Nonlinearity

PAPER • OPEN ACCESS

Solitary waves in dispersive evolution equations of Whitham type with nonlinearities of mild regularity

OPEN ACCESS
IOP Publishing | London Mathematical Society Nonlinearity
Nonlinearity **33** (2020) 1594–1624 https://doi.org/10.1088/1361-6544/ab60d5

Solitary waves in dispersive evolution equations of Whitham type with nonlinearities of mild regularity

Fredrik Hildrum

Department of Mathematical Sciences, NTNU – Norwegian University of Science and Technology, 7491 Trondheim, Norway

E-mail: fredrik.hildrum@ntnu.no

Received 7 March 2019, revised 27 November 2019
Accepted for publication 11 December 2019
Published 17 February 2020

CrossMark

Recommended by Dr Jean-Claude Saut

Abstract

We show existence of small solitary and periodic traveling-wave solutions in Sobolev spaces H^s, $s > 0$, to a class of nonlinear, dispersive evolution equations of the form

$$u_t + (Lu + n(u))_x = 0,$$

where the dispersion L is a negative-order Fourier multiplier whose symbol is of KdV type at low frequencies and has integrable Fourier inverse K and the nonlinearity n is inhomogeneous, locally Lipschitz and of superlinear growth at the origin. This generalises earlier work by Ehrnström, Groves and Wahlén on a class of equations which includes Whitham's model equation for surface gravity water waves featuring the exact linear dispersion relation. Tools involve constrained variational methods, Lions' concentration-compactness principle, a strong fractional chain rule for composition operators of low relative regularity, and a cut-off argument for n which enables us to go below the typical $s > \frac{1}{2}$ regime. We also demonstrate that these solutions are either waves of elevation or waves of depression when K is nonnegative, and provide a nonexistence result when n is too strong.

Keywords: solitary waves, Whitham-type equations, nonlinear dispersive equations
Mathematics Subject Classification numbers: 35A01; 35A15; 35Q35; 76B03; 76B15; 76B25

1. Introduction

1.1. Background

Many model equations for one-dimensional spacial evolution of water waves [20] may be written as

$$u_t + (Lu + n(u))_x = 0, \tag{1}$$

where L is a dispersive Fourier multiplier operator in space and n represents local nonlinear effects. Much effort has been put into answering whether (1) admits traveling-wave solutions—and in particular, *solitary waves*. Propagating with fixed speed ν and shape, these solutions take the form $(t, x) \mapsto u(x - \nu t)$ with $u(y) \to 0$ as $|y| \to \infty$, and satisfy

$$Lu - \nu u + n(u) = 0 \tag{2}$$

after integrating (1).

In 1967 Whitham [33, 34] proposed a shallow-water model of type (1) with $n(u) = u^2$ and

$$\mathscr{F}(Lu)(\xi) = \sqrt{\frac{\tanh \xi}{\xi}}\, \widehat{u}(\xi)$$

as an alternative to the Korteweg–de Vries (KdV) equation featuring the exact linear dispersion relation for unidirectional water waves influenced by gravity. As seen from

$$\mathfrak{m}(\xi) := \sqrt{\frac{\tanh \xi}{\xi}} = \underbrace{1 - \frac{1}{6}\xi^2}_{\text{KdV symbol}} + \mathcal{O}(\xi^4)$$

and figure 1, it is intuitively reasonable that Whitham's model should both perform better and on a wider range of wave numbers than the KdV equation.

Unfortunately, the nonlocal, singular nature of L—due to $\mathfrak{m}(\xi) \lesssim \langle\xi\rangle^{-\frac{1}{2}}$ being inhomogeneous and decaying very slowly at infinity—seems to have prevented people from rigorously studying the Whitham equation until recently. Significant breakthrough in the last decade, however, has put the original Whitham equation, and also other full-dispersion models, in the spotlight, beginning with the existence of periodic traveling waves by Ehrnström and Kalisch [9] in 2009 and solitary-wave solutions by Ehrnström, Groves and Wahlén [8] in 2012; see also [30]. Research has furthermore confirmed Whitham's conjectures for qualitative wave breaking (bounded wave profile with unbounded slope) in finite time [16] and the existence of highest, cusp-like solutions [10, 12]—now known to also have a convex profile between the stagnation points [13].

Additional analytical and numerical results for the Whitham equation include modulational instability of periodic waves [17, 29], local well-posedness in Sobolev spaces H^s, $s > \frac{3}{2}$, for both solitary and periodic initial data [11, 7, 19], non-uniform continuity of the data-to-solution map [1], symmetry and decay of traveling waves [3], analysis of modeling properties, dynamics and identification of scaling regimes [19], and wave-channel experiments and other numerical studies [2, 5, 18, 32].

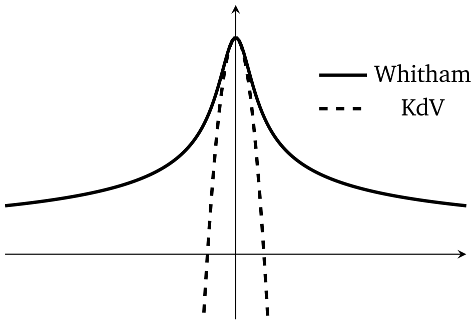


Figure 1. Whitham and KdV symbols.

In total, these investigations have demonstrated the potential usefulness of full-dispersion versions of traditional shallow-water models.

1.2. Assumptions and main results

In this paper we contribute to the longstanding mathematical program of fully understanding the interplay between dispersive and nonlinear effects for the formation of traveling waves. Specifically, we generalise [8], in which the authors proved the existence of small solitary and periodic traveling-wave solutions in the Sobolev space H^1 to a family of equations of the form (1) with 'Whitham-type' symbols—that is, negative-order, inhomogeneous symbols $\mathfrak{m}$ with KdV-type behaviour at low frequencies—and inhomogeneous nonlinearities n being at least quadratic near the origin. Under the following assumptions, we study the existence of solutions to (2) in fractional Sobolev spaces both on the real line and in the periodic setting, noting that $\sigma = -\frac{1}{2}$, $\ell = 1$ and $q = 1$ for the original Whitham equation.

A_1: *Linear, nonlocal dispersive term.*

(i) L is a Fourier multiplier operator with even, inhomogeneous symbol $\mathfrak{m} \colon \mathbb{R} \to \mathbb{R}$ of order $\sigma < 0$, that is,

$$\widehat{Lu} = \mathfrak{m}\,\widehat{u} \qquad \text{and} \qquad |\mathfrak{m}(\xi)| \lesssim \langle \xi \rangle^{\sigma},$$

where $\langle \xi \rangle := \sqrt{1+\xi^2}$.

(ii) $\mathfrak{m}$ is in the Wiener class W_0 of functions with absolutely integrable inverse Fourier transform, so that L is a convolution operator

$$Lu = \frac{1}{\sqrt{2\pi}} K * u$$

with kernel $K := \mathscr{F}^{-1}(\mathfrak{m}) \in \mathrm{L}^1$.

(iii) $\mathfrak{m}$ has a strictly positive unique global maximum at 0 and is $\mathrm{C}^{2\ell}$-regular around 0 for some $\ell \in \mathbb{Z}_+$, with $\mathfrak{m}^{(2\ell)}(0) < 0$. Thus $\mathfrak{m}$ has the Maclaurin expansion

$$\mathfrak{m}(\xi) = \mathfrak{m}(0) + \frac{\mathfrak{m}^{(2\ell)}(0)}{(2\ell)!}\xi^{2\ell} + \mathcal{O}\left(|\xi|^{2\ell+2}\right).$$

A_2: *Nonlinearity.*

$n \colon \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous ($n \in \mathrm{Lip}_{\mathrm{loc}}$) and of the form

$$n(x) = n_q(x) + n_{\mathrm{r}}(x),$$

where the leading-order term, with $q \in (0, 4\ell)$, equals

$$n_q(x) = \gamma|x|^{1+q} \qquad \text{or} \qquad n_q(x) = \gamma x|x|^q$$

for a constant $\gamma \neq 0$ or $\gamma > 0$, respectively, and the remainder satisfies

$$n_{\mathrm{r}}^{(j)}(x) = o\big(|x|^{1+q-j}\big)$$

as $x \to 0$ for all $j = 0, 1, \ldots, \lfloor\varsigma\rfloor$ if $n \in \mathrm{C}^{\varsigma}_{\mathrm{loc}}$ for some real $\varsigma < 1 + q$. In particular,

$$n^{(j)}(x) = \mathcal{O}\big(\,|x|^{1+q-j}\,\big) \qquad \text{for all } j = 0, \ldots, \lfloor\varsigma\rfloor.$$

When n is just in $\mathrm{Lip}_{\mathrm{loc}}$, we assume that $n'(x) = \mathcal{O}(|x|^q)$ almost everywhere as $x \to 0$.

Remark 1.1. We write $A \lesssim B$ or $B \gtrsim A$ if $A \leqslant cB$ for some constant $c > 0$ independent of A and B, and $A \eqsim B$ symbolises that $A \lesssim B \lesssim A$.

In comparison to [8] we consider more general symbols and nonlinearities. We allow for nonlinearities that are merely locally Lipschitz continuous and of superlinear growth ($q > 0$) at the origin, down from $n \in \mathrm{C}^2$ with at least quadratic growth ($q \geqslant 1$) in [8]. In order to allow $q \in (0, 1)$, we on the one hand make use of an order-optimal fractional chain rule; see (4) and section 2.3. On the other hand, we invoke, among other, the Gagliardo–Nirenberg inequality at a certain step, see sections 1.3 and 5, which both improves upon and simplifies the corresponding estimates in [8]. The upper bound $q < 4\ell$, however, is the same in both articles, and we establish that this bound is, in fact, optimal for small solitary waves with sufficiently high speed. Notice also in assumption A_2 that there is some decoupling of the regularity and the growth of n in the sense that $\varsigma < 1 + q$.

As regards the dispersive term, the KdV-type behaviour of $\mathfrak{m}$ at low frequencies in assumption A_1 (iii) coincides with that of [8]. When it comes to global regularity and decay, the authors of [8] assumed negative-order symbols $\mathfrak{m} \in \mathrm{S}^{\sigma}_{\infty}$, that is, $\mathfrak{m} \in \mathrm{C}^{\infty}$ and $\big|\mathfrak{m}^{(j)}(\xi)\big| \lesssim \langle\xi\rangle^{\sigma-j}$ for all $j \in \mathbb{N}_0$. This not only implies that $\mathfrak{m} \in \mathrm{W}_0$, but also that the kernel K is essentially very localised, which was used in [8] to control the nonlocal estimates. As an improvement, we show that all of these estimates, in fact, follow from general properties of convolution with an L^1 kernel, together with decay on $\mathfrak{m}$ itself—omitting any assumptions on its derivatives; see sections 1.3, 2.2, 4 and 6 for more details. For convenience, we include in the appendix a list of recent and practical sufficient conditions for symbols to be in W_0.

Under assumptions A_1 and A_2, we study (2) in the Sobolev space H^s on the real line and in the corresponding P-periodic analogue H^s_P in the periodic setting (see section 2.1 for definitions) for $s > 0$ satisfying

$$\frac{1}{2} - |\sigma| < s < \varsigma, \qquad \text{with } \varsigma < 1 + q, \tag{3}$$

and obtain the following main results.

Theorem 1.2 (Periodic traveling waves). *For each sufficiently small $\mu > 0$ there exists a period $P_\mu > 0$, such that for all $P \geqslant P_\mu$ equation (2) admits a nonconstant solution $u \in \mathrm{H}^s_P \cap \mathrm{L}^\infty$ with $\|u\|^2_{\mathrm{L}^2_P} = 2\mu$ and supercritical wave speed $\nu_P > \mathfrak{m}(0)$. Uniformly over $P \geqslant P_\mu$ these solutions satisfy*

$$\nu_P - \mathfrak{m}(0) \eqsim \mu^{q\alpha} \eqsim \|u\|^q_\infty\,,$$

where $\alpha := \frac{2\ell}{4\ell - q} > \frac{1}{2}$, and

$$\|u\|_{\mathrm{H}^s_P} \eqsim \mu^{\frac{1}{2}}.$$

Theorem 1.3 (Solitary waves). *For each sufficiently small $\mu > 0$ there exists a solution $u \in \mathrm{H}^s \cap \mathrm{L}^\infty$ to (2) with supercritical speed $\nu > \mathfrak{m}(0)$ and $\|u\|_0^2 = 2\mu$ satisfying*

$$\nu - \mathfrak{m}(0) \eqsim \mu^{q\alpha} \eqsim \|u\|_\infty^q ,$$

where α is as in theorem 1.2, and

$$\|u\|_s \eqsim \mu^{\frac{1}{2}}.$$

Remark 1.4. Theorems 1.2 and 1.3 also hold

(i) with no upper bound on s if n is a polynomial with least-order term of order $1 + q \in \mathbb{Z}_+$;
(ii) for $s = 1$ when n is just Lipschitz or C^1 around the origin.

Even if $n'(x) = \mathcal{O}(|x|^q)$ a.e. as $x \to 0$ does not hold in the $\mathrm{Lip}_{\mathrm{loc}}$ case, we still obtain solutions $u \in \mathrm{H}^s_P \cap \mathrm{L}^\infty$ satisfying, uniformly over $P \geqslant P_\mu$, the estimates

$$\nu_P - \mathfrak{m}(0) \eqsim \mu^{q/2} \qquad \text{and} \qquad \|u\|_{\mathrm{H}^s_P} \eqsim \mu^{\frac{1}{2}} \eqsim \|u\|_\infty .$$

The μ-dependent estimates on the wave speed and $\|u\|_\infty$ in theorems 1.2 and 1.3 involve the parameter α, which represents a balance between dispersive and nonlinear effects. Since $\alpha = \infty$ when $q = 4\ell$, one might expect that there are no nontrivial small solutions of (2) with speeds close to $\mathfrak{m}(0)$ if $q \geqslant 4\ell$. This is indeed the case in the solitary-wave setting, and is included in theorem 7.1.

We also demonstrate in theorem 7.2 that bounded solutions of (2) with supercritical speed are either waves of elevation or waves of depression in the special case when K is nonnegative, noting that this result is already known for the Whitham equation [12, corollary 4.4].

In working in fractional Sobolev spaces, both low- and high-order s come with technical difficulties. As in [8], we shall treat solutions of (2) as minimisers of a constrained variational problem, explained in details in section 1.3. When $s \leqslant \frac{1}{2}$, neither H^s nor H^s_P are embedded in L^∞, which unfortunately means that the minimisation problem is unbounded—even locally. We resolve this issue by a cut-off argument for n together with the lower bound $s > \frac{1}{2} - |\sigma|$ in (3). This implies that both $n(u)$ and Lu are in L^∞, and we have therefore essentially regained L^∞ control of (2).

Furthermore, we rely on the highly precise fractional chain rule

$$\|n(u)\|_s \lesssim \|u\|_\infty^q \|u\|_s \tag{4}$$

on $\mathrm{H}^s \cap \mathrm{L}^\infty$ by Runst and Sickel [28, theorem 5.3.4/1 (i)], which allows s to be *arbitrarily* close to ς, and does not seem to be well known. Apart from the immediate case $s \leqslant 1$, an elementary but tedious calculation using the classical higher-order chain rule (Faà di Bruno's formula) establishes (4) provided $u^{(\lfloor \varsigma \rfloor)} \in \mathrm{L}^\infty$, that is, when $s > \lfloor \varsigma \rfloor + \frac{1}{2}$. The general (high-order) result in [28], however, is based on technical harmonic analysis.

1.3. Outline of the variational method

We follow the variational approach in [8, 15], treating solitary-wave solutions as local minimisers of the functional

$$\mathcal{E}(u) := \underbrace{-\frac{1}{2}\int_{\mathbb{R}} uLu\,\mathrm{d}x}_{=:\mathcal{L}(u)} \underbrace{- \int_{\mathbb{R}} N(u)\,\mathrm{d}x}_{=:\mathcal{N}(u)},$$

subject to the constraint that $\mathcal{Q}(u) := \frac{1}{2}\int_{\mathbb{R}} u^2\,\mathrm{d}x$ is held fixed, where

$$N(x) := N_q(x) + N_{\mathrm{r}}(x), \qquad N_q(x) := \frac{xn_q(x)}{2+q} \qquad \text{and} \qquad N_{\mathrm{r}}(x) := \int_0^x n_{\mathrm{r}}(s)\,\mathrm{d}s$$

are primitives of n, n_q and n_{r} vanishing at 0. By Lagrange's multiplier principle, any such minimiser u satisfies

$$\mathcal{E}'(u) + \nu\mathcal{Q}'(u) = 0 \tag{5}$$

for some multiplier $\nu \in \mathbb{R}$, which implies that u solves (2) with wave speed ν. Here primes mean representatives of Fréchet derivatives in L^2; see section 2.4.

