.

2) Open-loop control: The control method from [1] amounts to predicting $u(0,s)$ for $s \in [0, \bar{d}(\bar{u}_0)]$, extending this prediction of $u(0,\cdot)$ by a function that becomes zero for all $t \geq T$ with $T > T_0 = \frac{1}{\lambda(0)}$ (these steps are equivalent to how it is done in this paper) and then computing $U(t)$ for all $t \in [0,T]$ by solving equation (1) in x-direction, with the values for $u(0,\cdot)$ as obtained from the prediction and extension steps as "initial" condition, and setting $U(\cdot) = u(1,\cdot)$. In a perfect example without uncertainty, these inputs are the same as the ones obtained from the algorithm in Section IV-D, and in case of linear systems even as the backstepping ones. However, the difference is that the input is computed for the whole interval $[0,T]$ (and subsequently for $[T,2T]$ etc) and cannot be updated based on state information. In contrast, in our method the state is fed back into the inputs every θ, where θ can be chosen arbitrarily small, and even continuously in time in the linear/backstepping case, which should lead to better robustness and performance in practice.

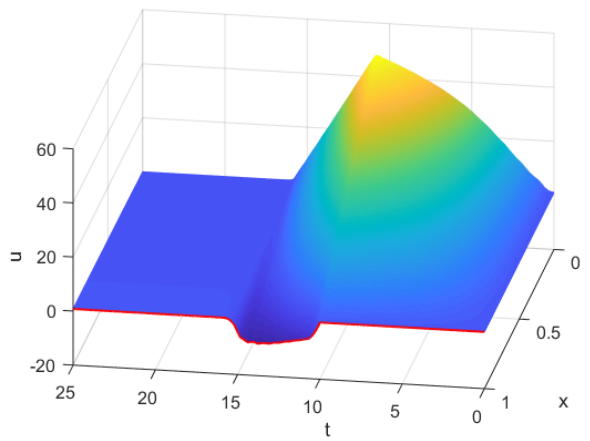

Fig. 2. Simulated trajectory of system (48)-(49) when the controller is switched on at $t = 10$ (using $U(t) = 1$ for $t < 10$, and $u_0 \equiv 1$). $U(t)$ is shown in solid red.

V. Numerical example

We consider an exemplary system with

$$\lambda = 1 + 0.3 \sin(0.05\, u(x,t)) \tag{48}$$

$$f = 1 - 2\cos(u(x,t)) + 1.2\, u(0,t) \tag{49}$$

and $u_0(x) = 1\ \forall x$. The simulated trajectories are depicted in Figure 2. To demonstrate the open-loop behavior, the controller is switched on at $t = 10$. For $t < 10$ we use $U(t) = 1$. As visible in Figure 2, the state diverges for this input. For $t \geq 10$ the feedback controller from Section IV-D is used with U^* as in (34) for $\delta = 10$. As predicted by theory, the controller steers the system to the origin in finite time.

VI. Conclusions

We presented a method for designing state-feedback boundary controllers for a class of 1-d quasilinear hyperbolic systems. The design exploits the predictability of trajectories on the maximum determinate domain, i.e. the part of the domain before the states are affected by the control input due to the delay. Then, subject to a compatibility condition depending on the state predictions, the boundary values at the uncontrolled boundary are designed to converge to zero in finite time with a desired rate. This ensures that the state in the whole domain converges to the origin. Finally, the actual inputs that ensure that the boundary values at the uncontrolled boundary are equal to the desired values, are computed by soving the ODE governing the dynamics on the characteristic lines *backwards in time*.

The method presented here is not limited to systems of form (1)-(3). In fact, the method can directly be applied to systems where λ and f additionally vary with x and t (in a smooth way) and, when evaluated at (x,t), depend on values of $u(\xi, t)$ for $\xi \in [0, x]$. This way, integral terms as in the system considered in [2] can be included.

Moreover, the design method can be further extended to systems with more than one state, and the same approach can also be used to design boundary observers and output feedback controllers. See also [6], [7], [8], [9] for corresponding results for semilinear systems. These extensions are the subject of current work.

Finally, with practical applications in mind, it is of high interest to investigate robustness. Under suitable assumptions on the coefficients one might expect that the trajectories depend on the system coefficients in a smooth way, which would ensure some degree of robustness of closed-loop stability and performance with respect to uncertainty in these coefficients. In this regard, it would be interesting to investigate design trade-offs in the sampling time θ, which determines both the computational load and the length of the prediction horizon.

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