Specifically, we minimise $\mathcal{E}$ over a 'constrained ball'

$$U_\mu^s := \left\{u \in \mathrm{H}^s : \|u\|_s < R \text{ and } \mathcal{Q}(u) = \mu\right\}$$

for small $\mu, R > 0$, and show in section 6 that any minimising sequence which stays away from the 'boundary' $\|\cdot\|_s = R$ converges—up to subsequences and translations—in H^{s-} to a nontrivial solution of (2) in H^s with help of Lions' concentration-compactness principle [25] adapted to the fractional setting [27, corollary 3.2].

One must of course confirm the existence of such a minimising sequence. Here the periodic traveling waves come into play. In section 3 we consider the corresponding variational problem for P-periodic traveling waves with functionals $\mathcal{E}_P$, $\mathcal{L}_P$, $\mathcal{N}_P$ and $\mathcal{Q}_P$, where the domain of integration now is $\left(-\frac{P}{2}, \frac{P}{2}\right)$. Both constructively and due to lack of coercivity, we penalise $\mathcal{E}_P$ so that minimising sequences do not come close to the 'boundary' in H_P^s. The (generalised) extreme value theorem yields solutions to the penalised problem, and *a priori* estimates show that the minimisers are unaffected by the penalisation. This establishes most of theorem 1.2, with *uniform* estimates in large P.

We next essentially show that

$$\left\{\begin{matrix}\text{the } P\text{-periodic traveling-wave problem} \\ \text{scaled, truncated and translated to } \left(-\frac{P}{2}, \frac{P}{2}\right)\end{matrix}\right\} \xrightarrow[P\to\infty]{} \text{the solitary-wave problem,}$$

and construct a 'boundary-distant' special minimising sequence for the latter with help of the periodic minimisers. Our approach simplifies and extends [8, lemma 3.3 and theorem 3.8] in that we only use that L is a convolution operator with integrable kernel K in order to deal with the nonlocal effects. In particular, we neither need to assume algebraic-type decay of Lu outside $\left(-\frac{P}{2}, \frac{P}{2}\right)$ for $u \in \mathrm{L}^2$ supported in $\left(-\frac{P}{2}, \frac{P}{2}\right)$ (see [8, proposition 2.1 (ii)]), nor that L commutes with 'the periodisation map' [8, proposition 2.5], although we note that this property remains true in our case. As a byproduct, we can also be less restrictive in the truncation process, as long as we have asymptotic control when $P \to \infty$.

This special minimising sequence, $\{\widetilde{u}_k\}_k$, also guarantees that the quantity

$$I_\mu := \inf\left\{\mathcal{E}(u) : u \in U_\mu^s\right\}$$

is *strictly subadditive*, meaning that

$$I_{\mu_1+\mu_2} < I_{\mu_1} + I_{\mu_2} \qquad \text{whenever } 0 < \mu_1, \mu_2 < \mu_1 + \mu_2 < \mu_\star \tag{6}$$

for some $\mu_\star > 0$, and is proved in section 5. For inhomogeneous n, this relies upon *a priori* estimates for the size and wave speed of $\widetilde{u}_k$. Whereas [8] decomposes $\widetilde{u}_k$ into low- and high-frequency components using sharp frequency cut-offs, we instead apply a smooth decomposition. This seems to be necessary for the estimates to work when $s \leqslant \frac{1}{2}$ in order to guarantee that the L^∞ norm of the high-frequency component is almost bounded by its H^s norm. Furthermore, in order to conclude the *a priori* estimates, the approach in [8] introduces some scaled Sobolev norms with weights depending on μ. The arguments [8, proof of theorem 4.4] seem to require $q \geqslant 1$, but with help of the Gagliardo–Nirenberg inequality, we found that $q > 0$ is possible; see specifically the proof of proposition 5.3.

Strict subadditivity also excludes the unwanted case of dichotomy in Lions' principle, where we again improve upon [8] by only taking into account that L is a convolution operator. Finally, *a priori* estimates for the size and speed of traveling waves then complete the proof of theorems 1.2 and 1.3.

2. Functional-analytic preliminaries

2.1. Spaces

Let

$$\widehat{\varphi}(\xi) := \mathscr{F}(\varphi)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(x)\, \mathrm{e}^{-\mathrm{i}\xi x}\, \mathrm{d}x$$

denote the unitary Fourier transform defined initially on the Schwartz space $\mathscr{S}$ and extended by duality to tempered distributions $\mathscr{S}'$. Define L^q, for $q \geqslant 1$, to be the space of real-valued functions on $\mathbb{R}$ whose norm $\|u\|_{\mathrm{L}^q} := \left(\int_{\mathbb{R}} |u|^q\, \mathrm{d}x\right)^{1/q}$ is finite, with $\|u\|_\infty := (\mathrm{ess})\sup_{x\in\mathbb{R}} |u(x)|$ in the (essentially) bounded L^∞ case. Plancherel's theorem shows that $\mathscr{F}$ is an isometric isomorphism between L^2 and $\left\{\widehat{u} \in \mathrm{L}^2(\mathbb{R} \to \mathbb{C}) : \widehat{u}(-\xi) = \overline{\widehat{u}(\xi)}\right\}$. Next define H^s, for any $s \geqslant 0$, to be the fractional Sobolev space of functions in L^2 with finite norm $\|u\|_s := \|\langle\cdot\rangle^s \widehat{u}\|_{\mathrm{L}^2(\mathbb{R}\to\mathbb{C})}$ and inner product $\langle u, v\rangle_s := \int_{\mathbb{R}} \langle\cdot\rangle^{2s}\, \widehat{u}\, \overline{\widehat{v}}\, \mathrm{d}\xi$, where $\langle\xi\rangle = \sqrt{1+\xi^2}$, and write L^2 for H^0. Since $\langle\xi\rangle^s \eqsim 1 + |\xi|^s$, it follows, in the sense of weak L^2-derivatives, that $\|u\|_s^2 \eqsim \|u\|_0^2 + \|u^{(s)}\|_0^2$ whenever $s \in \mathbb{Z}_+$. In the fractional case $s = k + \sigma$, with $k = \lfloor s \rfloor$ and $\sigma \in (0, 1)$, we also have the more 'local', finite-difference characterisation

$$\|u\|_s^2 \eqsim \|u\|_k^2 + \int_{|h|\leqslant\delta} \left\|\Delta_h^1 u^{(k)}\right\|_0^2 \frac{\mathrm{d}h}{|h|^{1+2\sigma}}$$

where $\Delta_h^1 f := f(\cdot + h) - f$ and $\delta > 0$ (commonly $\delta = \infty$, but only behaviour around $h = 0$ matters). All in all, we may therefore consider the space $\mathrm{H}^s(\Omega)$ of real functions defined on an open set $\Omega \subset \mathbb{R}$ whose norm equals that of H^s, except that L^2 integrals now go over Ω (and with δ appropriately).

In the periodic case, given any $P > 0$ and $q \geqslant 1$, let L^q_P be the space of P-periodic, locally q-integrable functions with norm $\|u\|_{\mathrm{L}^q_P} := \left(\int_{-\frac{P}{2}}^{\frac{P}{2}} |u|^q\, \mathrm{d}x\right)^{1/q}$. In particular, $u \in \mathrm{L}^2_P$ has the Fourier-series representation $u = \sum_{\xi\in\mathbb{Z}} \widehat{u}(\xi)\, e_\xi$, now with $\mathscr{F}$ as an isomorphism $\mathrm{L}^2_P \to \left\{\widehat{u} \in \ell^2(\mathbb{Z}) : \widehat{u}(-\xi) = \overline{\widehat{u}(\xi)}\right\}$, where

$$e_\xi(x) := \frac{\mathrm{e}^{2\pi\mathrm{i}\xi x/P}}{\sqrt{P}} \qquad \text{and} \qquad \widehat{u}(\xi) = \langle u, e_\xi\rangle_{\mathrm{L}^2_P} := \int_{-\frac{P}{2}}^{\frac{P}{2}} u\, \overline{e_\xi}\, \mathrm{d}x.$$

Similarly as above, we introduce the P-periodic real Sobolev space H^s_P, for $s \geqslant 0$, with inner product $\langle u, v\rangle_{\mathrm{H}^s_P} := \sum_{\xi\in\mathbb{Z}} \langle\xi\rangle_P^{2s}\, \widehat{u}(\xi)\, \overline{\widehat{v}(\xi)}$ and norm $\|u\|_{\mathrm{H}^s_P} := \langle u, u\rangle^{\frac{1}{2}}_{\mathrm{H}^s_P}$, where $\langle\xi\rangle_P := \left\langle \frac{2\pi\xi}{P}\right\rangle$. Again write L^2_P for H^0_P and note that

$$\|u\|^2_{\mathrm{H}^s_P} \eqsim \|u\|^2_{\mathrm{H}^k_P} + \int_{|h|\leqslant\delta} \left\|\Delta^1_h u^{(k)}\right\|^2_{\mathrm{L}^2_P} \frac{\mathrm{d}h}{|h|^{1+2\sigma}} \tag{7}$$

with $0 < \delta < \frac{P}{2}$, omitting the last term if $s \in \mathbb{Z}_+$. Thus $\|u\|_{\mathrm{H}^s_P} \eqsim \|u\|_{\mathrm{H}^s\left(-\frac{P}{2},\frac{P}{2}\right)}$ for $u \in \mathrm{H}^s_P$. Moreover, for any $\varphi_P \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R} \to [0,1])$ which is 1 in $\left(-\frac{P}{2}, \frac{P}{2}\right)$ and 0 in $\left\{|x| \geqslant \frac{P}{2} + \tau\right\}$ for fixed $\tau \lesssim P_{\min}$, we have

$$\|u\|_{\mathrm{H}^s_P} \eqsim \|\varphi_P u\|_s \tag{8}$$

uniformly in $P \geqslant P_{\min} > 0$. Equation (8) demonstrates that H^s_P is locally in H^s and that results for $\|\cdot\|_s$ carry over to $\|\cdot\|_{\mathrm{H}^s_P}$—in particular, we need not bother with the P-dependence in the hidden estimation constants. For example, when $s > \frac{1}{2}$, there is a continuous embedding of H^s into L^∞, and hence, $\mathrm{H}^s_P \hookrightarrow \mathrm{L}^\infty$ also.

2.2. Action of L on H^s *and* H^s_P

It follows immediately from $|\mathfrak{m}(\xi)| \lesssim \langle\xi\rangle^\sigma$ that L maps H^s continuously into $\mathrm{H}^{s+|\sigma|}$ for any s. Its action on periodic spaces, however, is less trivial. If $\mathfrak{m} \in \mathrm{C}^\infty$, then L maps $\mathscr{S}$ to itself and so it extends to a continuous operator $L\colon \mathscr{S}' \to \mathscr{S}'$ still satisfying $\widehat{Lu} = \mathfrak{m}\,\widehat{u}$. In particular,

$$\widehat{Lu}(\xi) = \mathfrak{m}\Big(\frac{2\pi\xi}{P}\Big)\,\widehat{u}(\xi), \qquad \xi \in \mathbb{Z}, \tag{9}$$

for P-periodic distributions, so that $L\colon \mathrm{H}^s_P \to \mathrm{H}^{s+|\sigma|}_P$ continuously. Fortunately, there is a more direct approach to the periodic case which also works for irregular symbols in W_0.

Proposition 2.1. *Convolution is a continuous bilinear operator* $\mathrm{L}^1 * \mathrm{L}^q_P \hookrightarrow \mathrm{L}^q_P$ *for all* $q \in [1,\infty]$. *In fact, if* $f \in \mathrm{L}^1$ *and* $u \in \mathrm{L}^q_P$, *then* $f * u = f_P *_P u$ *a.e., where*

$$f_P := \sum_{j\in\mathbb{Z}} f(\cdot + jP) \in \mathrm{L}^1_P \qquad \text{and} \qquad f_P *_P u := \int_{-\frac{P}{2}}^{\frac{P}{2}} f_P(y)\, u(\cdot - y)\,\mathrm{d}y.$$

Moreover,

$$\widehat{f_P}(\xi) = \sqrt{\frac{2\pi}{P}}\,\widehat{f}\Big(\frac{2\pi\xi}{P}\Big), \qquad \xi \in \mathbb{Z}, \tag{10}$$

relating the Fourier coefficients of f_P *with the Fourier transform of* f.

Proof. Intuitively, we reduce $\mathrm{L}^1 * \mathrm{L}^q_P \hookrightarrow \mathrm{L}^q_P$ to a special case of $\mathrm{L}^1_P *_P \mathrm{L}^q_P \hookrightarrow \mathrm{L}^q_P$. Since, in the most general case $q = 1$,

$$\int_{\mathbb{R}} \int_{-\frac{P}{2}}^{\frac{P}{2}} |f(y)\, u(x-y)|\,\mathrm{d}x\,\mathrm{d}y = \|f\|_{\mathrm{L}^1}\, \|u\|_{\mathrm{L}^1_P} < \infty,$$

we find from the Fubini–Tonelli theorem that $f * u$ exists a.e. and is in L^1_P. Subsequently we may then compute

$$\begin{aligned} f * u &= \sum_{j\in\mathbb{Z}} \int_{-\frac{P}{2}}^{\frac{P}{2}} f(y+jP)\, u(\cdot - y)\, \mathrm{d}y \\ &= \int_{-\frac{P}{2}}^{\frac{P}{2}} \sum_{j\in\mathbb{Z}} f(y+jP)\, u(\cdot - y)\, \mathrm{d}y = f_P *_P u \end{aligned}$$

by dominated convergence, periodicity of u plus the fact that $f_P \in \mathrm{L}^1_P$. With this representation Young's inequality gives

$$\|f * u\|_{\mathrm{L}^q_P} = \|f_P *_P u\|_{\mathrm{L}^q_P} \leqslant \|f_P\|_{\mathrm{L}^1_P} \|u\|_{\mathrm{L}^q_P},$$

and the result follows, noting that $\|f_P\|_{\mathrm{L}^1_P} \leqslant \|f\|_{\mathrm{L}^1}$. Similar reasoning also implies (10). □

Directly from proposition 2.1 and the convolution theorem for $\mathscr{F}$ we then obtain the following result.

Proposition 2.2. *L is a Fourier multiplier on L^2_P of the form (9), mapping H^s_P to $\mathrm{H}^{s+|\sigma|}_P$ continuously.*

Bear in mind that proposition 2.1 is by no means true for general $f \in \mathrm{L}^1$ if L^q_P is replaced by $\mathrm{L}^q_{\mathrm{loc}}$; it is the periodic structure that saves us.

2.3. Cut-off argument and estimates for n

In studying (5), we will need that n—or more precisely, the induced operator $n(u)(x) := n(u(x))$—is well-defined on $\mathrm{H}^s \cap \mathrm{L}^\infty$ and satisfies a 'fractional chain rule'. Specifically, the following result [28, theorem 5.3.4/1 (i)] holds. Its proof is based on a Taylor expansion of n and maximal-function techniques on dyadic scales to control the remainder.

Proposition 2.3 (Fractional chain rule). *Consider the case $n \in \mathrm{Lip}_{\mathrm{loc}}$ or $n \in \mathrm{C}^1_{\mathrm{loc}}$ with $s \in [0,1]$ in assumption A_2, or the case $n \in \mathrm{C}^\varsigma_{\mathrm{loc}}$ with $\varsigma \in (1, 1+q)$ and $s \in [0,\varsigma)$. Then n induces a composition operator on $\mathrm{H}^s \cap B$ satisfying*

$$\|n(u)\|_s \lesssim \|u\|_\infty^q \|u\|_s, \tag{11}$$

where B is a sufficiently small ball around 0 in L^∞. If n is a monomial of order $1+q \in \mathbb{Z}_+$, then (11) holds for all $s \geqslant 0$.

Chain rule-type results with gaps between s and $1+q$ are common in the literature, e.g. [6, section 3], but it does not seem to be well known that one can let s be arbitrarily close to the regularity index of the outer function.

Since we shall find solitary waves from the periodic problem as $P \to \infty$, it is very important that (11) extends to H^s_P and holds *uniformly* in $P \geqslant P_{\min}$. Estimating

$$\|n(u)\|_{\mathrm{H}^s_P} \eqsim \|n(\varphi_P u)\|_s \lesssim \|\varphi_P u\|_\infty^q \|\varphi_P u\|_s \eqsim \|u\|_\infty^q \|u\|_{\mathrm{H}^s_P} \tag{12}$$

with help of (8), shows that this is indeed the case. The first equivalence is a natural extension of (8) and proved in the same fashion using Leibniz' rule ($\lfloor s \rfloor$ times) plus the fact that $\|\varphi_P^{(k)}\|_\infty \lesssim \tau^{-k} \lesssim 1$ uniformly in P.

Corollary 2.4 (Fractional chain rule on H^s_P). *Suppose under assumption* A_2 *that* $n \in \mathrm{Lip}_{\mathrm{loc}}$ *or* $n \in \mathrm{C}^1_{\mathrm{loc}}$ *with* $s \in [0,1]$*, or* $n \in \mathrm{C}^{\varsigma}_{\mathrm{loc}}$ *with* $\varsigma \in (1, 1+q)$ *and* $s \in [0,\varsigma)$*. Then* n *induces a composition operator on* $\mathrm{H}^s_P \cap B$ *satisfying, uniformly in* P *bounded away from* 0,

$$\|n(u)\|_{\mathrm{H}^s_P} \lesssim \|u\|^q_\infty \, \|u\|_{\mathrm{H}^s_P}, \tag{13}$$

where B *is a sufficiently small ball around* 0 *in* L^∞*. If* n *is a monomial of order* $1+q \in \mathbb{Z}_+$*, then (13) holds for all* $s \geqslant 0$*.*

In the *a priori* unbounded case $s \leqslant \frac{1}{2}$, we also cut off the growth of n and consider instead

$$\widetilde{n}(x) = \begin{cases} n(x) & \text{if } |x| \leqslant A_\mu; \\ n(A_\mu \operatorname{sgn} x) & \text{if } |x| > A_\mu, \end{cases} \tag{14}$$

where $A_\mu \sim \mu^\theta$ and $\theta \in \left(0, \frac{1}{2}\right)$. Then

$$|\widetilde{n}(x)| \lesssim \mu^{\theta q} \, |x| \tag{15}$$

for *all* $x \in \mathbb{R}$ for μ sufficiently small. Moreover, now $\widetilde{n}$ is globally Lipschitz and satisfies, directly from (7),

$$\|\widetilde{n}(u)\|_{\mathrm{H}^s_P} \lesssim \mu^{\theta q} \, \|u\|_{\mathrm{H}^s_P}.$$

This estimate mimics the fractional chain rule (13) up to a small loss in the exponent q. We shall obtain that $\|u^\star_P\|^2_\infty \lesssim \mu$ for solutions $u^\star_P$ of the modified variational problem with $\widetilde{n}$ replaced by n. Therefore, since $\theta < \frac{1}{2}$, we get $\|u^\star_P\|_\infty \leqslant A_\mu$ for all sufficiently small μ. In other words, $\widetilde{n}(u^\star_P) = n(u^\star_P)$, and so $u^\star_P$ in fact solves the original problem. For the sake of brevity, write n for $\widetilde{n}$ from now on.

Proposition 2.3 and corollary 2.4 naturally restrict the range of feasible s from above. As regards a lower bound, we need $u^\star_P \in \mathrm{L}^\infty$. By construction $n(u^\star_P) \in \mathrm{L}^\infty$, and so from (2) it suffices that $Lu^\star_P \in \mathrm{L}^\infty$. This follows whenever $s > \frac{1}{2} - |\sigma|$ in light of $L\colon \mathrm{H}^s_P \to \mathrm{H}^{s+|\sigma|}_P$. Furthermore, (2) also yields

$$\left(\nu_P - \mu^{\theta q}\right) \|u^\star_P\|_\infty \lesssim \|Lu^\star_P\|_\infty \lesssim \|Lu^\star_P\|_{\mathrm{H}^{s+|\sigma|}_P} \lesssim \|u^\star_P\|_{\mathrm{H}^s_P}. \tag{16}$$

Hence, as we will establish that ν_P is uniformly bounded away from 0 and $\|u^\star_P\|^2_{\mathrm{H}^s_P} \lesssim \mu$ in lemmas 3.5 and 3.6, this gives $\|u^\star_P\|^2_\infty \lesssim \mu$ for all sufficiently small μ. Similar reasoning applies in the solitary-wave case.

2.4. *Properties of functionals*

Finally, we list some basic features of $\mathcal{L}$, $\mathcal{N}$, $\mathcal{Q}$ and their periodic counterparts. By weak continuity of an operator we mean that the operator maps weakly convergent sequences to strongly convergent sequences, which in the result below follows from the compact embedding of H^s_P in H^t_P whenever $s > t$.

Proposition 2.5. *If* $s \geqslant 0$*, then* $\mathcal{L}, \mathcal{Q}, \mathcal{N} \in \mathrm{C}^1(\mathrm{H}^s \to \mathbb{R})$ *and* $\mathcal{L}_P, \mathcal{N}_P, \mathcal{Q}_P \in \mathrm{C}^1(\mathrm{H}^s_P \to \mathbb{R})$ *have* L^2 *and* L^2_P *derivatives, respectively, given by*

$$\mathcal{L}'(u) := -Lu, \qquad \mathcal{N}'(u) := -n(u) \qquad \text{and} \qquad \mathcal{Q}'(u) := u.$$

Moreover, if $s > 0$*, then* $\mathcal{L}_P$*,* $\mathcal{N}_P$ *and thus also* $\mathcal{E}_P$ *are weakly continuous on* H^s_P*.*

3. Penalised variational problem for periodic traveling waves

In this section we prove theorem 1.2 by finding a constrained local minimiser of $\mathcal{E}_P$ satisfying the Lagrange multiplier principle. Specifically, we look for a minimiser $u_P^\star$ in the set

$$U_{P,\mu}^s := U_{P,\mu}^{s,R} := \left\{ u \in \mathrm{H}_P^s : \|u\|_{\mathrm{H}_P^s} < R \text{ and } \mathcal{Q}_P(u) = \mu \right\}$$

for which $\mathcal{E}_P'(u_P^\star) + \nu_P \mathcal{Q}_P'(u_P^\star) = 0$ for a multiplier $\nu_P \in \mathbb{R}$. Since $\mathcal{E}_P$ is noncoercive, however, minimising sequences may approach the 'boundary' $\|\cdot\|_{\mathrm{H}_P^s} = R$ of $U_{P,\mu}^s$, where Lagrange's principle might fail. In order to resolve this issue, we introduce a smooth, increasing *penaliser* $\varrho\colon \left[0, (2R)^2\right) \to [0, \infty)$ satisfying

$$\varrho(t) = 0 \text{ when } 0 \leqslant t \leqslant R^2 \qquad \text{and} \qquad \varrho(t) \nearrow \infty \text{ as } t \nearrow (2R)^2,$$

and instead minimise

$$\mathcal{E}_{P,\varrho}(u) := \mathcal{E}_P(u) + \varrho\left(\|u\|_{\mathrm{H}_P^s}^2 \right)$$

over the larger set $\widetilde{U}_{P,\mu} := U_{P,\mu}^{s,2R}$, see figure 2. For technical reasons, we also assume that for every $a \in (0,1)$ there exists $b > 1$ such that

$$\varrho'(t) \lesssim \varrho(t)^a + \varrho(t)^b \tag{17}$$

for all $t \in \left[R^2, (2R)^2\right)$. An example [8, section 3], up to appropriate scaling, is given by

$$t \mapsto \begin{cases} \frac{\mathrm{e}^{-1/(t-R^2)}}{(2R)^2 - t} & \text{if } t \in \left(R^2, (2R)^2\right); \\ 0 & \text{if } t \in \left[0, R^2\right]. \end{cases}$$

A priori estimates below show that ϱ is inactive at the minimum, and hence $u_P^\star \in U_{P,\mu}^s$, as desired.

Lemma 3.1. *$\mathcal{E}_{P,\varrho}$ admits a minimiser $u_P^\star \in \widetilde{U}_{P,\mu}$ satisfying the Euler–Lagrange equation*

$$\langle Lu_P^\star + n(u_P^\star) - \nu_P u_P^\star, w \rangle_{\mathrm{L}_P^2} = 2\varrho'\left(\|u_P^\star\|_{\mathrm{H}_P^s}^2\right) \langle u_P^\star, w \rangle_{\mathrm{H}_P^s} \tag{18}$$

for all $w \in \mathrm{H}_P^s$, where $\nu_P \in \mathbb{R}$ is the multiplier. If $\varrho' > 0$, then $u_P^\star \in \mathrm{H}_P^{3s}$.

Proof. Since ϱ is weakly lower semi-continuous and coercive, so is $\mathcal{E}_{P,\varrho}$ by proposition 2.5. Hence, it suffices to search for minimisers in the subset $\{u \in \widetilde{U}_{P,\mu} : \|u\|_{\mathrm{H}_P^s} \leqslant R'\}$ for some $R' < 2R$. This set is weakly closed by the compact embedding $\mathrm{H}_P^s \hookrightarrow \mathrm{L}_P^2$ for $s > 0$ together with the fact that closed balls are weakly closed (a consequence of Mazur's lemma). Existence of a minimiser $u_P^\star$ now follows from the generalised extreme value theorem ([31, theorem 1.2]). Evaluating

$$\langle \mathcal{U}_P'(u_P^\star), u_P^\star \rangle_{\mathrm{L}_P^2} = 2\mathcal{U}_P(u_P^\star) > 0$$

shows that $\langle \mathcal{U}_P'(u_P^\star), \cdot \rangle_{\mathrm{L}_P^2}$ does not vanish identically, and so Lagrange's principle gives (18).

As regards regularity, note that (18) especially holds for all w in the Fourier basis, implying that

$$\widehat{Lu_P^\star} + \widehat{n(u_P^\star)} - \nu_P \widehat{u_P^\star} = 2\varrho'\left(\|u_P^\star\|_{\mathrm{H}_P^s}^2\right) \langle \cdot \rangle_P^{2s} \widehat{u_P^\star} \tag{19}$$

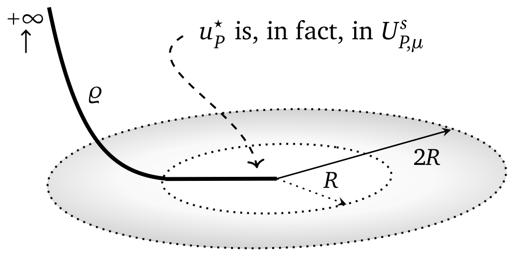


Figure 2. Illustrating the penalised problem.

pointwise in $\mathbb{Z}$. Since $u_P^\star, Lu_P^\star, n(u_P^\star) \in \mathrm{H}_P^s$, we get $\langle\cdot\rangle_P^{2s}\widehat{u_P^\star} \in \mathscr{F}(\mathrm{H}_P^s)$ if $\varrho' > 0$, that is, $u_P^\star \in \mathrm{H}_P^{3s}$. □

Perhaps $u_P^\star$ is just a constant solution of (2)? Due to the constraint $\mathcal{Q}_P(u) = \mu$, such solutions, if they exist, can only be of the form $u_{\text{trivial}} := \pm\sqrt{2\mu/P}$. Inserting u_{trivial} into (2) gives

$$(\nu_P - \mathfrak{m}(0))\, u_{\text{trivial}} = n(u_{\text{trivial}}),$$

and since n is superlinear near the origin, we observe that u_{trivial} will solve (2) when $\nu_P > \mathfrak{m}(0)$ for suitable μ and P with u_{trivial} small enough. In fact, constant solutions may also exist at subcritical speeds $\nu_P < \mathfrak{m}(0)$—for example if $u_{\text{trivial}} < 0$ and $n(x) \equiv n_q(x) = \gamma\, |x|^{1+q}$, with $\gamma > 0$. Fortunately, however, lemma 3.3 demonstrates that u_{trivial} does not minimise $\mathcal{E}_{P,\varrho}$ for sufficiently small μ and large P.

Lemma 3.2. *For all $q > 0$ it is true that*

$$\Gamma_q := \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sqrt{\frac{2}{3}}\,(1+\sin x)\right)^{2+q} \mathrm{d}x > 1.$$

Proof. Define $f(x) = \left(\sqrt{\frac{2}{3}}\,(1+\sin x)\right)^2$ and $\varphi(x) = x^{(2+q)/2}$. Then Jensen's inequality with strict convexity gives

$$\Gamma_q = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi\big(f(x)\big)\,\mathrm{d}x > \varphi\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x\right) = \varphi(1) = 1.$$

□

Lemma 3.3. *For all sufficiently small $\mu > 0$ there exists $P_\mu > 0$ such that u_{trivial} does not minimise $\mathcal{E}_{P,\varrho}$ on $\widetilde{U}_{\mu,P}$ and*

$$\inf\left\{\mathcal{E}_{P,\varrho}(u) : u \in \widetilde{U}_{P,\mu}\right\} < -\mu\left[\mathfrak{m}(0) + C\left(\frac{2\mu}{P}\right)^{q/2}\right] \tag{20}$$

whenever $P \geqslant P_\mu$, where $C > 0$. If $n = n_q$, we explicitly have $C = 2\,|\gamma|\,/(2+q)$.

Proof. Constructively,

$$u(x) := A\mathrm{sgn}(\gamma)\sqrt{\frac{2}{3}}\left[1 + \sin\left(\frac{2\pi x}{P}\right)\right],$$

scaled to obey $\mathcal{U}_P(u) = \mu$, where $A := \sqrt{2\mu/P}$, will be shown to satisfy both

$$\mathcal{E}_P(u) < -\mu\left(\mathfrak{m}(0) + CA^q\right) \qquad \text{and} \qquad \mathcal{E}_P(u) < \mathcal{E}_P(u_{\text{trivial}}) \tag{21}$$

for suitable μ, P, and $C > 0$. As u lies in $U^s_{P,\mu}$, where $\mathcal{E}_{P,\varrho} \equiv \mathcal{E}_P$, for sufficiently small μ, this proves the claim. (Note that it suffices to only consider positive u_{trivial}, because $\mathcal{E}_{P,\varrho}(A) \leqslant \mathcal{E}_{P,\varrho}(-A)$.)

Indeed,

$$\mathcal{E}_P(A\operatorname{sgn}\gamma) = -\mu\left[\mathfrak{m}(0) + \frac{2|\gamma|}{2+q}A^q + o(A^q)\right],$$

and

$$\mathcal{E}_P(u_{\text{trivial}}) \geqslant \mathcal{E}_P(A\operatorname{sgn}\gamma)$$

provided A is sufficiently small (this condition safeguards a possible issue when $n_q(x) = \gamma|x|^q$ and the signs of u_{trivial} and γ coincide). Nonzero Fourier coefficients of u are $\widehat{u}_0 = 2\sqrt{\mu/3}$ and $|\widehat{u}_{\pm 1}| = \sqrt{\mu/3}$, so that $\|u\|_s^2 = \frac{2}{3}\mu\left(2 + \langle 1\rangle_P^{2s}\right)$ is controlled by μ. Moreover, expanding $\mathfrak{m}$ gives that

$$\begin{aligned}\mathcal{L}_P(u) &= -\mu\left[\frac{2}{3}\mathfrak{m}(0) + \frac{1}{3}\mathfrak{m}\left(\frac{2\pi}{P}\right)\right] \\ &= -\mu\left[\mathfrak{m}(0) + cP^{-2\ell} + \mathcal{O}\left(P^{-2\ell-2}\right)\right]\end{aligned}$$

for $c := \mathfrak{m}^{(2\ell)}(0)/(2\ell)! < 0$. With Γ_q from lemma 3.2, this yields, after a change of variables in $\mathcal{N}_P(u)$, that

$$\mathcal{E}_P(u) = -\mu\left[\mathfrak{m}(0) + cP^{-2\ell} + \mathcal{O}\left(P^{-2\ell-2}\right) + \frac{2|\gamma|}{2+q}\Gamma_q A^q + o(A^q)\right].$$

Consequently, the first inequality in (21) then holds for A sufficiently small, while, since $\Gamma_q > 1$ and $q < 4\ell$, the second inequality becomes true for A sufficiently small and P large enough. □

Remark 3.4. Bound (20) has not optimal order with respect to q and has the defect of depending on P. By comparing with the solitary-wave problem, however, we can do better; see lemma 5.1.

Closely based on [8, lemmas 3.5–6] we next establish that $\varrho'\left(\|u_P^\star\|^2_{\mathrm{H}^s_P}\right)$ eventually vanishes based on a lower bound on ν_P and an *a priori* estimate for $\|u_P^\star\|_{\mathrm{H}^s_P}$.

Lemma 3.5. *With μ and P_μ as in lemma 3.3, the estimate*

$$\nu_P - \mathfrak{m}(0) > \widetilde{C}\left(\frac{2\mu}{P}\right)^{q/2} - c_\varrho\mu^\lambda + \begin{Bmatrix}\mathcal{O}\left(\mu^{\theta q}\right) & \text{if } s \leqslant \frac{1}{2} \\ o\left(\|u_P^\star\|_\infty^q\right) & \text{if } s > \frac{1}{2}\end{Bmatrix} \tag{22}$$

holds over the set of minimisers $u_P^\star$ of $\mathcal{E}_{P,\varrho}$ over $\widetilde{U}_{P,\mu}$ and $P \geqslant P_\mu$. Here $\widetilde{C} > 0$ (equals $|\gamma|$ if $n = n_q$), $\lambda > 0$, and $c_\varrho \geqslant 0$ vanishes when $\varrho = 0$.

Proof. Write $u := u_P^\star$ for clarity. We shall obtain (22) using the identity

$$\langle Lu + n(u), u\rangle_{\mathrm{L}^2_P} = -(2+q)\mathcal{E}_P(u) + q\mathcal{L}_P(u) - \int_{-\frac{P}{2}}^{\frac{P}{2}} \left[(2+q)N(u) - un(u)\right] \mathrm{d}x, \tag{23}$$

where the last integral vanishes if n is homogeneous.

First choose $w = u$ in (18) and observe that

$$2\nu_P\mu \geqslant \langle Lu + n(u), u\rangle_{\mathrm{L}^2_P} - \varrho'\left(\|u\|^2_{\mathrm{H}^s_P}\right) \cdot 4R^2.$$

Since

$$-\mathcal{E}_P(u) = -\mathcal{E}_{P,\varrho}(u) + \varrho\left(\|u\|^2_{\mathrm{H}^s_P}\right) > \mu\left[\mathfrak{m}(0) + C\left(\frac{2\mu}{P}\right)^{q/2}\right]$$

by (20) and $\varrho \geqslant 0$, and $\mathcal{L}_P(u) \geqslant -\mathfrak{m}(0)\mu$, we deduce from (23) that

$$\nu_P - \mathfrak{m}(0) > \underbrace{\frac{2+q}{2}C}_{=:\widetilde{C}}\left(\frac{2\mu}{P}\right)^{q/2} - \mu^{-1}\varrho'\left(\|u\|^2_{\mathrm{H}^s_P}\right)\cdot 4R^2 + \begin{Bmatrix} \mathcal{O}\left(\mu^{\theta q}\right) & \text{if } s \leqslant \frac{1}{2} \\ o\left(\|u\|^q_\infty\right) & \text{if } s > \frac{1}{2}\end{Bmatrix},$$

because

$$(2+q)N(u(x)) - u(x)n(u(x)) = \begin{Bmatrix} \mathcal{O}\left(|u(x)|^2\mu^{\theta q}\right) & \text{if } s \leqslant \frac{1}{2} \\ o\left(|u(x)|^{2+q}\right) & \text{if } s > \frac{1}{2}\end{Bmatrix}$$

uniformly over $u \in \widetilde{U}_{P,\mu}$ and $x \in \mathbb{R}$, where we used (15) when $s \leqslant \frac{1}{2}$.

It remains to establish that $\varrho'\left(\|u\|^2_{\mathrm{H}^s_P}\right) \lesssim \mu^{1+\lambda}$ for some $\lambda > 0$, and using (17), it suffices to prove that $\varrho\left(\|u\|^2_{\mathrm{H}^s_P}\right) \lesssim \mu^{1+\tilde{\lambda}}$ for some $\tilde{\lambda} > 0$. Crudely, we have $\mathcal{E}_{P,\varrho}(u) < -\mu\mathfrak{m}(0)$, and so

$$\varrho\left(\|u\|^2_{\mathrm{H}^s_P}\right) < -\mu\mathfrak{m}(0) - \mathcal{L}_P(u) - \mathcal{N}_P(u) \leqslant -\mathcal{N}_P(u).$$

If $s \leqslant \frac{1}{2}$, then $-\mathcal{N}_P(u) \lesssim \mu^{1+\theta q}$ directly from $|N(x)| \lesssim \mu^{\theta q}|x|^2$. In case $s > \frac{1}{2}$, then $-\mathcal{N}_P(u) \lesssim \mu\|u\|^q_\infty$. Choose $\vartheta \in (0,1)$ such that $\widetilde{s} := (1-\vartheta)s \in \left(\frac{1}{2}, s\right)$. By interpolation,

$$\|u\|_\infty \lesssim \|u\|_{\mathrm{H}^{\widetilde{s}}_P} \leqslant \|u\|^{\vartheta}_{\mathrm{L}^2_P}\|u\|^{1-\vartheta}_{\mathrm{H}^s_P} \lesssim \|u\|^{\vartheta}_{\mathrm{L}^2_P} \tag{24}$$

uniformly over $u \in \widetilde{U}_{P,\mu}$ and $P \geqslant P_\mu$, from which it follows that $\varrho\left(\|u\|^2_{\mathrm{H}^s_P}\right) \lesssim \mu^{1+\vartheta q}$. □

Lemma 3.6. *The estimate*

$$\|u_P^\star\|_{\mathrm{H}^s_P} \eqsim \mu^{\frac{1}{2}}$$

holds uniformly over the set of minimisers of $\mathcal{E}_{P,\varrho}$ over $\widetilde{U}_{P,\mu}$ and $P \geqslant P_\mu$.

Proof. Let $u := u_P^\star$ for convenience. Using $w := \mathscr{F}^{-1}\left(\langle\cdot\rangle_P^{2s}\widehat{\overline{u}}\right) \in \mathrm{H}^s_P$ in (18) if $\varrho' > 0$, or multiplying (19) by $\langle\cdot\rangle_P^{2s}\widehat{\overline{u}}$ and summing over $\mathbb{Z}$ if $\varrho' = 0$, we find—with the strong zero-convention $(0\cdot\infty = 0)$—that

$$\begin{aligned}\nu_P\|u\|^2_{\mathrm{H}^s_P} &= \langle Lu + n(u), u\rangle_{\mathrm{H}^s_P} - 2\varrho'\left(\|u\|^2_{\mathrm{H}^s_P}\right)\|u\|^2_{\mathrm{H}^{2s}_P} \\ &\leqslant \|u\|^2_{\mathrm{H}^{s+\frac{\sigma}{2}}_P} + \|n(u)\|_{\mathrm{H}^s_P}\|u\|_{\mathrm{H}^s_P},\end{aligned}$$

because $|\langle Lu, u\rangle_{\mathrm{H}^s_P}| \lesssim \|u\|^2_{\mathrm{H}_P^{s+\frac{\sigma}{2}}}$ by assumption on $\mathfrak{m}$. If $s > \frac{1}{2}$, the fractional chain rule (corollary 2.4) and (24) imply

$$\|n(u)\|_{\mathrm{H}^s_P} \lesssim \|u\|^q_\infty \|u\|_{\mathrm{H}^s_P} \lesssim \mu^{\vartheta q/2} \|u\|_{\mathrm{H}^s_P},$$

while if $s \leqslant \frac{1}{2}$, then

$$\|n(u)\|_{\mathrm{H}^s_P} \lesssim \mu^{\theta q} \|u\|_{\mathrm{H}^s_P}.$$

From lemma 3.5, combined with (24) when $s > \frac{1}{2}$, we find that ν_P is uniformly bounded away from 0 for all sufficiently small μ, uniformly over the set of minimisers of $\mathcal{E}_{P,\varrho}$ over $\widetilde{U}_{P,\mu}$ and $P \geqslant P_\mu$. Hence, with μ possibly even smaller,

$$\|u\|^2_{\mathrm{H}^s_P} \lesssim \|u\|^2_{\mathrm{H}_P^{s+\frac{\sigma}{2}}}.$$

Interpolating

$$\|u\|^2_{\mathrm{H}_P^{s+\frac{\sigma}{2}}} \leqslant \|u\|^{|\sigma|/s}_{\mathrm{L}^2_P} \|u\|^{2-(|\sigma|/s)}_{\mathrm{H}^s_P}$$

if $\sigma > -2s$, or using that $\|u\|^2_{\mathrm{H}_P^{s+\frac{\sigma}{2}}} \leqslant \|u\|^2_{\mathrm{L}^2_P}$ if $\sigma \leqslant -2s$, then gives $\|u\|_{\mathrm{H}^s_P} \lesssim \|u\|_{\mathrm{L}^2_P}$, and in combination with $\|u\|_{\mathrm{H}^s_P} \geqslant \|u\|_{\mathrm{L}^2_P}$ and $\|u\|_{\mathrm{L}^2_P} = (2\mu)^{\frac{1}{2}}$, this concludes the proof. □

According to lemma 3.6, ϱ vanishes for sufficiently small μ, and so $u_P^\star$ is in fact a minimiser for $\mathcal{E}_P$ over $U^s_{P,\mu}$ satisfying $\|u_P^\star\|_\infty \lesssim \|u_P^\star\|_{\mathrm{H}^s_P} \eqsim \mu^{\frac{1}{2}}$, where we remember estimate (16). In particular, $u_P^\star$ solves (2) with wave speed ν_P, noting that

$$\nu_P - \mathfrak{m}(0) \lesssim \|u_P^\star\|^q_\infty \lesssim \mu^{q/2} \tag{25}$$

uniformly over $P \geqslant P_\mu$, which follows from

$$(\nu_P - \mathfrak{m}(0))\, \|u_P^\star\|_{\mathrm{L}^2_P} \leqslant \|(\nu_P - L)u_P^\star\|_{\mathrm{L}^2_P} = \|n(u_P^\star)\|_{\mathrm{L}^2_P} \lesssim \|u_P^\star\|^q_\infty \|u_P^\star\|_{\mathrm{L}^2_P}.$$

In order to finish theorem 1.2, it remains to establish the improved bounds on ν_P and $\|u_P^\star\|_\infty$. This will be done in section 5; see the discussion following corollary 5.5.

4. From the periodic to the solitary-wave problem: a special minimising sequence

As outlined in section 1.3, we now construct a special minimising sequence for the solitary-wave problem with help of suitable scalings, truncations and translations of $u_P^\star$. To this end, we first establish a general asymptotic result as $P \to \infty$ for convolution operators with integrable kernels.

Lemma 4.1. *Let $f \in \mathrm{L}^1$ and $\{\widetilde{u}_P\}_P \subset \mathrm{H}^s$ be a bounded family of functions with $\operatorname{supp} \widetilde{u}_P \subset \left(-\frac{P}{2}, \frac{P}{2}\right)$, and associate, for each P, the periodic extension $u_P := \sum_{j\in\mathbb{Z}} \widetilde{u}_P(\cdot + jP) \in \mathrm{H}^s_P$ of $\widetilde{u}_P$. Then, as $P \to \infty$,*

$$\|f * (\widetilde{u}_P - u_P)\|_{\mathrm{H}^s\left(-\frac{P}{2},\frac{P}{2}\right)} \to 0 \qquad \text{and} \qquad \|f * \widetilde{u}_P\|_{\mathrm{H}^s\left(\left\{|x|>\frac{P}{2}\right\}\right)} \to 0.$$

Proof. Note first that $f * u_P = f_P *_P u_P$ by proposition 2.1, where $f_P = \sum_{j\in\mathbb{Z}} f(\cdot + jP) \in \mathrm{L}^1_P$. As such,

$$f * (\widetilde{u}_P - u_P)(x) = \int_{-\frac{P}{2}}^{\frac{P}{2}} \left[f(x-y) - f_P(x-y)\right] \widetilde{u}_P(y)\, \mathrm{d}y = (f - f_P) *_P \widetilde{u}_P(x) \tag{26}$$

for $x \in \left(-\frac{P}{2}, \frac{P}{2}\right)$, using that $u_P \equiv \widetilde{u}_P$ there. Young's inequality then gives

$$\|f * (\widetilde{u}_P - u_P)\|_{\mathrm{L}^2\left(-\frac{P}{2},\frac{P}{2}\right)} \leqslant \|f - f_P\|_{\mathrm{L}^1\left(-\frac{P}{2},\frac{P}{2}\right)} \|\widetilde{u}_P\|_{\mathrm{L}^2} \xrightarrow[P\to\infty]{} 0,$$

because $\{\widetilde{u}_P\}_P$ is bounded in L^2 and $\|f - f_P\|_{\mathrm{L}^1\left(-\frac{P}{2},\frac{P}{2}\right)} = \|f\|_{\mathrm{L}^1\left(\left\{|x|>\frac{P}{2}\right\}\right)} \to 0$ as $P \to \infty$.

Switching to $\|f * \widetilde{u}_P\|_{\mathrm{L}^2\left(\left\{|x|>\frac{P}{2}\right\}\right)}$, put $v_j := f * \widetilde{u}_P(\cdot + jP)$ and observe from dominated convergence that

$$\|f * \widetilde{u}_P\|^2_{\mathrm{L}^2\left(\left\{|x|>\frac{P}{2}\right\}\right)} = \sum_{|j|\geqslant 1} \int_{-\frac{P}{2}}^{\frac{P}{2}} |v_j|^2\, \mathrm{d}x = \int_{-\frac{P}{2}}^{\frac{P}{2}} \sum_{|j|\geqslant 1} |v_j|^2\, \mathrm{d}x \leqslant \int_{-\frac{P}{2}}^{\frac{P}{2}} |\sum_{|j|\geqslant 1} |v_j||^2\, \mathrm{d}x,$$

where the last estimate used $\|\|_{\ell^2(\mathbb{Z}\setminus\{0\})} \leqslant \|\|_{\ell^1(\mathbb{Z}\setminus\{0\})}$. Dominated convergence once more yields

$$\begin{aligned}
\sum_{|j|\geqslant 1} |v_j(x)| &\leqslant \sum_{|j|\geqslant 1} \int_{-\frac{P}{2}}^{\frac{P}{2}} |f(x + jP - y)|\ |\widetilde{u}_P(y)|\, \mathrm{d}y \\
&= \int_{-\frac{P}{2}}^{\frac{P}{2}} \sum_{|j|\geqslant 1} |f(x + jP - y)|\ |\widetilde{u}_P(y)|\, \mathrm{d}y \\
&= (|f|_P - |f|) *_P |\widetilde{u}_P|\, (x),
\end{aligned}$$

for $x \in \left(-\frac{P}{2}, \frac{P}{2}\right)$, where $|f|_P := \sum_{j\in\mathbb{Z}} |f(\cdot + jP)|$. Introducing $|\widetilde{u}_P|_P := \sum_{j\in\mathbb{Z}} |\widetilde{u}_P(\cdot + jP)|$ also, we have

$$(|f|_P - |f|) *_P |\widetilde{u}_P| = |f| * (|\widetilde{u}_P|_P - |\widetilde{u}_P|)$$

from (26), and so in total,

$$\|f * \widetilde{u}_P\|_{\mathrm{L}^2\left(\left\{|x|>\frac{P}{2}\right\}\right)} \leqslant \||f| * (|\widetilde{u}_P| - |\widetilde{u}_P|_P)\|_{\mathrm{L}^2\left(-\frac{P}{2},\frac{P}{2}\right)}.$$

Now note that the right-hand side vanishes as $P \to \infty$ by the first result applied to $|f|$ and $|\widetilde{u}_P|$.

With case $s = 0$ established, case $s \in \mathbb{Z}_+$ follows immediately since convolution commutes with differentiation, and so by interpolation it is true for any $s \geqslant 0$. □

Proposition 4.2. *Let $\{\widetilde{u}_P\}_P \subset \mathrm{H}^s$ be a bounded family of functions with $\operatorname{supp} \widetilde{u}_P \subset \left(-\frac{P}{2}, \frac{P}{2}\right)$, and define $u_P := \sum_{j\in\mathbb{Z}} \widetilde{u}_P(\cdot + jP) \in \mathrm{H}^s_P$. Then*

$$\mathcal{A}(\widetilde{u}_P) - \mathcal{A}_P(u_P) = 0, \quad \|\mathcal{A}'(\widetilde{u}_P) - \mathcal{A}'_P(u_P)\|_{\mathrm{H}^s\left(-\frac{P}{2},\frac{P}{2}\right)} = 0 \quad \text{and} \quad \|\mathcal{A}'(\widetilde{u}_P)\|_{\mathrm{H}^s\left(\left\{|x|>\frac{P}{2}\right\}\right)} = 0$$

for $\mathcal{A} \in \{\mathcal{Q}, \mathcal{N}\}$ and any P, whereas

$$\mathcal{L}(\widetilde{u}_P) - \mathcal{L}_P(u_P) \to 0, \quad \|\mathcal{L}'(\widetilde{u}_P) - \mathcal{L}'_P(u_P)\|_{\mathrm{H}^s\left(-\frac{P}{2},\frac{P}{2}\right)} \to 0 \quad \text{and} \quad \|\mathcal{L}'(\widetilde{u}_P)\|_{\mathrm{H}^s\left(\left\{|x|>\frac{P}{2}\right\}\right)} \to 0 \tag{27}$$

as $P \to \infty$. In particular, (27) also holds for $\mathcal{E}$, $\mathcal{E}_P$.

Proof. Since

$$n(\widetilde{u}_P(x)) = \begin{cases} n(u_P(x)) & \text{if } |x| < \frac{P}{2}; \\ 0 & \text{if } |x| \geqslant \frac{P}{2}, \end{cases}$$

and similarly for N, we readily obtain the result for $\mathcal{A} = \mathcal{N}$. Case $\mathcal{A} = \mathcal{Q}$ is analogous.

As L is a convolution operator with integrable kernel, lemma 4.1 gives the last two statements in (27). Observe then also that

$$|\mathcal{L}(\widetilde{u}_P) - \mathcal{L}_P(u_P)| = \left|\frac{1}{2}\int_{-\frac{P}{2}}^{\frac{P}{2}} \widetilde{u}_P\,(L\widetilde{u}_P - Lu_P)\,\mathrm{d}x\right| \leqslant \frac{1}{2}\,\|\widetilde{u}_P\|_0\,\|L\widetilde{u}_P - Lu_P\|_{\mathrm{L}^2\left(-\frac{P}{2},\frac{P}{2}\right)} \xrightarrow[P\to\infty]{} 0.$$

□

We now define the *special minimising sequence* for $\mathcal{E}$ over U^s_μ as follows. Since $\|u^\star_P\|_{\mathrm{H}^s_P} \eqsim \mu^{\frac{1}{2}}$ holds uniformly over $P \geqslant P_\mu$ by lemma 3.6, there must—argue by contradiction—be subintervals $\Omega_P := (x_P - \ell_P, x_P + \ell_P)$ of $\left(-\frac{P}{2}, \frac{P}{2}\right)$ such that $\|u^\star_P\|_{\mathrm{H}^s(\Omega_P)} \to 0$ and $\ell_P > 0$ satisfies $\ell_P/P \to 0$ as $P \to \infty$. We then translate and smoothly truncate $u^\star_P$ into

$$\widetilde{u}_P := A_P\chi_P u^{\star\,\mathrm{T}}_P \qquad \text{with } u^{\star\,\mathrm{T}}_P := u^\star_P\left(\cdot + x_P + \frac{P}{2}\right), \tag{28}$$

where $\chi_P \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R} \to [0,1])$ equals

$$\chi_P(x) = \begin{cases} 1 & \text{if } |x| \leqslant \frac{P}{2} - \ell_P; \\ 0 & \text{if } |x| \geqslant \frac{P}{2} - \epsilon, \end{cases}$$

for some fixed $\epsilon > 0$, and $A_P := \sqrt{2\mu}/\,\|\chi_P u^{\star\,\mathrm{T}}_P\|_0$, so that

$$\widetilde{u}_P \in U^s_\mu \qquad \text{and} \qquad \operatorname{supp}\widetilde{u}_P \subseteq \left\{|x| \leqslant \frac{P}{2} - \epsilon\right\} \subset \left(-\frac{P}{2}, \frac{P}{2}\right).$$

Moreover, let $u_P := \sum_{j\in\mathbb{Z}} \widetilde{u}_P(\cdot + jP) \in \mathrm{H}^s_P$ be the periodisation of $\widetilde{u}_P$; see figure 3 for illustration.

Intuitively, the more nonlocal L is—in the sense of 'distributing mass' of $\widetilde{u}_P$ from $\left(-\frac{P}{2}, \frac{P}{2}\right)$ into its complement—the faster ℓ_P likely should grow, because $\widetilde{u}_P$ is asymptotically negligible outside of $\left\{|x| \leqslant \frac{P}{2} - \ell_P\right\}$. In our case, it suffices in fact to let $\ell_P := \ell_\star$ be constant for all $P \geqslant P_\mu$. Note that [8] used $\ell_P \sim P^{\frac{1}{4}}$.

The *special minimising sequence* $\{\widetilde{u}_k\}_{k\in\mathbb{N}}$ is now defined as $\widetilde{u}_k := \widetilde{u}_{P_k}$, where $\{P_k\}_k$ is an increasing, unbounded sequence with $P_0 \geqslant P_\mu$. And in the following results extending [8, theorem 3.8], we show that $\{\widetilde{u}_k\}_k$ does indeed minimise $\mathcal{E}$ over U^s_μ, resembles $u^\star_P$ with $\|\widetilde{u}_k\|^2_s \eqsim \mu$, and approximates the traveling-wave equation (2) in H^s. For convenience, put $\Omega^\mathrm{T}_P := \left\{\frac{P}{2} - \ell_\star < |x| < \frac{P}{2}\right\}$, so that by construction, $\|u^{\star\,\mathrm{T}}_P\|_{\mathrm{H}^s(\Omega^\mathrm{T}_P)} \to 0$ as $P \to \infty$.

Lemma 4.3. $\|u_P - u^{\star\,\mathrm{T}}_P\|_{\mathrm{H}^s_P} \to 0$ *and* $\|\mathcal{E}'_P(u_P) - \mathcal{E}'_P(u^{\star\,\mathrm{T}}_P)\|_{\mathrm{H}^s_P} \to 0$ *as* $P \to \infty$.

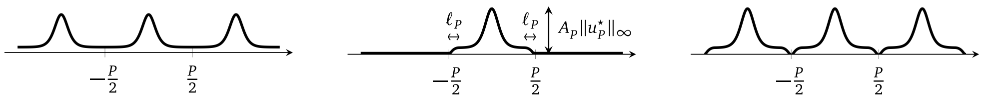
Proof. Since $A_P \to 1$, we find that

$$\|u_P - u^{\star\,\mathrm{T}}_P\|^2_{\mathrm{L}^2_P} = |A_P - 1|^2 \int_{\left(-\frac{P}{2},\frac{P}{2}\right)\setminus\Omega^\mathrm{T}_P} |u^{\star\,\mathrm{T}}_P|^2\,\mathrm{d}x + \int_{\Omega^\mathrm{T}_P} |(A_P\chi_P - 1)\,u^{\star\,\mathrm{T}}_P|^2\,\mathrm{d}x \xrightarrow[P\to\infty]{} 0,$$

because the first integral is less than $\|u^{\star\,\mathrm{T}}_P\|^2_{\mathrm{L}^2_P} = 2\mu$ whereas the latter is $\lesssim \|u^{\star\,\mathrm{T}}_P\|^2_{\mathrm{L}^2(\Omega^\mathrm{T}_P)}$, which vanishes. In a straightforward manner, this extends to H^s_P with help of (7), Leibniz' rule ($\lfloor s\rfloor$ times) plus the fact that $\|\chi^{(i)}_P\|_\infty \lesssim \ell^{-i}_\star \lesssim 1$ uniformly in P.

With the first result established, we then find that

$$\|\mathcal{L}'_P(u_P) - \mathcal{L}'_P(u^{\star\,\mathrm{T}}_P)\|_{\mathrm{H}^s_P} = \|L(u_P - u^{\star\,\mathrm{T}}_P)\|_{\mathrm{H}^s_P} \leqslant \mathfrak{m}(0)\,\|u_P - u^{\star\,\mathrm{T}}_P\|_{\mathrm{H}^s_P} \xrightarrow[P\to\infty]{} 0.$$



(a) (b) (c)

Figure 3. Illustrating the relationship between the periodic traveling waves $u_P^\star$ (real profile unknown), the truncated functions $\widetilde{u}_P$ converging to a solitary wave as $P \to \infty$, and the periodisations u_P of $\widetilde{u}_P$. (a) $u_P^{\star\,\mathrm{T}} = u_P^\star\left(\cdot + x_P + \frac{P}{2}\right)$. (b) $\widetilde{u}_P$. (c) u_P.

As regards

$$\|\mathcal{N}_P'(u_P) - \mathcal{N}_P'(u_P^{\star\,\mathrm{T}})\|_{\mathrm{H}_P^s} = \|n(u_P) - n(u_P^{\star\,\mathrm{T}})\|_{\mathrm{H}_P^s},$$

observe first that

$$\|n(u_P) - n(u_P^{\star\,\mathrm{T}})\|_{\mathrm{H}^s(-\frac{P}{2},\frac{P}{2})\setminus\Omega_P^{\mathrm{T}}} \xrightarrow[P\to\infty]{} 0,$$

essentially because $A_P \to 1$. Specifically, one may argue by the chain rule and dominated convergence—a linear combination of $n(u_P^{\star\,\mathrm{T}})$ and its $\lfloor s \rfloor$ derivatives, all of which are uniformly bounded in L_P^2, serves as a dominating function—because $(u_P - u_P^{\star\,\mathrm{T}})\,1_{(-\frac{P}{2},\frac{P}{2})\setminus\Omega_P^{\mathrm{T}}}$ and its $\lfloor s \rfloor$ derivatives converge pointwise to 0 a.e. as $P \to \infty$, and hence, also

$$\frac{\mathrm{d}^i}{\mathrm{d}x^i}\left[n(u_P(x)) - n(u_P^{\star\,\mathrm{T}}(x))\right] 1_{(-\frac{P}{2},\frac{P}{2})\setminus\Omega_P^{\mathrm{T}}}(x) \xrightarrow[P\to\infty]{\text{a.e.}} 0$$

for all $i = 0, \ldots, \lfloor s \rfloor$. Moreover,

$$\|n(u_P) - n(u_P^{\star\,\mathrm{T}})\|_{\mathrm{H}^s(\Omega_P^{\mathrm{T}})} \leqslant \|n(u_P)\|_{\mathrm{H}^s(\Omega_P^{\mathrm{T}})} + \|n(u_P^{\star\,\mathrm{T}})\|_{\mathrm{H}^s(\Omega_P^{\mathrm{T}})}.$$

On the right-hand side, the first term is controlled by the latter, rigorously due to Leibniz' rule and A_P being bounded. And, arguing similarly as (12), we also have

$$\|n(u_P^{\star\,\mathrm{T}})\|_{\mathrm{H}^s(\Omega_P^{\mathrm{T}})} \lesssim \|u_P^{\star\,\mathrm{T}}\|_{\mathrm{L}^\infty(\Omega_P^{\mathrm{T}})}^q \|u_P^{\star\,\mathrm{T}}\|_{\mathrm{H}^s(\Omega_P^{\mathrm{T}})} \xrightarrow[P\to\infty]{} 0,$$

with $\|u_P^{\star\,\mathrm{T}}\|_{\mathrm{L}^\infty(\Omega_P^{\mathrm{T}})} \leqslant \|u_P^\star\|_\infty \lesssim \|u_P^\star\|_{\mathrm{H}_P^s} < R$. Hence, $\|\mathcal{N}_P'(u_P) - \mathcal{N}_P'(u_P^{\star\,\mathrm{T}})\|_{\mathrm{H}_P^s} \to 0$ as $P \to \infty$, and the proof is complete. □

Proposition 4.4. *$\{\widetilde{u}_k\}_k$ is a minimising sequence for $\mathcal{E}$ over U_μ^s, and*

$$I_{P,\mu} \xrightarrow[P\to\infty]{} I_\mu,$$

where $I_{P,\mu} := \mathcal{E}_P(u_P^{\star\,\mathrm{T}})$ is the minimum of the periodic problem.

Proof. Writing $\mathcal{E}(\widetilde{u}_P) = (\mathcal{E}(\widetilde{u}_P) - \mathcal{E}_P(u_P)) + (\mathcal{E}_P(u_P) - \mathcal{E}_P(u_P^{\star\,\mathrm{T}})) + I_{P,\mu}$ and observing by proposition 4.2 and lemma 4.3 that

$$\mathcal{E}(\widetilde{u}_P) - \mathcal{E}_P(u_P) \xrightarrow[P\to\infty]{} 0$$

and

$$\mathcal{E}_P(u_P) - \mathcal{E}_P(u_P^{\star\,\mathrm{T}}) \leqslant \sup_{u \in U^s_{P,\mu}} \|\mathcal{E}_P'(u)\|_{\mathrm{L}^2_P} \|u_P - u_P^{\star\,\mathrm{T}}\|_{\mathrm{L}^2_P} \xrightarrow[P\to\infty]{} 0,$$

we get

$$I_\mu \leqslant \liminf_{P\to\infty} \mathcal{E}(\widetilde{u}_P) = \liminf_{P\to\infty} I_{P,\mu}.$$

Here we used that $\|\mathcal{E}_P'(u)\|_{\mathrm{L}^2_P}$ is uniformly bounded over $u \in U^s_{P,\mu}$, since $\|\mathcal{L}_P'(u)\|_{\mathrm{L}^2_P} \leqslant \mathfrak{m}(0)\, \|u\|_{\mathrm{L}^2_P} \lesssim \mu$ and

$$\|\mathcal{N}_P'(u)\|_{\mathrm{L}^2_P} = \|n(u)\|_{\mathrm{L}^2_P} \lesssim \|u\|_{\mathrm{L}^2_P} \begin{cases} \mu^{\theta q} & \text{if } s \leqslant \frac{1}{2}; \\ \|u\|_\infty^q & \text{if } s > \frac{1}{2}, \end{cases}$$

with $\|u\|_\infty \lesssim \|u\|_{\mathrm{H}^s_P} < R$.

Conversely, let $\widetilde{w} \in \mathrm{C}_\mathrm{c}^\infty$ satisfy $\mathcal{Q}(\widetilde{w}) = \mu$, and put $w_P := \sum_{j\in\mathbb{Z}} \widetilde{w}(\cdot + jP)$, so that $I_{P,\mu} \leqslant \mathcal{E}_P(w_P)$ and $\mathcal{E}_P(w_P) \to \mathcal{E}(\widetilde{w})$ as $P \to \infty$ by proposition 4.2. Then

$$\limsup_{P\to\infty} I_{P,\mu} \leqslant \mathcal{E}(\widetilde{w}),$$

and consequently also

$$\limsup_{P\to\infty} I_{P,\mu} \leqslant \inf\left\{\mathcal{E}(u) : u \in \mathrm{C}_\mathrm{c}^\infty \cap U^s_\mu\right\} = I_\mu$$

by continuity of $\mathcal{E}$ and density. □

Proposition 4.5. *The special minimising sequence $\{\widetilde{u}_k\}_k$ satisfies*

$$\sup_k \|\widetilde{u}_k\|_s \eqsim \mu^{\frac{1}{2}} \qquad \text{and} \qquad \|\mathcal{E}'(\widetilde{u}_k) + \nu_k \mathcal{Q}'(\widetilde{u}_k)\|_s \xrightarrow[k\to\infty]{} 0,$$

where $\nu_k := \nu_{P_k}$. In fact, we may assume that ν_k does not depend on k.

Proof. Theorem 1.2 and lemma 4.3 directly imply

$$\|\widetilde{u}_P\|_s \eqsim \|u_P\|_{\mathrm{H}^s_P} \leqslant \|u_P - u_P^{\star\,\mathrm{T}}\|_{\mathrm{H}^s_P} + \|u_P^{\star\,\mathrm{T}}\|_{\mathrm{H}^s_P} \lesssim \mu^{\frac{1}{2}}$$

for all $P \geqslant P_\mu$, where P_μ is replaced by a larger constant if necessary. Furthermore,

$$\|\mathcal{E}'(\widetilde{u}_P) + \nu_P \mathcal{Q}'(\widetilde{u}_P)\|_s \leqslant \underbrace{\|\mathcal{E}'(\widetilde{u}_P) + \nu_P \mathcal{Q}'(\widetilde{u}_P)\|_{\mathrm{H}^s\left(-\frac{P}{2},\frac{P}{2}\right)}}_{=:\mathfrak{I}_1} + \underbrace{\|\mathcal{E}'(\widetilde{u}_P) + \nu_P \mathcal{Q}'(\widetilde{u}_P)\|_{\mathrm{H}^s\left(\left\{|x|>\frac{P}{2}\right\}\right)}}_{=:\mathfrak{I}_2},$$

where

$$\mathfrak{I}_1 \leqslant \|\mathcal{E}'(\widetilde{u}_P) - \mathcal{E}_P'(u_P)\|_{\mathrm{H}^s\left(-\frac{P}{2},\frac{P}{2}\right)} + \nu_P \underbrace{\|\mathcal{Q}'(\widetilde{u}_P) - \mathcal{Q}_P'(u_P)\|_{\mathrm{H}^s\left(-\frac{P}{2},\frac{P}{2}\right)}}_{=0} \tag{29a}$$

$$+ \|\mathcal{E}_P'(u_P) - \mathcal{E}_P'(u_P^{\star\,\mathrm{T}})\|_{\mathrm{H}^s\left(-\frac{P}{2},\frac{P}{2}\right)} + \nu_P \|\mathcal{Q}'(u_P) - \mathcal{Q}_P'(u_P^{\star\,\mathrm{T}})\|_{\mathrm{H}^s\left(-\frac{P}{2},\frac{P}{2}\right)} \tag{29b}$$

$$+ \underbrace{\|\mathcal{E}_P'(u_P^{\star\mathrm{T}}) + \nu_P \mathcal{Q}_P'(u_P^{\star\mathrm{T}})\|_{\mathrm{H}^s(-\frac{P}{2},\frac{P}{2})}}_{=0}, \tag{29c}$$

vanishes as $P \to \infty$ due to proposition 4.2 for (29*a*); lemma 4.3 plus the fact that $\nu_P \mathcal{Q}_P'$ is a continuous linear operator on H_P^s—using that $\{\nu_P\}_P$ is bounded—for (29*b*); $u_P^{\star\mathrm{T}}$ solving (2) in H_P^s for (29*c*), and

$$\mathfrak{I}_2 = \|\mathcal{E}'(\widetilde{u}_P)\|_{\mathrm{H}^s(\{|x|>\frac{P}{2}\})} + \nu_P \underbrace{\|\mathcal{Q}'(\widetilde{u}_P)\|_{\mathrm{H}^s(\{|x|>\frac{P}{2}\})}}_{=0}$$

vanishes by proposition 4.2.

Finally, since $\{\nu_k\}_k$ is bounded, it admits a convergent subsequence, and we therefore conclude, noting that $\|\mathcal{Q}'(\widetilde{u}_k)\|_s = \|\widetilde{u}_k\|_s$ is uniformly bounded in k. □

5. Strict subadditivity and bounds in L^∞ and for the wave speed

In this section we establish that $\mu \mapsto I_\mu$ is strictly subadditive (6) on some interval $(0, \mu_\star)$ in order to rule out the case of dichotomy in Lion's principle, see section 6, and along the way also obtain improved lower bounds for the wave speed and upper bounds in L^∞. In fact, we prove that $\mu \mapsto I_\mu$ is *strictly subhomogeneous* on $(0, \mu_\star)$, meaning that

$$I_{a\mu} < aI_\mu \qquad \text{whenever } 0 < \mu < a\mu < \mu_\star, \tag{30}$$

which in turn implies strict subadditivity:

$$I_{\mu_1+\mu_2} < \left(\frac{\mu_1}{\mu_2} + 1\right) I_{\mu_2} = \frac{\mu_1}{\mu_2} I_{\frac{\mu_2}{\mu_1}\cdot\mu_1} + I_{\mu_2} \leqslant I_{\mu_1} + I_{\mu_2}.$$

Observe that if the nonlinearity n is homogeneous, then (30) follows directly from a scaling argument because $\mathcal{E}$ is homogeneous. In the presence of $\mathcal{N}_\mathrm{r}$, however, we need that $\mathcal{N}_\mathrm{r}(u) = o(\mu^{q\alpha})$. This would be guaranteed provided

$$\|u\|_\infty \lesssim \mu^\alpha \tag{31}$$

holds uniformly for a minimising sequence, which as we shall see, is the case for the special minimising sequence $\{\widetilde{u}_k\}_k$ in section 4.

As a first step toward (30) and (31), we require a μ-dependent upper bound on I_μ. Following [8], it seems natural to introduce the homogeneous, long-wave part $\mathcal{E}_\mathrm{lw} := \mathcal{L}_\mathrm{lw} + \mathcal{N}_q$ of $\mathcal{E}$, where

$$\mathcal{L}_\mathrm{lw}(u) := -\frac{\mathfrak{m}^{(2\ell)}(0)}{(2\ell)!} \int_{\mathbb{R}} \left|u^{(\ell)}\right|^2 \mathrm{d}x,$$

and consider scalings $S_\mathrm{lw} u := \mu^\alpha u(\mu^\beta \cdot)$ with $\alpha, \beta > 0$. We must have $2\alpha - \beta = 1$ in order for S_lw to map U_1^s into U_μ^s (for μ sufficiently small), whereas the condition $2\alpha + (2\ell - 1)\beta = (2 + q)\alpha - \beta$ arises naturally in balancing dispersion and nonlinear effects—that is, $\mathcal{L}_\mathrm{lw}$ and $\mathcal{N}_q$. This yields

$$\alpha = \frac{2\ell}{4\ell - q} \qquad \text{and} \qquad \beta = \frac{q}{4\ell - q}.$$

If $u \in U_1^{\ell+1}$, then a routine calculation using the scaling properties of $\mathscr{F}$ gives

$$\mathcal{E}(S_{\mathrm{lw}}u) + \mathfrak{m}(0)\mu = \mu^{1+q\alpha}\mathcal{E}_{\mathrm{lw}}(u) + o(\mu^{1+q\alpha}), \tag{32}$$

noting that the last term encaptures the effects of $\mathcal{N}_{\mathrm{r}}$ and the Taylor remainder of $\mathfrak{m}$. Note that when $s \leqslant \frac{1}{2}$, we implicitly choose μ so small that $S_{\mathrm{lw}}u$ does not see the cut-off (14) in n—this works because $\alpha > \frac{1}{2} > \theta$, where θ is as in (14). Almost verbatim from [8, corollary 3.4], we now obtain the following.

Lemma 5.1. *There exists a constant $I_\star > 0$ such that, for all sufficiently small μ,*

$$I_\mu < -\mathfrak{m}(0)\mu - I_\star\mu^{1+q\alpha} \tag{33}$$

and, uniformly over $P \geqslant P_\mu$,

$$I_{P,\mu} < -\mathfrak{m}(0)\mu - I_\star\mu^{1+q\alpha}. \tag{34}$$

Proof. Take any $\varphi \in \mathrm{C}_{\mathrm{c}}^\infty$ with $\mathcal{Q}(\varphi) = 1$ and define $u = \sqrt{\lambda}\varphi(\lambda\cdot)$. Then

$$\mathcal{E}_{\mathrm{lw}}(u) = \lambda^{2\ell}\mathcal{L}_{\mathrm{lw}}(\varphi) + \lambda^{q/2}\mathcal{N}_q(\varphi) < 0$$

for all sufficiently small λ provided that $q < 4\ell$ and $\mathcal{N}_q(\varphi) < 0$, the latter of which holds under assumption A_2 by choosing $\varphi > 0$ if $\gamma > 0$ and $\varphi < 0$ if $\gamma < 0$. Utilising (32) and proposition 4.4, this establishes both (33) and (34) for sufficiently small μ and large P_μ with $I_\star = -\frac{1}{4}\mathcal{E}_{\mathrm{lw}}(u)$, say. □

With proposition 4.5 and lemma 5.1 at hand, we now restrict our attention to 'special near-minimisers' $u \in U_\mu^s \cap \mathrm{L}^\infty$ of $\mathcal{E}$ satisfying

$$\mathcal{E}(u) < -\mathfrak{m}(0)\mu - I_\star\mu^{1+q\alpha} \qquad \text{and} \qquad \|\mathcal{E}'(u) + \nu\mathcal{Q}'(u)\|_{\mathrm{H}^s\cap\mathrm{L}^\infty} \lesssim \mu^M \tag{35}$$

for some $\nu \in \mathbb{R}$ and large number $M \geqslant \max\left\{\frac{1}{2} + q\alpha, \frac{1}{2}(1-q)^{-1}\right\}$ (with the last term present only when $q < 1$). Here $\|\cdot\|_{\mathrm{H}^s\cap\mathrm{L}^\infty} := \|\cdot\|_s + \|\cdot\|_\infty$. In close analogy to lemmas 3.5 and 3.6, with help of the identity

$$\nu u - Lu = n(u) + \mathcal{E}'(u) + \nu\mathcal{Q}'(u), \tag{36}$$

one obtains the following result.

Proposition 5.2. *The estimates $\|u\|_s \eqsim \mu^{\frac{1}{2}}$ and*

$$\nu - \mathfrak{m}(0) \gtrsim \mu^{q\alpha} + o\left(\|u\|_\infty^q\right) \tag{37}$$

hold uniformly over the set of special near minimisers (35).

Next we decompose u into its low and high-frequency components u_{lo} and u_{hi}, so that u_{lo} picks up the KdV-type behaviour of $\mathfrak{m}$ around 0 and the operator $\nu - L$ may be inverted in H^s with regards to u_{hi}. Specifically, choose $\xi_0 > 0$ in the interval around 0 where the expansion of $\mathfrak{m}$ in assumption A_1 (iii) holds such that $\mathfrak{m}(\xi) \leqslant \tau\mathfrak{m}(0)$ for $|\xi| \geqslant \xi_0 - \delta$, where $\tau \in (0, 1)$ and $0 < \delta \ll \xi_0$, and define operators $f \mapsto f_{\mathrm{lo}}$ and $f \mapsto f_{\mathrm{hi}}$ by

$$\widehat{f_{\mathrm{lo}}} = \varphi\widehat{f} \qquad \text{and} \qquad \widehat{f_{\mathrm{hi}}} = (1-\varphi)\widehat{f}, \tag{38}$$

where $\varphi \in C_c^\infty(\mathbb{R} \to [0,1])$ equals 1 for $|\xi| \leqslant \xi_0 - \delta$ and 0 for $|\xi| \geqslant \xi_0$. Now (36) splits into

$$(\nu - L)u_{\text{lo}} = n(u)_{\text{lo}} + (\mathcal{E}'(u) + \nu\mathcal{Q}'(u))_{\text{lo}} \tag{39}$$

$$(\nu - L)u_{\text{hi}} = n(u)_{\text{hi}} + (\mathcal{E}'(u) + \nu\mathcal{Q}'(u))_{\text{hi}}, \tag{40}$$

and this helps us to establish (31).

Proposition 5.3. *The estimate*

$$\|u\|_\infty \lesssim \mu^\alpha$$

holds uniformly over the set of special near minimisers (35).

Proof. Suppose first that the high-frequency component dominates in L^∞, that is, $\|u_{\text{hi}}\|_\infty \geqslant \|u_{\text{lo}}\|_\infty$, so that in particular, $\|u\|_\infty \lesssim \|u_{\text{hi}}\|_\infty$. When $s \leqslant \frac{1}{2}$, it is not clear *a priori* that $\|u_{\text{hi}}\|_\infty \lesssim \|u_{\text{hi}}\|_s$. It turns out to be almost true, as can be seen as follows. Young's inequality gives

$$\|n(u)_{\text{lo}}\|_\infty = \left\|(\mathscr{F}^{-1}\varphi) * n(u)\right\|_\infty \leqslant \left\|\mathscr{F}^{-1}\varphi\right\|_{L^1} \|n(u)\|_\infty \lesssim \|n(u)\|_\infty\,,$$

and likewise

$$\|(\mathcal{E}'(u) + \nu\mathcal{Q}'(u))_{\text{lo}}\|_\infty \lesssim \|\mathcal{E}'(u) + \nu\mathcal{Q}'(u)\|_\infty \lesssim \mu^M.$$

Hence

$$\begin{aligned}
\|n(u)_{\text{hi}}\|_\infty &\leqslant \|n(u)\|_\infty + \|n(u)_{\text{lo}}\|_\infty \\
&\lesssim \|n(u)\|_\infty \\
&\lesssim \mu^{\theta q}\|u\|_\infty \\
&\lesssim \mu^{\theta q}\|u_{\text{hi}}\|_\infty\,,
\end{aligned}$$

using (15), and similarly

$$\|(\mathcal{E}'(u) + \nu\mathcal{Q}'(u))_{\text{hi}}\|_\infty \lesssim \mu^M.$$

We find from (40) that

$$\begin{aligned}
(\nu - \mu^{\theta q})\|u_{\text{hi}}\|_\infty &\lesssim \|Lu_{\text{hi}}\|_\infty + \mu^M \\
&\lesssim \|Lu_{\text{hi}}\|_{s+|\sigma|} + \mu^M \\
&\lesssim \|u_{\text{hi}}\|_s + \mu^M,
\end{aligned}$$

and so for μ small enough it follows that $\|u_{\text{hi}}\|_\infty \lesssim \|u_{\text{hi}}\|_s + \mu^M$ when $s \leqslant \frac{1}{2}$.

Proposition 5.2 next implies that $\nu \geqslant (\tau + \epsilon)\mathfrak{m}(0)$ for all sufficiently small $\epsilon > 0$ and μ. Hence $\nu - \mathfrak{m}(\xi) \geqslant \epsilon\mathfrak{m}(0)$ on $\{|\xi| \geqslant \xi_0 - \delta\} \supseteq \operatorname{supp}(1 - \varphi)$, which means that the linear operator

$$\mathscr{F}^{-1}\left[(\nu - \mathfrak{m})^{-1}(1 - \varphi)\mathscr{F}\right] : H^s \to H^s$$

is uniformly bounded in norm over $\nu \geqslant (\tau + \epsilon)\mathfrak{m}(0)$. Consequently, (40) and the fractional chain rule (11) yield

$$\begin{aligned}\|u_{\mathrm{hi}}\|_s &\lesssim \|n(u)\|_s + \|\mathcal{E}'(u) + \nu \mathcal{Q}'(u)\|_s \\ &\lesssim \|u\|_s \|u\|_\infty^q + \mu^M \\ &\lesssim \mu^{\frac{1}{2}} \|u_{\mathrm{hi}}\|_\infty^q + \mu^M,\end{aligned}$$

and therefore also

$$\|u_{\mathrm{hi}}\|_\infty \lesssim \mu^{\frac{1}{2}} \|u_{\mathrm{hi}}\|_\infty^q + \mu^M. \tag{41}$$

Now note that

$$\|u_{\mathrm{hi}}\|_\infty \lesssim \|u_{\mathrm{hi}}\|_s + \mu^M \lesssim \mu^{\frac{1}{2}} + \mu^M.$$

If $q \geqslant 1$, then (41) shows that $\|u_{\mathrm{hi}}\|_\infty \lesssim \mu^M = o(\mu^\alpha)$ for sufficiently small μ. If $q < 1$, then (41) yields

$$\|u_{\mathrm{hi}}\|_\infty \lesssim \mu^{\frac{1}{2}(1-q)^{-1}} + \mu^M = o(\mu^\alpha)$$

for sufficiently small μ due to $M \geqslant \frac{1}{2}(1-q)^{-1} > \alpha$ and the fact that

$$x \leqslant a x^c + b \qquad \text{implies} \qquad x \lesssim a^{(1-c)^{-1}} + b \tag{42}$$

for $x, a, b > 0$ and $c \in (0,1)$. (To get (42), note first that $x \leqslant 2\max\{a x^c, b\}$. If b is the maximum, then $x \lesssim b \leqslant a^{(1-c)^{-1}} + b$. Otherwise, $x \leqslant (2a)^{(1-c)^{-1}}$, which gives $x \lesssim a^{(1-c)^{-1}} \leqslant a^{(1-c)^{-1}} + b$.)

Suppose instead that the low-frequency component dominates: $\|u_{\mathrm{lo}}\|_\infty > \|u_{\mathrm{hi}}\|_\infty$. By Maclaurin expansion of $\mathfrak{m}$ and (37) we have

$$\nu - \mathfrak{m}(\xi) > \nu - \mathfrak{m}(0) - \frac{c\mathfrak{m}^{(2\ell)}(0)}{(2\ell)!}\xi^{2\ell} > -\frac{c\mathfrak{m}^{(2\ell)}(0)}{(2\ell)!}\xi^{2\ell} + o\big(\|u\|_\infty^q\big) \tag{43}$$

for some $c > 0$ when $|\xi| < \xi_0$. Thus

$$\|u_{\mathrm{lo}}^{(2\ell)}\|_0 \lesssim \|(\nu - \mathfrak{m})\widehat{u_{\mathrm{lo}}}\|_0 + \|u_{\mathrm{lo}}\|_0\, o\big(\|u\|_\infty^q\big). \tag{44}$$

Equation (39) further gives

$$\begin{aligned}\|(\nu - \mathfrak{m})\widehat{u_{\mathrm{lo}}}\|_0 &\lesssim \|n(u)\|_0 + \|\mathcal{E}'(u) + \nu \mathcal{Q}'(u)\|_0 \\ &\lesssim \mu^{\frac{1}{2}} \|u\|_\infty^q + \mu^M \\ &\lesssim \mu^{\frac{1}{2}} \|u_{\mathrm{lo}}\|_\infty^q + \mu^M,\end{aligned} \tag{45}$$

and so we obtain

$$\|u_{\mathrm{lo}}^{(2\ell)}\|_0 \lesssim \mu^{\frac{1}{2}} \|u_{\mathrm{lo}}\|_\infty^q + \mu^M. \tag{46}$$

Gagliardo–Nirenberg's inequality then shows that

$$\|u_{\mathrm{lo}}\|_\infty \lesssim \|u_{\mathrm{lo}}\|_0^{1-\frac{1}{4\ell}} \|u_{\mathrm{lo}}^{(2\ell)}\|_0^{\frac{1}{4\ell}} \lesssim \mu^{\frac{1}{2}} \|u_{\mathrm{lo}}\|_\infty^{\frac{q}{4\ell}} + \mu^{\widetilde{M}},$$

where $\widetilde{M} := \frac{1}{2}\left(1 - \frac{1}{4\ell}\right) + \frac{M}{4\ell} \geqslant \alpha$, from which we finally deduce that

$$\|u_{\mathrm{lo}}\|_\infty \lesssim \mu^{\frac{1}{2}\left(1-\frac{q}{4\ell}\right)^{-1}} + \mu^{\widetilde{M}} = \mu^\alpha + \mu^{\widetilde{M}} \lesssim \mu^\alpha$$

with help of (42) for $c = q/4\ell$. □

Remark 5.4. Note that the estimates obtained in the case $\|u_{\mathrm{hi}}\|_\infty \geqslant \|u_{\mathrm{lo}}\|_\infty$ in the proof of proposition 5.3 are (slightly, when $q < 1$) better than in the low-frequency dominating scenario. For the actual solutions u in theorem 1.3, we must, at least when $q \geqslant 1$, have $\|u_{\mathrm{lo}}\|_\infty > \|u_{\mathrm{hi}}\|_\infty$, because $\|u\|_\infty \lesssim \|u_{\mathrm{hi}}\|_\infty \lesssim \mu^M$ with $M = \infty$ leads to the contradiction $u = 0$ in the high-frequency dominating case.

Propositions 5.2 and 5.3 now immediately imply the following result.

Corollary 5.5. *The estimate*

$$\nu - \mathfrak{m}(0) \gtrsim \mu^{q\alpha}$$

holds uniformly over the set of special near minimisers (35).

Moreover, $A_P \to 1$ as $P \to \infty$ in the construction (28) of $\widetilde{u}_k$ from $u_P^\star$, so proposition 5.3 also yields that $\|u_P^\star\|_\infty \lesssim \mu^\alpha$ uniformly in $P \geqslant P_\mu$ (possibly enlarged). But then, similarly as proposition 5.2, we get

$$\nu_P - \mathfrak{m}(0) \gtrsim \mu^{q\alpha} + o\left(\|u_P^\star\|_\infty^q\right) \gtrsim \mu^{q\alpha},$$

which leads to

$$\nu_P - \mathfrak{m}(0) \eqsim \mu^{q\alpha} \eqsim \|u_P^\star\|_\infty^q$$

with help of (25). This concludes the proof of theorem 1.2.

Lemma 5.6. *Special near minimisers satisfy*

$$\mathcal{N}(u) \lesssim -\mu^{1+q\alpha}, \qquad \mathcal{N}_q(u) \lesssim -\mu^{1+q\alpha} \qquad \text{and} \qquad \mathcal{N}_{\mathrm{r}}(u) = o(\mu^{1+q\alpha}).$$

Proof. Since $-\mathcal{L}(u) \leqslant \mathfrak{m}(0)\mu$, we find from (33) that

$$\mathcal{N}(u) = \mathcal{E}(u) - \mathcal{L}(u) \lesssim -\mu^{1+q\alpha},$$

and

$$|\mathcal{N}_{\mathrm{r}}(u)| = \int_{\mathbb{R}} o\left(|u|^{2+q}\right) \mathrm{d}x = o\left(\mu \,\|u\|_\infty^q\right) = o(\mu^{1+q\alpha})$$

by proposition 5.3. □

Proposition 5.7. *There exists $\mu_\star > 0$ such that $\mu \mapsto I_\mu$ is strictly subhomogeneous on $(0, \mu_\star)$.*

Proof. Fix $a > 1$ and note that $\|a^{\frac{1}{2}}u\|_s \lesssim \mu^{\frac{1}{2}} < R$ for any special near-minimiser u. Estimating

$$\begin{aligned} I_{a\mu} \leqslant \mathcal{E}(a^{\frac{1}{2}}u) &= \mathcal{L}(a^{\frac{1}{2}}u) + \mathcal{N}(a^{\frac{1}{2}}u) \\ &= a\mathcal{L}(u) + a^{\frac{1}{2}q}\mathcal{N}_q(u) + \mathcal{N}_{\mathrm{r}}(a^{\frac{1}{2}}u) \\ &= a\mathcal{E}(u) + \left(a^{\frac{1}{2}q} - a\right)\mathcal{N}_q(u) + \mathcal{N}_{\mathrm{r}}(a^{\frac{1}{2}}u) - a\mathcal{N}_{\mathrm{r}}(u) \\ &\leqslant a\mathcal{E}(u) - c\left(a^{\frac{1}{2}q} - a\right)\mu^{1+q\alpha} + o(\mu^{1+q\alpha}), \end{aligned}$$

where $c > 0$, we may finally choose $u = \widetilde{u}_k$ for the special minimising sequence $\{\widetilde{u}_k\}_k$ and let $k \to \infty$. It follows that

$$I_{a\mu} \leqslant aI_\mu - c\big(a^{\frac{1}{2}q} - a\big)\mu^{1+q\alpha} + o(\mu^{1+q\alpha}) < I_\mu. \qquad \square$$

6. Concentration-compactness argument for solitary waves

In this section we establish theorem 1.3 with help of Lions' concentration-compactness principle [25, lemma III.1 and remark III.3], stated in a suitable version below. Lions' principle, originally proved for H^s with $s \in \mathbb{N}$, generalises also to the fractional setting. Specifically, this concerns property (iii) under 'dichotomy', where we refer to [27, proposition 3.1 and corollary 3.2] for a derivation when $s \in (0, 1)$—which together with Lions' result extends to all $s > 0$.

Theorem 6.1 (Concentration-compactness principle). *Every bounded sequence $\{\eta_k\}_{k\in\mathbb{N}}$ in H^s satisfying*

$$\|\eta_k\|_0^2 \xrightarrow[k\to\infty]{} \lambda > 0$$

admits a subsequence, still denoted by $\{\eta_k\}_k$, for which one of the following phenomena takes place:

Concentration: There exists a sequence $\{x_k\}_k \subset \mathbb{R}$ such that

$$\inf_{k\in\mathbb{N}} \int_{B_r(x_k)} |\eta_k|^2 \,\mathrm{d}x \xrightarrow[r\to\infty]{} \lambda.$$

Vanishing: For all $r > 0$ it is true that

$$\sup_{y\in\mathbb{R}} \int_{B_r(y)} |\eta_k|^2 \,\mathrm{d}x \xrightarrow[k\to\infty]{} 0.$$

Dichotomy: There exist a value $\theta \in (0, \lambda)$, a sequence $\{x_k\}_k \subset \mathbb{R}$ and bounded sequences $\{\eta_k^{(1)}\}_k$, $\{\eta_k^{(2)}\}_k$ in H^s, such that

(i) $\left\|\eta_k - \eta_k^{(1)} - \eta_k^{(2)}\right\|_0 \to 0, \quad \left\|\eta_k^{(1)}\right\|_0^2 \to \theta, \quad$ and $\quad \left\|\eta_k^{(2)}\right\|_0^2 \to \lambda - \theta;$

(ii) $\operatorname{supp} \eta_k^{(1)} = \{|x - x_k| \leqslant A_k\}$ and $\operatorname{supp} \eta_k^{(2)} = \{|x - x_k| \geqslant B_k\}$ for $A_k, B_k \to \infty$ satisfying $\frac{A_k}{B_k} \to 0$; and

(iii) $\liminf_k \left(\left[\eta_k\right]_s^2 - \left[\eta_k^{(1)}\right]_s^2 - \left[\eta_k^{(2)}\right]_s^2\right) \geqslant 0$, *where* $[\cdot]_s^2 := \|\cdot\|_s^2 - \|\cdot\|_0^2$ *is a seminorm.*

Practically, we may rescale and assume that for all k,

$$\|\eta_k\|_0^2 = \lambda, \qquad \|\eta_k^{(1)}\|_0^2 = \theta, \qquad \text{and} \qquad \|\eta_k^{(2)}\|_0^2 = \lambda - \theta.$$

We apply theorem 6.1 to the special minimising sequence $\{\widetilde{u}_k\}_k$ for $\mathcal{E}$ over U_μ^s from section 4, dropping the tilde in $\widetilde{u}_k$ for clarity. Note that we may always assume that u_k is at least in U_μ^1, because we may let u_k be constructed from the periodic minimisers corresponding to $s = 1$, which is *a priori* best for Lipschitz nonlinearities.

Lemma 6.2. *Let $\tilde{s} \in (0, s)$ and suppose that a subsequence of $\{u_k\}_k$ 'concentrates'. Then a subsequence of $\{u_k(\cdot + x_k)\}_k$ converges in $\mathrm{H}^{\tilde{s}}$ to a minimiser of $\mathcal{E}$ over U^s_μ.*

Proof. Let $\epsilon > 0$ and define $v_k := u_k(\cdot + x_k)$, so that by assumption

$$\int_{|x|>r} v_k^2 \, \mathrm{d}x < \epsilon$$

for all sufficiently large $r > 0$, uniformly in k. Since $\{v_k\}_k \subset U^s_\mu$ is bounded in H^s, it converges weakly—up to a subsequence—in H^s to some $v \in U^s_\mu$. Moreover, boundedness implies L^2-concentration of the frequency spectrum, because

$$\int_{|\xi|>r'} |\widehat{v}_k|^2 \, \mathrm{d}\xi \leqslant \langle r' \rangle^{-2s} \|v_k\|_s^2 < \epsilon$$

for sufficiently large $r' > 0$, uniformly in k. This in turn yields equicontinuity in L^2 by estimating

$$\int_{\mathbb{R}} |v_k(\cdot + y) - v_k|^2 \, \mathrm{d}x = \int_{\mathbb{R}} \left| \left(\mathrm{e}^{\mathrm{i}y\xi} - 1\right) \widehat{v}_k(\xi) \right|^2 \mathrm{d}\xi \lesssim |y|^2 \int_{|\xi|\leqslant r'} |\widehat{v}_k|^2 \, \mathrm{d}\xi + \int_{|\xi|>r'} |\widehat{v}_k|^2 \, \mathrm{d}\xi < 2\epsilon,$$

valid uniformly for all sufficiently small y and uniformly in k. Kolmogorov–Riesz–Sudakov's compactness theorem then shows that $\{v_k\}_k$ converges, up to a subsequence, in L^2, with limit which must be v. Interpolating

$$\|w\|_{\tilde{s}} \leqslant \|w\|_0^{1-(\tilde{s}/s)} \|w\|_s^{\tilde{s}/s} \lesssim \|w\|_0^{1-(\tilde{s}/s)}$$

with $w := v_k - v$ for clarity, upgrades convergence to $\mathrm{H}^{\tilde{s}}$, and by continuity of $\mathcal{E}$ we are done. □

It remains to exclude vanishing and dichotomy. Note that there is an easily corrected flaw in the proof of vanishing in [8, lemma 5.2] (the fourth inequality); for example, one may use the Gagliardo–Nirenberg inequality as in the proof of lemma 6.3 below, or apply Hölder's inequality together with $\mathrm{H}^1 \hookrightarrow \mathrm{L}^\infty$.

Lemma 6.3. *Vanishing does not occur.*

Proof. Seeking to contradict lemma 5.6, we first observe that

$$|\mathcal{N}(u_k)| \lesssim \|u_k\|_{\mathrm{L}^{2+q}}^{2+q} \eqsim \sum_{j\in\mathbb{Z}} \|u_{k,j}\|_{\mathrm{L}^{2+q}}^{2+q},$$

where $u_{k,j} := u_k \varphi_j$ and $\{\varphi_j\}_j$ is a smooth partition of unity with $\varphi_j(x) \equiv 1$ for $|x - j| \leqslant \frac{1}{4}$ and $\operatorname{supp} \varphi_j = \left[j - \frac{3}{4}, j + \frac{3}{4}\right]$. Let v equal any $u_{k,j}$. Estimating

$$\|v\|_{\mathrm{L}^{2+q}}^{2+q} \lesssim \|v\|_s^{q/2s} \|v\|_0^{2+q-(q/2s)} \leqslant \|v\|_s^2 \|v\|_0^q,$$

by the Gagliardo–Nirenberg inequality, valid since $2s > q/(2+q)$ always holds for the chosen special minimising sequence, it then follows that

$$|\mathcal{N}(u_k)| \lesssim \left(\sup_{j\in\mathbb{Z}} \int_{\mathbb{R}} |u_{k,j}|^2 \, \mathrm{d}x \right)^{q/2} \underbrace{\sum_{j\in\mathbb{Z}} \|u_{k,j}\|_s^2}_{\eqsim \|u_k\|_s^2 < R^2} \xrightarrow[k\to\infty]{} 0$$

if $\{u_k\}_k$ vanishes, which is absurd. □

Suppose now that dichotomy occurs, so that $\{u_k\}_k$ admits decomposing sequences $\{u_k^{(1)}\}_k$, $\{u_k^{(2)}\}_k$, with

$$u_k^{(1)} \in U^s_{\frac{\theta}{2}}, \qquad u_k^{(2)} \in U^s_{\mu-\frac{\theta}{2}} \qquad \text{and} \qquad u_k^{(1)} + u_k^{(2)} \in U^s_\mu \qquad \text{for all } k; \tag{47}$$

see the proof of corollary 6.5. If separation of $u_k^{(1)}$ and $u_k^{(2)}$ leads to the energetic decomposition

$$\lim_k \left[\mathcal{E}\left(u_k^{(1)} + u_k^{(2)}\right) - \mathcal{E}\left(u_k^{(1)}\right) - \mathcal{E}\left(u_k^{(2)}\right)\right] = 0, \tag{48}$$

then subsequently

$$\lim_k \left[\mathcal{E}\left(u_k^{(1)}\right) + \mathcal{E}\left(u_k^{(2)}\right)\right] = \lim_k \mathcal{E}(u_k) = I_\mu,$$

using that

$$\left|\mathcal{E}(u_k) - \mathcal{E}\left(u_k^{(1)} + u_k^{(2)}\right)\right| \leqslant \sup_{u \in U^s_\mu} \|\mathcal{E}'(u)\|_0 \left\|u_k - u_k^{(1)} - u_k^{(2)}\right\|_0 \to 0$$

from property (i) and boundedness of $\|\mathcal{E}'(u)\|_0$ on U^s_μ. In light of strict subadditivity of $\mu \mapsto I_\mu$, we then get the contradiction

$$I_\mu < I_{\frac{\theta}{2}} + I_{\mu-\frac{\theta}{2}} \leqslant \lim_k \left[\mathcal{E}\left(u_k^{(1)}\right) + \mathcal{E}\left(u_k^{(2)}\right)\right] = I_\mu.$$

Accordingly, it suffices to establish (48). And to this end, note that since $\mathcal{N}$ is a local operator, it eventually splits as

$$\mathcal{N}\left(u_k^{(1)} + u_k^{(2)}\right) = \mathcal{N}\left(u_k^{(1)}\right) + \mathcal{N}\left(u_k^{(2)}\right),$$

whereas $\mathcal{L}$ satisfies

$$\mathcal{L}\left(u_k^{(1)} + u_k^{(2)}\right) = \mathcal{L}\left(u_k^{(1)}\right) + \mathcal{L}\left(u_k^{(2)}\right) - \left\langle Lu_k^{(1)}, u_k^{(2)}\right\rangle_0.$$

In order to show that the nonlocal interaction disappears as $k \to \infty$, one can introduce certain commutators and prove that their operator norms vanish [26]. Based on uniform continuity of $\xi \mapsto m(\xi)/\langle\xi\rangle^s$, which holds automatically in our case, this is applicable for a large class of symbols. For convolution operators, however, it seems more enlightening to work directly on the 'physical side', assuming just integrability of the kernel.

Lemma 6.4. *Let $f \in \mathrm{L}^1$ and $\{v_k\}_k, \{w_k\}_k \subset \mathrm{L}^2$ be bounded and satisfy*

$$\operatorname{supp} v_k = \{|x| \leqslant A_k\} \qquad \text{and} \qquad \operatorname{supp} w_k = \{|x| \geqslant B_k\}$$

for $0 \leqslant A_k, B_k \xrightarrow[k\to\infty]{} \infty$ with $B_k - A_k \to \infty$. Then $\langle f * v_k, w_k\rangle_0 \xrightarrow[k\to\infty]{} 0$.

Proof. An inspection of the proof of Young's inequality [14, 20.3.2 proposition] shows that

$$\left|\langle f * v_k, w_k\rangle_0\right|^2 \leqslant \|f * v_k\|^2_{\mathrm{L}^2(\operatorname{supp} w_k)} \|w_k\|_0^2 \leqslant \|f\|_{\mathrm{L}^1} \|w_k\|_0^2 \int_{\operatorname{supp} w_k} \int_{\operatorname{supp} v_k} |f(x-y)|\, |v_k(y)|^2 \,\mathrm{d}y\,\mathrm{d}x$$

with help of the Cauchy–Schwarz inequality. Changing the order of integration then yields

$$\int_{\{|x|\geqslant B_k\}}\int_{-A_k}^{A_k}|f(x-y)|\,|v_k(y)|^2\,\mathrm{d}y\,\mathrm{d}x=\int_{-A_k}^{A_k}|v_k(y)|^2\int_{\{|x+y|\geqslant B_k\}}|f(x)|\,\mathrm{d}x\,\mathrm{d}y,$$

and so, since $\{|x+y|\geqslant B_k\}\subseteq\{|x|\geqslant B_k-A_k\}$ for all $y\in[-A_k,A_k]$, we end up with

$$\left|\langle f*v_k,w_k\rangle_0\right|^2\leqslant\|f\|_{\mathrm{L}^1}\|v_k\|_0^2\|w_k\|_0^2\int_{\{|x|\geqslant B_k-A_k\}}|f(x)|\,\mathrm{d}x\xrightarrow[k\to\infty]{}0.$$

□

Corollary 6.5. *Dichotomy does not occur when $\mu\in(0,\mu_\star)$, with $\mu_\star$ as in proposition 5.7.*

Proof. Contrariwise, assume the existence of decomposing sequences $\{u_k^{(1)}\}_k$ and $\{u_k^{(2)}\}_k$ from theorem 6.1, rescaled to satisfy $\|u_k^{(1)}\|_0^2=\theta$ and $\|u_k^{(2)}\|_0^2=2\mu-\theta$ for all k. Flipping signs in property (iii) shows that

$$\limsup_k\left[u_k^{(1)}+u_k^{(2)}\right]_s^2\leqslant\limsup_k\left[u_k\right]_s^2$$

with help of the triangle inequality, which in combination with property (i) give

$$\limsup_k\left\|u_k^{(1)}+u_k^{(2)}\right\|_s\leqslant\limsup_k\|u_k\|_s<R.$$

Since $u_k^{(1)}$ and $u_k^{(2)}$ eventually separate (property (ii)), we also obtain

$$\limsup_k\left\|u_k^{(j)}\right\|_s\leqslant\limsup_k\left\|u_k^{(1)}+u_k^{(2)}\right\|_s,\qquad j=1,2,$$

and so, without loss of generality, we may assume (47).

Following the discussion prior to lemma 6.4, it remains to show that $\left\langle Lu_k^{(1)},u_k^{(2)}\right\rangle_0\xrightarrow[k\to\infty]{}0$. But this is immediate from lemma 6.4 and property (ii) after spatial translations $x\mapsto x-x_k$.

□

We conclude from lemmas 6.2, 6.3 and corollary 6.5 that $\mathcal{E}$ has a minimiser over U_μ^s. Combined with the estimates in propositions 5.2, 5.3 and corollary 5.5, we deduce, similarly as in the periodic case, that

$$\nu-\mathfrak{m}(0)\eqsim\mu^{q\alpha}\eqsim\|u\|_\infty^q.$$

This completes the proof of theorem 1.3.

7. Additional features

As a consequence of the analysis in the proof of proposition 5.3, we obtain a nonexistence result for small solitary waves in $\mathrm{H}^s\cap\mathrm{L}^\infty$ when the nonlinearity is too strong, which demonstrates the optimality of $q<4\ell$ in assumptions A_1 and A_2.

Theorem 7.1 (Nonexistence). *Let $s > 0$ be as in (3). If $q \geqslant 4\ell$ in assumptions A_1 and A_2, then there are no nonzero solutions $u \in \mathrm{H}^s \cap \mathrm{L}^\infty$ of equation (2) with speed ν satisfying $\nu - \mathfrak{m}(0) \gtrsim -\|u\|_\infty^q$ provided $\|u\|_s$ and $\|u\|_\infty$ are sufficiently small. In particular, this excludes small solitary waves in $\mathrm{H}^s \cap \mathrm{L}^\infty$ with supercritical speed when $q \geqslant 4\ell$.*

Proof. We split u into u_{lo} and u_{hi} exactly as in (38), so that (39) and (40) hold with $\mathcal{E}'(u) + \nu \mathcal{Q}'(u) \equiv 0$. Closely following the proof of proposition 5.3, suppose first that $\|u_{\mathrm{hi}}\|_\infty \geqslant \|u_{\mathrm{lo}}\|_\infty$. Without repeating the calculations we then obtain from (41) that

$$\|u_{\mathrm{hi}}\|_\infty \lesssim \|u\|_s \|u_{\mathrm{hi}}\|_\infty^q,$$

provided $\|u\|_\infty$ is sufficiently small. Since $q \geqslant 4\ell \geqslant 1$ in this scenario, we deduce that $u = 0$ if $\|u\|_s$ is sufficiently small.

Suppose instead that $\|u_{\mathrm{lo}}\|_\infty > \|u_{\mathrm{hi}}\|_\infty$. Due to $\nu - \mathfrak{m}(0) \gtrsim -\|u\|_\infty^q$, estimate (43) now becomes

$$\nu - \mathfrak{m}(\xi) + \frac{c\mathfrak{m}^{(2\ell)}(0)}{(2\ell)!}\xi^{2\ell} \gtrsim -\|u\|_\infty^q$$

for some $c > 0$ when $|\xi| < \xi_0$. By redoing estimates (44)–(46) with the appropriate modifications, one obtains

$$\|u_{\mathrm{lo}}^{(2\ell)}\|_0 \lesssim \|u\|_0 \|u\|_\infty^q \leqslant \|u\|_s \|u\|_\infty^q$$

for sufficiently small $\|u\|_\infty$, which implies that

$$\|u\|_\infty \lesssim \|u_{\mathrm{lo}}\|_\infty \lesssim \|u_{\mathrm{lo}}\|_0^{1-\frac{1}{4\ell}} \|u_{\mathrm{lo}}^{(2\ell)}\|_0^{\frac{1}{4\ell}} \lesssim \|u\|_s \|u\|_\infty^{\frac{q}{4\ell}}$$

by the Gagliardo–Nirenberg inequality. If $\|u\|_\infty \leqslant 1$, then for sufficiently small $\|u\|_s$ we conclude that $u = 0$ is the only possibility when $q \geqslant 4\ell$. $\square$

We finally establish with a basic argument that bounded solutions of (2) with supercritical speed are either waves of elevation or waves of depression in the special case when the convolution kernel K is nonnegative. This result is already known for the Whitham equation [12, corollary 4.4].

Theorem 7.2 (Sign of wave profile). *Suppose K is nonnegative and let $u \neq 0$ be a bounded solution of (2) with supercritical wave speed $\nu > \mathfrak{m}(0)$. If n is homogeneous, then u has a one-sided profile with $\operatorname{sgn} u = \operatorname{sgn} \gamma$ almost everywhere, where γ is as in assumption A_2. The same conclusion also holds for inhomogeneous n when $\|u\|_\infty$ is sufficiently small.*

Proof. It suffices to consider $n_q(u) = \gamma |u|^{1+q}$, as the sign-dependent case $n_q(u) = \gamma u |u|^q$ follows from $u \mapsto -u$ and arguing with the (essential) supremum of u instead of the infimum.

If $\gamma > 0$, suppose that $u_* := \operatorname{ess\,inf} u < 0$. Let $\epsilon > 0$ and—being slightly informal—let x_ϵ be any point such that $u(x_\epsilon) < u_* + \epsilon$. We find that $Lu(x_\epsilon) \geqslant L(u_*) = \widehat{K}(0)\, u_* = \mathfrak{m}(0)\, u_*$ because $K \geqslant 0$, and so

$$n(u(x_\epsilon)) = \nu u(x_\epsilon) - Lu(x_\epsilon) \leqslant (\nu - \mathfrak{m}(0))\, u_* + \epsilon\nu. \tag{49}$$

Since $\nu > \mathfrak{m}(0)$, the right-hand side in (49) becomes negative for ϵ sufficiently small. This is a contradiction if $n = n_q$, because $n_q(u(x_\epsilon)) > 0$, and also in the inhomogeneous case provided $\|u\|_\infty$ is sufficiently small.

If $\gamma < 0$, one may argue analogously with $\operatorname{ess\,sup} u$. $\square$

Acknowledgments

The author acknowledges the support by research Grant No. 250070 from The Research Council of Norway. Valuable suggestions from two anonymous referees that helped to improve the paper are gratefully acknowledged.

Appendix. Sufficient conditions for symbols to be in the Wiener class W_0

Sufficient conditions for symmetric symbols $\mathfrak{m}$ with weak decay to be in the Wiener class W_0 of functions with absolutely integrable inverse Fourier transform are for instance

- ⋆ $\mathfrak{m} \in AC_{\text{loc}}$ satisfying $|\mathfrak{m}(\xi)| \lesssim \langle\xi\rangle^{\sigma}$ and $|\mathfrak{m}'(\xi)| \lesssim \langle\xi\rangle^{\sigma'}$ almost everywhere for $\sigma < 0$ and $\sigma' \in \mathbb{R}$ with $\sigma + \sigma' < -1$; see [23, theorem 1] and [24, corollary 2.2]. This directly extends the S^{σ}_{∞} case. Here AC_{loc} is the space of locally absolutely continuous functions;
- ⋆ $\mathfrak{m} \in AC_{\text{loc}}$ satisfying $\mathfrak{m} \in L^{p_1}$ and $\mathfrak{m}' \in L^{p_2}$ for $1 \leqslant p_1 < \infty$, $1 < p_2 < \infty$ fulfilling $\frac{1}{p_1} + \frac{1}{p_2} > 1$ [21, theorem 1.1]; and
- ⋆ $\mathfrak{m}$ being *quasi-convex* on $(0, \infty)$, meaning that $\mathfrak{m} \in AC_{\text{loc}}$ with $\mathfrak{m}'$ locally of bounded variation and $\int_0^\infty \xi\, |\mathrm{d}\mathfrak{m}'(\xi)| < \infty$ (Riemann–Stieltjes integral). Example: $\mathfrak{m}(\xi) = (1 + \log(1 + |\xi|))^{-\alpha}$, for any $\alpha > 0$; see [4, theorem 6.3.11] and [22, theorem 5.4].

ORCID iDs

Fredrik Hildrum https://orcid.org/0000-0002-9905-1670

References

[1] Arnesen M N 2019 Non-uniform dependence on initial data for equations of Whitham type *Adv. Differ. Equ.* **24** 257–82 (https://projecteuclid.org/euclid.ade/1554256825)
[2] Borluk H, Kalisch H and Nicholls D P 2016 A numerical study of the Whitham equation as a model for steady surface water waves *J. Comput. Appl. Math.* **296** 293–302
[3] Bruell G, Ehrnström M and Pei L 2017 Symmetry and decay of traveling wave solutions to the Whitham equation *J. Differ. Equ.* **262** 4232–54
[4] Butzer P L and Nesse R J 1971 *Fourier Analysis and Approximation: One Dimensional Theory* (*Mathematische Reihe* vol 1) (Basel: Birkhäuser) (https://doi.org/10.1007/978-3-0348-7448-9)
[5] Carter J D 2018 Bidirectional Whitham equations as models of waves on shallow water *Wave Motion* **82** 51–61
[6] Christ F M and Weinstein M I 1991 Dispersion of small amplitude solutions of the generalized Korteweg–de Vries equation *J. Funct. Anal.* **100** 87–109
[7] Ehrnström M, Escher J and Pei L 2015 A note on the local well-posedness for the Whitham equation *Elliptic and Parabolic Equations* (*Springer Proceedings in Mathematics and Statistics* vol 119) (New York: Springer) pp 63–75
[8] Ehrnström M, Groves M D and Wahlén E 2012 On the existence and stability of solitary-wave solutions to a class of evolution equations of Whitham type *Nonlinearity* **25** 2903
[9] Ehrnström M and Kalisch H 2009 Traveling waves for the Whitham equation *Differ. Integral Equ.* **22** 1193–210 (https://projecteuclid.org/euclid.die/1356019412)
[10] Ehrnström M and Kalisch H 2013 Global bifurcation for the Whitham equation *Math. Modelling Nat. Phenom.* **8** 13–30
[11] Ehrnström M and Pei L 2018 Classical well-posedness in dispersive equations with nonlinearities of mild regularity, and a composition theorem in Besov spaces *J. Evol. Equ.* **18** 1147–71

[12] Ehrnström M and Wahlén E 2019 On Whitham's conjecture of a highest cusped wave for a nonlocal dispersive equation *Ann. l'Institut Henri Poincaré* C **36** 1603–37
[13] Enciso A, Gómez-Serrano J and Vergara B 2018 Convexity of Whitham's highest cusped wave (arXiv:1810.10935)
[14] Gasquet C and Witomski P 1998 *Fourier Analysis and Applications: Filtering, Numerical Computation, Wavelets* (*Texts in Applied Mathematics*) (New York: Springer)
[15] Groves M D and Wahlén E 2011 On the existence and conditional energetic stability of solitary gravity-capillary surface waves on deep water *J. Math. Fluid Mech.* **13** 593–627
[16] Hur V M 2017 Wave breaking in the Whitham equation *Adv. Math.* **317** 410–37
[17] Hur V M and Johnson M A 2015 Modulational instability in the Whitham equation for water waves *Stud. Appl. Math.* **134** 120–43
[18] Kalisch H, Moldabayev D and Verdier O 2017 A numerical study of nonlinear dispersive wave models with SpecTraVVave *Electron. J. Differ. Equ.* **2017** 1–23
[19] Klein C, Linares F, Pilod D and Saut J C 2018 On Whitham and related equations *Stud. Appl. Math.* **140** 133–77
[20] Lannes D 2013 *The Water Waves Problem: Mathematical Analysis and Asymptotics* (*Mathematical Surveys and Monographs* vol 188) (Providence, RI: American Mathematical Society)
[21] Liflyand E R 2010 On absolute convergence of Fourier integrals *Real Anal. Exch.* **36** 353–60
[22] Liflyand E R, Samko S and Trigub R M 2012 The Wiener algebra of absolutely convergent Fourier integrals: an overview *Anal. Math. Phys.* **2** 1–68
[23] Liflyand E R and Trigub R M 2010 On the representation of a function as an absolutely convergent Fourier integral *Proc. Steklov Inst. Math.* **269** 146–59
[24] Liflyand E R and Trigub R M 2011 Conditions for the absolute convergence of Fourier integrals *J. Approx. Theory* **163** 438–59
[25] Lions P L 1984 The concentration-compactness principle in the calculus of variations. The locally compact case. I *Ann. Inst. H. Poincaré* **1** 109–45
[26] Mæhlen O 2019 Solitary waves for weakly dispersive equations with inhomogeneous nonlinearities (arXiv:1902.05372)
[27] Parini E and Salort A 2019 Compactness and dichotomy in nonlocal shape optimization *Math. Nachr.* (accepted)
[28] Runst T and Sickel W 1996 *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations* (*De Gruyter Series in Nonlinear Analysis and Applications* vol 3) (Berlin: Walter de Gruyter)
[29] Sanford N, Kodama K, Carter J D and Kalisch H 2014 Stability of traveling wave solutions to the Whitham equation *Phys. Lett.* A **378** 2100–7
[30] Stefanov A and Wright J D 2018 Small amplitude traveling waves in the full-dispersion Whitham equation *J. Dyn. Differ. Equ.* (https://doi.org/10.1007/s10884-018-9713-8)
[31] Struwe M 2008 *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems* (*A Series of Modern Surveys in Mathematics*) 4th edn (New York: Springer)
[32] Trillo S, Klein M, Clauss G and Onorato M 2016 Observation of dispersive shock waves developing from initial depressions in shallow water *Physica* D **333** 276–84
[33] Whitham G B 1967 Variational methods and applications to water waves *Proc. R. Soc.* A **299** 6–25
[34] Whitham G B 1974 *Linear and Nonlinear Waves* (*Pure and Applied Mathematics*) (New York: Wiley)